



A study in quantum hydrodynamics
by Herbert A Zook

A THESIS Submitted to the Graduate Faculty in Partial fulfillment of the requirements for the degree
of Master of Science in Physics
Montana State University
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Abstract:

The experimental investigations of the properties of liquid helium II are reviewed. A chronological review of the attempts to explain theoretically the properties of liquid helium II is then given. Particular attention is paid to Landau's first paper on a quantum theory of liquid helium II. A short review of the boundary layer problem in hydrodynamics is then given with attention focused on the question of transition from laminar to turbulent flow in the boundary layer. In part IV, the roton concept postulated by Landau is extended and applied to the boundary layer problem. The velocity dependence of Emmons' g factor is derived and compared with the experimentally determined velocity dependence of g . Finally, a short discussion on the nature of rotons and phonons is given.

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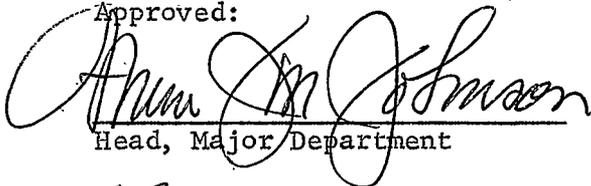
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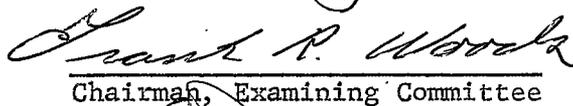
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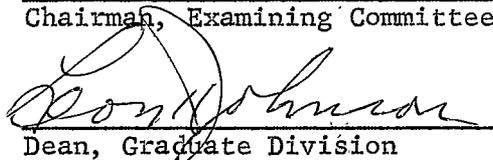
at

Montana State College

Approved:


Head, Major Department


Chairman, Examining Committee


Dean, Graduate Division

Bozeman, Montana
May, 1960

N378
Z 765
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ABSTRACT

The experimental investigations of the properties of liquid helium II are reviewed. A chronological review of the attempts to explain theoretically the properties of liquid helium II is then given. Particular attention is paid to Landau's first paper on a quantum theory of liquid helium II. A short review of the boundary layer problem in hydrodynamics is then given with attention focused on the question of transition from laminar to turbulent flow in the boundary layer. In part IV, the roton concept postulated by Landau is extended and applied to the boundary layer problem. The velocity dependence of Emmons' g factor is derived and compared with the experimentally determined velocity dependence of g . Finally, a short discussion on the nature of rotons and phonons is given.

I. INTRODUCTION

The classical dynamical theory of gases has proved to be an extremely powerful tool in that it has shown itself capable of predicting quantitatively the macroscopic behavior of large numbers of particles, namely, a gas.

When the density of points in phase space becomes large, however, a number of discrepancies arise between theory and observation. The introduction of quantum statistics has resolved most of these discrepancies. Notable successes are the Bose-Einstein statistics of a photon gas and the Fermi-Dirac theory of electrons. As would be expected, quantum statistics gives the same results as classical statistics when the density of points in phase space is small. Mathematical and conceptual difficulties appear in either the quantum or classical statistics when the gas being investigated is not in equilibrium.

For the case of a gas in motion, one generally applies the well developed theory of hydrodynamics. In particular, for the case of a gas flowing by an object, hydrodynamics specializes in an area known as boundary layer theory. A largely unresolved problem in boundary layer theory is the treatment of transition from laminar to turbulent flow in the boundary layer.

The purpose of this paper will be to demonstrate that quantum mechanical phenomena are probably responsible for the nature of this transition.

To do this, we first review the experimental and theoretical developments in liquid helium. In view of its success, we devote considerable attention to the quantum hydrodynamical theory of liquid helium II. Landau's

first paper on this subject is, in particular, studied in some detail.

A short review of the developments in boundary layer theory is then given. Special attention is given to the treatments on transition in the boundary layer.

