



Superfluid hydrodynamics in neutron stars
by Gregory Allen Mendell

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in
Physics

Montana State University

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Abstract:

Superfluidity is predicted to exist in neutron stars. Superfluid effects on the dynamics of these stars have not been investigated in depth in the past. In this thesis, superfluid hydrodynamics in neutron stars is developed extensively. It is shown that superfluidity has important effects on the oscillation modes; dissipative properties, and stability of these stars.

Very general hydrodynamic equations are derived which describe superfluid mixtures. The fluid equations are coupled to the electromagnetic and gravitational fields. Forces due to the quantized vortices of the superfluids are also included. It is shown that new vorticity-preserving forces can be introduced into the superfluid-mixture equations. The equations are then adapted to describe neutron stars composed primarily of superfluid neutrons, superconducting protons, and degenerate electrons and muons. The set of equations is closed by constructing a model of the total energy density and using it to express the dependent variables in terms of the independent variables. The low-frequency long-wavelength limit of the equations is determined. The results can be used to study superfluid effects on the global oscillations of neutron stars.

The equations are generalized further to include dissipative effects. Most important is a form of dissipation known as mutual friction, which occurs only in superfluids. In neutron stars, mutual friction is due to electron scattering off the neutron and proton vortices. An energy functional is constructed which determines the damping time of a mode due to the various forms of dissipation, including mutual friction. Plane-wave solutions are found to the equations. Mutual friction is shown to be the dominant form of dissipation in neutron stars for sufficiently large angular velocities.

Gravitational radiation tends to make all rotating stars unstable, while internal dissipation tends to counteract this instability. Thus, gravitational radiation can limit the maximum angular velocity of neutron stars. The most important conclusion of this thesis is that mutual friction completely suppresses the gravitational-radiation instability in rotating neutron stars cooler than the superfluid-transition temperature.

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A thesis submitted in partial fulfillment
of the requirements for the degree

of

Doctor of Philosophy

in

Physics

MONTANA STATE UNIVERSITY
Bozeman, Montana

August 1991

D378
M5225

APPROVAL

of a thesis submitted by

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This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

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ACKNOWLEDGMENTS

I wish to thank my adviser, Dr. Lee Lindblom, for many useful discussions and suggestions during the course of this work, and for providing me with financial support from his National Science Foundation and MONTS grants.

I also wish to thank my wife, Jeanette, for her support during my years in graduate school.

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ABSTRACT

Superfluidity is predicted to exist in neutron stars. Superfluid effects on the dynamics of these stars have not been investigated in depth in the past. In this thesis, superfluid hydrodynamics in neutron stars is developed extensively. It is shown that superfluidity has important effects on the oscillation modes, dissipative properties, and stability of these stars.

Very general hydrodynamic equations are derived which describe superfluid mixtures. The fluid equations are coupled to the electromagnetic and gravitational fields. Forces due to the quantized vortices of the superfluids are also included. It is shown that new vorticity-preserving forces can be introduced into the superfluid-mixture equations. The equations are then adapted to describe neutron stars composed primarily of superfluid neutrons, superconducting protons, and degenerate electrons and muons. The set of equations is closed by constructing a model of the total energy density and using it to express the dependent variables in terms of the independent variables. The low-frequency long-wavelength limit of the equations is determined. The results can be used to study superfluid effects on the global oscillations of neutron stars.

The equations are generalized further to include dissipative effects. Most important is a form of dissipation known as mutual friction, which occurs only in superfluids. In neutron stars, mutual friction is due to electron scattering off the neutron and proton vortices. An energy functional is constructed which determines the damping time of a mode due to the various forms of dissipation, including mutual friction. Plane-wave solutions are found to the equations. Mutual friction is shown to be the dominant form of dissipation in neutron stars for sufficiently large angular velocities.

Gravitational radiation tends to make all rotating stars unstable, while internal dissipation tends to counteract this instability. Thus, gravitational radiation can limit the maximum angular velocity of neutron stars. The most important conclusion of this thesis is that mutual friction completely suppresses the gravitational-radiation instability in rotating neutron stars cooler than the superfluid-transition temperature.

CHAPTER 1

INTRODUCTION

Purpose and Plan of This Thesis

Neutron stars are nature's grand tour of physics. Born from the collapsing core of a dying star in a supernova explosion, a typical neutron star has the mass of the sun compressed into the size of a city. The collapse can produce a rapid rotation rate and an intense magnetic field. Outside the star, the magnetic field interacts with a surrounding plasma, sometimes producing a corotating beam of electromagnetic radiation. When the earth lies within the beam's path we observe a pulsating star in the sky, a pulsar. Inside the star, the matter can be divided into several regions of crust and core, each corresponding to a different density regime. A large fraction of the matter is at a density greater than or equal to that of atomic nuclei. With such high densities, neutron stars are among the most gravitationally compact objects in the universe. They are also cold degenerate objects. Quantum mechanics is crucial to their description too.

Indeed, one of the most striking manifestations of macroscopic quantum effects to occur in nature — superfluidity — has been predicted to exist in neutron stars (for reviews see Pines and Alpar 1985; Sauls 1989). The interior neutrons are predicted to form superfluid condensates in the inner crust and in the outer core. The small fraction of protons in the outer core are expected to form a superconducting condensate. The transition temperatures to superfluidity and

superconductivity are of the order 10^9K (see Epstein 1988 for a summary of various calculations). While uncertainties exist in the transition temperatures, neutron stars are expected to cool well below 10^9K in a few months (e.g., see Tsuruta 1979; Nomoto and Tsuruta 1987). Furthermore, in rotating magnetized neutron stars the superfluid neutrons are expected to form an array of quantized vortices and the superconducting protons an array of quantized flux tubes. In general, superfluidity can be expected to have interesting effects on the dynamics of neutron stars. (Except when the distinction is important, superfluidity and superconductivity will be referred to generically as "superfluidity" and vortices and flux tubes will be referred to generically as "vortices" from here on.)

The dynamics of neutron stars are studied primarily by using hydrodynamics. Theoretical hydrodynamic models study the equilibrium configurations, oscillation modes, dissipative processes, and stability of these stars. Two significant questions these models attempt to answer are: what is the maximum mass and what is the maximum rotation rate of a neutron star? Accurate answers to these questions will involve new insights into the physics of matter at high densities. They will enhance our understanding of the universe in other ways as well. For example, the observation of a compact object with a mass exceeding the neutron star limit would probably indicate that the object is a black hole. On the other hand, the detection of an object spinning faster than the rotation limit of a neutron star would indicate that the matter has compressed far beyond nuclear density. This would probably mean that quark stars exist. An identification of one of these objects would be made more reliable by knowing more precisely the mass and rotation limits of a neutron star.

The theoretical determination of the mass and rotation limits of a neutron star requires a stability analysis of the equilibrium models of the star. In particular, the study of neutron star oscillations determines the stability of an equilibrium model. Dissipative effects either cause the amplitude of the oscillation to grow or to damp out. If the amplitude of a particular mode grows, then the star is unstable to this mode of oscillation. For example, it is known that gravitational radiation tends to make all rotating stars unstable, while internal dissipation (such as viscosity) tends to counteract this instability (for a review see Lindblom 1991). The onset of the gravitational-radiation instability, which determines the maximum rotation rate, is found by studying the nonaxisymmetric oscillations of the star. Studying neutron star oscillations may also be fruitful for other reasons. For example, these oscillations may be observable. (This possibility is discussed further in the next section of this chapter.)

One motivation for this thesis is that superfluidity can be expected to have important effects on neutron star oscillations. However, past studies in this area of research have only begun to explore the possibilities. Previously, Epstein (1988) has studied sound waves in nonrotating, nonmagnetized, superfluid neutron-star matter. However, most studies of neutron star oscillations have not used superfluid hydrodynamics (e.g., Meltzer and Thorne 1966; Thorne 1969a; Hansen and Cioffi 1980; Lindblom and Detweiler 1983; McDermott et al. 1985; Carroll et al. 1986; Cutler, Lindblom, and Splinter 1990; Ipser and Lindblom 1990; Strohmayer 1991). Also, a past study of neutron star magnetohydrodynamics by Easson and Pethick (1979) did not include dynamics for the superfluid neutrons and treated the protons as nonsuperconducting.

Another motivation for this thesis is that dissipation in superfluids is completely different from that in ordinary fluids. (For a review of dissipation in superfluids see Sonin 1987; Putterman 1974.) In a superfluid the condensate acts like a zero temperature nondissipative perfect fluid. However, the scattering of particles off the vortices can transmit forces to the superfluid condensate, leading to a form of dissipation known as mutual friction. Previous studies of neutron stars have investigated the effects of superfluidity on viscous dissipation, but have not included the effects of mutual friction (e.g., Cutler and Lindblom 1987, Cutler, Lindblom, and Splinter 1990, Ipser and Lindblom 1991).

The purpose of this thesis is to develop superfluid hydrodynamics in neutron stars. The results will be applicable to studies of oscillations of rotating and magnetized neutron stars, including dissipative effects. In particular, the focus is on neutron stars composed primarily of superfluid neutrons, superconducting protons, and degenerate electrons and muons. However, very general sets of equations are derived which can be easily adapted to other models of neutron stars (e.g., such as those containing a superfluid pion condensate), or to other situations of physical interest (e.g., such as mixtures of superfluid ^3He and ^4He). The equations are closed by relating the dependent variables to the independent variables via partial derivatives of a model of the energy density. The low-frequency-long-wavelength limit of the equations is found. The equations are generalized further to include dissipative effects. Plane wave solutions to the equations are found and compared with those for ordinary neutron-star matter. Finally, the effect of superfluidity on the gravitational-radiation instability is studied. *The most important conclusion of this thesis* is that mutual friction between the electrons and the quantized

neutron vortices completely suppresses the gravitational-radiation instability in rotating neutron stars cooler than the superfluid-transition temperature.

The plan of this thesis is discussed in the remainder of this section. In general, the approach used here is phenomenological. The underlying assumption is that the macroscopic properties of a neutron star can be described by a set of continuous fields. A complete set of such fields would be the number densities, entropy densities, and velocity fields of each species of particle in the star, plus the gravitational and electromagnetic fields specified everywhere inside and outside of the star. The temperature and pressure (and several other quantities) are specified in terms of the independent variables by an equation of state. (The equation of state is one place where the phenomenological theory touches base with the microphysics of the problem.) The equilibrium and evolution of neutron stars are thus described by the equations of fluid mechanics, electromagnetism, and gravitation.

Many approximations are made in the course of this thesis. These will be described in detail and justified when they occur. However, three major simplifying assumptions are made throughout this work and deserve extra mention here. First, in principle, the equations should be fully relativistic. However, Newtonian hydrodynamics is used since the fully relativistic treatment of the nonaxisymmetric oscillations of rotating stars has yet to be accomplished (e.g., see Lindblom 1991). Further development of relativistic superfluid hydrodynamics is also needed. Relativistic corrections will probably change the Newtonian results by no more than 20% (see Cutler and Lindblom 1991). Next, interactions between vortices are ignored in this thesis. Interaction between vortices of the same species will be

shown to have negligible effects on neutron star oscillations in the low-frequency-long-wavelength limit. However, in rotating magnetized neutron stars the neutron and proton vortices either will be forced to move through each other or they will become pinned onto or between each other. The details of this interaction are not yet understood (e.g., see Sauls 1989; Srinivasan 1990) but an estimate of the pinning force is given in Chapter 10. Lastly, the neutrons, which form Cooper pairs in the triplet state, are treated as if in the singlet state. This is not expected to have significant effects on the macroscopic behavior of the neutron superfluid (Sauls 1989). The equations presented here can give important results in spite of their limitations due to these simplifying approximations. An outline of what is accomplished in each chapter is given below.

In Chapter 2 superfluid hydrodynamic theory is reviewed extensively. First, the London theory is covered (e.g., see London 1950; 1954) which describes the macroscopic properties of a superfluid in terms of a macroscopic wave function. This is used to define the superfluid velocity. Next, the Landau two-fluid theory (Landau 1941) is derived phenomenologically from a set of “macroscopic” first principles. This illustrates the power of the phenomenological approach. The derivation given here is slightly different than that given elsewhere and provides a clearer starting point for the derivation of the hydrodynamics of superfluid mixtures, which comes in the next chapter. The last part of the Chapter 2 discusses how the Landau equations must be extended when vortices are present. Since the spacing between vortices is typically microscopic compared with the length scales of interest it is appropriate to perform a smooth-averaging of the equations over a volume containing many vortices. The smooth-averaging process is discussed in

terms of both a coarse-graining procedure and in terms of the phenomenological approach.

In Chapter 3 the general hydrodynamic theory for a mixture of N charged superfluids is derived. In this chapter the equations from past studies of superfluid mixtures (e.g., Holm and Kupershmidt 1987) are generalized still further by including a large class of vorticity-preserving interaction terms in the dynamical equations for the superfluid velocities. A Hamiltonian formulation for the equations is presented and used to couple the charged components of the mixture to the electromagnetic field. The new vorticity-preserving forces play a nontrivial dynamical role at the locations of vortices and have a profound effect on the electromagnetic coupling terms. By properly choosing these forces, it is possible to allow each component of the charged superfluid mixture to respond to the electromagnetic field via an appropriate Lorentz force law. The electromagnetic coupling proposed here is far more natural than that proposed in the previous studies of charge superfluid mixtures.

In Chapter 4 the equations are adapted to describe the outer-core region of a neutron star and are further modified in several important ways. A specific model of the energy density is constructed and its partial derivatives are used to relate the dependent variables to the independent variables. The equilibrium state is discussed and the linearized perturbation equations are found. The energy per unit length of the vortices is calculated. The characteristic frequencies associated with each of the force terms are discussed.

In Chapter 5 the low-frequency-long-wavelength limit of the equations is determined. In this limit the equations are also greatly simplified. For an appropriate frequency range it is shown that the number of independent velocities can

be reduced from three to two: The resulting equations will be applicable to studies of gravity modes (g-modes) and pressure modes (p-modes) (for a description of these modes see Chapter 4; Van Horn 1980), including electromagnetic, rotational, and vortex effects. A *new mode* is shown to exist in the superconducting-proton-degenerate-electron plasma, while Alfvén waves, found in ordinary plasmas, are shown not to exist. It is then further shown that the electromagnetic forces and the forces due to the underlying vortices can be ignored for sufficiently long length scales. The final set of equations will be applicable to studies of p-modes in rapidly rotating neutron stars. These equations generalize the work of Epstein (1988) and Easson and Pethick (1979). Furthermore, an error in Epstein's equations is corrected. Since the final results differ significantly from the normal magnetohydrodynamic (MHD) limit, the entire procedure used here will be referred to as the low-frequency-long-wavelength (LFLW) limit throughout this thesis.

In Chapter 6 the general theory of mutual friction is derived for a mixture of N charged superfluids. This generalizes the theory of mutual friction first developed by Hall and Vinen (1956), and later derived phenomenologically by Bekarevich and Khalatnikov (1961), in the context of superfluid ^4He . The dissipative coefficients which appear in the superfluid equations are then related to parameters which can be determined from microscopic calculations.

In Chapter 7 the first goal is to include mutual friction forces in the superfluid hydrodynamic equations which describe a neutron star. The LFLW limit of the mutual friction forces is taken. The dissipative coefficients in these forces are related to microphysical quantities by generalizing the method of Hall and Vinen (1956). Specifically, the remaining relevant coefficients are related to the velocity relaxation time between the charged fluids and the neutron vortices, as determined

by Alpar, Langer, and Sauls (1984). Outside of the vortices the main dissipative process is due to electron-electron scattering. This is incorporated into the theory by including viscous and thermal conductivity terms in the equations for the electron-muon fluid.

The second goal of this chapter is to estimate the effect of mutual friction on the damping times of p-modes. An energy functional is constructed which determines the damping times due to viscosity, thermal conductivity, and mutual friction. It is argued generically that the mutual friction damping times of the large scale p-modes are much shorter than the other damping times in rapidly rotating neutron stars.

In Chapter 8 the frequencies for plane symmetric solutions are found for several interesting cases and compared with those for ordinary neutron-star matter. Two special cases are considered: axial modes where the wave vector is parallel to the rotation axis, and inplane modes where the wave vector is perpendicular to the rotation axis. The results are compared with those for ordinary neutron-star matter. Errors in the previous study by Epstein (1988), in the case of a nonrotating star, are corrected here. As in terrestrial superfluids, the resulting frequencies are modified and the number of allowed modes is doubled when compared with the results for ordinary neutron-star matter. The damping times for these modes due to electron viscosity and mutual friction are discussed and compared with the results of Chapter 7.

The purpose of Chapter 9 is to investigate how the internal dissipation mechanisms that exist within superfluid neutron-star matter influence the gravitational-radiation instability. It is argued here that mutual friction between the electrons and the quantized neutron vortices completely suppresses the gravitational-

radiation instability in rotating neutron stars cooler than the superfluid-transition temperature.

In Chapter 10 the important results and conclusions of this thesis are summarized. Areas where further work is needed, and the likely outcomes of such studies, are discussed.

In the remaining sections of this chapter a background survey of neutron stars and superfluidity is given. The reader unfamiliar with either of these topics will find these sections useful and they serve as a guide to the literature. In addition: complete reviews of neutron stars can be found in Shapiro and Teukolsky (1983), Baym and Pethick (1975; 1979), Canuto (1978), Arnett and Bowers (1977), Backer and Kulkarni (1990), and Kundt (1990); superfluidity and superconductivity are covered in Khalatnikov (1989), Tilley and Tilley (1986), de Gennes (1966), Tinkham (1965, 1975), Parks (1969), and London (1950, 1954); superfluid hydrodynamics is dealt with extensively in Putterman (1974) and Sonin (1987).

Neutron Stars

Observational Properties

Soon after the discovery of the neutron by Chadwick in 1932, Zwicky and Baade (1934) proposed the idea of neutron stars. (There is a story, perhaps apocryphal, that Landau was actually the first to have this idea. See Graham-Smith 1990.) Oppenheimer and Volkoff (1939) generated the first neutron star stellar models. Neutron stars would be small degenerate bodies with roughly the mass of the sun compressed into a ball about a dozen kilometers across. Studies of neutron star cooling showed that their surfaces would emit x-rays, but that they would be too dim to be seen (e.g., Tsuruta and Cameron 1965). It seemed that neutron

stars, if they existed, were unobservable. But then in 1967 Bell discovered a radio source pulsing regularly every 1.3 seconds. It passed overhead once every sidereal day, as the stars do. (Bell referred to it as a Belisha beacon in her notes. The others in her group referred to the source as the LGM or the "little green men". See Wade 1975.) More such sources were discovered and it was soon realized that a new kind of star had been discovered—pulsars (Hewish et al. 1968).

Gold (1968) quickly identified pulsars as rotating neutron stars with intense magnetic fields. It had already been realized that the collapsing core of a dying star would be composed mostly of neutrons because of electron capture by the protons (Baade and Zwicky 1934): Conservation of angular momentum and magnetic flux would give large rotation rates and magnetic fields. (See Canuto 1978 for a derivation of flux conservation.) Thus, pulsars would form in supernova explosions. The discovery of pulsars in the Crab and Vela supernova remnants supported this idea. Today almost 500 pulsars are known, with periods ranging from 4.3s to 1.56ms (e.g., see Kundt 1990).

In the simplest model a neutron star possesses a dipole magnetic field which is not aligned with the rotation axis (e.g., see Shapiro and Teukolsky 1983). In this model a neutron star emits magnetic dipole radiation: (A crude calculation of the magnetic dipole radiation emitted by the Crab pulsar gives almost exactly the energy that is needed to keep the Crab nebula lit—a significant confirmation of the essential ideas of the model. See Gold 1969). Magnetic dipole radiation, though, is not the source of the pulses (this can be shown by analysis of the shape and frequency spectrum of the pulses). The exact pulse mechanism is still not understood. However, applying the boundary conditions for the electromagnetic

field at the star's surface, Goldreich and Julian (1969) show that a rotating magnetized neutron star in a vacuum would have a surface electric field strong enough to rip charges away from the surface. Their conclusion is that a rotating magnetized neutron star cannot exist in a vacuum, but must be surrounded by a plasma. The basic idea then of how the pulses are generated is that some sort of interaction between the plasma, the electromagnetic field, and/or the neutron star's surface creates a beam of electromagnetic radiation. (For example, one popular idea is that the beam is produced by a cascade of e^+e^- pairs.) This beam, however it is produced, corotates with the star. Anyone in the path of the beam detects a pulse each time the beam sweeps past them, once with each rotation of the star. (For a further discussion of pulsar models see Backer and Kulkarni 1990; Shapiro and Teukolsky 1983.)

Whatever the pulse mechanism is, the energy source of pulsars is rotation. Since they are radiating away energy they must slow down with time. In the simple dipole model the rate of rotational energy loss equals the rate that electromagnetic energy is radiated away from the star, i.e.,

$$\dot{E} = I\Omega\dot{\Omega} = -\frac{2|\ddot{\vec{m}}|}{3c^2}, \quad (1.1)$$

where $\dot{E}$ is the time derivative of the rotational energy, I the star's moment of inertia, Ω and $\dot{\Omega}$ the angular velocity of rotation and its time derivative, respectively, c the speed of light, and $\vec{m}$ the magnetic dipole moment. Using $|\vec{m}| \cong BR^3/2$, where B is the magnitude of the surface magnetic field at the pole and R is the neutron star radius, the time derivative of the pulse period can be related to the

magnetic field strength at the star's surface. Using typical neutron star numbers the result is

$$B \sin \theta \approx 10^{12} \left(\frac{P\dot{P}}{10^{-8} \frac{\text{s}^2}{\text{yr}}} \right)^{1/2} \text{ G}; \quad (1.2)$$

where θ is the angle $\vec{m}$ makes with the rotation axis, and $P = 2\pi/\Omega$ and $\dot{P}$ are the rotation period and its time derivative, respectively. In more complicated models the relationship between B and $P\dot{P}$ can be independent of the angle θ (see Shapiro and Teukolsky 1983). Typical observed values for $P\dot{P}$ are of the order $10^{-8} \text{ s}^2/\text{yr}$ and so typical pulsar magnetic field strengths are about 10^{12} G .

A simple estimate of the characteristic age of a pulsar is given by

$$t_{\text{age}} \approx \frac{1}{2} \frac{P}{\dot{P}}. \quad (1.3)$$

This assumes a constant magnetic field strength and that the observed period of a pulsar is much greater than its initial period at birth. For example, this gives an age of 1243yr for the Crab pulsar which agrees well with the Crab's known age of 937yr. (The crab pulsar was formed in the supernova explosion of 1054 A.D.) A typical value for t_{age} is 10^7 yr . However, this over estimates the true age. Equation (1.3) could also be interpreted as the approximate spindown time of a pulsar if $\dot{\Omega} = \text{constant}$ held true, but pulsars spin down more slowly than this linear approximation suggests. The actual time it would take a pulsar to radiate away all of its rotational energy probably exceeds the present age of the universe. However, pulsars stop pulsing before they stop spinning. This is because the e^+e^- pair cascade that is believed to be the source of the pulses turns off when the voltage between the surface of the star and the interstellar medium drops below the value necessary to produce the pairs. The voltage is generated by the

star's rotation. Thus, it can be shown that e^+e^- pairs cannot be created when a pulsar's magnetic field and period lie below the curve $B/P^2 = 2 \times 10^{11} \text{G} \cdot \text{s}^{-2}$. All known pulsars have observed values for B and P above this curve. Typical pulsars probably evolve below this curve, and thus turn off, in approximately 10^7yr . (For a further discussion of these ideas see Backer and Kulkarni 1990).

Comparison of pulsar magnetic fields with their ages suggest that the magnetic field decays exponentially on a time scale of $10^6 - 10^7 \text{yr}$ (Ostriker and Gunn 1969; Gunn and Ostriker 1970; Lyne Manchester and Taylor 1985; Stollman 1987). The observations could alternatively be interpreted as decay of the alignment angle θ to zero (e.g., see Kundt 1990). However observations do not support this alternative theory (Kulkarni 1986; Srinivasan 1989). Thus, the decay is presumedly due to ohmic dissipation in the neutron star's crust (see Shapiro and Teukolsky 1983). On the other hand, Sang and Chanmugam (1987), and Sang, Chanmugam, and Tsuruta (1990) argue that the magnetic field does not decay exponentially and its decrease could be only a few orders of magnitude during the lifetime of the universe.

Even less is known about a neutron star's interior magnetic field. Its strength is unknown, but could be several orders of magnitude larger or smaller than the surface field. Early estimates suggested it would persist for the lifetime of the universe (Baym, Pethick, and Pines 1969a) and some evidence supports this suggestion (Kulkarni 1986). However, Jones (1987) calculates that the interior field could be expelled to the crust in less than 10^7yr . A more recent and completely different theory of flux expulsion from the interior is given by Srinivasan et al. (1990). (For a summary of other theories of neutron star magnetic field evolution see Sang, Chanmugam, and Tsuruta 1990; for a further list of references see

Kundt 1990.) In any case, the main point is that the interior magnetic field is poorly understood.

Neutron stars are not only associated with pulsars. Accretion of matter from a companion onto a neutron star is the accepted explanation of x-ray binaries (e.g., see Bahcall 1978; Verbunt 1990). Neutron stars are also believed to be the sources of gamma ray bursts (See Ruderman 1989). Observations of x-ray binaries give mass estimates of neutron stars in the range of $1M_{\odot} - 2M_{\odot}$ where $M_{\odot}$ is the mass of the sun (see Avni 1978; Rappaport and Joss 1983; also see Shapiro and Teukolsky 1983).

The most accurate determination of the mass of a neutron star is from the Hulse-Taylor binary pulsar (PSR 1913+16). This pair consists of two orbiting neutron stars, one pulsing. High precision measurements of the pulse doppler shift plus measurements of the periastron advance (predicted by general relativity) give masses of (Taylor and Weisberg 1989)

$$\begin{aligned} M_{\text{pulsar}} &= 1.442 \pm 0.003 M_{\odot}, \\ M_{\text{companion}} &= 1.386 \pm 0.003 M_{\odot}. \end{aligned} \tag{1.4}$$

The binary pulsar is, of course, even more famous for the decay of its orbital period. This provides strong observational evidence for the existence of gravitational radiation (e.g., see Taylor and Weisberg 1989).

Unlike masses, which can in principle be determined from Kepler's third law, neutron star radii are hard to measure. Observations of the flux and spectrum of x-ray bursts can be used to give an estimate of the size of the emitting area. Assuming this is roughly the surface area of a neutron star gives an estimate for the star's radius. Observations during gamma-ray bursts of the redshift of the e^+e^- pair annihilation line have also been used to determine neutron star radii.

The results are in agreement with radii of order 10km (see Van Paradijs and Lewin 1990).

Neutron star oscillations may also be observable. Quasiperiodicity in the microstructure and subpulse of pulsar pulses, in x-ray binaries and x-ray bursters and in gamma ray bursts may be due to such oscillations (see Epstein 1988 for a list of references). Explanations of these observations as neutron star oscillations are at present far from convincing, though Van Horn (1980) has made some progress along these lines.

Finally, advancements in x-ray observations may make the detection of thermal radiation from neutron star surfaces possible. Young neutron stars ($t_{\text{age}} < 10^5 \text{yr}$) are expected to have surface temperatures of the order 10^6K . The surfaces glow in the soft x-ray part of the spectrum. Interior temperatures will be about one to two orders of magnitude higher. (See Tsuruta 1979 for a review of early calculations.) Recently, data from the *Einstein Observatory* has set upper limits on the temperatures of several supernova remnants, including four that are known to contain neutron stars. The data has been compared with theory in several studies (e.g., see Nomoto and Tsuruta 1986, 1987; Page and Baron 1990). Most of the data points are explainable by standard cooling theories. Standard cooling means cooling without the presence of exotic matter such as pions or a quark-gluon plasma. However, the temperature of the Vela pulsar lies below the standard cooling curve. This indicates that some neutron stars may contain exotic matter.

Evidence of Superfluidity

Neutron stars are very cold objects even though their temperature seems high

by terrestrial standards. The reason is that at nuclear density the Fermi energy expressed as a temperature is about 10^{12}K . Neutron star temperatures are far below this, which means neutron-star matter is very nearly in its lowest energy state. Such matter is called degenerate. It has been shown by Cooper (1956) and Bardeen, Cooper, and Schrieffer (1957) that an attractive interaction between degenerate fermions near the Fermi surface can result in the formation of Cooper pairs and a phase transition to superfluidity. (This is discussed in detail in the next section.) If the particles are charged then a superconductor is formed. The degenerate nucleons in a neutron star do attract each other via the strong nuclear force. Thus, one might expect the neutron star's neutrons to become superfluid and the protons to become superconducting at low enough temperatures. (This possibility was first suggested by Migdal 1959.) Evidence which supports this conjecture comes from the study of what are known as pulsar glitches.

A sudden increase in the pulsation rate—known as a “glitch”—has been observed on several occasions in several pulsars. For example, three glitches have been observed in the Crab pulsar with $\Delta\Omega/\Omega \sim 10^{-9} - 10^{-8}$, and seven giant glitches have been observed in the Vela pulsar with $\Delta\Omega/\Omega \sim 10^{-6}$. (Glitches are measurable because pulsars are amazingly stable clocks. Some pulse periods are known to thirteen decimal places.) The glitch rise time is less than a day and the pulsar relaxes back to its normal slow down rate on a time scale of days to years (see Sauls 1989; Pines and Alpar 1975; Shapiro and Teukolsky 1983).

The most accepted theory of glitches is that they represent a transfer of angular momentum from a portion of the superfluid interior to the remainder of the star. Rotating superfluids carry their angular momentum in an array of quantized vortices. (In equilibrium, the array itself rotates rigidly with the rest

of the star.) In the inner crust these vortices tend to pin to the heavy nuclei. As the star spins down the pinned vortices are forced to slow down as well, but their density remains the same. Thus, the angular momentum of the superfluid, which is carried by the vortices, does not decrease with the star as long as the vortices remain pinned. However, an outward pressure gradient is created across the pinned vortices. When this pressure gradient reaches a critical value the vortices will violently unpin transferring angular momentum to the rest of the star, spinning it up slightly. A glitch occurs. (Alternatively, a theory in which crust breaking, rather than vortex unpinning, occurs has recently been proposed by Ruderman 1991.) In a more detailed version of the model the vortices are not absolutely pinned but can creep along the pinning sites because of thermal fluctuations. The motion of the vortices is dissipative and acts as an internal heat source. The theory of vortex motion is referred to as vortex-creep theory. This theory was first used to explain glitches by Alpar et al. 1984. (For a review see Pines and Alpar 1985; for more recent work see Epstein and Baym 1988; Bildsten and Epstein 1989.)

The measurement of the interval between glitches and the post-glitch relaxation time for one pulsar sets the parameters of vortex-creep theory. The theory applied to another pulsar then only depends on the temperature and spindown rate of that star. Using measured spindown rates and temperature estimates, vortex-creep theory gives a reasonable fit to the glitch data of various pulsars. Conversely, the theory can be used to give temperature estimates for the neutron stars (Alpar, Nandkumar, and Pines 1985). The temperature estimate for the Vela neutron star concurs with the suggestion that it contains some exotic matter, though not enough data is available to make any firm conclusions.

Another source of evidence for superfluidity in neutron stars may come in the future from studies of neutron star temperatures. The profile of cooling curves, which show the star's surface temperature as a function of time, are sensitive to the presence of superfluidity. However, measurements much more precise than the currently known upper bounds on the surface temperatures will be needed.

Equilibrium Neutron Star Models

The theoretical equilibrium mass and radius of a neutron star is found by solving the equations of hydrostatic equilibrium for a specific nuclear matter equation of state. In the nonrotating case, a one parameter family of stellar models can be generated by specifying the central density of each model and then integrating the pressure outwards from the center of the star until it drops to zero.

Figure 1 shows the mass of a neutron star as a function of central density for three equations of state of varying stiffness. (Roughly speaking, a stiffer equation of state gives a higher pressure for a given density.) In Figure 2 the density is plotted as a function of the radial coordinate from the center of the star for the same three equations of state as in Figure 1. The plots in Figure 2 are for $1.4M_{\odot}$ stars. The results shown here were obtained using the fully relativistic equations for hydrostatic equilibrium (See Arnett and Bowers 1977; Shapiro and Teukolsky 1983). It can be seen in these figures that the mass and radius of a neutron star are very dependent on the equation of state of nuclear matter. The equations of state used are given in Pandharipande (1971), Bethe and Johnson (1974), and Serot (1979). These equations of state can be described as soft, intermediate, and stiff, respectively, and are representative of the range of equations of state for nuclear matter found in the literature (e.g., see Arnett and Bowers 1977).

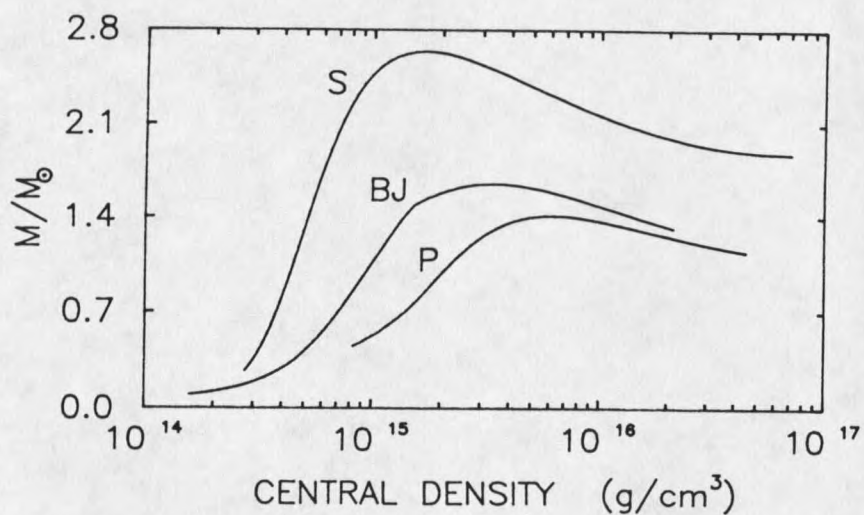


Figure 1. Mass versus central density. Results are shown for equations of state S (Serot 1979), BJ (Bethe and Johnson 1974) and P (Pandharipande 1971).

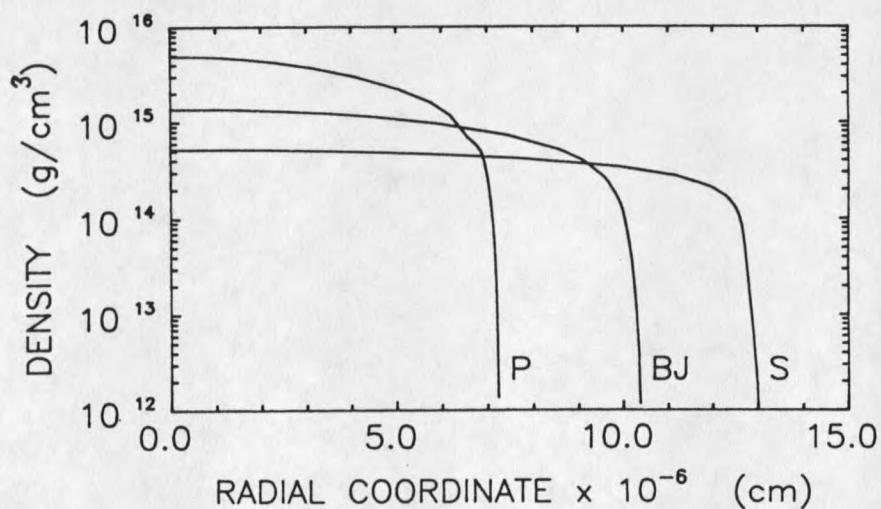


Figure 2. Density versus radial coordinate. Results are shown for equations of state S (Serot 1979), BJ (Bethe and Johnson 1974) and P (Pandharipande 1971).

In Figure 1 it can be seen that, in general; as the central density increases the mass of the star increases until a maximum mass is reached. Stars with central densities larger than that of the maximum mass star are unstable against collapse, whereas stars with central densities less than that of the maximum mass star are stable. The maximum mass increases in magnitude, but occurs at a lower central density, as the equation of state stiffens. For example the soft Pandharipande (1971) equation of state results in a maximum mass of about $1.4M_{\odot}$ at a central density of roughly $6 \times 10^{15} \text{ g/cm}^3$. The intermediate Bethe and Johnson (1974) equation of state results in a maximum mass of about $1.6M_{\odot}$ at a central density of $3 \times 10^{15} \text{ g/cm}^3$. Finally, the stiff Serot (1979) equation of state results in a maximum mass of about $2.6M_{\odot}$ at a central density of only $1.7 \times 10^{15} \text{ g/cm}^3$.

In Figure 2 the star with the softest equation of state has the smallest radius and the largest central density. The star with the stiffest equation of state has the largest radius and the smallest central density. Also, the density falls off very slowly from the central density except in the outer most part of the star. The important result is that most of the mass, about 80%, is at or above nuclear densities.

Arnett and Bowers (1977) argue that all supernova remnants, and thus all neutron stars, should have a mass of roughly $1.4M_{\odot}$, since above the Chandrasekhar limit (equal to $1.4M_{\odot}$) degenerate stellar cores become unstable against collapse. However, this does not mean that $1.4M_{\odot}$ is the maximum mass of a neutron star. An absolute upper limit on this mass, based on the minimal requirements of microscopic stability and causality, is about $3.6M_{\odot}$ (see Shapiro and Teukolsky 1983). Also, rotating stars can support more mass for a given central density. Friedman, Ipser, and Parker (1984) show that rotation can increase

the maximum mass of a neutron star up to 30% above its maximum nonrotating value:

Finally, the equilibrium composition of the star as a function of the total density is determined by calculating the ground state of nuclear matter at the various densities (e.g., see Baym, Bethe and Pethick 1971, Baym and Pethick 1975). The result is that the star is divided into several regions. The crustal regions are solid, although neutrons begin to drip out of the nuclei at a density of $4.3 \times 10^{11} \text{g/cm}^3$. These neutrons form a dilute gas in the inner crust. For densities above $2.8 \times 10^{14} \text{g/cm}^3$ the nuclei merge into a uniform fluid outer core of neutrons, protons, electrons, and muons. The greatest uncertainty is at densities greater than two or three times nuclear density. At these high densities an inner core is expected to form as the matter undergoes a transition to some new exotic phase. Solid neutrons, a pion condensate, or a quark-gluon plasma are just three possibilities.

Typical neutron star interiors are shown in Figure 3 for the stiff Serot equation of state (on the left) and for the soft Pandharipande equation of state (on the right). The stars shown each have a mass of $1.4M_{\odot}$. The regions where the neutrons and protons are expected to form superfluids are indicated. The important result shown in Figure 3 is that neutron stars with a stiff equation of state are primarily composed of superfluid neutrons, superconducting protons, and degenerate electrons and muons. There is some evidence in favor of a stiff equation of state as will be described in the next section.

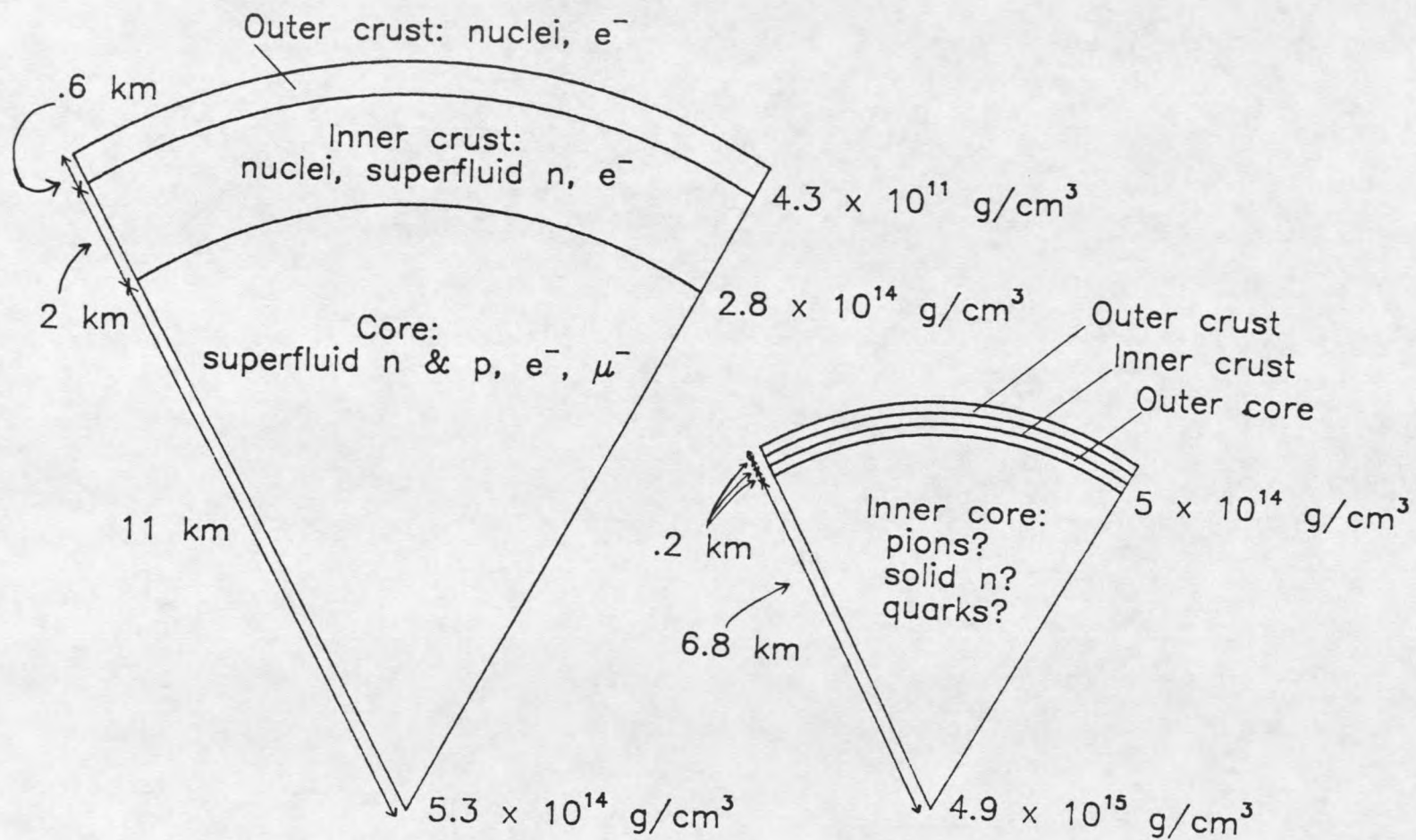


Figure 3. Cross sections of $1.4M_{\odot}$ neutron stars. Results shown are for Serot (on the left) and Pandharipande (on the right) equations of state.

How Fast Can a Neutron Star Rotate?

The discovery of millisecond pulsars (e.g., see Backer and Kulkarni 1990) naturally suggests the question: "How fast can a neutron star rotate?" The two fastest millisecond pulsars have periods of 1.56ms and 1.61ms which differ by only 3% from each other, while the next shortest period is longer by a factor of two. This suggests that $P \sim 1.6\text{ms}$ represents some sort of minimum rotation period (Friedman et al. 1988).

