

Embeddings of inverse limit spaces
by James Lee Kassebaum

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematical Sciences
Montana State University
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Abstract:
Barge and Martin have shown that the inverse limit of any interval map may be topologically realized as an attractor of a planar homeomorphism H in such a way that H restricted to the attractor is conjugate to the induced homeomorphism on the inverse limit. They extend the interval map to a near homeomorphism of a disk containing the interval and use a theorem of Morton Brown to obtain their result. We present similar results for degree -1, 0, or 1 circle maps and give necessary conditions for a map to be extended to a near homeomorphism. Then, it is shown that for a given finite, connected, planar, graph, G, containing a branch point, the set of surjective, continuous maps of G which cannot be extended to a near homeomorphism on any neighborhood of G are open and dense in the set of all such maps, with the C^0-topology. However, an example of such a map for which the inverse limit can be embedded as an attractor in the plane is given. Next, we prove the inverse limit of any n-od ($n \geq 3$) can be embedded as an attractor in 3-space. We then give necessary and sufficient conditions for the inverse limit of a finite, connected, planar, graph with a surjective bonding map to be chainable. Since any chainable continuum may be embedded in the plane, such an inverse limit is planar. Finally, we give an example of a chainable continuum for which there exists a homeomorphism which is not essentially extendable.

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A thesis submitted in partial fulfillment
of the requirements for the degree

of

Doctor of Philosophy

in

Mathematical Sciences

MONTANA STATE UNIVERSITY
Bozeman, Montana

October 1995

D378
K1543

APPROVAL

of a thesis submitted by

James Lee Kassebaum

This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

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Date 10/31/95

ACKNOWLEDGEMENTS

I would first like to thank Jesus Christ for His leading me to Montana State University to work with such an impressive, yet accessible, group of mathematicians. I count myself most fortunate to have had the opportunity to work with Marcy Barge, without whose help the following work would not have been possible. Detailing his efforts in full would require more paper than the thesis itself. The members of my Graduate Committee, Marcy Barge, Russell Walker, Richard Swanson, Gary Bogar, Richard Gillette, and Ray Ditterline, also deserve much praise for their efforts. Finally, I would like to thank my parents, Tim Snyder, Mary Oman, and the many friends that I have made while at Montana State University for their support.

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ABSTRACT

Barge and Martin have shown that the inverse limit of any interval map may be topologically realized as an attractor of a planar homeomorphism H in such a way that H restricted to the attractor is conjugate to the induced homeomorphism on the inverse limit. They extend the interval map to a near homeomorphism of a disk containing the interval and use a theorem of Morton Brown to obtain their result. We present similar results for degree -1, 0, or 1 circle maps and give necessary conditions for a map to be extended to a near homeomorphism. Then, it is shown that for a given finite, connected, planar, graph, G, containing a branch point, the set of surjective, continuous maps of G which cannot be extended to a near homeomorphism on any neighborhood of G are open and dense in the set of all such maps, with the C^0-topology. However, an example of such a map for which the inverse limit can be embedded as an attractor in the plane is given. Next, we prove the inverse limit of any n-od ($n \geq 3$) can be embedded as an attractor in 3-space. We then give necessary and sufficient conditions for the inverse limit of a finite, connected, planar, graph with a surjective bonding map to be chainable. Since any chainable continuum may be embedded in the plane, such an inverse limit is planar. Finally, we give an example of a chainable continuum for which there exists a homeomorphism which is not essentially extendable.

CHAPTER 1

INTRODUCTION

We begin our discussion with the following preliminary definitions:

Definition 1.1 *Let Z be a compact metric space and $g : Z \to Z$ be continuous. Define $\varprojlim g = \{(z_0, z_1, \ldots) | \, z_i \in Z \text{ and } g(z_{i+1}) = z_i\}$. Then $\varprojlim g$ is a subset of the product space $\{(z_0, z_1, \ldots) | z_i \in Z\}$ with the product topology.*

Definition 1.2 *Let $h : X \to X$ be a map and $U \subset X$ be relatively compact with $h(cl(U)) \subset U$. Then $\Lambda = \cap_{n \geq 0} h^n(U)$ is called an attractor for h.*

Inverse limit spaces provide interesting examples of continua. The pseudo-arc, a homogeneous indecomposable plane continuum (Bing, [5]), is homeomorphic to the inverse limit of an interval map ([11], [18]). Much attention has been paid to the embeddability of inverse limit spaces in Euclidean space. Ralph Bennett ([4]), R. H. Bing ([6], [7]), J. R. Isbell ([12]), James Keesling ([13], [14]), Michael C. McCord ([17]), and David C. Wilson ([14]) have all published on this topic. In addition, inverse limit spaces have been used to model attractors of homeomorphisms. For example, the solenoid (Smale, [23]) is homeomorphic to the inverse limit space of a circle map.

Definition 1.3 *Define the* **induced homeomorphism** $\hat{g} : \varprojlim g \to \varprojlim g$ *by* $\hat{g}((z_0, z_1, \ldots)) = (g(z_0), z_0, z_1, \ldots)$.

Definition 1.4 *Let G be a finite, connected, planar graph and $f : G \to G$ be a continuous map. If there exists a homeomorphism $H : X \to X$ such that K is an attractor of H, K is homeomorphic to $\varprojlim f$, and $H|_K$ is conjugate to the shift homeomorphism, $\hat{f}$, on $\varprojlim f$, then we will say that $\varprojlim f$ can be embedded as an attractor in X.*

Barge and Martin ([1]) have shown that for <u>any</u> interval map f, $\varprojlim f$ can be embedded as an attractor in $\mathbb{R}^2$. They have thus insured a plethora of interesting planar attractors. In the proof of their result, they use the following definitions and theorem of Morton Brown ([10]).

Definition 1.5 *(Brown [10]) Let X be a metric space. A map $f : X \to X$ is a* **near homeomorphism** *if for any $\epsilon > 0$ there is a homeomorphism H_ϵ of X onto itself such that $\|H_\epsilon - f\| < \epsilon$.*

Definition 1.6 *(Brown [10]) Let X_i be a sequence of compact metric spaces, and for $i \geq 2$ let f_i map X_i into X_{i-1}. Let $f_{ij} = f_{i+1}f_{i+2}\ldots f_j$ and $f_{ii} = 1$. If z is a point of X_i then z_i will always denote the ith coordinate of z. Hence $z = (z_i)$. Then the subspace $S = \{z \in \prod_1^\infty X_i | f_{ij}(z_j) = z_i\}$ of $\prod_1^\infty X_i$ is the limit space of the inverse system (X_i, f_i).*

Theorem 1.1 *(Brown [10]) Let $S = Lim(X_i, f_i)$ where the X_i are all homeomorphic to a compact metric space X, and for all i, f_i is a near homeomorphism. Then S is homeomorphic to X.*

The method of Barge and Martin is to extend the interval map f to F–a near homeomorphism of a disk containing the interval in its interior–and use the theorem of Morton Brown.

In chapter 2, we obtain similar results for degree -1,0, or 1 circle maps. Also, two results on extendibility to a near homeomorphism are given. In chapter 3, we show that if f is a continuous, onto map of a triod which can be approximated by planar embeddings, $\varprojlim f$ is at least planar. Now let G be a finite, connected planar graph. If G contains a branch point, the set of continuous maps, f, of G onto itself such that f cannot be approximated by planar embeddings is open and dense in the set of all maps of G onto itself. Since such a map f cannot be extended to a near homeomorphism on any planar neighborhood of G, the Barge-Martin construction cannot be used to embed $\varprojlim f$ as an attractor in $\mathbb{R}^2$. However, an example of a map $f : G \to G$ (onto, with G a triod) which cannot be approximated by planar embeddings, but for which $\varprojlim f$ can be embedded as an attractor in $\mathbb{R}^2$ is supplied. Sanford and Walker have shown that "a positive entropy map of the product of a Cantor Set and an arc (which covers a homeomorphism) cannot be "embedded" into a near homeomorphism of the 2-disk. Thus, a theorem of M. Brown cannot be used to embed the induced shift map on the corresponding inverse limit space into a 2-disk homeomorphism" ([22]).

In chapter 4, the inverse limit of any n-od ($n \geq 3$) is embedded as an attractor in $\mathbb{R}^3$ and in chapter 5 we return to the question of which maps $f : G \to G$ have the property that $\varprojlim f$ can be embedded as an attractor in the plane. As a first step toward that goal, we give necessary and sufficient conditions that $\varprojlim f$, $f : G \to G(onto)$, be snake-like or chainable–a condition insuring that $\varprojlim f$ is planar. Finally, after a few examples, propositions, and a corollary, we turn to the question of extending a homeomorphism of a given planar continuum to a homeomorphism of all of $\mathbb{R}^2$. For, once we embed $\varprojlim f$ in the plane, we need to be able to extend $\hat{f}$ to a homeomor-

phism of all of $\mathbb{R}^2$ in order to embed $\varprojlim f$ as an attractor in the plane. Brechner ([9]) has shown that there exists a chainable continuum $M \subseteq S^2$ and a homeomorphism $h : M \to M(onto)$ such that h is not "essentially extendable". We give a similar example of a chainable continuum K and a homeomorphism $f : K \to K(onto)$ such that f is not essentially extendable to all of $\mathbb{R}^2$. Barge and Walker also have such an example ([2]). Brechner ([8]) and Wayne Lewis ([15]) both have results that imply that if K is any chainable continuum and $f : K \to K(onto)$ is any homeomorphism, f is essentially extendable to all of $\mathbb{R}^3$. We then close with an example that shows that even if $\varprojlim f$ is planar the shift homeomorphism, $\hat{f}$, may not be essentially extendable to $\mathbb{R}^2$.

CHAPTER 2

EMBEDDING CIRCLE-LIKE CONTINUA AS ATTRACTORS

In the following, it is shown that for a circle map f, $\varprojlim f$ can be embedded as an attractor in the plane for $\deg(f) = 0$ and the plane less the origin for $\deg(f) \in \{-1, 1\}$. Dr. James Keesling has pointed out that Michael C. McCord has shown the following related result ([17], pg. 323):

Proposition 2.1 *(McCord [17]) A circle-like continuum X can be embedded in the plane if and only if $H^1(X) = 0$ or $H^1(X) \approx \mathbb{Z}$.*

($H^1(X)$ denotes the 1-dimensional Čech cohomology $\mathbb{Z}$-module of X.)

Finally, two theorems on extending a map to a near homeomorphism are given. The importance of these results lies in the fact that in order to embed $\varprojlim f$ as an attractor in a space Y, we first need to extend $f : S^1 \to S^1$ to a near homeomorphism of Y.

Definition 2.7 *Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and $A = \{z \in \mathbb{C} \mid \frac{1}{2} \leq |z| \leq \frac{3}{2}\}$ and define the* **covering maps:**
$\pi : \mathbb{R} \to S^1$ *by $\pi(x) = e^{2\pi ix}$ and $P : \mathbb{R} \times [\frac{1}{2}, \frac{3}{2}] \to A$ by $P(x, y) = ye^{2\pi ix}$.*

Definition 2.8 *Let $f : S^1 \to S^1$ be continuous. A* **lift** *of f is a continuous function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that $\pi \circ \tilde{f} = f \circ \pi$.*

Definition 2.9 *$\deg(f) = \tilde{f}(1) - \tilde{f}(0)$, where $\tilde{f}$ is a lift of f.*

Lemma 2.1 *Let $f : S^1 \to S^1$ be continuous with $\deg(f) = 0$. Then there is an interval I and a map $g : I \to I$ so that $\varprojlim f$ is homeomorphic to $\varprojlim g$.*

Proof: Let $\tilde{f}$ be a lift of f, $a = \min_{t\in\mathbb{R}}\{\tilde{f}(t)\}$, $b = max\{a+1, \max_{t\in\mathbb{R}}\{\tilde{f}(t)\}\}$, and $g = \tilde{f}|_{[a,b]}$. Note that since $deg(f) = 0$, $\tilde{f}(x+k) = \tilde{f}(x)$ for each real number x and integer k. Define $\Pi : \varprojlim g \to \varprojlim f$ via $\Pi(t_0, t_1, \ldots) = (\pi(t_0), \pi(t_1), \ldots)$, where π is the covering map $\pi(x) = e^{2\pi ix}$. $(t_0, t_1, \ldots) \in \varprojlim g \Rightarrow \tilde{f}(t_{n+1}) = t_n$

$\Rightarrow \pi \circ \tilde{f}(t_{n+1}) = \pi(t_n) \Rightarrow f(\pi(t_{n+1})) = \pi(t_n)$, so $(\pi(t_0), \pi(t_1), \ldots) \in \varprojlim f$.

Π **is one-to-one:** Suppose that $\Pi(t_0, t_1, \ldots) = \Pi(t'_0, t'_1, \ldots)$. For each positive integer n, $\pi(t_{n+1}) = \pi(t'_{n+1}) \Rightarrow t_{n+1} = t'_{n+1} + k$, k an integer.

So, $\tilde{f}(t_{n+1}) = t_n = \tilde{f}(t'_{n+1} + k) = \tilde{f}(t'_{n+1}) = t'_n$. Therefore, $t_n = t'_n$, for each positive integer n.

Π **is onto:** Let $(s_0, s_1, \ldots) \in \varprojlim f$ and $T_n = \{\pi^{-1}(s_n)\} \cap [a, b]$, for each positive integer n. Then for $n \geq 1$, $\tilde{f}(T_n) \in T_{n-1}$, since $f(s_n) = s_{n-1}$, $\pi \circ \tilde{f} = f \circ \pi$, and $\{\pi^{-1}(s_n)\} = \{t_n + k | t_n \in [0, 1], k \text{ an integer}\}$. Construct an element of $\varprojlim g$ by letting $t_0 = \tilde{f}(T_1), t_1 = \tilde{f}(T_2), t_2 = \tilde{f}(T_3), \ldots$. Then $(t_0, t_1, \ldots) \in \varprojlim g$ and $\Pi((t_0, t_1, \ldots)) = (s_0, s_1, \ldots)$.

Π **is continuous:** Since the metric topology on $\varprojlim f$ is equivalent to the product topology and for each $\tilde{t} \in \varprojlim f$, $\Pi(\tilde{t}) = (\pi(\pi_j(\tilde{t})))_{j\in N}$ (π_j is projection in the jth coordinate), Π is continuous.

$\varprojlim g$ is compact and $\varprojlim f$ is Hausdorff, so Π is a homeomorphism.

PTL

Lemma 2.2 *($\varprojlim g, \tilde{g}$) is conjugate to ($\varprojlim f, \tilde{f}$).*

Proof: $\Pi : \varprojlim g \to \varprojlim f$ is a homeomorphism and $(\tilde{f} \circ \Pi)((t_0, t_1, \ldots)) = \tilde{f}(e^{2\pi i t_0}, e^{2\pi i t_1}, \ldots)$ $= (e^{2\pi i \tilde{f}(t_0)}, e^{2\pi i t_0}, e^{2\pi i t_1}, \ldots) = \Pi(\tilde{f}(t_0), t_0, t_1, \ldots) = (\Pi \circ \tilde{g})((t_0, t_1, \ldots))$

PTL

Theorem 2.2 *If $deg(f) = 0$, then $\varprojlim f$ can be embedded as an attractor in the plane.*

Proof: By a result of Barge and Martin [1], there is an embedding h from $\varprojlim g$ into the plane, $\mathbb{R}^2$, a homeomorphism $k = h \circ \tilde{g} \circ h^{-1} : h(\varprojlim g) \to h(\varprojlim g)$, and a homeomorphism, K of the plane, so that $K|_{h(\varprojlim g)} = k$, and $h(\varprojlim g)$ is a global attractor for K. So, we have the following commutative diagram:

$$\begin{array}{ccccccc}
\varprojlim f & \xrightarrow{\Pi^{-1}} & \varprojlim g & \xrightarrow{h} & h(\varprojlim g) & \xrightarrow{i} & \mathbb{R}^2 \\
\downarrow \tilde{f} & & \downarrow \tilde{g} & & \downarrow k & & \downarrow K \\
\varprojlim f & \xrightarrow{\Pi^{-1}} & \varprojlim g & \xrightarrow{h} & h(\varprojlim g) & \xrightarrow{i} & \mathbb{R}^2
\end{array}$$

where i is the inclusion map. $(h \circ \Pi^{-1})(\varprojlim f) = h(\varprojlim g)$ embeds $\varprojlim f$ in $\mathbb{R}^2$ and $h \circ \Pi^{-1} \circ \tilde{f} \circ \Pi \circ h^{-1} = h \circ \tilde{g} \circ h^{-1} = k$, so that the shift map on this embedding is just k and the result follows.

PTL

Theorem 2.3 *Let $f : S^1 \to S^1$ be continuous with $deg(f) \in \{-1, 1\}$. Then $\varprojlim f$ can be embedded as an attractor in the plane less the origin.*

Proof: Let $\tilde{f} : \mathbb{R} \to \mathbb{R}$ be a lift of f. Since $\tilde{f}(x+1) = \tilde{f}(x) + 1$ for every $x \in \mathbb{R}$, there is a $b > 0$ so that $graph(\tilde{f}) \subset S = \{(x, y) \in \mathbb{R}^2 | y \in \mathbb{R}, y - b < x < y + b\}$. For every $y \in \mathbb{R}$, define a homeomorphism $h_y : [y - b, y + b] \times \{y\} \to [y - b, y + b] \times \{y\}$ by

$$h_y(x, y) = \begin{cases} (\tilde{f}(y) + \frac{\tilde{f}(y)-y+b}{b}(x - y), y), x \in [y - b, y] \\ (\tilde{f}(y) + \frac{y+b-\tilde{f}(y)}{b}(x - y), y), x \in [y, y + b] \end{cases}$$

Now, define $H : cl(S) \to cl(S)$ by $H(x, y) = h_y(x, y)$. Note that $H(x, x) = (\tilde{f}(x), x)$, for every $x \in \mathbb{R}$. H is one-to-one and onto, since each h_y is. Let $\tilde{S} = \{(x, y) \in \mathbb{R}^2 | y \in [0, 1], y - b \leq x \leq y + b\}$. Then, let $\tilde{H} = H|_{\tilde{S}}$.

$\tilde{H}$ is continuous: Fix $(x, y) \in \tilde{S}$. Since any two norms on a finite dimensional vector space are equivalent, we'll use the one-norm, i.e., $\|(x, y)\| = |x| + |y|$.

Case I: $x < y$. If $\|(x, y) - (x_1, y_1)\|$ is small, $x_1 < y_1$ and $\|\tilde{H}(x, y) - \tilde{H}(x_1, y_1)\|$
$= \|(\tilde{f}(y) + \frac{\tilde{f}(y)-y+b}{b}(x - y), y) - (\tilde{f}(y_1) + \frac{\tilde{f}(y_1)-y_1+b}{b}(x_1 - y_1), y_1)\| \leq |\tilde{f}(y) - \tilde{f}(y_1)|$
$+ |\frac{y(x-y)-b(x-y)-\tilde{f}(y)(x-y)-y_1(x_1-y_1)+b(x_1-y_1)+\tilde{f}(y_1)(x_1-y_1)}{b}| + |y - y_1| \leq |\tilde{f}(y) - \tilde{f}(y_1)|$
$+ \frac{1}{b}|y(x-y)-y_1(x_1-y_1)|+|(x-y)-(x_1-y_1)|+\frac{1}{b}|\tilde{f}(y_1)(x_1-y_1)-\tilde{f}(y)(x-y)|+|y-y_1|$
$\leq |\tilde{f}(y) - \tilde{f}(y_1)| + \frac{1}{b}(|y||(x - y) - (x_1 - y_1)| + |x_1 - y_1||y - y_1|) + |(x - y) - (x_1 - y_1)|$
$+ \frac{1}{b}(|\tilde{f}(y_1)||(x_1 - y_1) - (x - y)| + |x - y||\tilde{f}(y_1) - \tilde{f}(y)|) + |y - y_1|$ which can be made small by the continuity of $\tilde{f}$ and boundedness of the region $cl(S)$.

Case II: $x = y$ and without loss of generality $x_1 > y_1$. $\|\tilde{H}(x, y) - \tilde{H}(x_1, y_1)\|$
$= \|(\tilde{f}(y), y) - (\tilde{f}(y_1) + \frac{y_1+b-\tilde{f}(y_1)}{b}(x_1 - y_1), y_1)\| \leq |\tilde{f}(y) - \tilde{f}(y_1)| + |\frac{y_1+b-\tilde{f}(y_1)}{b}||x_1 - y_1|$
$+ |y - y_1|$ is small if $\|(x, y) - (x_1, y_1)\|$ is small.

Case III: $x > y$.

Similar to case I.

Therefore, $\tilde{H} : \tilde{S} \to \tilde{S}$ is continuous. Since $\tilde{H}$ is bijective continuous and $\tilde{S}$ is closed and compact, $\tilde{H}$ is a homeomorphism. Since $cl(S)$ can be partitioned into closed sets $\{\tilde{S}_i\}$ so that $H|_{\tilde{S}_i}$ is a homeomorphism for every i, $H : cl(S) \to cl(S)$ is a homeomorphism by the pasting lemma ([19], pg. 108).

$\underline{deg(f) = 1}$: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation on $\mathbb{R}^2$ induced by the matrix $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Also, let $\alpha : \mathbb{R} \times [\frac{1}{2}, \frac{3}{2}] \to \mathbb{R} \times [-b, b]$ be the homeomorphism defined by $\alpha(x, y) = (x, 2b(y - 1))$. Define $K : A \to A$ by $K(ye^{2\pi ix})$
$= (P \circ \alpha^{-1} \circ T^{-1} \circ H \circ T \circ \alpha \circ P^{-1})(ye^{2\pi ix})$.