We then extend the concept of rotons and phonons as developed by Landau so as to apply to gases at ordinary temperatures. On this basis we are able to derive the velocity dependence of Emmons' g factor.

A complete quantum mechanical description of transition in the boundary layer has not yet been found. This paper, it is hoped, should provide a starting point for such a description.

II. LIQUID HELIUM

A. Experimental Phenomena in Liquid Helium. A research group at the University of Leiden (Holland), under the direction of Kammerlingh-Onnes, first liquified helium gas in 1908. Measurements were difficult at these low temperatures and it took a number of years to obtain accurate measurements of the properties of liquid helium--such as its density, surface tension, dielectric constant, etc.

In 1927 Keesom and Wolke¹¹ made a rather startling discovery, which is best told in their own words:

"When measuring the dielectric constant of liquid helium between the boiling point and 1.9 °K, we observed that at a temperature almost corresponding with that at which Kammerlingh-Onnes had found a maximum in the density curve, the dielectric constant shows a sudden jump, or at least a jump made in a very small temperature region. The thought suggested itself that, at that temperature, the liquid helium transforms into another phase, liquid as well. If we call the liquid stable at the higher temperatures "liquid helium I", the liquid stable at the lower temperatures "liquid helium II", then the dielectric constant of liquid helium I should be greater than that of liquid helium II."

Thus it was first recognized that helium had two liquid phases depending upon whether the temperature was above or below a certain critical temperature T_λ , now called the " λ -point".

Recent measurements have shown that helium first liquified under its own vapor pressure at about 4.2 °K, exists as liquid helium I down to $T_\lambda = 2.19$ °K, at which point it undergoes a transition to a quite different state

of matter called liquid helium II. Unless it is subjected to a pressure of about 25 atmospheres, helium II remains a liquid down to the lowest temperatures so far achieved--about 0.003 °K.

The above phenomena motivated an intense experimental study into the properties of liquid helium II and by 1940 nearly everything of importance that is now known experimentally about liquid helium II had already been observed.

Although an interesting subject in itself, we shall refrain from reviewing the voluminous experimental work on helium II and present only some of the experimental results that have aided the theoretical understanding of helium II.

At the λ -point, many of the physical properties of liquid helium undergo an abrupt and often discontinuous change. Figures 1, 2, and 3 show the experimental changes of three of these quantities near the λ -point.

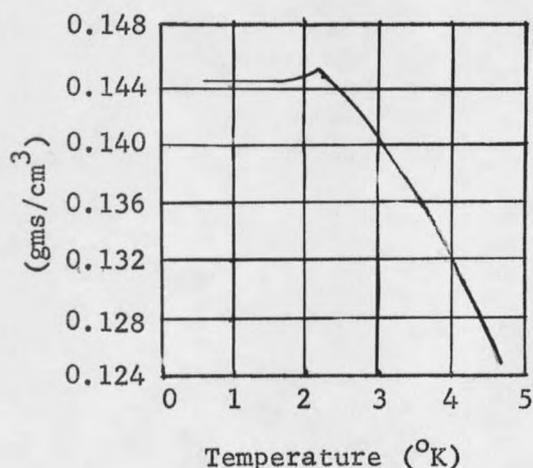


Figure 1. Density of liquid helium under its own vapor pressure

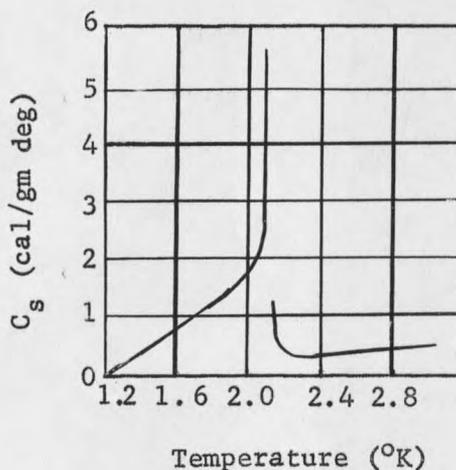


Figure 2. Specific heat of liquid helium under its own vapor pressure.