The absolute limit on the rotation rate of a neutron star is determined by the Keplerian limit, at which point mass on the equator rotates with its orbital velocity. A star which rotates faster than this would be unstable to mass shedding at the equator. It is believed, however, that millisecond pulsars are spun up from dead pulsars by accretion (Alpar et al 1982) or are born spinning fast due to accretion induced collapse of white dwarfs (Chanmugam and Brechet 1987). Thus, it has been suggested that the limiting rotation rate is a property of the accretion process (Alpar et al. 1982; Wasserman and Cordes 1988). On the other hand, Friedman et al. (1988) argue that models which require accretion to turn off at $P \sim 1.6\text{ms}$ must fine tune the values of magnetic field and the accretion rate. They argue that it is more plausible that the maximum rotation rate is determined by the equation of state and the mass of the star.

Following the arguments of Friedman et al. (1988), for a given mass star, the larger the radius the smaller the Keplerian limit on the angular velocity of rotation. Figure 2, showing the radii of $1.4M_{\odot}$ stars, then indicates that the stiffer the equation of state the smaller the maximum rotation rate. Since it is plausible that most neutron stars have a mass of at least $1.4M_{\odot}$, the equation of state is inferable from the maximum rotation rate. Furthermore, only the stiffest

equations of state have a minimum rotation period of $P \sim 1.6\text{ms}$. Thus, it is reasonable to hypothesize a stiff equation of state for nuclear matter. Redshift data of the pair annihilation line also suggest a stiff equation of state (Friedman et al. 1988; Lindblom 1984). However, the existence of the Vela pulsar below the standard cooling curve suggests that it contains exotic matter and thus has a high central density. If nuclear matter is described by a stiff equation of state, then the Vela neutron star would need to have a mass significantly larger than $1.4M_{\odot}$.

The upper limit on the rotation rate as a function of equation of state may not depend only on the Keplerian limit. The studies of Chandrasekhar (1970a,b), Friedman and Schutz (1977), and Friedman (1978) show that all rotating stars are unstable to the emission of gravitational radiation. However, internal dissipation can restabilize the star. (See Lindblom and Hiscock 1983; Lindblom and Detweiler 1977; Friedman 1983; Lindblom 1986; Lindblom 1991). In ordinary neutron stars, the critical angular velocity at which the instability sets in is within 10% of the Keplerian limit. Thus, the above arguments suggesting a stiff equation of state for nuclear matter are not significantly changed by the gravitational-radiation instability. However, this instability is important to determining the precise upper limit on the rotation rate of neutron stars.

Superfluidity

Observational Properties

Superfluids and superconductors are distinguished by their ability to maintain persistent mass or charge currents without dissipation. For example Reppy and Depatie (1964) rotated a vessel packed with a porous material and containing superfluid ^4He . After the vessel was brought to rest the ^4He continued to flow,

showing no reduction in its angular velocity over the twelve-hour period of the experiment. Experiments also show that superfluid ^4He can flow through narrow channels without viscosity. Similarly, in an experiment at MIT, a current circulated in a ring without a driving force for more than a year (see Tinkham 1965). A persistent current can be set up in a superconductor by bringing a magnet nearby. The repulsive force between the magnet and the current in the superconductor will cause the magnet to levitate above the superconductor indefinitely. (For further references see Tilley and Tilley 1986.)

Another similarity between superfluids and superconductors is the formation of vortices. Rotating superfluids do not rotate rigidly but form arrays of quantized vortices. (The array is stationary in a frame which corotates with the container.) Similarly, a type II superconductor forms an array of quantized flux-carrying vortices when placed in a magnetic field. (A type I superconductor completely expels all magnetic flux from its interior. This is known as the Meissner effect.) The array of vortices in rotating ^4He has been photographed by Yarmchuk and Packard (1982). The array of vortices in type II superconductors has been photographed by many investigators by spraying magnetic particles onto the superconducting surface (e.g., see Essmann and Träuble 1967.) The currents around these vortices are further examples of nondissipative flows in superfluids and superconductors.

Not all superfluid behavior corresponds to that of a perfect fluid. Experiments using piles of thinly spaced discs suspended by a torsion fiber in superfluid ^4He show that the oscillations of the discs are damped out (see Tilley and Tilley 1986 for references). Evidently a superfluid is capable of both viscous and non-viscous motion at the same time. This observation led Tisza (1938) to suggest a two-fluid description of superfluid ^4He . In the two-fluid model a superfluid is a

mixture of a superfluid which flows without dissipation, and a normal fluid which flows with ordinary dissipative effects. It should be pointed out that the two-fluid model is a paradigm for describing the phenomenological properties of the fluid. The two fluids cannot be separated. In the phenomenological approach of this thesis this subtle property of superfluidity will not be important.

A superconductor could also be described by the two-fluid model. However, in the case of a laboratory superconductor, the normal fluid would be composed of normal electrons which would be clamped to the atomic lattice on ordinary resistive time scales. Thus, the two-fluid behavior of a superconductor would not be as apparent as it is in ^4He . The fact that ^4He atoms are bosons and the electrons in a superconducting metals are fermions would also make superfluidity and superconductivity seem distinct.

London (1950, 1954) was the first to emphasize the similarities between superfluidity and superconductivity. He suggested that both superfluidity and superconductivity were due to the condensation of a significant fraction of the particles into a single quantum state. The particles in the ground state would be responsible for superfluid mass and charge currents. The main difference between a superfluid and a superconductor would be that in the latter the particles are charged. The fact that ^3He atoms, which are fermions, form a superfluid confirms the unifying description advocated by London (see Tilley and Tilley 1986). In a Fermi fluid, such as the neutron and proton fluids in a neutron star, the condensation occurs via the formation of Cooper pairs and the ground-state energy is separated from excited states by an energy gap.

Cooper Pairs

A remarkable paper by Cooper (Cooper 1956) showed the essence of the microscopic process that leads to superfluidity in a system of fermions. The calculation is based on the theory of an ideal Fermi gas and it is simple enough to repeat here. The treatment presented here follows that given by Baym (1973).

Consider a collection of spin 1/2 fermions in a box of volume L^3 . The Pauli exclusion principle states that only one fermion can exist in each quantum state available to the system. The momentum of the particles is quantized. For example, the momentum of the particle in the x direction is given by $p_x = 2\pi\hbar n_x/L$, where $n_x = 1, 2, 3, \dots$. The number of momentum states per unit volume of momentum space is then $L^3/(8\pi^3\hbar^3)$. Thus, in an ideal Fermi gas there can be only two fermions in each of the momentum states of the system, one spin up and the other spin down. In the ground state, the particles will fill all the momentum states up to a maximum value called the Fermi momentum, p_F . The filled states form a sphere in momentum space called the Fermi sea, with the top of this sea called the Fermi surface. The Fermi momentum is found by integrating the number of momentum states over the Fermi sea and equating twice this value (for the two spin states) with the total number of particles in the system. The result is $p_F = \hbar(3\pi^2 n)^{1/3}$, where n is the number density of particles. The energy of a particle at the top of the Fermi sea is called the Fermi energy and it is given by $E_F = p_F^2/2m$, where m is the mass of the particle.

Cooper realized that superconductivity might be due to an attractive interaction between particles at the top of the Fermi sea. If the particles were to form bound pairs the binding energy of the pairs would lower the overall energy of the system, as was observed in superconductors. To study this possibility, Cooper

solved the problem of two particles above the Fermi sea which interact via an attractive potential with the following matrix elements, expressed in the momentum basis for the two particles:

$$\langle \vec{p}_1 \vec{p}_2 | V | \vec{p}_1' \vec{p}_2' \rangle = \begin{cases} -V_0 & p_F < p_1, p_2, p_1', p_2' < p_1 \\ 0 & \text{otherwise} \end{cases} \quad (1.4)$$

For this potential the pair of particles will feel an attractive interaction proportional to V_0 if their momenta lie within an interaction zone above the Fermi sea given by $p_F < p < p_1$. To conserve momentum the restriction $\vec{p}_1 + \vec{p}_2 = \vec{p}_1' + \vec{p}_2'$ must also hold.

The procedure now is to solve for the ground-state energy of the pair of particles for the potential given in Eq. (1.4). The state vector for the system can be written as an expansion of free particle momentum states, i.e., as

$$|\psi\rangle = \sum_{\vec{p}_1 \vec{p}_2} a_{\vec{p}_1 \vec{p}_2} |\vec{p}_1, \vec{p}_2\rangle. \quad (1.5)$$

The Schrödinger equation is given by

$$H|\psi\rangle = \left(\frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V \right) |\psi\rangle = E|\psi\rangle. \quad (1.6)$$

Taking $\langle \vec{p}_1 \vec{p}_2 | H | \psi \rangle = E \langle \vec{p}_1 \vec{p}_2 | \psi \rangle$ gives

$$\frac{p_1^2 + p_2^2}{2m} a_{\vec{p}_1 \vec{p}_2} - V_0 \sum_{\vec{p}_1' \vec{p}_2'}' a_{\vec{p}_1' \vec{p}_2'} = E a_{\vec{p}_1 \vec{p}_2}, \quad (1.7)$$

where the prime on the summation sign means that the sum is to take place only over the interaction zone. Solving this equation for $a_{\vec{p}_1 \vec{p}_2}$ gives

$$a_{\vec{p}_1 \vec{p}_2} = \left(\frac{-V_0}{E - \frac{p_1^2 + p_2^2}{2m}} \right) \sum_{\vec{p}_1' \vec{p}_2'}' a_{\vec{p}_1' \vec{p}_2'}. \quad (1.8)$$

Summing both sides of Eq. (1.8) over the interaction zone results in the following transcendental equation for energy:

$$1 = \sum_{\vec{p}_1 \vec{p}_2} ' \left[\frac{-V_o}{E - \frac{p_1^2 + p_2^2}{2m}} \right]. \quad (1.9)$$

It can be shown that the lowest energy solution to Eq. (1.9) occurs when the sum over the interaction zone in this equation is further restricted by the condition $\vec{p}_1 + \vec{p}_2 = 0$. An approximate solution can be found by replacing the sum with an integral over the interaction zone. The integral can be performed over energy states by using the density of free particle states

$$g_o(E) = \left(\frac{L^3}{4\pi^2} \right) \left[\frac{(2m)^{3/2}}{\hbar^3} \right] \sqrt{E}, \quad (1.10)$$

where $g_o(E)dE$ is the number of states between E and $E + dE$. It is also useful to define the energy range associated with the interaction zone: $E_I \equiv p_I^2/2m - p_F^2/2m$. The result for the ground-state energy E_{GS} can then be approximated by

$$E_{GS} \approx 2E_F - 2\Delta, \quad (1.11)$$

where

$$\Delta \approx E_I \exp \left[\frac{-2}{g_o(E_F)V_o} \right]. \quad (1.12)$$

Similarly, the result for $a_{\vec{p}_1 \vec{p}_2}$ given by Eq. (1.8) along with Eq. (1.5) can be used to find the ground-state wave function. The result as a function of the distance between the two particles $r = |\vec{r}_1 - \vec{r}_2|$ is roughly

$$\psi(r) \propto (1/r) \sin[(\Delta/2\hbar v_F)r] \sin(p_F r/\hbar), \quad (1.13)$$

where $v_F = p_F/m$ is the Fermi velocity. Thus, the size of the bound pair (the width of the wave function) is of the order

$$\xi \equiv \frac{\hbar v_F}{\pi \Delta}. \quad (1.14)$$

The length scale ξ is known as the coherence length and plays an important role in the theory of superfluidity.

The important conclusion of this calculation is that the ground-state energy per particle of the pair is less than the Fermi energy and is separated from the Fermi energy by a gap Δ . This means that the Fermi sea will be unstable to the formation of these pairs, which have since become known as Cooper pairs. Thus, a Fermi fluid which has undergone a phase transition to superfluidity will form a condensate of Cooper pairs. However, to get the correct expression for the energy gap one needs to use many-body theory. Setting up the correct second quantized Hamiltonian and solving it was one of the major accomplishments of Bardeen, Cooper, and Schrieffer (1957). Their theory (from now on referred to as BCS theory) is discussed in the next part of this chapter.

BCS Theory

The main shortcoming of Cooper's calculations is that it treats two particular particles quite differently from the rest. Bardeen, Cooper, and Schrieffer (1957) remedy this by using the standard many-body theory approach of second quantization (e.g., see Baym 1973). The method involves introducing creation and annihilation operators. Only a simple overview of the theory will be given here. (For more extensive treatments, besides the 1957 paper, see Tilley and Tilley 1986; Rickayzen 1969; Amundsen and Østgaard 1985a.) Let $c_{\vec{p}\sigma}^\dagger$ and $c_{\vec{p}\sigma}$ create

and annihilate a particle with momentum $\vec{p}$ and spin σ , respectively. A simple second quantized Hamiltonian which generalizes the problem solved by Cooper to a many-body problem is given by

$$H = \sum_{\vec{p}\sigma} \varepsilon_{\vec{p}} c_{\vec{p}\sigma}^{\dagger} c_{\vec{p}\sigma} + \sum_{\vec{p}\vec{p}'} V_{\vec{p}\vec{p}'} c_{\vec{p}\uparrow}^{\dagger} c_{-\vec{p}\downarrow}^{\dagger} c_{\vec{p}'\uparrow} c_{-\vec{p}'\downarrow}, \quad (1.15)$$

where $V_{\vec{p}\vec{p}'}$ is an arbitrary function of $\vec{p}$ and $\vec{p}'$, and $\varepsilon_{\vec{p}} = p^2/2m - E_F$ is the energy of a particle measured from the Fermi surface. The up and down arrow refer to spin up and spin down states, respectively. To find the lowest energy state, Bardeen et al. (1957) assume the ground state is a superposition of many-body states in which the pair states with $\vec{p}_1 + \vec{p}_2 = 0$ are either completely occupied or completely empty (i.e., no half occupied pairs are allowed). The state vector can then be written as

$$|\text{BCS}\rangle = \prod_{\vec{p}} (u_{\vec{p}}^* + v_{\vec{p}} c_{\vec{p}\uparrow}^{\dagger} c_{-\vec{p}\downarrow}^{\dagger}) |0\rangle. \quad (1.16)$$

In this equation $u_{\vec{p}}$ is the probability amplitude that a pair is unoccupied and $v_{\vec{p}}$ is the probability amplitude that a pair is occupied (an asterisk refers to complex conjugation). Normalization of the state vector requires that

$$|u_{\vec{p}}|^2 + |v_{\vec{p}}|^2 = 1. \quad (1.17)$$

The energy is then given by

$$E = \langle \text{BCS} | H | \text{BCS} \rangle = \sum_{\vec{p}} 2\varepsilon_{\vec{p}} |v_{\vec{p}}|^2 + \sum_{\vec{p}} \Delta_{\vec{p}}^* u_{\vec{p}} v_{\vec{p}}. \quad (1.18)$$

where the generalized gap is defined by

$$\Delta_{\vec{p}} = \sum_{\vec{p}'} V_{\vec{p}\vec{p}'} u_{\vec{p}'} v_{\vec{p}'}. \quad (1.19)$$

The ground-state energy is then found by the variation principle. That is, the amplitudes $u_{\vec{p}}$ and $v_{\vec{p}}$ are determined by minimizing the energy. The energy is extremized when $dE/dv_{\vec{p}}^* = 0$ while holding $v_{\vec{p}}$ and $u_{\vec{p}}^*$ fixed and subject to the constraint given by Eq. (1.17). The general solution is a bit messy and won't be given here. (The curious can see Tilley and Tilley 1986.)

In the simple case that $V_{\vec{p}\vec{p}'} = -V_0$ for the interaction zone $-E_1 < \varepsilon_{\vec{p}}; \varepsilon_{\vec{p}'} < E_1$, a simple solution for the gap can once again be found, given by

$$|\Delta| \approx 2E_1 \exp \left[\frac{-1}{g_0(E_F)V_0} \right]. \quad (1.20)$$

In this case the ground-state energy of all the particles is given by $E_{GS} = E_n - \frac{1}{2}g_0(E_F)\Delta^2$, where E_n is the energy the particles would have if all of them were in the normal state.

The energy of excited states are then found by realizing that an excited state is formed by breaking a pair. The energy of an excited state measured from the ground state then is the energy of the particle in the broken pair minus the energy of the corresponding occupied pair. The result is that the energy of elementary excitations away from the BCS ground state are given by $E_{\vec{p}} = \sqrt{\varepsilon_{\vec{p}}^2 + |\Delta|^2}$ (see Tilley and Tilley 1986). This has the important consequence that the number of states available to the system in the vicinity of the Fermi energy is given by $g(E) \cong g_0(E_F)d\varepsilon/dE$ which gives the following result:

$$g(E) \cong \begin{cases} 0, & |E| \leq |\Delta|, \\ \frac{g_0(E_F)E}{\sqrt{E^2 - |\Delta|^2}}, & |E| > |\Delta|. \end{cases} \quad (1.21)$$

The result is plotted in Figure 4. This shows the well known result that the minimum energy required to create an excitation above the ground state is 2Δ . The gap has very important consequences for the behavior of the superfluid.

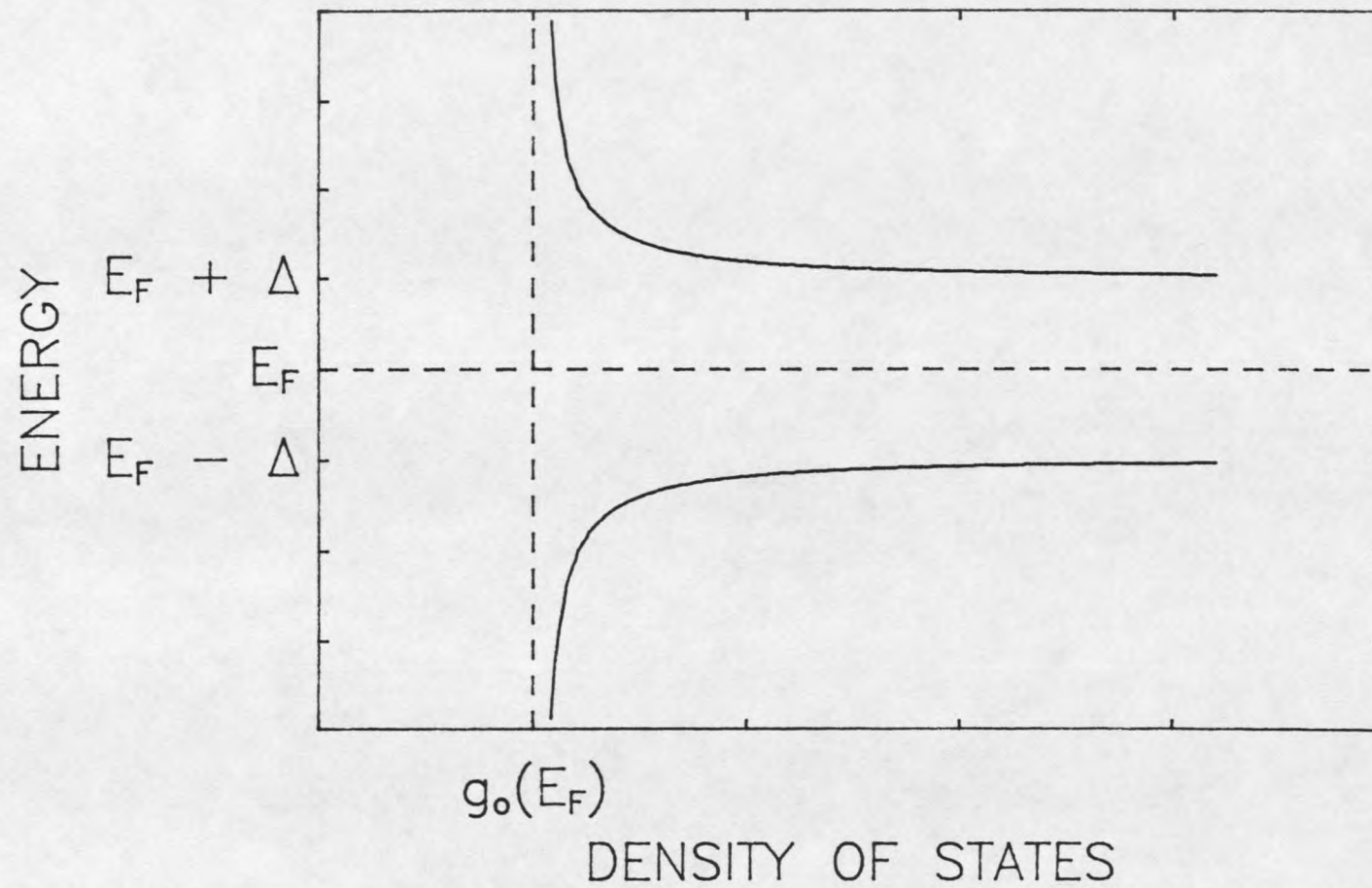


Figure 4. Density of states in a BCS superfluid.

Gap in Neutron Stars

The transition temperature to superfluidity, T_c , is related to the gap by roughly $k_B T_c \approx \Delta$, where k_B is the Boltzman constant. This means superfluidity is favored when the energy associated with thermal excitations is smaller than the energy gap. The calculation of the transition temperature to superfluidity and superconductivity in neutron stars then involves calculating the various quantities which appear in Eq. (1.20) (or more sophisticated equations for the gap). The calculation has been performed by a number of researchers. The results obtained by Amundsen and Østgaard (1985a,b) are shown in Figures 5 and 6. (For a summary of other results see Epstein 1988.) At low densities, corresponding to those of the inner crust, the neutrons form $^1S_0$ Cooper pairs. At higher densities, the short range repulsive part of the nuclear interaction causes the neutrons to form $^3P_2$ Cooper pairs. Since only a small fraction of the total number of particles is protons, the protons still form $^1S_0$ Cooper pairs at these higher total densities. (Also note that neutron stars are probably the highest T_c superconductors in the present universe.)

Figures 5 and 6 show that the gap vanishes (i.e., the transition temperature falls below a typical interior temperature) outside limited density ranges. However, for a stiff equation of state, a $1.4M_\odot$ star will have most of its mass within these ranges. In this case, most of the interior will consist of superfluid neutrons and superconducting protons, along with degenerate electrons and muons.

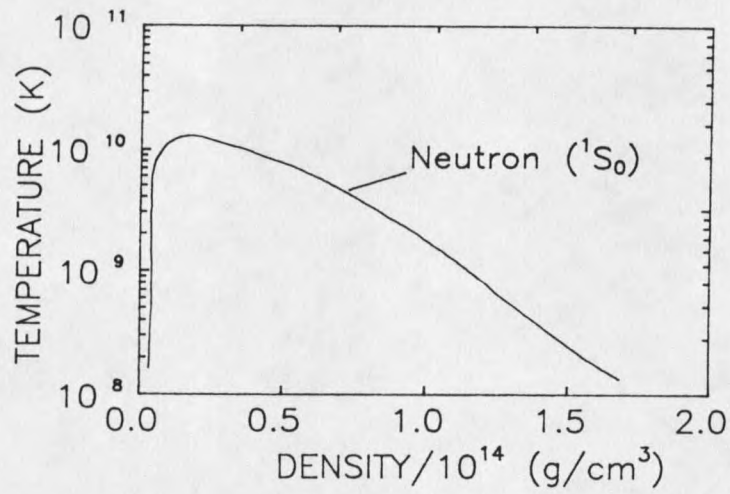


Figure 5. Superfluid-transition temperature versus density in the crust.

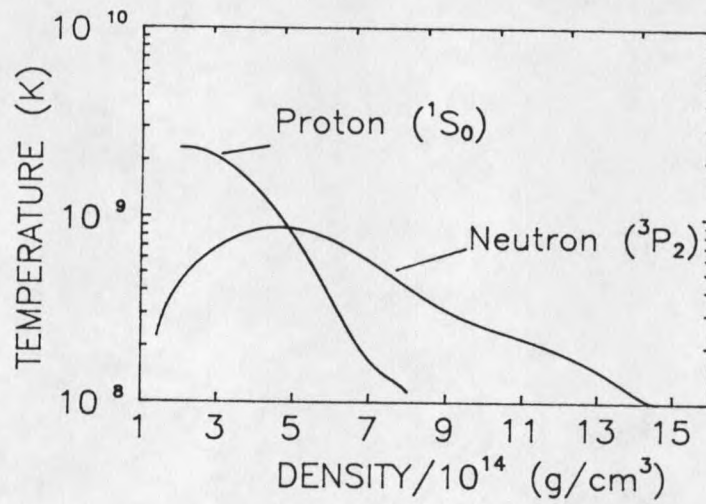


Figure 6. Superfluid-transition temperature versus density in the core.

CHAPTER 2

SUPERFLUID HYDRODYNAMICS

London Theory

In Chapter 1 it was stated that the quantum process underlying the transition to superfluidity (or superconductivity) is the condensation of a significant fraction of the particles into a ground state. In the case that the transition takes place in a Fermi fluid, e.g., such as the neutron and proton fluids in a neutron star, the microscopic properties are described by BCS theory. However, to describe the macroscopic behavior of a superfluid a macroscopic phenomenological approach is needed.

One of the most successful approaches to describing the macroscopic properties of superfluids was developed by the Londons (London and London 1935; London 1950, 1954; see also Tilley and Tilley 1986). It was proposed, using the analogy of electrons which orbit an atom without dissipation, that the superfluid condensate should be described by a wave function Ψ . However, in the London theory Ψ is not given a probabilistic interpretation, but is rather to be interpreted as a classical complex scalar field. (This idea was also suggested by Onsager 1949 and Penrose 1951.) The wave function can be written as

$$\Psi = \psi_0 e^{iS}. \quad (2.1)$$

In the classical interpretation given here, the wave function is not normalized to give the probability density but is normalized to give the density of particles in

the condensate ρ_s , i.e.,

$$\Psi^* \Psi = \psi_0^2 = \rho_s. \quad (2.2)$$

In the London theory ρ_s is assumed to be a slowly varying function over the distance scales of interest and is thus treated as a constant. When this assumption holds the canonical momentum divided by the mass m of the particles (or Cooper pairs) in the condensate is given by

$$v_s^a + aA^a = \frac{\hbar}{m} \nabla^a S. \quad (2.3)$$

The right-hand side of this equation is found by operating on Ψ with $-i(\hbar/m)\nabla^a$, where ∇^a is the covariant derivative operator, e.g., in Cartesian coordinates ∇^a is just the partial derivative $\partial/\partial x_a$. (Note: throughout this thesis the latin indices a, b, c will be used to refer to the spatial components of vectors and tensors and summation over these repeated indices is assumed. Vector notation using over-arrows will also be used when convenient.) Equation (2.3) is one form of what is commonly called the London equation. On the left-hand side of this equation it defines the superfluid velocity v_s^a as the kinematic portion of the canonical momentum of the superfluid particles (or Cooper pairs) per unit mass. Also in Eq. (2.3), $a = q/mc$ is the charge-to-mass ratio of each particle (or Cooper pair) divided by c , the speed of light, and A^a is the electromagnetic vector potential. (Gaussian units are used throughout this thesis for all electromagnetic quantities.) Thus, Eq. (2.3) includes the possibility that the particles are charged.

The very important results of circulation and flux quantization can now be derived from the London equation. (Quantization of circulation was first considered by Onsager 1949 and Feynman 1955. Quantization of flux was first considered by Abrikosov 1957.) First consider integrating Eq. (2.3) around a closed contour.

The result on the right-hand side will be $\hbar/m$ times the change in the phase S around the contour. However, the requirement that the wave function be single-valued requires that the change in the phase be an integer multiple of 2π . The result is

$$\oint (v_s^a + aA^a) d\ell_a = \frac{nh}{m} \quad \text{where } n = 0, \pm 1, \pm 2, \dots \quad (2.4)$$

An equivalent conclusion to that of Eq. (2.4) can be reached by taking the curl of both sides of Eq. (2.3). The curl of the left side can be written as $\omega_s^a + aB^a$, where $\omega_s^a \equiv (\vec{\nabla} \times \vec{v}_s)^a$ is the vorticity of the superfluid and $B^a \equiv (\vec{\nabla} \times \vec{A})^a$ is the magnetic induction. Now the right-hand side of Eq. (2.3) is equal to a constant times a gradient. As a consequence, the vorticity and the magnetic induction are linked everywhere by $\omega_s^a = -aB^a$, except where the curl of the right-hand side of Eq. (2.3) becomes singular. On the basis of microscopic theory and laboratory experiments, the singular regions are found to form arrays of one-dimensional quantized vortices (see Chapter 1). In the ground state $n = 1$ for each vortex, where n is the quantum number in Eq. (2.4). Thus, the curl of Eq. (2.3) takes on the form (see Sonin 1987)

$$\omega_s^a + aB^a = \frac{\hbar}{m} \sum_i \int \nu_i^a(\ell) \delta^3[\vec{r} - \vec{r}_i(\ell)] d\ell, \quad (2.5)$$

where $r^a(\ell)$ is a vector giving the location of the points (parameterized by ℓ) along the i th vortex, and ν_i^a is the unit vector tangent to that vortex.

In the case of an uncharged superfluid the result of either Eq. (2.4) or Eq. (2.5) is that in the ground state each vortex carries one quantum of circulation given by

$$\oint v_s^a d\ell_a = \int \omega_s^a dS_a = \frac{\hbar}{m} \equiv \kappa. \quad (2.6)$$

The quantum of circulation κ depends on the mass of the particles (or Cooper pairs). For the superfluid neutrons in a neutron star $m = 2m_n$, and thus $\kappa = 1.98 \times 10^{-3} \text{ cm}^2/\text{s}$.

In the case of a charged superfluid (i.e., a superconductor) the current around the vortex will generate a magnetic induction along the vortex. Outside the vortex, the equations to be solved are the curl of the London equation and Faraday's law, i.e.,

$$\omega_s^a = -\frac{q}{mc} B^a \quad (2.7)$$

and

$$(\vec{\nabla} \times \vec{B})^a = \frac{4\pi q \rho_s}{c} v_s^a. \quad (2.8)$$

Taking the curl of Eq. (2.8) and substituting Eq. (2.7) gives the result

$$\nabla^2 B^a + \frac{4\pi q^2 \rho_s}{m^2 c^2} B^a = 0. \quad (2.9)$$

This is the well known vector Helmholtz equation. By studying solutions to Eqs. (2.7)–(2.9) it can easily be shown that far from a vortex the magnetic induction and the superfluid velocity will diminish exponentially in a characteristic distance Λ known as the London depth. (The various fields associated with a vortex will be studied in more detail at the end of Chapter 4, in the context of neutron stars.) The London depth is given by

$$\Lambda = \sqrt{\frac{m^2 c^2}{4\pi q^2 \rho_s}}. \quad (2.10)$$

Thus, the contour in Eq. (2.4) can be extended out to where the superfluid velocity essentially vanishes. The result is that in the ground state each vortex in a charged superfluid carries one quantum of flux given by

$$\oint A^a d\ell_a = \int B^a dS_a = \frac{hc}{q} \equiv \Phi_0. \quad (2.11)$$

In most cases (e.g., the superconducting protons in a neutron star) the charge is given by $q = 2e$, where e is the absolute value of the charge of an electron. In this case, the quantum of flux is given by $\Phi_0 = 2 \times 10^{-7} \text{G} \cdot \text{cm}^2$.

In general, the London equation, Eq. (2.3), is valid only outside the core of a vortex. The core of a vortex can be defined as the region within which the superfluid density ρ_s falls rapidly to zero. It is found that the radius of the core is roughly equal to the coherence length of the superfluid ξ [see Eq. (1.14); Tilley and Tilley 1986]. When ρ_s varies rapidly on the scale of the coherence length ξ a theory more general than the London theory is needed. One theory, which can handle more general cases, such as the core of a vortex, is the Ginzburg-Landau theory (Ginzburg and Landau 1950). In this theory Ψ is identified as an order parameter which describes the phase transition to superfluidity. The basic idea of the Ginzburg-Landau theory is that the free energy density of the system can be expanded in powers of the order parameter. The requirement that the equilibrium phase corresponds to a minimum of the free energy then results in a set of Euler-Lagrange equations for the order parameter. The details of theory will not be given here. However, in the static limit, the Ginzburg-Landau equations are basically identical to those which appear in the standard quantum field theory of the Higg's field. Superfluidity can be treated then as an example of spontaneous symmetry breaking (see Ryder 1985). In addition, it is interesting to note that the Landau-Ginzburg theory can be derived from the microscopic BCS theory (see Werthamer 1969; Tilley and Tilley 1986). Thus, when ρ_s is a slowly varying function which can be treated as a constant, the London theory is also justified by microscopic theory. In some ways then the London theory

is similar to the WKB approximation in quantum mechanics (see also the next section; Putterman 1974):

So far it has been shown that quantized vortices can exist in superfluids. These vortices will appear when their presence minimizes a properly chosen thermodynamic potential, such as the free energy of the system. The result is that it is energetically favorable for vortices to appear only under certain conditions. The conditions given here will be restricted to the case of zero temperature, since the main interest in this thesis is in neutron stars which have cooled far below their superfluid-transition temperatures. In a rotating uncharged superfluid vortices appear above a critical angular velocity $\Omega_{c1} = (\hbar/m)(1/R_o^2)\ln(R_o/\xi)$, where R_o is the size of the container. In a neutron star a typical value for Ω_{c1} is 10^{-14}s^{-1} . However, a large enough angular velocity will cause the density of vortices to become so great that the vortices touch each other and superfluidity is destroyed. This occurs at a second critical angular velocity given by $\Omega_{c2} = (\hbar/m)(1/\xi^2)$. In a neutron star this corresponds to $\Omega_{c2} \approx 10^{20}\text{s}^{-1}$. Thus, in all neutron stars, except those with a negligible angular velocity, the superfluid neutrons can be expected to be threaded with an array of vortices. The situation is more complicated in a superconductor. Type I superconductors obey the Meissner effect, completely expelling all magnetic flux until a critical field $H_c = [(2m)^{3/4}/\hbar^{3/2}]\sqrt{E_F}|\Delta|$ is reached which destroys the superconductivity. On the other hand, a Type II superconductor obeys the Meissner effect below a certain critical field $H_{c1} = (\Phi_o/4\pi\Lambda^2)\ln(\Lambda/\xi)$. Above this field exists a vortex state in which the magnetic flux is carried by the array of vortices described above. Finally, when a second critical field $H_{c2} = \Phi_o/2\pi\xi^2$ is reached the vortices touch

each other and type II superconductivity is destroyed. In a neutron star the protons are expected to form a type II superconductor and the typical values for H_{c1} and H_{c2} are 10^{15}G and 10^{16}G , respectively. Since the interior magnetic field of a neutron star is probably of the order 10^{12}G , one might expect that all magnetic flux should then be expelled from the region of superconducting protons. However, it has been argued that the flux expulsion time is sufficiently long that it will still be favorable for the protons to exist in the vortex state (See Baym, Pethick, and Pines 1969b).

Lastly, at high densities, the neutrons are expected to form $^3P_2$ Cooper pairs. For spin triplet pairing a more complicated order parameter than a complex scalar field is needed. However, this is not expected to affect the macroscopic description of the neutron superfluid. Thus, the complex scalar field description of the neutron condensate will be adequate for the purposes of this thesis (see Sauls 1989).

Landau Equations

While the London theory does describe superfluid vortices, it does not describe the macroscopic dynamics of a superfluid. This was accomplished by Landau (1941), who used the two-fluid concept to derive a set of superfluid hydrodynamic equations. The derivation is based on a phenomenological approach. (See also Khalatnikov 1957; Putterman 1974.) All hydrodynamics theories assume that the system can be described completely by a set of continuous classical fields. Landau's approach then makes use of the following conditions: 1) the usual conservation laws are to be obeyed, 2) the equations are to be covariant under a Galilean transformation, and 3) the laws of thermodynamics are to be obeyed. The first two conditions should hold for any classical theory. The third condition

applies only to macroscopic systems. Thus, Landau's phenomenological approach could be described as a derivation from "macroscopic" first principles. It is also assumed that the usual laws of classical mechanics and electrodynamics hold. The power of Landau's method is that it is independent of any microscopic theory.

Beginning with the assumption that a superfluid is describable by a set of continuous fields, Landau postulated that the description of a superfluid should include an additional velocity degree of freedom along with the fields which describe an ordinary fluid. This extra velocity is called the superfluid velocity, v_s^a . The other independent variables can be taken to be, as in an ordinary fluid, the mass density, ρ , the entropy density, s , and the momentum density, P^a . The time evolution of these variables is described by the usual conservation laws for mass, entropy (in the nondissipative limit), and momentum. These are given, respectively, by

$$\partial_t \rho + \nabla_b M^b = 0, \quad (2.12)$$

$$\partial_t s + \nabla_b s^b = 0, \quad (2.13)$$

$$\partial_t P^a + \nabla_b \pi^{ab} = 0. \quad (2.14)$$

In these equations, the mass current M^a , the entropy current s^a , and the stress tensor π^{ab} are yet to be determined in terms of the independent variables. Next, for the superfluid velocity, Landau postulated that it obey the following form of Newton's second law:

$$\partial_t v_s^a + v_s^b \nabla_b v_s^a = -\nabla^a \chi, \quad (2.15)$$

where χ is an as yet undetermined scalar field. The form of Eq. (2.15) is chosen such that the superfluid velocity v_s^a obeys the Kelvin circulation theorem, $\Gamma = \oint v_s^a d\ell_a = \text{constant}$. A constant entropy fluid is known to obey this theorem.

[That v_s^a obey this theorem is also consistent with the quantization of circulation, given by Eq. (2.6).] Since the particles in a superfluid condensate can be thought of as occupying a single quantum state, the entropy of the condensate is zero. The idea then is that the superfluid velocity describes a zero entropy (and therefore constant entropy) perfect fluid. One final remarkable property of Eq. (2.14) is that if the London condensate wave function Ψ is a solution to the Schrödinger equation, then equation Eq. (2.15) can be shown to follow from the standard WKB approximation used in quantum mechanics (see Putterman 1974). Finally, energy should be conserved. The energy conservation law in terms of the energy density U and the energy current U^a is given by

$$\partial_t U + \nabla_b U^b = 0. \quad (2.16)$$

To close the set of equations, Eqs. (2.12)–(2.16), the dependent variables, M^a , s^a , π^{ab} , χ , U , and U^a , must be determined in terms of the independent variables ρ , s , P^a , and v_s^a . This is accomplished by noting that Eqs. (2.12)–(2.15) constitute eight independent equations which describe the time evolution of the eight independent variables. Thus, these equations must imply the energy conservation law, Eq. (2.16). This requirement, along with the principle of Galilean covariance, can be used to determine the dependent variables.

The first step is to require Galilean covariance. Equations (2.12)–(2.16) are covariant under Galilean transformations if the variables measured in a frame travelling with velocity v^a are related to the variables in the “lab” frame by

$$\rho' = \rho, \quad (2.17)$$

$$M^{a'} = M^a + \rho v^a, \quad (2.18)$$

$$s' = s, \quad (2.19)$$

$$s^{a'} = s^a + sv^a, \quad (2.20)$$

$$P^{a'} = P^a + \rho v^a, \quad (2.21)$$

$$\pi^{ab'} = \pi^{ab} + P^a v^b + v^a M^b + \rho v^a v^b \quad (2.22)$$

$$v_s^{a'} = v_s^a + v^a, \quad (2.23)$$

$$\chi' = \chi, \quad (2.24)$$

$$U' = U + P^b v_b + \frac{1}{2} \rho v^b v_b, \quad (2.25)$$

$$U^{a'} = U^a + (\frac{1}{2} \rho v^b v_b + P^b v_b + U) v^a + \frac{1}{2} v^b v_b M^a + v_b \pi^{ba}. \quad (2.26)$$

A form for the energy density which transforms correctly is

$$U = \tilde{U}_o + v_s^b P_b - \frac{1}{2} \rho v_s^b v_{sb}, \quad (2.27)$$

where $\tilde{U}_o$ is the energy density in a frame which moves with velocity v_s^a and is thus a Galilean invariant. It is possible to form only three independent Galilean invariants from the independent variables. These can be taken to be ρ , s , and $|P^a - \rho v_s^a|$. Any Galilean invariant of the system must be expressible in terms of these variables. In particular, the differential of $\tilde{U}_o$ can be expressed as

$$d\tilde{U}_o = \mu d\rho + T ds + (v_r^b - v_s^b) d(P_b - \rho v_{sb}). \quad (2.28)$$

(Note that, in general, the dependent variables could also depend on derivatives of the independent variables. In this case the list of Galilean invariants would need to be extended. However, here it is assumed that all the dependent variables discussed so far can be expressed as algebraic functions of the independent variables. This assumption is the usual one made for fluids, e.g., see Putterman 1974.)