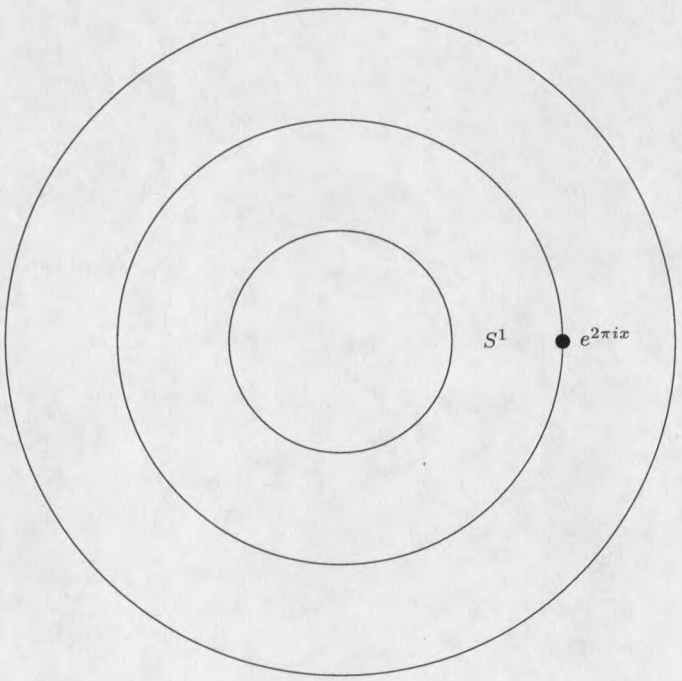


Figure 1: A

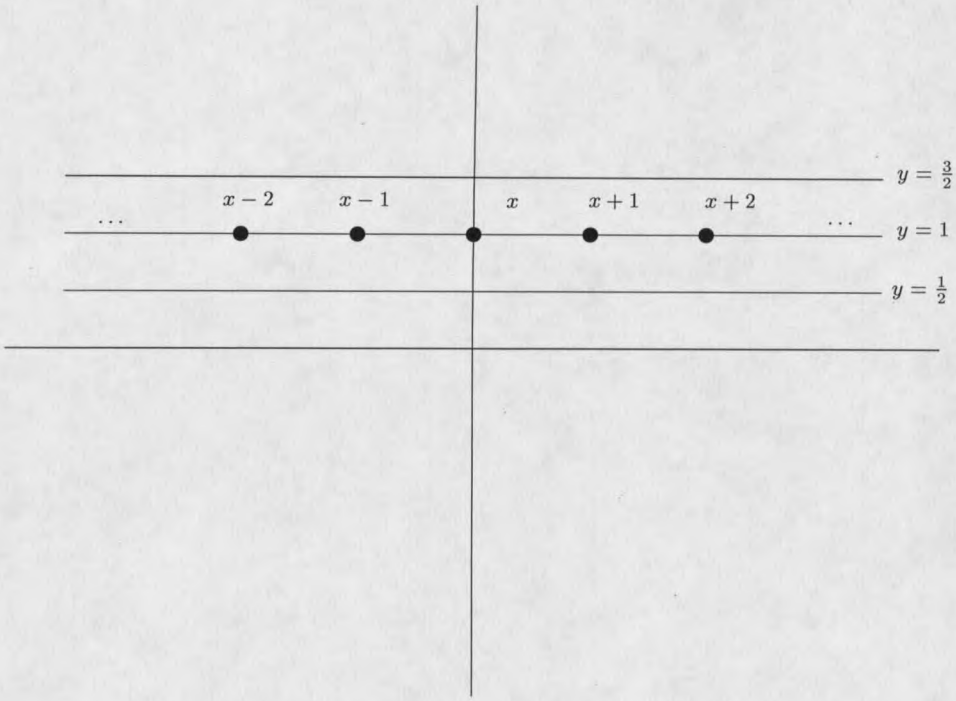


Figure 2: $P^{-1}(A)$

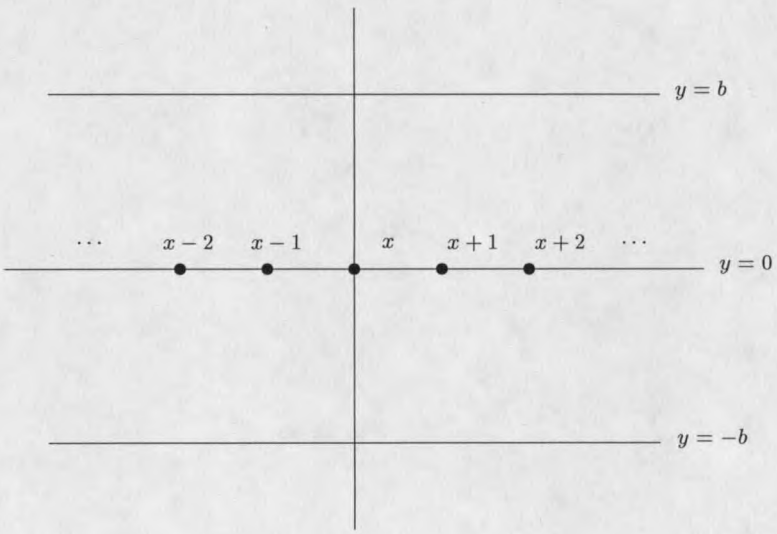


Figure 3: $\alpha \circ P^{-1}(A)$

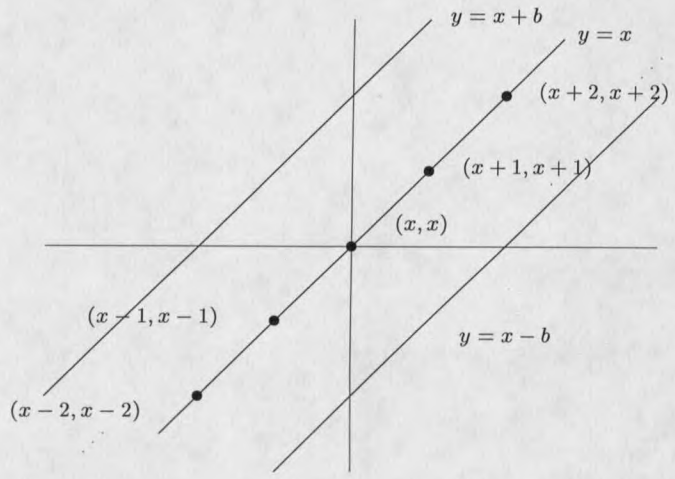


Figure 4: $T \circ \alpha \circ P^{-1}(A)$

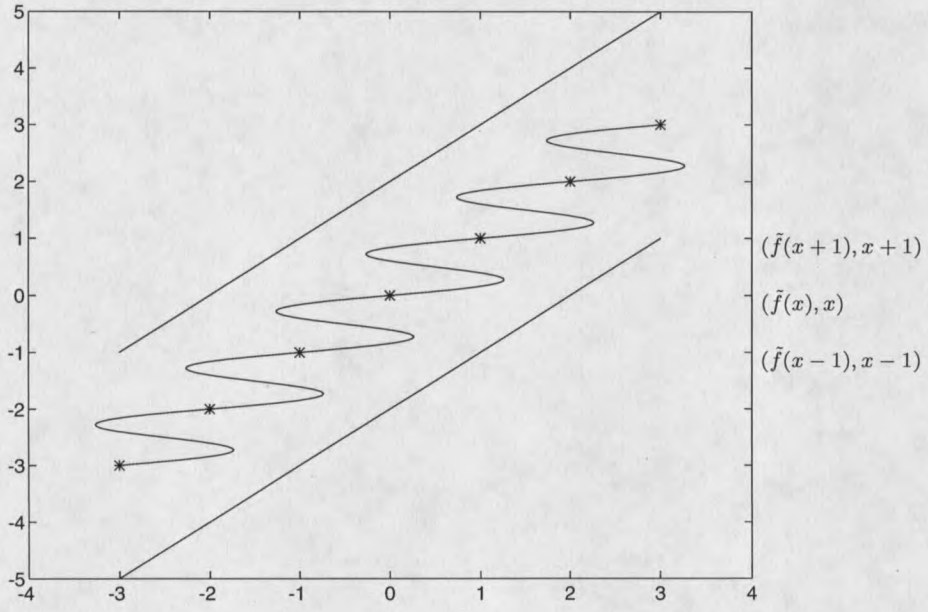


Figure 5: $H \circ T \circ \alpha \circ P^{-1}(A)$

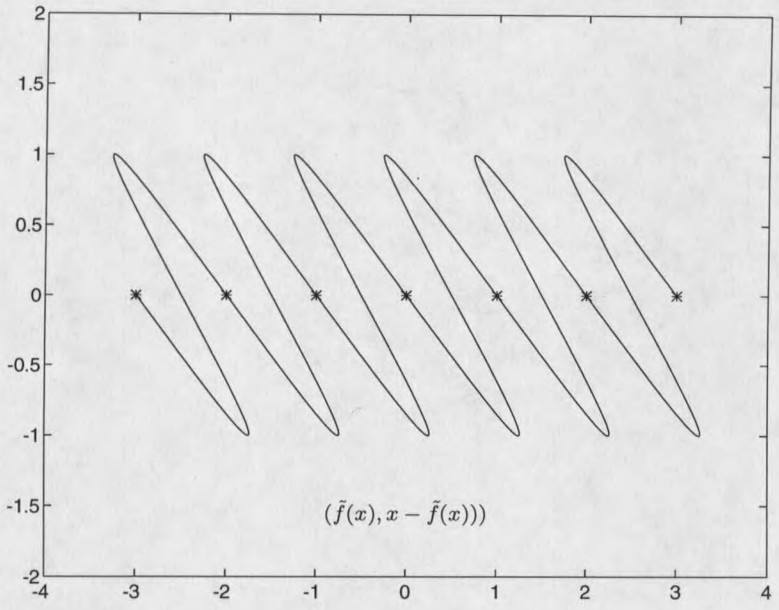


Figure 6: $T^{-1} \circ H \circ T \circ \alpha \circ P^{-1}(A)$

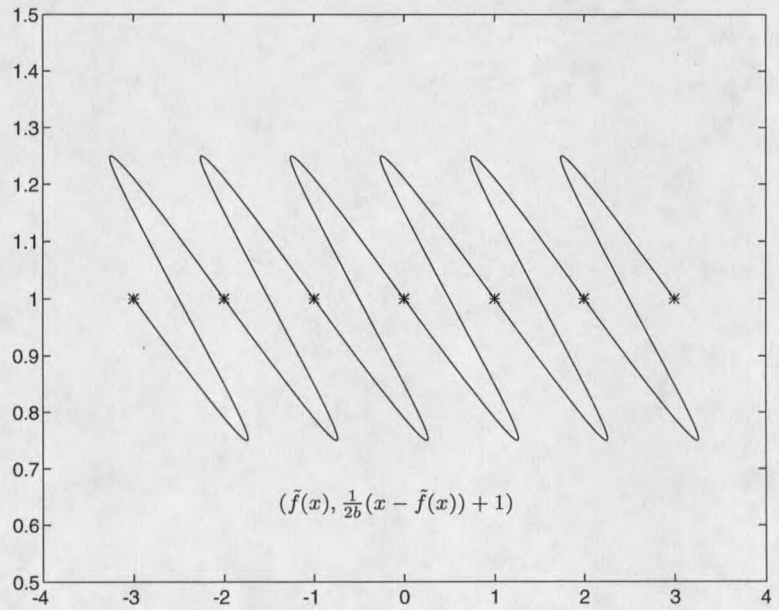
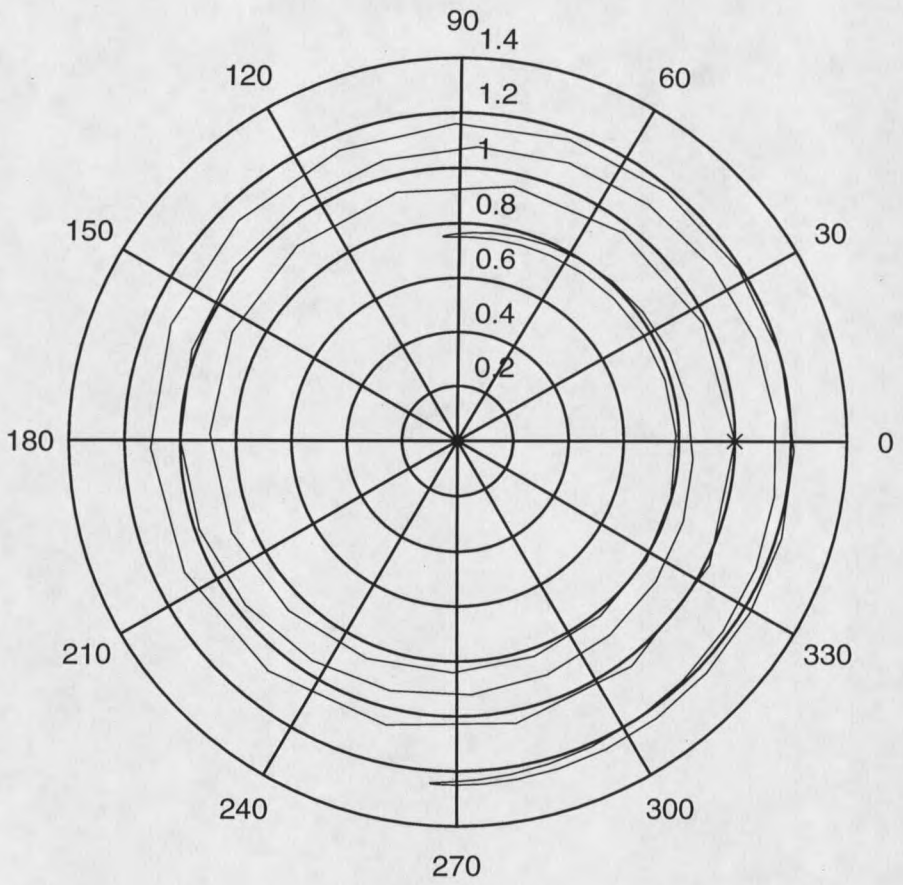


Figure 7: $\alpha^{-1} \circ T^{-1} \circ H \circ T \circ \alpha \circ P^{-1}(A)$



$\frac{1}{2b}(x-\tilde{f}(x))+1)e^{2\pi i\tilde{f}(x)}$

Figure 8: $P \circ \alpha^{-1} \circ T^{-1} \circ H \circ T \circ \alpha \circ P^{-1}(A)$

K is a homeomorphism:

well-defined: Suppose that $y_1 e^{2\pi i x_1} = y_2 e^{2\pi i x_2} \in A$. Then $y_1 = y_2$ and

$x_1 = x + k_1, x_2 = x + k_2, x \in [0,1), k_1, k_2 \in \mathbb{Z}$.

$K(y_1 e^{2\pi i x_1}) = (P \circ \alpha^{-1} \circ T^{-1} \circ H \circ T \circ \alpha)(\{(x+k, y_1) | k \in \mathbb{Z}\})$

$= (P \circ \alpha^{-1} \circ T^{-1} \circ H)(\{(x+k, x+k+2by_1-2b) | k \in \mathbb{Z}\})$

(Without loss of generality, $x+k+2by_1-3b \leq x+k \leq x+k+2by_1-2b$. Let $w = x+2by_1-2b$)

$= (P \circ \alpha^{-1} \circ T^{-1})(\{(\tilde{f}(w)+k+(\frac{\tilde{f}(w)-w+b}{b})(2b-2by_1), w+k) | k \in \mathbb{Z}\})$

$= P(\{(\tilde{f}(w) + k + (\frac{\tilde{f}(w)-w+b}{b})(2b - 2by_1), \frac{w-\tilde{f}(w)}{2b} - (\frac{\tilde{f}(w)-w+b}{b})(1 - y_1) + 1) | k \in \mathbb{Z}\})$

$= (\frac{w-\tilde{f}(w)}{2b} - (\frac{\tilde{f}(w)-w+b}{b})(1 - y_1) + 1) \exp[2\pi i(\tilde{f}(w) + (\frac{\tilde{f}(w)-w+b}{b})(2b - 2by_1))] \in A.$

$K(y_2 e^{2\pi i x_2}) = (P \circ \alpha^{-1} \circ T^{-1} \circ H \circ T \circ \alpha)(\{(x+k, y_1) | k \in \mathbb{Z}\}) = K(y_1 e^{2\pi i x_1})$. Therefore, $K : A \to A$ is well-defined.

One-to one: $K^{-1} = P \circ \alpha^{-1} \circ T^{-1} \circ H^{-1} \circ T \circ \alpha \circ P^{-1}$ is well-defined and $K \circ K^{-1} = K^{-1} \circ K = idy$.

Onto: $P^{-1}(A) = \mathbb{R} \times [\frac{1}{2}, \frac{3}{2}]$. H, T, and α are homeomorphisms and P is a covering map. Therefore $K(A) = A$.

Continuous: Let U be an open set in A. Then $K^{-1}(U)$

$= (P \circ \alpha^{-1} \circ T^{-1} \circ H^{-1} \circ T \circ \alpha)(P^{-1}(U)) = P((\alpha^{-1} T^{-1} \circ H^{-1} \circ T \circ \alpha)(P^{-1}(U)))$ which is open in A, since P, being a covering map is open. Therefore, K is continuous.

Since A is compact Hausdorff, and K is bijective continuous, K is a homeomorphism. Also, for any $x \in \mathbb{R}, K(e^{2\pi i x}) = (\frac{x-\tilde{f}(x)}{2b} + 1) e^{2\pi i \tilde{f}(x)}$. Extend K to $\tilde{K} : \tilde{A} = \{z \in \mathbb{C} \, | \frac{1}{4} \leq |z| \leq \frac{7}{4}\} \to \tilde{A}$ by letting $\tilde{K} = idy$ on $\tilde{A} - A$. Define a near homeomorphism $G : \tilde{A} \to \tilde{A}$ piecewise linearly along radial line segments as follows:

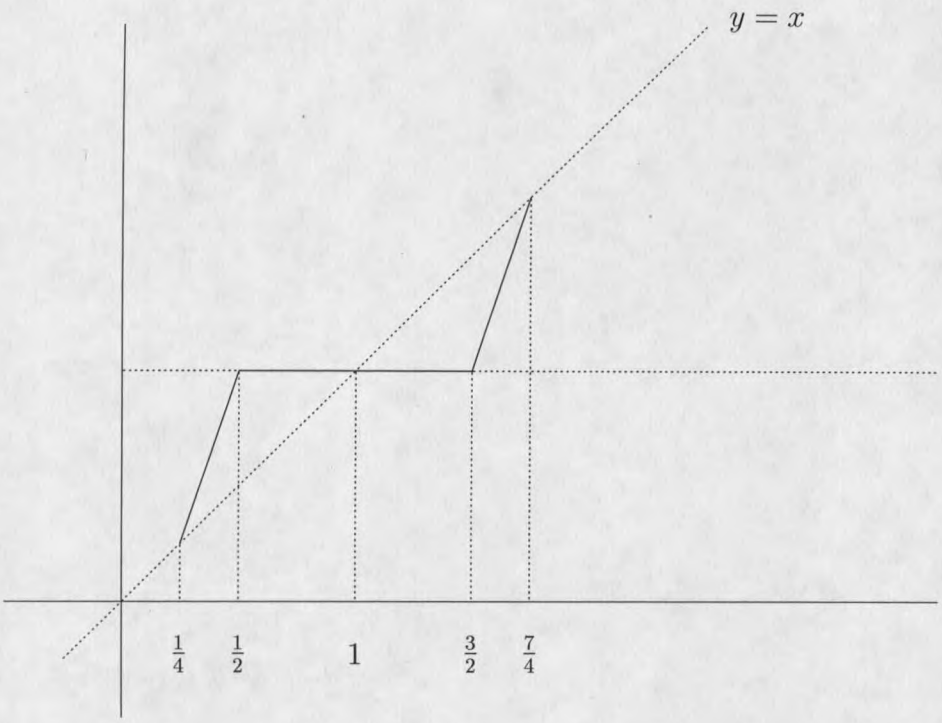


Figure 9:

Let $F = G \circ \tilde{K}$. F is a near homeomorphism, so that $X = \varprojlim F$ is homeomorphic to $\tilde{A}$. Also, $F|_{S^1} = f$ so that $\hat{F}|_{\varprojlim f}$ is a global attractor for $int(X)$ under $\hat{F}$. The result follows.

$\underline{deg(f) = -1}$: Choose $S = \{(x, y) \in \mathbb{R}^2 | y \in \mathbb{R}, -y - b < x < -y + b\}$ so that $graph(f) \subset S$ and a homeomorphism $H : cl(S) \to cl(S)$ so that $H(x, -x) = (\tilde{f}(x), -x)$. Finally, let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation on $\mathbb{R}^2$ induced by the matrix $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and proceed as in the $deg(f) = 1$ case.

PTL

Definition 2.10 *(Munkres [19]) Let $P : E \to B$ be a map. If f is a continuous mapping of some space X into B, a* **lifting** *of f is a map $\tilde{f} : X \to E$ so that $P \circ \tilde{f} = f$.*

Lemma 2.3 *Let $f : S^1 \to S^1$ be continuous and assume that $\{h_n\}_{n=1}^{\infty}$ is a sequence of embeddings, $h_n : S^1 \hookrightarrow A = \{z \in \mathbb{C} \mid \frac{1}{2} \leq |z| \leq \frac{3}{2}\}$, with $\|h_n - f\|_\infty \to 0$ as $n \to \infty$. Let $E = \mathbb{R} \times [\frac{1}{2}, \frac{3}{2}]$ and $P : E \to A$ be defined by $P(x, y) = ye^{2\pi ix} \in A$. Define $\bar{f} : I = [0, 1] \to S^1$ by $\bar{f}(t) = f(e^{2\pi it})$ and $\bar{h}_n : I \to A$ by $\bar{h}_n(t) = h_n(e^{2\pi it})$. Let $\tilde{f} : I \to E$ be a lifting of $\bar{f}$. Then there exists a sequence $\{\tilde{h}_n\}_{n=1}^{\infty}$ so that $\tilde{h}_n$ is a lifting of $\bar{h}_n$ for each n and $\|\tilde{h}_n - \tilde{f}\|_\infty \to 0$ as $n \to \infty$.*

Proof: Let $\epsilon > 0$ be given. Cover A with open sets $\{U_\alpha\}_{\alpha \in \varphi}$ where each U_α is evenly covered by P. Since $\tilde{f}([0, 1])$ is compact, there exists a compact set $K \subset E$ so that $\tilde{f}([0, 1]) \subset int(K)$ and $P(int(K)) = A$. $P^{-1}(\cup_{\alpha \in \varphi} U_\alpha) = \cup_{\beta \in ß} V_\beta$ where V_β is open and $P|_{V_\beta} : V_\beta \to A$ is a homeomorphism, for each $\beta \in ß$. $\{V_\beta\}_{\beta \in ß}$ is an open cover of K. Therefore, there is a Lebesgue number δ for this open cover. Choose $n \in \mathbb{Z}^+, n > 2$, so that $\frac{\epsilon}{n} < \delta$. The balls of rational radii less than $\frac{\epsilon}{n}$, the coordinates of whose centers are rational, form a countable basis for E. Cover $int(K)$ with these balls, say $int(K) = \cup_{i=1}^{\infty} B_{r_i}(x_i)$, each $B_{r_i}(x_i) \subset int(K)$ and open in E. $P|_{B_{r_i}(x_i)} : B_{r_i}(x_i) \to A$ is a homeomorphism for every i. $\tilde{f}^{-1}[(\cup_{i=1}^{\infty} B_{r_i}(x_i)) \cap \tilde{f}(I)]$ is an open cover of I. Choose a Lebesgue number ξ for this cover and a partition $s_1 < s_2 < \ldots < s_k$ of I with $s_1 = 0, s_k = 1$, and $s_{j+1} - s_j < \frac{\xi}{2}$ for $1 \leq j \leq k-1$. Then for each $1 \leq j \leq k-1, \tilde{f}([s_j, s_{j+1}]) \subset B_{r_{i_j}}(x_{i_j})$ and $\bar{f}([s_j, s_{j+1}]) \subset P(B_{r_{i_j}}(x_{i_j}))$, for some $B_{r_{i_j}}(x_{i_j})$. Since only finitely many $B_{r_{i_j}}(x_{i_j})$'s are required and $\|h_i - f\|_\infty \to 0$ as $i \to \infty$, there exists an N so that $i \geq N$ implies that for every $1 \leq j \leq k-1$, there exists a $B_{r_{i_j}}(x_{i_j})$ so that $\bar{f}([s_j, s_{j+1}]), \bar{h}_i([s_j, s_{j+1}]) \subset P(B_{r_{i_j}}(x_{i_j}))$ and $\tilde{f}([s_j, s_{j+1}]) \subset B_{r_{i_j}}(x_{i_j})$. Fix $i_o > N$. Construct the lifting $\tilde{h}_{i_0}$ of $\bar{h}_{i_0}$ as follows:

$\tilde{h}_{i_0}|_{[s_j,s_{j+1}]} \equiv (P|_{B_{r_{i_j}}(x_{i_j})})^{-1} \circ \bar{h}_{i_0}|_{[s_j,s_{j+1}]}$ for each $1 \leq j \leq k-1$, where for each j, $B_{r_{i_j}}(x_{i_j})$ is chosen so that $\bar{f}([s_j, s_{j+1}]), \bar{h}_{i_0}([s_j, s_{j+1}]) \subset P(B_{r_{i_j}}(x_{i_j}))$ and $\tilde{f}([s_j, s_{j+1}]) \subset B_{r_{i_j}}(x_{i_j})$. The fact that $\tilde{h}_{i_0}$ is a lifting follows exactly as in the proof of the path lifting lemma ([19], pg. 337). The fact that $\|\tilde{f} - \tilde{h}_{i_0}\|_\infty < \epsilon$ follows from the fact that $\tilde{h}_{i_0}([s_j, s_{j+1}]), \tilde{f}([s_j, s_{j+1}]) \subset B_{r_{i_j}}(x_{i_j})$ for each j. Therefore, $\|\tilde{f} - \tilde{h}_i\|_\infty < \epsilon$, for every $i \geq N$. Thus, $\|\tilde{f} - \tilde{h}_i\|_\infty \to 0$ as $i \to \infty$.

PTL

Theorem 2.4 *Let $f : S^1 \to S^1$ be continuous with $deg(f) = n \in \mathbb{Z} - \{-1, 0, 1\}$. Then there does not exist a near homeomorphism $G : D^2 \to D^2$ so that $G|_{S^1} = f$, where $D^2 = \{z \in \mathbb{C} \mid |z| \leq \frac{3}{2}\}$.*

Proof: Suppose that G is such a near homeomorphism. Let $\bar{f}$ and $P : E \to A$ be defined as in the previous lemma. Let $\tilde{f}$ be a lifting of $\bar{f}$ and by the previous lemma, choose a homeomorphism $h : S^1 \hookrightarrow A$ so that $\|\tilde{h} - \tilde{f}\|_\infty < \frac{1}{4}$, $\tilde{h}$ a lifting of $\bar{h}$. $P(\tilde{h}(0)) = \bar{h}(0) = \bar{h}(1) = P(\tilde{h}(1))$, so that $\pi_1(\tilde{h}(1)) - \pi_1(\tilde{h}(0)) \in \mathbb{Z}$, where $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is defined by $\pi_1(x, y) = x$. Since $deg(f) = n, \pi_1(\tilde{f}(1)) - \pi_1(\tilde{f}(0)) = n$. Therefore, $\pi_1(\tilde{h}(1)) - \pi_1(\tilde{h}(0)) = n$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y)$ $= (x + 1, y)$. Then $T^n(\tilde{h}(I)) \cap \tilde{h}(I) \neq \phi$. Therefore, by [16], $T(\tilde{h}(I)) \cap \tilde{h}(I) \neq \phi$. So, there exists $(x, y) \in \tilde{h}(I)$ so that $(x + 1, y) \in \tilde{h}(I)$. Say, $\tilde{h}(s_1) = (x, y)$ and $\tilde{h}(s_2) = (x + 1, y)$, where $s_1, s_2 \in I$. Since $\tilde{h}(1) = T^n(\tilde{h}(0))$, it can't be the case that $s_1 = 0$ and $s_2 = 1$ or vice versa. Then $\bar{h}(s_1) = \pi(\tilde{h}(s_1)) = \pi(\tilde{h}(s_2)) = \bar{h}(s_2)$, so that $h(e^{2\pi i s_1}) = h(e^{2\pi i s_2})$, which contradicts the fact that h is a homeomorphism. Thus, there does not exist a near homeomorphism $G : D^2 \to D^2$ so that $G|_{S^1} = f$.

PTL

Remark: This result also follows from Proposition 2.8. For, if such a near homeomorphism G exists, then $\varprojlim f \subset \varprojlim G \approx D^2$. So, $\varprojlim f$ is embedded in the plane, contradicting Proposition 2.8.

Proposition 2.2 *Let $x \in X$, X a topological space. If V_x is a neighborhood of x such that $i : \pi_1(V_x, x) \hookrightarrow \pi_1(X, x)$ is trivial (i is the inclusion map and $\pi_1(V_x, x)$ represents the fundamental group of V_x relative to the base point x, [19], pg. 326.) and U_x is a path connected neighborhood of x such that $U_x \subset V_x$, then for every $y \in U_x$, $[P] \in \pi_1(V_x, y)$ implies that $[P] = [e_y]$, as an element of $\pi_1(X, y)$.*

Proof: Let $y \in U_x$ and $[P] \in \pi_1(V_x, y)$ be given. Since U_x is path connected, there exists a path α in U_x from x to y. $[\bar{\alpha}] \star [P] \star [\alpha] \in \pi_1(V_x, x)$, so $[\bar{\alpha}] \star [P] \star [\alpha] = [e_x]$ in $\pi_1(X, x)$. Thus, $[P] = [\alpha] \star [e_x] \star [\bar{\alpha}] = [e_y]$ in $\pi_1(X, y)$.

PTL

Theorem 2.5 *Suppose that X is a compact, semi-locally simply connected, path connected, locally path connected metric space such that $\pi_1(X, x_0)$ is finitely generated for some $x_0 \in X$. Let $F : X \to X$ be a near homeomorphism (F is surjective, since X is compact). Then $F_\star : \pi_1(X, x_0) \to \pi_1(X, F(x_0))$ is an isomorphism of fundamental groups.*

Proof: F a near homeomorphism implies that there exists a sequence $\{H_k\}_{k=1}^{\infty}$ of homeomorphisms of X so that $\|H_k - F\|_\infty \to 0$ as $k \to \infty$. Let $[g_1], \ldots, [g_m]$ generate $\pi_1(X, x_0)$. For every $x \in X$, there is an open neighborhood V_x of x so that $i : \pi_1(V_x, x) \hookrightarrow \pi_1(X, x)$ is trivial, where i is the inclusion map. Since X is locally path connected, for every x, there exists a path connected neighborhood U_x of x so that $U_x \subset V_x$. X compact implies that there exists a Lebesgue number δ for $\{U_x\}_{x \in X}$. Denote $F_\star([g_i]) \equiv [f_i], 1 \leq i \leq m$. $f_i^{-1}(f_i(I) \cap (\cup_{x \in X} B_{\frac{\delta}{6}}(x)))$ is an open cover of I, for every $1 \leq i \leq m$, so for every i, there exists a Lebesgue number η_i. Let $\eta = \frac{1}{2} \min_{1 \leq i \leq m} \{\eta_i\}$. Partition I, say $t_0 = 0 < t_1 < \ldots < t_n = 1$, with

$t_j - t_{j-1} < \eta, 1 \leq j \leq m$. Then for each i and $[t_{j-1}, t_j], f_i([t_{j-1}, t_j]) \subset B_{\frac{\delta}{6}}(x)$ some $x \in X$. About each point $f_i(0) = F(x_0) \equiv y_0, f_i(t_j) \equiv y_{ij}, 1 \leq i \leq m, 1 \leq j \leq n-1$ choose a path connected neighborhood W_0, W_{ij}, respectively with diameter less than $\frac{\delta}{3}$. Since $\|H_k - F\|_\infty \to 0$ as $k \to \infty$, we can choose a homeomorphism H of X such that for each $1 \leq i \leq m$ and $[t_{j-1}, t_j], 1 \leq j \leq n, f_i([t_{j-1}, t_j])$ and $H(g_i([t_{j-1}, t_j]))$ are contained in the same $\frac{\delta}{6}$-ball and $F(x_0), H(x_0) \in W_0; y_{ij}, H(g_i(t_j))$
$\in W_{ij} (1 \leq j \leq n-1)$. Denote $H \circ g_i$ by h_i and $H(g_i(t_j))$ by $z_{ij}, 1 \leq i \leq m$ and
$1 \leq j \leq n-1$. Let $z_0 \equiv H(x_0)$. Choose paths α_0, α_{ij} with diameters less than $\frac{\delta}{3}$ from z_0 to y_0 and z_{ij} to y_{ij} respectively. Then for each $1 \leq i \leq m$ and
$2 \leq j \leq n-1, diam(\alpha_{i(j-1)} \star \bar{h}_i|_{[s_{j-1}, s_j]} \star \bar{\alpha}_{ij} \star f_i|_{[s_{j-1}, s_j]} < \delta$. So by the previous proposition, $f_i|_{[s_{j-1}, s_j]} \approx_P \alpha_{ij} \star h_i|_{[s_{j-1}, s_j]} \star \bar{\alpha}_{i(j-1)}$. When $j = 1, f_i|_{[s_0, s_1]}$
$\approx_P \alpha_{i1} \star h_i|_{[s_0, s_1]} \star \bar{\alpha}_0$ and when $j = n, f_i|_{[s_{n-1}, s_n]} \approx_P \alpha_0 \star h_i|_{[s_{n-1}, s_n]} \star \bar{\alpha}_{i(n-1)}$. Hence, for every $1 \leq i \leq m$, $f_i \approx_P f_i|_{[s_{n-1}, s_n]} \star f_i|_{[s_{n-2}, s_{n-1}]} \star \ldots \star f_i|_{[s_0, s_1]}$
$\approx_P \alpha_0 \star h_i|_{[s_{n-1}, s_n]} \star \bar{\alpha}_{i(n-1)} \star \alpha_{i(n-1)} \star h_i|_{[s_{n-2}, s_{n-1}]} \star \bar{\alpha}_{i(n-2)} \star \ldots \star \alpha_{i1} \star h_i|_{[s_0, s_1]} \star \bar{\alpha}_0$
$\approx_P \alpha_0 \star h_i \star \bar{\alpha}_0$ That is $[f_i] = [\alpha_0] \star [h_i] \star [\bar{\alpha}_0]$ for every i. Thus, since $\pi_1(X, x_0)$ is finitely generated, if $[P] \in \pi_1(X, x_0), F_\star([P]) = [\alpha_0] \star H_\star([P]) \star [\bar{\alpha}_0]$. Since H is a homeomorphism, $H_\star : \pi_1(X, x_0) \to \pi_1(X, H(x_0))$ is an isomorphism ([19], pg. 330).

$F_\star$ is one-to-one: Let $[P], [Q] \in \pi_1(X, x_0)$. Suppose that $F_\star([P]) = F_\star([Q])$
$\in \pi_1(X, F(x_0))$, then $[\alpha_0] \star H_\star([P]) \star [\bar{\alpha}_0] = [\alpha_0] \star H_\star([Q]) \star [\bar{\alpha}_0]$, which implies that $H_\star([P]) = H_\star([Q])$ so that $[P] = [Q]$.

$F_\star$ is onto: Let $[Q] \in \pi_1(X, F(x_0))$, then $[\bar{\alpha}_0] \star [Q] \star [\alpha_0] \in \pi_1(X, H(x_0))$. $H_\star$ onto implies that there exists a $[P] \in \pi_1(X, x_0)$ such that $H_\star([P]) = [\bar{\alpha}_0] \star [Q] \star [\alpha_0]$. Whence, $[\bar{\alpha}_0] \star F_\star([P]) \star [\alpha_0] = [\bar{\alpha}_0] \star [Q] \star [\alpha_0]$ which implies that $F_\star([P]) = [Q]$. Thus, $F_\star$ is an isomorphism.