For example, the heat conductivity increases by a factor of nearly 10^6 with little more than a degree drop in temperature; the viscosity is enormously less below the λ -point; the specific heat depends upon the temperature in an entirely different manner below the λ -point than it does above it. Other quantities that show anomalies are: compressibility, sound velocity, surface tension, density, internal energy, and latent heats of fusion and vaporization.

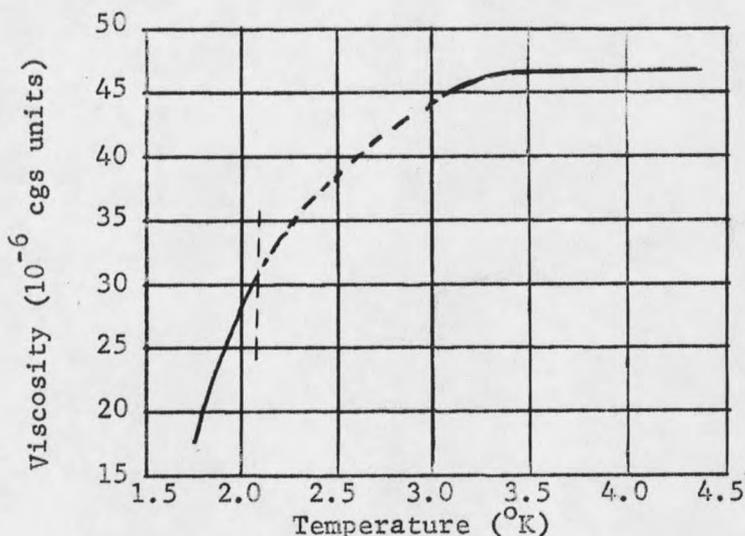


Figure 3. Viscosity of liquid helium I and II.

B. Theoretical Developments in Liquid Helium II up to 1941. The theoretical significance of the properties of liquid helium II just discussed lies in the fact that they seemed not to obey the hydrodynamical or thermodynamical equations that had been so successful in the classical treatments of other fluids.

This led to the suspicion that quantum phenomena might be playing a major role. F. London then made the fruitful observation in 1938 that the condensation phenomenon of an ideal Bose-Einstein gas gives a specific heat

versus temperature curve similar to that of liquid helium II (see Figure 4). A number of papers then appeared as a consequence of this suggestion¹⁵. L. Tisza²³ further suggested that superfluidity, heat capacity, and the "fountain effect" are properties that a partly condensed Bose-Einstein gas might be expected to have. Therefore a partly condensed Bose-Einstein gas should behave similarly to liquid helium II near the λ -point. He later published a detailed treatment of his "two fluid" hypothesis.*

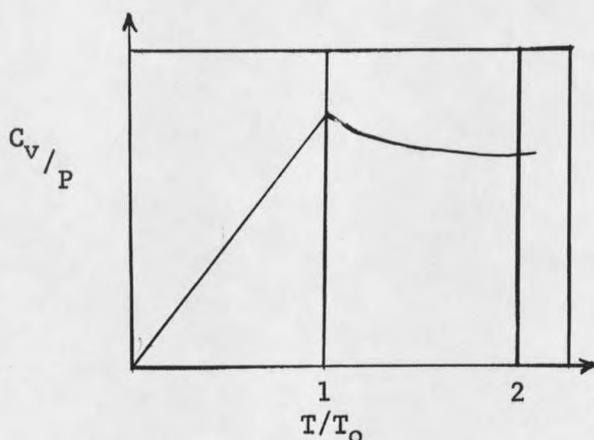


Figure 4. The condensation phenomenon of a Bose-Einstein gas.

In 1941, a brilliant paper by Landau appeared, which gave a fairly quantitative explanation of the properties of liquid helium II. His idea was that, instead of looking at the behavior of the individual atoms, one should look at the cooperative behavior of the liquid as a whole and enumerate the quantum states of this behavior¹³.