Note that Eq. (2.28) is a generalization of the first law of thermodynamics to the case where there is an additional velocity degree of freedom. The partial derivatives of $\tilde{U}_0$ thus define the new variable given by the coefficient of each term on the right-hand side of Eq. (2.28). The new variables introduced are the chemical potential μ , the temperature T , and the normal velocity v_r^a . Galilean invariance requires that v_r^a transform as a velocity field, i.e., $v_r^{a'} = v_r^a + v^a$. (The subscript r will be used for the normal velocity to distinguish it from the superfluid velocity of the neutrons in subsequent chapters. It may be helpful to think of r as standing for regular, as opposed to super.) Since the velocity v_r^a is a vector field, it must be expressible in terms of the two independent vector fields P^a and v_s^a . Alternatively, P^a can be expressed in terms of v_s^a and v_r^a by

$$P^a = \rho_s v_s^a + \rho_r v_r^a, \quad (2.29)$$

where ρ_s and ρ_r are called the superfluid mass density and the normal mass density, respectively. The Galilean transformation properties of P^a , v_s^a , and v_r^a require that

$$\rho_s + \rho_r = \rho. \quad (2.30)$$

A useful step that will simplify the calculation somewhat (and which is not included in previous derivations of the Landau equations) is to add and subtract $(v_r^b - v_s^b)(P_b - \rho v_{sb}) - \frac{1}{2}\rho|\vec{v}_s - \vec{v}_r|^2$ from the right-hand side of Eq. (2.27). The result is

$$U = U_0 + v_r^b P_b - \frac{1}{2}\rho v_r^b v_{rb} \quad (2.31)$$

where

$$U_0 = \tilde{U}_0 - (v_r^b - v_s^b)(P_b - \rho v_{sb}) + \frac{1}{2}\rho|\vec{v}_s - \vec{v}_r|^2. \quad (2.32)$$

Thus, U_o is the energy density in a frame which moves with velocity v_r^a . Its differential is

$$dU_o = (\mu + \frac{1}{2}|\vec{v}_r - \vec{v}_s|^2)d\rho + Tds + (P_b - \rho v_{rb})d(v_s^b - v_r^b). \quad (2.33)$$

The partial time derivative of U can now be found using Eqs. (2.31), (2.33), and (2.12)–(2.15). Rearranging the result to give it the form of a conservation law and simplifying gives

$$\begin{aligned} \partial_t U + \nabla_b \left[(\mu + \frac{1}{2}|\vec{v}_s - \vec{v}_r|^2 - \frac{1}{2}v_r^b v_{rb})M^b + Ts^b + v_{ra}(\pi^{ab} - pg^{ab}) \right] \\ = (M^b - \rho v_r^b)\nabla_b \mu - (P^b - \rho v_r^b)\nabla_b \chi \\ + (s^b - sv_r^b)\nabla_b T \\ + (\pi^{ab} - pg^{ab} - v_s^a M^b - P^a v_r^b + \rho v_s^a v_r^b)\nabla_b v_{ra} \\ + [(M^b - P^b)(v_s^a - v_r^a)]\nabla_b v_{sa}. \end{aligned} \quad (2.34)$$

Note that g^{ab} , the inverse of the Euclidean metric tensor g_{ab} , appears in this equation. In Cartesian coordinates, or any orthonormal basis, this tensor is just the identity matrix. The more general metric tensor is used here since it is often convenient to work in a nonorthonormal basis. Also, in arriving at Eq. (2.34) it has been convenient to introduce the pressure p defined by

$$p + U_o \equiv (\mu + \frac{1}{2}|\vec{v}_s - \vec{v}_r|^2)\rho + Ts. \quad (2.35)$$

The dependent quantities are now determined by requiring that energy be conserved. First note that energy is conserved if the right-hand side of Eq. (2.34) vanishes. Furthermore, note that the right-hand side of Eq. (2.34) can be written as a sum of terms proportional to the quantities $\nabla_b \mu$, $\nabla_b T$, $\nabla_b v_{ra}$, and $\nabla_b v_{sa}$. Since for an arbitrary configuration of the fluid these quantities can be chosen

independently the coefficient of each must vanish separately. In general, first one needs to expand $\nabla_b \chi$ in terms of these quantities and collect terms. However, simple inspection of the terms on the right-hand side of Eq. (2.34) leads to the following solutions for the dependent variables [after simplifying using Eqs. (2.29) and (2.30)]:

$$M^a = P^a = \rho_s v_s^a + \rho_r v_r^a, \quad (2.36)$$

$$\pi^{ab} = \rho_s v_s^a v_s^b + \rho_r v_r^a v_r^b, \quad (2.37)$$

$$s^a = s v_r^a, \quad (2.38)$$

$$\chi = \mu. \quad (2.39)$$

The terms in square brackets on the left-hand side of Eq. (2.34) can be identified as the energy current U^a . It can be shown further that these solutions transform correctly under a Galilean transformation. It can also be seen that calling v_r^a the normal velocity is appropriate, since it is proportional to the entropy current. The velocity v_r^a thus will be subjected to the normal effects of dissipation. Also, it is appropriate that $\chi = \mu$, since experiments show that superfluids flow in response to gradients in the chemical potential rather than pressure gradients (e.g., see Putterman 1974). For completeness, the chemical potential, the temperature, and the superfluid mass density are specified by

$$\mu = \frac{\partial U_o}{\partial \rho} - \frac{1}{2} |\vec{v}_s - \vec{v}_r|^2, \quad (2.40)$$

$$T = \frac{\partial U_o}{\partial s}, \quad (2.41)$$

$$\rho_s = \frac{\partial U_o}{\partial \frac{1}{2} |\vec{v}_s - \vec{v}_r|^2}. \quad (2.42)$$

The normal mass density ρ_r and the pressure can be computed from Eqs. (2.30) and (2.35). The one place that the Landau equations touch base with microscopic theory is that such a theory is needed to construct the energy density U_0 .

Finally, it may seem possible to construct more elaborate solutions than the ones given by Eqs. (2.36)–(2.39). For example, one can take χ to be an arbitrary function of μ , T , and $\frac{1}{2}|\vec{v}_s - \vec{v}_r|^2$, and this can result in other solutions. Such a possibility is considered in Chapter 1, Appendix I, and Appendix III of Putterman (1974). He shows that no other solution is possible as long as the superfluid velocity is required to remain an independent velocity field which carries no entropy.

Smooth-averaged Superfluid Hydrodynamics

The Landau equations are very successful in describing superfluid hydrodynamics in regions where $\vec{\nabla} \times \vec{v}_s = 0$. However, when vortices are present, such as in a rotating superfluid or a magnetized superconductor, the superfluid velocity is a very rapidly varying function on the scale of the intervortex separation. This separation is usually microscopic compared with the length scales of interest, and thus it is appropriate to perform a smooth-averaging of the variables over a volume containing many vortices. The smooth-averaged hydrodynamics can be derived either by a coarse-graining procedure or by using the phenomenological macroscopic first principles method employed by Landau. This thesis will use the latter method. However, both methods will be discussed briefly here. For a more complete discussion of the coarse-graining procedure see Baym and Chandler (1983). For a further discussion of the phenomenological approach in the context

of rotating superfluids see Bekarevich and Khalatnikov (1961). Both methods are reviewed in Sonin (1987).

The hydrodynamics of a superfluid threaded with vortices is described, in principle, by the Landau equations in the region outside the vortices, while the vortices themselves are described by the curl of the London equation, Eq. (2.5). One way to describe the macroscopic behavior over length scales much larger than the intervortex separation is to perform a coarse-grained average of these equations (e.g., see Baym and Chandler 1983). The coarse-grained average of a function $f(\vec{r})$ is defined by

$$\langle f(\vec{r}) \rangle = \int d^3 r' W(\vec{r}') f(\vec{r} - \vec{r}'), \quad (2.43)$$

where $W(\vec{r}')$ is a weight function normalized such that

$$\int d^3 r' W(\vec{r}') = 1. \quad (2.44)$$

The exact nature of the weight function is not important as long as it is continuous and smooth on distance scales larger than the microscopic scales which are being averaged over. (For a further discussion of this point see Jackson Chapter 6 and reference therein.) Applying this averaging to the curl of the London equation, Eq. (2.5), gives

$$\langle \omega_s^a + aB^a \rangle = \frac{\hbar}{m} \int d^3 r' W(\vec{r}') \left\{ \sum_i \int \nu_i^a(\ell) \delta^3[\vec{r} - \vec{r}' - \vec{r}_i(\ell)] d\ell \right\}. \quad (2.45)$$

Interchanging the order of integration and summation and employing the delta function gives

$$\langle \omega_s^a + aB^a \rangle = \frac{\hbar}{m} \sum_i' \int \nu_i^a(\ell) W[(\vec{r} - \vec{r}_i(\ell))] d\ell. \quad (2.46)$$

The prime on the summation sign and the integral sign means that these operations are now only to take place over the averaging volume $= \int d^3r'$. One way to proceed further is to assume that the vortex array can be foliated by a set of $z_i = \text{constant}$ surfaces. Then choosing $\ell = z_i$, making the substitution $\vec{r}' = \vec{r} - \vec{r}_i$, and integrating both sides of Eq. (2.46) over the remainder of the averaging volume, approximately gives

$$\langle \omega_s^a + aB^a \rangle = \frac{\hbar}{m} \frac{1}{A} \sum_i' \langle \nu_i^a \rangle = \frac{\hbar}{m} \frac{N}{A} \sum_i' \frac{\langle \nu_i^a \rangle}{N}, \quad (2.47)$$

where A is the cross sectional area of the averaging volume and N is the number of vortices which cross through the averaging volume. The average number of vortices per unit area, n_v , is then given by $n_v \equiv N/A$. Furthermore, let the average local direction of the vortex array be defined by $\nu^a \equiv \sum_i' \langle \nu_i^a \rangle / N$. Then Eq. (2.47) becomes

$$\langle \omega_s^a + aB^a \rangle = \frac{\hbar}{m} n_v \nu^a. \quad (2.48)$$

Since vortices in equilibrium are found to form triangular arrays, the vortex density is related to the intervortex separation d by $n_v = (2/\sqrt{3})(1/d^2)$. Thus, using Eq. (2.48) the intervortex separation is given by

$$d = \left(\frac{2\hbar}{\sqrt{3}m|\omega^a + aB^a|} \right)^{1/2}. \quad (2.49)$$

The coarse-graining procedure can also be applied to the Landau equations of the last section. The only important change occurs in the equation for the superfluid velocity, v_s^a . Using the vector identity $v^b \nabla_b v^a = \frac{1}{2} \nabla^a v^2 - (\vec{v} \times \vec{\omega})^a$ the Landau equation can be written as

$$\partial_t v_s^a + \frac{1}{2} \nabla^a v_s^2 - (\vec{v}_s \times \vec{\omega}_s)^a = \nabla^a \mu. \quad (2.50)$$

Now the vorticity in this equation is singular, given by the curl of the London equation, Eq. (2.5). Thus, when the coarse-graining is performed on this equation the result is (see Sonin 1987; Baym and Chandler 1983)

$$\partial_t \langle v_s^a \rangle + \frac{1}{2} \nabla^a \langle v_s \rangle^2 - (\vec{v}_L \times \langle \vec{\omega}_s \rangle)^a = \nabla^a \langle \mu \rangle. \quad (2.51)$$

Note that the coarse-graining procedure commutes with derivatives (e.g., see Jackson 1975). Also, because of the singular nature of the vorticity, the average *local* velocity of the vortex array, given by v_L^a , replaces v_s^a in the cross product term (see Sonin 1987). Equation (2.51) can be rearranged to give

$$\partial_t \langle v_s^a \rangle + \langle v_s^b \rangle \nabla_b \langle v_s^a \rangle = f^a / \rho_s, \quad (2.52)$$

where f^a is defined by

$$f^a = -\rho_s [(\langle \vec{v}_s \rangle - \vec{v}_L) \times \langle \vec{\omega}_s \rangle]^a. \quad (2.53)$$

The right-hand side of Eq. (2.53) has the form of the well known Magnus force which acts on a moving vortex. (The Magnus force arises due to the pressure gradient which acts across the boundary of a moving vortex core because of the Bernoulli effect.) The interpretation of Eq. (2.52) is that in smooth-averaged superfluid hydrodynamics the average superfluid velocity field responds to an additional force f^a , which depends on the vortices. However, note that for the coarse-grained approach to be useful an equation for the velocity v_L^a needs to be found. This problem can be avoided by adopting instead the phenomenological approach.

The power of the phenomenological approach is that one can completely forget about the coarse-graining procedure. Just as the Landau equations were

derived without any appeal to an averaging process, hydrodynamic equations describing the smooth-averaged properties of a superfluid threaded with vortices can be similarly derived. However, the coarse-graining described above does provide a clue as to the starting assumptions. In this thesis, the assumptions are the same as those made in the deriving of the Landau equations, with three important changes. First, the vorticity of a superfluid and the magnetic induction are assumed to obey the smooth-averaged curl of the London equation

$$\omega_s^a + aB^a = \frac{h}{m}n_v\nu^a. \quad (2.54)$$

Second, the superfluid velocity equation is taken to be given by

$$\partial_t v_s^a + v_s^b \nabla_b v_s^a = F^a, \quad (2.55)$$

In Eq. (2.55) the force F^a (actually an acceleration) is not assumed a priori to be related to the force f^a in Eq. (2.52). Instead F^a is to be determined in the same manner as the other dependent variables were found in the derivation of the Landau equations. The third assumption is that the Galilean invariant energy density U_o must also depend on the vortex density n_v , since it too is a Galilean invariant. Bekarevich and Khalatnikov (1961) were the first to make these assumptions for a single uncharged superfluid.

In Eqs. (2.54) and (2.55), and in the remainder of this thesis (except when the vortex energies are derived in Chapter 4), all quantities are to be interpreted as the smooth-averages over a volume containing many vortices. No averaging symbols are used—the quantities are introduced phenomenologically. This approach forms the basis of the rest of this thesis.

CHAPTER 3

GENERAL THEORY OF CHARGED SUPERFLUID MIXTURES

Introduction

The goal of this chapter is to derive a system of equations that describes the dynamics of mixtures of charged superfluids on length scales larger than their intervortex separations. The discussion presented here is based on Mendell and Lindblom (1991). A mixture of two superfluids has been considered previously by Khalatnikov (1957) and by Andreev and Bashkin (1976). Vardanyan and Sedrakyan (1981) generalized the mixture equations to include charged superfluids coupled to the electromagnetic field, while Holm and Kupershmidt (1987) extended the theory to a mixture of N charged superfluids and developed a Hamiltonian formalism for the resulting equations. Here a derivation of the uncharged versions of these equations is presented based on the macroscopic first principles discussed in Chapter 2. In this derivation, however, a large class of new vorticity-preserving forces is included in the superfluid velocity equations. Special cases of these forces have been considered previously by Bekarevich and Khalatnikov (1961). A Hamiltonian formulation of the uncharged equations is also presented which generalizes the work of Holm and Kupershmidt (1987). Using this Hamiltonian formulation the coupling of the charged components of the superfluid mixture to the electromagnetic field is found. The new vorticity-preserving force

terms have a profound effect on the electromagnetic coupling. By properly choosing these forces, it is possible to allow each component of the charged superfluid mixture to respond to the electromagnetic field via an appropriate Lorentz force law. The electromagnetic coupling proposed here is far more natural than those proposed by Vardanyan and Sedrakyan (1981) or Holm and Kupershmidt (1987).

A fluid is considered consisting of a mixture of N species of superfluids, a single "normal" fluid consisting of the excited states of all the superfluid species, and an additional "ordinary" fluid of other particles. To avoid confusion use of the terms "normal" and "ordinary" will refer, respectively, to the latter two fluids. The specific physical system of interest here is the superfluid-neutron, superconducting-proton interior of a neutron star. However, very general equations are derived here which may be applicable to a variety of systems. For example, the final equations presented here could be adapted to describe superfluidity in pion or quark matter, as well the nuclear matter, in a neutron star. Other systems to which this chapter may apply include mixtures of He^3 in He^4 and mixtures of protons and neutrons in a heavy-metal crystal (for references see Holm and Kupershmidt 1987).

The macroscopic superfluid equations of the type introduced by Landau (1941) are intended to describe the behavior of the large scale properties of the fluid (e.g., density, temperature, velocity). The theory of superfluid mixtures being considered here is described by macroscopic variables associated with each component of the fluid: e.g., the total mass density and superfluid velocity, ρ_α and v_α^a , associated with the α th species of superfluid, the velocity, v_r^a , of the normal fluid, and the density and velocity, ρ_e and v_e^a , of the ordinary fluid. Note that Greek indices are used to denote the species of the superfluids in the mixture. The

subscript r is used to refer to the normal fluid to avoid confusion with the neutron superfluid in subsequent chapters. The ordinary fluid is composed primarily of electrons in a neutron star, so the subscript e is used to refer to this fluid whatever its composition might be. Note that the normal and ordinary fluids are allowed to have independent dynamics at this point in order to allow for the possibility of macroscopic electromagnetic interactions between these fluids. In contrast, all of the excited superfluid states are combined into a single normal fluid. In a neutron-star superfluid this equilibrium should be maintained by the strong interactions of the various species on very short time scales.

For each superfluid species the macroscopic velocity, v_α^a , is chosen to be proportional to the kinematic portion of the canonical momentum of the particles (or Cooper pairs) which have condensed into the superfluid state. The canonical momentum for each species is given, therefore, by $m_\alpha(v_\alpha^a + a_\alpha A^a)$, where m_α is the mass, $a_\alpha = q_\alpha/m_\alpha c$ is the charge-to-mass ratio of each particle (or Cooper pair) divided by c , the speed of light, and A^a is the electromagnetic vector potential. In general, vortices of each species of superfluid are assumed to be present in the fluid and the intervortex separation is expected to be much smaller than typical macroscopic length scales of interest in this thesis. Thus it will be appropriate to consider the variables which describe the macroscopic properties of the fluid to be the smooth-averaged version of these variables, with the averaging performed over a volume containing many vortices. As discussed in Chapter 2, and following Bekarevich and Khalatnikov (1961) and Sonin (1987), the smooth-averaged curl of the London equation is given by

$$\omega_\alpha^a + a_\alpha B^a = \frac{2\pi\hbar}{m_\alpha} n_{v\alpha} v_\alpha^a, \quad (3.1)$$

where $\omega_\alpha^a = (\vec{\nabla} \times \vec{v}_\alpha)^a$ is the vorticity, $B^a = (\vec{\nabla} \times \vec{A})^a$ is the magnetic induction, and $n_{v\alpha}$ is the number of vortices per unit area perpendicular to $\vec{v}_\alpha$, (the unit vector parallel to the *average* local direction of the vortices). (Reminder: Latin indices, a, b, c , refer to the spatial components of vectors and tensors, and summation over these repeated indices is assumed. Summation over Greek indices is not intended unless explicitly noted.) To complete the smooth-averaging process, the expression for the energy of the fluid must be modified to include the internal energy associated with the vortices (e.g., the kinetic and/or magnetic energy associated with the circulation of fluid about each vortex). In the derivation of the superfluid equations in the following sections a simple model is included for this vortex energy which is a natural generalization of the model proposed by Bekarevich and Khalatnikov (1961).

Uncharged Superfluid Mixtures

The macroscopic dynamical equations for mixtures of superfluids are derived here using the method developed by Landau (1941) (see also Khalatnikov 1957). One begins by imposing the appropriate conservation laws for this system: mass conservation for each species of particle, the conservation of total momentum, and conservation of entropy, for the ordinary and the normal fluid. These equations together with an assumed form for the variation in the energy density (the first law of thermodynamics) imply an equation for the time evolution of the energy density of the fluid. The requirement that this energy-evolution equation be a conservation law is used to fix the remaining undetermined quantities in the theory: the form of the stress tensor and the forces that appear in the equations for the fluid velocities.

The equations that describe the evolution of this fluid mixture include the conservation of mass of each species of particle, the conservation of entropy and the total momentum conservation equations. These are given by

$$\partial_t \rho_\alpha + \nabla_b (M_\alpha^b) = 0, \quad (3.2)$$

$$\partial_t \rho_e + \nabla_b (\rho_e v_e^b) = 0, \quad (3.3)$$

$$\partial_t s_r + \nabla_b (s_r v_r^b) = 0, \quad (3.4)$$

$$\partial_t s_e + \nabla_b (s_e v_e^b) = 0, \quad (3.5)$$

$$\partial_t P^a + \nabla_b \pi^{ab} = 0. \quad (3.6)$$

In these equations the mass current of each species of superfluid particle (including both superfluid and normal phases) is denoted M_α^a , the entropy densities of the ordinary and normal fluids are denoted as s_e and s_r , respectively, and the stress tensor is denoted π^{ab} . The total momentum density of the fluid P^a is taken to be equal to the sum of the mass currents of each species of particle, i.e.,

$$P^a = \rho_e v_e^a + \sum_\alpha M_\alpha^a. \quad (3.7)$$

In addition to the conservation laws, Eqs. (3.2)–(3.6), dynamical equations for the evolution of the superfluid velocities v_α^a and the ordinary fluid velocity v_e^a must be specified. Without loss of generality these equations are taken to have the forms

$$\partial_t v_\alpha^a + v_\alpha^b \nabla_b v_\alpha^a + \nabla^a \tilde{\mu}_\alpha = F_\alpha^a, \quad (3.8)$$

$$\partial_t v_e^a + v_e^b \nabla_b v_e^a + \nabla^a \tilde{\mu}_e + \frac{s_e}{\rho_e} \nabla^a T_e = F_e^a. \quad (3.9)$$

The left sides of Eqs. (3.8)–(3.9) are respectively the Landau equation for the evolution of the superfluid velocity and the Euler equation for the evolution of an

ordinary fluid. The forces F_α^a and F_e^a which appear on the right sides of Eqs. (3.8)–(3.9) are yet to be determined. The F_α^a must either be curl-free or they must vanish when the superfluid velocities v_α^a are curl-free, however, if the superfluid equations are to insure that the v_α^a evolve in a curl-free manner consistent with the Landau equation in the absence of vortices. In these equations T_e and T_r (see below) are the temperatures of the ordinary and normal fluids, while $\tilde{\mu}_\alpha$ and $\tilde{\mu}_e$ are the chemical potentials measured in a frame moving with velocity v_r^a (the tildes are used to distinguish these chemical potentials from those defined in Chapter 4). More precisely, if U_o is the energy density of the fluid in the frame moving with velocity v_r^a then these quantities are defined as the indicated coefficients in the first law of thermodynamics

$$\begin{aligned} dU_o = & T_e ds_e + (\tilde{\mu}_e + \tfrac{1}{2}|\vec{v}_e - \vec{v}_r|^2) d\rho_e + \tfrac{1}{2}\rho_e d|\vec{v}_e - \vec{v}_r|^2 + T_r ds_r \\ & + \sum_\alpha \left\{ (\tilde{\mu}_\alpha + \tfrac{1}{2}|\vec{v}_\alpha - \vec{v}_r|^2) d\rho_\alpha + (M_{\alpha b} - \rho_\alpha v_{rb}) d(v_\alpha^b - v_r^b) + \lambda_\alpha^b d\omega_{\alpha b} \right\}. \end{aligned} \quad (3.10)$$

This equation is similar to Eq. (2.33) in Chapter 2, but has been extended to describe a mixture of fluids and to account for the vortex energies. The terms, $\lambda_\alpha^b d\omega_{\alpha b}$, which appear in Eq. (3.10), generalize the terms proposed by Bekarevich and Khalatnikov (1961) to describe the energy associated with the vortices in the fluid. Since $\omega_{\alpha b}$ is proportional to the number density of vortices [when $a_\alpha = 0$ in Eq. (3.1)] the magnitude of λ_α^b is a measure of the energy per vortex. When λ_α^a is set to zero the equations return to their original Landau form in which the superfluid velocities satisfy Eq. (2.3) and have singular vorticities. When λ_α^a is not zero, the superfluid velocities are to be interpreted as smooth-averaged quantities that satisfy Eq. (3.1). Bekarevich and Khalatnikov (1961) argue that λ_α^a should be taken to be parallel to ω_α^a in this case and have a magnitude such that $\vec{\lambda}_\alpha \cdot \vec{\omega}_\alpha$

is the energy density associated with the fluid circulating about the vortices. Here λ_α^a is allowed to be an arbitrary Galilean invariant vector field.

The superfluid mass currents M_α^a are also defined by the first law of thermodynamics, Eq. (3.10). Nepomnyashchii (1976) shows (on the basis of a particular microscopic theory) that these currents must be the same as those that appear in the mass conservation laws, Eq. (3.2). Here it is assumed that the M_α^a as defined by Eq. (3.10) are equal to those that appear in the mass conservation laws, Eq. (3.2), in general. Furthermore, the currents $M_\alpha^a - \rho_\alpha v_r^a$ are Galilean invariant vectors and therefore could be written as some linear combinations of the Galilean invariant vector fields in the problem: $v_\alpha^a - v_r^a$, $v_e^a - v_r^a$, and ω_α^a . While no detailed physical motivation for including terms in $M_\alpha^a - \rho_\alpha v_r^a$ proportional to $v_e^a - v_r^a$ or ω_α^a has ever been proposed, Khalatnikov (1989) has speculated that terms proportional to ω_α^a might be needed when the fluid velocities are large. The precise form of these currents in terms of the fundamental variables of the problem will not play a significant role in the analysis. It is probably most appropriate, nevertheless, to think of $M_\alpha^a - \rho_\alpha v_r^a$ in terms of Andreev and Bashkin's expression (Andreev and Bashkin 1976)

$$M_\alpha^a - \rho_\alpha v_r^a = \sum_\beta \rho_{\alpha\beta} (v_\beta^a - v_r^a). \quad (3.11)$$

The mass density matrix $\rho_{\alpha\beta}$ must be evaluated for any physical system of interest on the basis of some microphysical model for that system. Explicit expressions for $\rho_{\alpha\beta}$ have been derived by several authors (Andreev and Bashkin 1976; Vardanyan and Sedrakyan 1981; Alpar, Langer, and Sauls 1984; Sauls 1989). They find that $\rho_{\alpha\beta}$ must, quite generally, be symmetric in $\alpha\beta$, but that it is not diagonal for

many cases of interest (including the neutron-proton superfluids of neutron-star interiors).

The dynamical evolution of an uncharged superfluid mixture would be completely determined by Eqs. (3.2)–(3.10) if expressions for the stress tensor π^{ab} and the forces F_α^a and F_e^a were known. The method for determining these quantities is to demand that the evolution of the energy predicted by these equations is in fact a conservation law. The energy density of the fluid U is related to U_o , the energy density measured in the frame of reference moving at velocity v_r^a , by the expression (see Chapter 2)

$$U = U_o + P_b v_r^b - \frac{1}{2} \rho v_r^b v_{rb}, \quad (3.12)$$

where ρ denotes the total mass density of the fluid, $\rho = \rho_e + \sum_\alpha \rho_\alpha$. The time evolution of this quantity can be computed with the aid of Eqs. (3.2)–(3.10). The resulting expression is

$$\begin{aligned} \partial_t U + \nabla_b U^b = & \rho_e (v_{eb} - v_{rb}) F_e^b + \nabla_b v_{ra} \left\{ \pi^{ab} - p g^{ab} - \rho_e v_e^a v_e^b \right. \\ & - \sum_\alpha \left[v_\alpha^a (M_\alpha^b - \rho_\alpha v_r^b) + M_\alpha^a v_r^b + g^{ab} \lambda_\alpha^c \omega_{\alpha c} - \lambda_\alpha^a \omega_\alpha^b \right] \Big\} \\ & + \sum_\alpha \left[M_{\alpha b} - \rho_\alpha v_{rb} + (\vec{\nabla} \times \vec{\lambda}_\alpha)_b \right] \left[F_\alpha^b + 2(v_{ra} - v_{\alpha a}) \nabla^{[a} v_\alpha^{b]} \right], \end{aligned} \quad (3.13)$$

where the energy current U^a and the pressure p are defined by the expressions

$$\begin{aligned} U^a = & T_e s_e v_e^a + T_r s_r v_r^a + (\tilde{\mu}_e + \frac{1}{2} |\vec{v}_e - \vec{v}_r|^2 - \frac{1}{2} |\vec{v}_r|^2) \rho_e v_e^a + (\pi^{ba} - p g^{ab}) v_{rb} \\ & + \sum_\alpha (\tilde{\mu}_\alpha + \frac{1}{2} |\vec{v}_e - \vec{v}_r|^2 - \frac{1}{2} |\vec{v}_r|^2) M_\alpha^a + \sum_\alpha \left\{ \vec{\lambda}_\alpha \times [\vec{F}_\alpha + (\vec{v}_\alpha - \vec{v}_r) \times \vec{\omega}_\alpha] \right\}^a, \end{aligned} \quad (3.14)$$

and

$$p = -U_o + T_e s_e + T_r s_r + \rho_e (\tilde{\mu}_e + \frac{1}{2} |\vec{v}_e - \vec{v}_r|^2) + \sum_\alpha \rho_\alpha (\tilde{\mu}_\alpha + \frac{1}{2} |\vec{v}_\alpha - \vec{v}_r|^2). \quad (3.15)$$

Square brackets surrounding a pair of indices [e.g., in Eq. (3.13)] indicates antisymmetrization, i.e., $\nabla^{[a}v_{\alpha}^{b]} = \frac{1}{2}(\nabla^a v_{\alpha}^b - \nabla^b v_{\alpha}^a)$.

Equation (3.13) will guarantee the local conservation of energy of the fluid if the right side vanishes. It is natural to require that the term proportional to $\nabla_b v_{ra}$ on the right vanishes separately by defining the stress tensor as

$$\pi^{ab} = pg^{ab} + \rho_e v_e^a v_e^b + \sum_{\alpha} \left[v_{\alpha}^a (M_{\alpha}^b - \rho_{\alpha} v_r^b) + M_{\alpha}^a v_r^b + g^{ab} \lambda_{\alpha}^c \omega_{\alpha c} - \lambda_{\alpha}^a \omega_{\alpha}^b \right]. \quad (3.16)$$

This expression for π^{ab} is a symmetric tensor if the mass currents M_{α}^a are given by Eq. (3.11), with $\rho_{\alpha\beta}$ symmetric in $\alpha\beta$ and $\lambda_{\alpha}^a = \lambda_{\alpha} \omega_{\alpha}^a$ (where λ_{α} is any scalar function). It is also natural to set the force F_e^a equal to zero. While it is possible that there could exist some nonelectromagnetic force acting on the ordinary fluid, there is no present physical reason to include a force of this type (other than dissipative forces) for the case of the electron fluid in neutron-star matter. Since dissipation is ignored in this chapter and the electromagnetic forces are ignored in this section, it is reasonable to take $F_e^a = 0$. This leaves only the term containing the forces F_{α}^a . This term will vanish if F_{α}^a is given by

$$F_{\alpha}^a = \sum_{\beta} K_{\alpha\beta}^{ab} \left[M_{\beta b} - \rho_{\beta} v_{rb} + (\vec{\nabla} \times \vec{\lambda}_{\beta})_b \right] + 2(v_{rb} - v_{\alpha b}) \nabla^{[a} v_{\alpha}^{b]}, \quad (3.17)$$

where $K_{\alpha\beta}^{ab}$ is any tensor that is antisymmetric in the sense that $K_{\alpha\beta}^{ab} = -K_{\beta\alpha}^{ba}$. The forces F_{α}^a must vanish, however, when the vorticity of the superfluid velocities vanishes if Eq. (3.8) is to return to the Landau form in the absence of vortices. The tensor $K_{\alpha\beta}^{ab}$ must vanish, therefore, whenever the vorticity of v_{α}^a vanishes. A simple example of a tensor that meets these criteria is

$$K_{\alpha\beta}^{ab} = \sum_{\gamma} 2K_{\alpha\beta\gamma} \nabla^{[a} v_{\gamma}^{b]}, \quad (3.18)$$

where $K_{\alpha\beta\gamma}$ is symmetric in $\alpha\beta$. For the remainder of this chapter attention is restricted to this case with $K_{\alpha\beta\gamma}$ an arbitrary function of s_r and ρ_μ . The forces given by Eqs. (3.17)–(3.18) are generalizations of those included by Bekarevich and Khalatnikov (1961) in the nondissipative limit of their equations. It is worth noting that these forces do not vanish when the coefficients λ_α^a are set to zero. In this case the forces become singular at the locations of vortices. By introducing λ_α^a the forces F_α^a may be interpreted as the smooth-averaged forces the vortices exert on the smooth-averaged flow of the superfluids.

Hamiltonian Formulation

One major goal of this chapter is to determine the coupling of charged superfluid mixtures to the electromagnetic field. This is most easily accomplished by introducing a Hamiltonian formulation of the equations. Holm and Kupershmidt (1987) give such a formulation for the special case of the fluid equations presented above when $K_{\alpha\beta\gamma}$ and λ_α^a are zero. Here their work is generalized to include nonzero values for these quantities. The Hamiltonian density is taken to be the energy density of the fluid; thus, the Hamiltonian of the fluid, H , is given by

$$H = \int d^3x U. \quad (3.19)$$

It will be convenient (primarily for the discussion on charged fluids that follows) to relabel the fluid variables, v_e^a , v_α^a , and P^a when they occur in the Hamiltonian form of the equations by

$$u_e^a = v_e^a, \quad (3.20)$$

$$u_\alpha^a = v_\alpha^a, \quad (3.21)$$

$$\Upsilon^a = P^a; \quad (3.22)$$

Thus, using Eqs. (3.10) and (3.12), the variation in the Hamiltonian may be written as

$$\begin{aligned} \delta H = \int d^3x \bigg\{ & T_e \delta s_e + \left(\tilde{\mu}_e + \frac{1}{2} |\vec{v}_e - \vec{v}_r|^2 - \frac{1}{2} |\vec{v}_r|^2 \right) \delta \rho_e + \rho_e (u_e^b - v_r^b) \delta u_{eb} \\ & + T_r \delta s_r + \sum_{\alpha} \left(\tilde{\mu}_{\alpha} + \frac{1}{2} |\vec{v}_{\alpha} - \vec{v}_r|^2 - \frac{1}{2} |\vec{v}_r|^2 \right) \delta \rho_{\alpha} \\ & + \sum_{\alpha} \left[M_{\alpha}^b - \rho_{\alpha} v_r^a + (\vec{\nabla} \times \vec{\lambda}_{\alpha})^b \right] \delta u_{\alpha b} + v_r^b \delta \Upsilon_b \bigg\}. \end{aligned} \quad (3.23)$$

In addition to the expressions for the Hamiltonian and its variation, the Hamiltonian formulation of the fluid equations must include the definition of a Poisson bracket. In terms of this bracket the evolution equations could then be expressed in the standard Hamiltonian form

$$\partial_t F = -[F, H], \quad (3.24)$$

where F is an arbitrary smooth function of the dynamical fluid variables (i.e., s_e , ρ_e , u_e^a , s_r , ρ_{α} , u_{α}^a , and Υ^a). It is reasonably straightforward to find a bracket through which Eq. (3.24) reproduces the fluid equations (3.2)–(3.9) and which satisfies the anti-symmetry property $[F, G] = -[G, F]$ for arbitrary smooth F and G . The following bracket satisfies these conditions:

$$[F, G] = (F, G) - 2 \int d^3x \sum_{\alpha\beta\gamma} \frac{\delta F}{\delta u_{\alpha}^a} \frac{\delta G}{\delta u_{\beta}^b} K_{\alpha\beta\gamma} \nabla^{[a} u_{\gamma}^{b]}, \quad (3.25)$$

where (F, G) is defined by

$$\begin{aligned}
 (F, G) = \int d^3x \left\{ \frac{\delta F}{\delta s_e} \nabla^b \left[\frac{s_e}{\rho_e} \frac{\delta G}{\delta u_e^b} + s_e \frac{\delta G}{\delta \Upsilon^b} \right] + \frac{\delta F}{\delta \rho_e} \nabla^b \left[\frac{\delta G}{\delta u_e^b} + \rho_e \frac{\delta G}{\delta \Upsilon^b} \right] \right. \\
 + \frac{\delta F}{\delta u_e^a} \left[\frac{s_e}{\rho_e} \nabla^a \frac{\delta G}{\delta s_e} + \nabla^a \frac{\delta G}{\delta \rho_e} - \frac{2}{\rho_e} \frac{\delta G}{\delta u_e^b} \nabla^{[a} u_e^{b]} - 2 \frac{\delta G}{\delta \Upsilon^b} \nabla^{[a} u_e^{b]} \right] \\
 + \frac{\delta F}{\delta s_r} \nabla^b \left[s_r \frac{\delta G}{\delta \Upsilon^b} \right] + \sum_{\alpha} \frac{\delta F}{\delta \rho_{\alpha}} \nabla^b \left[\frac{\delta G}{\delta u_{\alpha}^b} + \rho_{\alpha} \frac{\delta G}{\delta \Upsilon^b} \right] \\
 + \sum_{\alpha} \frac{\delta F}{\delta u_{\alpha}^a} \left[\nabla^a \frac{\delta G}{\delta \rho_{\alpha}} + u_{\alpha}^b \nabla^a \frac{\delta G}{\delta \Upsilon^b} + \frac{\delta G}{\delta \Upsilon^b} \nabla^b u_{\alpha}^a \right] \\
 + \frac{\delta F}{\delta \Upsilon^a} \left[s_e \nabla^a \frac{\delta G}{\delta s_e} + \rho_e \nabla^a \frac{\delta G}{\delta \rho_e} + \nabla^b \left(u_e^a \frac{\delta G}{\delta u_e^b} \right) - \frac{\delta G}{\delta u_e^b} \nabla^a u_e^b \right. \\
 + \sum_{\alpha} \left(\rho_{\alpha} \nabla^a \frac{\delta G}{\delta \rho_{\alpha}} + \nabla^b \left[u_{\alpha}^a \frac{\delta G}{\delta u_{\alpha}^b} \right] - \frac{\delta G}{\delta u_{\alpha}^b} \nabla^a u_{\alpha}^b \right) \\
 \left. \left. + s_r \nabla^a \frac{\delta G}{\delta s_r} + \Upsilon^b \nabla^a \frac{\delta G}{\delta \Upsilon^b} + \nabla^b \left(\Upsilon^a \frac{\delta G}{\delta \Upsilon^b} \right) \right] \right\}. \tag{3.26}
 \end{aligned}$$

The representation of the bracket in Eq. (3.25) is noncanonical. To establish that it is a Poisson bracket, it is necessary to verify that the Jacobi "identity,"

$$[E, [F, G]] + [F, [G, E]] + [G, [E, F]] = 0, \tag{3.27}$$

is satisfied for arbitrary smooth E , F , and G . The bracket (F, G) is known to satisfy the Jacobi identity because it is the direct sum of the Poisson bracket for an ordinary perfect fluid (Dzyaloshinskii and Volovick 1980; Holm, Marsden, and Ratiu 1986) and the Poisson bracket for a mixture of superfluids given by Holm and Kupershmidt (1987) (up to simple algebraic changes of variables). Thus, (F, G) is a Poisson bracket. Knowing this will aid in determining the conditions under which $[F, G]$ is a Poisson bracket.

The plan is to investigate the conditions under which terms can be added to a Poisson bracket and have the result continue to satisfy the Jacobi identity.

Consider the Poisson bracket

$$(F, G) = \int d^3x \frac{\delta F}{\delta z^A} \left[X^{AB} \left(\frac{\delta G}{\delta z^B} \right) \right]. \quad (3.28)$$

In this expression F and G are arbitrary functions of the fields z^A , and the index A runs over the complete collection of these fields. The operator X^{AB} acts to the right (on the argument in parenthesis) and is antisymmetric in the sense that $(F, G) = -(G, F)$. Summation over repeated indices A, B, C , etc. is assumed. It is also assumed that this bracket satisfies the Jacobi identity,

$$0 = \sum_{(EFG)} (E, (F, G)), \quad (3.29)$$

where the indicated sum is to be performed by adding the terms obtained by cyclically permuting the arbitrary smooth functions E, F , and G .

Next the bracket (F, G) is modified as follows:

$$[F, G] = (F, G) + \int d^3x \frac{\delta F}{\delta z^A} \left[Y^{AB} \left(\frac{\delta G}{\delta z^B} \right) \right]. \quad (3.30)$$

The operator Y^{AB} is also antisymmetric so that $[F, G] = -[G, F]$. The conditions under which the combined bracket, $[F, G]$, satisfies the Jacobi identity are now investigated. A straightforward calculation yields the following identity:

$$\begin{aligned} \sum_{(EFG)} [E, [F, G]] = & - \sum_{(EFG)} \int \int d^3x d^3x' \left\{ \left[Y^{AB} \left(\frac{\delta E}{\delta z^B} \right) \right] \frac{\delta F}{\delta z^C} \left[\frac{\delta X^{CD}}{\delta z^A} \left(\frac{\delta G}{\delta z^D} \right) \right] \right. \\ & \left. + \left[\left(X^{AB} + Y^{AB} \right) \left(\frac{\delta E}{\delta z^B} \right) \right] \frac{\delta F}{\delta z^C} \left[\frac{\delta Y^{CD}}{\delta z^A} \left(\frac{\delta G}{\delta z^D} \right) \right] \right\}. \end{aligned} \quad (3.31)$$

The combined bracket, $[F, G]$, satisfies the Jacobi identity if and only if this expression vanishes for all smooth E, F , and G .

In general, it requires an extremely tedious calculation to determine whether or not the Jacobi identity is satisfied for a given bracket. Equation (3.31), however, provides a relatively simple method of checking the Jacobi identity whenever one is interested in determining whether a relatively simple addition to a given Poisson bracket results in a bracket that also satisfies the Jacobi identity. The Poisson bracket for a mixture of superfluids falls into this general category. The bracket (F, G) defined in Eq. (3.26) is quite complicated but it is known to satisfy the Jacobi identity. The combined bracket $[F, G]$ in Eq. (3.25), involving $K_{\alpha\beta\gamma}$, is a relatively trivial modification of the original bracket. Using Eq. (3.31) then to test whether the brackets defined in Eqs. (3.25) and (3.26) satisfy the Jacobi identity gives the result

$$\begin{aligned}
 \sum_{(EFG)} [E, [F, G]] = & \sum_{(EFG)} \sum_{\mu\nu} \int d^3x \left\{ 2 \sum_{\gamma} \left[s_r \frac{\partial K_{\mu\nu\gamma}}{\partial s_r} + \sum_{\alpha} \rho_{\alpha} \frac{\partial K_{\mu\nu\gamma}}{\partial \rho_{\alpha}} + K_{\mu\nu\gamma} \right] \nabla^{[c} u_{\gamma}^{d]} \frac{\delta E}{\delta u_{\mu}^c} \frac{\delta F}{\delta u_{\nu}^d} \nabla^b \frac{\delta G}{\delta u_{\beta}^b} \right. \\
 & + 2 \sum_{\alpha\beta\gamma} \left[\delta_{\alpha\beta} \frac{\partial K_{\mu\nu\gamma}}{\partial \rho_{\alpha}} + K_{\beta\mu\alpha} K_{\alpha\nu\gamma} \right] \nabla^{[c} u_{\gamma}^{d]} \frac{\delta E}{\delta u_{\mu}^c} \frac{\delta F}{\delta u_{\nu}^d} \nabla^b \frac{\delta G}{\delta u_{\beta}^b} \\
 & + 2 \sum_{\alpha\beta\gamma} \left[K_{\nu\beta\alpha} K_{\alpha\mu\gamma} - K_{\nu\mu\alpha} K_{\alpha\beta\gamma} \right] \nabla^{[c} u_{\gamma}^{d]} \frac{\delta E}{\delta u_{\mu}^d} \frac{\delta F}{\delta u_{\nu}^b} \nabla^b \frac{\delta G}{\delta u_{\beta}^c} \\
 & + \sum_{\alpha\beta\gamma} \left[K_{\mu\nu\alpha} K_{\alpha\beta\gamma} - K_{\mu\beta\alpha} K_{\alpha\nu\gamma} \right] \nabla^c \nabla^d u_{\gamma}^b \frac{\delta E}{\delta u_{\mu}^c} \frac{\delta F}{\delta u_{\nu}^d} \frac{\delta G}{\delta u_{\beta}^b} \\
 & \left. + 2 \sum_{\alpha\beta\gamma} \left[K_{\alpha\nu\gamma} \nabla^b K_{\beta\mu\alpha} - K_{\beta\nu\alpha} \nabla^b K_{\alpha\mu\gamma} \right] \nabla^{[c} u_{\gamma}^{d]} \frac{\delta E}{\delta u_{\mu}^c} \frac{\delta F}{\delta u_{\nu}^d} \frac{\delta G}{\delta u_{\beta}^b} \right\}. \quad (3.32)
 \end{aligned}$$

This expression must vanish if the Poisson bracket in Eq. (3.25) is to satisfy the Jacobi identity. The integral clearly vanishes if the coefficient of each term in the

integrand vanishes separately. Thus, sufficient conditions which guarantee that the integral vanishes for all smooth E , F , and G are

$$0 = s_r \frac{\partial K_{\mu\nu\gamma}}{\partial s_r} + \sum_{\alpha} \rho_{\alpha} \frac{\partial K_{\mu\nu\gamma}}{\partial \rho_{\alpha}} + K_{\mu\nu\gamma}, \quad (3.33)$$

$$0 = \frac{\partial K_{\mu\nu\gamma}}{\partial \rho_{\beta}} + \sum_{\alpha} K_{\beta\mu\alpha} K_{\alpha\nu\gamma}, \quad (3.34)$$

$$0 = \sum_{\alpha} (K_{\nu\beta\alpha} K_{\alpha\mu\gamma} - K_{\nu\mu\alpha} K_{\alpha\beta\gamma}). \quad (3.35)$$

$$0 = \sum_{\alpha} (K_{\alpha\nu\gamma} \nabla^{\alpha} K_{\beta\mu\alpha} - K_{\beta\nu\alpha} \nabla^{\alpha} K_{\alpha\mu\gamma}). \quad (3.36)$$

Since Eq. (3.36) is a consequence of Eqs. (3.33)–(3.35), it is not an independent condition. Also note that the first term in the integral in Eq. (3.32) involves the variations with respect to Υ^a . Since the entire integral must vanish for arbitrary smooth functions E , F , and G it follows that the coefficient of this first term must vanish separately. Thus Eq. (3.33) is a necessary condition. All of the remaining terms involve only variations with respect to u_{α}^a . Numerous integrations by parts have been performed on the expression given here, but these have not produced a form of this equation from which the remaining necessary conditions may easily be extracted. The weakest conditions found, however, are Eqs. (3.33)–(3.35); they may in fact be the necessary conditions.