PTL

CHAPTER 3

EMBEDDING G-LIKE CONTINUA AS ATTRACTORS

In the following we give three results concerning inverse limits of maps of finite, connected, planar graphs (G). The first shows that if f is any triod map (or a particular type of n-od map) which can be uniformly approximated by an embedding, $\varprojlim f$ is planar, and, in some instances, may be embedded as an attractor of a homeomorphism of the plane extending $\hat{f}$. Secondly, we show that there exists an open and dense class of maps in the set of all maps of a given finite, connected, planar graph for which the Barge-Martin construction cannot be used to embed their inverse limits as planar attractors. Finally, we close with an example of a triod map for which the Barge-Martin construction does not apply, but the inverse limit can be embedded as a planar attractor.

Let T denote a triod with edges l_1, l_2, and l_3. Let 0 be the intersection of the edges and a, b, and c be the extreme points of the edges l_1, l_2, and l_3 respectively. Let D be a closed disk and $h : T \subset int(D) \hookrightarrow int(D)$ be an embedding. Our first goal is to extend h to a homeomorphism $H' : D' \to D'$, where D' is a closed disk with $D \subset int(D')$ and $H'|_{\partial D'} = idy$.

Lemma 3.4 *$D - \{h(T) \cup \partial D\}$ is connected.*

Proof: $D-(\partial D \cup \{h(0)\})$ is connected, since it is an open annulus. $h(l_1)$ is an end-cut in $D-(\partial D \cup \{h(0)\})$ and so by Theorem 11.6, pg. 118, [20], $[D-(\partial D \cup \{h(0)\})]-h(l_1)$ is a domain. $h(l_2)$ is an end-cut in this domain, and so removing it does not destroy connectivity. Similarly, $h(l_3)$ is an end-cut in this domain, and so removing it does not

destroy connectivity. Therefore, $D - \{h(T) \cup \partial D\}$ is connected.

PTL

Definition 3.11 *(Whyburn [26]) A locally compact connected set is called a generalized continuum.*

Lemma 3.5 *There exists a sequence* $\{L_i\}_{i=1}^{\infty}$ *of locally connected generalized continua contained in int(D) such that* $h(a) \in \partial L_i, L_i \cap h(T) = \emptyset$, *and* $L_{i+1} \subset L_i$ *for every* i. *Also* $diam(L_i) \to 0$ *as* $i \to \infty$.

Proof: Choose $\epsilon > 0$ so that $B_\epsilon(h(a)) \cap \partial D = \emptyset$. Choose $x \in l_1 - \{0, a\}$ so that $h([x, a]) \subset B_\epsilon(h(a))$. Consider $\varphi = B_\epsilon(h(a)) - h(T - (x, a])$–an open subset of $\mathbb{R}^2$. Consider the components of φ. $h((x, a])$ is connected and $h((x, a]) \subset \varphi$. So, $h((x, a]) \subset K$, where K is a component of φ. Also, K is open in $\mathbb{R}^2$. Then $h((x, a])$ is an end-cut in K and so $K - h((x, a])$ is open in $\mathbb{R}^2$ and connected (Theorem 11.6, pg. 118, [20]). Therefore, $L_1 = K - h((x, a])$ is a locally connected generalized continuum such that $h(a) \in \partial L_1, L_1 \cap (h(T) \cup \partial D) = \emptyset$ and $diam(L_1) < 2\epsilon$. $B_{\frac{\epsilon}{2}}(h(a)) \cap \partial D = \emptyset$. Choose $y > x$ so that $y \in l_1 - \{0, a\}$ and $h((y, a]) \subset B_{\frac{\epsilon}{2}}(h(a))$. Let $\hat{\varphi} = B_{\frac{\epsilon}{2}}(h(a)) - h(T - (y, a])$ and $h((y, a]) \subset \hat{K}$, $\hat{K}$ a component of $\hat{\varphi}$. $\hat{\varphi} \subset \varphi$ an open subset of $\mathbb{R}^2$ implies that each component of $\hat{\varphi}$ is contained in exactly one component of φ. Also, $h((y, a]) \subset K$ implies that $\hat{K} \subset K$. Therefore, $L_2 = \hat{K} - h((y, a]) \subset L_1$, where L_2 is a locally connected generalized continuum such that $h(a) \in \partial L_2, L_2 \cap (h(T) \cup \partial D) = \emptyset$, and $diam(L_2) < \epsilon$. Continue inductively to get the sequence $\{L_i\}_{i=1}^{\infty}$.

PTL

Lemma 3.6 *Let $x \in \partial D$. Then there exists a sequence $\{K_i\}_{i=1}^{\infty}$ of locally connected generalized continua $\subset$ int(D) such that $x \in \partial K_i, K_{i+1} \subset K_i$, and $K_i \cap h(T) = \emptyset$, for every i. Also, $diam(K_i) \to 0$ as $i \to \infty$.*

Proof: Choose $\epsilon > 0$ such that $B_\epsilon(x) \cap h(T) = \emptyset$. Let K_1be a component of $(B_\epsilon(x) \cap D) - \partial D$ such that $x \in \partial K_1$, K_2 be a component of $(B_{\frac{\epsilon}{2}}(x) \cap D) - \partial D$ such that $x \in \partial K_2$ and $K_2 \subset K_1, \ldots$.

PTL

Lemma 3.7 *Let $x \in \partial D$. Then there exists a simple arc from $h(a)$ to x lying entirely within $(D - (h(T) \cup \partial D)) \cup \{h(a), x\}$.*

Proof: $D - (h(T) \cup \partial D)$ is a locally connected generalized continuum and so is arc-wise connected (Corollary 4.11, pg. 27, [25]). Also, by the preceding two lemmas, there exist sequences $\{L_i\}_{i=1}^{\infty}$ and $\{K_i\}_{i=1}^{\infty}$ of locally connected generalized continua contained within $int(D)$ such that $K_i \cap h(T) = L_i \cap h(T) = \emptyset, K_{i+1} \subset K_i$, $L_{i+1} \subset L_i, h(a) \in \partial L_i$, and $x \in \partial K_i$, for every i. Also, $diam(L_i), diam(K_i) \to 0$ as $i \to \infty$. Choose points $x_1 \in K_1, y_1 \in L_1$. Let T_1 be a simple arc in $D - (h(T) \cup \partial D)$ connecting x_1 and y_1. Select a sequence of points $\{x_n\}_{n=2}^{\infty}$ such that $x_n \in K_n$, for every n and a sequence of points $\{y_n\}_{n=2}^{\infty}$ such that $y_n \in L_n$, for every n. Obtain sequences of simple arcs $\{S_n\}_{n=2}^{\infty}$ and $\{T_n\}_{n=2}^{\infty}$ so that $S_n \subset K_{n-1}, T_n \subset L_{n-1}, S_n$ connects x_n and x_{n-1}, and T_n connects y_n and y_{n-1}. Let $M = \{x, h(a)\} \cup T_1 \cup (\cup_{n=2}^{\infty} T_n) \cup (\cup_{n=2}^{\infty} S_n)$. Then (*) M is a locally connected generalized continuum in $(D - (h(T) \cup \partial D)) \cup \{x, h(a)\}$. Therefore M and hence also

$(D - (h(T) \cup \partial D)) \cup \{x, h(a)\}$ contains a simple arc joining x and $h(a)$.

PTL

(*) $M = \{x, h(a)\} \cup T_1 \cup (\cup_{n=2}^{\infty} T_n) \cup (\cup_{n=2}^{\infty} S_n)$ is a locally connected continuum.

Proof: Define $f : [0,1] \to M$ as follows:

$[0,1] = [\frac{1}{3}, \frac{2}{3}] \cup (\cup_{n=2}^{\infty} [\frac{1}{n+2}, \frac{1}{n+1}]) \cup (\cup_{n=2}^{\infty} [1 - \frac{1}{n+1}, 1 - \frac{1}{n+2}]) \cup \{0,1\}$. T_1 a simple arc implies that there exists a homeomorphism $f_1 : [\frac{1}{3}, \frac{2}{3}] \to T_1$. Similarly, there exist homeomorphisms $f_n : [\frac{1}{n+2}, \frac{1}{n+1}] \to T_n, n \geq 2$ and $g_n : [1 - \frac{1}{n+1}, 1 - \frac{1}{n+2}] \to S_n, n \geq 2$. Define

$$f(t) = \left\{ \begin{array}{l} f_1(t), t \in [\frac{1}{3}, \frac{2}{3}] \\ f_n(t), t \in [\frac{1}{n+2}, \frac{1}{n+1}], n \geq 2 \\ g_n(t), t \in [1 - \frac{1}{n+1}, 1 - \frac{1}{n+2}], n \geq 2 \\ h(a), t = 0 \\ x, t = 1. \end{array} \right\}.$$

f is continuous at any $t \in (0,1)$, by the pasting lemma ([19], pg. 108). Suppose that $\{t_i\}_{i=1}^{\infty} \subset [0,1]$ and $t_i \to 0$ as $i \to \infty$. $f(t_i) \in cl(\cup_{n=2}^{\infty} T_n) = \cup_{n=2}^{\infty} T_n \cup \{h(a)\}$, i large enough. For every j, there exists an N_j such that $i \geq N_j$ implies that $f(t_i) \in \cup_{n=j}^{\infty} T_n \cup \{h(a)\} \subset cl(L_j)$. $diam(cl(L_j)) \to 0$ as $j \to \infty$, so $f(t_i) \to y \in \cap_{j=1}^{\infty} cl(L_j) = \{h(a)\}$. Therefore, $f(t_i) \to h(a)$. Thus, f is continuous at $t = 0$. Similarly, f is continuous at $t = 1$. So, by Theorem 8.2, pg. 89, [20], M is compact and locally connected. Also $T_1 \cup (\cup_{n=2}^{\infty} T_n) \cup (\cup_{n=2}^{\infty} S_n)$ is connected, since it's the union of overlapping connected sets. Whence, its closure M is connected.

PTL

Theorem 3.6 *There exists a homeomorphism $H' : D' \to D'$ such that $H'|_T = h$ and $H'|_{\partial D'} = idy$.*

Proof: Let α be a simple arc in $\{D - (h(T) \cup \partial D)\} \cup \{h(a), x\}$ connecting $h(a)$ and x, where $x \in \partial D$. Then $U = D - (h(T) \cup \partial D \cup \alpha)$ is an open, connected set in $\mathbb{R}^2$.

Similarly, we can choose a simple arc β in U from $h(b)$ to $y \in \partial D, y \neq x$. The points x and y split ∂D into two disjoint simple arcs γ_1 and γ_2. $h(l_1) \cup \alpha \cup \gamma_1 \cup \beta \cup h(l_2)$ and $h(l_1) \cup \alpha \cup \gamma_2 \cup \beta \cup h(l_2)$ are two simple closed curves in D which intersect along $\alpha \cup h(l_1) \cup h(l_2) \cup \beta$. This splits D into two disks, say D_1' and D_2', respectively. $h(l_3 - \{0\})$ must lie entirely within the interior of one of these two disks. Without loss of generality, $h(l_3 - \{0\}) \subset int(D_1')$. Consider $D - (D_2' \cup \partial D_1') = int(D_1')$. $h(l_3 - \{0\})$ is an end-cut in $int(D_1')$, so, by Theorem 11.6, pg. 118, [20], $(int(D_1')) - h(l_3 - \{0\})$ is connected. Then let $z \in \gamma_1 - (\alpha \cup \beta)$. $z \in \partial D_1'$. Choose a simple arc γ from $h(c)$ to z in $(int(D_1')) - h(l_3 - \{0\})$. Write D as the union of three disks D_1', D_2', and D_3', where D_1' and D_3' are obtained from the disk previously labeled D_1' by removing $h(l_3) \cup \gamma_0$. D_1', D_2', and D_3' correspond to the disks D_1, D_2, and D_3 in D obtained by joining a, b, and c to ∂D with radial line segments. We choose three homeomorphisms $h_1 : \partial D_1 \to \partial D_1', h_2 : \partial D_2 \to \partial D_1'$ and $h_3 : \partial D_3 \to \partial D_3'$, which agree with one another on the common boundaries. By Corollary 3, pg 173, of [20], we can extend these to homeomorphisms $H_1 : D_1 \to D_1', H_2 : D_2 \to D_2'$, and $H_3 : D_3 \to D_3'$ which agree with one another on the common boundary points. Therefore, by the pasting lemma ([19], pg. 108), we get a homeomorphism $H : D \to D$ which extends h. Recall that $D \subset int(D')$. Then, by the annulus theorem ([21], pg. 11), there exists a homeomorphism $H' : D' \to D'$ such that $H'|_D = H$ and $H'|_{\partial D'} = idy$.

PTL

The previous Theorem does not hold in general for any n-od, $n \geq 4$, as noted by Barge. For example, consider a homeomorphism of a 4-od which fixes two of the legs and switches the other two.

Definition 3.12 *Let G be any finite, connected, planar graph. Define $C(G) = \{f : G \to G | f$ is continuous and surjective$\}$ and $\eta(G) = \{f \in C(G) | f$ cannot*

be uniformly approximated by planar embeddings$\}$.

Theorem 3.7 *Let* $f \in C(T) - \eta(T)$ *and* $\{h_n\}_{n=1}^{\infty}$ *be a sequence of embeddings,* $h_n : T \hookrightarrow int(D)$ *such that* $\|h_n - f\|_\infty \to 0$ *as* $n \to \infty$. *Let* $H_n' : D' \to D'$ *be a sequence of homeomorphisms such that* $H_n'|_T = h_n$ *and* $H_n'|_{\partial D'} = idy$. *Then if* $\{H_{n_i}'\}_{i=1}^{\infty}$ *is a subsequence of* $\{H_n'\}_{n=1}^{\infty}$ *such that* $\{H_{n_i}'\}_{i=1}^{\infty}$ *converges uniformly to* $F : D' \to D'$, $\varprojlim f$ *can be topologically realized as a global attractor in the plane.*

Proof: F is a near homeomorphism. $F|_T = \lim_{i\to\infty} H_{n_i}'|_T = \lim_{i\to\infty} h_{n_i} = f$. Let D_1 and D_2 be two closed disks such that $D' \subset int(D_1)$ and $D_1 \subset int(D_2)$. let $p : D_2 \to D_2$ be a near homeomorphism such that $p|_T = idy, p(D') = T, p(D_1) = D'$, and if $x \in int(D_2)$, there exists an $n \in \mathbb{Z}^+$ such that $p^n(x) \in D'$. Extend F to $\tilde{F} : D_2 \to D_2$ by

$$\tilde{F}(x) = \left\{ \begin{array}{l} F(x), x \in D' \\ x, x \notin D' \end{array} \right\}.$$

Let $G : D_2 \to D_2$ be defined by $G(x) = p(\tilde{F}(x))$. Then G is a near homeomorphism such that $G|_T = f$ and if $x \in int(D_2)$, there exists and $n \in \mathbb{Z}^+$ such that $G^n(x) \in T$. The proof would then proceed as in the construction of Barge and Martin ([1]).

PTL

If it is not the case that the sequence of homeomorphisms $\{H_n\}_{n=1}^{\infty}$ of the preceeding theorem have a uniformly convergent subsequence, we can use the following theorem by McCord [17] to embed $\varprojlim f$ in $\mathbb{R}^2$. First, we need a definition of McCord's.

Definition 3.13 *(McCord [17]) Let* (E, d) *be a locally compact metric space and let* $f : X \to Y$ *be a map where* X *and* Y *are compact subsets of* E. *Then* f *can be approximated by embeddings in* E, *provided that for each* $\epsilon > 0$ *there exists an open set* U *with* $X \subset U \subset E$ *and a* $1-1$ *map* $\mu : U \to E$ *such that* $d(\mu(x), f(x)) < \epsilon$ *for all* x *in* X.

Theorem 3.8 *(McCord [17]) Let (E, d) be a locally compact metric space and let (X, f) be an inverse limit sequence of compact subsets of E such that each bonding map $f_n^{n+1} : X_{n+1} \to X_n$ can be approximated by embeddings in E. Then the limit X_∞ can be embedded in E.*

Note that $f \in C(T) - \eta(T)$ is distinctly different from the case where f can be approximated by embeddings in $\mathbb{R}^2$. $f \in C(T) - \eta(T)$ implies that for every $\epsilon > 0$, there exists an embedding $g : T \hookrightarrow \mathbb{R}^2$ such that $\|f - g\|_\infty < \epsilon$. however, f can be approximated by embeddings in $\mathbb{R}^2$ implies that for every $\epsilon > 0$, there exists an embedding μ of a neighborhood of T into $\mathbb{R}^2$ such that $\|f - \mu|_T\|_\infty < \epsilon$. The example, given by Barge, of a homeomorphism of a 4-od, G, which switches two of the legs and fixes the other two is an example of a map which is an element of $C(G) - \eta(G)$ but which cannot be approximated by embeddings in $\mathbb{R}^2$.

Theorem 3.9 *If $f \in C(T) - \eta(T)$, then $\varprojlim f$ is planar.*

Proof: We have shown that for every $\epsilon > 0$, there exists a homeomorphism $H' : D' \to D'$ such that $H'|_{\partial D'} = idy$ and $\|f - H'|_T\| < \epsilon$. Hence, f can be approximated by embeddings in $\mathbb{R}^2$ and by McCord's theorem, $\varprojlim f$ is planar.

PTL

Next, we show that there exists an open and dense class of maps in the set of all maps of a given finite, connected, planar graph for which the Barge-Martin construction cannot be used to embed their inverse limits as planar attractors. We begin with some notation.

Denote elements of l_1 by $\{s|0 \leq s \leq 1\}$, elements of l_2 by $\{s'|0 \leq s' \leq 1\}$, and elements of l_3 by $\{s''|0 \leq s'' \leq 1\}$.

Proposition 3.3 *$\eta(T)$ is open in $(C(T), \|\cdot\|_\infty)$.*

Proof: Suppose not. Then there exists an $f \in \eta(t)$ such that for every $\epsilon > 0$, there is an $h \in C(T) - \eta(T)$ with $\|f - h\|_\infty < \epsilon$. Since $f \in \eta(T)$, there is an $\epsilon_0 > 0$ so that there is no embedding $g : T \to \mathbb{R}^2$ with $\|f - g\|_\infty < \epsilon_0$. Choose $h \in C(T) - \eta(T)$ so that $\|h - f\|_\infty < \frac{\epsilon_0}{2}$ and $g : T \to \mathbb{R}^2$ an embedding with $\|h - g\|_\infty < \frac{\epsilon_0}{2}$. Then $\|f - g\|_\infty < \epsilon_0$, a contradiction.

PTL

Lemma 3.8 *Let $\epsilon > 0$ be given. Let $I = [a, b] \subset l_i - \{0\}, i \in \{1, 2, 3\}$. Let $t_0 = a < t_1 < t_2 < t_3 < t_4 < t_5 = b$ be a partition of $[a, b]$ into five equal subintervals. Define a piecewise linear function $k : I \to T$ as follows:*
$k(t_0) = \frac{\epsilon}{4}, k(t_1) = 0, k(\frac{1}{2}(t_1 + t_2)) = \frac{\epsilon}{8}'', k(t_2) = 0, k(\frac{1}{2}(t_2 + t_3)) = \frac{\epsilon}{4}', k(t_3) = 0,$
$k(\frac{1}{2}(t_3 + t_4)) = \frac{\epsilon}{8}, k(t_4) = 0, k(t_5) = \frac{\epsilon}{4}''$. *Then there does not exist a sequence of embeddings $\{\beta_n\}_{n=1}^{\infty}, \beta_n : I \to \mathbb{R}^2$ such that, $\|\beta_n - k\|_\infty \to 0$ as $n \to \infty$.*

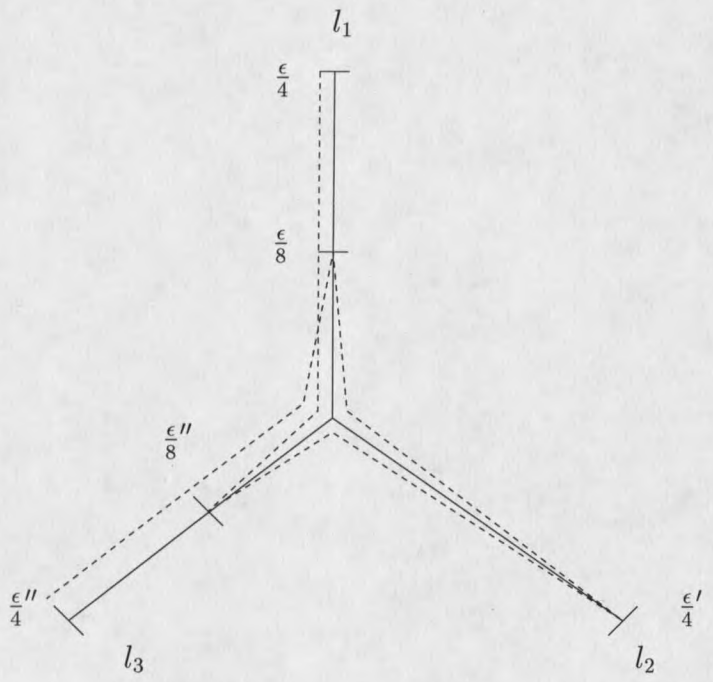


Figure 10: image of k

Proof: Assume that there exists such a sequence of embeddings $\{\beta_n\}_{n=1}^{\infty}$. Choose one, say β, such that $\|\beta - k\|_\infty < \frac{\epsilon}{32}$. Then let $N = B_{\frac{\epsilon}{32}}([0, \frac{\epsilon}{4}] \cup [0, \frac{\epsilon'}{4}] \cup [0, \frac{\epsilon''}{8}])$, $L_1 = cl(N) \cap cl(B_{\frac{\epsilon}{32}}(\frac{\epsilon}{4}))$, $L_2 = cl(N) \cap cl(B_{\frac{\epsilon}{32}}(\frac{\epsilon''}{8})))$, and $L_3 = cl(N) \cap cl(B_{\frac{\epsilon}{32}}(\frac{\epsilon'}{4}))$. $\beta(t_0) \in B_{\frac{\epsilon}{32}}(\frac{\epsilon}{4})$. Let $x_1 \in L_1$ and $\gamma_1 : [0,1] \to cl(B_{\frac{\epsilon}{32}}(\frac{\epsilon}{4}))$ be defined by $\gamma_1(t) = (1-t)x_1 + t\beta(t_0)$. Consider $\gamma_1^{-1}(cl(B_{\frac{\epsilon}{32}}(\frac{\epsilon}{4})) \cap \beta([t_0, t_4]))$–a closed subset of $[0,1]$ not containing 0. Therefore, the least element of this set, say t^* is positive. Then $\alpha_1 = \gamma_1([0, t^*])$ is a simple arc from x_1 to an element, say $\beta(s_1) \in \beta([t_0, t_4])$, such that $\alpha_1 \cap \beta([t_0, t_4]) = \{\beta(s_1)\}$ and $\alpha_1 \subset cl(B_{\frac{\epsilon}{32}}(\frac{\epsilon}{4}))$. Let $x_2 \in L_2, x_3 \in L_3$ and choose simple arcs α_2 and α_3 from $\beta(s_2) \in \beta([t_0, t_4])$ to x_2 and $\beta(s_3) \in \beta([t_0, t_4])$ to x_3 respectively

such that $\alpha_2 \cap \beta([t_0, t_4]) = \{\beta(s_2)\}, \alpha_3 \cap \beta([t_0, t_4]) = \{\beta(s_3)\}, \alpha_2 \subset B_{\frac{\epsilon}{32}}(\frac{\epsilon}{8}'')$, and $\alpha_3 \subset B_{\frac{\epsilon}{32}}(\frac{\epsilon}{4}')$. Then $s_1 < s_2 < s_3$ and $N - \{\alpha_1 \cup \beta([s_1, s_2]) \cup \alpha_2\}$ consists of the two disks D_1 and D_2 as follows:

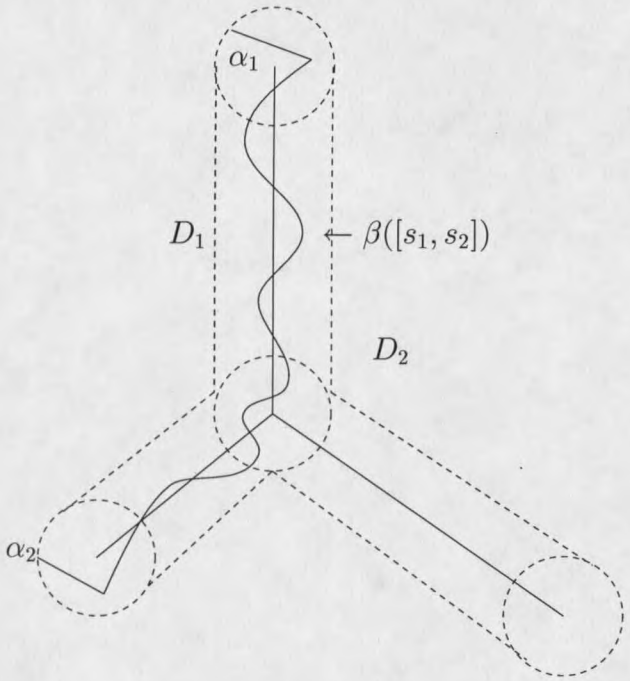


Figure 11:

$\beta((s_2, t_4])$ is connected and doesn't intersect $\partial D_1 \cup \partial D_2$ and so must lie entirely within D_1 or D_2. But, $\beta(\frac{1}{2}(t_2 + t_3)) \in B_{\frac{\epsilon}{32}}(\frac{\epsilon}{4}')$, so $\beta((s_2, t_4]) \subset D_2$.

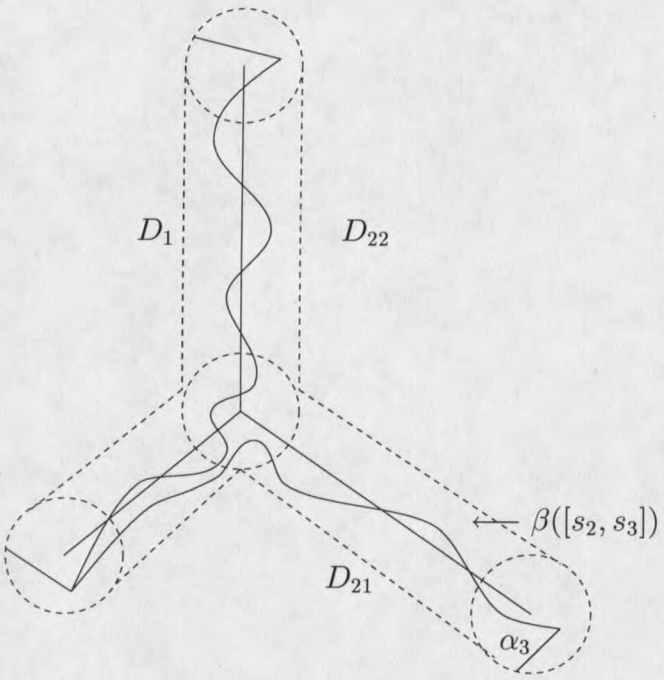


Figure 12:

As pictured, $\beta([s_2, s_3]) \cup \alpha_3$ splits D_2 into two sub-disks D_{21} and D_{22}. $\beta((s_3, t_4])$ must lie entirely within one of these disks. But, $\beta(\frac{1}{2}(t_3 + t_4)) \in B_{\frac{\epsilon}{32}}(\frac{\epsilon}{8})$, so $\beta((s_3, t_4]) \subset D_{22}$. Since $\beta((t_4, t_5]) \cap \beta([s_1, s_3]) = \emptyset$ and $\beta(t_4) \in B_{\frac{\epsilon}{32}}(0) \cap D_{22}$, there is an $s \in (t_4, t_5]$ so that either $\beta(s) \in B_{\frac{\epsilon}{32}}(\frac{\epsilon}{4})$ or $\beta(s) \in B_{\frac{\epsilon}{32}}(\frac{\epsilon}{4}')$, a contradiction.

PTL

Theorem 3.10 *$\eta(T)$ is open and dense in $C(T)$.*

Proof: The fact that $\eta(T)$ is open in $C(T)$ is shown in Propositon 3.32. Let $h \in C(T) - \eta(T)$ and $\epsilon > 0$ be given. There exists an $s \in T - \{0\}$ and a $\delta > 0$

such that $0 \notin B_\delta(s)$ and $\|h(t)\| < \frac{\epsilon}{4}$, for every $t \in B_\delta(s)$. Without loss of generality, $s \in l_1 - \{0\}$. Define $\hat{h}$ piecewise linearly on $cl(B_\delta(s))$ as follows:
$\hat{h}(s-\delta) = h(s-\delta), \hat{h}(s-\frac{3}{4}\delta) = 0, \hat{h}(s-\frac{\delta}{2}) = \frac{\epsilon}{4}, \hat{h}(s-\frac{3\delta}{10}) = 0, \hat{h}(s-\frac{\delta}{5}) = \frac{\epsilon''}{8}$,
$\hat{h}(s-\frac{\delta}{10}) = 0, \hat{h}(s) = \frac{\epsilon'}{4}, \hat{h}(s+\frac{\delta}{10}) = 0, \hat{h}(s+\frac{\delta}{5}) = \frac{\epsilon}{8}, \hat{h}(s+\frac{3\delta}{10}) = 0, \hat{h}(s+\frac{\delta}{2})$
$= \frac{\epsilon''}{4}, \hat{h}(s+\frac{3}{4}\delta) = 0$, and $\hat{h}(s+\delta) = h(s+\delta)$. If $t \in T - cl(B_\delta(s))$, let $\hat{h}(t) = h(t)$. $\hat{h}$ is continuous by the pasting lemma ([19], pg. 108). $\hat{h}(cl(B_\delta(s))) \supset h(cl(B_\delta(s)))$, so $\hat{h}$ is onto. Also, by the way that $\hat{h}$ is defined on $cl(B_{\frac{\delta}{2}}(s))$ and the previous lemma, $\hat{h} \in \eta(T)$. If $t \in cl(B_\delta(s)), \|h(t) - \hat{h}(t)\| < \epsilon$. Therefore, $\|h - \hat{h}\|_\infty \leq \frac{\epsilon}{2}$. Since, $\epsilon > 0$ was arbitrary, $\eta(T)$ is dense in $C(T)$.