As we shall make use of his theory later on it will be of value to us to study it in some detail.

* A survey of this and other aspects of the theory up to the end of 1955 is given by J. Wilks, Repts. Prog. in Phys. 20, 38 (1957).

C. Landau's Theory of Liquid Helium. The first problem undertaken by Landau was to determine the form of the operators for the physical observables. Following Landau, we now treat this problem.

An arbitrary system of interacting particles (a liquid) can be described in classical theory by means of the density ρ and the flow of mass \vec{J} , which are determined in the following manner. Let \vec{R} be the radius vector of an arbitrary point in space and \vec{r}_α -- the radius vector of a particle with a mass m_α . Then ρ is taken as

$$\rho = \sum_{\alpha} m_{\alpha} \delta(\vec{r}_{\alpha} - \vec{R}),$$

where $\delta(\vec{r}_{\alpha} - \vec{R}) = \delta(x_{\alpha} - X) \delta(y_{\alpha} - Y) \delta(z_{\alpha} - Z)$ and the δ 's are the Dirac δ -functions. The summation is extended over all the particles in the system. We note that $\int \rho dV$ gives the total mass of the system. Similarly, the density \vec{J} of the flow of mass is given by

$$\vec{J} = \sum_{\alpha} m_{\alpha} \vec{v}_{\alpha} \delta(\vec{r}_{\alpha} - \vec{R}) = \sum_{\alpha} \vec{P}_{\alpha} \delta(\vec{r}_{\alpha} - \vec{R})$$

(\vec{v}_{α} , \vec{P}_{α} are the velocity and momentum of the particle m_{α}). It is to be emphasized that this description treats the microscopic motion and not the macroscopic motion of the liquid.

He then finds that the corresponding quantum operators for the above classical observables are

$$\hat{\rho} = \sum_{\alpha} m_{\alpha} \delta(\vec{r}_{\alpha} - \vec{R}) \tag{2.1}$$

and

$$\hat{J} = \frac{1}{2} \sum_{\alpha} \vec{P}_{\alpha} \delta(\vec{r}_{\alpha} - \vec{R}) + \delta(\vec{r}_{\alpha} - \vec{R}) \vec{P}_{\alpha} \tag{2.2}$$

where $P_\alpha = \frac{\hbar}{i} \nabla_\alpha$, the familiar momentum operator for a single particle. \hat{J} has been symmetrized with respect to \hat{P} and δ so as to be Hermitean. It is easily shown for a one particle system that these operators give the expected results.

To derive some of the other operators that we shall need, we first determine the commutation rules.

Remembering that these operators are always to be thought of as operating on some wave function and in view of the δ -function, we have

$$\hat{P}_1 \hat{P}_2 - \hat{P}_2 \hat{P}_1 = 0 \quad (2.3)$$

where $\hat{P}_1 = \hat{P}(\vec{R}_1)$, $\hat{P}_2 = \hat{P}(\vec{R}_2)$.

For the sake of brevity let us consider only one term from each of the sums (2.1) and (2.2) when determining the commutation rules, as the operators corresponding to different particles commute with each other. To determine the commutation of \hat{P} with \hat{J} for a single particle we write

$$\hat{J}_1 \hat{P}_2 - \hat{P}_2 \hat{J}_1 = \frac{m\hbar}{2i} \left\{ \begin{array}{l} [\nabla \delta(\vec{r} - \vec{R}_1) + \delta(\vec{r} - \vec{R}_1) \nabla] \delta(\vec{r} - \vec{R}_2) - \\ - \delta(\vec{r} - \vec{R}_2) [\nabla \delta(\vec{r} - \vec{R}_1) + \delta(\vec{r} - \vec{R}_1) \nabla] \end{array} \right\}.$$