That there exist nontrivial solutions to Eqs. (3.33)–(3.35) can be illustrated for the simple “diagonal” case in which $K_{\alpha\beta\gamma}$ is proportional to Kronecker deltas: $K_{\alpha\beta\gamma} \propto \delta_{\alpha\gamma} \delta_{\beta\gamma}$. In this case the general solution to Eqs. (3.33)–(3.35) is given by

$$K_{\alpha\beta\gamma} = \frac{\delta_{\alpha\gamma} \delta_{\beta\gamma}}{\rho_{\alpha} + k_{\alpha} s_r}, \quad (3.37)$$

where the k_{α} are arbitrary constants. The case $k_{\alpha} = 0$ is the generalization to mixtures of the “momentum representation” bracket introduced by Holm and

Kupershmidt (1982) for a single component "nonrotating" superfluid, written here in a somewhat different choice of variables:

Electromagnetic Coupling

In this section the coupling of the charged components of the superfluid mixture to the electromagnetic field is found. The most efficient way to accomplish this coupling uses the procedure, based on the Hamiltonian formulation of the equations, that is discussed by Holm and Kupershmidt (1987). In this approach, modeled after the electromagnetic coupling to a charged particle, the physical momenta in the problem are replaced by their "canonical" counterparts (which are formed by adding terms to the physical momenta that are proportional to the electromagnetic vector potential). The Hamiltonian is rewritten in terms of these "canonical" variables, but its value (for a given fluid state) is modified only by the addition of the terms needed to describe the energy of the electromagnetic field. The Poisson bracket is unchanged (when written in terms of these "canonical" variables) except for the addition of the terms needed to describe the dynamics of the electromagnetic field itself. This procedure has been described as "minimal coupling," since it limits the electromagnetic-interaction terms in the fluid equations to those obtained by appropriately replacing the physical momenta of the uncharged theory with the corresponding "canonical" momenta. The fluid equations generated in this way interact with the electromagnetic field only through forces that are qualitatively similar to the Lorentz force. These equations guarantee that the appropriate conservation laws are satisfied identically.

The state of the superfluid mixture is described by the dynamical variables s_e , ρ_e , v_e^a , s_r , ρ_α , v_α^a , and P^a . Each of the vector fields among these dynamical

variables is proportional to the physical momentum density of its corresponding constituent particles: P^a is the total momentum density of the fluid, v_α^a is proportional to the momentum density of the superfluid particles (or Cooper pairs) of species α (appropriately averaged when λ_α^a is nonzero), and v_e^a is proportional to the momentum density of the ordinary fluid. Thus, it is appropriate to define the following "canonical" variables:

$$\Upsilon^a = P^a + \left[a_e \rho_e + \sum_\alpha a_\alpha \rho_\alpha \right] A^a, \quad (3.38)$$

$$u_\alpha^a = v_\alpha^a + a_\alpha A^a, \quad (3.39)$$

$$u_e^a = v_e^a + a_e A^a, \quad (3.40)$$

where A^a is the electromagnetic vector potential and $a_\alpha = q_\alpha/m_\alpha c$ is the charge to mass ratio of species α divided by the speed of light (as in the introduction to this chapter). (Note that these variables cannot be chosen as "canonical momenta" in the usual terminology of Hamiltonian mechanics, since they are not mutually commuting with respect to the Poisson bracket. This is because these variables are proportional to momentum densities. See Landau 1941 for a further discussion of this point.)

The next step in this minimal coupling procedure is to add to the Hamiltonian the terms that describe the energy of the electromagnetic field. The standard macroscopic expression for the variation of this energy is

$$dU_{\text{EM}} = \frac{1}{4\pi} \left[E^a dD_a + H^a dB_a \right], \quad (3.41)$$

where D^a is the displacement, B^a is the magnetic induction. This expression serves as a definition for E^a and H^a , the electric and magnetic fields. When vortices are present in the superfluid there exist microscopic electrical currents

circulating about each vortex. The vortex averaged equations neglect these currents and the magnetic energy that is associated with them. While this energy could be included implicitly in the definition of H^a given in Eq. (3.41), it is more convenient to include an expression for this additional energy explicitly, i.e.,

$$dU_{EM} = \frac{1}{4\pi} \left[E^a dD_a + H^a dB_a \right] + \sum_{\alpha} a_{\alpha} \lambda_{\alpha}^a dB_a. \quad (3.42)$$

This additional energy term, when combined with the term $\lambda_{\alpha}^a d\omega_{\alpha a}$ that was included in the expression for the fluid energy in Eq. (3.23), results in the following expression for the internal energy associated with the vortices: $\lambda_{\alpha}^a d(\omega_{\alpha a} + a_{\alpha} B_a)$. Since the vortex averaged version of the London equation, Eq. (3.1), dictates that $\omega_{\alpha}^a + a_{\alpha} B^a$ is proportional to the density of vortices, it is appropriate that the variation of this combination yields the internal vortex energy λ_{α}^a . Note that the term, $a_{\alpha} \lambda_{\alpha}^a dB_a$, is also consistent with the expression for the magnetic energy density of a flux line lattice as computed by de Gennes and Matricon (1964). When the additional terms in Eq. (3.42) are added to the expression for the variation of the fluid Hamiltonian in Eq. (3.23) and when the result is reexpressed in terms of the "canonical" variables of Eqs. (3.38)–(3.40), the following expression for the variation of the total Hamiltonian is obtained:

$$\begin{aligned} \delta H = \int d^3x \Big\{ & T_e \delta s_e + \left(\tilde{\mu}_e + \frac{1}{2} |\vec{v}_e - \vec{v}_r|^2 - \frac{1}{2} |\vec{v}_r|^2 - a_e v_r^b A_b \right) \delta \rho_e \\ & + \rho_e \left(u_e^b - a_e A^b - v_r^b \right) \delta u_{eb} \\ & + \sum_{\alpha} \left(\tilde{\mu}_{\alpha} + \frac{1}{2} |\vec{v}_{\alpha} - \vec{v}_r|^2 - \frac{1}{2} |\vec{v}_r|^2 - a_{\alpha} v_r^b A_b \right) \delta \rho_{\alpha} \\ & + v_r^b \delta \Upsilon_b + T_r \delta s_r + \sum_{\alpha} \left[M_{\alpha}^b - \rho_{\alpha} v_r^b + (\vec{\nabla} \times \vec{\lambda}_{\alpha})^b \right] \delta u_{\alpha b} \\ & + \frac{1}{4\pi} E^b \delta D_b + \left[\frac{1}{4\pi} (\vec{\nabla} \times \vec{H})^b - \frac{1}{c} J^b \right] \delta A_b \Big\}, \end{aligned} \quad (3.43)$$

where the total macroscopic electrical current, J^a , is defined by

$$J^a = a_e c \rho_e v_e^a + \sum_{\alpha} a_{\alpha} c M_{\alpha}^a. \quad (3.44)$$

The final step in this minimal-coupling procedure is to construct the appropriate Poisson bracket. By assumption, the fluid portions of the bracket are the same (when written in terms of the "canonical" variables) as the uncharged-superfluid bracket, $[F, G]$, of Eq. (3.25). One simply adds to $[F, G]$, the standard Poisson bracket for the electromagnetic field:

$$\{F, G\} = [F, G] + 4\pi c \int d^3x g^{ab} \left(\frac{\delta F}{\delta A^a} \frac{\delta G}{\delta D^b} - \frac{\delta F}{\delta D^a} \frac{\delta G}{\delta A^b} \right). \quad (3.45)$$

Since $\{F, G\}$ is defined as the direct sum of $[F, G]$ and the standard bracket for the electromagnetic field, it will satisfy the Jacobi identity whenever $[F, G]$ does.

Having specified the desired form of the Hamiltonian in Eq. (3.43); along with the bracket, $\{F, G\}$ in Eq. (3.45), the time evolution of any quantity is determined via Hamilton's equations, given by

$$\partial_t F = -\{F, H\}. \quad (3.46)$$

In particular, the time evolutions of the dynamical variables s_e , ρ_e , v_e^a , s_r , ρ_{α} , v_{α}^a , P^a , A^a , and D^a are determined. The resulting evolution equations for the mass and entropy densities are unchanged from their uncharged analogues, Eqs. (3.2)–(3.5) (i.e., the corresponding conservation laws). In contrast the total fluid momentum density, P^a , is no longer conserved as it was in Eq. (3.6). This quantity now evolves as follows:

$$\partial_t P^a + \nabla_b \pi^{ab} = \sigma E^a + \left[\left(\frac{1}{c} \vec{J} + \sum_{\alpha} a_{\alpha} \vec{\nabla} \times \vec{\lambda}_{\alpha} \right) \times \vec{E} \right]^a, \quad (3.47)$$

where P^a is the total fluid momentum density of Eq. (3.7), π^{ab} is the fluid stress tensor of Eq. (3.16), and the total electrical charge density σ is defined by

$$\sigma = a_e c \rho_e + \sum_{\alpha} a_{\alpha} c \rho_{\alpha}. \quad (3.48)$$

Note that the total fluid momentum responds to a Lorentz force in which the electric current includes contributions of the form $ca_{\alpha} \vec{\nabla} \times \vec{\lambda}_{\alpha}$. The momentum evolution equation, Eq. (3.47), can also be written in a form in which the Lorentz force on its right side involves only the macroscopic current J^a . This is accomplished by redefining the stress tensor (including the pressure) in Eq. (3.16) by making the substitution $\vec{\omega}_{\alpha} \rightarrow \vec{\omega}_{\alpha} + a_{\alpha} \vec{B}$. Since the combination $\vec{\omega}_{\alpha} + a_{\alpha} \vec{B}$ represents the density of vortices, by Eq. (3.1), this reexpressed version of the stress tensor might be considered to be the more natural one.

The equations for the evolution of v_{α}^a and v_e^a , Eqs. (3.8) and (3.9), are also transformed by the addition of electromagnetic terms. In particular the forces $\vec{F}_{\alpha}$ and $\vec{F}_e$ become,

$$\begin{aligned} \vec{F}_{\alpha} = & -\vec{v}_{\alpha} \times \vec{\omega}_{\alpha} + ca_{\alpha} \vec{E} + \vec{v}_r \times (\vec{\omega}_{\alpha} + a_{\alpha} \vec{B}) \\ & + \sum_{\beta\gamma} K_{\alpha\beta\gamma} (\vec{M}_{\beta} - \rho_{\beta} \vec{v}_r + \vec{\nabla} \times \vec{\lambda}_{\beta}) \times (\vec{\omega}_{\gamma} + a_{\gamma} \vec{B}), \end{aligned} \quad (3.49)$$

and

$$\vec{F}_e = a_e (c \vec{E} + \vec{v}_e \times \vec{B}). \quad (3.50)$$

While the ordinary fluid responds to the standard Lorentz force, the force on the superfluid is rather more complicated. Note that this force is influenced profoundly by the presence of the new vorticity preserving forces that are proportional to $K_{\alpha\beta\gamma}$. Also note that this force reduces to the expression given by Holm and Kupershmidt (1987) for the case $K_{\alpha\beta\gamma} = \lambda_{\alpha}^a = 0$. However, this force agrees

with that of Vardanyan and Sedrakyan (1981) only when the vortex-free London equation, $\vec{\omega}_\alpha + a_\alpha \vec{B} = 0$, is satisfied or when $K_{\alpha\beta\gamma}$ has the value given in Eq. (3.58) and the mass density tensor, $\rho_{\alpha\beta}$ of Eq. (3.11), is diagonal. Their equations do not appear to be consistent with energy conservation under any other circumstances.

The Hamiltonian formalism also produces the equations for the evolution of the dynamical electromagnetic fields, A^a and D^a . These are simply the time dependent Maxwell equations, given by

$$\partial_t A^a = -cE^a, \quad (3.51)$$

and

$$\partial_t D^a = c(\vec{\nabla} \times \vec{H})^a - 4\pi J^a. \quad (3.52)$$

The divergence of Eq. (3.52) guarantees that the constraint equation,

$$\nabla_a D^a = 4\pi\sigma, \quad (3.53)$$

is preserved as the fluid evolves. The definition of the magnetic field H^a given in Eq. (3.42) allows the equations for electromagnetic fields to be written in a form that includes only the sources, σ and J^a , associated with the macroscopic motions of the fluid.

Hamilton's equations, Eq. (3.46), determine the evolution of all physical quantities, including the total momentum and energy of the combined superfluid and electromagnetic system. Equation (3.47) for the evolution of the momentum can be rewritten in a form that makes the conservation of the total momentum self-evident:

$$0 = \partial_t \left[P^a + \frac{1}{4\pi c} (\vec{D} \times \vec{B})^a \right] + \nabla_b \left\{ \pi^{ab} - U_{EM} g^{ab} - (g^{ac} g^{bd} - g^{ab} g^{cd}) \left[\frac{1}{4\pi} (E_c D_d + H_c B_d) + \sum_\alpha a_\alpha \lambda_{\alpha c} B_d \right] \right\}. \quad (3.54)$$

Similarly, the equation for the evolution of the total energy of the system can be written in the form of a conservation law:

$$0 = \partial_t (U + U_{EM}) + \nabla_b \left[U^b + \frac{c}{4\pi} (\vec{E} \times \vec{H})^b + c \sum_{\alpha} a_{\alpha} (\vec{E} \times \vec{\lambda}_{\alpha})^b \right]. \quad (3.55)$$

That these conservation laws are satisfied is unrelated to the issue of whether the bracket $\{F, G\}$ in Eq. (3.45) satisfies the Jacobi identity or not. Thus; the evolution equations presented here guarantee momentum and energy conservation for any values of $K_{\alpha\beta\gamma}$ even if they cannot be expressed as a rigorous Hamiltonian system.

Up to this point the discussion of the electromagnetic coupling to mixtures of superfluids has been conducted without restricting the values of the coefficients $K_{\alpha\beta\gamma}$ that appear in the superfluid force, Eq. (3.49). Ultimately, the choice of these coefficients must be based on experimental criteria, or at the very least on a microscopic model of the material. Here a suggested natural choice for the $K_{\alpha\beta\gamma}$ is given, from the viewpoint of the macroscopic analysis developed here. Since these coefficients participate in the superfluid-electromagnetic coupling, it is reasonable to choose them to insure that this coupling has the form of an appropriate Lorentz force. In analogy with the force that acts on the total momentum of the fluid, Eq. (3.47), it seems natural to require that $\vec{F}_{\alpha}$ take the form of the Lorentz force with an electric current arising from the macroscopic flow of the superfluid condensate plus a term which depends on the vortices via λ_{α}^a . The macroscopic current associated with the particles of species α that have condensed into the superfluid state is given by

$$J_{s\alpha}^a = ca_{\alpha} (M_{\alpha}^a - \rho_{r\alpha} v_r^a), \quad (3.56)$$

where the normal component of the mass density is denoted $\rho_{r\alpha}$. The super component of the mass density can then be defined by $\rho_{s\alpha} \equiv \rho_\alpha - \rho_{r\alpha}$. The superfluid force equation will reduce to the desired form

$$\begin{aligned} \vec{F}_\alpha = & \frac{1}{\rho_{s\alpha}} \left[\frac{1}{ca_\alpha} \vec{J}_{s\alpha} - \rho_{s\alpha} \vec{v}_\alpha + \vec{\nabla} \times \vec{\lambda}_\alpha \right] \times \vec{\omega}_\alpha \\ & + ca_\alpha \vec{E} + \frac{1}{c\rho_{s\alpha}} \left(\vec{J}_{s\alpha} + ca_\alpha \vec{\nabla} \times \vec{\lambda}_\alpha \right) \times \vec{B}, \end{aligned} \quad (3.57)$$

when $K_{\alpha\beta\gamma}$ is chosen as follows:

$$K_{\alpha\beta\gamma} = \frac{1}{\rho_{s\alpha}} \delta_{\alpha\gamma} \delta_{\beta\gamma}. \quad (3.58)$$

This choice generalizes to charged mixtures of superfluids the interaction proposed by Volovik and Dotsenko (1980) and by Khalatnikov and Lebedev (1980) for a single component uncharged superfluid. It is interesting to note that different approaches have lead to a similar result. Here it is required that the superfluid velocity respond to the electromagnetic field by an appropriate Lorentz force law. Using similar reasoning, Khalatnikov and Levedev (1980) make a choice equivalent to Eq. (3.58) in order to eliminate the normal fluid velocity from the force on the superfluid velocity. The more microscopic derivation of Volovik and Dotsenko (1980), on the other hand, can be related to the study of the underlying vortex motion made by Rasetti and Regge (1975). Also note that the choice in Eq. (3.58) does not in general have the form, Eq. (3.37), needed for the bracket to satisfy the Jacobi identity. Except for the case of a zero-temperature fluid (when $s_r = 0$ and $\rho_{s\alpha} = \rho_\alpha$) these fluid equations do not, therefore, have a rigorous Hamiltonian formulation. Volovik and Dotsenko (1980), however, indicate a Hamiltonian formulation is possible for nonzero temperatures when the superfluid component of

the mass density is treated as an independent variable. While this may be a necessary assumption for certain applications, especially near the superfluid-transition temperature (e.g., see Ginzburg and Sobaynin 1976), here the superfluid component of the mass density, $\rho_{s\alpha}$, is regarded as being specified in terms of the other variables by an appropriate equation of state [e.g., see Eq. (2.42)].

The superfluid force, Eq. (3.57), can be reexpressed in a form that illustrates more clearly the physical significance of its various terms. Let us consider the case when the superfluid mass currents $\vec{M}_\alpha$ are given by the Andreev and Bashkin (1976) expression, Eq. (3.11). Defining the density of normal particles $\rho_{r\alpha}$ as the coefficient of $\vec{v}_r$ in the mass current, then the superfluid density $\rho_{s\alpha}$ is related to the mass density matrix $\rho_{\alpha\beta}$ by

$$\rho_{s\alpha} = \sum_{\beta} \rho_{\alpha\beta}. \quad (3.59)$$

Using this relation, the superfluid force can be written as

$$\begin{aligned} \rho_{s\alpha} \vec{F}_\alpha = & \sum_{\beta} \rho_{\alpha\beta} (\vec{v}_\beta - \vec{v}_\alpha) \times \vec{\omega}_\alpha + (\vec{\nabla} \times \vec{\lambda}_\alpha) \times (\vec{\omega}_\alpha + a_\alpha \vec{B}) \\ & + ca_\alpha \rho_{s\alpha} \vec{E} + \frac{1}{c} \vec{J}_{s\alpha} \times \vec{B}. \end{aligned} \quad (3.60)$$

In this expression, the first term is a mutual drag force between the flow of one species of superfluid past another. This term vanishes if the mass density matrix is diagonal. The next term is analogous to the force, introduced by Hall (1958) to describe the elasticity of the vortices. When the vortex energy λ_α^a is taken to be proportional to $\omega_\alpha^a + a_\alpha B^a$, this term is the natural charged-fluid generalization of Bekarevich and Khalatnikov's (1961) expression for this force. The last terms are the Lorentz force with an electric current produced by the macroscopic flow of superfluid particles. Note that the current that appears in this Lorentz force

contains Fermi-liquid effects (by way of the mass density matrix $\rho_{\alpha\beta}$) analogous to those computed by Easson and Pethick (1979) to describe nonsuperfluid protons in a neutron-star interior. This expression also reduces to the standard Lorentz force when there is only a single component charged superfluid, which agrees with the result given by Putterman (1974).

Gravitational Coupling

In this section it is briefly shown how Newtonian gravitational forces can be incorporated into the derivation of the hydrodynamic equations given in this chapter. First, Eq. (3.12) for the energy density should be modified to become:

$$U = U_o + P_b v_r^b - \frac{1}{2} \rho v_r^b v_{rb} + \frac{1}{2} \rho \varphi. \quad (3.61)$$

The last term in this equation is the gravitational potential energy of the fluids in terms of the gravitational potential φ and total mass density ρ . The gravitational potential obeys the equation

$$\nabla_b \nabla^b \varphi = 4\pi G \rho, \quad (3.62)$$

where G is Newton's gravitational constant. The partial time derivative of U can then be computed and energy will be conserved (i.e., $\partial_t U + \nabla_b U^b = 0$) if the following replacements are made in the corresponding quantities:

$$F_\alpha^a \rightarrow F_\alpha^a - \nabla^a \varphi, \quad (3.63)$$

$$F_e^a \rightarrow F_e^a - \nabla^a \varphi, \quad (3.64)$$

$$\pi^{ab} \rightarrow \pi^{ab} + \frac{1}{2} \rho \varphi g^{ab} + \frac{1}{8\pi G} \left[(\nabla^a \varphi)(\nabla^b \varphi) - \varphi \nabla^a \nabla^b \varphi \right], \quad (3.65)$$

$$\begin{aligned}
U^a \rightarrow U^a + \varphi \rho_e v_e^a + \varphi \sum_{\alpha} \left[M_{\alpha}^a + (\vec{\nabla} \times \vec{\lambda}_{\alpha})^a \right] \\
- \frac{1}{2} \rho \varphi v_r^a - \frac{1}{8\pi G} \left[(\nabla^a \varphi)(\nabla^b \varphi) - \varphi \nabla^a \nabla^b \varphi \right] v_{rb} \\
- \frac{1}{8\pi G} \left[(\partial_t \varphi)(\nabla^a \varphi) - \varphi \nabla^a \partial_t \varphi \right].
\end{aligned} \tag{3.66}$$

Thus, the gravitational forces acting on the superfluids, equal to $-\nabla^a \varphi$, are exactly as would be expected from the equivalence principle.

CHAPTER 4

ADAPTING THE EQUATIONS TO NEUTRON STARS

Introduction

The purpose of this chapter is to develop equations which can be used to investigate the effects of superfluidity on the global oscillations of neutron stars. In general, the star is assumed to be rotating and magnetized. In particular, this chapter focuses on the outer-core region. The densities of this region are defined to be above that in which nuclei dissolve into a liquid of mostly neutrons (with a small fraction of protons, electrons, and even fewer muons) and below that in which nuclear matter theory is not yet fully understood, i.e., the density range approximately specified by $2.8 \times 10^{14} \text{g/cm}^3 < \rho < 5 \times 10^{14} \text{g/cm}^3$ (Baym, Bethe, and Pethick 1971). Most of the matter in a $1.4M_{\odot}$ neutron star has a density in this range if the equation of state for nuclear matter is very stiff. As described in Chapter 1, some evidence exists suggesting that the equation of state for nuclear matter is indeed very stiff.

In this chapter, the equations for a general mixture of superfluids derived in Chapter 3 are used to describe the outer-core region of a neutron star. The results are further extended in two important ways: a specific model of the energy density is constructed and partial derivatives of the energy density are used to relate the dependent variables to the independent variables. The vortex energy per unit length appears in these equations. This is calculated next, using

the *nonsmooth-averaged* London equations and Ampere's law. After this, the equilibrium properties of a typical neutron star are discussed and the linearized perturbation equation are presented. Finally, the characteristic frequencies associated with each of the forces which act on the fluids are summarized. This sets the stage for chapter 5 in which the low-frequency-long-wavelength limit of the equations is taken. The discussion in this and the next chapter is based on Mendell (1991a).

Nonlinear Hydrodynamic Equations

Here, the equations in Chapter 3 are adapted to describe a mixture of superfluid neutrons, superconducting protons, and degenerate electrons and muons. The low-temperature limit is taken, in that the excitations of the neutrons and protons are ignored (a valid approximation except near the superfluid-transition temperatures). The independent hydrodynamic variables which describe the properties of the fluid mixture then can be taken to be (for the neutrons, protons, electrons, and muons, respectively) the mass densities, ρ_n , ρ_p , ρ_e , ρ_μ , the velocity fields, v_n^a , v_p^a , $v_e^a = v_\mu^a$, and the entropy carried by the electron-muon fluid, $\tilde{s}_e = s_e + s_\mu$ (the superfluid condensates carry no entropy). Note that the velocities of the electrons and muons are set equal to each other (and only the total entropy of this fluid is considered), since scattering between these two fluids causes them to flow together on very short time scales on the order of

$$\tau_s \approx \left(\frac{\hbar}{k_B T_F} \right) \left(\frac{T_F}{T} \right)^2 \sim 8 \times 10^{-14} \text{ s} \left(\frac{T_F}{10^{12} \text{ K}} \right) \left(\frac{10^7 \text{ K}}{T} \right)^2, \quad (4.1)$$

where T_F is the Fermi temperature of the electrons (Sauls 1988; see also Pines and Nozières 1966).

In a typical rotating magnetized neutron star it is expected that the neutrons and protons form arrays of quantized vortices (see Baym, Pethick, and Pines 1969; Chapter 1). The vortices are microscopic in size with radii roughly equal to the coherence length ξ . The typical spacing between vortices, d , is much larger than the coherence length but still microscopic compared with the radius of a neutron star. For example, the spacing between neutron vortices in the Vela pulsar is expected to be about 10^{-3} cm and the spacing between proton vortices for a typical magnetic field of 10^{12} G is expected to be about 10^{-10} cm (Sauls 1988). Thus the larger of these numbers is usually the spacing between neutron vortices. Assuming a triangular array in equilibrium (as is observed in terrestrial superfluids) the intervortex spacing between neutron vortices, d_n , is given by

$$d_n = \left[\frac{h}{2\sqrt{3}m_n\Omega} \right]^{1/2} \approx 3.4 \times 10^{-3} \left(\frac{10^2 \text{ s}^{-1}}{\Omega} \right)^{1/2} \text{ cm}, \quad (4.2)$$

where Ω is the angular velocity of rotation. To investigate the hydrodynamic behavior of superfluid neutron stars on scales much larger than the intervortex spacing, it is therefore appropriate to use smooth-averaged hydrodynamics described in Chapters 2 and 3. This hydrodynamics is valid for length scales much larger than the intervortex separation given in Eq. (4.2).

Beginning with the smooth-averaged hydrodynamic equations for the mass and entropy densities, the mass density of each species will be taken to be conserved separately, since weak-interaction time scales are much longer than typical neutron star oscillation time scales (see Epstein 1988; Easson and Pethick 1979). In the nondissipative limit, the entropy of the electron-muon fluid is also conserved. These conservation laws are then given by

$$\partial_t \rho_n + \nabla_b (\rho_{nn} v_n^b + \rho_{np} v_p^b) = 0, \quad (4.3)$$

$$\partial_t \rho_p + \nabla_b (\rho_{pn} v_n^b + \rho_{pp} v_p^b) = 0, \quad (4.4)$$

$$\partial_t \rho_e + \nabla_b (\rho_e v_e^b) = 0, \quad (4.5)$$

$$\partial_t \rho_\mu + \nabla_b (\rho_\mu v_e^b) = 0, \quad (4.6)$$

$$\partial_t \tilde{s}_e + \nabla_b (\tilde{s}_e v_e^b) = 0, \quad (4.7)$$

Note that, as pointed out by Sonin (1987), small corrections to the mass densities of the condensates due to the smooth-averaging process over vortices can be ignored since they involve only small terms of the order $\xi^2/d^2 \ll 1$, which is roughly the fractional volume occupied by the vortices. (See Baym and Chandler 1983 for a theory which includes these terms.) Also note that in Eqs. (4.3) and (4.4) the mass currents of the superfluids depend on the superfluid velocities of both the neutrons and protons via the mass density matrix ρ_{nn} , $\rho_{np} = \rho_{pn}$, and ρ_{pp} . The resulting effect is often referred to as a Fermi liquid "drag" between species and was first introduced into the theory of superfluid mixtures by Andreev and Bashkin (1976). Calculations by Andreev and Bashkin (1976) (see also Alpar, Langer, and Sauls 1984; Sauls 1988) show that in the low-temperature limit considered here these quantities are given by

$$\rho_{nn} = \rho_n \left(\frac{m_n}{m_n^*} \right), \quad (4.8)$$

$$\rho_{np} = \rho_{pn} = \rho_n \left(\frac{m_n^* - m_n}{m_n^*} \right) = \rho_p \left(\frac{m_p^* - m_p}{m_p^*} \right), \quad (4.9)$$

$$\rho_{pp} = \rho_p \left(\frac{m_p}{m_p^*} \right), \quad (4.10)$$

where m_n^* and m_p^* are the effective masses of the neutron and proton, respectively.

Note that these equations obey the constraints, $\rho_{nn} + \rho_{np} = \rho_n$ and $\rho_{pp} + \rho_{pn} = \rho_p$,

required by Galilean invariance of Eqs (4.3) and (4.4). Estimates of the effective masses as a function of the density appear in Chao, Clark, and Yang (1972), Takatsuka (1973), and Sjöberg (1976). The results in these studies do not agree exactly. However, they indicate that the effective mass of the proton falls roughly in the range $0.3 \leq (m_p^*/m_p) \leq 0.7$ for the density range of a neutron-star outer core. [Also, the results in Chao et al. (1972) and Sjöberg (1976) violate the constraint between (m_n/m_n^*) , (m_p/m_p^*) , and (ρ_p/ρ_n) given by Eq. (4.9). Takatsuka 1973 does not give values for all of these quantities. This discrepancy needs to be sorted out in the future but will not affect the results of this thesis.]

Next, the fluid velocities are considered. First, the superfluid velocities v_n^a and v_p^a can be defined by the smooth-averaged curl of the London equations, given by

$$\omega_n^a = \frac{h}{2m_n} n_{vn} \nu_n^a, \quad (4.11)$$

$$\omega_p^a + a_p B^a = \frac{h}{2m_p} n_{vp} \nu_p^a, \quad (4.12)$$

where n_{vn} and n_{vp} are the average number of neutron and proton vortices, respectively, per unit area perpendicular to the local average direction of these vortices given by unit vectors ν_n^a and ν_p^a . The smooth averaged vorticities in these equations are given by $\omega_n^a = (\vec{\nabla} \times \vec{v}_n)^a$ and $\omega_p^a = (\vec{\nabla} \times \vec{v}_p)^a$, where v_n^a and v_p^a are to be interpreted as smooth-averaged superfluid velocities. In Eq. (4.12) B^a is to be interpreted as the smooth-averaged magnetic induction. Second, the smooth-averaged hydrodynamic equations for the velocities are given by

$$\partial_t v_n^a + v_n^b \nabla_b v_n^a + \nabla^a \tilde{\mu}_n + \nabla^a \varphi = \frac{\rho_{np}}{\rho_n} [(\vec{v}_p - \vec{v}_n) \times \vec{\omega}_n]^a + \frac{1}{\rho_n} [(\vec{\nabla} \times \vec{\lambda}_n) \times \vec{\omega}_n]^a, \quad (4.13)$$

$$\begin{aligned}
\partial_t v_p^a + v_p^b \nabla_b v_p^a + \nabla^a \tilde{\mu}_p + \nabla^a \varphi = & \frac{\rho_{np}}{\rho_p} [(\vec{v}_n - \vec{v}_p) \times \vec{\omega}_p]^a \\
& + \frac{1}{\rho_p} [(\vec{\nabla} \times \vec{\lambda}_p) \times (\vec{\omega}_p + a_p \vec{B})]^a \\
& + \frac{\sigma_p}{\rho_p} \left[\vec{E} + \frac{1}{\rho_p} \left(\frac{\rho_{pp} \vec{v}_p + \rho_{pn} \vec{v}_n}{c} \right) \times \vec{B} \right]^a,
\end{aligned} \quad (4.14)$$

$$\partial_t v_e^a + v_e^b \nabla_b v_e^a + \frac{\rho_e}{\tilde{\rho}_e} \nabla^a \tilde{\mu}_e + \frac{\rho_\mu}{\tilde{\rho}_e} \nabla^a \tilde{\mu}_\mu + \frac{\tilde{s}_e}{\tilde{\rho}_e} \nabla^a T_e + \nabla^a \varphi = \frac{\sigma_e}{\tilde{\rho}_e} \left(\vec{E} + \frac{\vec{v}_e}{c} \times \vec{B} \right)^a. \quad (4.15)$$

The left-hand sides of Eqs. (4.13) and (4.14) have the form of the Landau equation for ordinary superfluid flow (with the “chemical potentials” in these equations, $\tilde{\mu}_n$, $\tilde{\mu}_p$, $\tilde{\mu}_e$, and $\tilde{\mu}_\mu$ to be determined from the energy density; see the next two sections of this chapter). Of the forces on the right-hand sides of Eqs. (4.13) and (4.14), the first terms, proportional to ρ_{np} , represent a drag force between the neutrons and protons, and the second terms, involving λ_n^a and λ_p^a (also determined from the energy density) represent an elastic force acting on the smooth-averaged flow of the superfluids due to the underlying vortices (see Chapter 3; Bekarevich and Khalatnikov 1961; Mendell and Lindblom 1991). On the other hand, Eq. (4.15), which describes the time evolution of the velocity of the electron-muon fluid, is obtained by averaging the Euler equations for the electrons and muons by taking $\partial_t(\rho_e v_e^a + \rho_\mu v_\mu^a)$, dividing by $\tilde{\rho}_e \equiv \rho_e + \rho_\mu$, and setting $v_e^a = v_\mu^a$. This is appropriate because of the short time between scattering events for the particles of these two fluids.

Turning to the electromagnetic coupling terms in Eqs. (4.14) and (4.15), the electric field E^a and magnetic induction B^a in these equations evolve according to the macroscopic Maxwell equations:

$$\nabla_b D^b = 4\pi\sigma, \quad (4.16)$$

$$\nabla_b B^b = 0, \quad (4.17)$$

$$(\vec{\nabla} \times \vec{E})^a = -\frac{1}{c} \partial_t B^a, \quad (4.18)$$

$$(\vec{\nabla} \times \vec{H})^a = \frac{4\pi}{c} J^a + \frac{1}{c} \partial_t D^a. \quad (4.19)$$

The magnetic field H^a and displacement field D^a are determined from the energy density. The total charge density and charge current in these equations are given by

$$\sigma = \sigma_p + \sigma_e, \quad (4.20)$$

$$J^a = \sigma_p \left(\frac{\rho_{pp}}{\rho_p} v_p^a + \frac{\rho_{pn}}{\rho_p} v_n^a \right) + \sigma_e v_e^a, \quad (4.21)$$

where the charge densities of the proton and electron-muon fluids appearing in Eqs. (4.14), (4.15), (4.20), and (4.21) are given, respectively, by

$$\sigma_p = (e/m_p) \rho_p, \quad (4.22)$$

$$\sigma_e = -(e/m_e) \rho_e - (e/m_\mu) \rho_\mu. \quad (4.23)$$

Finally, the Newtonian gravitational potential φ is specified by

$$\nabla_b \nabla^b \varphi = 4\pi G \rho, \quad (4.24)$$

where G is Newton's gravitational constant, and $\rho = \rho_n + \rho_p + \rho_e + \rho_\mu$ is the total mass density of the fluid.

Energy Density

The total energy density can be specified as the sum of terms

$$U = U_I + U_G + U_{KE} + U_V + U_{EM}, \quad (4.25)$$

where U_I is the internal energy density of the fluid, U_G the gravitational potential energy density, U_{KE} is the kinetic energy density, U_V the vortex energy density,

and U_{EM} the energy density of the electromagnetic field. The internal energy density is taken to be a function of the mass and entropy densities only and has been calculated from nuclear matter theory by many investigators (e.g., see Baym and Pethick 1975 for a review). The gravitational potential energy density is given by the usual expression, $U_G = \frac{1}{2}\rho\varphi$. Next, following Alpar et al. (1984), the macroscopic kinetic energy of the fluid, including the drag effect, is accounted for by $U_{\text{KE}} = \frac{1}{2}\rho_{nn}v_n^2 + \rho_{np}v_n^b v_{pb} + \frac{1}{2}\rho_{pp}v_p^2 + \frac{1}{2}\tilde{\rho}_e v_e^2$. Then, following Bekarevich and Khalatnikov (1961), terms are added which account for the energy density of the underlying vortices given by $U_V = \varepsilon_n n_{vn} + \varepsilon_p n_{vp}$. In this expression the product is taken between the energy per unit length of the neutron and proton vortices, ε_n and ε_p [calculated in this chapter; see Eqs. (4.56) and (4.62)], and the corresponding number of vortices per unit area specified in Eq. (4.11) and (4.12). This simple model ignores interactions between the vortices. Vortex interactions are not expected to play a significant role in the small amplitude oscillations of neutron stars of interest in this thesis. (See the discussion of the vortex frequencies in the last section of this chapter; see Sonin 1987; but see also Chapter 10.)

Finally, terms must be added which account for the energy in the electromagnetic field. The magnetic energy density is complicated by the fact that part of the magnetic induction is confined to the vortices, whereas an additional non-quantized magnetic field known as the London field (London 1950) can exist in a rotating superconductor. Furthermore, Alpar et al. (1984) have shown that the drag between the neutrons and protons causes the neutron vortices to carry a magnetic flux as well as the proton vortices (see also the section of this chapter on the vortex energies). Taking into account these considerations, the smooth-average magnetic induction can be related to the magnetic flux of the vortices and

the London field by

$$B^a = n_{vp}\Phi_p\nu_p^a + n_{vn}\Phi_n\nu_n^a + B_L^a. \quad (4.26)$$

In this equation the magnetic flux of the proton and neutron vortices are specified by Φ_p and Φ_n , respectively. The magnetic flux carried by a proton vortex is just equal to the usual quantum of flux $\Phi_0 = (hc/2e) = 2 \times 10^{-7} \text{ G} \cdot \text{cm}^2$. In contrast, $\Phi_n = (m_p/m_n)(\rho_{np}/\rho_{pp})\Phi_0$, which vanishes when the drag vanishes, i.e., when $\rho_{np} \rightarrow 0$. The smooth-averaged London field is specified by B_L^a . The average London field can be related to the average vorticity of the proton and neutron fluids by using Eqs. (4.11), (4.12), and (4.26). The result is

$$B_L^a = -\frac{m_p c}{e} \omega_p^a - \frac{m_p c}{e} \frac{\rho_{np}}{\rho_{pp}} \omega_n^a. \quad (4.27)$$

The first term is the usual expression for the London field generalized slightly to the case of arbitrary proton vorticity. The second term on the right-hand side is due to the drag effect.

Now, to see how to account for the magnetic energy density, the magnetic induction in the vicinity of a vortex can be written as the sum of two parts, the magnetic induction of the vortex, B_{vortex}^a and the London field, B_L^a . The magnetic energy density then looks like

$$(1/8\pi)|\vec{B}_{\text{vortex}} + \vec{B}_L|^2 = (1/8\pi)(B_{\text{vortex}}^2 + 2\vec{B}_L \cdot \vec{B}_{\text{vortex}} + B_L^2). \quad (4.28)$$

The first term has already been accounted for, as it is included in the energy per unit length of the vortices, ε_n and ε_p . The second term is the interaction between the London field and the vortices and can be written as

$$2(1/8\pi)B_L^b(\nu_{pb}n_{vp}\Phi_p + \nu_{nb}n_{vn}\Phi_n). \quad (4.29)$$

In Eq. (4.29) and the B_L^2 term in Eq. (4.28), Eq. (4.27) can be used to specify the London field and equations (4.11) and (4.12) can be used to specify the vortex densities.

The energy density of the electric field is not so complicated. Since the charges in a neutron star are not bound, it is natural to assume $D^a = E^a$ and thus that the energy density of the electric field is $(1/8\pi)D^2$.

Summing the expressions for the various parts of the energy density and simplifying the result is

$$\begin{aligned}
 U = & U_I + \frac{1}{2}\rho\varphi + \frac{1}{2}\rho_{nn}v_n^2 + \rho_{np}v_n^b v_{pb} + \frac{1}{2}\rho_{pp}v_p^2 + \frac{1}{2}\tilde{\rho}_e v_e^2 \\
 & + \frac{2m_n}{h}\epsilon_n|\vec{\omega}_n| + \frac{2m_p}{h}\epsilon_p|\vec{\omega}_p + a_p\vec{B}| \\
 & + \frac{1}{8\pi}D^2 + \frac{1}{8\pi}B^2 - \frac{1}{8\pi}\frac{m_p^2 c^2}{e^2}|\vec{\omega}_p + a_p\vec{B}|^2 \\
 & - \frac{1}{8\pi}\frac{m_p^2 c^2}{e^2}\frac{\rho_{np}^2}{\rho_{pp}^2}|\vec{\omega}_n|^2 - \frac{1}{4\pi}\frac{m_p^2 c^2}{e^2}\frac{\rho_{np}}{\rho_{pp}}\omega_n^b(\omega_{pb} + a_p B_b).
 \end{aligned} \tag{4.30}$$

Dependent Variables

To have a closed set of equations, the dependent quantities appearing in Eqs (4.13)–(4.19), $\tilde{\mu}_n$, $\tilde{\mu}_p$, $\tilde{\mu}_e$, $\tilde{\mu}_\mu$, λ_n^a , λ_p^a , E^a , and H^a , must be specified in terms of the independent variables. These quantities are defined by the following differential [adapted from those given in Eqs. (3.10) and (3.42) in Chapter 3]:

$$\begin{aligned}
 dU_o + dU_{EM} = & [\tilde{\mu}_n + \frac{1}{2}|\vec{v}_n - \vec{v}_r|^2]d\rho_n + [\tilde{\mu}_p + \frac{1}{2}|\vec{v}_p - \vec{v}_r|^2]d\rho_p \\
 & + [\tilde{\mu}_e + \frac{1}{2}|\vec{v}_e - \vec{v}_r|^2]d\rho_e + [\tilde{\mu}_\mu + \frac{1}{2}|\vec{v}_e - \vec{v}_r|^2]d\rho_\mu \\
 & + T_e ds_e + T_\mu ds_\mu + T_r ds_r + \frac{1}{2}\tilde{\rho}_e d|\vec{v}_e - \vec{v}_r|^2 \\
 & + (M_n^b - \rho_n v_r^b) d(v_{nb} - v_{rb}) + (M_p^b - \rho_p v_r^b) d(v_{pb} - v_{rb}) \\
 & + \lambda_n^b d\omega_{nb} + \lambda_p^b d(\omega_{pb} + a_p B_b) + \frac{1}{4\pi} E^b dD_b + \frac{1}{4\pi} H^b dB_b,
 \end{aligned} \tag{4.31}$$

where $U_o + U_{\text{EM}} = U - \frac{1}{2}\rho\varphi - P_b v_r^b + \frac{1}{2}\rho v_r^b v_r^b$. In this expression $P^a = \rho_n v_n^a + \rho_p v_p^a + \tilde{\rho}_e v_e^a$ is the total momentum density of the fluid and U_o is the energy density of the fluid in a frame which comoves with the velocity of the neutron and proton excitations given by v_r^a . Note that in the low-temperature limit considered here the excitations are ignored, which corresponds to setting $T_r = 0$ in Eq. (4.31). The velocity of the excitations, v_r^a , will not appear in the final set of equations presented in this chapter. Using Eq. (4.30) for the energy density, an expression can be easily obtained for $U_o + U_{\text{EM}}$. The dependent quantities can then be found by taking the appropriate partial derivatives indicated by Eq. (4.31).