PTL

Corollary 3.1 *If G is any finite, connected, planar graph containing a branch point, $\eta(G)$ is open and dense in $C(G)$.*

Proof: The previous proposition holds to show that $\eta(G)$ is open in $C(G)$ and the above construction can be used at a branch point of G to show that $\eta(G)$ is dense in $C(G)$.

PTL

Note that if $f \in \eta(T)$, the Barge-Martin ([1]) construction cannot be used to embed $\varprojlim f$ as an attractor in the plane. However, as the following example shows, there is a map $f \in \eta(T)$ such that $\varprojlim f$ can be embedded as an attractor in the plane.

Example 3.1 *Let $f : T \to T$ be defined piecewise linearly on $l_1 \cup l_2$ as follows:*

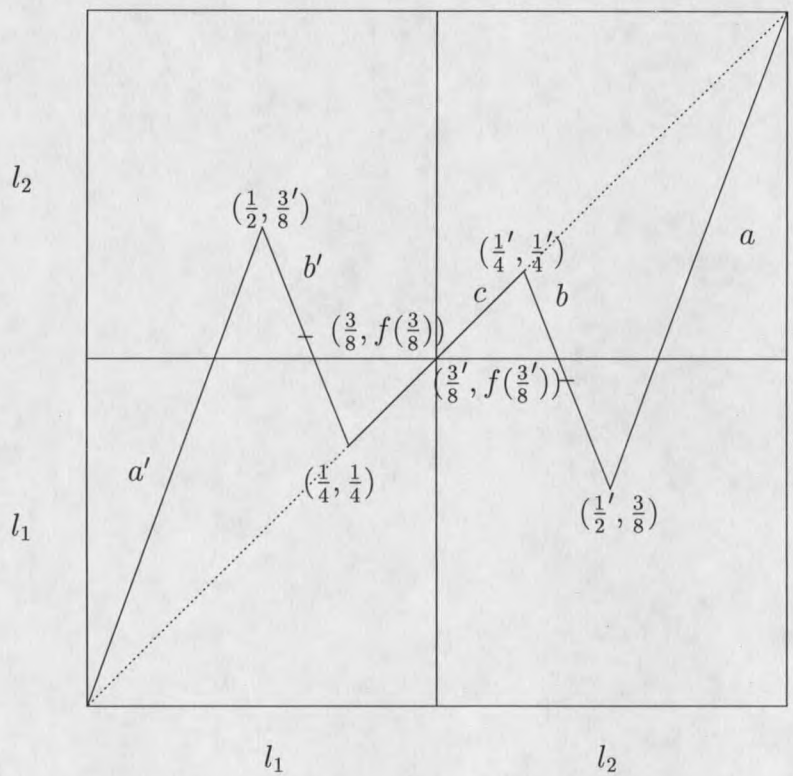


Figure 13: $f|_{l_1 \cup l_2}$

Define $f|_{l_3} \equiv idy$.

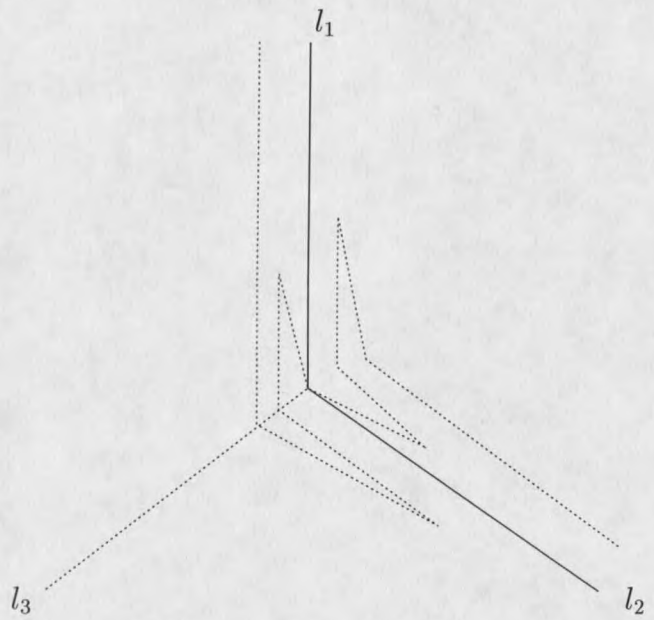


Figure 14: image of f

Then $f \in \eta(T)$ *and* $\varprojlim f$ *can be embedded as a global attractor in the plane.*

Proof: $\underline{f \in \eta(T):}$ Suppose that $f \in C(T) - \eta(T)$ and let $h : T \to B_{\frac{\epsilon}{32}}(T)$ be an embedding such that $\|h - f\|_\infty < \frac{\epsilon}{32}$. Note that $h([0, \frac{1}{2}']) \cap B_{\frac{\epsilon}{32}}(\frac{3}{8}') = \emptyset$. Let $N = B_{\frac{\epsilon}{32}}(l_1 \cup l_3 \cup [0, \frac{3}{8}']), L_1 = cl(N) \cap cl(B_{\frac{\epsilon}{32}}(1)), L_2 = cl(N) \cap cl(B_{\frac{\epsilon}{32}}(\frac{3}{8}'))$, and $L_3 = cl(N) \cap cl(B_{\frac{\epsilon}{32}}(1''))$. Let $p_1 \in L_1, p_2 \in L_2$, and $p_3 \in L_3$ and as in the proof that $\eta(T)$ is open and dense in $C(T)$, choose simple arcs $\alpha_1, \alpha_2, \alpha_3$ from p_1 to $h(s_1)$, $h(s_2)$ to p_2, and $h(s_3)$ to p_3 respectively such that $\alpha_1 \cap h(T) = \{h(s_1)\}$, $\alpha_1 \subset cl(B_{\frac{\epsilon}{32}}(1)), \alpha_2 \cap h(l_1 \cup l_3 \cup [0, \frac{1}{2}']) = \{h(s_2)\}, \alpha_2 \subset cl(B_{\frac{\epsilon}{32}}(\frac{3}{8}')), \alpha_3 \cap h(T) = \{h(s_3)\}$, $\alpha_3 \subset cl(B_{\frac{\epsilon}{32}}(1'')), s_1, s_2 \in l_1, 0 < s_2 < s_1 < 1$ and $s_3 \in l_3$. $\alpha_1 \cup h([s_1, s_2]) \cup \alpha_2$ divides N into two sub-disks D_1 and D_2. $h([0, s_2))$ and $h([0, \frac{1}{2}'])$ are entirely contained within one of these two disks. But, $h(l_3) \subset D_1$, so $h([0, s_2)), h([0, \frac{1}{2}']) \subset D_1$.

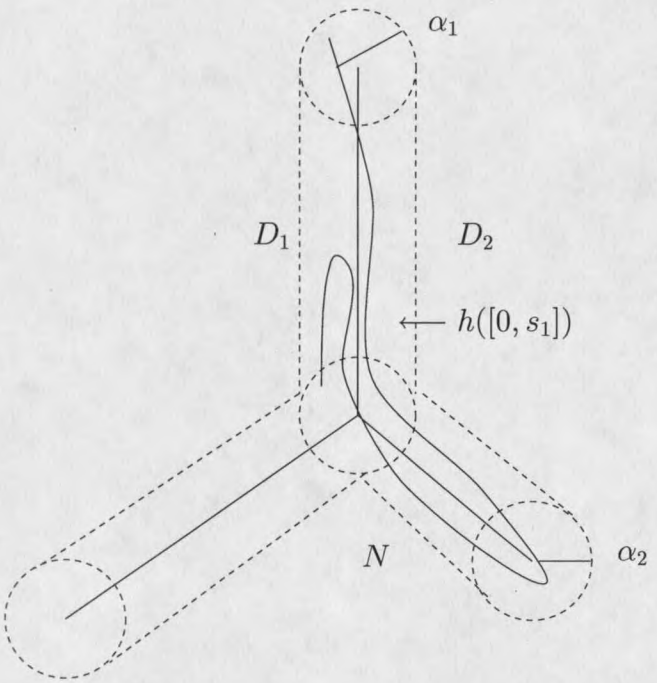


Figure 15:

$h([0, s_1]) \cup h([0, s_3]) \cup \alpha_3$ breaks D_1 up into two sub-disks D_{11} and D_{12}.

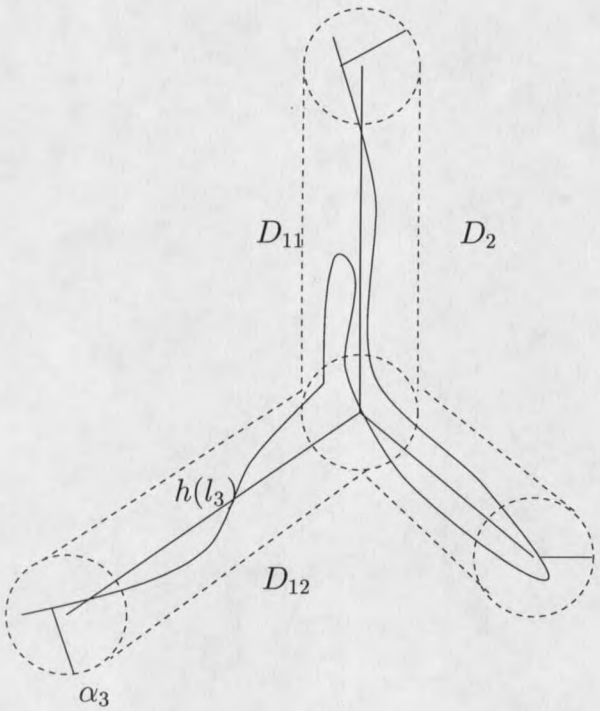


Figure 16:

Since $h(\frac{1}{2}') \in D_1$ and $h(\frac{1}{2}') \in B_{\frac{\epsilon}{32}}(\frac{3}{8}), h(\frac{1}{2}') \in D_{11}$. $h([\frac{1}{2}', 1'])$ cannot intersect ∂D_{11}, so $h([\frac{1}{2}', 1']) \subset D_{11}$ but $h(1') = 1'$, a contradiction. Thus, $f \in \eta(T)$.

Let $x \in [\frac{3}{8}, 1']$, then by choosing all pre-images of x from the segment labeled "a" on the graph of $f|_{l_1 \cup l_2}$, we get a simple arc $S_0 = \{(x, a_1, a_2, a_3, \ldots) | x \in [\frac{3}{8}, 1']\}$ in $\varprojlim f$ with endpoints $(1', 1', \ldots)$ and $(\frac{3}{8}, \frac{1}{2}', a_2^1, a_3^1, \ldots)$. In similar fashion, we complete the following table:

SIMPLE ARC	*ENDPOINTS*
$S_0 = \{(x, a_1, a_2, a_3, \ldots) \mid x \in [\frac{3}{8}, 1']\}$	$(1', 1', 1', \ldots)$ $(\frac{3}{8}, \frac{1}{2}', a_2^1, a_3^1, \ldots)$
$S_1 = \{(x, b_1, a_2, a_3, \ldots) \mid x \in [\frac{3}{8}, \frac{1}{4}']\}$	$(\frac{1}{4}', \frac{1}{4}', a_2^2, a_3^2, \ldots)$ $(\frac{3}{8}, \frac{1}{2}', a_2^1, a_3^1, \ldots)$
$S_2 = \{(x, c_1, a_2, a_3, \ldots) \mid x \in [\frac{1}{4}, \frac{1}{4}']\}$	$(\frac{1}{4}', \frac{1}{4}', a_2^2, a_3^2, \ldots)$ $(\frac{1}{4}, \frac{1}{4}, a_2^3, a_3^3, \ldots)$
$S_3 = \{(x, b_1', a_2, a_3, \ldots) \mid x \in [\frac{1}{4}, f(\frac{3}{8})]\}$	$(f(\frac{3}{8}), \frac{3}{8}, \frac{1}{2}', a_3^4, a_4^4, \ldots)$ $(\frac{1}{4}, \frac{1}{4}, a_2^3, a_3^3, \ldots)$
$S_4 = \{(x, b_1', b_2, a_3, a_4, \ldots) \mid x \in [\frac{1}{4}, f(\frac{3}{8}]\}$	$(f(\frac{3}{8}), \frac{3}{8}, \frac{1}{2}', a_3^4, a_4^4, \ldots)$ $(\frac{1}{4}, \frac{1}{4}, b_2^5, a_3^5, a_4^5, \ldots)$
$S_5 = \{(x, c_1, b_2, a_3, a_4, \ldots) \mid x \in [\frac{1}{4}, \frac{1}{4}']\}$	$(\frac{1}{4}', \frac{1}{4}', \frac{1}{4}', a_3^6, a_4^6, \ldots)$ $(\frac{1}{4}, \frac{1}{4}, b_2^5, a_3^5, a_4^5, \ldots)$
$S_6 = \{(x, c_1, c_2, a_3, a_4, \ldots) \mid x \in [\frac{1}{4}, \frac{1}{4}']\}$	$(\frac{1}{4}', \frac{1}{4}', \frac{1}{4}', a_3^6, a_4^6, \ldots)$ $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, , a_3^7, a_4^7, \ldots)$
$S_7 = \{(x, c_1, b_2', a_3, a_4, \ldots) \mid x \in [\frac{1}{4}, f(\frac{3}{8})]\}$	$(f(\frac{3}{8}), f(\frac{3}{8}), \frac{3}{8}, \frac{1}{2}', a_4^8, a_5^8, \ldots)$ $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, a_3^7, a_4^7, a_5^7, \ldots)$
$S_8 = \{(x, c_1, b_2', b_3, a_4, a_5, \ldots) \mid x \in [\frac{1}{4}, f(\frac{3}{8})]\}$	$(f(\frac{3}{8}), f(\frac{3}{8}), \frac{3}{8}, \frac{1}{2}', a_4^8, a_5^8, \ldots)$ $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, b_3^9, a_4^9, a_5^9, \ldots)$
$S_9 = \{(x, c_1, c_2, b_3, a_4, a_5, \ldots) \mid x \in [\frac{1}{4}, \frac{1}{4}']\}$	$(\frac{1}{4}', \frac{1}{4}', \frac{1}{4}', \frac{1}{4}', a_4^{10}, a_5^{10}, \ldots)$ $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, b_3^9, a_4^9, a_5^9, \ldots)$
$S_{10} = \{(x, c_1, c_2, c_3, a_4, a_5, \ldots) \mid x \in [\frac{1}{4}, \frac{1}{4}']\}$	$(\frac{1}{4}', \frac{1}{4}', \frac{1}{4}', \frac{1}{4}', a_4^{10}, a_5^{10}, \ldots)$ $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, a_4^{11}, a_5^{11}, \ldots)$
$\vdots$	$\vdots$

Replace c_3 with b_3' in S_{10} to get S_{11}, replace a_4 with b_4 in S_{11} to get S_{12}, replace b_3' with c_3 in S_{12} to get S_{13}, replace b_4 with c_4 in S_{13} to get S_{14}, ... By symmetry, $(l_1 \cup l_2, f|_{l_1 \cup l_2})$ also contains simple arcs:

$S_0' = \{(x, a_1', a_2', a_3', \ldots) | x \in [1, \frac{3}{8}']\}$

$S_1' = \{(x, b_1', a_2', a_3', \ldots) | x \in [\frac{1}{4}, \frac{3}{8}']\}$

$S_2' = \{(x, c_1, a_2', a_3', \ldots) | x \in [\frac{1}{4}, \frac{1}{4}']\}$

$\vdots \qquad\qquad \vdots.$

In addition, it contains the simple arc:

$S = \{(x, c_1, c_2, c_3, \ldots) | x \in [\frac{1}{4}, \frac{1}{4}'], c_i = x, \text{for all } i\}$. We embed the arcs $S, S_0, S_1, S_2, \ldots$ in $\mathbb{R}^2$ as follows:

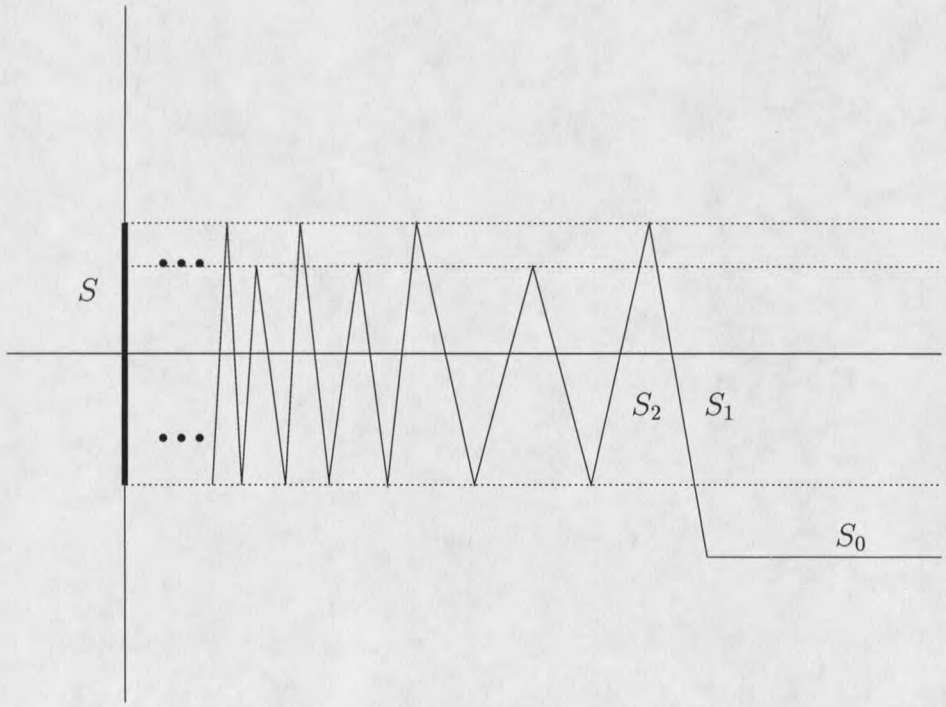


Figure 17:

The shift map on this embedding stretches S_0 across $S_0 \cup S_1 \cup S_2 \cup S_3$, S_1 across $S_4 \cup S_5$, S_2 maps across S_6, S_3 across $S_7, \ldots$. Each "W" to the left of $S_0 \cup S_1$ gets shifted onto the "W" to its immediate left. In similar fashion, we embed the simple arcs $S, S'_0, S'_1, S'_2, \ldots$ in $\mathbb{R}^2$ as follows:

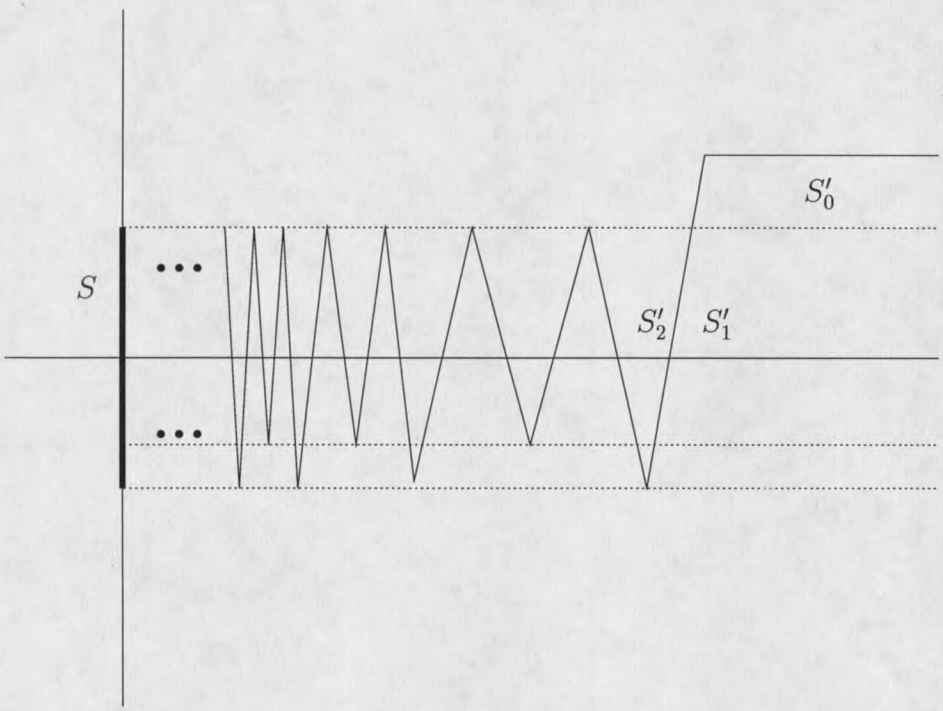


Figure 18:

$\hat{f}$ on this embedding stretches $S'_0 \cup S'_1$ across the "M" $S'_2 \cup S'_3 \cup S'_4 \cup S'_5$ and shifts each "M" onto the "M" to its immediate left. We alter the above embedding, so that when combined with the previous embedding, we have the following planar embedding of $\varprojlim f$:

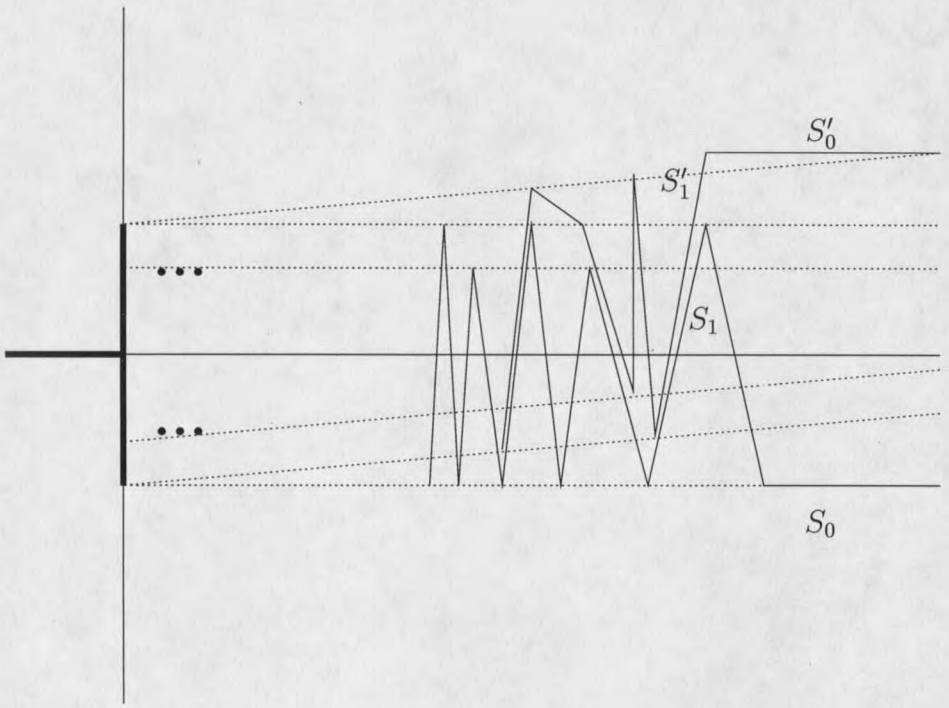


Figure 19:

We add arcs to the previous planar embedding of $\varprojlim f$ to get the following disk D and subdisks $\{\dot{D}_i\}_{i=1}^{\infty}$, $\{D_i\}_{i=1}^{\infty}$, and $\{\ddot{D}_i\}_{i=1}^{\infty}$.

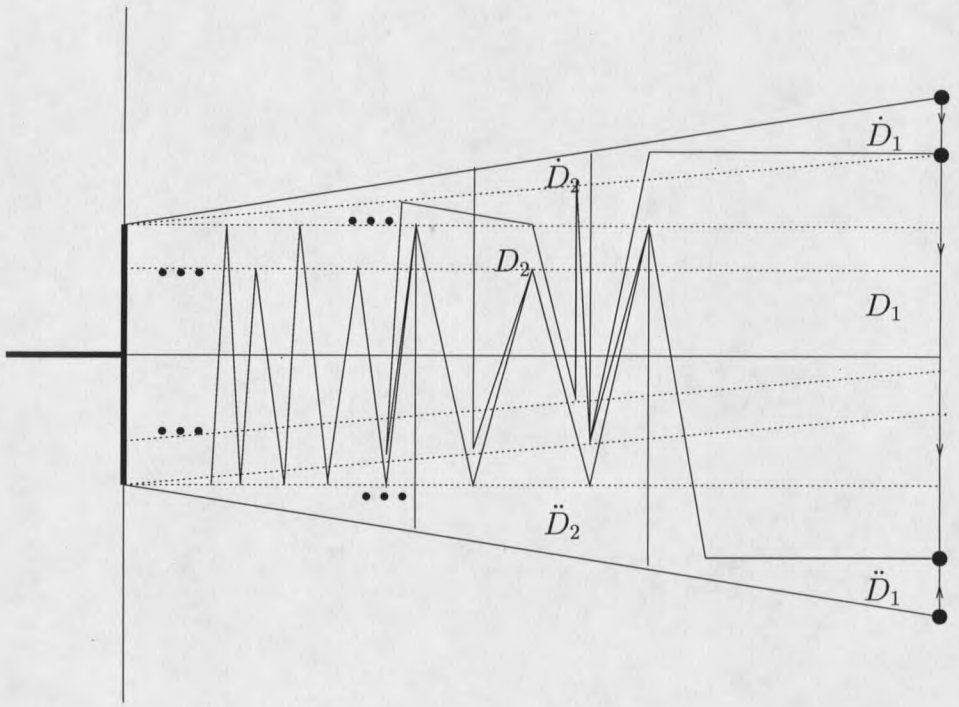


Figure 20: Disk D

Let g be $\hat{f}$, the shift homeomorphism, on our planar embedding of $\varprojlim f$. Define a homeomorphism $H : D \to D$ as follows:

Vertically folliate each $\dot{D}_i$ and $\ddot{D}_i$. Arrange that H takes the leaf intersecting (x_0, y_0) in $\dot{D}_i$ intersected with the embedding of $\varprojlim f$ onto the vertical leaf intersecting $g((x_0, y_0))$, similarly for each leaf in $\ddot{D}_i$. We define $H|_{\partial D_1} : \partial D_1 \to \partial(D_1 \cup D_2)$ in the obvious way and use Corollary 3, pg. 173, of [20] to extend to a homeomorphism $H|_{D_1} : D_1 \to D_1 \cup D_2$. We also need to insure that the right edge of D is invariant under H and that it contains exactly the four labeled fixed points–the interior two lying on our embedding of $\varprojlim f$. Finally, if x lies on this edge but is not one of the four fixed points we require that $H^n(x)$ converges monotonically to one of the two interior fixed points, as indicated. Define H on the limit bar to be the identity map. $H|_{D_i}$ basically shifts and horizontally shrinks each disk D_i onto D_{i+1} for $i \geq 2$. We add four arcs to the disk D to get the disks $\tilde{D}_1, \tilde{D}_2, \tilde{D}_3$, and $\tilde{D}_4$. Now, use the following lemma to define H on $\tilde{D}_1, \tilde{D}_2$, and $\tilde{D}_3$ and the lemma following it to define H on $\tilde{D}_4$ so that H is a homeomorphism of the disk $D \cup \tilde{D}_1 \cup \tilde{D}_2 \cup \tilde{D}_3 \cup \tilde{D}_4$ such that H is the identity on the boundary of this disk, our embedding of $\varprojlim f$ is a global attractor, and H restricted to this attractor is conjugate to $\hat{f}$. Since the interior of this disk is homeomorphic to $\mathbb{R}^2$, the result follows.