We simplify the right hand expression by noticing that operators of the form

$$\delta(\vec{r} - \vec{R}_1) \nabla \delta(\vec{r} - \vec{R}_2)$$

can be transformed in the following way:

$$\delta(\vec{r}-\vec{R}_1)\nabla(\vec{r}-\vec{R}_2) = \delta(\vec{r}-\vec{R}_1)\delta(\vec{r}-\vec{R}_2)\nabla + \delta(\vec{r}-\vec{R}_1)[\nabla\delta(\vec{r}-\vec{R}_2)]$$

where the brackets indicate that ∇ operates only on $\delta(\vec{r}-\vec{R}_2)$ and not on the wave function directly. Similarly

$$\nabla\delta(\vec{r}-\vec{R}_1)\delta(\vec{r}-\vec{R}_2) = \delta(\vec{r}-\vec{R}_2)\nabla\delta(\vec{r}-\vec{R}_1) + \delta(\vec{r}-\vec{R}_1)[\nabla\delta(\vec{r}-\vec{R}_2)].$$

The result is

$$\hat{J}_1\hat{p}_2 - \hat{p}_2\hat{J}_1 = \frac{\hbar}{i}m\delta(\vec{r}-\vec{R}_1)[\nabla\delta(\vec{r}-\vec{R}_2)].$$

and in view of the factor $\delta(\vec{r}-\vec{R}_1)$ we may write this as

$$\hat{J}_1\hat{p}_2 - \hat{p}_2\hat{J}_1 = \frac{m\hbar}{i}\delta(\vec{r}-\vec{R}_1)[\nabla\delta(\vec{R}_1-\vec{R}_2)]$$

where ∇ is the derivative with respect to the \vec{R}_1 coordinates. For an arbitrary system we may now write

$$\hat{J}_1\hat{p}_2 - \hat{p}_2\hat{J}_1 = \frac{\hbar}{i}\hat{p}[\nabla\delta(\vec{R}_1-\vec{R}_2)] \quad (2.4)$$

(We note that it makes no difference whether we write \hat{p}_1 or \hat{p}_2 on the right hand side of the equation).

Let us now introduce the operator \vec{v} of the velocity of the liquid according to

$$\hat{J} = \frac{1}{2}(\hat{p}\vec{v} + \vec{v}\hat{p}) \quad (2.5)$$

$$\vec{v} = \frac{1}{2}\left(\frac{1}{\hat{p}}\hat{J} + \hat{J}\frac{1}{\hat{p}}\right) \quad (2.6)$$

as it will be more convenient to use the operator \hat{v} instead of the flow \hat{v} .

Other commutation rules are found to be

$$\hat{v}_i \hat{p}_2 - \hat{p}_2 \hat{v}_i = \frac{\hbar}{i} [\nabla S(\vec{R}_1 - \vec{R}_2)] \quad (2.7)$$

$$\hat{v}_{ii} \hat{v}_{2k} - \hat{v}_{2k} \hat{v}_{ii} = \frac{\hbar}{i} \delta(\vec{R}_1 - \vec{R}_2) \frac{1}{\rho} (\widehat{\text{rot } \vec{v}})_{ik} \quad (2.8)$$

where $(\widehat{\text{rot } \vec{v}})_{ik}$ denotes the difference

$$\frac{\partial v_k}{\partial x_i} - \frac{\partial v_i}{\partial x_k}$$

we also find that

$$(\widehat{\text{rot } \vec{v}})_i \hat{p}_2 - \hat{p}_2 (\widehat{\text{rot } \vec{v}})_i = 0. \quad (2.9)$$

We now show that if we apply the formulae obtained to the macroscopic motion of the liquid we get, as we would expect, the usual hydrodynamical equations of motion. The energy of a unit volume of a classical liquid is

$$\rho \frac{v^2}{2} + \rho \mathcal{E}(\rho)$$

where $\mathcal{E}(\rho)$ is the internal energy of a unit mass of the liquid. We suppose that the energy \mathcal{E} depends only on the density ρ of the liquid; this corresponds to the macroscopic character of the consideration and is connected with a statistical averaging. For a microscopic investigation this supposition is, of course, invalid.