The results for the chemical potentials in Eqs. (4.13)–(4.15) are given by

$$\begin{aligned} \tilde{\mu}_n = \mu_n - \frac{1}{2} \frac{\partial \rho_{np}}{\partial \rho_n} |\vec{v}_n - \vec{v}_p|^2 + \frac{2m_n}{h} \frac{\partial \varepsilon_n}{\partial \rho_n} |\vec{\omega}_n| + \frac{2m_p}{h} \frac{\partial \varepsilon_p}{\partial \rho_n} |\vec{\omega}_p + a_p \vec{B}| \\ - \frac{1}{8\pi} \frac{m_p^2 c^2}{e^2} \left[\frac{\partial(\rho_{np}^2/\rho_{pp}^2)}{\partial \rho_n} \right] |\vec{\omega}_n|^2 - \frac{1}{4\pi} \frac{m_p^2 c^2}{e^2} \left[\frac{\partial(\rho_{np}/\rho_{pp})}{\partial \rho_n} \right] \omega_n^b (\omega_{pb} + a_p B_b), \end{aligned} \quad (4.32)$$

$$\begin{aligned} \tilde{\mu}_p = \mu_p - \frac{1}{2} \frac{\partial \rho_{np}}{\partial \rho_p} |\vec{v}_n - \vec{v}_p|^2 + \frac{2m_n}{h} \frac{\partial \varepsilon_n}{\partial \rho_p} |\vec{\omega}_n| + \frac{2m_p}{h} \frac{\partial \varepsilon_p}{\partial \rho_p} |\vec{\omega}_p + a_p \vec{B}| \\ - \frac{1}{8\pi} \frac{m_p^2 c^2}{e^2} \left[\frac{\partial(\rho_{np}^2/\rho_{pp}^2)}{\partial \rho_p} \right] |\vec{\omega}_n|^2 - \frac{1}{4\pi} \frac{m_p^2 c^2}{e^2} \left[\frac{\partial(\rho_{np}/\rho_{pp})}{\partial \rho_p} \right] \omega_n^b (\omega_{pb} + a_p B_b), \end{aligned} \quad (4.33)$$

$$\tilde{\mu}_e = \mu_e, \quad (4.34)$$

$$\tilde{\mu}_\mu = \mu_\mu. \quad (4.35)$$

In these equations new chemical potentials have been introduced, defined as

$$\mu_A \equiv \left(\frac{\partial U_I}{\partial \rho_A} \right), \quad (4.36)$$

with A an index taking on the values $A = n, p, e, \mu$.

The temperature of the electron-muon fluid is related to the other variables (assuming that the electrons and muons are in thermal equilibrium) by

$$T_e = T_\mu = \frac{\partial U_I}{\partial s_e} = \frac{\partial U_I}{\partial s_\mu}. \quad (4.37)$$

This equation also serves as a constraint, relating the entropy of the electrons to the entropy of the muons.

Next, computing λ_n^a and λ_p^a , it is found that

$$\begin{aligned} \lambda_n^a = & \frac{2m_n}{h} \varepsilon_n \left[1 - \frac{1}{2 \ln(d_n/\xi_n)} \right] \frac{\omega_n^a}{|\vec{\omega}_n|} - \frac{1}{4\pi} \left(\frac{m_p c}{e} \right)^2 \left(\frac{\rho_{np}}{\rho_{pp}} \right)^2 \omega_n^a \\ & - \frac{1}{4\pi} \left(\frac{m_p c}{e} \right)^2 \left(\frac{\rho_{np}}{\rho_{pp}} \right) (\omega_p^a + a_p B^a), \end{aligned} \quad (4.38)$$

$$\lambda_p^a = \frac{2m_p}{h} \varepsilon_p \frac{\omega_p^a + a_p B^a}{|\vec{\omega}_p + a_p \vec{B}|} - \frac{1}{4\pi} \left(\frac{m_p c}{e} \right)^2 (\omega_p^a + a_p B^a) - \frac{1}{4\pi} \left(\frac{m_p c}{e} \right)^2 \left(\frac{\rho_{np}}{\rho_{pp}} \right) \omega_n^a. \quad (4.39)$$

Note that besides terms proportional to the energy per unit length of the vortices, the other terms appear due to the presence of the London field and the drag between the neutrons and protons. Also, the term proportional to $[2 \ln(d_n/\xi_n)]^{-1}$ results from the dependence of ε_n on d_n (see the next section), which depends on $|\vec{\omega}_n|$.

Lastly, the magnetic field H^a and electric field E^a , in terms of the magnetic induction B^a and displacement field D^a , are given simply by

$$E^a = D^a, \quad (4.40)$$

$$H^a = B^a. \quad (4.41)$$

Vortex Energy Per Unit Length

Introduction

The vortex energies per unit length, ε_n and ε_p , which appear in the equations for $\tilde{\mu}_n$, $\tilde{\mu}_p$, λ_n^a , and λ_n^b , have yet to be determined. This is accomplished in this section using superfluid hydrodynamics which describes a single vortex, i.e., superfluid hydrodynamics which has *not* been smooth-averaged. These equations have been discussed in general terms in Chapter 2 and are not used elsewhere in this thesis *except* in this section. Though calculations similar to those given here exist elsewhere (see Khalatnikov 1989, de Gennes 1966), the purpose here is to see how the drag between the neutrons and protons affects the results. The system of equations to be solved is given by the London equations (not smooth-averaged; see Chapter 2),

$$v_n^a = \frac{\hbar}{2m_n} \nabla^a S_n, \quad (4.42)$$

$$v_p^a + a_p A^a = \frac{\hbar}{2m_p} \nabla^a S_p, \quad (4.43)$$

and Ampere's law for steady state motion in the rest frame of the electron-muon fluid:

$$(\vec{\nabla} \times \vec{B})^a = \frac{e}{m_p} (\rho_{pp} v_p^a + \rho_{np} v_n^a). \quad (4.44)$$

These equations are valid outside a vortex (ignoring the spin-triplet pairing of the neutrons). In general, inside a vortex the full Ginzburg-Landau equations for the order parameters must be solved (e.g., see Tilley and Tilley 1986). The complications associated with the Ginzburg-Landau theory are avoided here by using approximate solutions.

Neutron Vortex

Following Fetter and Hohenberg (1969) and Alpar et al. (1984), the spatial dependence of the mass density matrix, given in cylindrical coordinates (ϖ, ϕ, z) for a neutron vortex in equilibrium along the z -axis, can be approximated by

$$\rho_{nn}, \rho_{np} \propto \begin{cases} \varpi^2/\xi_n^2, & (\varpi < \xi_n), \\ \text{constant}, & (\varpi \geq \xi_n), \end{cases} \quad (4.45)$$

$$\rho_{pp} \propto \text{constant everywhere.} \quad (4.46)$$

Equations (4.42), (4.43), and (4.44) can be solved to obtain the following expressions, some of which are given in Alpar et al. (1984):

$$v_n^a = \frac{\hbar}{2m_n} \frac{1}{\varpi} \hat{\phi}^a, \quad (4.47)$$

$$A^a = -\frac{m_{pc}}{e} v_p^a, \quad (4.48)$$

for all ϖ , and

$$B^a = \begin{cases} \left(\frac{\Phi_n}{\pi \xi_n^2} \right) \left[1 - (\xi_n/\Lambda_n) K_1(\xi_n/\Lambda_n) I_0(\varpi/\Lambda_n) \right] \hat{z}^a, & \varpi < \xi_n, \\ \left(\frac{\Phi_n}{\pi \xi_n^2} \right) \left(\frac{\xi_n}{\Lambda_n} \right) I_1(\xi_n/\Lambda_n) K_0(\varpi/\Lambda_n) \hat{z}^a, & \varpi \geq \xi_n, \end{cases} \quad (4.49)$$

$$v_p^a = \begin{cases} \left[\left(\frac{e}{m_p c} \right) \left(\frac{\Phi_n}{\pi \xi_n^2} \right) K_1(\xi_n/\Lambda_n) I_1(\varpi/\Lambda_n) - \left(\frac{\hbar}{2m_n} \right) \left(\frac{\rho_{np}}{\rho_{pp}} \right) \left(\frac{\varpi}{\xi_n^2} \right) \right] \hat{\phi}^a, & \varpi < \xi_n, \\ \left[\left(\frac{e}{m_p c} \right) \left(\frac{\Phi_n}{\pi \xi_n^2} \right) I_1(\xi_n/\Lambda_n) K_1(\varpi/\Lambda_n) - \left(\frac{\hbar}{2m_n} \right) \left(\frac{\rho_{np}}{\rho_{pp}} \right) \left(\frac{1}{\varpi} \right) \right] \hat{\phi}^a, & \varpi \geq \xi_n. \end{cases} \quad (4.50)$$

In these equations the coherence length for the neutrons is related to the corresponding Fermi velocity, v_{Fn} , and the superfluid-transition temperature, T_{cn} , by (Tilley and Tilley 1986)

$$\xi_n \approx \frac{\hbar v_{Fn}}{1.75 \pi k_B T_{cn}} \approx 1.1 \times 10^{-11} \left(\frac{m_n}{m_n^*} \right) \left(\frac{\rho_n}{10^{14} \text{g/cm}^3} \right)^{1/3} \left(\frac{10^9 \text{K}}{T_{cn}} \right) \text{cm}. \quad (4.51)$$

The London depth is given by

$$\Lambda_n = \left(\frac{m_p^2 c^2}{4\pi e^2 \rho_{pp}} \right)^{1/2} \approx 9.3 \times 10^{-12} \left(\frac{m_p^*}{m_p} \right)^{1/2} \left(\frac{10^{13} \text{ g/cm}^3}{\rho_p} \right)^{1/2} \text{ cm}, \quad (4.52)$$

which is the distance over which the magnetic induction falls off exponentially far from a vortex. The K 's and I 's which appear in these equations are the modified Bessel functions as defined in Abramowitz and Stegun (1972). Thus each neutron vortex carries a magnetic flux equal to $\Phi_n = \oint A^b \hat{\phi}_b \omega d\phi = (m_p/m_n)(\rho_{np}/\rho_{pp})\Phi_0$, where $\Phi_0 = (hc/2e) = 2 \times 10^{-7} \text{ G} \cdot \text{cm}^2$ is the quantum of flux for an ordinary superconductor.

The energy per unit length of a neutron vortex, ε_n , is then specified by the kinetic energy in the condensate plus the energy in the magnetic field, i.e.,

$$\varepsilon_n = \int_0^{2\pi} \int_0^{d_n} \left(\frac{1}{2} \rho_{nn} v_n^2 + \rho_{np} v_n^b v_{pb} + \frac{1}{2} \rho_{pp} v_p^2 + \frac{1}{8\pi} B^2 \right) \omega d\omega d\phi, \quad (4.53)$$

where the cutoff on the radial integral corresponds to the distance between vortices. The result is,

$$\begin{aligned} \varepsilon_n = & \pi \frac{\varrho^2}{\rho_{pp}} \frac{\hbar^2}{4m_n^2} \ln(d_n/\xi_n) + \frac{\pi}{4} \left[\frac{\rho_{nn}\rho_{pp} + \varrho^2}{\rho_{pp}} \right] \frac{\hbar^2}{4m_n^2} \\ & + \frac{1}{8} \frac{\Phi_n^2}{\pi^2 \xi_n^2} \frac{\xi_n^2}{\Lambda_n^2} I_1^2(\xi_n/\Lambda_n) [K_0(\xi_n/\Lambda_n) K_2(\xi_n/\Lambda_n) - K_0^2(\xi_n/\Lambda_n)] \\ & + \frac{1}{8} \frac{\Phi_n^2}{\pi^2 \xi_n^2} \left\{ 1 + \frac{\xi_n^2}{\Lambda_n^2} K_1^2(\xi_n/\Lambda_n) [I_0^2(\xi_n/\Lambda_n) - I_0(\xi_n/\Lambda_n) I_2(\xi_n/\Lambda_n)] \right. \\ & \left. - 4K_1(\xi_n/\Lambda_n) I_1(\xi_n/\Lambda_n) \right\}, \end{aligned} \quad (4.54)$$

where terms which are exponentially small by a factor of $\exp(-d_n/\Lambda_n)$ have been ignored (this factor follows from using asymptotic expansions for the modified Bessel functions). In this equation ϱ^2 is the determinant of the mass density matrix, defined by

$$\varrho^2 \equiv \rho_{nn}\rho_{pp} - \rho_{np}^2. \quad (4.55)$$

The largest of the terms in Eq. (4.54) is the first one, and it corresponds to the kinetic energy of the condensate outside the vortex core, modified very slightly by the drag since $(\varrho^2/\rho_{pp}) \approx \rho_n$. The second term is the kinetic energy of the condensate inside the core and is about 20 times smaller than the first term and can be neglected. The rest of the terms are due to the magnetic energy. Since $\xi_n > \Lambda_n$, in contrast to the case of an ordinary type II superconductor, the contribution to the magnetic energy inside the core is comparable to that outside. However, these terms are about 10^{-4} times smaller than the first term and can be neglected as well. Thus, approximately,

$$\varepsilon_n \approx \pi \frac{\hbar^2}{4m_n^2} \frac{\varrho^2}{\rho_{pp}} \ln(d_n/\xi_n) \sim 10^9 \text{ erg/cm}, \quad (4.56)$$

where $\rho_n \sim 2.8 \times 10^{14} \text{ g/cm}^3$, $\rho_p \sim 10^{13} \text{ g/cm}^3$, $m_p^*/m_p \sim 0.5$, $\Omega \sim 10^2 \text{ s}^{-1}$, and $T_{cn} \sim 10^9 \text{ K}$ have been used in the estimate of ε_n .

Proton Vortex

For a single proton vortex in equilibrium the neutron condensate is uniform and $v_n^a = 0$. Thus the energy per unit length of a proton vortex will be unaffected by the drag effect except that the mass density associated with the proton condensate will be ρ_{pp} instead of ρ_p . Although, as in an ordinary superconductor, the magnetic energy now will be comparable to the kinetic energy, the energy inside the core can be ignored compared with the contribution from outside, as in an ordinary type II superconductor (see de Gennes 1966). In this case one can treat the vortex as a one dimensional singularity with $(\vec{\nabla} \times \vec{\nabla} S_p)^a = \delta(x)\delta(y)\hat{z}^a$, having expressed the delta function in Cartesian coordinates (x, y, z) . The resulting solutions for $\varpi \geq \xi_p$ are

$$A^a = -\frac{m_p c}{e} v_p^a + \frac{\hbar c}{2e \varpi} \hat{\phi}^a, \quad (4.57)$$

$$B^a = \frac{\Phi_p}{2\pi\Lambda_p^2} K_0(\varpi/\Lambda_p) \hat{z}^a, \quad (4.58)$$

$$v_p^a = \frac{e}{m_p c} \frac{\Phi_p}{2\pi\Lambda_p} K_1(\varpi/\Lambda_p) \hat{\phi}^a, \quad (4.59)$$

where $\Lambda_p = \Lambda_n$ and $\Phi_p = \oint A^b \hat{\phi}_b \varpi d\phi = \Phi_o$. The energy per unit length of a proton vortex, ε_p , is then computed by the integral in Eq. (4.53) except that the lower bound on the radial integral must now be cut off at the radius of the vortex core, i.e., at $\varpi = \xi_p$, whereas the result is relatively insensitive to the upper bound since both v_p^a and B^b fall off exponentially far from the vortex. The result is

$$\varepsilon_p = \frac{\Phi_p^2}{16\pi^2\Lambda_p^4} \frac{\xi_p^2}{2} K_0(\xi_p/\Lambda_p) [K_2(\xi_p/\Lambda_p) - K_0(\xi_p/\Lambda_p)], \quad (4.60)$$

where the coherence length for the protons is related to the corresponding Fermi velocity, v_{Fp} , and superfluid-transition temperature, T_{cp} , by (Tilley and Tilley 1986)

$$\xi_p \approx \frac{\hbar v_{Fp}}{1.75\pi k_B T_{cp}} \approx 4.9 \times 10^{-12} \left(\frac{m_p}{m_p^*} \right) \left(\frac{\rho_p}{10^{13} \text{g/cm}^3} \right)^{1/3} \left(\frac{10^9 \text{K}}{T_{cp}} \right) \text{cm}. \quad (4.61)$$

Using the expansion of the K 's for small argument in Abramowitz and Stegun (1971), ε_p can be approximated as

$$\varepsilon_p \approx \frac{\Phi_p^2}{16\pi^2\Lambda_p^2} \ln(\Lambda_p/\xi_p) \sim 10^7 \text{erg/cm}, \quad (4.62)$$

where $\rho_p \sim 10^{13} \text{g/cm}^3$, $m_p^*/m_p \sim 0.5$, and $T_{cp} \sim 5 \times 10^9 \text{K}$ have been used to estimate the size of ε_p .

As a final note, the assumption of type II superconductivity is valid when $\xi_p/\Lambda_p < \sqrt{2}$ (see Baym et al. 1969; de Gennes 1966). Using the expressions in

Eqs. (4.61) and (4.52) for these quantities (note $\Lambda_p = \Lambda_n$) the condition for type II superconductivity is

$$0.53 \left(\frac{m_p}{m_p^*} \right)^{3/2} \left(\frac{\rho_p}{10^{13} \text{g/cm}^3} \right)^{5/6} \left(\frac{10^9 \text{K}}{T_{cp}} \right) < \sqrt{2}. \quad (4.63)$$

Otherwise type I superconductivity is favored and the protons would be in the intermediate state instead of the vortex state. Since $(m_p/m_p^*) \sim 2$ the protons could form a type I superconductor, though calculations of the proton gap (e.g., see Chao et al. 1972; Takatsuka 1973; Amundsen and Østgaard 1985) suggest that T_{cp} may be larger than 10^9K by a factor large enough to negate this possibility. Another intriguing possibility is that there is a density at which a transition takes place from type II to type I superconductivity.

Equilibrium Conditions and Linear Perturbation Equations

The general equilibrium conditions appropriate to rotating magnetized neutron stars are discussed briefly here. The equilibrium mass and entropy densities will be taken to be stationary and axisymmetric. The equilibrium temperature will be taken as uniform, since core temperatures are expected to be isothermal because of the large thermal conductivity of the degenerate electrons (Tsuruta 1979). All of the equilibrium velocities will be taken to corotate rigidly (recall that this refers to the smooth-averaged velocities), i.e., in cylindrical coordinates (ϖ, ϕ, z) , for rotation about the z -axis, the equilibrium velocities are given by $\Omega \varpi \hat{\phi}$ (with the caret denoting unit vectors). Notation will be simplified slightly by defining $v_o^a \equiv \Omega \varpi \hat{\phi}^a$ and $(\vec{\nabla} \times \vec{v}_o)^a = 2\Omega \hat{z}^a \equiv \omega_o^a$. On the other hand, the equilibrium magnetic induction will be taken as an arbitrary stationary vector

field, B_o^a . As in a rotating conducting magnet, it will be assumed that an electric field is induced inside a rotating neutron star such that the Lorentz force on the charged particles vanishes, i.e., the equilibrium electric field E_o^a is related to the equilibrium magnetic induction B_o^a by (see Rosser 1968; Goldreich and Julian 1969)

$$E_o^a = - \left(\frac{\vec{v}_o}{c} \times \vec{B}_o \right)^a. \quad (4.64)$$

The divergence of E_o^a implies a small equilibrium charge density given by

$$\sigma_o \approx \frac{-\Omega B_o^b \hat{z}_b}{2\pi c(1 - \Omega^2 \varpi^2/c^2)}. \quad (4.65)$$

For small perturbations away from equilibrium, Eqs. (4.3)–(4.24) can be linearized to first order in the perturbations. The resulting equations are (using δ to prefix all perturbed quantities)

$$\partial_t \delta \rho_n + v_o^b \nabla_b \delta \rho_n + \nabla_b (\rho_{nn} \delta v_n^b + \rho_{np} \delta v_p^b) = 0, \quad (4.66)$$

$$\partial_t \delta \rho_p + v_o^b \nabla_b \delta \rho_p + \nabla_b (\rho_{np} \delta v_n^b + \rho_{pp} \delta v_p^b) = 0, \quad (4.67)$$

$$\partial_t \delta \rho_e + v_o^b \nabla_b \delta \rho_e + \nabla_b (\rho_e \delta v_e^b) = 0, \quad (4.68)$$

$$\partial_t \delta \rho_\mu + v_o^b \nabla_b \delta \rho_\mu + \nabla_b (\rho_\mu \delta v_e^b) = 0, \quad (4.69)$$

$$\partial_t \delta \tilde{s}_e + v_o^b \nabla_b \delta \tilde{s}_e + \nabla_b (\tilde{s}_e \delta v_e^b) = 0, \quad (4.70)$$

$$\begin{aligned} \partial_t \delta v_n^a = & -v_o^b \nabla_b \delta v_n^a - \delta v_n^b \nabla_b v_o^a - \nabla^a \delta \tilde{\mu}_n - \nabla^a \delta \varphi \\ & + \frac{\rho_{np}}{\rho_n} [(\delta \vec{v}_p - \delta \vec{v}_n) \times \vec{\omega}_o]^a + \delta \left[\frac{1}{\rho_n} (\vec{\nabla} \times \vec{\lambda}_n) \times \vec{\omega}_n \right]^a, \end{aligned} \quad (4.71)$$

$$\begin{aligned} \partial_t \delta v_p^a = & -v_o^b \nabla_b \delta v_p^a - \delta v_p^b \nabla_b v_o^a - \nabla^a \delta \tilde{\mu}_p - \nabla^a \delta \varphi \\ & + \frac{\rho_{np}}{\rho_p} [(\delta \vec{v}_n - \delta \vec{v}_p) \times \vec{\omega}_o]^a + \delta \left[\frac{1}{\rho_p} (\vec{\nabla} \times \vec{\lambda}_p) \times (\vec{\omega}_p + a_p \vec{B}) \right]^a \\ & + \frac{\sigma_p}{\rho_p} \left(\delta \vec{E} + \frac{\vec{v}_o}{c} \times \delta \vec{B} + \frac{1}{\rho_p} \frac{\rho_{pp} \delta \vec{v}_p + \rho_{np} \delta \vec{v}_n}{c} \times \vec{B}_o \right)^a, \end{aligned} \quad (4.72)$$

$$\begin{aligned} \partial_t \delta v_e^a = & -v_o^b \nabla_b \delta v_e^a - \delta v_e^b \nabla_b v_o^a - \delta \left(\frac{\rho_e}{\tilde{\rho}_e} \nabla^a \tilde{\mu}_e \right) - \delta \left(\frac{\rho_\mu}{\tilde{\rho}_e} \nabla^a \tilde{\mu}_\mu \right) \\ & - \frac{\tilde{s}_e}{\tilde{\rho}_e} \nabla^a \delta T_e - \nabla^a \delta \varphi + \frac{\sigma_e}{\tilde{\rho}_e} \left(\delta \vec{E} + \frac{\vec{v}_o}{c} \times \delta \vec{B} + \frac{\delta \vec{v}_e}{c} \times \vec{B}_o \right)^a, \end{aligned} \quad (4.73)$$

$$\nabla_b \delta E^b = 4\pi \delta \sigma, \quad (4.74)$$

$$\nabla_b \delta B^b = 0, \quad (4.75)$$

$$(\vec{\nabla} \times \delta \vec{E})^a = -\frac{1}{c} \partial_t \delta B^a, \quad (4.76)$$

$$(\vec{\nabla} \times \delta \vec{B})^a = \frac{4\pi}{c} \delta J^a + \frac{1}{c} \partial_t \delta E^a, \quad (4.77)$$

$$\nabla_b \nabla^b \delta \varphi = 4\pi G \delta \rho. \quad (4.78)$$

Characteristic Frequencies

In this section the characteristic frequencies associated with each of the force terms in the velocity equations, Eqs. (4.71)–(4.73), are discussed. The frequencies can be found by studying plane-wave solutions to the equations. In these solutions, the space and time dependence of all the variables is completely specified by $\exp(i\vec{k} \cdot \vec{x} - i\omega t)$, where ω is the angular frequency, $k = (2\pi/\lambda)$ is the magnitude of the wave vector k^a , and λ the wavelength of the oscillation. Assuming a plane-wave form for the space and time dependence of the variables is convenient when one wants to compare the magnitudes of various terms, since in this case the derivatives in the equations are replaced by the following algebraic substitutions: $\partial_t V^a \rightarrow -i\omega V^a$ and $\nabla^a f^b \rightarrow ik^a V^b$. These relations hold exactly for plane waves and hold qualitatively for all types of oscillation in general.

Since plane-waves have been studied extensively in ordinary fluids, superfluids, and plasmas by others (although not in neutron stars) the details of the solutions are not given here. Instead, the purpose of this section is to summarize

the size of the characteristic frequencies in neutron stars. These frequencies will be used to compare the sizes of various terms when taking the LFLW limit in the next chapter. (Plane-wave solutions to the LFLW equations, including dissipative effects, will be given in Chapter 8.)

Two frequencies associated with the electromagnetic forces (well known in plasma physics) are the plasma frequency, ω_p , and the cyclotron frequency, Ω_c (e.g., see Boyd and Sanderson 1969). For motion of the protons these frequencies are given by

$$\omega_p = \left(\frac{4\pi e^2 \rho_p}{m_p^2} \right)^{1/2} \approx 3.2 \times 10^{21} \left(\frac{\rho_p}{10^{13} \text{g/cm}^3} \right)^{1/2} \text{ s}^{-1}, \quad (4.79)$$

$$\Omega_c = \frac{eB_o}{m_p c} \approx 9.6 \times 10^{15} \left(\frac{B_o}{10^{12} \text{G}} \right) \text{ s}^{-1}. \quad (4.80)$$

A third characteristic frequency associated with the electromagnetic forces is the helicon frequency, ω_h , which is given by

$$\omega_h = \Omega_c \frac{c^2 k^2}{\omega_p^2} \approx 3.3 \times 10^{-17} \left(\frac{B_o}{10^{12} \text{G}} \right) \left(\frac{10^{13} \text{g/cm}^3}{\rho_p} \right) \left(\frac{10^6 \text{cm}}{\lambda} \right)^2 \text{ s}^{-1}. \quad (4.81)$$

For low frequencies and large magnetic fields this oscillation is relevant in superconductors (see de Gennes and Matricon 1964; Abrikosov, Kemoklidze, and Khalatnikov 1965).

Van Horn (1980) has previously discussed Alfvén waves in the context of neutron stars (see also Boyd and Sanderson 1969). Interestingly, it will be shown in the next chapter that Alfvén waves (which occur in ordinary plasmas) do not occur in the superconducting-proton-degenerate-electron plasma considered here. The Lorentz force $(\delta \vec{J}/c \times \vec{B}_o)^a$ (which acts on the average velocity of the charged fluids; see Chapter 5) gives rise to Alfvén waves in the usual case. However, in

the present case it is canceled by a term arising from $-(1/4\pi)(m_p c/e)^2 a_p B^a$ in the expression for λ_p^a given by Eq. (4.39).

There are several characteristic frequencies associated with oscillations of the underlying vortices. (See Sonin 1987 for a review of vortex oscillations in superfluids; see Friedel, de Gennes, and Matricon 1963 for a calculation of vortex oscillations in superconductors.) The frequency of circularly polarized vortex oscillations which travel parallel to the vortices, known as Kelvin waves, are given by

$$\omega_{vn} = \frac{2m_n \varepsilon_n k^2}{\hbar \rho_n} \approx 2.0 \times 10^{-13} \left(\frac{\varepsilon_n}{10^9 \text{erg/cm}} \right) \left(\frac{10^{14} \text{g/cm}^3}{\rho_n} \right) \left(\frac{10^6 \text{cm}}{\lambda} \right)^2 \text{ s}^{-1}, \quad (4.82)$$

$$\omega_{vp} = \frac{2m_p \varepsilon_p k^2}{\hbar \rho_p} \approx 2.0 \times 10^{-14} \left(\frac{\varepsilon_p}{10^7 \text{erg/cm}^3} \right) \left(\frac{10^{13} \text{g/cm}^3}{\rho_p} \right) \left(\frac{10^6 \text{cm}}{\lambda} \right)^2 \text{ s}^{-1}, \quad (4.83)$$

for the neutron and proton vortices, respectively. These vortex frequencies can be shown to arise from the first terms in Eqs. (4.38) and (4.39) for λ_n^a and λ_p^a .

The term proportional to $\varepsilon_p B^a$ in Eq. (4.39) for λ_p^a has not been included in previous studies of vortex oscillations. Careful analysis of this term (which will be given in the next Chapter) shows that it gives rise to a new circularly polarized mode which travels parallel to the magnetic axis. The frequency involves the cyclotron frequency and the vortex oscillation frequency of the protons. For this reason this frequency will be called the cyclotron-vortex frequency, and it is given by

$$\omega_{cv} = \left[\frac{\rho_{pp}}{\rho_p} \frac{2m_p \varepsilon_p \Omega_C}{\hbar \rho_p} \right]^{1/2} k$$

$$\approx 14. \left[\left(\frac{m_p}{m_p^*} \right) \left(\frac{\varepsilon_p}{10^7 \text{erg/cm}} \right) \left(\frac{10^{13} \text{g/cm}^3}{\rho_p} \right) \left(\frac{B_0}{10^{12} \text{G}} \right) \right]^{1/2} \left(\frac{10^6 \text{cm}}{\lambda} \right) \text{ s}^{-1}. \quad (4.84)$$

This frequency may be observable in laboratory superconductors if the conductivity of electrons in excited states is high enough so that the resistive damping time of this mode is not too short.

Turning to the Coriolis "force" and the drag force, these are proportional to the angular velocity of the neutron star:

$$\Omega \sim 1 - 10^4 \text{ s}^{-1}. \quad (4.85)$$

The lower bound is roughly the angular velocity observed for the longest period pulsar, and the upper bound corresponds to the most extreme estimate of the Keplerian limit at which point mass shedding occurs at the equator. Previously modes associated with rotation have been termed r-modes by Papaloizou and Pringle (1978) (see also Van Horn 1980).

A frequency characterizing the size of the gravitational force (and roughly equal to the Keplerian limit of rotation) is defined by

$$\Omega_o \equiv \sqrt{\pi G \bar{\rho}_o} \sim 10^4 \text{ s}^{-1}, \quad (4.86)$$

where $\bar{\rho}_o$ is the average density of the star (see Ipser and Lindblom 1990).

The pressure terms (involving gradients of the chemical potentials and the temperature) give rise to p-modes when mass density perturbations dominate. These modes have a typical frequency related to the sound velocity, v_s , by $\omega_s = v_s k$ where typically $v_s \sim .1c$ (see Chapter 8). The subscript "s" will be used for these modes to avoid confusion with the plasma frequency which already employs the subscript p. For wavelengths on the order of a typical neutron star radius, $R \sim 10^6 \text{ cm}$,

$$\omega_s \sim 10^4 \text{ s}^{-1}. \quad (4.87)$$

In nonzero temperature neutron stars g-modes can arise due to thermally induced buoyancy perturbations with gravitation acting as a restoring force. Following Van Horn (1980), the typical frequency of the fundamental g-modes is

$$\omega_g \sim 30 \text{ s}^{-1}. \quad (4.88)$$

Lastly, there are two oscillation frequencies which can occur in neutron stars that cannot be investigated by the equations in this thesis. Interactions between the neutron vortices can give rise to collective oscillations of the vortex array known as Tkachenko waves (Tkachenko 1966). However, interactions between vortices are ignored in the equations in this thesis. Tkachenko waves have been considered previously in the context of neutron stars by Ruderman (1970) (but see also Fetter and Stauffer 1970; Sonin 1987). They have a frequency $\omega_T = \sqrt{(\hbar\Omega/16\pi m_n)}k$. This gives very low frequencies of the order 10^{-7}s^{-1} for wavelengths comparable to a typical neutron star radius of 10^6cm and angular velocities of order 10^2s^{-1} . (Tkachenko oscillations of the proton vortices are probably insignificant since the interactions between these vortices are much weaker than those between neutron vortices.) Also ignored in this thesis is the spin-triplet pairing of the neutrons, which gives the neutron vortices a spin degree of freedom. However, an investigation of the hydrodynamics of superfluid $^3P_2$ neutrons by Brand and Pleiner 1981 concludes that the spin waves are overdamped in neutron stars. Although Tkachenko waves and spin waves are beyond the scope of the equations presented here, this will not affect finding the LFLW equations which describe the oscillations of interest in this thesis.

CHAPTER 5

LOW-FREQUENCY-LONG-WAVELENGTH LIMIT

Introduction

For appropriate low-frequencies and long-wavelengths the linearized perturbation equations given in the last chapter, Eqs. (4.66)–(4.78), can be greatly simplified. Following the usual steps used to obtain the magnetohydrodynamic (MHD) limit (well known in plasma physics), for an appropriate frequency range it is shown that the number of independent velocities can be reduced from three to two. The resulting equations will be applicable to studies of gravity modes (g-modes) and pressure modes (p-modes), including electromagnetic, rotational, and vortex effects. (See the definitions at the end of Chapter 4.) Here these equations are used to show that a previously unknown mode, which oscillates with the cyclotron-vortex frequency (defined in Chapter 4), exists in the superconducting-proton-degenerate-electron plasma. Finally, the equations are further simplified by showing that the electromagnetic forces and the forces due to the underlying vortices can be ignored for a sufficiently large angular velocity of rotation. This can be reexpressed to give a lower bound on the length scale of the oscillation. The final set of equations presented in this chapter will be applicable to studies of the global p-modes in rapidly rotating neutron stars.

Reducing the Number of Independent Velocities

The g-modes and p-modes have frequencies which are much smaller than the cyclotron and plasma frequencies. On the other hand, the g-modes can have frequencies similar in size to the cyclotron-vortex frequency. However, the frequencies of both these mode types are much larger than the frequencies of the other types of vortex oscillation and the helicon mode. Thus it will be useful to restrict attention to the frequency range

$$\omega_h, \omega_{vp}, \omega_{vn} \ll \omega \ll \Omega_C, \omega_p. \quad (5.1)$$

This frequency range specifies what is meant here by low frequency (as opposed to the ultralow frequencies of the lower bounds in this equation).

The upper bounds in Eq. (5.1) correspond to those imposed in the ordinary MHD limit (see Boyd and Sanderson 1969; Freidberg 1982; Easson and Pethick 1979). Thus, following the usual MHD procedure, the standard low-frequency modifications of Maxwell's equations, Eqs. (4.74)–(4.77), are given by

$$(\vec{\nabla} \times \delta \vec{B})^a = \frac{4\pi}{c} \delta J^a, \quad (5.2)$$

$$\delta \sigma = 0. \quad (5.3)$$

Equation (5.3) follows from the divergence of Eq. (5.2) and charge conservation. Equation (5.2) is justified when the displacement current, $(1/c)\partial_t \delta E^a \sim (\omega/c)\delta E$, is small compared with $(\vec{\nabla} \times \delta \vec{B})^a \sim k\delta B$ (dropping vector notation when using ω and k to estimate magnitudes, as in the last chapter). This condition is met as long as $(\omega/ck)^2 \ll 1$ [having used Faraday's law, Eq. (4.76), to show $\delta B \sim (ck/\omega)\delta E$], which is valid for the modes of interest.

The advantage of Eq. (5.2) is that the current density is now completely determined from the magnetic induction. This in turn gives a constraint on the perturbations of the velocities which are related to δJ^a by,

$$\delta J^a = \sigma_p \left(\frac{\rho_{pp}}{\rho_p} \delta v_p^a + \frac{\rho_{np}}{\rho_p} \delta v_n^a \right) + \sigma_e \delta v_e^a. \quad (5.4)$$

Terms involving equilibrium velocities and charge density perturbations do not appear since these terms cancel when $\delta\sigma = 0$.

An equation is still needed for the electric field, relating it to the other variables, since it has been eliminated from Ampere's law in Eq. (5.2). This is accomplished in the standard way by calculating $\partial_t \delta J^a$ and solving for δE^a . The result is

$$\delta E^a = - \left(\frac{\delta \vec{v}_e}{c} \times \vec{B}_o \right)^a - \left(\frac{\vec{v}_o}{c} \times \delta \vec{B} \right)^a \quad (5.5a)$$

$$+ \frac{4\pi}{\tilde{\omega}_{pe}^2} \left\{ \left[\partial_t \delta J^a + v_o^b \nabla_b \delta J^a + \delta J^b \nabla_b v_o^a \right] + \sigma_o \nabla^a \delta \varphi \right. \quad (5.5b)$$

$$+ \sigma_p \frac{\rho_{np}}{\rho_p} \delta \left[\frac{1}{\rho_n} (\vec{\nabla} \times \vec{\lambda}_n) \times \vec{\omega}_n \right]^a + \sigma_p \frac{\rho_{pp}}{\rho_p} \delta \left[\frac{1}{\rho_p} (\vec{\nabla} \times \vec{\lambda}_p) \times (\vec{\omega}_p + a_p \vec{B}) \right]^a \quad (5.5c)$$

$$+ \sigma_p \frac{\rho_{np}}{\rho_p} \frac{\varrho^2}{\rho_n \rho_p} [(\delta \vec{v}_n - \delta \vec{v}_p) \times \vec{\omega}_o]^a + \frac{\rho_{pp}}{\rho_p} [(\delta \vec{J} - \sigma_o \delta \vec{v}_e) \times a_p \vec{B}_o]^a \quad (5.5d)$$

$$+ \sigma_p \left[\frac{\rho_{np}}{\rho_p} \nabla^a \delta \tilde{\mu}_n + \frac{\rho_{pp}}{\rho_p} \nabla^a \delta \tilde{\mu}_p + \frac{\sigma_e}{\sigma_p} \delta \left(\frac{\rho_e}{\tilde{\rho}_e} \nabla^a \tilde{\mu}_e \right) + \frac{\sigma_e}{\sigma_p} \delta \left(\frac{\rho_\mu}{\tilde{\rho}_e} \nabla^a \tilde{\mu}_\mu \right) \right] \quad (5.5e)$$

$$+ \sigma_e \frac{\tilde{\sigma}_e}{\tilde{\rho}_e} \nabla^a \delta T_e \left. \right\}, \quad (5.5f)$$

where in this equation $\tilde{\omega}_{pe}^2 \equiv 4\pi[(\rho_{pp}/\rho_p)(\sigma_p^2/\rho_p) + (\sigma_e^2/\tilde{\rho}_e)] \approx (\rho_p/\tilde{\rho}_e)\omega_p^2$, and ϱ^2 is the determinant of the mass density matrix defined in Eq. (4.55). Of the terms on the right-hand side, the first term in Eq. (5.5a) is usually the largest. The second term in Eq. (5.5a) could be equally important in very rapidly rotating neutron stars. The remaining terms in Eq. (5.5b)–(5.5f) are proportional to the

small factor $(4\pi/\tilde{\omega}_{pe}^2)$. These terms are smaller than the first term in Eq. (5.5a) by the following factors: in Eq. (5.5b) the current terms are smaller by at least a factor of $(\tilde{\rho}_e/\rho_p)(c^2k^2/\omega_p^2)$ and the gravitational force term is smaller by at least a factor of $(\tilde{\rho}_e/\rho_p)(\Omega/\omega)(\Omega_o^2/\omega_p^2)$; in Eq. (5.5c) the vortex forces are smaller by factors of $(\tilde{\rho}_e/\rho_p)(\omega_{vn}/\Omega_C)$ and $(\tilde{\rho}_e/\rho_p)(\omega_{cv}/\omega)(\omega_{cv}/\Omega_C)$; in Eq. (5.5d) the drag term is smaller by $(\tilde{\rho}_e/\rho_p)(\Omega/\Omega_C)$ and the Lorentz force terms are smaller by factors of $(\tilde{\rho}_e/\rho_p)(\omega_h/\omega)$ and $(\tilde{\rho}_e/\rho_p)(\Omega/\omega_p)(\Omega_C/\omega_p)$; finally in Eqs. (5.5e) and (5.5f) the pressure terms are smaller by at least a factor of $(\tilde{\rho}_e/\rho_p)(\omega_s/\omega)(\omega_s/\Omega_C)$. Thus it is usually acceptable to take Eq. (5.5a) as the expression for δE^a . On the other hand, if the equilibrium magnetic induction vanishes, then the pressure terms will dominate, at least for p-modes. In all cases, to be consistent, it must be checked that the terms dropped from Eqs. (5.5a)–(5.5f) give terms smaller than all of the terms kept in the equations into which an expression for δE^a is substituted.

Next, it is useful to introduce the average velocity of the charged fluid which can be defined as

$$u^a \equiv \frac{\rho_p}{\tilde{\rho}_p} v_p^a + \frac{\tilde{\rho}_e}{\tilde{\rho}_p} v_e^a, \quad (5.6)$$

where $\tilde{\rho}_p = \rho_p + \tilde{\rho}_e$. For small perturbations away from equilibrium,

$$\delta u^a = \frac{\rho_p}{\tilde{\rho}_p} \delta v_p^a + \frac{\tilde{\rho}_e}{\tilde{\rho}_p} \delta v_e^a. \quad (5.7)$$

Taking $\partial_t \delta u^a$ with the aid of Eqs. (4.72) and (4.73) results in

$$\partial_t \delta u^a = -v_o^b \nabla_b \delta u^a - \delta u^b \nabla_b v_o^a - \frac{\rho_p}{\tilde{\rho}_p} \nabla^a \delta \tilde{\mu}_p - \frac{\tilde{\rho}_e}{\tilde{\rho}_p} \delta \left(\frac{\rho_e}{\tilde{\rho}_e} \nabla^a \tilde{\mu}_e \right) \quad (5.8a)$$

$$- \frac{\tilde{\rho}_e}{\tilde{\rho}_p} \delta \left(\frac{\rho_\mu}{\tilde{\rho}_e} \nabla^a \tilde{\mu}_\mu \right) - \frac{\tilde{s}_e}{\tilde{\rho}_p} \nabla^a \delta T_e - \nabla^a \delta \varphi + \frac{\rho_{np}}{\tilde{\rho}_p} [(\delta \vec{v}_n - \delta \vec{v}_p) \times \vec{\omega}_o]^a \quad (5.8b)$$

$$+ \frac{\rho_p}{\tilde{\rho}_p} \delta \left[\frac{1}{\rho_p} (\vec{\nabla} \times \vec{\lambda}_p) \times (\vec{\omega}_p + a_p \vec{B}) \right]^a + \frac{1}{\tilde{\rho}_p} \left(\frac{\delta \vec{J}}{c} \times \vec{B}_o \right)^a \quad (5.8c)$$

$$+ \frac{\sigma_o}{\tilde{\rho}_p} \left(\delta \vec{E} + \frac{\vec{v}_o}{c} \times \delta \vec{B} \right)^a. \quad (5.8d)$$

One advantage of introducing δu^a is that the terms in Eq. (5.8d) are proportional to σ_o . Since $\sigma_o \sim (\Omega \Omega_C / \omega_p^2) \sigma_p$ is exceedingly small (and in fact zero in nonrotating neutron stars), the terms in Eq. (5.8d) usually can be ignored.