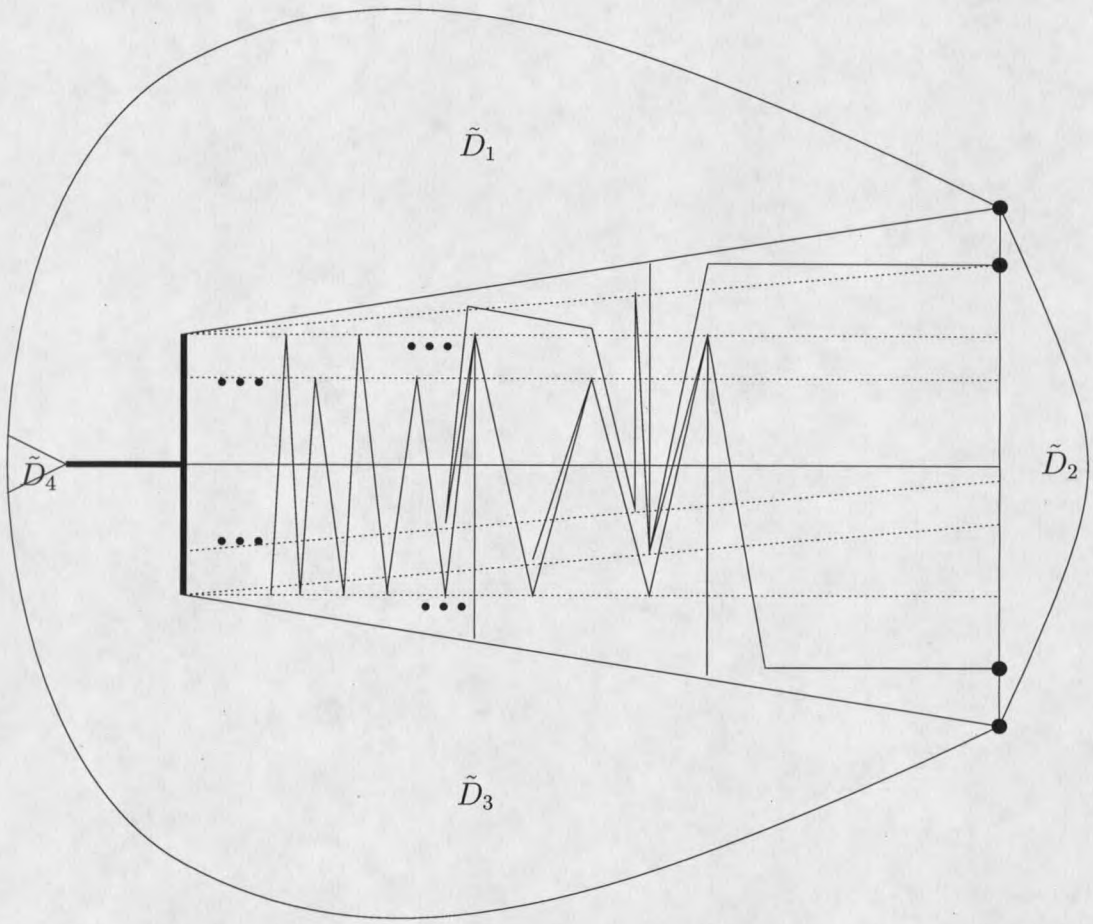


Figure 21: $D \cup \tilde{D}_1 \cup \tilde{D}_2 \cup \tilde{D}_3 \cup \tilde{D}_4$

PTL

Lemma 3.9 *Let $S = [0,1] \times [0,1]$ be the unit square in the xy-plane. Let $K|_{\partial S} : \partial S \to \partial S$ be defined by $K(x,0) = (\alpha(x), 0), K(x,1) = (x,1)$, $K(0,y) = (0, \beta(y))$, and $K(1,y) = (1, \beta(y))$, where $\alpha, \beta : [0,1] \to [0,1]$ are homeomorphisms with $\alpha(0) = \beta(0) = 0$ and $\alpha(1) = \beta(1) = 1$. Then $K : S \to S$ defined by $K((x,y)) = ((1-\beta(y))(\alpha(x) - x) + x, \beta(y))$ is a homeomorphism.*

Proof: K is continuous by the continuity of α and β. Suppose that $K((x_1,y_1)) = K((x_2,y_2))$ for $(x_1,y_1),(x_2,y_2) \in S$. Then $\beta(y_1) = \beta(y_2)$, so that $y_1 = y_2$. The map K can be thought of as a composition of two maps, the first of which takes (x,y) to $(x,\beta(y))$ and the second horizontally projects $(x,\beta(y))$ onto the line through the points $(x,1)$ and $(\alpha(x),0)$ in S. Since α is an orientation preserving homeomorphism of $[0,1]$, if $x_1 \neq x_2$, the lines l_1, through the points $(x_1,1)$ and $(\alpha(x_1),0)$, and l_2, through the points $(x_2,1)$ and $(\alpha(x_2),0)$, do not intersect. This contradicts the assumption that $K(x_1,y_1) = K(x_2,y_2)$. Therefore, $x_1 = x_2$ and K is one-to-one. By definition, K maps ∂S onto ∂S and $K(S) \subset S$. Therfore, K must be onto, otherwise one could construct a continuous map of S onto ∂S that is homotopically nontrivial when restricted to ∂S. Since K is bijective continuous and S is compact Hausdorff, K is a homeomorphism.

PTL

Lemma 3.10 *Let $S = \{(r,\theta) | 0 \leq r \leq 1, \theta = \frac{\pi}{2}t, 0 \leq t \leq 1\}$ be a quarter disk in the xy-plane and $\alpha, \beta : [0,1] \to [0,1]$ be homeomorphisms with $\alpha(0) = \beta(0) = 0$ and $\alpha(1) = \beta(1) = 1$. Then $K : S \to S$ defined by $K(r, \frac{\pi}{2}t) = (1-t)\alpha(r) + t\beta(r)$ is a homeomorphism.*

Proof: K is continous, since α and β are. If $K(r_1, \frac{\pi}{2}t_1) = K(r_2, \frac{\pi}{2}t_2)$, then $t_1 = t_2$, so that $(1-t_1)[\alpha(r_1) - \alpha(r_2)] + t_1[\beta(r_1) - \beta(r_2)] = 0$. If $t_1 = 0$ or $t_1 = 1$, then

we're done. So, assume that $(1-t_1), t_1 > 0$ and that $r_2 > r_1$. Since α and β are orientation preserving homeomorphisms of $[0,1]$, they must be strictly increasing. Thus, $\alpha(r_1) - \alpha(r_2), \beta(r_1) - \beta(r_2) < 0$, which contradicts the fact that

$$(1-t_1)[\alpha(r_1) - \alpha(r_2)] + t_1[\beta(r_1) - \beta(r_2)] = 0.$$

We reach a similar contradiction if we assume that $r_2 < r_1$. Hence, K is one-to-one. K maps ∂S onto itself, thus, as in the previous lemma, K is onto. Therefore, K is a homeomorphism.

PTL

CHAPTER 4

EMBEDDING n-OD-LIKE CONTINUA AS ATTRACTORS

We show that if T is any n-odd ($n \geq 3$) and $f : T \to T$ is continuous, then $\varprojlim f$ can be embedded as an attractor in $\mathbb{R}^3$. For the sake of introduction, assume that T is a triod with legs l_1, l_2, and l_3 of unit length and that $l_1 \cap l_2 \cap l_3 = \{0\}$. Also, assume that $f(0) = t_0 \in l_1, 0 \leq t_0 \leq 1$. To obtain our result, we create three coordinate systems Γ_1, Γ_2, and Γ_3–each containing a triod with legs $\tilde{L}_1, L_{12}, L_{13}$; $\tilde{L}_1, L_{22}, L_{23}$; and $\tilde{L}_1, L_{32}, L_{33}$ respectively so that $\Gamma_1 \cap \Gamma_2 \cap \Gamma_3 = \tilde{L}_1$. Next, we embed T in $M = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ so that l_1 is embedded in Γ_1, l_2 is embedded in Γ_2, l_3 is embedded in Γ_3, and 0 gets mapped onto $t_0 \in \tilde{L}_1$. In each of the coordinate systems Γ_1, Γ_2, and Γ_3 we select homeomorphisms that take l_1 onto $graph(f|_{l_1})$, l_2 onto $graph(f|_{l_2})$, and l_3 onto $graph(f|_{l_3})$, respectively. Choose near homeomorphisms in each of Γ_1, Γ_2, and Γ_3 so that $graph(f|_{l_1})$ is collapsed "properly" onto the triod $\tilde{L}_1 \cup L_{12} \cup L_{13}$; $graph(f|_{l_2})$ is collapsed "properly" onto $\tilde{L}_1 \cup L_{22} \cup L_{23}$; and $graph(f|_{l_3})$ is collapsed "properly" onto $\tilde{L}_1 \cup L_{32} \cup L_{33}$, respectively. Now, pick near homeomorphisms that map $\tilde{L}_1$ "properly" across l_1; L_{12}, L_{22}, and L_{32} "properly" across l_2; and L_{13}, L_{23}, and L_{33} "properly" across l_3. We end up with a near homeomorphism of $cl(B_8(\vec{0}))$ in $\mathbb{R}^3$ that restricted to T is f, and restricted to $\partial B_8(\vec{0}) = idy$. Finally, we choose a near homeomorphism of $cl(B_9(\vec{0}))$ in $\mathbb{R}^3$ that restricted to T is f, and so that T is an attractor for this near homeomorphism. The result will then follow by the Barge-Martin argument ([1]).

More generally, let T denote an n-odd ($n \geq 3$) with legs, $l_1, l_2, \ldots, l_n$, of unit length and let 0 denote the intersection of the legs. Orient T in the plane so that l_1 makes

an angle of $\frac{\pi}{2n}$ in the counter-clockwise direction from the positive x-axis, 0 corresponds to the origin, and the angle between consecutive legs is $\frac{2\pi}{n}$. Label the remaining legs $l_2, \ldots, l_n$, in the counter-clockwise direction. Let $\bar{T}$ be an n-odd with legs $L_1, L_2, \ldots, L_n$ of length 6, situated in the plane so that $L_1 \supset l_1, L_2 \supset l_2, \ldots, L_n \supset l_n$. Connect the extreme points of consecutive legs of $\bar{T}$ with line segments to form a regular n-gon, K. Label the edge joining L_1 and L_2 by e_1, L_2 and L_3 by $e_2, \ldots$, L_n and L_1 by e_n.

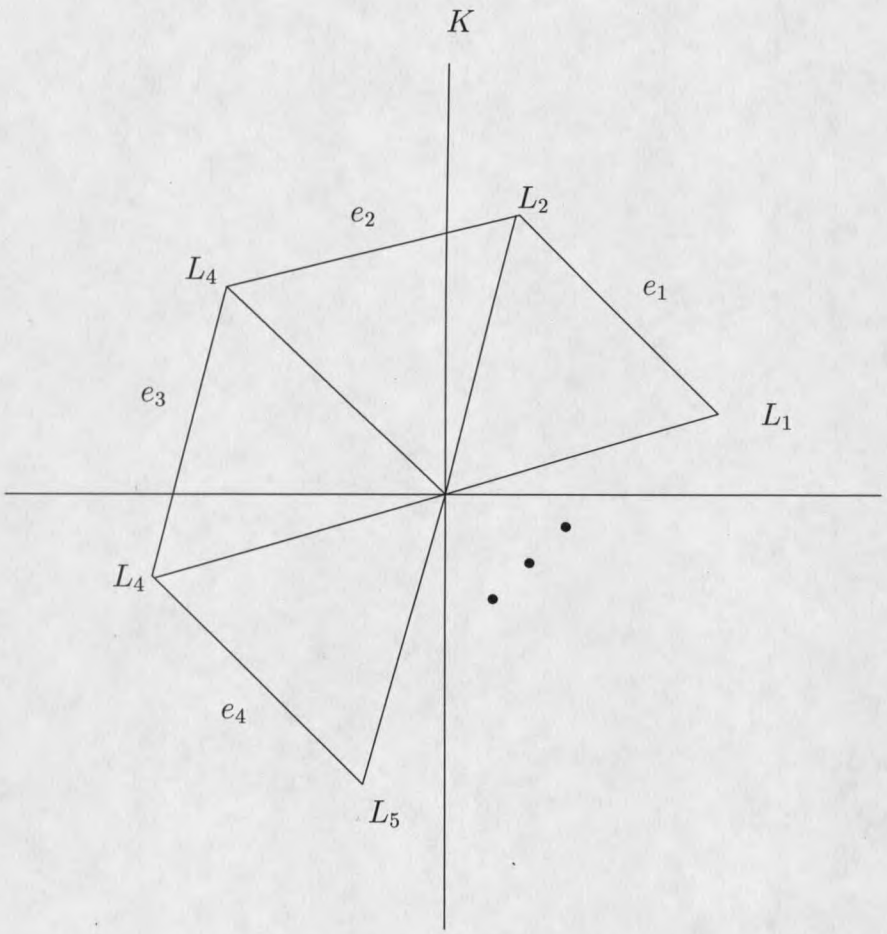


Figure 22: K

Choose an embedding

$\beta : K \to \{(\rho, \phi, \theta) \subset \mathbb{R}^3 | \rho \geq 0, 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \theta \leq \frac{\pi}{n}\}$ so that:

1. β is a translation on $L_1, L_2, \ldots, L_n$,

2. $\beta(L_1)$ is the interval $[0, 6]$ on the z-axis,

3. β linearly contracts between consecutive legs,

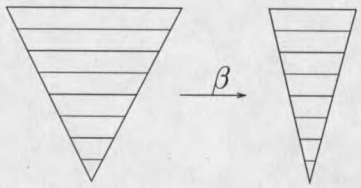


Figure 23: β

4. The projection onto the xy-plane of the images of $e_1, e_2, \ldots, e_n$ under β are as follows ($e_i' = \pi_{xy}(\beta(e_i))$):

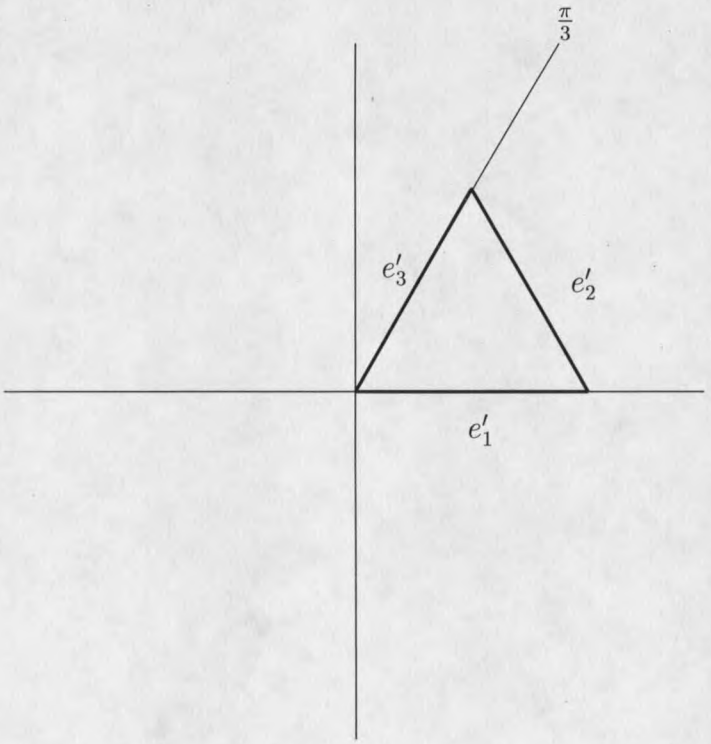


Figure 24: $n = 3$:

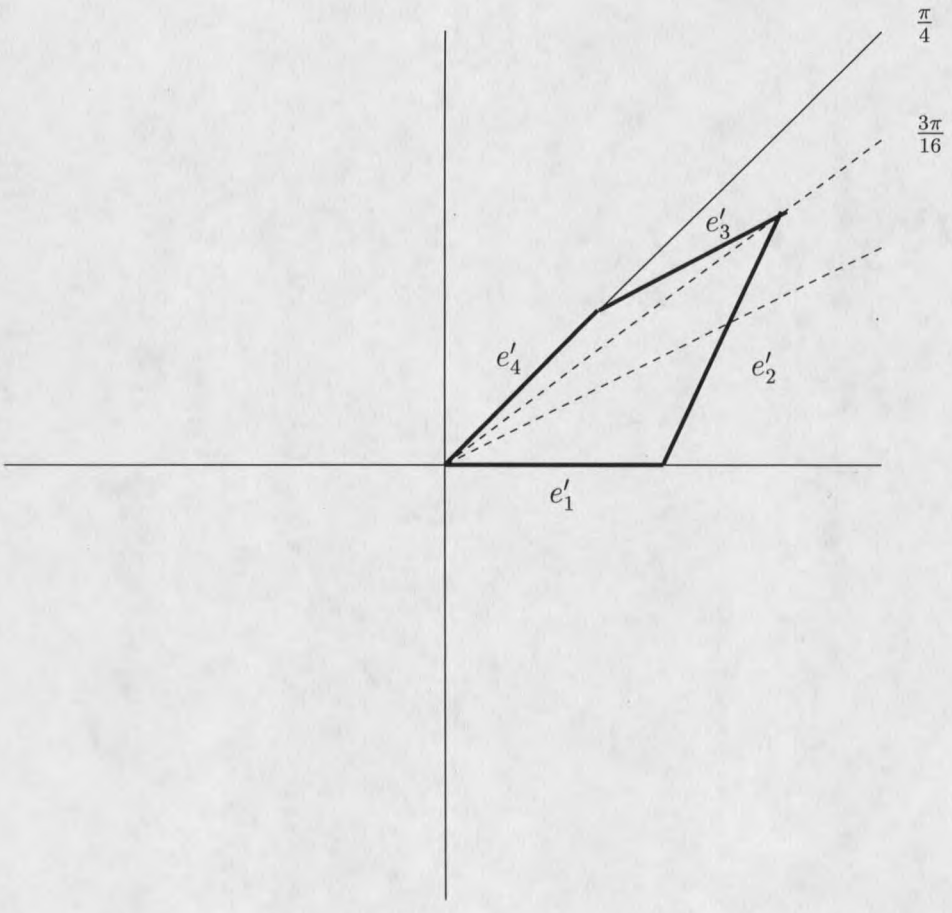


Figure 25: $n = 4$:

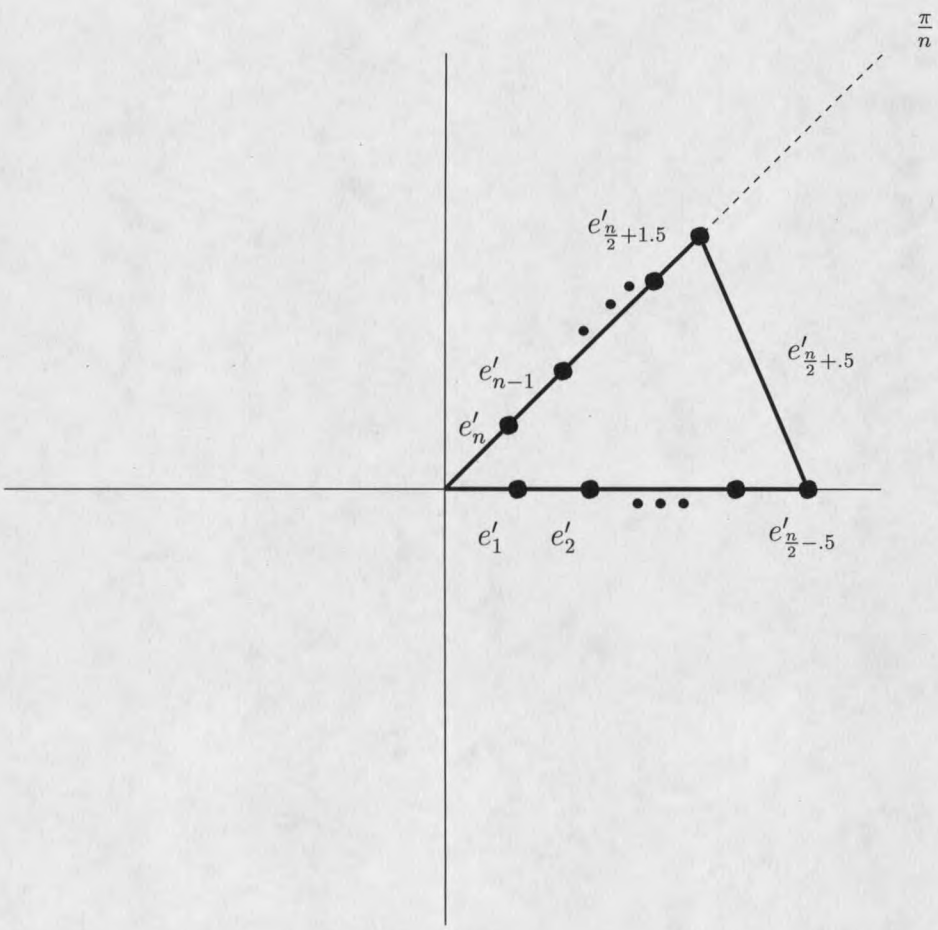


Figure 26: $n > 4, n$ odd:

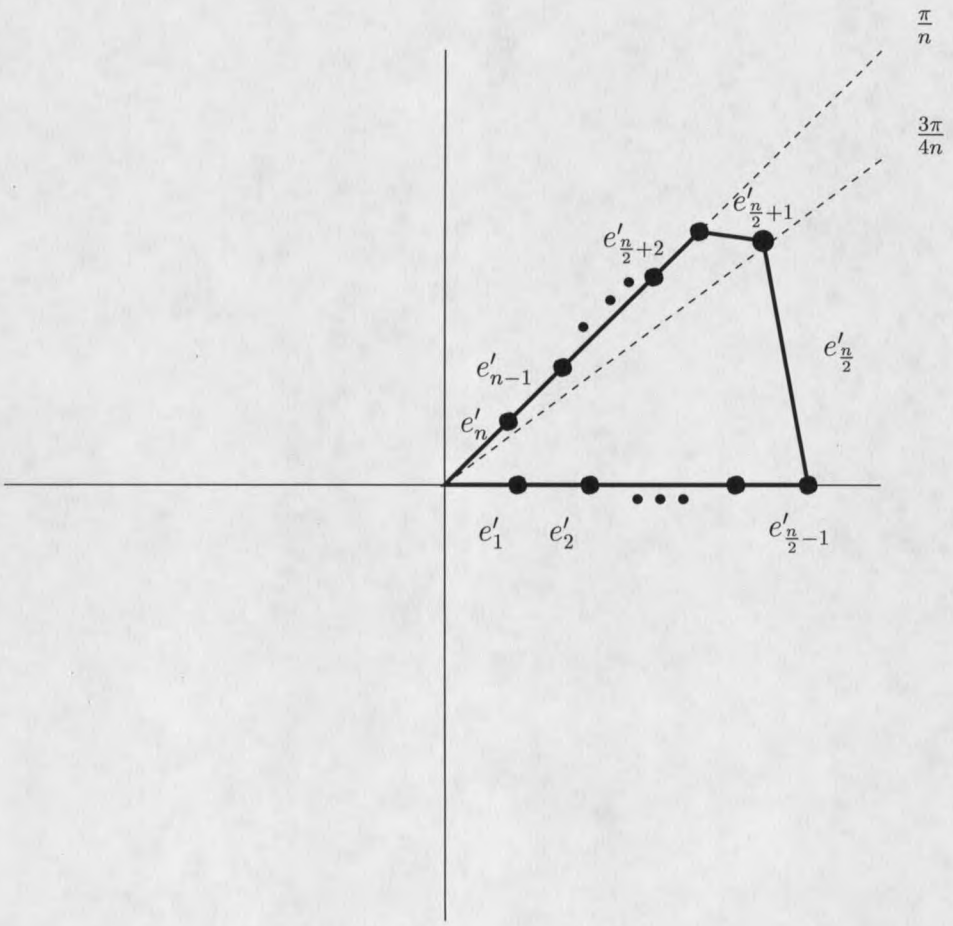


Figure 27: $n > 4, n$ even:

Let $\bar{K}$ be the solid $K \times [0, \frac{3}{2}]$. Define a homeomorphism $\bar{\beta} : \bar{K} \hookrightarrow \mathbb{R}^3$ by $\bar{\beta}(k, z) = \beta(k) + z\vec{v}$, where $k \in K, z \in [0, \frac{3}{2}], \vec{v}$ is in the convex hull of $L_1, \ldots, L_n, \|\vec{v}\| = 1$, and $\vec{v}$ is chosen so that $\bar{\beta}(\bar{K}) \subset \{(\rho, \phi, \theta) | \rho \geq, 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \theta \leq \frac{\pi}{n}\}$. Place $n-1$ copies of $\bar{\beta}(\bar{K})$ in the $n-1$ sectors $\{(\rho, \phi, \theta) | \rho \geq 0, 0 \leq \phi \leq \frac{\pi}{2}, \frac{2\pi}{n} \leq \theta \leq \frac{3\pi}{n}\}$, $\{(\rho, \phi, \theta) | \rho \geq 0, 0 \leq \phi \leq \frac{\pi}{2}, \frac{4\pi}{n} \leq \theta \leq \frac{5\pi}{n}\}, \ldots,$ $\{(\rho, \phi, \theta) | \rho \geq 0, 0 \leq \phi \leq \frac{\pi}{2}, \frac{(2n-2)\pi}{n} \leq \theta \leq \frac{(2n-1)\pi}{n}\}$, by composing $\bar{\beta}(\bar{K})$ with a counter-clockwise rotation about the z-axis through angles of $\frac{2\pi}{n}, \frac{4\pi}{n}, \ldots,$ and $\frac{(2n-2)\pi}{n}$, respectively. Let M be the space consisting of the union of these n solids. Note that the intersection of these n solids is precisely the interval $[0, 6]$ on the z-axis and that $M \subset B_8(\vec{0})$ irregardless of n. Now, $T \subset (-\frac{3}{2}, \frac{3}{2}) \times (-\frac{3}{2}, \frac{3}{2})$ and $[-2, 2] \times [-2, 2] \subset K$. Let $f : T \to T$ be continuous and without loss of generality, $f(0) = (x_0, y_0) \in l_1 \subset \mathbb{R}^2$. Since $(x_0, y_0) \in l_1, 0 \leq t_0 = \|(x_0, y_0)\| \leq 1$, we can label the point (x_0, y_0) by t_0 without ambiguity. We can do similarly for any point $(x, y) \in L_i$, as long as L_i is specified. Let $S = [-2, 2] \times [-2, 2]$, and $\bar{S} = [-2, 2] \times [-2, 2] \times [0, \frac{3}{2}]$.

Proposition 4.4 *Let $l_i, i \in \{1, 2, \ldots, n\}$ be given. There exists a homeomorphism $H : \bar{S} \to \bar{S}$ such that $H(t_0, z) = (f|_{l_i}(z), z)$, for $z \in [0, 1]$, and $H|_{\partial \bar{S}} = idy$. (i.e. H takes $\{t_0\} \times [0, 1]$ onto the graph of $f|_{l_i}$).*

Proof: Denote elements of l_i by t, where $0 \leq t \leq 1$ and $t = 0$ corresponds to $l_1 \cap \ldots \cap l_n = \{0\}$. Then $f|_{l_i} : [0, 1] \to T$ and $f(1) \in l_j, j \in \{1, 2, \ldots, n\}$. Extend $f|_{l_i}$ piecewise linearly to a map $\bar{f} : [0, \frac{3}{2}] \to T$, by $\bar{f}(1) = f|_{l_i}(1), \bar{f}(\frac{5}{4}) = 0$, and $\bar{f}(\frac{3}{2}) = t_0$. Define a homeomorphism $H_z : S \times \{z\} \to S \times \{z\}, z \in [0, \frac{3}{2}]$ as follows.