The corresponding quantum operator is

$$\frac{\hat{v} \rho \hat{v}}{2} + \rho \mathcal{E}(\rho)$$

(The operators for ρ and \mathcal{E} corresponds to multiplication by these quantities.)

The Hamiltonian \hat{H} of the liquid is an integral over the volume

$$\hat{H} = \int \left\{ \frac{\hat{v} \rho \hat{v}}{2} + \rho \mathcal{E}(\rho) \right\} dV \quad (2.10)$$

where dV integrates only over the coordinate \vec{R} in the δ -function and not over the particle coordinates \vec{r} .

For the derivative of the density ρ with respect to time one has the operator

$$\hat{\rho} = \frac{i}{\hbar} (\hat{H} \rho - \rho \hat{H}).$$

Denote temporarily the coordinates of the point at which ρ is taken by the index 1, and the coordinates of the variable point in the region of integration in (2.10) by the index 2. Then

$$\hat{\rho}_1 = \int \left\{ [\hat{v}_2 \rho_2 \hat{v}_2 \rho_1 - \rho_1 \hat{v}_2 \rho_2 \hat{v}_2] + [\rho_2 \mathcal{E}(\rho_2) \rho_1 - \rho_1 \rho_2 \mathcal{E}(\rho_2)] \right\} dV_2$$

In view of (2.3) the second term under the integral sign vanishes, and the first can be written

$$\frac{1}{2} [\hat{v}_2 \rho_2 (\hat{v}_2 \rho_1 - \rho_1 \hat{v}_2) + (\hat{v}_2 \rho_1 - \rho_1 \hat{v}_2) \rho_2 \hat{v}_2]$$

or, by introducing (2.7):

$$\frac{\hbar}{2i} [\nabla \delta(\vec{R}_2 - \vec{R}_1)] (\hat{v}_2 \rho_2 + \rho_2 \hat{v}_2) = \frac{\hbar}{i} [\nabla \delta(\vec{R}_2 - \vec{R}_1)] \hat{J}_2.$$

In this way

$$\hat{p}_1 = \int [\nabla \delta(\vec{R}_1 - \vec{R}_2)] \hat{J}_2 dV_2 = - \int \delta(\vec{R}_2 - \vec{R}_1) \text{div} \hat{J}_2 dV_2 = - \text{div} \hat{J}_1$$

i.e. we come to the continuity equation in operational form:

$$\frac{\partial \hat{p}}{\partial t} + \text{div}(\rho \hat{v} + \hat{v} \rho) = 0. \quad (2.11)$$

In a similar way the derivative

$$\hat{v} = \frac{i}{\hbar} (\hat{H} \hat{v} - \hat{v} \hat{H})$$

can be calculated, which brings us to the equation

$$\frac{\partial \hat{v}_i}{\partial t} + \frac{1}{2} (\hat{v}_k \frac{\partial \hat{v}_i}{\partial x_k} + \frac{\partial \hat{v}_i}{\partial x_k} \hat{v}_k) = - \frac{1}{\rho} \frac{\partial}{\partial x_i} \left(\frac{dE}{d\rho} \right) \quad (2.12)$$

i.e. Euler's equation in an operational form ($\frac{dE}{d\rho}$ is the pressure P of the liquid).

We note that the equations (2.11), (2.12) are less general than the commutation rules (2.3)-(2.9), which are also valid for an exact microscopical investigation of the liquid.

These equations would be extremely difficult, if not impossible, to solve directly for the equations of motion of the fluid, however, an analysis of the operators that we have developed provides the clue to the spectrum of eigenvalues that we should expect. We give Landau's analysis of these operators.