The remaining task is to invert Eqs. (5.4) and (5.7). The result is

$$\delta v_e^a \approx \frac{\rho_{pp}}{\rho_p} \delta u^a + \frac{\rho_{np}}{\rho_p} \delta v_n^a - \frac{m_p}{e \rho_p} \delta J^a, \quad (5.9)$$

$$\delta v_p^a \approx \delta u^a - \frac{\tilde{\rho}_e \rho_{np}}{\rho_p \rho_p} \delta v_n^a + \frac{m_p \tilde{\rho}_e}{e \rho_p^2} \delta J^a. \quad (5.10)$$

In obtaining these equations, the following approximations have been made: First, $(\sigma_e / \sigma_p) \approx -1$ was used, ignoring a term smaller by the factor $\sigma_o / \sigma_p \sim (\Omega \Omega_C / \omega_p^2)$. Second, each of the coefficients of δu^a , δv_n^a , and δJ^a have been simplified by ignoring terms smaller by a factor of order $(\tilde{\rho}_e / \rho_p)$. The terms proportional to the current density δJ^a usually can be ignored also, since they are smaller than the other terms in Eqs. (5.9) and (5.10) by a factor of (ω_h / ω) . (Keeping the terms proportional to δJ^a leads to what is commonly called the Hall term; see the next section.)

Equations (5.9) and (5.10) reduce the set of independent velocities from (v_n^a, v_p^a, v_e^a) to (v_n^a, u^a) . The dynamical equations for this reduced set of variables will be useful to studies of the g-modes and p-modes.

Cyclotron-vortex Frequency

In Chapter 4 it was noted that a new mode, which has not been found by other authors, can be shown to exist in the superconducting-proton-degenerate-electron plasma. The new frequency associated with this mode has been named the cyclotron-vortex frequency. A straightforward derivation of this frequency is given in this section using the results of the last section. Also, Alfvén waves, which occur in ordinary plasmas, will be shown not to occur here, as was also noted in Chapter 4.

To begin with, consider uniform superconducting neutron-star matter. For simplicity assume that the matter is not rotating and that there is a uniform equilibrium magnetic induction which defines the z -axis, i.e., take $\Omega = 0$ and $B^a = B_o \hat{z}^a$, where $\hat{z}^a$ is the unit vector in the z direction. Next, look for plane wave solutions with the wave vector parallel to the z -axis, i.e., let $k^a = k \hat{z}^a$. The space-time dependence of all the perturbed quantities can then be taken to be completely specified by $\exp(ikz - i\omega t)$. Partial derivatives can be replaced by the expressions: $\partial_t \delta f^a \rightarrow -i\omega \delta f^a$, and in Cartesian coordinates $\nabla^a f^b \rightarrow \partial^a f^b \rightarrow ik^a f^b$. Making these substitutions the equations become purely algebraic. The equations also decouple into components which are parallel and transverse to the z -axis. The resulting set of equations, taken from the last section, for the transverse components are

$$ik(\hat{z} \times \delta \vec{E})^a = \frac{i\omega}{c} \delta B^a, \quad (5.11)$$

$$ik(\hat{z} \times \delta \vec{B})^a = \frac{4\pi}{c} \delta J^a, \quad (5.12)$$

$$\delta E^a = -\left(\frac{\delta \vec{v}_e}{c} \times B_o \hat{z}\right)^a, \quad (5.13)$$

$$\delta v_{n\perp}^a = 0, \quad (5.14)$$

$$-i\omega\delta u_{\perp}^a = \frac{ik}{\tilde{\rho}_p}[(\hat{z} \times \delta\vec{\lambda}_p) \times a_p B_o \hat{z}]^a + \frac{1}{\tilde{\rho}_p} \left(\frac{\delta\vec{J}}{c} \times B_o \hat{z} \right)^a. \quad (5.15)$$

In these equations the symbol $\perp$ indicates the projections of these vectors perpendicular to the z -axis. In terms involving cross products with $\hat{z}$ this symbol is dropped for convenience. Also, in Eq. (5.13) only the largest term in Eq. (5.5) has been kept, as is usual in the MHD limit. Equations (5.11) and (5.12) can then be used to give the useful result

$$\delta E^a = \frac{4\pi i\omega}{c^2 k^2} \delta J^a. \quad (5.16)$$

The next step will be to find a useful expression for the vortex force acting on $\delta u_{\perp}^a$. Using Eq. (4.39) it is easy to show that $\delta\lambda_p^a$ is given by

$$\delta\lambda_p^a = \left[\frac{2m_p}{h} \frac{\varepsilon_p}{\Omega_C} - \frac{1}{4\pi} \frac{m_p^2 c^2}{e^2} \right] [ik(\hat{z} \times \delta\vec{v}_p) + a_p \delta\vec{B}]^a. \quad (5.17)$$

The vortex force in Eq. (5.15) can then be simplified to become

$$\begin{aligned} \frac{ik}{\tilde{\rho}_p}[(\hat{z} \times \delta\vec{\lambda}_p) \times a_p B_o \hat{z}]^a &= \frac{\rho_p}{\tilde{\rho}_p}(\omega_{vp} - \omega_h)(\delta v_p \times \hat{z})^a + \frac{1}{\tilde{\rho}_p} \frac{\omega_{vp}}{\Omega_C} \frac{\omega_p^2}{c^2 k^2} \left(\frac{\delta\vec{J}}{c} \times B_o \hat{z} \right)^a \\ &\quad - \frac{1}{\tilde{\rho}_p} \left(\frac{\delta\vec{J}}{c} \times B_o \hat{z} \right)^a. \end{aligned} \quad (5.18)$$

Note that the last term on the right-hand side of Eq. (5.18) will cancel the last term on the right-hand side of Eq. (5.15). Since this is the term which usually gives rise to Alfvén waves, these waves do not occur in the superconducting plasma considered here.

Finally, a useful expression for $(\delta\vec{u} \times \hat{z})^a$ will be found. Using Eqs. (5.9) and (5.10) given in Chapter 4 and noting that $\delta v_{n\perp} = 0$ gives

$$\delta v_{e\perp}^a = \frac{\rho_{pp}}{\rho_p} \delta u_{\perp} - \frac{m_p}{e\rho_p} \delta J^a, \quad (5.19)$$

$$\delta v_{p\perp}^a = \delta u_{\perp} + \frac{m_p \tilde{\rho}_e}{e \rho_p^2} \delta J^a. \quad (5.20)$$

Using Eq. (5.19), Eq. (5.13) can then be expressed as

$$\delta E^a = -\frac{\rho_{pp}}{\rho_p} \left(\frac{\delta \vec{u}}{c} \times B_o \hat{z} \right)^a + \frac{m_p}{e \rho_p} \left(\frac{\delta \vec{J}}{c} \times B_o \hat{z} \right)^a. \quad (5.21)$$

The second term on the right-hand side of Eq. (5.21) is commonly called the Hall term. Equations (5.16) and (5.21) can then be used to find the useful expression

$$(\delta \vec{u} \times \hat{z})^a = -\frac{\rho_p}{\rho_{pp}} \frac{\omega_p^2}{c^2 k^2} \frac{i\omega}{\Omega_C^2} \frac{1}{\rho_p} \frac{\delta J^a}{c} B_o + \frac{\rho_p}{\rho_{pp}} \frac{1}{\Omega_C} \frac{1}{\rho_p} \left(\frac{\delta \vec{J}}{c} \times B_o \hat{z} \right)^a. \quad (5.22)$$

Substituting Eqs. (5.18), (5.20), and (5.22) into Eq. (5.15) and simplifying results in the following eigenvalue equation:

$$\left[\omega^2 - \omega_h(\omega_{vp} - \omega_h) - \frac{\rho_{pp}}{\rho_p} \Omega_C \omega_{vp} \right] (\delta \vec{J} \times \hat{z})^a + i\omega(\omega_{vp} - 2\omega_h) \delta J^a = 0. \quad (5.23)$$

Solving Eq. (5.23), the eigenvalues are given by

$$\omega^2 - \omega_h(\omega_{vp} - \omega_h) - \frac{\rho_{pp}}{\rho_p} \Omega_C \omega_{vp} = \pm \omega(\omega_{vp} - 2\omega_h), \quad (5.24)$$

and the components of the eigenvectors are related by $\delta J_x = \mp i \delta J_y$. The solution corresponds to a circularly polarized wave travelling along the z -axis. Equation (5.24) can be solved using the quadratic formula. The results, corresponding to the two sign choices in Eq. (5.24), are

$$\omega = \begin{cases} \pm \sqrt{\frac{\rho_{pp}}{\rho_p} \Omega_C \omega_{vp} - \frac{\omega_{vp}}{2}} + \omega_h, \\ \pm \sqrt{\frac{\rho_{pp}}{\rho_p} \Omega_C \omega_{vp} + \frac{\omega_{vp}}{2}} - \omega_h. \end{cases} \quad (5.25)$$

where terms smaller than those kept by at least a factor of ω_{vp}/Ω_C have been dropped. The first term in Eq. (5.25) has not been found by other authors. Since it depends on the cyclotron frequency Ω_C and the vortex oscillation frequency

ω_{vp} it is natural to call it the cyclotron-vortex frequency. As already discussed in Chapter 4, in neutron-star matter this frequency will be considerably larger than either the vortex oscillation frequency or the helicon frequency which also appear in Eq. (5.25). In a terrestrial superconductor the superfluid electrons and the normal electrons should also behave as a superconducting plasma. Thus the cyclotron-vortex frequency may be observable in terrestrial superconductors.

Ignoring Electromagnetic and Vortex Forces

The equations can be simplified further in many ways depending on the specific modes of interest. Here attention will be restricted to the large scale p-modes in rapidly rotating neutron stars. Specifically, the frequencies will now be restricted to be much larger than the frequency of the cyclotron-vortex mode and thus lie in the range

$$\omega_{cv} \ll \omega \ll \Omega_C, \omega_p. \quad (5.26)$$

The full advantage of using the variable u^a can now be exploited. For the frequency range in Eq. (5.26) the electromagnetic forces and the forces due to the underlying vortices in Eq. (5.8c) and (5.8d) can be dropped from the equation for δu^a . The condition for this to hold is that the largest of these forces, which is the vortex force proportional to λ_p^a , is smaller than the smallest of the remaining forces, which is the drag force between the neutrons and protons. It can be shown the ratio of these forces is roughly $(\omega_{cv}/\omega)(\omega_{cv}/\Omega)$. A sufficient condition that this ratio is small can then be taken to be $\Omega > \omega_{cv}$. This condition makes precise what is meant here by rapidly rotating. This inequality can be rearranged

to give a lower bound to the length scale of the oscillation by using Eq. (4.84) to relate ω_{CV} to the wavelength λ . The result is

$$\lambda > 10^5 \left[\left(\frac{m_p}{m_p^*} \right) \left(\frac{\varepsilon_p}{10^7 \text{ erg/cm}} \right) \left(\frac{10^{13} \text{ g/cm}^3}{\rho_p} \right) \left(\frac{B_o}{10^{12} \text{ G}} \right) \right]^{\frac{1}{2}} \left(\frac{10^2 \text{ s}^{-1}}{\Omega} \right) \text{ cm.} \quad (5.27)$$

Obviously, the lower bound on the wavelengths must be less than the typical neutron star radius, $R \sim 10^6 \text{ cm}$, if the above range is to be physically relevant. However, this bound need only be applied when the aim is to study the effects of rotation but ignore the vortex force proportional to λ_p^a .

Similarly, the force due to the neutron vortices (proportional to λ_n^a) can be dropped from Eq. (4.71) for the neutron velocity when $\Omega > \omega_{Vn}$, which is much less restrictive than $\Omega > \omega_{CV}$.

Thus, when the preceding conditions hold the complete set of equations become:

$$\partial_t \delta \rho_n + v_o^b \nabla_b \delta \rho_n + \nabla_b (\rho_{nn} \delta v_n^b + \rho_{np} \delta u^b) = 0, \quad (5.28)$$

$$\partial_t \delta \rho_p + v_o^b \nabla_b \delta \rho_p + \nabla_b (\rho_{pp} \delta u^b + \rho_{np} \delta v_n^b) = 0, \quad (5.29)$$

$$\partial_t \delta \rho_e + v_o^b \nabla_b \delta \rho_e + \nabla_b \left[\left(\frac{\rho_e}{\rho_p} \right) (\rho_{pp} \delta u^b + \rho_{np} \delta v_n^b) \right] = 0, \quad (5.30)$$

$$\partial_t \delta \rho_\mu + v_o^b \nabla_b \delta \rho_\mu + \nabla_b \left[\left(\frac{\rho_\mu}{\rho_p} \right) (\rho_{pp} \delta u^b + \rho_{np} \delta v_n^b) \right] = 0, \quad (5.31)$$

$$\partial_t \delta \tilde{s}_e + v_o^b \nabla_b \delta \tilde{s}_e + \nabla_b \left[\left(\frac{\tilde{s}_e}{\rho_p} \right) (\rho_{pp} \delta u^b + \rho_{np} \delta v_n^b) \right] = 0, \quad (5.32)$$

$$\begin{aligned} \partial_t \delta v_n^a = & -v_o^b \nabla_b \delta v_n^a - \delta v_n^b \nabla_b v_o^a - \nabla^a \delta \mu_n \\ & - \nabla^a \delta \varphi + \frac{\rho_{np}}{\rho_n} [(\delta \vec{u} - \delta \vec{v}_n) \times \vec{\omega}_o]^a, \end{aligned} \quad (5.33)$$

$$\begin{aligned} \partial_t \delta u^a = & -v_o^b \nabla_b \delta u^a - \delta u^b \nabla_b v_o^a - \nabla^a \delta \mu_p - \frac{\tilde{\rho}_e}{\rho_p} \delta \left(\frac{\rho_e}{\tilde{\rho}_e} \nabla^a \mu_e \right) \\ & - \frac{\tilde{\rho}_e}{\rho_p} \delta \left(\frac{\rho_\mu}{\tilde{\rho}_e} \nabla^a \mu_\mu \right) - \frac{\tilde{s}_e}{\rho_p} \nabla^a \delta T_e - \nabla^a \delta \varphi + \frac{\rho_{np}}{\rho_p} [(\delta \vec{v}_n - \delta \vec{u}) \times \vec{\omega}_o]^a, \end{aligned} \quad (5.34)$$

$$\nabla_b \nabla^b \delta\varphi = 4\pi G \delta\rho. \quad (5.35)$$

A couple of other approximations have been made to simplify these equations even further. The terms proportional to the current density δJ^a in Eqs. (5.9) and (5.10) have been ignored when using these equations to substitute for δv_p^a and δv_e^a , since they are smaller than the terms kept by a factor of (ω_h/ω) . Only the chemical potentials defined by Eq. (4.36) are kept in Eqs. (5.33) and (5.34). This ignores terms which are smaller by at least a factor of $(\omega_{cv}^2/\omega_s^2)$ in the full definitions of the chemical potentials, given by Eqs. (4.32) and (4.33). The mass currents in Eqs. (5.28) and (5.29) and the drag terms in Eqs. (5.33) and (5.34) have been simplified ignoring terms smaller by a factor of $(\tilde{\rho}_e/\rho_p)$ than those kept.

CHAPTER 6

GENERAL THEORY OF MUTUAL FRICTION

Introduction

Dissipation in superfluids is completely different from that in ordinary fluids in at least two important ways. (For a review of dissipation in superfluids see Sonin 1987; Putterman 1974.) First, a superfluid condensate acts like a zero temperature nondissipative perfect fluid. Thus, ordinary dissipative processes, such as viscosity, occur only in the excitations of the superfluid and in any other ordinary fluid present. Second, the scattering of particles off the quantized vortices, which can exist in a superfluid, leads to a form of dissipation known as mutual friction. Thus, mutual friction is a unique dissipative process which exists only in superfluids. The theory of mutual friction was first developed by Hall & Vinen (1956) in the context of superfluid ^4He . The theory was first given a phenomenological derivation by Bekarevich and Khalatnikov (1961). (See Sonin 1987 for a review of other theories of mutual friction.) However, mutual friction has not been studied previously in mixtures of superfluids (although Onuki 1983 has studied mutual friction in a mixture of nonsuperfluid ^3He and superfluid ^4He).

In this chapter the general theory of mutual friction for a mixture of N superfluids (each with an arbitrary electrical charge) plus an additional ordinary fluid is studied in detail. The discussion presented here is based on Mendell (1991b). The

approach of this chapter will be to build on the work of Chapter 3 in which generalized nondissipative forces were included in the theory. Here these generalized forces are extended to include mutual friction forces. The excitations of the superfluids will be ignored (a valid approximation except near the superfluid-transition temperatures and appropriate for neutron stars). In this case mutual friction will be due to the scattering of the additional ordinary fluid off the superfluid vortices. In the first part of this chapter the mutual friction forces will be found by following the phenomenological approach of Bekarevich and Khalatnikov (1961) (see also Chapter 2; Landau 1941; Khalatnikov 1957; Putterman 1974; Andreev and Bashkin 1976; Mendell 1991a,b). The resulting forces will contain a large number of coefficients. In the second part of this chapter these coefficients are related to parameters which can be determined by specific microphysical calculations by generalizing the analysis of Hall and Vinen (1956):

Mutual Friction Forces

In the phenomenological method of Bekarevich and Khalatnikov (1961) the general form of the mutual friction forces is found by requiring that the theory be consistent with conservation of energy (the first law of thermodynamics) and by requiring that the entropy production be nonnegative (the second law of thermodynamics). The first step is to introduce mutual friction forces into the equations. This is easily done by making the following extension of the generalized forces in Chapter 3:

$$F_{\alpha}^a \rightarrow F_{\alpha}^a + F_{\alpha(\text{MF})}^a, \quad (6.1)$$

$$F_e^a \rightarrow F_e^a + F_{e(\text{MF})}^a. \quad (6.2)$$

(Reminder: In these equations greek indices α, β, γ , etc., are used as species indices for each of the N species of superfluids. Sums over these indices are not intended unless explicitly specified. The index e refers to the ordinary fluid, e.g., the electron-muon fluid in neutron stars.) In this extension of the theory the forces on the left-hand sides of Eqs: (6.1) and (6.2) which appear in Chapter 3 are simply replaced by the forces on the right-hand sides. The mutual friction forces in these equations, $F_{\alpha(\text{MF})}^a$ and $F_{e(\text{MF})}^a$, are to be interpreted as the average forces transmitted between the ordinary fluid and the superfluid condensates due to the scattering to the ordinary fluid off the superfluid vortices. Since $F_{e(\text{MF})}^a$ is the reaction force acting on the ordinary fluid due to each of the condensates, it is natural to impose the following constraint on these forces:

$$\rho_e F_{e(\text{MF})}^a = - \sum_{\alpha} \rho_{\alpha} F_{\alpha(\text{MF})}^a, \quad (6.3)$$

where ρ_e and ρ_{α} are the mass densities of the ordinary fluid and the superfluids respectively. (Note that the "forces" in Chapter 3 and in this chapter actually have the units of acceleration.)

The next step is to generalize the entropy equation to include the effects of dissipation. The entropy is no longer conserved in this case. Rather, dissipation can increase the entropy of the system. To account for this an entropy production term Σ is introduced such that

$$\partial_t s_e + \nabla_b (s_e v_e^b) = \frac{\Sigma}{T}. \quad (6.4)$$

Since the focus here is on mutual friction, the heat flux, which could also be included in this equation, will be ignored in this chapter.

The first requirement of the phenomenological approach is now applied: consistency with conservation of energy. Making the substitutions specified in Eqs. (6.1)–(6.3), and including the entropy production term Σ in the entropy equation, the result for the time evolution of the energy density U is easily obtained from Eq. (3.13) in Chapter 3. The result is

$$\partial_t U + \nabla_b U^b = \Sigma + \sum_{\alpha} \rho_{\alpha} F_{\alpha(\text{MF})}^a X_{\alpha a}, \quad (6.5)$$

where the energy current U^a can be determined from Eqs. (3.14) and (3.55) in Chapter 3, and X_{α}^a is given by

$$X_{\alpha}^a \equiv \left[\sum_{\beta} \frac{1}{\rho_{\alpha}} \rho_{\alpha\beta} v_{\beta}^a + \frac{1}{\rho_{\alpha}} (\vec{\nabla} \times \vec{\lambda}_{\alpha})^a \right] - v_e^a. \quad (6.6)$$

The mass density matrix in Eq. (6.6), $\rho_{\alpha\beta}$, is discussed in Chapter 3 and Chapter 4 (see also Andreev and Bashkin 1976). (The off-diagonal terms in this matrix account for the possibility of Fermi liquid “drag” effects between the different superfluid species.) Energy conservation will hold then when the right-hand side of Eq. (6.5) vanishes. This gives

$$\Sigma = - \sum_{\alpha} \rho_{\alpha} F_{\alpha(\text{MF})}^a X_{\alpha a} \geq 0. \quad (6.7)$$

The inequality in Eq. (6.7) is due to the second requirement of the phenomenological approach: nonnegative entropy production. This gives a strong constraint on the mutual friction forces $F_{\alpha(\text{MF})}^a$. In particular, these forces must be written in terms of the relative mass currents X_{α}^a to ensure nonnegative entropy production. Furthermore, it is natural to assume that the mutual friction forces should also be proportional to the number of vortices per unit area: $n_{v\alpha}$ for each species α . This vortex density can be related to the smooth-averaged vorticity

of the superfluids ω_α^a and to the magnetic induction B^a via the smooth-averaged curl of the London equation [see Chapter 2 and Chapter 3; Eq. (3.1)] by

$$W_\alpha^a \equiv \omega_\alpha^a + a_\alpha B^a = \frac{h}{M_\alpha} n_{v\alpha} v_\alpha^a. \quad (6.8)$$

Thus, assuming that the mutual friction forces depend on the vortex densities is equivalent to assuming that these forces depend on the vector fields W_α^a defined by this equation.

Taking into account the above considerations, the general form of the mutual friction forces which can be constructed from the vector fields X_α^a and W_α^a , is given by

$$F_{\alpha(\text{MF})}^a = - \sum_{\beta\gamma} \left[A_{\alpha\beta\gamma} (W_\gamma^b X_{\beta b}) \frac{W_\gamma^a}{|\vec{W}_\gamma|} + B'_{\alpha\beta\gamma} (\vec{X}_\beta \times \vec{W}_\gamma)^a + B_{\alpha\beta\gamma} W_\gamma^{ab} X_{\beta b} \right], \quad (6.9)$$

where

$$W_\gamma^{ab} = |\vec{W}_\gamma| \left[g^{ab} - \frac{W_\gamma^a W_\gamma^b}{|\vec{W}_\gamma|^2} \right]. \quad (6.10)$$

The mutual friction coefficients $A_{\alpha\beta\gamma}$, $B'_{\alpha\beta\gamma}$, and $B_{\alpha\beta\gamma}$ are dimensionless functions. Equation (6.9) is just the natural generalization of the mutual friction forces in Eq. (18) of Bekarevich and Khalatnikov (1961), and the coefficients $B'_{\alpha\beta\gamma}$, and $B_{\alpha\beta\gamma}$ are just a natural extension of the coefficients B' and B given by Hall and Vinen (1956). Note that it has not been assumed a priori that the coefficients are diagonal, since from the phenomenological point of view off-diagonal terms could appear as a result of Fermi liquid effects, just as they do in the mass density matrix $\rho_{\alpha\beta}$.

Each of the mutual friction coefficients in Eq. (6.10) has to be determined by a microscopic calculation. (This is discussed in the second part of this chapter.)

However, all of the coefficients are not independent but are restricted by,

$$\Sigma = \sum_{\alpha\beta\gamma} \left[\rho_{\alpha} A_{\alpha\beta\gamma} \frac{(W_{\gamma}^a X_{\alpha a})(W_{\gamma}^b X_{\beta b})}{|\vec{W}_{\gamma}|} + \rho_{\alpha} B'_{\alpha\beta\gamma} X_{\alpha a} (\vec{X}_{\beta} \times \vec{W}_{\gamma})^a \right. \\ \left. + \rho_{\alpha} B_{\alpha\beta\gamma} W_{\gamma}^{ab} X_{\alpha a} X_{\beta b} \right] \geq 0. \quad (6.11)$$

A set of sufficient conditions under which the inequality holds is

$$\frac{1}{2}(\rho_{\alpha} A_{\alpha\beta\gamma} + \rho_{\beta} A_{\beta\alpha\gamma}) = \sum_i a_{\gamma}^i A_{\alpha\gamma}^i A_{\beta\gamma}^i ; \quad a_{\gamma}^i \geq 0, \quad (6.12)$$

$$\rho_{\alpha} B'_{\alpha\beta\gamma} = \rho_{\beta} B'_{\beta\alpha\gamma} \quad (6.13)$$

$$\frac{1}{2}(\rho_{\alpha} B_{\alpha\beta\gamma} + \rho_{\beta} B_{\beta\alpha\gamma}) = \sum_i b_{\gamma}^i B_{\alpha\gamma}^i B_{\beta\gamma}^i ; \quad b_{\gamma}^i \geq 0, \quad (6.14)$$

where, for each γ , a_{γ}^i and b_{γ}^i are the eigenvalues and $A_{\alpha\gamma}^i$ and $B_{\alpha\gamma}^i$ are the eigenvectors of the symmetric matrices on the left-hand sides of Eq. (6.12) and (6.14), respectively. The mutual friction coefficients are now discussed further in the next part of this chapter.

Mutual Friction Coefficients

In this section the mutual friction coefficients are now related to more easily calculable microphysical parameters. The method will be to generalize the derivation of the mutual friction coefficients first given by Hall and Vinen (1956). In their method, the mutual friction coefficients are related to microphysical quantities by examining the forces per unit length on the vortices. This approach, which has proved to be extremely useful in studies of ^4He , is generalized here to the case of N superfluids.

The first force to be discussed is taken to be the force due to the scattering of the ordinary fluid off the vortices. Hall and Vinen (1956) argue that the average

of this force acting on the vortices of species γ (and adapted to the notation of this thesis) can be written as

$$f_{e\gamma}^a = C_\gamma(v_e^a - v_{L\gamma}^a). \quad (6.15)$$

In this equation v_e^a is the smooth-averaged velocity of the ordinary fluid and $v_{L\gamma}^a$ is the local average velocity of the vortices of species γ . Equation (6.15) then states that the average force per unit length on the vortices due to the scattering of the ordinary fluid off the vortices is proportional to the relative velocity between the ordinary fluid and the vortices. The usefulness of Eq. (6.15) is that the proportionality factor C_γ can be calculated from the microphysics of the problem. For example, the factor C_γ has been related to the scattering cross section for rotons by Hall and Vinen (1956) in the case of ^4He . Of more interest here is that dissipation due to the scattering will cause the ordinary fluid to relax to the vortices of species γ in a characteristic relaxation time $\tau_{V\gamma}$. Thus, the factor C_γ is related to the relaxation time by approximately

$$C_\gamma \approx \frac{\rho_{\text{eff}}}{n_{V\gamma}\tau_{V\gamma}}, \quad (6.16)$$

where ρ_{eff} is the effective mass density of the fluid which relaxes to the vortices, and the relaxation time $\tau_{V\gamma}$ can be determined by a microscopic calculation based on the physics of the scattering process (e.g., see the next chapter).

Hall and Vinen (1956) also argue that the condensates can transmit a force per unit length to moving vortices which has the form of the Magnus force (see Chapter 2; Sonin 1987). Also, Nozières and Vinen (1966) show that this type of force acts on the vortices of charge superfluids as well. (Alternative forces have been proposed, leading to different theories of mutual friction. See Sonin

1987 for references. However, Jones 1987 has argued that the Magnus form of the force is the correct one for neutron stars.) The Magnus force is given by $f_M^a = \rho_s[(\vec{v}_s - \vec{v}_L) \times \vec{K}]^a$, where K^a is the circulation of the vortices, $v_s^a - v_L^a$ is the average relative velocity between the superfluid condensate and the vortices, and ρ_s is the mass density of the condensate. Physically, this force can be understood to arise via the pressure gradient which is created across the vortices (due to the Bernoulli effect) when there is a relative velocity $v_s^a - v_L^a$. The resulting force is perpendicular to both this relative velocity and the direction of the circulation vector K^a . Generalizing this force to a mixture of N superfluids (and allowing for the possibility of the drag effect between the species) the Magnus-type force due to the condensate of species α acting on the vortices of species γ can be written as

$$f_{\alpha\gamma}^a = \rho_\alpha \sum_\beta R_{\alpha\beta\gamma} \left[\left(\frac{1}{\rho_\beta} \vec{M}_{s\beta} - \vec{v}_{L\gamma} \right) \times \vec{K}_\gamma \right]^a, \quad (6.17)$$

where $K_\gamma^a \equiv W_\gamma^a / n_{V\gamma}$, and the superfluid mass currents $M_{s\alpha}^a$ in this equation are given by

$$M_{s\alpha}^a = \sum_\beta \rho_{\alpha\beta} v_\beta^a. \quad (6.18)$$

The dimensionless coefficients $R_{\alpha\beta\gamma}$ can be determined by repeating the microscopic derivation of the Magnus force (e.g., as described by Sonin 1987) for the fluid mixture under study here. This derivation has not yet been undertaken. However, a physically reasonable expression for these coefficients will be given at the end of this chapter.

It is now assumed that the net force acting on the vortices must vanish. This treats the vortices as massless objects. Note that the mass per unit length of a vortex is not zero, but given approximately by the small quantity $\rho_\alpha \pi \xi_\alpha^2$, where ξ_α

is the radius of the vortex core (and also the coherence length of the superfluid). Thus the net force on the vortices, approximately specified by $\rho_\alpha \pi \xi_\alpha^2 (dv_{L\alpha}^a/dt)$, can be ignored as long as it is small compared with $f_{e\gamma}^a$. For an oscillation mode with frequency ω this condition is given by $\rho_\alpha \pi \xi_\alpha^2 \omega v_{L\alpha}^a \ll f_{e\alpha}^a$. Using Eq. (6.15) and $C_\gamma \approx \rho_{\text{eff}}/(n_{v\gamma} \tau_{v\gamma})$ gives $\omega \ll (1/\tau_{v\gamma})(\rho_{\text{eff}}/\rho_\alpha)(1/n_{v\gamma} \pi \xi_\alpha^2)(\delta v_e/\delta v_{L\gamma} - 1) \sim 10^{14} \text{s}^{-1}$. The numerical estimate is typical for neutron star oscillations. Thus, for the frequencies of interest in this thesis, the inertia of the vortices can be ignored. The condition that the net force vanish on the vortices of species γ is then given by

$$f_{e\gamma}^a + \sum_{\alpha} f_{\alpha\gamma}^a = 0. \quad (6.18)$$

Equations (6.15)–(6.18) form the fundamental equations of the Hall and Vinen (1956) analysis. This set of equations can be solved to find the mutual friction force acting on a condensate of species α . The first step is to substitute Eqs. (6.15) and (6.17) into Eq. (6.18) and solve for $v_{L\gamma}^a$. The result is

$$\begin{aligned} v_{L\gamma}^a = \frac{1}{D_\gamma} \Big\{ & C_\gamma^2 v_e^a + \sum_{\mu\nu} \rho_\mu R_{\mu\nu\gamma} C_\gamma \left[\left(\frac{1}{\rho_\nu} \vec{M}_{s\nu} - \vec{v}_e \right) \times \vec{K}_\gamma \right]^a \\ & + \sum_{\mu\nu\sigma\delta} \rho_\mu \rho_\sigma R_{\mu\nu\gamma} R_{\sigma\delta\gamma} K_\gamma^2 \frac{1}{\rho_\delta} M_{s\delta}^a \\ & - \sum_{\mu\nu\sigma\delta} \rho_\mu \rho_\sigma R_{\mu\nu\gamma} R_{\sigma\delta\gamma} \left[K_{\gamma b} (M_{s\delta}^b + v_{L\gamma}^b) \right] K_\gamma^a \Big\}, \end{aligned} \quad (6.19)$$

where

$$D_\gamma = C_\gamma^2 + \sum_{\mu\nu\sigma\delta} \rho_\mu \rho_\sigma R_{\mu\nu\gamma} R_{\sigma\delta\gamma} K_\gamma^2. \quad (6.20)$$

Substituting Eq. (6.19) into Eq. (6.17) then gives, for the force per unit length due to condensate α acting on vortex species γ ,

$$f_{\alpha\gamma}^a = \rho_\alpha \sum_\beta R_{\alpha\beta\gamma} \left\{ - \sum_{\mu\nu} \frac{\rho_\mu R_{\mu\nu\gamma} C_\gamma}{D_\gamma} \left\{ \left[\left(\frac{1}{\rho_\nu} \vec{M}_{s\nu} - \vec{v}_e \right) \times \vec{K}_\gamma \right] \times \vec{K}_\gamma \right\}^a \right. \\ \left. + \left[\left(\frac{1}{\rho_\beta} \vec{M}_{s\beta} - \sum_{\mu\nu\sigma\delta} \frac{\rho_\mu \rho_\sigma R_{\mu\nu\gamma} R_{\sigma\delta\gamma}}{\rho_\delta D_\gamma} \vec{M}_{s\delta} \right) \times \vec{K}_\gamma \right]^a \right. \\ \left. - \frac{C_\gamma^2}{D_\gamma} (\vec{v}_e \times \vec{K}_\gamma)^a \right\}. \quad (6.21)$$

The total force acting on the condensate of species α , assumed to be the mutual friction force $F_{\alpha(\text{MF})}^a$, is then given by the forces opposite to those in Eq. (6.21) summed over all vortex species, i.e.,

$$F_{\alpha(\text{MF})}^a = - \sum_\gamma \frac{n_{V\gamma}}{\rho_\alpha} f_{\alpha\gamma}^a. \quad (6.22)$$

The factor $n_{V\gamma}/\rho_\alpha$ appears in this equation since $F_{\alpha(\text{MF})}^a$ has the units of acceleration and $f_{\alpha\gamma}^a$ has the units of force per unit length. After simplifying, the result is

$$F_{\alpha(\text{MF})}^a = - \sum_{\beta\gamma} \left[\frac{R_{\alpha\beta\gamma} C_\gamma^2}{D_\gamma} (\vec{X}_\beta \times \vec{W}_\gamma)^a + \sum_{\mu\nu} \frac{\rho_\mu R_{\alpha\nu\gamma} R_{\mu\beta\gamma} |\vec{K}_\gamma| C_\gamma}{D_\gamma} W_\gamma^{ab} X_{\beta b} \right] \quad (6.23a)$$

$$+ \sum_{\beta\gamma\mu\nu\sigma\delta} \frac{\rho_\mu \rho_\sigma R_{\mu\nu\gamma}}{D_\gamma} (R_{\alpha\delta\gamma} R_{\sigma\beta\gamma} - R_{\alpha\beta\gamma} R_{\sigma\delta\gamma}) \left(\frac{K_\gamma^2}{\rho_\beta} \vec{M}_{s\beta} \times \vec{W}_\gamma \right)^a. \quad (6.23b)$$

Note that in obtaining this equation the approximation, $(1/\rho_\beta) \vec{M}_{s\beta}^a - v_e^a \approx X_\beta^a$, has been used. This ignores the term due to the elastic properties of the vortices proportional to $\vec{\nabla} \times \vec{\lambda}_\beta$ in Eq. (6.6). This is consistent with ignoring the inertia of the vortices and the assumption of Eq. (6.18).

The mutual friction coefficients can now be determined in terms of microphysical parameters C_γ and $R_{\alpha\beta\gamma}$. Comparing Eqs. (6.23a) and (6.9), the mutual friction coefficients can be clearly identified. The results are

$$A_{\alpha\beta\gamma} = 0, \quad (6.24)$$

$$B'_{\alpha\beta\gamma} = R_{\alpha\beta\gamma} \frac{C_\gamma^2}{D_\gamma}, \quad (6.25)$$

$$B_{\alpha\beta\gamma} = \frac{|\vec{K}_\gamma| C_\gamma}{D_\gamma} \sum_{\nu\mu} \rho_\mu R_{\alpha\nu\gamma} R_{\mu\beta\gamma}. \quad (6.26)$$

The term in Eq. (6.23b), which depends on $M_{s\alpha}^a$ instead of X_α^a , has been ignored. This is consistent if

$$\sum_{\mu\nu\sigma\delta} \rho_\mu \rho_\sigma R_{\mu\nu\gamma} (R_{\alpha\delta\gamma} R_{\sigma\beta\gamma} - R_{\alpha\beta\gamma} R_{\sigma\delta\gamma}) = 0. \quad (6.27)$$

Alternatively, it could be argued that the term in Eq. (6.23b) should not be considered a part of the mutual friction force. Instead it could be interpreted as a drag force between the superfluid species which should be ignored when comparing Eqs (6.23a) and (6.23b) with Eq. (6.9).

The expressions for $B'_{\alpha\beta\gamma}$ and $B_{\alpha\beta\gamma}$ can be simplified by making the assumption that $R_{\alpha\beta\gamma}$ is diagonal and that its diagonal values equal unity, i.e., $R_{\alpha\beta\gamma} = \delta_{\alpha\beta} \delta_{\alpha\gamma}$. This assumption is plausible for the following reasons. First, this gives the correct result in the case of a single superfluid, and it is a natural generalization of this case to assume that the mutual friction force on each species depends only on the vortices of that species and the relative mass current between that species and the ordinary fluid. Second, the diagonal form for $R_{\alpha\beta\gamma}$ satisfies

the condition given by equation (6.27). Then the independent coefficients are reduced to just two, which can be specified as B'_α and B_α . These are then given by

$$B'_\alpha = \frac{C_\alpha^2}{C_\alpha^2 + \rho_\alpha^2 K_\alpha^2}, \quad B_\alpha = \frac{\rho_\alpha |\vec{K}_\alpha| C_\alpha}{C_\alpha^2 + \rho_\alpha^2 K_\alpha^2}. \quad (6.28)$$

CHAPTER 7

DISSIPATIVE EFFECTS IN SUPERFLUID NEUTRON STARS

Introduction

The purpose of this chapter is to extend the superfluid hydrodynamic equations which describe neutron stars, developed in Chapters 4 and 5, to include dissipative effects. The discussion presented here is based on Mendell (1991b). Dissipation in a superfluid is completely different from that in an ordinary fluid, as explained in the last chapter. In the last chapter the general theory of mutual friction was derived for an arbitrary mixture of superfluids. In the case of a neutron star, the term "mutual friction" will be used to refer to the scattering of electrons off the neutron and proton vortices. In particular, Alpar, Langer, and Sauls (1984) have shown that this scattering is primarily due to the magnetic flux of these vortices (see also Sauls, Stein, and Serene 1982). (The neutron vortices carry a substantial magnetic flux due to the Fermi liquid drag effect between the neutrons and protons. This result is derived in Alpar et al. 1984 and is discussed in Chapter 4.)

The first goal of this chapter is to include mutual friction forces in the equations of motion. Previous studies of neutron stars have investigated the effects of superfluidity on viscous dissipation, but have not included the effects of mutual friction (Cutler and Lindblom 1987, Cutler, Lindblom, and Splinter 1990, Ipser

and Lindblom 1991). (Also, previously Mineev 1975 has considered dissipation in mixtures of superfluids, not including mutual friction.) Here, the general form of the mutual friction forces are derived following the phenomenological approach of Bekarevich and Khalatnikov (1961). The low-frequency-long-wavelength (LFLW) limit of these forces is taken. The dissipative coefficients in these forces are related to microphysical quantities by generalizing the method of Hall and Vinen (1956). Specifically, the remaining relevant coefficients are related to the velocity relaxation time between the charged fluids and the neutron vortices as determined by Alpar et al. (1984) (see also Sauls et al. 1982). On the other hand, outside the vortices the main dissipative process is due to electron-electron scattering. This is incorporated into the theory by including viscous and thermal conductivity terms in the equations for the electron-muon fluid.

The second goal of this chapter is to estimate the effect of mutual friction on the damping times of p-modes. An energy functional is constructed which determines the damping times due to viscosity, thermal conductivity, and mutual friction. It is argued generically that the mutual friction damping times are much shorter than the other damping times for angular velocities of rotation $\Omega > 100\text{s}^{-1}(10^7\text{K}/T)^2$, for length scales of the order of a typical neutron star radius $R \sim 10^6\text{cm}$, and for temperatures in the range $10^7\text{K} \leq T < T_c$, where $T_c \sim 10^9\text{K}$ is the superfluid-transition temperature. The effect of this on the gravitational-radiation instability known to exist in the $l = m$ p-modes is discussed in Chapter 9.

Dissipative Effects

Dissipative Forces, Including Mutual Friction

The equations of motion are generalized here to include the effects of dissipation. In particular the focus will be on determining the mutual friction forces which act between the neutron and proton condensates and the electron-muon fluid. The method used here follows the approach of Bekarevich and Khalatnikov (1961). In this method the form of the mutual friction forces is determined phenomenologically by requiring that the equations be consistent with energy conservation and that the entropy production be nonnegative. The derivation is similar to that given in Chapter 6, except dissipation due to electron viscosity and thermal conductivity is also included. The results of Chapter 6 will be used to determine the mutual friction coefficients.

The first step is to include dissipative forces in the velocity equations. The equations for the neutron and proton velocities are altered as follows:

$$\partial_t v_n^a = \dots + F_{n(\text{MF})}^a, \quad (7.1)$$

$$\partial_t v_p^a = \dots + F_{p(\text{MF})}^a. \quad (7.2)$$

In these equations the ellipses refer to the nondissipative forces that can be found in Eqs. (4.13) and (4.14) of Chapter 4. Here, the forces, $F_{n(\text{MF})}^a$ and $F_{p(\text{MF})}^a$, are included. These forces represent the mutual friction forces acting on the neutron and proton condensates.