Vertically foliate $S \times \{z\}$:

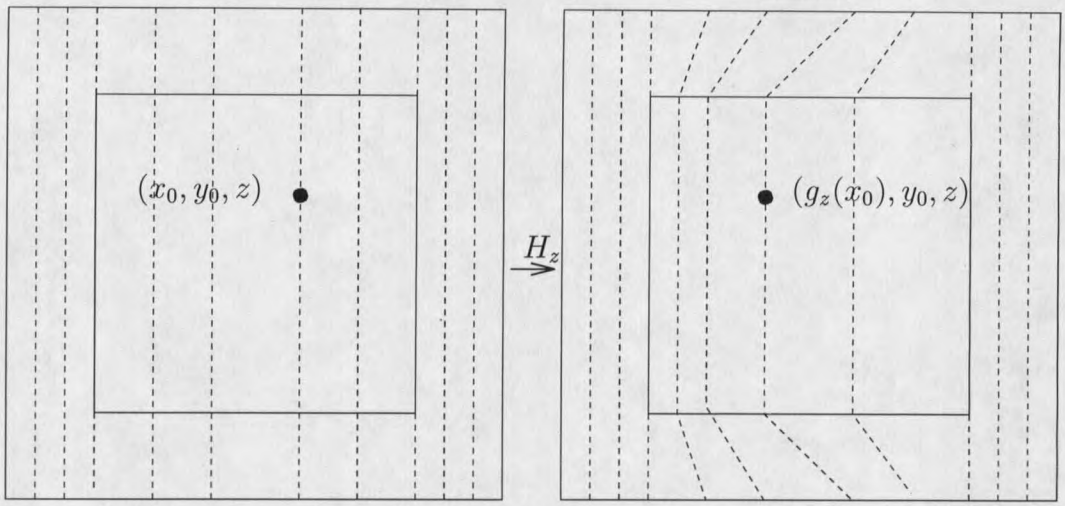


Figure 28: H_z

H_z translates each vertical line segment $\{x\}\times[-\frac{3}{2},\frac{3}{2}]\times\{z\}$ inside $[-\frac{3}{2},\frac{3}{2}]\times[-\frac{3}{2},\frac{3}{2}]\times\{z\}$ onto the vertical line segment $\{g_z(x)\}\times[-\frac{3}{2},\frac{3}{2}]\times\{z\}$ inside $[-\frac{3}{2},\frac{3}{2}]\times[-\frac{3}{2},\frac{3}{2}]\times\{z\}$. In the preceeding, $g_z : [-\frac{3}{2},\frac{3}{2}] \to [-\frac{3}{2},\frac{3}{2}]$ is defined as follows:

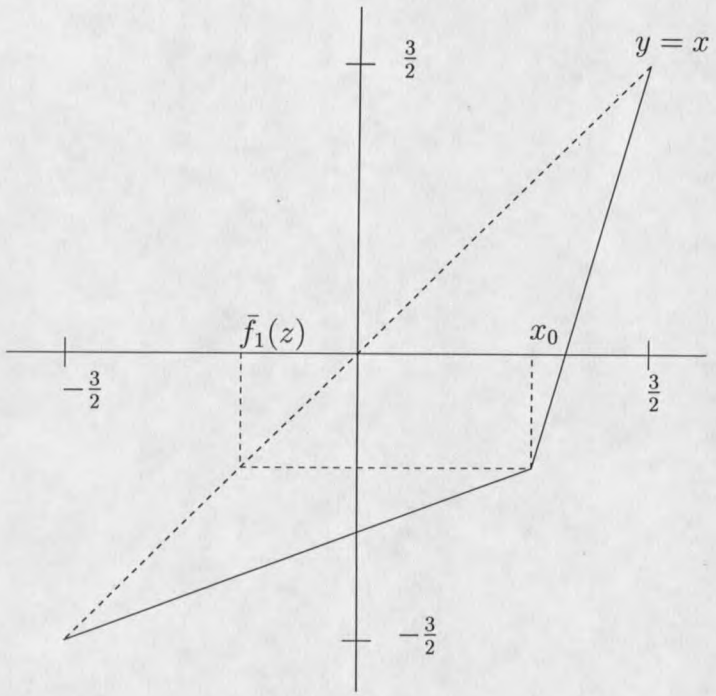


Figure 29: $y = g_z(x)$

where $\bar{f}(z) = (\bar{f}_1(z), \bar{f}_2(z))$. $H_z = idy$ on $([-2, -\frac{3}{2}] \times [-2, 2] \times \{z\}) \cup ([\frac{3}{2}, 2] \times [-2, 2] \times \{z\})$ and $H_z(x, y, z) = (2(y-2)(x - g_z(x)) + x, y, z)$ on $[-\frac{3}{2}, \frac{3}{2}] \times [\frac{3}{2}, 2] \times \{z\}$; i.e. H_z linearly stretches each vertical segment $\{x\} \times [\frac{3}{2}, 2] \times \{z\}$ onto the segment joining $(g_z(x), \frac{3}{2}, z)$ and $(x, 2, z)$. Similarly, on $[-\frac{3}{2}, \frac{3}{2}] \times [-2, -\frac{3}{2}] \times \{z\}$, $H_z(x, y, z) = (2(y + 2)(g_z(x) - x) + x, y, z)$; H_z linearly stretches each vertical segment $\{x\} \times [-2, -\frac{3}{2}] \times \{z\}$ onto the segment joining $(x, -2, z)$ and $(g_z(x), -\frac{3}{2}, z)$. Define $H : \bar{S} \to \bar{S}$ via $H(x, y, z) = H_z(x, y, z)$. Since each $H_z : S \times \{z\} \to S \times \{z\}$ is a homeomorphism, H is one-to-one and onto. H is continuous on $[-\frac{3}{2}, \frac{3}{2}] \times [-\frac{3}{2}, \frac{3}{2}] \times [0, \frac{3}{2}]$, since $H(x, y, z) = H_z(x, y, z) = (g_z(x), y, z)$, each g_z is continuous, and $\{g_z\}$ vary continuously in z.

Similarly, H is continuous on $([-\frac{3}{2},\frac{3}{2}]\times[\frac{3}{2},2]\times[0,\frac{3}{2}])\cup([-\frac{3}{2},\frac{3}{2}]\times[-2,-\frac{3}{2}]\times[0,\frac{3}{2}])$. Since $H = idy$ off the aforementioned sets, H is continuous by the pasting lemma. Because $\bar{S}$ is compact-Hausdorff, H is a homeomorphism.

Define a homeomorphism $V_z : S\times\{z\}\to S\times\{z\}, z\in[0,\frac{3}{2}]$ as follows. Horizontally foliate $S\times\{z\}$ as in figure 30.

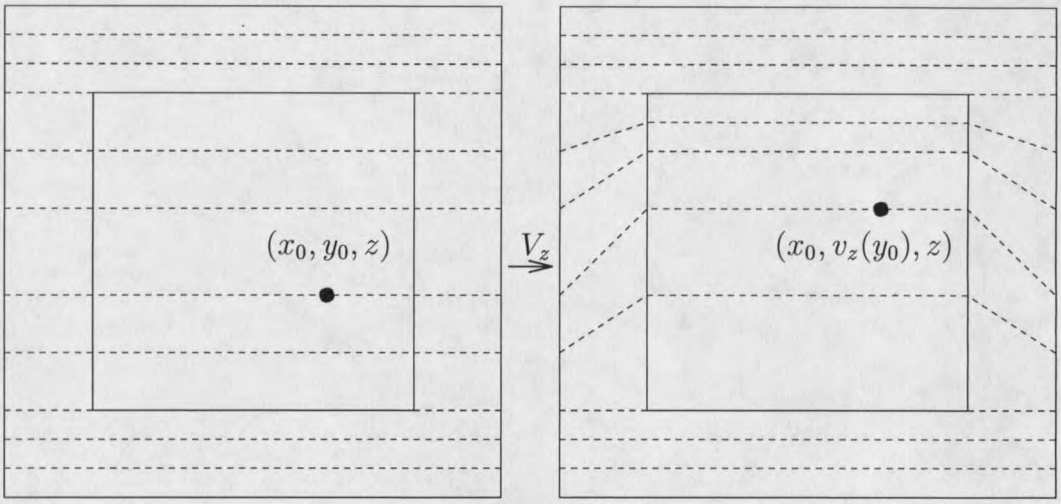


Figure 30: V_z

V_z translates each horizontal line segment $[-\frac{3}{2},\frac{3}{2}]\times\{y\}\times\{z\}$ inside $[-\frac{3}{2},\frac{3}{2}]\times[-\frac{3}{2},\frac{3}{2}]\times\{z\}$ onto the horizontal line segment $[-\frac{3}{2},\frac{3}{2}]\times\{v_z(y)\}\times\{z\}$ inside $[-\frac{3}{2},\frac{3}{2}]\times[-\frac{3}{2},\frac{3}{2}]\times\{z\}$, where $v_z : [-\frac{3}{2},\frac{3}{2}]\to[-\frac{3}{2},\frac{3}{2}]$ is defined as follows as in figure 31.

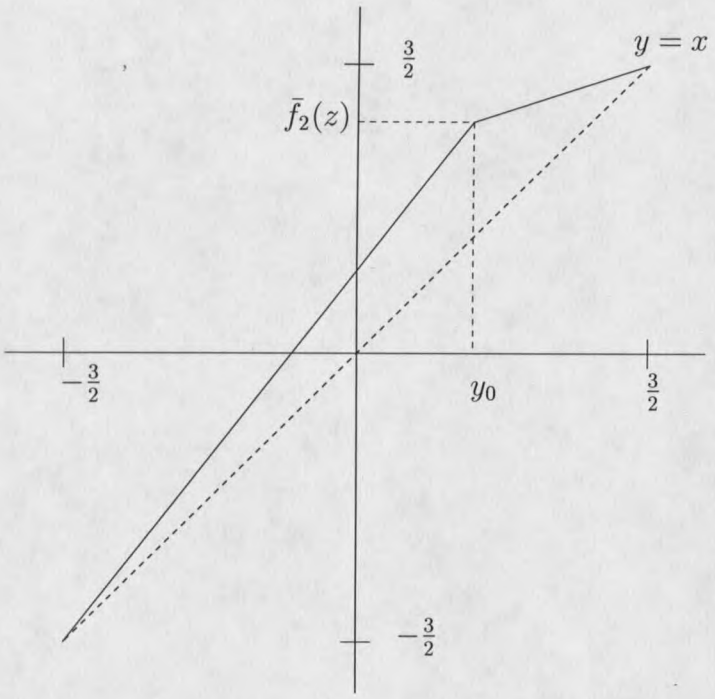


Figure 31: $y = v_z(x)$

$V_z = idy$ on $([-2,2]\times[-2,-\frac{3}{2}]\times\{z\})\cup([-2,2]\times[\frac{3}{2},2]\times\{z\})$ and $V_z(x,y,z) = (x, 2(v_z(y)-y)(x+2)+y, z)$; i.e. V_z linearly stretches each horizontal segment $[-2,-\frac{3}{2}]\times\{y\}\times\{z\}$ onto the segment joining $(-2,y,z)$ and $(-\frac{3}{2}, v_z(y), z)$. Similarly, on $[\frac{3}{2},2]\times[-\frac{3}{2},\frac{3}{2}]\times\{z\}, V_z(x,y,z) = (x, 2(y-v_z(y))(x-2)+y, z)$; i.e. V_z linearly stretches each horizontal segment $[\frac{3}{2},2]\times\{y\}\times\{z\}$ onto the segment joining $(\frac{3}{2}, v_z(y), z)$ and $(2,y,z)$. Define the homeomorphism $V : \bar{S} \to \bar{S}$ via $V(x,y,z) = V_z(x,y,z)$. Then $H \circ V$ is the required homeomorphism.

PTL

Now, label the n solids whose union is M as follows:

Let $\Gamma_1 = \bar{\beta}(\bar{K})$ and label the others $\Gamma_2, \Gamma_3, \ldots, \Gamma_n$ counterclockwise, where $\Gamma_i = \bar{\beta}_i(\bar{K}), i \geq 2, \bar{\beta}_i = R_i \circ \bar{\beta}$, R_i = counterclockwise rotation by $\frac{(2i-2)}{n}\pi$. We can write $\bar{\beta}_1 = \bar{\beta}$. Each Γ_i contains a copy of $\bar{T}$–the n-odd with legs $L_1, L_2, \ldots, L_n$, each of length 6. Moreover, $\cap_{i=1}^{n}\Gamma_i$ contains a copy $\tilde{L}_1 = \bar{\beta}_1(L_1) = \bar{\beta}_2(L_1) = \ldots = \bar{\beta}_n(L_1)$ of L_1 (the interval $[0,6]$ on the positive z-axis). On Γ_i, label the copies of $L_2, L_3, \ldots, L_n$ by $L_{i2} = \bar{\beta}_i(L_2), L_{i3} = \bar{\beta}_i(L_3), \ldots, L_{in} = \bar{\beta}_i(L_n)$, respectively. Note that $M \subset B_8(\vec{0})$. Choose a homeomorphism $\Psi : cl(B_8(\vec{0})) \to cl(B_8(\vec{0}))$ so that $\Psi(\vec{0}) = (0,0,t_0), \Psi(t_i,0) = \bar{\beta}_i(x_0,y_0,t_i)$ where $0 \leq t_i \leq 1, t_i \in L_i$, and $\Psi = idy$ on $\partial(cl(B_8(\vec{0}))$. By the previous proposition, for every $i = 1,2,\ldots,n$, there exists a homeomorphism $H_i : [-2,2] \times [-2,2] \times [0,\frac{3}{2}] = \bar{S} \to \bar{S}$ such that $H_i(x_0,y_0,z) = (f|_{l_i}(z),z)$, for $z \in [0,1]$ and $H_i|_{\partial\bar{S}} = idy$. As $\bar{S} \subset \bar{K}$, we can extend H_i to $\bar{H}_i : \bar{K} \to \bar{K}$, for every i, by letting $\bar{H}_i = idy$ on $\bar{K} - \bar{S}$. Choose a near homeomorphism $P : \bar{K} \to \bar{K}$ so that $P = idy$ on $\partial\bar{K}$ and $P^{-1}((x,y,0)) = (x,y) \times [0,1]$, for each $(x,y) \in [-2,2] \times [-2,2]$. Now, define a near homeomorphism $\Omega : M \to M$ as follows:

$\Omega|_{\Gamma_i} : \Gamma_i \to \Gamma_i$ is defined by $\Omega|_{\Gamma_i}(x,y,z) = \bar{\beta}_i \circ P \circ \bar{H}_i \circ \bar{\beta}_i^{-1}(x,y,z)$. Extend to $\bar{\Omega} : cl(B_8(\vec{0})) \to cl(B_8(\vec{0}))$, by letting $\bar{\Omega} = idy$ on $cl(B_8(\vec{0})) - M$. Then $\bar{\Omega} \circ \Psi : cl(B_8(\vec{0})) \to cl(B_8(\vec{0}))$ is a near homeomorphism such that if $(x,y) \in l_i \subset L_i \subset \mathbb{R}^2, \|(x,y)\| = t$, then $(\bar{\Omega} \circ \Psi)(x,y,0) = \bar{\Omega}(\bar{\beta}_i(x_0,y_0,t)) = \bar{\beta}_i \circ P \circ \bar{H}_i(x_0,y_0,t) = \bar{\beta}_i \circ P(f|_{l_i}(t),t) = \bar{\beta}_i(f(x,y))$. So, if $(\bar{\Omega} \circ \Psi)$ is composed with a near homeomorphism of $cl(B_8(\vec{0}))$ which "properly" takes $\tilde{L}_1$ onto L_1; $L_{22}, L_{23}, \ldots, L_{2n}$ onto L_2; …; and $L_{n2}, L_{n3}, \ldots, L_{nn}$ onto L_n, we will have succeeded in extending f to a near homeomorphism of $cl(B_8(\vec{0}))$. Toward that end, we prove the next two results.

Proposition 4.5 *Let* $Q = \{(\rho,\phi,\theta) \in \mathbb{R}^3 | 0 \leq \rho \leq r, 0 \leq \phi \leq \phi_1, \theta_1 \leq \theta \leq \theta_2$, *where* $0 < \phi_1 < \pi$ *and* $0 < \theta_2 - \theta_1 < 2\pi\}$, $S_1 = \{(\rho,\phi_0,\theta_0) | 0 \leq \rho \leq r^*, 0 < \phi_0 < \phi_1, \theta_1 <$

$\theta_0 < \theta_2, r^* < r, \phi_0, \theta_0$ *fixed*$\}$, *and* $S_2 = \{(\rho, \phi_1, \theta_0) | 0 \leq \rho \leq r^*, \theta_1 < \theta_0 < \theta_2\}$. *Then there exists a near homeomorphism* $g : Q \to Q$ *such that* $g|_{\partial Q} = idy, g(S_1) = S_2$, *and* $g(\rho, \phi_0, \theta_0) = (\rho, \phi_1, \theta_0) \in S_2$, *for all* $(\rho, \phi_0, \theta_0) \in S_1$. *Let* $S_3 = \{(\rho, 0, 0) | 0 \leq \rho \leq r^*\}$, *then there exists a near homeomorphism* $g^* : Q \to Q$ *such that* $g^*|_{\partial Q} = idy, g^*(S_1) = S_3$, *and* $g^*(\rho, \phi_0, \theta_0) = (\rho, 0, 0) \in S_3$, *for all* $(\rho, \phi_0, \theta_0) \in S_1$.

Proof: First choose a homeomorphism expanding/contracting in the ϕ-direction and rotating in the θ-direction, so that S_1 is taken onto $S_1' = \{(\rho, \phi_0; 0) | 0 \leq \rho \leq r^*\}$ and S_2 is taken onto $S_2' = \{(\rho, \frac{\pi}{2}, 0) | 0 \leq \rho \leq r^*\}$. Then select a homeomorphism of this sector fixing those points on the yz-plane and expanding/contracting in the θ-direction to map onto the sector $\{(\rho, \phi, \theta) | 0 \leq \rho \leq r, 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \theta \leq \pi\}$.

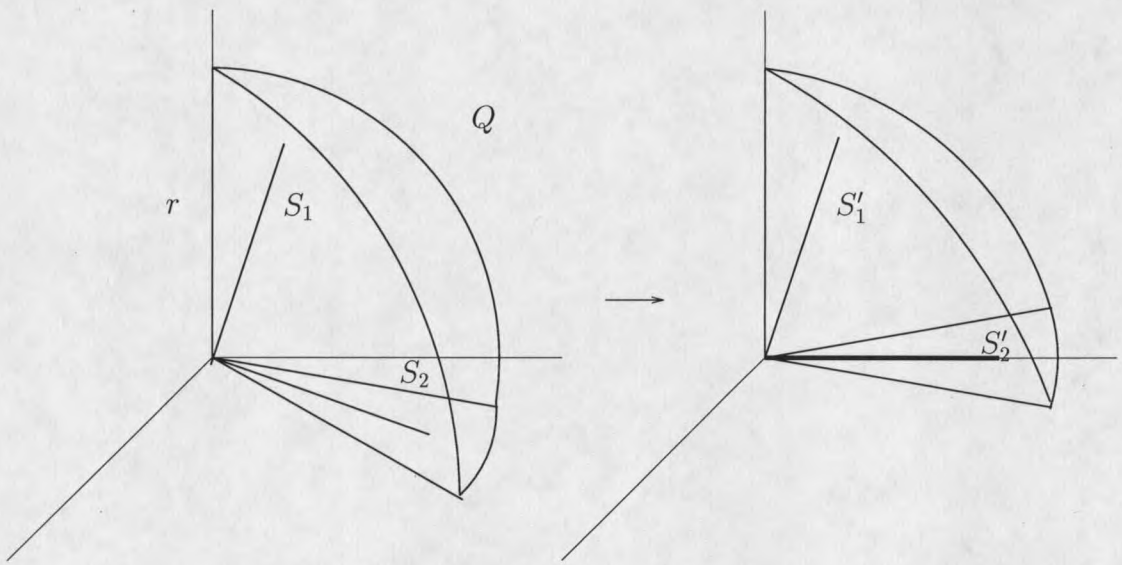


Figure 32:

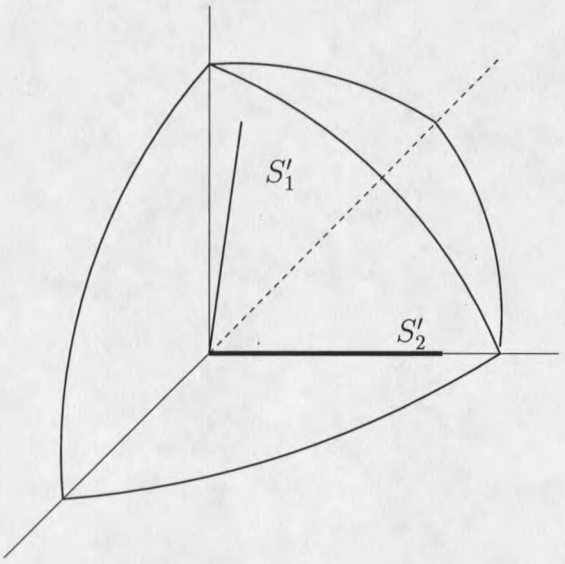


Figure 33:

Compose this homeomorphism, with a homeomorphism taking the sector $\{(\rho, \phi, \theta) | 0 \leq \rho \leq r, 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \theta \leq \pi\}$ onto the rectangular parallelpiped $[-r, r] \times [0, r] \times [0, r]$ and fixing S_1' and S_2'. Label the composition of all these homeomorphisms β.

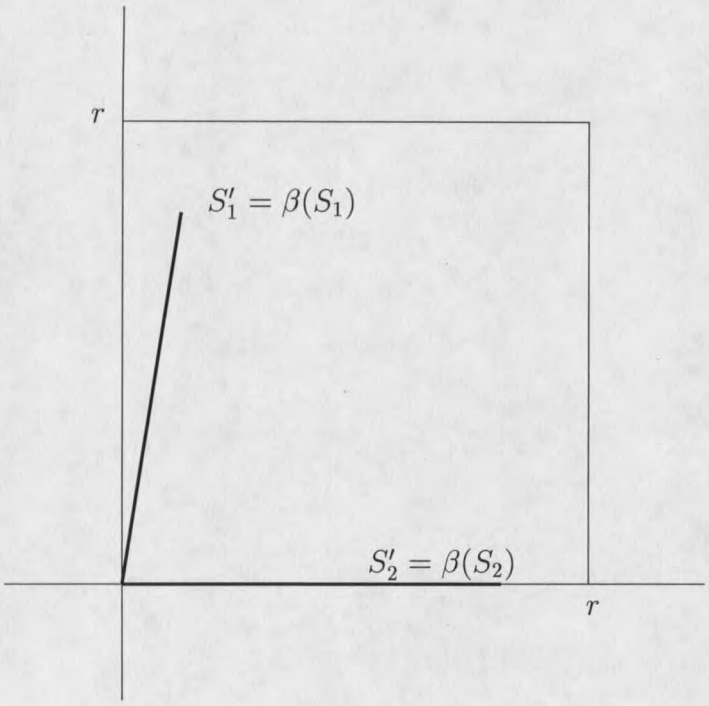


Figure 34: $\{0\} \times [0, r] \times [0, r]$

Let $S_0 = \{0\} \times [0, r] \times [0, r]$ and construct a continuous function $G : S_0 \times [0, 1] \to S_0$. Denote $G|_{S_0 \times \{t\}}$ by G_t. We will have the following properties:

1. $G_1 = idy$;
2. G_t is a homeomorphism if $0 < t \leq 1$;
3. for every $t, G_t|_{\partial S_0} = idy$;

4. if $(\rho,\phi,\theta)\in S_1', G_0((\rho,\phi,\theta))=(\rho,\frac{\pi}{2},0)\in S_2'$.

Then G_0 is a near homeomorphism by property 2. Define a map $G^\star:[-r,r]\times[0,r]\times[0,r]\to[-r,r]\times[0,r]\times[0,r]$ by $G^*(x,y,z)=G_{\frac{|x|}{r}}(0,y,z)+(x,0,0)$.

Claim: $G^\star$ is a near homeomorphism.

Proof of Claim: Let $\epsilon>0$ be given. Since any two norms on a finite dimensional vector space are equivalent, we will use the 1-norm, i.e. $\|(x,y,z,w)\|=|x|+|y|+|z|+|w|$. Since $S_0\times[0,1]$ is compact, G is uniformly continuous. So, there exists $0<\delta<1$ such that $\|(0,y_1,z_1,t_1)-(0,y_2,z_2,t_2)\|<\delta$ implies that $\|G_{t_1}(0,y_1,z_1)-G_{t_2}(0,y_2,z_2)\|<\frac{\epsilon}{2}$. For each $-r\le x\le r$, denote $\{x\}\times[0,r]\times[0,r]$ by S_x. Choose $0<\eta<\frac{\delta r}{1-\delta}$ and define $H^\star:S=\cup_{-r\le x\le r}S_x\to S$ by $H^*(x,y,z)=G_{\frac{|x|+\eta}{r+\eta}}(0,y,z)+(x,0,0)$. For fixed x, $H^\star|_{S_x}$ is a homeomorphism of S_x, so $H^\star$ is one-to-one and onto. S being compact-Hausdorff, in order to show $H^\star$ is a homeomorphism we need only establish that $H^\star$ is continuous. So, fix $(x,y,z)\in S$ and choose $0<\delta^*<\frac{\epsilon}{2}$ so that $(\max\{1,\frac{1}{r}\})\delta^*<\delta$. Then if $(x_1,y_1,z_1)\in B_{\delta^*}(x,y,z)\cap S$, $\|H^*(x,y,z)-H^*(x_1,y_1,z_1)\|\le\|G_{\frac{|x|+\eta}{r+\eta}}(0,y,z)-G_{\frac{|x_1|+\eta}{r+\eta}}(0,y_1,z_1)\|+\|(x,0,0)-(x_1,0,0)\|\le\|G_{\frac{|x|+\eta}{r+\eta}}(0,y,z)-G_{\frac{|x_1|+\eta}{r+\eta}}(0,y_1,z_1)\|+\frac{\epsilon}{2}$. But, $\|(0,y,z,\frac{|x|+\eta}{r+\eta})-(0,y_1,z_1,\frac{|x_1|+\eta}{r+\eta})\|\le|y-y_1|+|z-z_1|+\frac{|x-x_1|}{r+\eta}|\le(\max\{1,\frac{1}{r}\})\|(x,y,z)-(x_1,y_1,z_1)\|<(\max\{1,\frac{1}{r}\})\delta^*<\delta$. Therefore, $\|G_{\frac{|x|+\eta}{r+\eta}}(0,y,z)-G_{\frac{|x_1|+\eta}{r+\eta}}(0,y_1,z_1)\|<\frac{\epsilon}{2}$ and so $H^\star$ is continuous. Hence, $H^\star$ is a homeomorphism. Let $(x,y,z)\in S$, then $\|G^*(x,y,z)-H^*(x,y,z)\|=\|G_{\frac{|x|}{r}}(0,y,z)-G_{\frac{|x|+\eta}{r+\eta}}(0,y,z)\|<\frac{\epsilon}{2}$, since $|\frac{|x|}{r}-\frac{|x|+\eta}{r+\eta}|=|\frac{|x|(r+\eta)}{r(r+\eta)}-\frac{r(|x|+\eta)}{r(r+\eta)}|=|\frac{\eta(|x|-r)}{r(r+\eta)}|\le\frac{\eta}{r+\eta}=\frac{1}{1+\frac{r}{\eta}}<\frac{1}{1+\frac{r}{\frac{\delta r}{1-\delta}}}=\frac{1}{1+\frac{1-\delta}{\delta}}=\frac{\delta}{\delta+1-\delta}=\delta$. Thus, $\|G^*-H^*\|_\infty<\epsilon$, where $\|\cdot\|_\infty$ denotes the sup norm. Hence, $G^\star$ is a near homeomorphism, and the claim is proved.//

Now, $\beta^{-1} \circ G^{\star} \circ \beta$ is the required near homeomorphism g and the existence of $g^{\star}$ is proven similarly. This concludes the proof of the proposition.

PTL

The following proposition will complete our extension of f to a near homeomorphism of $cl(B_8(\vec{0}))$.

Proposition 4.6 *There exists a near homeomorphism* $K : cl(B_8(\vec{0})) \rightarrow cl(B_8(\vec{0}))$ *such that:*

(i) $K|_{\partial(cl(B_8(\vec{0})))} = idy$,

(ii) $K = idy$ *on a neighborhood of the negative z-axis intersect* $cl(B_8(\vec{0}))$ *of the form* $\{(\rho, \phi, \theta) | \rho \geq 0, \phi_0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi,$ *where* $\pi > \phi_0 > \frac{\pi}{2}$ *is fixed*$\} \cap cl(B_8(\vec{0}))$,

(iii) $K|_T = idy$,

(iv) $K(\tilde{L_1}) = L_1$,

(v) $K(L_{ij}) = L_i, 1 \leq i \leq n, 2 \leq j \leq n$,

(vi) if $t \in \tilde{L}_1, K(t) = t \in L_1$,

(vii) if $t \in L_{ij}, K(t) = t \in L_i$.

Proof: Induction on n:

$n = 3$:

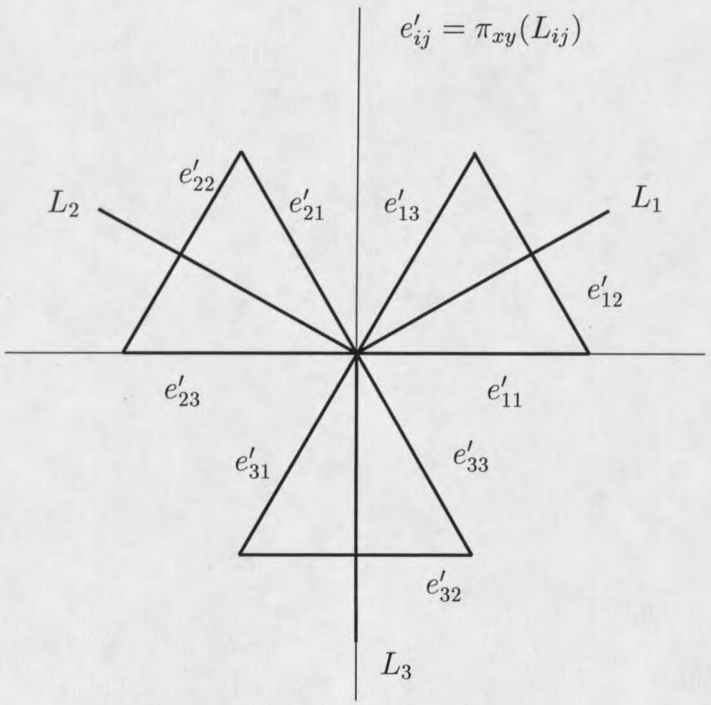


Figure 35:

$\tilde{L}_1$ lies along the interval $[0, 6]$ on the positive z-axis. Choose a translate Q_1 of a sector of form Q (as in the previous proposition) so that $\tilde{L}_1 \subset int(Q_1), L_1 \subset \partial Q_1, Q_1 \cap ((\cup_{1 \leq i \leq 3, 2 \leq j \leq 3} L_{ij}) \cup (\cup_{2 \leq i \leq 3} L_i)) = \{\vec{0}\}$, and so that the last proposition applies to give a homeomorphism $g_1 : Q_1 \to Q_1$ with $g_1(\tilde{L}_1) = L_1$, where $g_1(t) = t \in L_1$, for every $t \in \tilde{L}_1$. Extend g_1 to $G_1 : cl(B_8(\vec{0})) \to cl(B_8(\vec{0}))$ by letting $G_1 = idy$ on $cl(B_8(\vec{0})) - Q_1$.