In the classical hydrodynamics of ideal liquids it is shown that if, at a certain moment of time, the motion is irrotational ($\text{rot } \vec{v} = 0$) in the whole volume of the liquid, it will be irrotational for all other moments of time (Lagrange's theorem)*. It appears that this classical theorem finds its analogy in quantum mechanics. According to the commutation rules (2.9), $\text{rot } \hat{v}$ always commutes with the density ρ . The components of $\text{rot } \hat{v}$, however do not commute, generally speaking, either with each other or with the velocity \hat{v} (when the operation rot is applied to the equation (2.8) the right hand side does not vanish). Therefore $\text{rot } \hat{v}$ does not, generally speaking, commute with the Hamiltonian, i.e. is not conserved.

An exception is the case when over the whole volume $\text{rot } \hat{v} = 0$. In this case we have zero in the right hand side of (2.8) and $\text{rot } \hat{v}$ commutes with ρ and \hat{v} and, therefore, also with the Hamiltonian.

In this way $\text{rot } \hat{v}$ is conserved if it is zero. In other words a quantum liquid always possesses stationary states in which $\text{rot } \hat{v}$ equals zero over the whole volume of the liquid. Such a state might be called by analogy to classical hydrodynamics, a state of irrotational motion of the liquid.

Concerning these results an analogy can be made with the angular momentum \vec{M} in quantum mechanics**. The commutation of two components of \vec{M} with each other leads to the third component of \vec{M} , with the result that all of the components of \vec{M} commute with each other if they are all equal to zero. It is also known that there exist no states with an infinitely

* e.g. see Leigh Page, Introduction to Theoretical Physics, Third edition, (D. Van Nostrand Co., 1952) p. 248.

** A discussion of this is given by Landau and Lifschitz, Quantum Mechanics (Pergamon Press LTD., 1958) pp. 78-80.

small angular momentum, its first non-zero eigenvalues are of the order of \hbar . This is a consequence of the fact that the commutation rules are inhomogeneous--their left-hand sides are quadratic in \vec{M} and the right hand sides are linear.

A similar statement can be advanced concerning $\text{rot}\vec{v}$ in quantum hydrodynamics. Namely, no states can exist in which $\text{rot}\vec{v}$ would be non-zero, but arbitrarily small over the whole volume of the liquid. In other words, between the states of potential ($\text{rot}\vec{v} = 0$) and vortex ($\text{rot}\vec{v} \neq 0$) motions of a quantum liquid there is no continuous transition.

From this the principle features of the energy spectrum of a liquid directly follow. The presence of a gap between the states of the potential and vortex motions means that between the lowest energy levels of vortex and potential motions a certain finite energy level must exist. As to the question which of these two lies lower apparently both cases are logically possible. It will be shown below that we get the phenomenon of superfluidity if we suppose that the normal level of the potential motions lies lower than the normal level of the vortex motions. Hence we must suppose that this very case exists in liquid helium. It must be remarked, however, that, as only one quantum liquid exists, liquid helium, the question as to whether such a distribution of the levels and hence the property of superfluidity is a general property of a quantum liquid cannot be solved experimentally.

This brings us to the following picture of the distribution of the levels in the energy spectrum of liquid helium (it must be emphasized that we do not here refer to the levels for single helium atoms but to the levels

corresponding to the states of the whole liquid). This spectrum is made up of two superimposed continuous spectra. One of them corresponds to the potential motions and the other to vortex motions. The lowest level of the vortex spectrum is situated above the lowest level of the potential spectrum, this latter level being the normal, unexcited state of a liquid; the energy interval between these two levels we denote by Δ .

The value of the energy gap Δ cannot be calculated exactly. Its order of magnitude is

$$\Delta \sim \hbar^2 \left(\frac{\rho^2}{m^3} \right)^{1/3} \quad (2.13)$$

(m being the mass of a helium atom and ρ the density of the liquid). This is the only quantity with the dimensions of energy which can be built up from m , ρ and \hbar .

Landau then supposes that every weakly excited state of the fluid can be regarded as an aggregate of "elementary excitations"--the elementary excitations of the potential motion he calls "phonons" and the elementary excitations of the vortex motion he calls "rotons". He further supposes that if the roton and phonon density is not too great that they may be treated as an ideal gas which obeys Bose statistics.