Next, the equation for the electron-muon velocity is altered as follows:

$$\partial_t v_e^a = \dots + F_{e(\text{MF})}^a - \frac{1}{\tilde{\rho}_e} \nabla_b \tau^{ab}. \quad (7.3)$$

In this case the ellipses refer to the nondissipative forces which can be found in Eq. (4.15) of Chapter 4. The mutual friction force, $F_{e(\text{MF})}^a$, acting on the electron-muon fluid is just the reaction force due to the scattering of this fluid off the vortices of each of the condensates. Thus it is natural to impose the constraint

$$\tilde{\rho}_e F_{e(\text{MF})}^a = -\left(\rho_n F_{n(\text{MF})}^a + \rho_p F_{p(\text{MF})}^a\right). \quad (7.4)$$

(Note that the mutual friction "forces" actually have the units of acceleration.) Furthermore, the viscous forces acting on the electron-muon fluid are accounted for by the viscous stress tensor, τ^{ab} . Assuming the standard equation for this tensor, it is given by (e.g., see Putterman 1974).

$$\tau^{ab} = -\eta_e [\nabla^a v_e^b + \nabla^b v_e^a - \frac{2}{3} g^{ab} \nabla_c v_e^c] - \zeta_e g^{ab} \nabla_c v_e^c. \quad (7.5)$$

(Reminder: in this equation g^{ab} is the inverse of the three dimensional Euclidean metric g_{ab} , i.e., in Cartesian coordinates or in any orthonormal basis g^{ab} and g_{ab} are each just the identity matrix.) The shear viscosity, η_e , and the bulk viscosity, ζ_e , of the electron-muon fluid are positive thermodynamic functions which must be determined from microscopic theory.

Another modification from Chapter 4 is that the entropy of the electron-muon fluid is no longer conserved, but instead a term for the entropy production, Σ , is included. Also, the heat flux, q^b , is included. This gives

$$\partial_t \tilde{s}_e + \nabla_b \left[\tilde{s}_e v_e^b + \frac{q^b}{T_e} \right] = \frac{\Sigma}{T_e}. \quad (7.6)$$

(Note that the superfluid condensates carry no entropy and the excitations of the neutrons and protons are ignored in this chapter, as in Chapters 4 and 5.)

Assuming the heat flux follows the standard equation (e.g., see Putterman 1974) gives

$$q^a = -\kappa_e \nabla^a T_e; \quad (7.7)$$

where the thermal conductivity, κ_e , of the electron-muon fluid is a positive thermodynamic function which must be determined from microscopic theory.

This leaves unspecified only the mutual friction forces, $F_{n(\text{MF})}^a$ and $F_{p(\text{MF})}^a$. These forces are now determined by requiring that the equations of motion be consistent with conservation of energy and that the entropy production be non-negative.

The partial time derivative of the total energy density U [see Eq. (4.30) in Chapter 4] can be computed with the aid of Eq. (4.3)–(4.67), (4.13)–(4.24), in Chapter 4 and the modifications to those equations given here by Eqs. (7.1)–(7.7). The results can be expressed in the following form:

$$\begin{aligned} \partial_t U + \nabla_b U^b = & \Sigma - 2\eta_e \Theta^{ab} \Theta_{ab} - \zeta_e \Theta^2 - \frac{\kappa_e}{T_e} (\nabla^b T_e) (\nabla_b T_e) \\ & + \rho_n \left[\frac{\rho_{nn}}{\rho_n} v_{nb} + \frac{\rho_{np}}{\rho_n} v_{pb} + \frac{1}{\rho_n} (\vec{\nabla} \times \vec{\lambda}_n)_b - v_{eb} \right] F_{n(\text{MF})}^b \\ & + \rho_p \left[\frac{\rho_{pp}}{\rho_p} v_{pb} + \frac{\rho_{np}}{\rho_p} v_{nb} + \frac{1}{\rho_p} (\vec{\nabla} \times \vec{\lambda}_p)_b - v_{eb} \right] F_{p(\text{MF})}^b, \end{aligned} \quad (7.8)$$

where the energy current, U^b , is given by

$$\begin{aligned} U^a = & v_{eb} \tau^{ab} + q^a + \left[\rho_e \tilde{\mu}_e + \rho_\mu \tilde{\mu}_\mu + \tilde{s}_e T_e + \frac{1}{2} \tilde{\rho}_e v_e^2 + \tilde{\rho}_e \varphi \right] v_e^a \\ & + \left[\rho_{nn} (\tilde{\mu}_n + \frac{1}{2} v_n^2) + \rho_{np} (\tilde{\mu}_p + \frac{1}{2} v_p^2) + \rho_n \varphi \right] v_n^a \\ & + \left[\rho_{pp} (\tilde{\mu}_p + \frac{1}{2} v_p^2) + \rho_{np} (\tilde{\mu}_n + \frac{1}{2} v_n^2) + \rho_p \varphi \right] v_p^a \\ & + \left\{ \vec{\lambda}_n \times \left[\vec{F}_{n(\text{MF})} + \frac{1}{\rho_n} (\rho_{nn} \vec{v}_n + \rho_{np} \vec{v}_p + \vec{\nabla} \times \vec{\lambda}_n) \times \vec{\omega}_n \right] \right\}^a \\ & + \left\{ \vec{\lambda}_p \times \left[\vec{F}_{p(\text{MF})} + \frac{1}{\rho_p} (\rho_{pp} \vec{v}_p + \rho_{np} \vec{v}_n + \vec{\nabla} \times \vec{\lambda}_p) \times (\vec{\omega}_p + a_p \vec{B}) \right] \right\}^a \\ & + \frac{c}{4\pi} (\vec{E} \times \vec{H})^a + \frac{1}{8\pi G} \left[\varphi \nabla^a \partial_t \varphi - (\partial_t \varphi) \nabla^a \varphi \right], \end{aligned} \quad (7.9)$$

and the shear, Θ^{ab} , and expansion, Θ , are given by

$$\Theta^{ab} = \frac{1}{2}[\nabla^a v_e^b + \nabla^b v_e^a - \frac{2}{3}g^{ab}\nabla_c v_e^c], \quad (7.10)$$

$$\Theta = \nabla_c v_e^c. \quad (7.11)$$

(The other quantities in these equations are defined in Chapter 4.) The requirement that energy be conserved is satisfied when the right-hand side of Eq. (7.8) vanishes, i.e., when

$$\begin{aligned} \Sigma = & 2\eta_e \Theta^{ab} \Theta_{ab} + \zeta_e \Theta^2 + \frac{\kappa_e}{T_e} (\nabla^b T_e)(\nabla_b T_e) \\ & - \rho_n \left[\frac{\rho_{nn}}{\rho_n} v_{nb} + \frac{\rho_{np}}{\rho_n} v_{pb} + \frac{1}{\rho_n} (\vec{\nabla} \times \vec{\lambda}_n)_b - v_{eb} \right] F_{n(\text{MF})}^b \\ & - \rho_p \left[\frac{\rho_{pp}}{\rho_p} v_{pb} + \frac{\rho_{np}}{\rho_p} v_{nb} + \frac{1}{\rho_p} (\vec{\nabla} \times \vec{\lambda}_p)_b - v_{eb} \right] F_{p(\text{MF})}^b. \end{aligned} \quad (7.12)$$

The requirement that the entropy production Σ be nonnegative for every fluid state then provides a strong constraint on the form of the mutual friction forces given by [compare with Eq. (6.7)]

$$\begin{aligned} - \left\{ \rho_n \left[\frac{\rho_{nn}}{\rho_n} v_{nb} + \frac{\rho_{np}}{\rho_n} v_{pb} + \frac{1}{\rho_n} (\vec{\nabla} \times \vec{\lambda}_n)_b - v_{eb} \right] F_{n(\text{MF})}^b \right. \\ \left. + \rho_p \left[\frac{\rho_{pp}}{\rho_p} v_{pb} + \frac{\rho_{np}}{\rho_p} v_{nb} + \frac{1}{\rho_p} (\vec{\nabla} \times \vec{\lambda}_p)_b - v_{eb} \right] F_{p(\text{MF})}^b \right\} \geq 0. \end{aligned} \quad (7.13)$$

The terms multiplying the mutual friction forces in this expression represent the relative velocities between the mass flow of the condensates and the electron-muon fluid, with "corrections" for the drag effect (via the mass density matrix ρ_{nn} , ρ_{np} , and ρ_{pp}) and for the elastic properties of the vortices (via the vector fields λ_n^a and λ_p^a). The mutual friction forces must be expressible in terms of these relative velocities in order to form a general quadratic form which then satisfies the required inequality. Furthermore, mutual friction should depend on the vortex densities, n_{vn} and n_{vp} . The vortex densities are proportional to the smooth-averaged vector

fields ω_n^a and $\omega_p^a + a_p B^a$ (where ω_n^a and ω_p^a are the smooth-averaged vorticities of the neutrons and protons, respectively, and B^a is the smooth-averaged magnetic induction; see Chapter 4). Thus it is natural to assume that the mutual friction forces should depend on these vector fields as well.

Making the reasonable assumption that the mutual friction force acting on each condensate depends only on the vortex densities (via ω_n^a and $\omega_p^a + a_p B^a$) and relative velocities (discussed above) of the corresponding species, general expressions for the mutual friction forces are given by [compare with Eq. (6.9)]

$$\begin{aligned}
 F_{n(\text{MF})}^a = & -A_n \left[\left(\frac{\rho_{nn}}{\rho_n} v_n^b + \frac{\rho_{np}}{\rho_n} v_p^b + \frac{1}{\rho_n} (\vec{\nabla} \times \vec{\lambda}_n)^b - v_e^b \right) \vec{\omega}_{nb} \right] \frac{\omega_n^a}{|\vec{\omega}_n|} \\
 & - B'_n \left[\left(\frac{\rho_{nn}}{\rho_n} \vec{v}_n + \frac{\rho_{np}}{\rho_n} \vec{v}_p + \frac{1}{\rho_n} \vec{\nabla} \times \vec{\lambda}_n - \vec{v}_e \right) \times \vec{\omega}_n \right]^a \\
 & - B_n \left\{ \vec{\omega}_n \times \left[\left(\frac{\rho_{nn}}{\rho_n} \vec{v}_n + \frac{\rho_{np}}{\rho_n} \vec{v}_p + \frac{1}{\rho_n} \vec{\nabla} \times \vec{\lambda}_n - \vec{v}_e \right) \times \frac{\vec{\omega}_n}{|\vec{\omega}_n|} \right] \right\}^a,
 \end{aligned} \tag{7.14}$$

$$\begin{aligned}
 F_{p(\text{MF})}^a = & -A_p \left[\left(\frac{\rho_{pp}}{\rho_p} v_p^b + \frac{\rho_{pn}}{\rho_p} v_n^b + \frac{1}{\rho_p} (\vec{\nabla} \times \vec{\lambda}_p)^b - v_e^b \right) W_{pb} \right] \frac{W_p^a}{|\vec{W}_p|} \\
 & - B'_p \left[\left(\frac{\rho_{pp}}{\rho_p} \vec{v}_p + \frac{\rho_{pn}}{\rho_p} \vec{v}_n + \frac{1}{\rho_p} \vec{\nabla} \times \vec{\lambda}_p - \vec{v}_e \right) \times \vec{W}_p \right]^a \\
 & - B_p \left\{ \vec{W}_p \times \left[\left(\frac{\rho_{pp}}{\rho_p} \vec{v}_p + \frac{\rho_{pn}}{\rho_p} \vec{v}_n + \frac{1}{\rho_p} \vec{\nabla} \times \vec{\lambda}_p - \vec{v}_e \right) \times \frac{\vec{W}_p}{|\vec{W}_p|} \right] \right\}^a,
 \end{aligned} \tag{7.15}$$

where $W_p^a = \omega_p^a + a_p B^a$. These equations are just the natural generalization of those derived by Bekarevich and Khalatnikov (1961) in the case of ^4He . That these forces result in a nonnegative quadratic form for the entropy production is easily verified by direct substitution into the inequality (7.13). (Note, however, that a more general expression for the mutual friction forces is possible. This is given in the Chapter 6, where "cross terms" between species are included. The microscopic calculation needed to determine whether such cross terms are necessary has not yet

been undertaken and is beyond the scope of the smooth-averaged hydrodynamics considered in this thesis.)

Finally, the mutual friction coefficients A_n , B'_n , B_n , A_p , B'_p , and B_p in Eqs. (7.14) and (7.15) are positive functions which need to be determined. These coefficients were related to more easily calculated microphysical parameters in Chapter 6 by extending the analysis of Hall and Vinen (1956) to a mixture of superfluids. The results are discussed after taking the LFLW limit which reduces the number coefficients which must be considered by one-half.

LFLW Limit

The LFLW limit of the mutual friction forces is now found, simplifying the expression for this form of dissipation greatly. As long as the dissipative time scales are long compared with the oscillation time scales of interest then the LFLW arguments given in Chapter 5 can be used here as well.

As in Chapter 5, it is convenient to use the average velocity of the charged fluids, u^a , which is given by Eq. (5.6) in Chapter 5. Recomputing $\partial_t \delta u^a$, including the dissipative forces, then gives

$$\begin{aligned} \partial_t \delta u^a &= \dots + \frac{\rho_p}{\tilde{\rho}_p} \delta F_{p(\text{MF})}^a + \frac{\tilde{\rho}_e}{\tilde{\rho}_p} \delta F_{e(\text{MF})}^a - \frac{1}{\tilde{\rho}_p} \nabla_b \delta \tau^{ab} \\ &= \dots - \frac{\rho_n}{\tilde{\rho}_p} \delta F_{n(\text{MF})}^a - \frac{1}{\tilde{\rho}_p} \nabla_b \delta \tau^{ab}, \end{aligned} \quad (7.16)$$

where the ellipses refer to the nondissipative forces given in Eq. (5.8a)–(5.8d) in Chapter 5. Note that of the mutual friction forces, only $\delta F_{n(\text{MF})}^a$ appears in the final expression on the right-hand side of Eq. (7.16). Thus only $\delta F_{n(\text{MF})}^a$ need be considered further here.

Following the LFLW approximations made in Chapter 5, the relative velocity which will appear in the expression for $\delta F_{n(\text{MF})}^a$ can be further simplified by

$$\frac{\rho_{nn}}{\rho_n} \delta v_n^a + \frac{\rho_{np}}{\rho_n} \delta v_p^a + \delta \frac{1}{\rho_n} (\vec{\nabla} \times \vec{\lambda}_n)^a - \delta v_e^a \approx \frac{\varrho^2}{\rho_n \rho_p} (\delta v_n^a - \delta u^a), \quad (7.17)$$

where ϱ^2 is the determinant of the mass density matrix, given by Eq. (4.55). In obtaining Eq. (7.17), Eqs. (5.9) and (5.10) in Chapter 5 have been used to express δv_p^a and δv_e^a in terms of δv_n^a and δu^a . Terms smaller than those kept by a factor of $(\tilde{\rho}_e/\rho_p)$ have been dropped. Furthermore, the elastic force of the neutron vortices (i.e., the term $\vec{\nabla} \times \vec{\lambda}_n$) has been dropped. This is a valid approximation as long as the angular velocity satisfies $\Omega \gg \omega_{vn}$ (where ω_{vn} , which is of the order 10^{-13} s^{-1} , is the vortex oscillation frequency discussed in Chapter 4). Thus, the mutual friction force $\delta F_{n(\text{MF})}^a$ simplifies in the LFLW limit to become

$$\begin{aligned} \delta F_{n(\text{MF})}^a = & -A_n \left(\frac{\varrho^2}{\rho_n \rho_p} \right) \left[(\delta v_n^b - \delta u^b) \omega_{ob} \right] \frac{\omega_o^a}{|\vec{\omega}_o|} \\ & - B_n' \left(\frac{\varrho^2}{\rho_n \rho_p} \right) \left[(\delta \vec{v}_n - \delta \vec{u}) \times \vec{\omega}_o \right]^a \\ & - B_n \left(\frac{\varrho^2}{\rho_n \rho_p} \right) |\vec{\omega}_o| \left(g^{ab} - \frac{\omega_o^a \omega_o^b}{|\vec{\omega}_o|^2} \right) (\delta v_{nb} - \delta u_b), \end{aligned} \quad (7.18)$$

where ω_o^a is the equilibrium vorticity of the neutron fluid.

Mutual Friction Coefficients

The results obtained in Chapter 6 for the remaining relevant mutual friction coefficients A_n , B_n' , and B_n which appear in Eq. (7.18) are discussed in detail here. The calculation used in Chapter 6 is based on the well established analysis of Hall and Vinen (1956). (For a review of other methods of derivation see Sonin 1987.)

Using Eqs. (6.24) and (6.28) in the Chapter 6, the results for these coefficients are given by

$$A_n = 0; \quad B'_n = \frac{C_n^2}{C_n^2 + \rho_n^2 K_n^2}, \quad B_n = \frac{\rho_n |\vec{K}_n| C_n}{C_n^2 + \rho_n^2 K_n^2}. \quad (7.19)$$

In these equations $\vec{K}_n \equiv \vec{\omega}_o/n_{vn}$ and C_n is the proportionality factor in the relation $f_{en}^a = C_n(v_e^a - v_{Ln}^a)$, where f_{en}^a is the average force per unit length acting on the neutron vortices due to electron scattering off these vortices and v_{Ln}^a is the average local velocity of the neutron vortices [see Eq. (6.15) in Chapter 6]. This scattering process will cause the electrons to relax to the neutron vortices on a time scale which will be defined as τ_v . Thus, on dimensional grounds $C_n \approx (\rho_{\text{eff}}/n_{vn}\tau_v)$, where ρ_{eff} is the effective mass density which relaxes to the neutron vortices on a time scale equal to τ_v . (An analogous expression for C_n is given in Alpar and Sauls 1988 in a different context.) It is appropriate to take $\rho_{\text{eff}} = \tilde{\rho}_p$, since Alpar et al. (1984) have shown that electromagnetic forces (and the scattering of electrons off the proton vortices) will cause the charged fluids to flow together on time scales much shorter than τ_v .

Thus, the expressions for the nonzero mutual friction coefficients can be simplified to give

$$B_n^2 \approx B'_n \approx \left(\frac{\tilde{\rho}_p}{\rho_n} \right)^2 \left(\frac{1}{2\Omega\tau_v} \right)^2, \quad (7.20)$$

where $|\vec{\omega}_o| = 2\Omega = n_{vn}|\vec{K}|$ and only terms to lowest order in $(\tilde{\rho}_p/\rho_n)(1/\Omega\tau_v) \approx 10^{-4}$ have been kept.

Finally, the expression for τ_v derived by Alpar et al. (1984) (see also Sauls et al. 1982) can be used. It is relatively insensitive to the neutron gap Δ_n and the

temperature. For the density range of a neutron star outer core it can be approximated to within $\sim 3\%$ of the more complicated (though more exact) expression given in Alpar et al. (1984) by

$$\frac{1}{\tau_V} \approx 0.022 \Omega x^{1/6} \left(\frac{m_p^* - m_p}{m_p} \right)^2 \left(\frac{m_p}{m_p^*} \right)^{1/2} \left(\frac{\rho}{10^{14} \text{g/cm}^3} \right)^{1/6}, \quad (7.21)$$

where m_p^* is the effective mass of the protons discussed in Chapter 4 and $x = \rho_p/(\rho_n + \rho_p)$. The dependence on the effective mass occurs since τ_V is mainly due to the scattering of electrons off the magnetic flux of the neutron vortices and the largest contribution to this magnetic flux is proportional to $\rho_{np} = \rho_p(1 - m_p/m_p^*)$ [see Chapter 4]. Using Eqs. (7.20) and (7.21) then gives for B_n :

$$B_n \approx 0.011 \left(\frac{m_p^* - m_p}{m_p} \right)^2 \left(\frac{m_p}{m_p^*} \right)^{1/2} \left(\frac{x^{7/6}}{1-x} \right) \left(\frac{\rho}{10^{14} \text{g/cm}^3} \right)^{1/6}, \quad (7.22)$$

where $\tilde{\rho}_p \approx \rho_p$ has been used. Note that B_n vanishes when $m_p^* = m_p$, i.e., when the Fermi liquid drag vanishes (actually mutual friction would still exist in this case because of other scattering processes, but the magnitude of B_n would be significantly reduced).

LFLW Dissipative Equations

The LFLW equations suitable to studies of the large scale p-modes are given here, including the effects of dissipation. As noted previously, as long as the dissipative time scales are long compared with the oscillation time scales of interest, then the dissipative terms will not modify the LFLW arguments given in Chapter 5. Thus, using the results of this chapter it is trivial to include the dissipative forces in the nondissipative LFLW equations given by Eqs. (5.28)–(5.35) in Chapter 5. The results are

$$\partial_t \delta \rho_n + v_o^b \nabla_b \delta \rho_n + \nabla_b (\rho_{nn} \delta v_n^b + \rho_{np} \delta u^b) = 0, \quad (7.23)$$

$$\partial_t \delta \rho_p + v_o^b \nabla_b \delta \rho_p + \nabla_b (\rho_{pp} \delta u^b + \rho_{np} \delta v_n^b) = 0, \quad (7.24)$$

$$\partial_t \delta \rho_e + v_o^b \nabla_b \delta \rho_e + \nabla_b \left[\rho_e \left(\frac{\rho_{pp}}{\rho_p} \delta u^b + \frac{\rho_{np}}{\rho_p} \delta v_n^b \right) \right] = 0, \quad (7.25)$$

$$\partial_t \delta \rho_\mu + v_o^b \nabla_b \delta \rho_\mu + \nabla_b \left[\rho_\mu \left(\frac{\rho_{pp}}{\rho_p} \delta u^b + \frac{\rho_{np}}{\rho_p} \delta v_n^b \right) \right] = 0, \quad (7.26)$$

$$\partial_t \delta \tilde{s}_e + v_o^b \nabla_b \delta \tilde{s}_e + \nabla_b \left[\tilde{s}_e \left(\frac{\rho_{pp}}{\rho_p} \delta u^b + \frac{\rho_{np}}{\rho_p} \delta v_n^b \right) - \frac{\kappa_e}{T_e} \nabla^b \delta T_e \right] = 0 \quad (7.27)$$

$$\begin{aligned} \partial_t \delta v_n^a = & -v_o^b \nabla_b \delta v_n^a - \delta v_n^b \nabla_b v_o^a - \nabla^a \delta \mu_n \\ & - \nabla^a \delta \varphi + \frac{\rho_{np}}{\rho_n} \left[(\delta \vec{u} - \delta \vec{v}_n) \times \vec{\omega}_o \right]^a \\ & - B_n \frac{\varrho^2}{\rho_n \rho_p} |\vec{\omega}_o| \left(g^{ab} - \frac{\omega_o^a \omega_o^b}{|\vec{\omega}_o|^2} \right) (\delta v_{nb} - \delta u_b), \end{aligned} \quad (7.28)$$

$$\begin{aligned} \partial_t \delta u^a = & -v_o^b \nabla_b \delta u^a - \delta u^b \nabla_b v_o^a - \nabla^a \delta \mu_p - \frac{\tilde{\rho}_e}{\rho_p} \delta \left(\frac{\rho_e}{\tilde{\rho}_e} \nabla^a \mu_e \right) \\ & - \frac{\tilde{\rho}_e}{\rho_p} \delta \left(\frac{\rho_\mu}{\tilde{\rho}_e} \nabla^a \mu_\mu \right) - \frac{\tilde{s}_e}{\rho_p} \nabla^a \delta T_e - \nabla^a \delta \varphi + \frac{\rho_{np}}{\rho_p} \left[(\delta \vec{v}_n - \delta \vec{u}) \times \vec{\omega}_o \right]^a \\ & + B_n \frac{\rho_n}{\rho_p} \frac{\varrho^2}{\rho_n \rho_p} |\vec{\omega}_o| \left(g^{ab} - \frac{\omega_o^a \omega_o^b}{|\vec{\omega}_o|^2} \right) (\delta v_{nb} - \delta u_b) - \frac{1}{\rho_p} \nabla_b \delta \tau^{ab}, \end{aligned} \quad (7.29)$$

$$\nabla_b \nabla^b \delta \varphi = 4\pi G \delta \rho. \quad (7.30)$$

Again, $\tilde{\rho}_p \approx \rho_p$ has been used to simplify some of the terms. Also, Eqs. (7.28) and (7.29) have been simplified slightly by ignoring the term in the mutual friction force proportional to B'_n . This term would give only a negligible correction to the drag force [proportional to $(\delta \vec{v}_n - \delta \vec{u}) \times \vec{\omega}_o$], which would be a factor of roughly $B'_n \sim 10^{-8}$ smaller than drag force terms kept in Eqs. (7.28) and (7.29).

Effect of Mutual Friction on The Damping Times

In this section the damping times of the large scale p-mode oscillations of rapidly rotating neutron stars are discussed. The primary goal is to estimate the effect mutual friction will have on the damping times of these modes. A

mode is defined to be a solution to the equations having a time dependence given by $\exp(i\omega t - t/\tau)$ where ω is the real part of the frequency and τ is the damping time of the mode. However, computing the damping time by solving the equations directly is highly nontrivial. Instead, the usual approach is first to solve the simpler nondissipative equations (which results in only the real part of the frequencies). The damping times are then more conveniently calculated by introducing a real energy functional E whose time derivative gives the damping time of the modes. Such an energy functional has been found in the present case to be given by (generalizing a similar energy given in Ipser and Lindblom 1991)

$$E = \frac{1}{2} \int \left\{ \rho_{nn} \delta v_n^b \delta v_{nb}^* + \rho_{np} \delta v_n^b \delta u_b^* + \rho_{np} \delta u^b \delta v_{nb}^* + \rho_{pp} \delta u^b \delta u_b^* \right. \\ \left. + \sum_{AB} \frac{\partial^2 U_I}{\partial y_A \partial y_B} \delta y_A \delta y_B^* - \frac{1}{4\pi G} (\nabla^b \delta \varphi) (\nabla_b \delta \varphi^*) \right\} d^3 x. \quad (7.31)$$

In this equation y_A represents the variables $(\rho_n, \rho_p, \rho_e, \rho_\mu, \tilde{s}_e)$ (A and B are indices which sum over these variables), and the internal energy of the fluid, U_I , is taken to be a function of the same set of variables. The nonperturbative quantities are to be evaluated in equilibrium. The time derivative of E , using Eqs. (7.23)–(7.30) and Eqs. (4.36) and (4.37) of Chapter 4 (which define the partial derivatives of U_I), is

$$\frac{dE}{dt} = - \int \left\{ 2\eta_e \delta \Theta^{ab} \delta \Theta_{ab}^* + \zeta_e \delta \Theta \delta \Theta^* + \frac{\kappa_e}{T_e} (\nabla^b \delta T_e) (\nabla_b \delta T_e^*) \right. \\ \left. + \rho_n B_n \left(\frac{\rho^4}{\rho_n^2 \rho_p^2} \right) |\vec{\omega}_o| (g^{ab} - \frac{\omega_o^a \omega_o^b}{|\vec{\omega}_o|^2}) (\delta u_a - \delta v_{na}) (\delta u_b^* - \delta v_{nb}^*) \right\} d^3 x, \quad (7.32)$$

[where the shear $\delta \Theta^{ab}$ and expansion $\delta \Theta$ can be expressed in terms of δv_n^a and δu^a using Eqs. (7.10) and (7.11) and Eq. (5.9) in Chapter 5]. The damping time

τ of a mode can then be calculated using

$$\frac{1}{\tau} = -\frac{1}{2E} \frac{dE}{dt}. \quad (7.33)$$

Each of the terms on the right-hand side of Eq. (7.32) is proportional to one of the dissipative coefficients associated with the different forms of dissipation. Thus, the inverse damping time can be thought of as a sum of inverse damping times, each due to one of the forms of dissipation, i.e.,

$$\frac{1}{\tau} = \frac{1}{\tau_\eta} + \frac{1}{\tau_\zeta} + \frac{1}{\tau_\kappa} + \frac{1}{\tau_{\text{MF}}}. \quad (7.34)$$

For example, τ_{MF} represents the mutual friction damping time associated with the term proportional to B_n in Eq. (7.32). Previously it has been shown that the bulk viscosity and thermal conductivity of ordinary nuclear matter plays a negligible role in the damping times for temperatures below $10^{10} K$ (Cutler and Lindblom 1987, Cutler, Lindblom, and Splinter 1990, Ipser and Lindblom 1991). Superfluidity should not alter these basic conclusions. This leaves us with the shear damping time and the mutual friction damping time given by

$$\frac{1}{\tau_\eta} = \frac{1}{E} \int \eta_e \delta \Theta^{ab} \delta \Theta_{ab}^* d^3x, \quad (7.35)$$

$$\frac{1}{\tau_{\text{MF}}} = \frac{1}{2E} \int \rho_n B_n \left(\frac{\varrho^4}{\rho_n^2 \rho_p^2} \right) |\vec{\omega}_o| \left(g^{ab} - \frac{\omega_o^a \omega_o^b}{|\vec{\omega}_o|^2} \right) (\delta u_a - \delta v_{na}) (\delta u_b^* - \delta v_{nb}^*) d^3x. \quad (7.36)$$

The remaining task here will be to discuss the relative size of these two damping times. Qualitatively, this can be done by making the following substitution appropriate for plane wave solutions: $\nabla^a \delta v^b \rightarrow i k^a \delta v^b$, where k^a is the wave vector. The magnitude of the wave vector is related to the wavelength, λ , by $k = (2\pi/\lambda)$. Furthermore, the background will be approximated as uniform (ignoring the effects of gravitation and rotation on the equilibrium mass densities).

Then, for a longitudinal plane wave traveling perpendicular to the rotation axis, the ratio of the mutual friction damping time to the shear damping time can be shown to be

$$\frac{\tau_{\text{MF}}}{\tau_{\eta}} \approx \frac{(8\pi^2/3)\eta_e[(\rho_{pp}/\rho_p) + (\rho_{np}/\rho_p)(\delta v_n/\delta u)]^2(1/\lambda^2)}{\rho_n(\varrho^4/\rho_n^2\rho_p^2)(\Omega B_n)[1 - (\delta v_n/\delta u)]^2}. \quad (7.37)$$

In this equation δv_n and δu are the components of these velocities parallel to the wave vector $\vec{k}$. The ratio $\delta v_n/\delta u$ will not equal unity in general. In Chapter 8 it will be shown, for sound waves in superfluid neutron-star matter, that $|\delta v_n/\delta u| < 0.7$ for densities above nuclear density.

Mutual friction will be the dominant damping mechanism whenever the condition $(\tau_{\text{MF}}/\tau_{\eta_e}) < 1$ holds. This condition will always hold for sufficiently large angular velocities and sufficiently long wavelengths, i.e., in particular it holds for

$$\Omega\lambda^2 > \Gamma, \quad (7.38)$$

where

$$\Gamma = \frac{(8\pi^2/3)\eta_e[(\rho_{pp}/\rho_p) + (\rho_{np}/\rho_p)(\delta v_n/\delta u)]^2}{\rho_n(\varrho^4/\rho_n^2\rho_p^2)(B_n)[1 - (\delta v_n/\delta u)]^2}. \quad (7.39)$$

The goal will now be to find an upper bound on Γ (i.e., the greatest lower bound on $\Omega\lambda^2$.) It can be shown that Γ is basically wavelength and angular velocity independent. However, because of the shear viscosity, it is sensitive to the temperature. The shear viscosity η_e is due to electron-electron scattering (the small fraction of muon will be ignored in this factor). An approximate expression for the electron shear viscosity given by Cutler and Lindblom (1987) is

$$\eta_e = 6.0 \times 10^{20} \left(\frac{\rho}{10^{14} \text{ g/cm}^{-3}} \right)^2 \left(\frac{10^7 \text{ K}}{T_e} \right)^2 \text{ g} \cdot \text{cm}^{-1} \cdot \text{s}^{-1}. \quad (7.40)$$

This expression fits the calculation by Flowers and Itoh (1976) using the data in Baym, Bethe, and Pethick (1971) to within $\sim 5\%$. In this equation ρ is the total mass density. Using this equation for η_e and Eq. (7.22) for B_n , gives

$$\Gamma \approx 10^{10} \left(\frac{10^7 \text{K}}{T_e} \right)^2 \left\{ \frac{\rho_{14}^{5/6} x^{-7/6} \left[\frac{m_p}{m_p^*} + \left(1 - \frac{m_p}{m_p^*} \right) \frac{\delta v_n}{\delta u} \right]^2}{\frac{\rho^4}{\rho_n^2 \rho_p^2} \left[\frac{(m_p^* - m_p)}{m_p} \right]^2 \left(\frac{m_p}{m_p^*} \right)^{1/2} \left[1 - \frac{\delta v_n}{\delta u} \right]^2} \right\} \frac{\text{cm}^2}{\text{s}}, \quad (7.41)$$

where $\rho_{14} \equiv (\rho/10^{14} \text{g/cm}^3)$. Uncertainties exist in the various quantities in this equation, but they range over the following values for outer core densities $2.8 \times 10^{14} \text{g/cm}^3 < \rho < 5 \times 10^{14} \text{g/cm}^3$ (Baym et al. 1971): the fraction of protons increases with increasing density and falls in the range $0.038 < x < 0.051$ interpolating the data in Table 6 of Baym, et al. (1971); the proton effective mass decreases with increasing density and falls in the range $.3 < m_p^*/m_p < .7$ (see Chapter 4 for references); Chapter 8 indicates $\delta v_n/\delta u < .7$; the range of values for x and m_p^* gives, $1.4 < (\rho^2/\rho_n \rho_p) < 3.5$. The factor Γ is extremized when $(\rho/10^{14} \text{g/cm}^3) = 5$, $x = .038$, $m_p^*/m_p = .7$, and $(\rho^2/\rho_n \rho_p) = 1.4$. The result is

$$\Gamma < 10^{14} \left(\frac{10^7 \text{K}}{T_e} \right)^2 \text{cm}^2 \text{s}^{-1}. \quad (7.42)$$

A conservative condition on the angular velocity of rotation in terms of the length scale of the oscillation and the temperature under which mutual friction dominates the damping time is then given by

$$\Omega > 100 \left(\frac{10^6 \text{cm}}{\lambda} \right)^2 \left(\frac{10^7 \text{K}}{T_e} \right)^2 \text{s}^{-1}. \quad (7.43)$$

Thus it can be seen from Eq. (7.43) that mutual friction will be most important for long wavelength oscillations; when $\lambda \sim 10^6 \text{cm}$, which is of the order of a typical neutron star radius. For $T = 10^8 - 10^7 \text{K}$ mutual friction will dominate

then for $\Omega \geq 1\text{--}100\text{s}^{-1}$. But for $T = 10^6\text{K}$, Eq. (7.38) gives $\Omega > 1 \times 10^4\text{s}^{-1}$. This angular velocity is roughly the maximum Keplerian limit thought possible [corresponding to a .5ms pulsar; the fastest known pulsar has $\Omega = 4.03 \times 10^3\text{s}^{-1}$, (e.g., see Chapter 1; Kundt 1990)]. Thus, the shear viscosity will dominate for temperatures $T \leq 10^6\text{K}$. On the other hand, as the temperature approaches the transition temperature to superfluidity, $T_c \sim 10^9\text{K}$, superfluidity is destroyed and mutual friction can no longer act. Thus, mutual friction will be the dominant damping mechanism for the temperature range $10^7\text{K} < T < T_c$, in rapidly rotating neutron stars.

So far the above conclusion has been based on the analysis of a plane wave traveling perpendicular to the rotation axis (fluid motion parallel to the rotation axis produces no mutual friction). This conclusion will be verified in Chapter 8, in which plane symmetric modes are studied. However, this conclusion also can be extended to the global p-modes of a neutron star. The situation is analogous to coupled pendulums of different lengths: two modes will occur, and the pendulums will not oscillate together in either mode, since the pendulum lengths are unequal. In a superfluid neutron star the number of allowed modes will split into two branches compared with those found in a neutron star composed of nonsuperfluid matter. For the upper branch the neutron fluid and charged fluids will oscillate opposite to each other. For the lower branch the neutron fluid and charged fluids oscillate in the same direction, but with different amplitudes. The amplitudes will be different since the equations of state (giving the pressure terms) differ for the neutron fluid and the charged fluids. Also, the boundary condition will require that a component of the fluid motions lie in the plane perpendicular to the rotation axis. Thus, generically, it can be expected that these modes will be damped by

mutual friction. Thus mutual friction is expected to dominate the damping of the global p-modes in rapidly rotating neutron stars for the temperature range indicated in the previous paragraph.

CHAPTER 8

PLANE SYMMETRIC MODES

Introduction

This chapter describes the plane-wave solutions of the LFLW perturbation equations [Eqs. (7.24)–(7.30)] found in the last chapter. These solutions exist when the equilibrium state is approximated as uniform density matter. In this case gravitational and centrifugal effects on the equilibrium mass densities are ignored. Perturbations of the gravitational field are also ignored. The term $-(\tilde{s}_e/\rho_p)\nabla^a\delta T$ in Eq. (7.29) is ignored too, since it is typically smaller than the gradients in the chemical potentials which appear in this equation by a factor of $(T/T_F)^2$. However, the effects of rotation on solutions to the perturbation equations will be included to illustrate the importance of mutual friction in rotating superfluid neutron stars. The approximations made here are valid for wavelengths which are small compared with a typical neutron star radius. However, in the numerical results of this chapter wavelengths comparable to a typical neutron star radius will be used. The numerical results then will be indicative of the qualitative behavior of the global oscillations of rotating neutron stars.

In the case of a uniform rotating fluid, plane wave solutions can be found in the rotating frame. Let the rotation axis define the z -axis of the system. The transformation from Cartesian coordinates in the inertial frame to Cartesian

coordinates which corotate with the fluid is given by

$$x' = x \cos(\Omega t) + y \sin(\Omega t), \quad (8.1)$$

$$y' = -x \sin(\Omega t) + y \cos(\Omega t), \quad (8.2)$$

$$z' = z, \quad (8.3)$$

$$t' = t. \quad (8.4)$$

The equilibrium velocity field observed in the inertial frame of the fluid is given by $\vec{v}_o = -\Omega y \hat{x} + \Omega x \hat{y} = -\Omega y' \hat{x}' + \Omega x' \hat{y}'$, where a caret refers to a unit vector parallel to the indicated direction. Under this transformation the partial derivatives of a scalar field S transforms as

$$\partial_t S + v_o^b \nabla_b S = \partial_{t'} S, \quad (8.5)$$

$$\nabla^a S = \nabla^{a'} S, \quad (8.6)$$

and partial derivatives of vector field V^a transform as

$$\partial_t V^a + v_o^b \nabla_b V^a = \partial_{t'} V^{a'} - \frac{1}{2} [(\vec{V} \times (\vec{\nabla} \times \vec{v}_o))]^{a'}, \quad (8.7)$$

$$\nabla^a V^b = \nabla^{a'} V^{b'}. \quad (8.8)$$

Each component of the perturbed quantities is assumed to vary in space and time as $\exp(i\vec{k} \cdot \vec{x}' - i\omega t')$ in the $\hat{x}'$, $\hat{y}'$, $\hat{z}'$ basis. Equations (8.5)–(8.8) then can be replaced with the following algebraic expressions:

$$\partial_t S + v_o^b \nabla_b S = -i\omega S, \quad (8.9)$$

$$\nabla^a S = ik^{a'} S, \quad (8.10)$$

$$\partial_t V^a + v_o^b \nabla_b V^a = -i\omega V^{a'} - \Omega(\vec{V} \times \hat{z})^{a'}, \quad (8.11)$$

$$\nabla^a V^b = k^{a'} V^{b'}. \quad (8.12)$$

Using the substitutions in Eqs. (8.9)–(8.12) the linear perturbation equations become a purely algebraic eigenvalue problem for the complex frequencies ω . The problem can be solved using the standard methods of linear algebra. However, in general, this still involves taking the determinate of a 10×10 matrix. Thus the general results are very complicated and hardly enlightening. Instead, attention will be restricted to two special cases: axial modes where the wave vector is parallel to the rotation axis and inplane modes where the wave vector is perpendicular to the rotation axis. The resulting eigenvectors for each mode of the system are given by the relative amplitudes of the various perturbed quantities and the solutions for the frequencies can be written as a dispersion relation $\omega(\vec{k})$. The real part of this frequency gives the oscillation frequency of the mode. The damping time of the mode can be defined in terms of the imaginary part of the frequency, ω_I , by

$$\frac{1}{\tau} = -\omega_I. \quad (8.13)$$

To compute numerical quantities the internal energy of the fluid will be modeled using the equation of state developed by Baym, Bethe, and Pethick (1971) (from now on referred to as BBP). The internal energy density in this case is given by

$$U_I = n_b W(\rho_n, \rho_p) + E_e(\rho_e) + E_\mu(\rho_\mu), \quad (8.14)$$

where n_b is the number density of baryons and W the energy per baryon given by BBP's Eq. (3.14), whereas E_e and E_μ are the energy densities of ideal electron and muon gases, respectively, appearing in BBP's Eq. (8.1). The following definitions will be used:

$$\mu_n = \frac{\partial U_I}{\partial \rho_n} = \frac{\partial(n_b W)}{\partial \rho_n}, \quad (8.15)$$

$$\mu_p = \frac{\partial U_I}{\partial \rho_p} = \frac{\partial(n_b W)}{\partial \rho_p}, \quad (8.16)$$

$$\mu_e = \frac{\partial U_I}{\partial \rho_e} = \frac{\partial E_e}{\partial \rho_e} = \frac{c}{m_e} \left[\hbar^2 (3\pi^2 n_e)^{2/3} + m_e^2 c^2 \right]^{1/2}, \quad (8.17)$$

$$\mu_\mu = \frac{\partial U_I}{\partial \rho_\mu} = \frac{\partial E_\mu}{\partial \rho_\mu} = \frac{c}{m_\mu} \left[\hbar^2 (3\pi^2 n_\mu)^{2/3} + m_\mu^2 c^2 \right]^{1/2}, \quad (8.18)$$

where n_e , and n_μ are the number densities of the electrons and muons, respectively. Also, the baryon number density is the sum of the neutron number density, n_n , and the proton number density, n_p , i.e., $n_b = n_n + n_p$. Use of the BBP equation of state will allow direct comparison of the results presented in this chapter with those of Epstein (1988). Note, however, that the definitions of the chemical potentials in Eqs. (8.15)–(8.18) differ slightly from those given by Epstein (1988), by a factor equal to the mass of the corresponding particle.