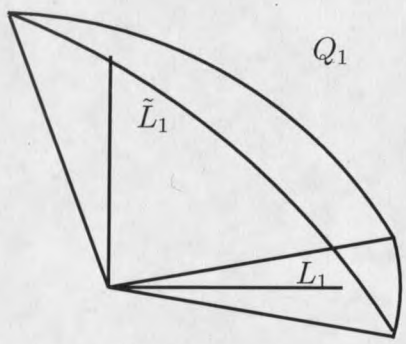


Figure 36:

Let l_z denote the interval $[0, 6]$ on the positive z-axis. Select a translate Q_{12} of a sector of form Q so that $l_z \subset \partial Q_{12}, L_{12} \subset int(Q_{12}), Q_{12} \cap ((\cup_{1 \leq i \leq 3, 2 \leq j \leq 3} L_{ij}) \cup (\cup_{1 \leq i \leq 3} L_i)) - L_{12}) = \{\vec{0}\}$, and such that the previous proposition guarantees a homeomorphism $g_{12} : Q_{12} \to Q_{12}$ with $g(L_{12}) = l_z$ and $g(t) = (0, 0, t) \in l_z$, for all $t \in L_{12}$.

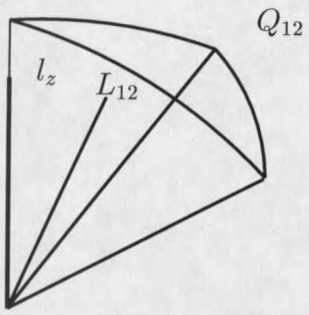


Figure 37:

Extend to a homeomorphism $G_{12} : cl(B_8(\vec{0})) \to cl(B_8(\vec{0}))$ by letting $G_{12} = idy$ on $cl(B_8(\vec{0})) - Q_{12}$. Similarly map L_{22} and L_{32} onto l_z with appropriate near homeomorphisms and then map l_z onto L_2 with a near homeomorphism. Finally, use the previous proposition to map each of L_{13}, L_{23}, and L_{33} onto l_z with a near homeomorphism and then l_z onto L_3. Note that we can choose each of these near homeomorphisms to be the identity on $\{(\rho, \phi, \theta) | \rho \geq 0, \frac{3}{4}\pi \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}$. Then the composition, K, of all these near homeomorphisms is the required near homeomorphism. Assume that the result is true for $n = k - 1$.

$\underline{n = k :}$ As in the $n = 3$ case, choose a homeomorphism $G_1 : cl(B_8(\vec{0})) \to cl(B_8(\vec{0}))$ such that $G_1(\tilde{L}_1) = L_1$ with $G_1(t) = t \in L_1$, for every $t \in \tilde{L}_1, G_1 = idy$ on $\partial(cl(B_8(\vec{0}))), G_1$ disturbs only the leg $\tilde{L}_1$, and $G_1 = idy$ on $cl(B_8(\vec{0})) \cap \{(\rho, \phi, \theta) | \rho \geq 0, \frac{3\pi}{4} \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}$. By the induction hypothesis, the near homeomorphism $K_{n-1} : cl(B_8(\vec{0})) \to cl(B_8(\vec{0}))$ for the $(k-1)$-odd is the identity on a set $\Gamma = cl(B_8(\vec{0})) \cap \{(\rho, \phi, \theta) | \rho \geq 0, \phi_0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$, where $\pi > \phi_0 > \frac{\pi}{2}$ is fixed$\}$. Choose a near homeomorphism taking L_1 onto the ray $\phi = \phi_0 + \frac{1}{4}(\pi - \theta_0), \theta = \frac{\pi}{2n}, 0 \leq$

$\rho \leq 6$, that is the identity on $\tilde{\Gamma} = \{(\rho, \phi, \theta) | \rho \geq 0, \phi_0 + \frac{1}{4}(\pi - \phi_0) \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\} \cap cl(B_8(\vec{0}))$. Now select appropriate near homeomorphisms taking L_{k2} onto L_2, L_{k3} onto $L_3, \ldots, L_{kk}$ onto L_k, and then $L_{12}, L_{22}, \ldots, L_{k-1\,2}$ onto l_z. Next, pick a homeomorphism of $cl(B_8(\vec{0}))$ that is the identity on $\tilde{\Gamma}$ and so that when composed with K_{n-1} maps L_z onto L_2; $L_{13}, L_{23}, \ldots, L_{k-1\,3}$ onto L_3; ...; and $L_{1k}, L_{2k}, \ldots, L_{k-1\,k}$ onto L_k. Choose a homeomorphism of $cl(B_8(\vec{0}))$ that is the identity on $\partial(cl(B_8(\vec{0}))) \cup \tilde{\Gamma}$ that takes $L_2, \ldots, L_k$ back to their original positions. Finally, choose a near homeomorphism taking L_1 back to its original position that is the identity off a sector not containing $\{(\rho, \phi, \theta) | \rho \geq 0, \phi_0 + \frac{1}{2}(\pi - \phi_0) \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}$. The composition of all these maps is the required map K and the result follows by induction.

PTL

$K \circ (\bar{\Omega} \circ \Psi)$ is then an extension of f to a near homeomorphism of $cl(B_8(\vec{0}))$ that is the identity on $\partial cl(B_8(\vec{0}))$. Extend $K \circ (\bar{\Omega} \circ \Psi)$ to a near homeomorphism γ of $cl(B_9(\vec{0}))$ by letting $\gamma = idy$ on $cl(B_9(\vec{0})) - cl(B_8(\vec{0}))$. Choose a homeomorphism $\hat{\gamma} : cl(B_9(\vec{0})) \to cl(B_9(\vec{0}))$ so that $\hat{\gamma} = idy$ on $\partial(cl(B_9(\vec{0}))), \hat{\gamma}^{-1}(cl(B_{\frac{1}{4}}(T))) = cl(B_8(\vec{0}))$, and so that for any $\vec{x} \in cl(B_9(\vec{0}))$, there exists an $n \in \mathbb{Z}^+$ so that $\hat{\gamma}^n(\vec{x}) \in cl(B_8(\vec{0}))$. Also, select a near homeomorphism $\bar{\gamma} : cl(B_9(\vec{0})) \to cl(B_9(\vec{0}))$ so that $\bar{\gamma} = idy$ on $cl(B_9(\vec{0})) - B_{\frac{1}{2}}(T)$ and so that $\bar{\gamma}^{-1}(T) = cl(B_{\frac{1}{4}}(T))$. Let $F = \bar{\gamma} \circ \hat{\gamma} \circ \gamma$. Then F is a near homeomorphism of $cl(B_9(\vec{0}))$ so that $F|_T = f$ and T is a global attractor for $cl(B_9(\vec{0}))$ under F. By the theorem of Morton Brown [10], $\varprojlim F$ is homeomorphic to $cl(B_9(\vec{0}))$. Finally, as in the paper by Barge-Martin ([1]), we have the following result:

Theorem 4.11 *If $f : T \to T$ is a continuous function, T an n-odd ($n \geq 3$), then $\varprojlim f$ can be embedded as an attractor in $\mathbb{R}^3$.*

CHAPTER 5

CHAINABILITY AND EXTENDIBILITY

As a first step in seeing which maps $f : G \to G$ have the property that $\varprojlim f$ can be embedded as an attractor in the plane, we give necessary and sufficient conditions that $\varprojlim f$ be snake-like or chainable. First, we need a definition and a theorem by Bing [7].

Definition 5.14 *(Bing [7]) A chain is a finite collection of open sets $d_1, d_2, \ldots, d_j$ such that d_i intersects d_j if and only if $i = j-1, j$ or $j+1$. We do not impose the condition that the d_i's are connected. If the links of a chain are of diameter less than ϵ, the chain is called an ϵ-chain. A compact continuum is called snake-like (or chainable) if for each positive number ϵ it can be covered by an ϵ-chain.*

Theorem 5.12 *(Bing [7]) Each snake-like continuum is homeomorphic with a plane continuum.*

Thus, if $f : G \to G$ is such that $\varprojlim f$ is chainable, $\varprojlim f$ is planar.

Theorem 5.13 *$f \in C(G)$. $\varprojlim f$ is chainable if and only if for every $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$ such that if $n \geq N$, there exist maps g_n from G onto $I = [0,1]$ and $h_n : I \to G$ so that the following diagram ϵ-commutes:*

$$\begin{array}{ccc} & f^n & \\ G & \longrightarrow & G \\ g_n \searrow & & \nearrow h_n \\ & I & \end{array}$$

Proof: ($\Rightarrow$) We prove the result for $f \in C(T)$, T a triod with each leg of unit length, the general case being very similar. Suppose that $\varprojlim f$ is chainable and let $\epsilon > 0$ be given. Let $L_1, \ldots, L_m$ be an $\frac{\epsilon}{2}$-chain of $\varprojlim f$ such that the distance between non-adjacent links is bounded below by some positive number and each link intersects $\varprojlim f$ (Barrett, [3]). Let η be a Lebesque number for the open cover $\{L_1, \ldots, L_m\}$ of $\varprojlim f$. Choose $N \in \mathbb{Z}^+$ such that $\sum_{i=N+1}^{\infty} \frac{diam(T)}{2^i} \leq \sum_{N+1}^{\infty} \frac{1}{2^{i-1}} < \frac{\eta}{2}$. Then for every $n \geq N$, there exists a $\delta(n) > 0$ such that if $t \in T, diam\pi_n^{-1}(B_\delta(t)) < \eta$. Fix $n \geq N$ and cover T with δ-balls as follows:

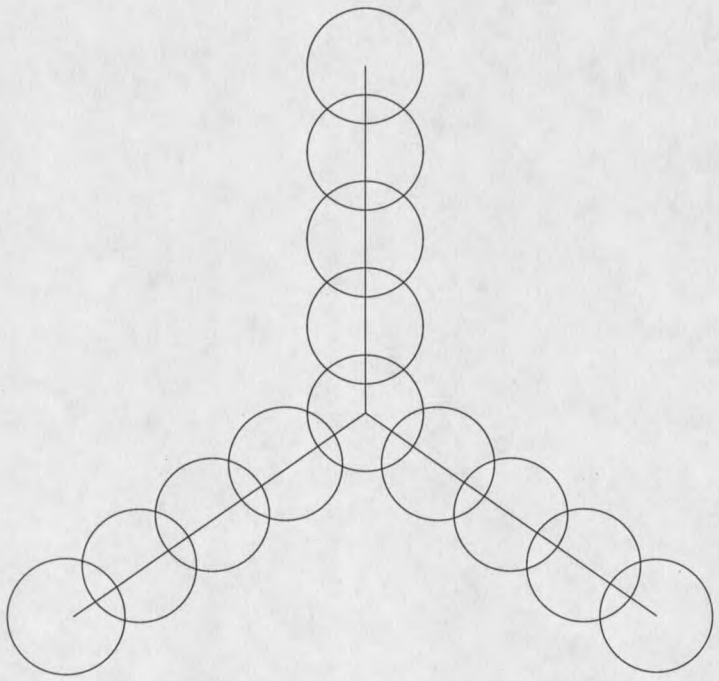


Figure 38:

$\cup_{i=1}^{l} B_\delta(x_i)$ where the distance between non-adjacent balls is bounded below by some

positive number and each ball intersects T. For each $i, \pi_n^{-1}(B_\delta(x_i)) \subset L_j$, some $1 \leq j \leq m$. For each i, let $j_i = \min_{1 \leq j \leq m} \{j | \pi_n^{-1}(B_\delta(x_i)) \subset L_j\}$. Let $j_\star = \min_{1 \leq i \leq l} \{j_i\}$ and $j^\star = \max_{1 \leq i \leq l} \{j_i\}$. $j_\star \leq j^\star$ by definition. If $j_\star = j^\star, \pi_n^{-1}(B_\delta(x_i)) \subset L_{j^\star}$, for every $1 \leq i \leq l$ which implies that $diam f^n(T) \leq \frac{\epsilon}{2}$–which can't happen if $\frac{\epsilon}{2} < diam T$, since f is onto. So, assuming that $\frac{\epsilon}{2} < diam T$, $j_\star < j^\star$. Let $d_{j_\star} = \cup_{\{i | j_i = j_\star\}} B_\delta(x_i), d_{j_\star + 1} = \cup_{\{i | j_i = j_\star + 1\}} B_\delta(x_i), \ldots, d_{j^\star} = \cup_{\{i | j_i = j^\star\}} B_\delta(x_i)$.

Proposition 5.7 *If $k \in \mathbb{Z}, j_\star < k < j^\star$, there exists $j_i, 1 \leq i \leq l$, such that $j_i = k$.*

Proof: If $d_k = \emptyset$, some $j_\star < k < j^\star$, $\varprojlim f = \pi_n^{-1}(T) \subset \{L_1, \ldots, L_{k-1}, L_{k+1}, \ldots, L_m\}$. Then, $\{A, B\}$ forms a separation of $\varprojlim f$, where $A = \cup_{i=1}^{k-1} L_i$ and $B = \cup_{i=k+1}^{m} L_i$, a contradiction.

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For ease of notation, assume that $j_\star = 1$ and $j^\star = m$. Chain $I = [0, 1]$ with m open intervals as follows:

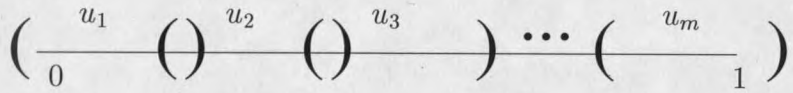

Figure 39:

i.e. each open interval intersects I and the distance between non-adjacent intervals is bounded below by a positive number. Then $\cup_{i=1}^{m-1}(u_i \cap u_{i+1})$ consists of $m-1$ disjoint open intervals, say $v_1, v_2, \ldots v_{m-1}$–ordered according to increasing left-hand endpoints. Also, $I - (\cup_{i=1}^{m-1} v_i)$ consists of m disjoint closed intervals $J_1, \ldots, J_m$–again ordered according to increasing left-hand endpoints.

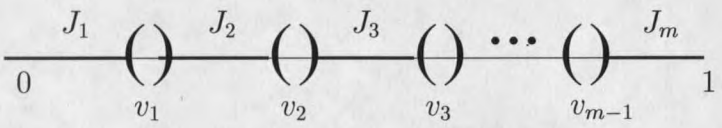

Figure 40:

Similarly, $\cup_{i\neq j}(d_i \cap d_j)$ is a finite collection of disjoint open intervals in T. $T-(\cup_{i\neq j}(d_i \cap d_j))$ is a finite disjoint union of closed sets–all but one of which is a closed interval.

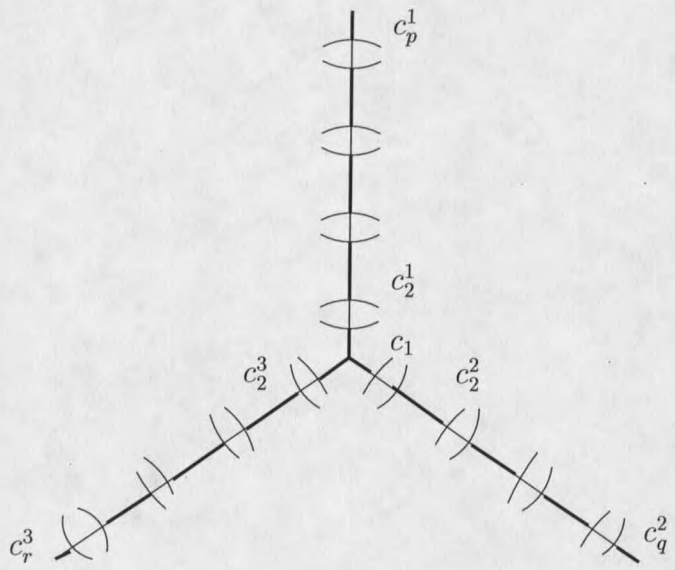

Figure 41:

Let c_1 be the closed set containing 0. Label the remaining closed intervals in l_1 according to increasing left-hand endpoints $c_2^1, c_3^1, \ldots, c_p^1$; the closed intervals in l_2 similarly as $c_2^2, c_3^2, \ldots, c_q^2$; and finally those in l_3 as $c_2^3, c_3^3, \ldots, c_r^3$. Note that c_1 is contained in exactly one d_k, and that the same is true of c_i^j, all i and j. Let $c_1^1 = c_1 \cap l_1$ and choose a continuous function $\Phi_1 : l_1 \to I$ such that if $c_i^1 \subset d_k, \Phi_1(c_i^1) = J_k$. $c_i^1 \subset d_k$ implies that $c_{i+1}^1 \subset d_{k-1}$ or d_{k+1}. If $c_{i+1}^1 \subset d_{k-1}$, we require that Φ_1 map the open interval in T whose closure intersects both c_i^1 and c_{i+1}^1 onto v_{k-1}. Finally, if $c_{i+1}^1 \subset d_{k+1}$, we require that $\Phi_1(c_{i+1}^1) = J_{k+1}$ and that Φ_1 map the open interval in T whose closure intersects both c_i^1 and c_{i+1}^1 onto v_k. Similarly, define maps $\Phi_2 : l_2 \to I$ and $\Phi_3 : l_3 \to I$ with $\Phi_2(0) = \Phi_3(0) = \Phi_1(0)$. Define a continuous function g_n from T onto I by

$g_n(t) = \left\{ \begin{array}{l} \Phi_1(t), t \in l_1 \\ \Phi_2(t), t \in l_2 \\ \Phi_3(t), t \in l_3 \end{array} \right\}$. Define a map $h_n : I \to T$ as follows:

for each $1 \leq k \leq m$ choose one of the disjoint closed subsets c_1 or c_i^j of $T - (\cup_{i \neq j}(d_i \cap d_j))$ contained in d_k and a continuous function Ψ_k from J_k onto $\{c_1 \text{ or } c_i^j\} \subset d_k$. Define $h_n|_{J_k} = f^n \circ \Psi_k$. This defines h_n on $\cup_{i=1}^m J_i$. Let $J_i = [a_i, b_i], i = 1, \ldots, m$, then $v_i = (b_i, a_{i+1}), i = 1, \ldots, m-1$. For $i = i, \ldots, m-1$, let $\alpha_i : I \to T$ be a path from $h_n(b_i)$ to $h_n(a_{i+1})$ of least diameter. Define $h_n|_{v_i}(t) = \alpha(\frac{t-b_i}{a_{i+1}-b_i})$. The h_n thusly defined is continuous.

Proposition 5.8 *The following diagram ϵ-commutes:*

$$\begin{array}{ccc} T & \overset{f^n}{\longrightarrow} & T \\ {\scriptstyle g_n}\searrow & & \nearrow{\scriptstyle h_n} \\ & I & \end{array}$$

Proof: <u>case 1</u>: Suppose that $t \in c_1$ or c_i^j, some i, j. Let $t \in c_1 \subset d_k$, some $1 \leq k \leq m$, the other cases being similar. Then $g_n(t) \in J_k$ and $h_n \circ g_n(t) = h_n|_{J_k} \circ g_n(t) = f^n \circ \Psi_k \circ g_n(t) = f^n(x)$, where $x \in d_k$. But, $diam\pi_n^{-1}(d_k) \leq diam L_k < \frac{\epsilon}{2}$, since $\pi_n^{-1}(d_k) \subset L_k$. Therefore, $diam f^n(d_k) \leq diam\pi_n^{-1}(d_k) < \frac{\epsilon}{2}$. Hence, $\|f^n(t) - h_n g_n(t)\| < \frac{\epsilon}{2}$.//

For the last two cases, assume that $t \in T - (\cup_{i \neq j}(d_i \cap d_j))$, where, without loss of generality, t is in the open interval whose closure intersects both c_i^1 and $c_{i+1}^1, 1 \leq i \leq p-1$.

<u>case 2</u>: Assume that $c_i^1 \subset d_k$ and $c_{i+1}^1 \subset d_{k-1}$. Then $f^n(t) \in f^n(d_k), g_n(t) \in v_{k-1}$, and $h_n(g_n(t)) = \alpha_{k-1}(\frac{g_n(t)-b_{k-1}}{a_k-b_{k-1}})$–where α_{k-1} is a path from $h_n(b_{k-1}) \in f^n(d_{k-1})$ to $h_n(a_k) \in f^n(d_k)$ in T of least diameter. Also, $f^n(d_{k-1}) \cap f^n(d_k) \neq \emptyset, diam f^n(d_{k-1}) < \frac{\epsilon}{2}, diam f^n(d_k) < \frac{\epsilon}{2}$ and α_{k-1} of least diameter imply that $\|f^n(t) - h_n g_n(t)\| \leq diam f^n(d_k) + diam f^n(d_{k-1}) < \epsilon$.//

case 3: Assume that $c_i^1 \subset d_k$ and $c_{i+1}^1 \subset d_{k+1}$.

Similar to case 2.//

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Therefore, $\|f^n - h_n g_n\|_\infty < \epsilon$. Whence, ($\Rightarrow$) is proven.

($\Leftarrow$) Suppose that for every $\epsilon > 0$, there exists an $N \in \mathbb{Z}^+$ such that if $n \geq N$, there exist maps g_n from T onto $I = [0,1]$ and $h_n : I \to T$ such that the following diagram ϵ-commutes:

$$\begin{array}{ccc} & f^n & \\ T & \longrightarrow & T \\ g_n \searrow & & \nearrow h_n \\ & I & \end{array}$$

Let $\epsilon > 0$ be given. Choose $m \in \mathbb{Z}^+$ such that $\sum_{i=m+1}^{\infty} \frac{2}{2^i} < \frac{\epsilon}{4}$ and $\alpha > 0$ such that $\sum_{i=0}^{m} \frac{\alpha}{2^i} < \frac{\epsilon}{4}$. Choose $k > 0$ such that $diam A < k$ implies that $diam f^j(A) < \alpha, 0 \leq j \leq m$. Choose $N \in \mathbb{Z}^+$ such that $\|f^N - h_N g_N\|_\infty < \frac{k}{4}$ and $\eta > 0$ so that $diam h_N(B_\eta(t)) < \frac{k}{4}$, for every $t \in I$. If $t \in I$ and $y, z \in g_N^{-1}(B_\eta(t))$, $\|f^N(y) - f^N(z)\|$ $= \|f^N(y) - h_N g_N(y) + h_N g_N(y) - h_N g_N(z) + h_N g_N(z) - f^N(z)\| \leq \|f^N(y) - h_N g_N(y)\| +$ $\|h_N g_N(y) - h_N g_N(z)\| + \|h_N g_N(z) - f^N(z)\| < 2\frac{k}{4} + diam h_N(B_\eta(t)) < \frac{3k}{4}$. Therefore, $diam f^N(g_N^{-1}(B_\eta(t))) \leq \frac{3k}{4} < k$. Let $L_1, \ldots, L_k$ be a 2η-chain of I, where $L_1 = B_\eta(t_1), \ldots, L_k = B_\eta(t_k); t_1, \ldots, t_k \in I$. Then $C = \{\pi_{N+m}^{-1}(g_N^{-1}(L_1)), \ldots, \pi_{N+m}^{-1}(g_N^{-1}(L_k))\}$ is a chain of $\varprojlim f$. Consider $\pi_{N+m}^{-1}(g_N^{-1}(L_1))$. Let $\vec{y}, \vec{z} \in \pi_{N+m}^{-1}(g_N^{-1}(L_1))$. Then $y_{N+m}, z_{N+m} \in g_N^{-1}(B_\eta(t))$ implies that $\|f^N(y_{N+m}) - f^N(z_{N+m})\| < k$ which shows that $\|f^{N+j}(y_{N+m}) - f^{N+j}(z_{N+m})\| < \alpha$, for every $0 \leq j \leq m$. Therefore, $d(\vec{y}, \vec{z}) < \sum_{i=0}^{m} \frac{\alpha}{2^i} + \sum_{i=m+1}^{\infty} \frac{2}{2^i} < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}$. Thus, $diam \pi_{N+m}^{-1}(g_N^{-1}(L_1)) < \epsilon$. Since the same argument applies to $\pi_{N+m}^{-1}(g_N^{-1}(L_2)), \ldots, \pi_{N+m}^{-1}(g_N^{-1}(L_k))$, C is an ϵ-chain of $\varprojlim f$.

Hence, $\varprojlim f$ is chainable.

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added in proof. As noted by Piotr Minc, the following, more general result, is known:

Theorem 5.14 *Let X_n be a locally connected continuum for $n = 1, 2, \ldots$. For $m < n$, let f_{mn} be a map of X_n to X_m. Let X be the inverse limit of $\{X_n, f_{mn}\}$ and let π_n denote the projection of X into X_n.*

The following two conditions are equivalent:

(i) For each $\epsilon > 0$ and for each positive integer m there is an integer $n > m$ and there are maps $g : X_n \to [0,1]$ and $h : [0,1] \to X_m$ such that $|hg - f_{mn}| < \epsilon$.

(ii) X is chainable.

However, the previously proven theorem was arrived at independently and its proof is different than the one that I've seen for the above theorem.

Our goal is then to embed such an inverse limit as an attractor in $\mathbb{R}^2$, which we will return to after the following brief interlude. The following gives an example of a map $f \in C(T)$ such that no iterate of f factors through an interval map, but $\varprojlim f$ is chainable.

Example 5.2 *Define f piecewise linearly on $l_1 \cup l_2$ as in the following figure and $f|_{l_1 \cup l_3}$ in similar fashion so that image(f) is as pictured. Then $\varprojlim f$ is chainable, but no iterate of f factors through an interval map.*

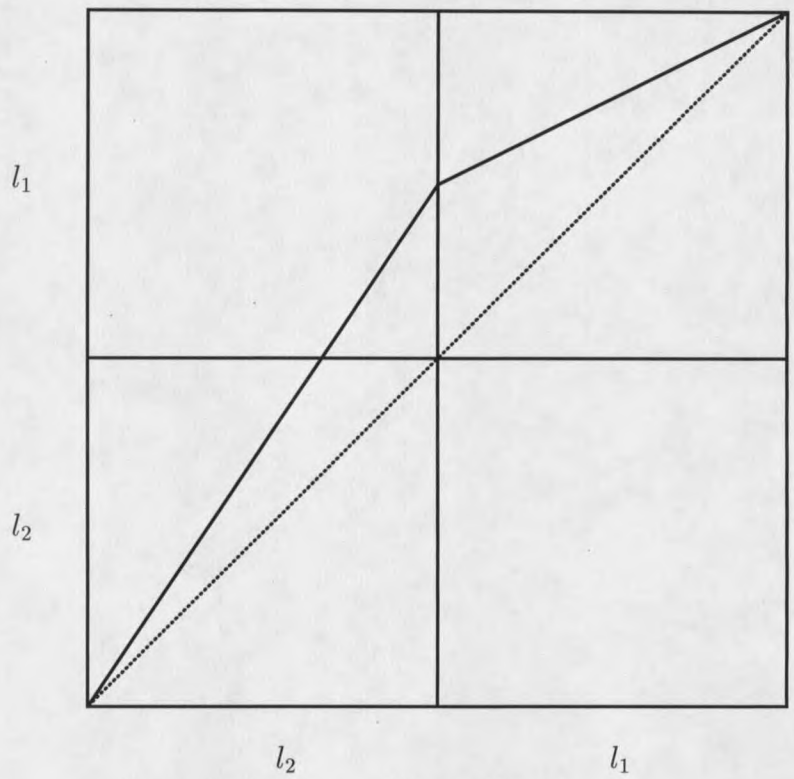

Figure 42: $f|_{l_1 \cup l_2}$

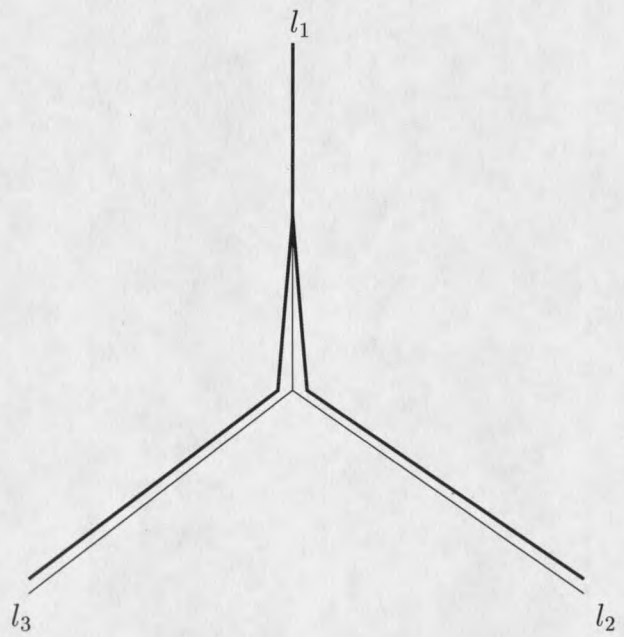

Figure 43: Image of f

Note : We can modify the above example to obtain an $f \in C(T)$ such that $\varprojlim f$ is chainable; f does not factor through an interval map, but some iterate of f factors through an interval map. We arrange that a neighborhood of the fixed point in l_1 is collapsed onto the fixed point.