To find the temperature dependence of the heat capacity of the fluid it is supposed that the energy added is used only to excite potential and vortex motions in the fluid. Thus, additional energy must either create new phonons and rotons or it must raise the old ones to new levels of excitation. From this it is found, that the heat capacity of liquid helium consists of two parts--a phonon part (C_{ph}) and a roton part (C_r). The

phonon part turns out to be the Debye heat capacity

$$C_{Ph} = 4.4 \times 10^{-3} T^3 \text{ cal/deg.}$$

The roton part of the heat capacity (C_r) has, owing to the presence of the energy gap Δ between the normal and vortex levels, an exponential temperature dependence. If we consider the roton density as small then Bose-Einstein statistics reduce to Maxwell-Boltzmann statistics (for sufficiently low temperatures this should be true). The free energy of the gas with an undetermined number of particles (the number of rotons is itself a temperature function determined from the condition of the minimum of the free energy) is

$$F_r = -KT V \int e^{-\frac{\epsilon}{KT}} \frac{d\vec{T}_p}{(2\pi\hbar)^3}$$

$$(d\vec{T}_p = dP_x dP_y dP_z)$$

(V is the volume, K = Boltzmann's constant, T is the absolute temperature and $\epsilon = \Delta + \frac{\vec{P}^2}{2\mu}$ where \vec{P} and μ are the roton momentum and effective mass respectively). Below we shall refer the free energy to 1 gm of helium corresponding to which we shall put $V = \frac{1}{\rho}$ (ρ being the helium density), so that

$$F_r = -\frac{KT}{\rho(2\pi\hbar)^3} \int e^{-\frac{\epsilon}{KT}} d\vec{T}_p \quad (2.14)$$

the number of rotons is

$$N_r = \frac{1}{\rho(2\pi\hbar)^3} \int e^{-\frac{\epsilon}{KT}} d\vec{T}_p \quad (2.15)$$

and using $\epsilon = \Delta + \frac{p^2}{2\mu}$ we get

$$F_r = - \frac{kT(\mu kT)}{\rho(2\pi\hbar^2)^{3/2}} e^{-\frac{\Delta}{kT}} \quad (2.16)$$

$$N_r = \frac{1}{\rho} \left(\frac{\mu kT}{2\pi\hbar^2} \right)^{3/2} e^{-\frac{\Delta}{kT}} \quad (2.17)$$

The entropy is

$$S_r = - \frac{\partial F_r}{\partial T} = \frac{1}{\rho} \left(\frac{\mu k}{2\pi\hbar^2} \right)^{3/2} T^{1/2} \Delta e^{-\frac{\Delta}{kT}} \left(1 + \frac{5kT}{2\Delta} \right) \quad (2.18)$$

and the roton heat capacity is

$$C_r = T \frac{\partial S_r}{\partial T} = \frac{1}{\rho} \left(\frac{\mu}{2\pi\hbar^2} \right)^{3/2} \frac{k^{1/2} \Delta^2}{T^{1/2}} e^{-\frac{\Delta}{kT}} \left[1 + 3 \frac{kT}{\Delta} + \frac{15}{4} \left(\frac{kT}{\Delta} \right)^2 \right] \quad (2.19)$$

Using experimental data¹⁰ for specific heat one finds for μ and Δ the values:

$$\frac{\Delta}{k} = 8-9 \text{ } ^\circ\text{K}, \quad \mu \approx 8 m_{\text{He}}$$

(m_{He} is the mass of a helium atom).

From this foundation Landau then considers some conceptual experiments from which he is able to derive the minimum velocities with which the fluid must flow past an object in order to excite the phonon (elementary sound excitation) and roton (elementary vortex excitation) spectra.

Although the derivations are elementary, they are lengthy, so we shall only state the results. To excite the first level of the phonon spectrum he finds that the relative object-to-fluid velocity must exceed that of the velocity of sound in the liquid $|\vec{v}| > C$, where C is the velocity of