Definition 5.15 *Let* $C_h(G) = \{f \in C(G) | \varprojlim f$ *is chainable* $\}$.

By modifying the above example, we obtain a sequence $\{f_n\}_{n=1}^{\infty} \subset C(T) - C_h(T)$ such that $\{f_n\}$ converges uniformly to f (as in the above example). We do this by expanding the fixed point in l_1 to a fixed interval of length $\frac{1}{n}, n \geq 3$, to obtain f_n. In addition, given a fixed $f_{n_0} \in C(T) - C_h(T), n_0 \geq 3$, we can construct a sequence $\{g_m\}_{m=1}^{\infty} \subset C_h(T)$ such that g_m converges uniformly to f_{n_0}. We then have the following result:

Proposition 5.9 $C_h(T)$ *is neither open nor closed in* $C(T)$.

Proposition 5.10 $C(T) - C_h(T)$ *is not dense in* $C(T)$.

Proof: Define $f : T \to T$ piecewise linearly so that $f(l_1 \cup l_2) = 1' \in l_2$ and $f(l_3)$ maps across T, as pictured. Then $f \in C_h(T)$ and there exists an $\epsilon_0 > 0$ such that if $\|g - f\|_\infty < \epsilon_0, g \in C(T)$, then $g \in C_h(T)$.

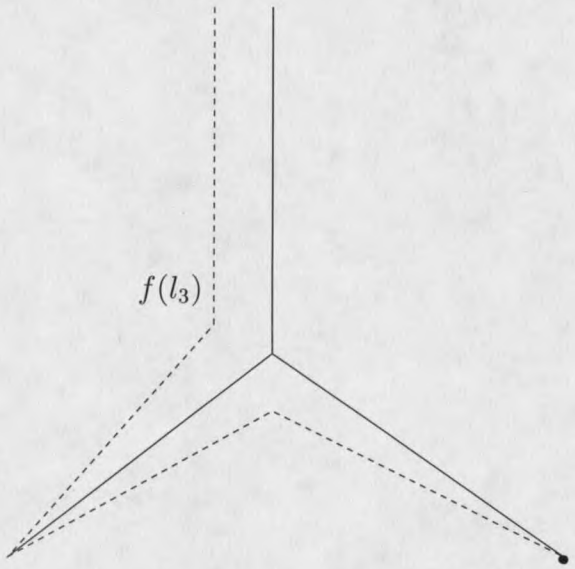

Figure 44: $image(f)$

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Proposition 5.11 *$C_h(T)$ is not dense in $C(T)$.*

Proof: Define $f \in C(T) - C_h(T)$ piecewise linearly along each leg $l_i (i = 1, 2, 3)$ as follows:

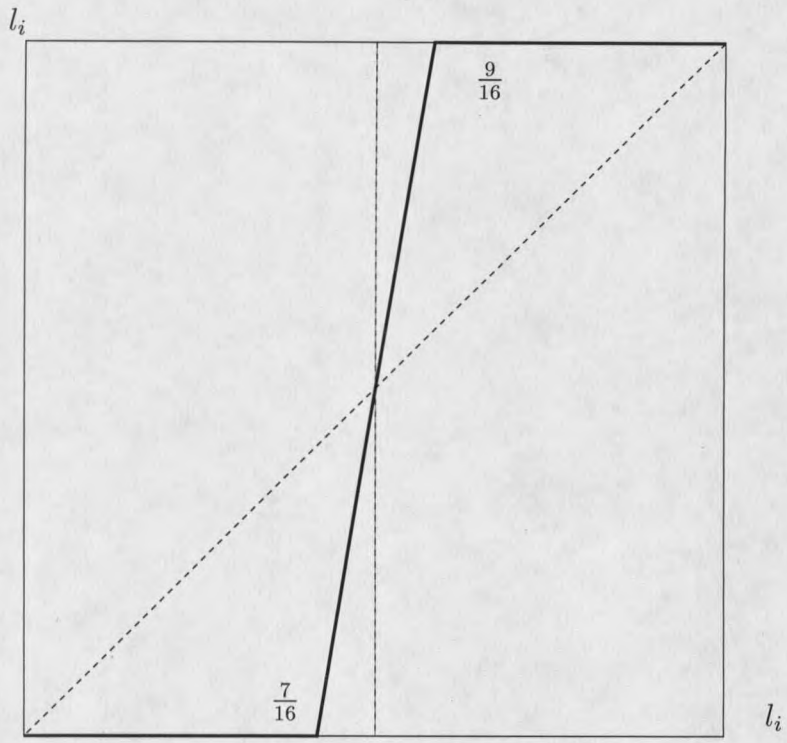

Figure 45: $f|_{l_i}, i = 1, 2, 3.$

We will show that if $\|f - g\| < \frac{1}{32}$, then $g \in C(T) - C_h(T)$. g has a fixed point in each of the intervals $(\frac{15}{32}, \frac{17}{32}) \subset l_1, (\frac{15}{32}', \frac{17}{32}') \subset l_2$, and $(\frac{15}{32}'', \frac{17}{32}'') \subset l_3$. If $x \in (\frac{15}{32}, \frac{17}{32})$, $g(x) < x$. Let $\bar{x}_1 = \inf\{x \in [\frac{15}{32}, \frac{17}{32}] | g(x) = x\}$. Then $g(\bar{x}_1) = \bar{x}_1$. $g([0, \bar{x}_1]) \subset [0, \epsilon_0'] \cup [0, \epsilon_0''] \cup [0, \bar{x}_1]$. Similarly, define $\bar{x}_2 \in l_2$ and $\bar{x}_3 \in l_3$ and let $T_1 = [0, \bar{x}_1] \cup [0, \bar{x}_2] \cup [0, \bar{x}_3] \subset T$. Then $g(T_1) \subseteq T_1, g(\bar{x}_1) = \bar{x}_1, g(\bar{x}_2) = \bar{x}_2$, and $g(\bar{x}_3) = \bar{x}_3$. If $g(T_1) \neq T_1$, let $p \in T_1 - g(T_1)$. $T - \{p\}$ consists of two open

sets which form a separation of $g(T_1)$–a contradiction. Therefore, $g(T_1) = T_1$. So, $N = \varprojlim g|_{T_1}$ is a subcontinuum of $\varprojlim g$. Let $O_1 = (\frac{15}{32}, 1] \subset l_1, O_2 = (\frac{15}{32}', 1'] \subset l_2$, and $O_3 = (\frac{15}{32}'', 1''] \subset l_3$. O_1, O_2, and O_3 are open in T. So, $\hat{O}_i = \cup_{j=0}^{\infty} \pi_j^{-1}(O_i), 1 \leq i \leq 3$, are open in $\varprojlim g$. Also $\hat{O}_i \neq \emptyset$ for each $1 \leq i \leq 3$, since O_i contains a fixed point of g for each $1 \leq i \leq 3$. Next, suppose that $\hat{O}_1 \cap \hat{O}_2 \neq \emptyset$. i.e. there exists a $\underline{z} \in \varprojlim g$ such that $z_j \in O_1$ and $z_k \in O_2$, where without loss of generality, $j < k$. $g^{k-j}(z_k) = z_j \in O_1$, where $z_k \in O_2 \subset l_2$. But, $g(B_{\frac{1}{32}}(l_i)) \subset B_{\frac{1}{32}}(l_i)$, so that $g^n(B_{\frac{1}{32}}(l_i)) \subset B_{\frac{1}{32}}(l_i)$, for every $n \in \mathbb{Z}^+$ and $1 \leq i \leq 3$. This contradicts the fact that $g^{k-j}(z_k) \in O_1$. Therefore, $\hat{O}_1 \cap \hat{O}_2 = \emptyset$. In similar fashion, we obtain that $\hat{O}_1, \hat{O}_2$, and $\hat{O}_3$ are mutually disjoint. Finally, $\varprojlim g - N \subset \hat{O}_1 \cup \hat{O}_2 \cup \hat{O}_3$. Thus, $\varprojlim g - N$ consists of three mutually separated sets and so $\varprojlim g$ is a triod ([24]). Whence, according to Bing ([7]), $\varprojlim g$ is not chainable.

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It is clear that the above examples and results can be generalized to other finite, connected, planar graphs. So, the following appears to be the best general result posssible.

Corollary 5.2 *$C_h(G)$ is a G_δ set in $C(G)$.*

Proof: Let $A_n = \{f \in C(G) |$ there exists an $m \in \mathbb{Z}^+$ such that $f^m \frac{1}{2^n}$–factors through an interval map $\}$, for each $n \geq 1$. Then A_n is open for each $n \geq 1$ and $C_h(G) = \cap_{n=1}^{\infty} A_n$.

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We return to some results related to our goal. In order to embed $\varprojlim f$ as an attractor in $\mathbb{R}^2$, where $\varprojlim f$ is chainable, we need to be able to extend a homeomorphism of an embedding of $\varprojlim f$ to a homeomorphism of $\mathbb{R}^2$. The following discussion shows

that, in general, given an arbitrary chainable continuum K, embedded in $\mathbb{R}^2$, and a homeomorphism $f : K \to K$, f cannot be extended to a homeomorphism of all of $\mathbb{R}^2$. We start with a definition and a theorem by Brechner ([9]).

Definition 5.16 *(Brechner [9]) We call a homeomorphism h of a planar continuum M onto itself, essentially extendable if and only if there exists an imbedding $\phi : M \to E^2$ such that*

$$\phi h \phi^{-1} : \phi(M) \to \phi(M)$$

can be extended to a homeomorphism $\overline{\phi h \phi^{-1}} : E^2 \to E^2$. ($S^2$ may replace E^2 in the definition).

Theorem 5.15 *(Brechner [9]) There exists a chainable continuum $M \subseteq S^2$ and a homeomorphism $h : M \to M$ such that h is not essentially extendable.*

Dr. Brechner proves her theorem by "gluing" two pseudo-arcs together at a common endpoint and constructing a homeomorphism which is period two on one of the pseudo-arcs and fixes the other. She uses some sophisticated prime end theory to show that this homeomorphism is not essentially extendable. We construct a similar example in the following theorem for which elementary techniques can be used to show that the homeomorphism is not essentially extendable.

Theorem 5.16 *There exists a chainable continuum $K \subset \mathbb{R}^2$ and a homeomorphism $f : K \to K$ such that f is not essentially extendable.*

Proof: Let K be the following planar, chainable continuum:
$l = \{-2\}\times[-1,1], l_1 = \{-1\}\times[-1,1], l_n = \{-2+\frac{1}{n}\}\times[-1,1], \hat{l}_n = \{-\frac{1}{n}\}\times[-\frac{1}{n},\frac{1}{n}], r = \{2\}\times[-1,1], r_1 = \{1\}\times[-1,1], r_n = \{2-\frac{1}{n}\}\times[-\frac{1}{n},\frac{1}{n}]$, and $\hat{r}_n = \{\frac{1}{n}\}\times[-\frac{1}{n},\frac{1}{n}]$– where $n \geq 2$ at each occurrence. Also, place a double "$\sin\frac{1}{x}$" curve between each set of vertical bars as pictured. Define a homeomorphism $f : K \to K(onto)$ so

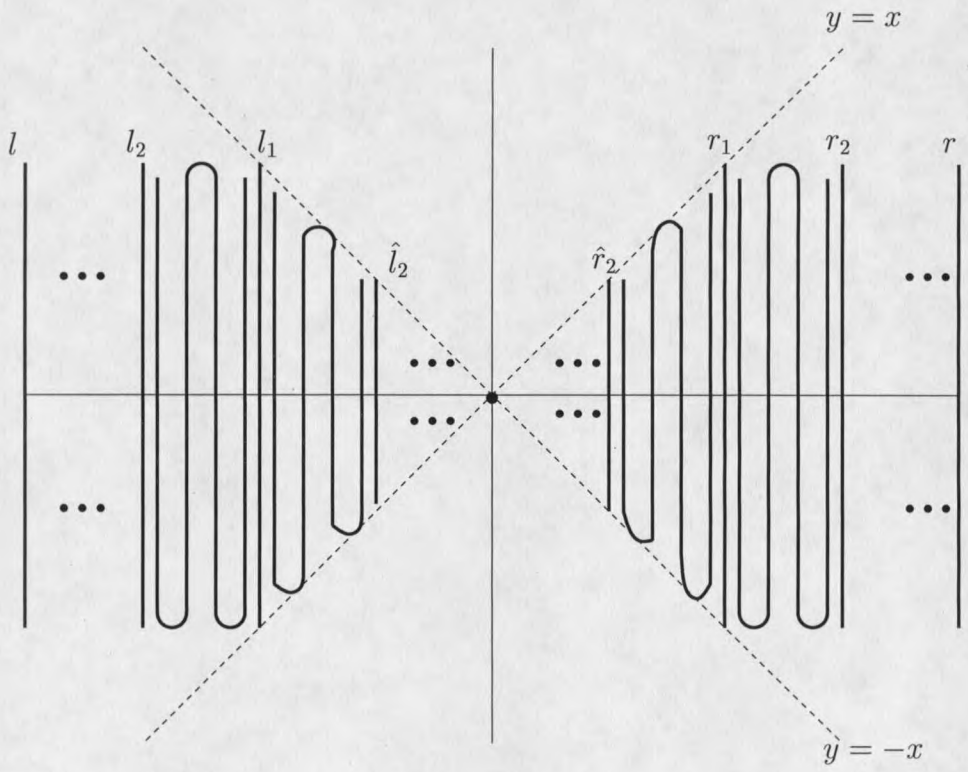

Figure 46: K

that each vertical bar and each double "$\sin \frac{1}{x}$" curve is left invariant, $f(0) = 0$, $f = idy$ on that part of K lying in the right half-plane, and f flips each vertical bar in the left half-plane about the x-axis–i.e. if (x, y) is an element of one of those vertical bars, $f((x, y)) = (x, -y)$. Now, let $L = \alpha(K)$ be any planar embedding of K. Suppose that $G : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism such that $G|_L = \alpha f \alpha^{-1}$. We show that G is orientation preserving. Then, in similar fashion, it can be shown that G is orientation reversing–a contradiction. Let $H : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving homeomorphism such that $H\alpha|_l = idy|_l$. Then $HGH^{-1}|_{H(L)} = H\alpha f \alpha^{-1} H^{-1}|_{H(L)}$, i.e. HGH^{-1} extends $H\alpha f \alpha^{-1} H^{-1}|_{H(L)}$. Let $l_i^\star = H(\alpha(l_i)), i = 1, 2, \ldots$. Since $H\alpha|_l = idy, HGH^{-1}|_{H(L)}(H\alpha(l))$ flips $H\alpha(l) = l$ about its midpoint. Also, each $l_i^\star$ is invariant under HGH^{-1}. Choose $0 < \delta < \frac{1}{32}$ such that $HGH^{-1}(B_\delta((-2, t))) \subset B_{\frac{1}{32}}(B_\delta((-2, -t)))$, for every $t \in [-1, 1]$. Then choose $N \geq 2$ such that $(H\alpha)((-2 + \frac{1}{N}, t)) \in B_\delta((-2, t))$, for every $t \in [-1, 1]$. Suppose, without loss of generality, that the horizontal line $y = 0$ intersects $l_N^\star$ in the half-plane $\{(x, y) | x > -2\}$, then no line $y = t, t \in [-\frac{3}{4}, \frac{3}{4}]$ intersects $l_N^\star$ in the half-plane $\{(x, y) | x < -2\}$. The line $y = \frac{1}{2}$ intersects $(l_N^\star \cap B_\delta((-2, \frac{1}{2})))$, by the connectivity of $l_N^\star$. Let w_1 be the point of the intersection with least x-coordinate. Since $l_N^\star$ is closed, $w_1 \in l_N^\star$. Similarly, let w_2 be the point of intersection of $(l_N^\star \cap B_\delta((-2, -\frac{1}{2})))$ with least x-coordinate. Let α_1 be a simple arc from $(-2, -\frac{1}{2})$ to $(-2, \frac{1}{2})$ along $x = -2$, let α_2 be a simple arc from $(-2, \frac{1}{2})$ to w_1 along $y = \frac{1}{2}$, let α_3 be a simple arc in $l_N^\star$ from w_1 to w_2, and let α_4 be a simple arc from w_2 to $(-2, -\frac{1}{2})$ along $y = -\frac{1}{2}$. Then $\Gamma = \alpha_1 \star \alpha_2 \star \alpha_3 \star \alpha_4$ is a simple closed curve in $\mathbb{R}^2$ oriented clockwise so that $HGH^{-1}(\Gamma)$ is oriented counterclockwise. Therefore, HGH^{-1} is orientation reversing. Since H is orientation preserving, G is orientation reversing. We can do a similar argument, using r, to show that G is orientation preserving.

This contradicts the existence of such a G.

PTL

The following theorem shows that even if $\varprojlim f$ is planar, $\hat{f}$, the shift homeomorphism, may not be extendable to $\mathbb{R}^2$.

Theorem 5.17 *There exist an $f \in C(T), T$ a triod, so that $\varprojlim f$ is planar, but $\hat{f}$ is not essentially extendable to $\mathbb{R}^2$.*

Proof: Define $f \in C(T)$ piecewise linearly as follows:

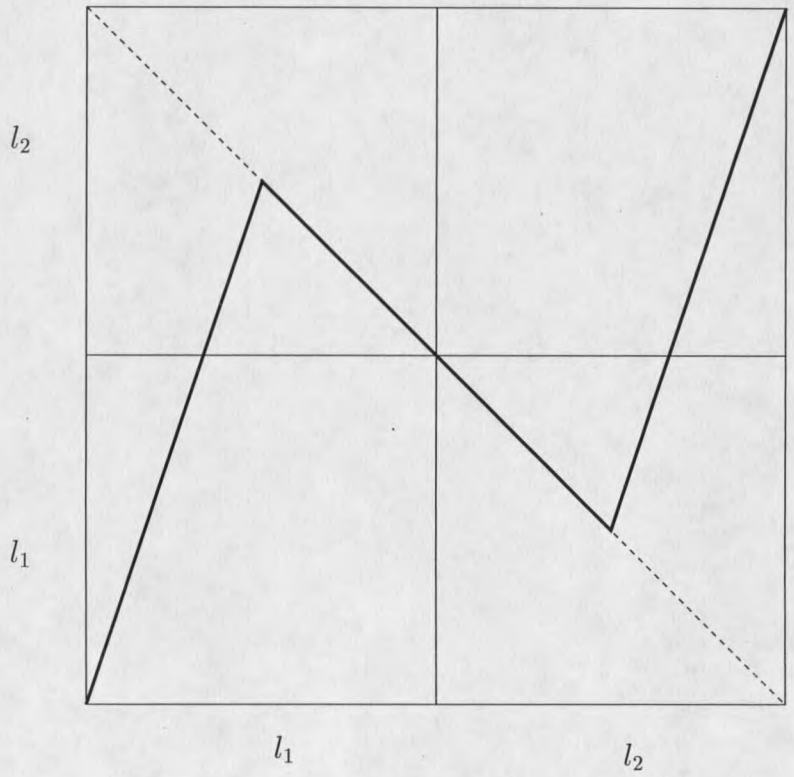

Figure 47: $f|_{l_1 \cup l_2}$

Define $f|_{l_3} = idy$. Then $\varprojlim f|_{l_1 \cup l_2}$ can be embedded in $\mathbb{R}^2$ as follows:

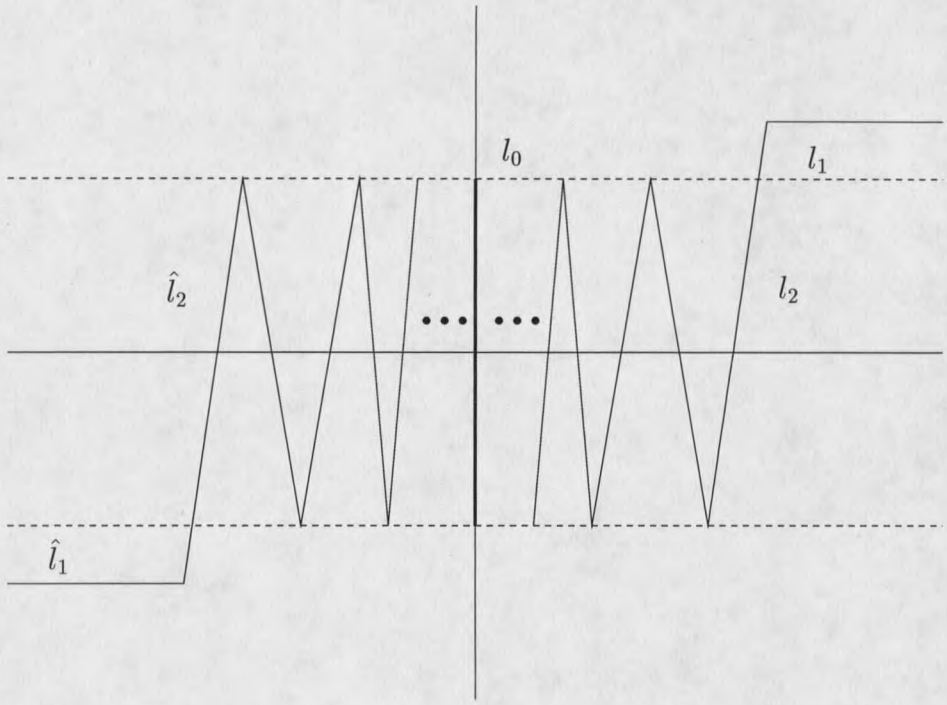

Figure 48:

$\hat{f}$ restricted to $\varprojlim f|_{l_1 \cup l_2}$ shifts l_0 across $l_0 \cup l_1$, each l_i across $l_{i+1}, i \geq 1, \hat{l}_0$ across $\hat{l}_0 \cup \hat{l}_1, \hat{l}_i$ across $\hat{l}_{i+1}, i \geq 1$, and flips l about the x-axis. So, let $K = \alpha(\varprojlim f)$ be the following planar embedding of $\varprojlim f$ (where the interval from $(-1, 0)$ to $(0, 0)$ corresponds to $\varprojlim f|_{l_3}$).

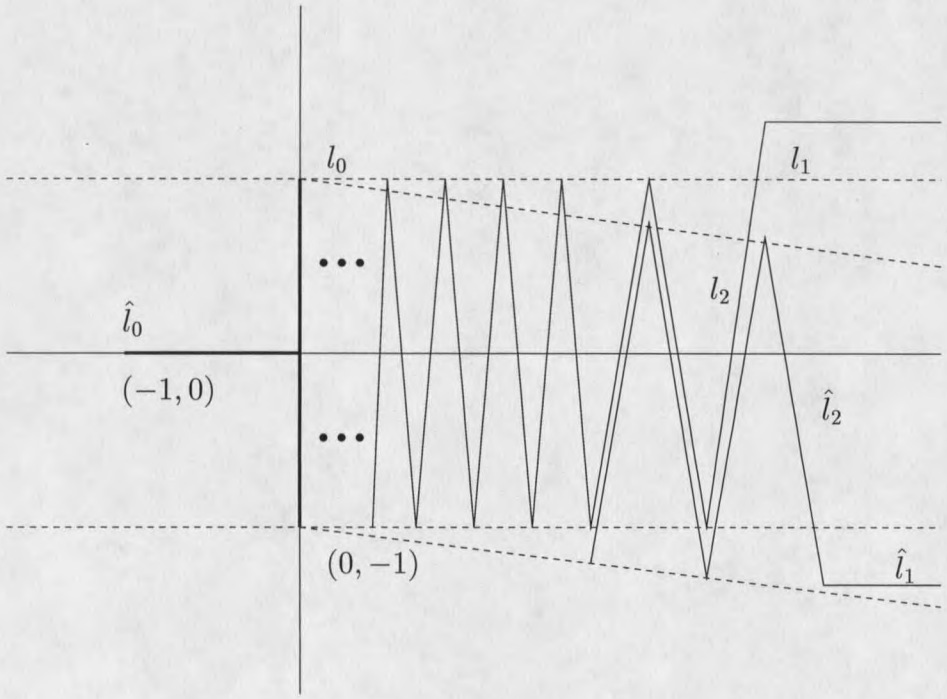

Figure 49:

Showing that $\hat{f}$ is not essentially extendable to $\mathbb{R}^2$ is then equivalent to showing that $g = \alpha\hat{f}\alpha^{-1} : K \to K$ is not essentially extendable to $\mathbb{R}^2$. Suppose that g is essentially extendable to $\mathbb{R}^2$. Then there exists an embedding $\beta : K \to \mathbb{R}^2$ such that $\beta g\beta^{-1}$ extends to a homeomorphism G of $\mathbb{R}^2$. Let H be and orientation preserving homeomorphism of $\mathbb{R}^2$ such that $H\beta|_{l_0} = idy|_{l_0}$. Then $HGH^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism such that $HGH^{-1}|_{H(\beta(K))} = H\beta\alpha\hat{f}\alpha^{-1}\beta^{-1}H^{-1}|_{H(\beta\alpha(K))}$–i.e. HGH^{-1} extends $\hat{f}$ on $H(\beta(K))$. Let $H\beta(\hat{l}_i) = \hat{l}_i^\star$, for all $i \geq 0$, and $H\beta(l_i) = l_i^\star$, for all $i \geq 1$. Thus $HGH^{-1}|_{\hat{l}_0^\star} = idy|_{\hat{l}_0^\star}$ and $HGH^{-1}|_{l_0}$ flips l_0 about the x-axis. There exists a $\delta > 0$ so that $H(\beta([-\delta, 0), 0)) \subset B_{\frac{1}{16}}((0,0))$. Without loss of generality, assume that $H(\beta([-\delta, 0), 0))$ is contained in the left half-plane $P = \{(x,y)|x < 0\}$. There exists an $\eta > 0$ so that $HGH^{-1}(B_\eta((0,t))) \subset B_{\frac{1}{16}}((0,-t))$ for every $-1 \leq t \leq 1$. Since $\hat{l}_i^\star$ and $l_i^\star$ limit on l_0 as $i \to \infty$ and $H(\beta([-\delta, 0), 0)) \subset P$, there exists an N such that $i \geq N$ implies that $\hat{l}_i^\star, l_i^\star \subset int(D)$, where $D = cl(B_\eta((0,1))) \cup cl(B_\eta((0,-1))) \cup (cl(B_\eta(l_0)) \cap \{(x,y)|x \geq 0\})$. Let $\gamma_1 : [0,\infty) \to \cup_{n=1}^\infty l_n^\star$ be a homeomorphism and $t_1 = \sup\{t \geq 0|\gamma_1(t) \in \partial D\}$. Let $l^\star$ be the half ray $\gamma_1([t_1, \infty)))$. Also, let $\gamma_2 : [0,\infty) \to \cup_{n=1}^\infty \hat{l}_n^\star$ be a homeomorphism, $t_2 = \sup\{t \geq 0|\gamma_2(t) \in \partial D\}$, and $\hat{l}^\star = \gamma_2((t_2, \infty))$. $int(D) - l^\star$ consists of two components, say D_u containing $(0,1)$ and D_l containing $(0,-1)$. Because $\hat{l}^\star$ is connected, without loss of generality, $\hat{l}^\star \subset D_u$. $HGH^{-1}(int(D) - l^\star)$ consists of two components, say D_u' containing $(0,1)$ and D_l' containing $(0,-1)$. Since $HGH^{-1}((0,1)) = (0,-1)$ and vice versa, $HGH^{-1}(D_u) = D_l'$ and $HGH^{-1}(D_l) = D_u'$. But, $HGH^{-1}(\hat{l}^\star) \subset \hat{l}^\star$ so that $HGH^{-1}(\hat{l}^\star) \cap D_u' \neq \emptyset$, contradicting the fact that $HGH^{-1}(\hat{l}^\star) \subset D_l'$.

PTL

In this chapter, we have given necessary and sufficient conditions that $\varprojlim f$ be chainable for $f : G \to G$ continuous and onto with G a finite, connected, planar graph. We have also shown that, in general, a homeomorphism of a chainable continuum is

not essentially extendable to the plane. However, it is still open as to whether or not $\varprojlim f$ chainable implies that $\hat{f}$ is essentially extendable to the plane. Brechner and Lewis ([8], [15]) both have results implying that such an $\hat{f}$ is essentially extendable to $\mathbb{R}^3$. Another question is "for which $f : G \to G$ with $\varprojlim f$ planar is $\hat{f}$ essentially extendable to the plane (or $\mathbb{R}^3$)?" Finally, if $f : G \to G$, can $\varprojlim f$ be embedded in $\mathbb{R}^3$ in such a way that $\hat{f}$ on this embedding extends to a homeomorphism of all of $\mathbb{R}^3$?

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