The fractional concentration of the protons, electrons, and muons, can be found from data in BBP's Table 6. The data in this table is not smooth, but can be fit to the following curves:

$$\frac{n_p}{n_n + n_p} = 0.096 n_b^{1/2}, \quad (8.19)$$

$$\frac{n_e}{n_\mu} = 1.988 n_b - 0.2916. \quad (8.20)$$

These equations reproduce the data in Table 6 of BBP to within 5% for $n_b \geq 0.170 \text{ fm}^{-3}$. Furthermore, ignoring a negligible equilibrium charge density (see Chapter 4), charge neutrality requires,

$$n_p = n_e + n_\mu. \quad (8.21)$$

Lastly, to compute the mass density matrix, $\rho_{nn}, \rho_{np}, \rho_{pp}$ the effective mass of the proton must be known. The effective mass is a very slowly varying function

of the density. Since its value is uncertain (see Chapter 4), although it probably lies in range $0.3 \leq m_p^*/m_p \leq 0.7$, the median value $m_p^*/m_p = 0.5$ will be used throughout this chapter. In this case the mass density matrix is given by

$$\rho_{nn} = \rho_n + \rho_p, \quad (8.22)$$

$$\rho_{np} = -\rho_p, \quad (8.23)$$

$$\rho_{pp} = 2\rho_p. \quad (8.24)$$

Axial Modes

In the case of axial modes the wave vector $\vec{k}$ is chosen parallel to the rotation axis (taken to be the z -axis) the equations split into transverse (perpendicular to $\vec{k}$) and longitudinal (parallel to $\vec{k}$) parts. The resulting dispersion relation for the solution of the transverse equations is

$$\left(\frac{\omega}{2\Omega}\right)^2 + \left(\frac{\rho_{nn}}{\rho_n} + \frac{\rho_{pp}}{\rho_p}\right) \left(\frac{\omega}{2\Omega}\right) + \left(\frac{\varrho^2}{\rho_n \rho_p}\right) = I_{\perp}, \quad (8.25)$$

where

$$\begin{aligned} I_{\perp} = & \left[i \frac{\rho_{pp}}{\rho_p} \frac{\eta_e k^2}{2\Omega \rho_p} + i \frac{\rho_n}{\rho_p} \frac{\rho_n + \rho_p}{\rho_n} \frac{\varrho^2}{\rho_n \rho_p} B_n \right] \left(\frac{\omega}{2\Omega}\right) \\ & + i \frac{\varrho^2}{\rho_n \rho_p} \frac{\eta_e k^2}{2\Omega \rho_p} + i \frac{\rho_n}{\rho_p} \frac{\rho_n + \rho_p}{\rho_n} \frac{\varrho^2}{\rho_n \rho_p} B_n + i \frac{\varrho^2}{\rho_n \rho_p} \frac{\eta_e k^2}{2\Omega \rho_p} B_n. \end{aligned} \quad (8.26)$$

The nondissipative limit is taken by setting $I_{\perp} = 0$. The resulting frequencies are

$$\omega_0 = \pm \begin{cases} 2\Omega \frac{\varrho^2}{\rho_n \rho_p}, \\ 2\Omega. \end{cases} \quad (8.27)$$

Each of these modes corresponds to a circularly polarized wave travelling along the z -axis, i.e., $\delta v_x = \pm i \delta v_y$ for each velocity field. (Primes on the coordinates

will be dropped in the remainder of this chapter, but all quantities are understood to be measured in the corotating frame.) The lower branch corresponds to the frequency of what is called the inertial mode in an ordinary perfect fluid (see Sonin 1987). This was to be expected since the inertial mode is independent of the composition of fluid. Thus a mode always exists in a mixture of rotating fluids where all the fluids oscillate together with frequency 2Ω , i.e., for this mode $\delta \vec{v}_n = \delta \vec{u}$. The upper branch represents the case where the fluids oscillate opposite to each other and is modified by the "drag" factor $(\varrho^2/\rho_n \rho_p) \approx 2$. For this mode $\delta v_n^a = -(\rho_p/\rho_n)\delta u^a$.

The damping time for the above modes can be found by expanding $\omega/2\Omega$ out to first order in $(\eta_e k^2/2\Omega \rho_p)$ and B_n . The results are:

$$\frac{1}{\tau} \approx \begin{cases} \frac{\rho_n + \rho_p}{\rho_p} \frac{\varrho^2}{\rho_n \rho_p} 2\Omega B_n + \frac{\varrho^2}{\rho_p (\rho_n + \rho_p)} \frac{\eta_e k^2}{\rho_p}, \\ \frac{\eta_e k^2}{\rho_n + \rho_p}. \end{cases} \quad (8.28)$$

The lower branch in Eq. (8.27) ($\omega_o = 2\Omega$) is not damped by mutual friction since $\delta v_n^a = \delta u^a$ for this mode. However, this is a very special case which occurs because of the cylindrical symmetry of the problem. Inclusion of gravity will break this symmetry. Thus, generically, it can be expected there will be no global oscillation for which the neutron and charged fluids will oscillate exactly together.

The resulting dispersion relation for the longitudinal component is equivalent to the dispersion relation for sound waves in an isotropic (nonrotating) neutron star, since rotation has no effect on sound waves travelling along the z -axis. The

result is

$$\begin{aligned} \left(\frac{\omega}{k}\right)^4 & - \left[\rho_{nn} \frac{\partial \mu_n}{\partial \rho_n} + \rho_{pp} \frac{\partial \mu_p}{\partial \rho_p} + \rho_{np} \left(\frac{\partial \mu_n}{\partial \rho_p} + \frac{\partial \mu_p}{\partial \rho_n} \right) \right. \\ & \quad \left. + \rho_{pp} \left(\frac{\rho_e^2}{\rho_p^2} \frac{\partial \mu_e}{\partial \rho_e} + \frac{\rho_\mu^2}{\rho_p^2} \frac{\partial \mu_\mu}{\partial \rho_\mu} \right) \right] \left(\frac{\omega}{k}\right)^2 \\ & + \varrho^2 \left(\frac{\partial \mu_n}{\partial \rho_n} \frac{\partial \mu_p}{\partial \rho_p} - \frac{\partial \mu_n}{\partial \rho_p} \frac{\partial \mu_p}{\partial \rho_n} + \frac{\rho_e^2}{\rho_p^2} \frac{\partial \mu_n}{\partial \rho_n} \frac{\partial \mu_e}{\partial \rho_e} + \frac{\rho_\mu^2}{\rho_p^2} \frac{\partial \mu_n}{\partial \rho_n} \frac{\partial \mu_\mu}{\partial \rho_\mu} \right) = I_{||}, \end{aligned} \quad (8.29)$$

where

$$I_{||} = -i \frac{4}{3} \frac{\rho_{pp}}{\rho_p} \frac{1}{k} \frac{\eta_e k^2}{\rho_p} \left(\frac{\omega}{k}\right)^3 + i \frac{4}{3} \frac{\varrho^2}{\rho_n \rho_p} \frac{1}{k} \frac{\eta_e k^2}{\rho_p} \rho_n \frac{\partial \mu_n}{\partial \rho_n} \left(\frac{\omega}{k}\right). \quad (8.30)$$

This result corrects that given by Epstein (1988) and generalizes it to include dissipative effects. His calculation is inconsistent as he does not include the drag terms in the mass currents, but does include them in his definition of the kinetic energy density. Since Epstein uses the variation of the kinetic energy with respect to the mass currents to determine equations for the fluid velocities, these too are incorrect in his paper.

The nondissipative limit of Eq. (8.29) is obtained by setting $I_{||} = 0$. The superfluid sound velocities, $v_{s1}, v_{s2} = (\omega/k)$, can be computed using the quadratic formula. The fluid velocities are related by

$$\delta v_{nz} = \left[\rho_{np} \frac{\partial \mu_n}{\partial \rho_n} + \rho_{pp} \frac{\partial \mu_p}{\partial \rho_p} \right] \left[\frac{\omega^2}{k^2} - \rho_{nn} \frac{\partial \mu_n}{\partial \rho_n} - \rho_{np} \frac{\partial \mu_n}{\partial \rho_p} \right]^{-1} \delta u_z. \quad (8.31)$$

The viscous damping time, τ_{η_e} , can be computed by expanding ω/k out to first order in $(\eta_e k^2 / \rho_p)$ with the result

$$\frac{1}{\tau_{\eta_e 1,2}} = \frac{2}{3} \frac{\eta_e k^2}{\rho_p} \left[\frac{\varrho^2}{\rho_n \rho_p} \rho_n \frac{\partial \mu_n}{\partial \rho_n} - \frac{\rho_{pp}}{\rho_p} v_{s1,2}^2 \right] \left[v_{s1}^2 + v_{s2}^2 - 2v_{s1,2}^2 \right]^{-1}. \quad (8.32)$$

Mutual friction does not have an effect on these modes since the motion of the fluids is parallel to the vortices. Although the theory of mutual friction can be

extended to include a component parallel to the vortices (see Bekarevich and Khalatnikov 1961; Chapter 6), it seems reasonable to expect such a contribution would be small. (Recall that the dominant contribution to mutual friction in a neutron star comes from the scattering of electrons off the magnetic flux carried by the vortices, and the magnetic force on electrons travelling parallel to the vortices vanishes.) It should be noted that the fluid motion for the global oscillations of a neutron star will not be confined to motion parallel to the vortices. Thus, the global modes of a neutron star will be damped by mutual friction.

The dispersion relation for the sound velocity in the case of no superfluidity can be found by taking ρ_n times Eq. (7.28), adding it to ρ_p times Eq. (7.29), and setting $\delta v_n^a = \delta u^a$. Furthermore, the viscosity of normal neutron-star matter differs from that of the electrons. Thus, in the viscous stress tensor the replacement $\eta_e \rightarrow \eta_n$ should be made, where the viscosity of normal neutron-star matter, η_n , is mostly due to neutron scattering. The resulting dispersion relation for sound waves in normal neutron-star matter is given by,

$$\left(\frac{\omega}{k}\right)^2 - v_{\text{SN}}^2 = -i \frac{4}{3} \frac{1}{k} \frac{\eta_n k^2}{\rho_n + \rho_p} \left(\frac{\omega}{k}\right), \quad (8.33)$$

where the normal sound velocity is given by

$$v_{\text{SN}}^2 = \frac{1}{\rho_n + \rho_p} \left[\rho_n^2 \frac{\partial \mu_n}{\partial \rho_n} + \rho_n \rho_p \left(\frac{\partial \mu_n}{\partial \rho_p} + \frac{\partial \mu_p}{\partial \rho_n} \right) + \rho_p^2 \frac{\partial \mu_p}{\partial \rho_p} + \rho_e^2 \frac{\partial \mu_e}{\partial \rho_e} + \rho_\mu^2 \frac{\partial \mu_\mu}{\partial \rho_\mu} \right]. \quad (8.34)$$

The damping time of the normal sound waves is given by

$$\frac{1}{\tau_{\eta_n}} = \frac{2}{3} \frac{\eta_n k^2}{\rho_n + \rho_p}. \quad (8.35)$$

The value of η_n has been calculated by Flowers and Itoh (1976) and the results can be fit to within a few percent by (see Cutler and Lindblom 1987)

$$\eta_n = 1.11 \times 10^{16} \left(\frac{\rho}{10^{14} \text{g/cm}^3} \right)^{9/4} \left(\frac{T}{10^9 \text{K}} \right)^{-2} \text{g} \cdot \text{cm}^{-1} \cdot \text{s}^{-1}. \quad (8.36)$$

The results for the sound velocities and relative amplitudes is given in Table 1. The results for the sound velocities and viscous damping times are given in Table 2 for a temperature $T = 10^7 \text{K}$ and for a wavelength $\lambda = 2\pi/k$ equal to 10^6cm . In these tables the first column is the total mass density taken from the first column of Table 6 in BBP. It has been convenient to define $\rho_{14} \equiv (\rho/10^{14} \text{g/cm}^3)$. One important result is that correctly including the drag terms in the mass currents greatly increases the second sound velocities in comparison with those computed by Epstein (1988).

To compute the viscous damping times in Table 2 the viscosity of electron matter, given by Eq. (7.40) in Chapter 7 has been used. In the case of no superfluidity, the shear viscosity for normal neutron-star matter, given by Eq. (8.36), has been used. Both τ_{η_n} and $\tau_{\eta_{e1,2}}$ scale with the temperature as T^2 and with wavelength as λ^2 .

Table 1. Axial Sound Wave Velocities and Relative Amplitudes.

ρ_{14}	v_{SN}/c	v_{S1}/c	$(\delta v_{nz}/\delta u_z)_1$	v_{S2}/c	$(\delta v_{nz}/\delta u_z)_2$
2.89	0.131	0.128	0.747	0.311	-0.0701
3.02	0.133	0.130	0.737	0.313	-0.0731
3.15	0.136	0.132	0.727	0.315	-0.0765
3.22	0.137	0.134	0.722	0.316	-0.0783
3.43	0.142	0.137	0.707	0.320	-0.0839
3.65	0.146	0.141	0.693	0.324	-0.0895
3.89	0.150	0.145	0.680	0.328	-0.0952

Table 2. Axial Sound Wave Velocities and Damping Times (in Seconds).

ρ_{14}	v_{SN}/c	$\tau_{\eta n}$	v_{s1}/c	$\tau_{\eta e1}$	v_{s2}/c	$\tau_{\eta e2}$
2.89	0.131	9400.	0.128	822.	0.311	46.7
3.02	0.133	8930.	0.130	761.	0.313	46.0
3.15	0.136	8450.	0.132	699.	0.315	45.1
3.22	0.137	8231.	0.134	671.	0.316	44.7
3.43	0.142	7574.	0.137	592.	0.320	43.6
3.65	0.146	7010.	0.141	526.	0.324	42.5
3.89	0.150	6510.	0.145	471.	0.328	41.6

Inplane Modes

Modes with the wave vector perpendicular to the rotation axis are now considered. In this case the equations cannot be separated into transverse and longitudinal parts. The full dispersion relation including the dissipative terms is given by,

$$\left(\frac{\omega}{2\Omega}\right)^6 - \left[1 + \left(\frac{\varrho^2}{\rho_n \rho_p}\right)^2 + \frac{(v_{s1}^2 + v_{s2}^2)k^2}{4\Omega^2}\right] \left(\frac{\omega}{2\Omega}\right)^4 + \left[\left(\frac{\varrho^2}{\rho_n \rho_p}\right)^2 \left(1 + \frac{V_1^2 k^2}{4\Omega^2}\right) + \frac{v_{s1}^2 v_{s2}^2 k^4}{16\Omega^4}\right] \left(\frac{\omega}{2\Omega}\right)^2 = \sum_{i=1}^5 C_i \left(\frac{\omega}{2\Omega}\right)^i, \quad (8.37)$$

where

$$C_5 = -2i \frac{\varrho^2}{\rho_n \rho_p} \frac{\rho_n + \rho_p}{\rho_p} B_n - i \frac{7}{3} \frac{\rho_{pp}}{\rho_p} \frac{\eta_e k^2}{2\Omega \rho_p} \quad (8.38)$$

$$C_4 = \left[\frac{\rho_n + \rho_p}{\rho_p} \frac{\varrho^2}{\rho_n \rho_p}\right]^2 B_n^2 + \frac{7}{3} \frac{\varrho^2}{\rho_n \rho_p} \left[1 + \frac{\rho_{pp}}{\rho_p} \frac{\rho_n + \rho_p}{\rho_p}\right] \frac{\eta_e k^2}{2\Omega \rho_p} B_n + \frac{4}{3} \left(\frac{\rho_{pp}}{\rho_p}\right)^2 \left(\frac{\eta_e k^2}{2\Omega \rho_p}\right)^2, \quad (8.39)$$

$$\begin{aligned}
C_3 = i \left\{ \frac{\varrho^2}{\rho_n \rho_p} \left[\frac{V_2^2 k^2}{4\Omega^2} + \frac{\rho_n + \rho_p}{\rho_p} \left[2 + \frac{(v_{s1}^2 + v_{s2}^2)k^2}{4\Omega^2} \right] \right] B_n \right. \\
+ \frac{\varrho^2}{\rho_n \rho_p} \left[\frac{4 V_4^2 k^2}{3 \cdot 4\Omega^2} + \frac{\rho_n \rho_{pp}}{\varrho^2} \frac{(v_{s1}^2 + v_{s2}^2)k^2}{4\Omega^2} + \frac{7 \rho_{nn}}{3 \rho_n} \right] \frac{\eta_e k^2}{2\Omega \rho_p} \\
\left. + \frac{\varrho^2}{\rho_n \rho_p} \left[\frac{7 \varrho^2}{3 \rho_n \rho_p} \frac{\rho_n + \rho_p}{\rho_p} B_n + \frac{8 \rho_{pp}}{3 \rho_p} \frac{\eta_e k^2}{2\Omega \rho_p} \right] \frac{\eta_e k^2}{2\Omega \rho_p} B_n \right\}, \quad (8.40)
\end{aligned}$$

$$\begin{aligned}
C_2 = - \left(\frac{\varrho^2}{\rho_n \rho_p} \right)^2 \left\{ \left[\frac{\rho_n + \rho_p}{\rho_p} \frac{V_2^2 k^2}{4\Omega^2} + \left(\frac{\rho_n + \rho_p}{\rho_p} \right)^2 \right] B_n^2 \right. \\
+ \left[\frac{\rho_n \rho_{pp}}{\varrho^2} \frac{V_2^2 k^2}{4\Omega^2} + \frac{4 \rho_n + \rho_p}{3 \rho_p} \frac{V_4^2 k^2}{4\Omega^2} \right. \\
+ \left. \frac{\rho_n \rho_p}{\varrho^2} \frac{(v_{s1}^2 + v_{s2}^2)k^2}{4\Omega^2} + \frac{7 \rho_n}{3 \rho_p} \right] \frac{\eta_e k^2}{2\Omega \rho_p} B_n \\
\left. + \left[\frac{4 \rho_n \rho_{pp}}{3 \varrho^2} \frac{V_4^2 k^2}{4\Omega^2} + \frac{4}{3} \right] \left(\frac{\eta_e k^2}{2\Omega \rho_p} \right)^2 + \frac{4}{3} \left(\frac{\eta_e k^2}{2\Omega \rho_p} \right)^2 B_n^2 \right\} \quad (8.41)
\end{aligned}$$

$$\begin{aligned}
C_1 = -i \left(\frac{\varrho^2}{\rho_n \rho_p} \right)^2 \left\{ \left[\frac{\rho_n \rho_p}{\varrho^2} \frac{\rho_n + \rho_p}{\rho_p} \frac{v_{s1}^2 v_{s2}^2 k^4}{16\Omega^4} + \frac{V_3^2 k^2}{4\Omega^2} \right] B_n \right. \\
+ \left[\frac{\rho_n^2 \rho_p \rho_{pp}}{\varrho^4} \frac{v_{s1}^2 v_{s2}^2 k^4}{16\Omega^4} + \frac{V_5^2 k^2}{4\Omega^2} \right] \frac{\eta_e k^2}{2\Omega \rho_p} \\
\left. + \left[\frac{V_2^2 k^2}{4\Omega^2} B_n + \frac{4 V_4^2 k^2}{3 \cdot 4\Omega^2} \frac{\eta_e k^2}{2\Omega \rho_p} \right] \frac{\eta_e k^2}{2\Omega \rho_p} B_n \right\}, \quad (8.42)
\end{aligned}$$

$$C_0 = \frac{\varrho^2}{\rho_n \rho_p} \frac{v_{s1}^2 v_{s2}^2 k^4}{16\Omega^4} \frac{\eta_e k^2}{2\Omega \rho_p} B_n. \quad (8.43)$$

In these equations v_{s1} and v_{s2} are the axial sound velocities, and the other velocities are given by

$$\begin{aligned}
V_1^2 = \frac{\rho_n \rho_p}{\varrho^2} \left[\rho_n \frac{\rho_{pp}}{\rho_p} \frac{\partial \mu_n}{\partial \rho_n} + \rho_p \frac{\rho_{nn}}{\rho_n} \frac{\partial \mu_p}{\partial \rho_p} - \rho_{np} \left(\frac{\partial \mu_n}{\partial \rho_p} + \frac{\partial \mu_p}{\partial \rho_n} \right) \right. \\
\left. + \rho_p \frac{\rho_{nn}}{\rho_n} \frac{\rho_e^2}{\rho_p^2} \frac{\partial \mu_e}{\partial \rho_e} + \rho_p \frac{\rho_{nn}}{\rho_n} \frac{\rho_\mu^2}{\rho_p^2} \frac{\partial \mu_\mu}{\partial \rho_\mu} \right]. \quad (8.44)
\end{aligned}$$

$$V_2^2 = \left[\frac{\rho_n^2}{\rho_p} \frac{\partial \mu_n}{\partial \rho_n} + \rho_p \frac{\partial \mu_p}{\partial \rho_p} + \rho_n \left(\frac{\partial \mu_n}{\partial \rho_p} + \frac{\partial \mu_p}{\partial \rho_n} \right) + \rho_e \frac{\rho_e}{\rho_p} \frac{\partial \mu_e}{\partial \rho_e} + \rho_\mu \frac{\rho_\mu}{\rho_p} \frac{\partial \mu_\mu}{\partial \rho_\mu} \right], \quad (8.45)$$

$$V_3^2 = \frac{\rho_n}{\rho_p} \left[\rho_p \frac{\partial \mu_n}{\partial \rho_n} + \rho_p \frac{\partial \mu_p}{\partial \rho_p} - \rho_p \left(\frac{\partial \mu_n}{\partial \rho_p} + \frac{\partial \mu_p}{\partial \rho_n} \right) + \rho_e \frac{\rho_e}{\rho_p} \frac{\partial \mu_e}{\partial \rho_e} + \rho_\mu \frac{\rho_\mu}{\rho_p} \frac{\partial \mu_\mu}{\partial \rho_\mu} \right], \quad (8.46)$$

$$V_4^2 = \rho_n \frac{\partial \mu_n}{\partial \rho_n}, \quad (8.47)$$

$$V_5^2 = \rho_p \frac{\partial \mu_p}{\partial \rho_p} + \rho_e \frac{\rho_e}{\rho_p} \frac{\partial \mu_e}{\partial \rho_e} + \rho_\mu \frac{\rho_\mu}{\rho_p} \frac{\partial \mu_\mu}{\partial \rho_\mu}. \quad (8.48)$$

Setting the dissipative coefficients equal to zero; frequencies can be computed from Eq. (8.37) using the quadratic formula. The zero frequency modes can be shown to be quickly damped when the dissipative coefficients are included and will not be considered further in this chapter. The relationship between the velocity fields is most easily expressed in terms of the mass currents given in the LFLW limit by,

$$\delta M_n^a = \rho_{nn} \delta v_n^a + \rho_{np} \delta u^a, \quad (8.49)$$

$$\delta M_p^a = \rho_{np} \delta v_n^a + \rho_{pp} \delta u^a. \quad (8.50)$$

The resulting relationship between these vectors for the inplane modes is

$$\begin{aligned} 2\Omega \left[\frac{\rho_n \rho_{pp} + \rho_p \rho_{nn}}{\varrho^2} \omega^2 + \frac{\rho_{nn}}{\rho_{np}} \rho_p \frac{\partial \mu_n}{\partial \rho_p} k^2 - \rho_n \frac{\partial \mu_n}{\partial \rho_n} k^2 \right] \delta M_{n\parallel} \\ = i \left[\frac{\rho_n \rho_p}{\varrho^2} \omega^3 + 4\Omega^2 \omega + \frac{\rho_n}{\rho_{np}} \rho_p \frac{\partial \mu_n}{\partial \rho_p} k^2 \omega \right] \delta M_{n\perp}, \end{aligned} \quad (8.51)$$

$$\begin{aligned} \left[i\omega \frac{\rho_n \rho_{np}}{\varrho^2} + i\rho_n \frac{\partial \mu_n}{\partial \rho_p} \frac{k^2}{\omega} \right] \delta M_{p\parallel} \\ = \left[\left(i\omega \frac{\rho_n \rho_{pp}}{\varrho^2} - i\rho_n \frac{\partial \mu_n}{\partial \rho_n} \frac{k^2}{\omega} \right) \delta M_{n\parallel} + 2\Omega \delta M_{n\perp} \right], \end{aligned} \quad (8.52)$$

$$\delta M_{p\perp} = -\frac{2\Omega}{i\omega} \frac{\varrho^2}{\rho_n \rho_{np}} \delta M_{n\parallel} + \frac{\rho_{pp}}{\rho_{np}} \delta M_{n\perp}, \quad (8.53)$$

where $\parallel$ refers to the component of the mass currents parallel to the wave vector $\vec{k}$ and $\perp$ refers to the component perpendicular to both the wave vector and the z -axis. These equations can be inverted to relate δv_n^a to δu^a .

Using the complete expression for the dispersion relation given in Eq. (8.37) and expanding $(\omega/2\Omega)$ out to first order in B_n and $(\eta_e k^2/2\Omega\rho_p)$, the following expression for the damping time of these modes is obtained:

$$\frac{1}{\tau} = \frac{1}{\tau_{MF}} + \frac{1}{\tau_{\eta_e}}, \quad (8.54)$$

where,

$$\frac{1}{\tau_{MF}} = |2\Omega|B_n \left(\frac{\Gamma_{MF}}{\Gamma} \right), \quad (8.55)$$

$$\frac{1}{\tau_{\eta_e}} = \frac{\eta_e k^2}{\rho_p} \left(\frac{\Gamma_{\eta}}{\Gamma} \right), \quad (8.56)$$

and where Γ_{MF} , Γ_{η} , and Γ are given by

$$\begin{aligned} \Gamma_{MF} = & \frac{2\rho^2}{\rho_n\rho_p} \frac{\rho_n + \rho_p}{\rho_p} \left(\frac{\omega_o}{2\Omega} \right)^4 \\ & - \frac{\rho^2}{\rho_n\rho_p} \left\{ \frac{V_2^2 k^2}{4\Omega^2} + \frac{\rho_n + \rho_p}{\rho_p} \left[2 + \frac{(v_{s1}^2 + v_{s2}^2)k^2}{4\Omega^2} \right] \right\} \left(\frac{\omega_o}{2\Omega} \right)^2 \\ & + \left[\frac{\rho^2}{\rho_n\rho_p} \frac{\rho_n + \rho_p}{\rho_p} \frac{v_{s1}^2 v_{s2}^2 k^4}{16\Omega^4} + \left(\frac{\rho^2}{\rho_n\rho_p} \right)^2 \frac{V_3^2 k^2}{4\Omega^2} \right], \end{aligned} \quad (8.57)$$

$$\begin{aligned} \Gamma_{\eta} = & \frac{7}{3} \frac{\rho_{pp}}{\rho_p} \left(\frac{\omega_o}{2\Omega} \right)^4 - \left[\frac{4}{3} \frac{\rho^2}{\rho_n\rho_p} \frac{V_4^2 k^2}{4\Omega^2} + \frac{\rho_{pp}}{\rho_p} \frac{(v_{s1}^2 + v_{s2}^2)k^2}{4\Omega^2} + \frac{7}{3} \frac{\rho^2}{\rho_n\rho_p} \frac{\rho_{nn}}{\rho_n} \right] \left(\frac{\omega_o}{2\Omega} \right)^2 \\ & + \left[\frac{\rho_{pp}}{\rho_p} \frac{v_{s1}^2 v_{s2}^2 k^4}{16\Omega^4} + \left(\frac{\rho^2}{\rho_n\rho_p} \right)^2 \frac{V_5^2 k^2}{4\Omega^2} \right], \end{aligned} \quad (8.58)$$

$$\begin{aligned} \Gamma = & 6 \left(\frac{\omega_o}{2\Omega} \right)^4 - 4 \left[1 + \left(\frac{\rho^2}{\rho_n\rho_p} \right)^2 + \frac{(v_{s1}^2 + v_{s2}^2)k^2}{4\Omega^2} \right] \left(\frac{\omega_o}{2\Omega} \right)^2 \\ & + 2 \left[\left(\frac{\rho^2}{\rho_n\rho_p} \right)^2 \left(1 + \frac{V_1^2 k^2}{4\Omega^2} \right) + \frac{v_{s1}^2 v_{s2}^2 k^4}{16\Omega^4} \right]. \end{aligned} \quad (8.59)$$

In Eq. (8.57)–(8.59) ω_o is the frequency in the limit the dissipative coefficients are set equal to zero.

The results for the phase velocities scaled with the speed of light c and the relative amplitudes are given Table 3 for $\Omega = 10^2 \text{s}^{-1}$ and $\lambda = 10^6 \text{cm}$. The results for the phase velocities scaled with the speed of light c and the damping times are given in Table 4 for $\Omega = 10^2$, $\lambda = 10^6 \text{cm}$, and $T = 10^7 \text{K}$. As can be seen in these tables the angular velocity has a negligible effect on the frequencies, and thus the phase velocities of these modes do not differ significantly from those for the axial sound waves.

One result to note in Table 3 is that the neutron and charged fluids do not flow exactly together for either mode (this can also be seen in Table 1). For the mode labelled "1" the amplitudes of the velocity components parallel to the wave vector are related by $\delta v_{n\parallel} \sim .7\delta u_{\parallel}$. For the mode labelled "2" the amplitudes of the velocity components parallel to the wave vector are related by $\delta v_{n\parallel} \sim -.08\delta u_{\parallel}$. The neutron and charged fluids do not flow together since the restoring forces acting on the neutron and charged fluids differ via the different chemical potentials of these two fluids. Thus, it can be expected that these two fluids will oscillate with unequal amplitudes for any mode which is sensitive to the equation of state.

In Table 4 it can be seen that $\tau_{\text{MF}} < \tau_{\eta_e}$. Thus mutual friction dominates the damping for $\Omega = 10^2 \text{s}^{-1}$, $\lambda = 10^6 \text{cm}$, and $T = 10^7 \text{K}$. As pointed out in the last section, the viscous damping time, τ_{η_e} , scales with the temperature as T^2 and with wavelength as λ^2 . The mutual friction damping time, τ_{MF} , scales with the angular velocity as Ω^{-1} . Thus mutual friction will also dominate at higher temperatures or larger angular velocities, while viscosity will dominate at shorter wavelengths. This is in agreement with the estimates given in Chapter 7.

Table 3: Inplane Phase Velocities and Relative Amplitudes.

ρ_{14}	$(\omega/ck)_1$	$(\delta v_{n }/\delta u_{ })_1$ $(\delta v_{n\perp}/\delta u_{\perp})_1$ $(i\delta v_{n\perp}/\delta v_{n })_1$	$(\omega/ck)_2$	$(\delta v_{n }/\delta u_{ })_2$ $(\delta v_{n\perp}/\delta u_{\perp})_2$ $(i\delta v_{n\perp}/\delta v_{n })_2$
2.89	0.128	0.747	0.311	-0.0701
		0.587		-0.0551
		0.00821		0.00558
3.02	0.130	0.737	0.313	-0.0731
		0.575		-0.0570
		0.00807		0.00551
3.15	0.132	0.727	0.315	-0.0765
		0.562		-0.0592
		0.00791		0.00542
3.22	0.134	0.722	0.316	-0.0783
		0.556		-0.0602
		0.00784		0.00538
3.43	0.137	0.707	0.320	-0.0839
		0.537		-0.0637
		0.00760		0.00526
3.65	0.141	0.693	0.324	-0.0895
		0.520		-0.0671
		0.00739		0.00515
3.89	0.145	0.680	0.328	-0.0952
		0.504		-0.0705
		0.00719		0.00504

Table 4: Inplane Phase Velocities and Damping Times (in Seconds).

ρ_{14}	$(\omega/ck)_1$	$\tau_{\eta e1}$	τ_{MF1}	τ_1	$(\omega/ck)_2$	$\tau_{\eta e2}$	τ_{MF2}	τ_2
2.89	0.128	822.	202.	162.	0.311	46.7	1.75	1.69
3.02	0.130	761.	178.	144.	0.313	46.0	1.74	1.67
3.15	0.132	699.	156.	127.	0.315	45.1	1.72	1.65
3.22	0.134	671.	146.	120.	0.316	44.7	1.71	1.65
3.43	0.137	592.	120.	100.	0.320	43.6	1.68	1.62
3.65	0.141	526.	101.	84.7	0.324	42.5	1.65	1.59
3.89	0.145	471.	86.1	72.8	0.328	41.6	1.63	1.57

CHAPTER 9

STABILITY OF ROTATING SUPERFLUID NEUTRON STARS

Introduction

Consider the perturbations of a neutron star having a time dependence given by $\exp(i\omega t - t/\tau)$, where ω and τ are real. Such a perturbation is stable whenever $1/\tau \geq 0$. Thus, in a sequence of rotating stars parameterized by the angular velocity Ω , those stars rotating slower than a critical angular velocity Ω_{CR} are stable, where Ω_{CR} is defined to be the smallest root of $1/\tau(\Omega_{\text{CR}}) = 0$. (Assuming of course that the star having $\Omega = 0$ is stable.) The critical angular velocity then represents the onset of an instability in the sequence of rotating stars (assuming that Ω_{CR} is not a local minimum). However, the sequence always terminates at the Keplerian limit, Ω_{MAX} , at which point mass on the equator rotates with its orbital velocity. A star rotating faster than this would be unstable to mass shedding, and thus no equilibrium models exist with angular velocities greater than Ω_{MAX} . Thus, the maximum stable angular velocity of a star is given by Ω_{CR} if solutions to $1/\tau(\Omega_{\text{CR}}) = 0$ exist with $\Omega_{\text{CR}} < \Omega_{\text{MAX}}$, otherwise it is given by Ω_{MAX} .

The maximum stable angular velocity of a neutron star can be limited by a gravitational-radiation instability. Chandrasekhar (1970a,b) showed that certain rotating stars are unstable to the emission of gravitational radiation. Friedman and Schutz (1977) showed that this was in fact true for all rotating perfect fluid

configurations. They also showed that the modes most sensitive to the onset of the instability were nonaxisymmetric with an angular dependence $\exp(im\phi)$, where ϕ is the azimuthal coordinate and m is an integer. On the other hand, internal dissipation in the star tends to counteract this instability. (See also Friedman and Schutz 1977; Lindblom and Detweiler 1977; Friedman 1978, 1983; Lindblom and Hiscock 1983; Lindblom 1986; Lindblom and Mendell 1991; Lindblom 1991).

To understand what causes the gravitational-radiation instability to occur in rotating stars consider a star viewed from above one of its rotational poles, and define the angular momentum of the star to be positive. If the star begins to oscillate with angular and time dependence $\exp(im\phi + i\omega t)$, it will emit gravitational radiation because of the time varying mass distribution (e.g., see Schutz 1985). Such a star is depicted in Figure 7. If the bumps shown in Figure 7 travel in the same sense as the equilibrium angular velocity, when viewed from a distant inertial frame, then the gravitational radiation emitted from the star will carry away positive angular momentum. To conserve angular momentum the star must then lose positive angular momentum. To see how this will occur, consider the bumps when viewed in a frame which corotates with the equilibrium angular velocity. If the bumps, when viewed in this frame, still move in the same sense as the equilibrium angular velocity the bumps add positive angular momentum to the star. In this case, the star can lose positive angular momentum by losing the bumps, i.e., gravitational-radiation reaction causes the bumps to damp out. On the other hand, if the bumps, when viewed in the corotating frame, move in the opposite direction of the equilibrium angular velocity (i.e., they are attempting to swim up stream so to speak) then the bumps subtract angular momentum from the star. In this case, the star can lose positive angular momentum by increasing the

amplitude of the bumps, i.e., gravitational-radiation reaction causes the bumps to grow—the mode is unstable. Note that two conditions must be satisfied for the instability to occur: the bumps must travel in the same sense as the equilibrium angular velocity when viewed in an inertial frame and they must travel in the opposite direction when viewed in the corotating frame. Thus, the onset point of the instability occurs when the bumps appear stationary in the inertial frame. At least this would be true if no other forms of dissipation were present. To determine the damping or growth time of a mode all forms of dissipation affecting the mode must be included in the calculation.

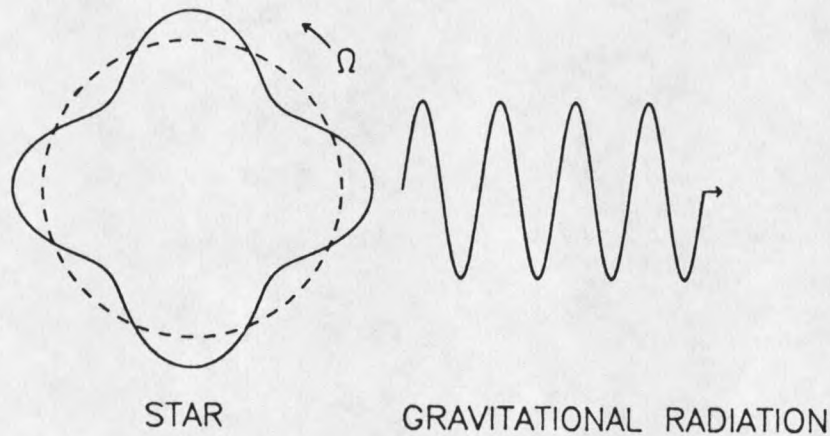


Figure 7. Equatorial region of an oscillating star in a $m=4$ mode. Such a star will emit gravitational radiation.

In the next section, using the results of Thorne (1969b), an expression is found for the damping (or growth) time of a mode due to gravitational-radiation reaction in a superfluid neutron star. The last section of this chapter then investigates the effects of superfluidity on the gravitational-radiation instability. It will be shown that mutual friction completely suppresses this instability in neutron stars cooler than the superfluid-transition temperature.

Gravitational-radiation Reaction

Let F_G^a be the gravitational force per unit mass acting on each element of fluid. In the case of Newtonian hydrodynamics this force is given in terms of the gravitational potential φ by

$$F_G^a = -\nabla^a \varphi. \quad (9.1)$$

In the weak-field limit of general relativity Thorne (1969b) showed that Newtonian hydrodynamics could be extended to include the effects of gravitational-radiation reaction by replacing the gravitational force in Eq. (9.1) with

$$F_G^a = -\nabla^a \varphi + \nabla^a \varphi_{\text{GR}}, \quad (9.2)$$

where

$$\varphi_{\text{GR}} = - \sum_{l=2}^{\infty} \sum_{m=-l}^{m=l} (-1)^l N_l r^l Y_l^m \frac{d^{2l+1} D_l^m}{dt^{2l+1}}. \quad (9.3)$$

In Eq. (9.3) r is the radial coordinate in spherical coordinates, Y_l^m the spherical harmonic function, and N_l and D_l^m are given by

$$N_l = \frac{4\pi G}{c^{2l+1}} \frac{(l+1)(l+2)}{l(l-1)[(2l+1)!!]^2}, \quad (9.4)$$

$$D_l^m = \int \rho r^l Y_l^{*m} d^3x. \quad (9.5)$$

(The notation used here follows that of Ipser and Lindblom 1991.) The extended gravitational force given by Eq. (9.2) acts on the fluid velocities. Otherwise, Thorne (1969b) showed that the other equations of Newtonian hydrodynamics remain unchanged in the limit he considered.

To include the effects of gravitational-radiation reaction in the case of the fluid mixture which exists in a superfluid neutron star, the gravitational force

acting on each of the superfluids and the electron-muon fluid should also be given by Eq. (9.2). The reason is the equivalence principle, which states that all forms of mass and energy couple to gravity with equal strength. If the gravitational-radiation reaction force was different for each species of particle in the star then the damping time of a mode due to gravitation-radiation reaction would depend on the concentrations of the various species, in violation of this principle.

The formalism for finding the damping time of a mode given in Chapter 7 can now be easily extended to include the effects of gravitational-radiation reaction. The energy functional which gives the damping times can be chosen exactly the same as in Eq. (7.31), i.e.,

$$E = \frac{1}{2} \int \left[\rho_{nn} \delta v_n^b \delta v_{nb}^* + \rho_{np} \delta v_n^b \delta u_b^* + \rho_{np} \delta u^b \delta v_{nb}^* + \rho_{pp} \delta u^b \delta u_b^* \right. \\ \left. + \sum_{AB} \frac{\partial^2 U_I}{\partial y_A \partial y_B} \delta y_A \delta y_B^* - \frac{1}{4\pi G} (\nabla^b \delta \varphi) (\nabla_b \delta \varphi^*) \right] d^3 x. \quad (9.6)$$

However, when the gravitational forces in Eq. (9.2) are included in the velocity equations the time derivative of E is modified. In this case it is given by

$$\frac{dE}{dt} = \left(\frac{dE}{dt} \right)_o + \left(\frac{dE}{dt} \right)_{\text{GR}}, \quad (9.7)$$

where

$$\left(\frac{dE}{dt} \right)_o = - \int \left[2\eta_e \delta \Theta^{ab} \delta \Theta_{ab}^* + \zeta_e \delta \Theta \delta \Theta^* + \frac{\kappa_e}{T_e} (\nabla^b \delta T_e) (\nabla_b \delta T_e^*) \right. \\ \left. + \rho_n B_n \left(\frac{\varrho^4}{\rho_n^2 \rho_p^2} \right) |\vec{\omega}_o| (g^{ab} - \frac{\omega_o^a \omega_o^b}{|\vec{\omega}_o|^2}) (\delta u_a - \delta v_{na}) (\delta u_b^* - \delta v_{nb}^*) \right] d^3 x, \quad (9.8)$$

is identical to that given in Eq. (7.32) and where

$$\left(\frac{dE}{dt} \right)_{\text{GR}} = \frac{1}{2} \int \left[\rho \delta \left(\frac{P^b}{\rho} \right)^* \nabla_b \delta \varphi_{\text{GR}} + \rho \delta \left(\frac{P^b}{\rho} \right) \nabla_b \delta \varphi_{\text{GR}}^* \right] d^3 x. \quad (9.9)$$

In Eq. (9.9) P^b is the total momentum density of the fluid, which is also the sum of the mass currents. Thus, the sum of the mass conservation laws can be written as

$$\partial_t \delta \rho + v_o^b \nabla_b \delta \rho + \nabla_b \left[\rho \delta \left(\frac{P^b}{\rho} \right) \right] = 0. \quad (9.10)$$

For modes which have space time dependence $\exp(im\phi + i\omega t)$ this becomes

$$(i\omega + im\Omega)\delta \rho + \nabla_b \left[\rho \delta \left(\frac{P^b}{\rho} \right) \right] = 0. \quad (9.11)$$

Thus, integrating Eq. (9.9) by parts and using Eq. (9.3) to find $\delta \varphi_{\text{GR}}$ gives

$$\left(\frac{dE}{dt} \right)_{\text{GR}} = -(\omega + m\Omega) \sum_{l=2}^{\infty} \sum_{m'=-l}^{m'=l} N_l \omega^{2l+1} \delta D_l^{m'} \delta D_l^{*m'}, \quad (9.12)$$

where

$$\delta D_l^{m'} = \int \delta \rho r^l Y_l^{*m'} d^3x. \quad (9.13)$$

The damping time due to gravitational-radiation reaction can then be defined by

$$\frac{1}{\tau_{\text{GR}}} = -\frac{1}{2E} \left(\frac{dE}{dt} \right)_{\text{GR}}. \quad (9.14)$$

Thus, $1/\tau_{\text{GR}}$ is given by

$$\frac{1}{\tau_{\text{GR}}} = -(\omega + m\Omega) \sum_{l=l_{\min}}^{\infty} N_l \omega^{2l+1} \delta D_l^m \delta D_l^{*m}. \quad (9.15)$$

The sum in Eq. (9.12) has been simplified in Eq. (9.15) by noting that $\delta D_l^{m'}$ vanishes unless $m' = m$, since $\delta \rho$ varies with ϕ as $\exp(im\phi)$ and $Y_l^{*m'}$ varies with ϕ as $\exp(-im'\phi)$. Thus, in Eq. (9.15) $l_{\min}$ is the larger of 2 or $|m|$.

Superfluidity Effects on the Gravitational-radiation Instability

The total damping time τ of a mode can then be calculated from

$$\frac{1}{\tau} = -\frac{1}{2E} \frac{dE}{dt}. \quad (9.16)$$

As in Chapter 7, the inverse damping time can be thought of as a sum of inverse damping times, each due to one of the forms of dissipation, i.e.,

$$\frac{1}{\tau} = \frac{1}{\tau_K} + \frac{1}{\tau_\zeta} + \frac{1}{\tau_\eta} + \frac{1}{\tau_{MF}} + \frac{1}{\tau_{GR}}. \quad (9.17)$$

This equation is similar to that given in Chapter 7 except that now τ_{GR} has also been included. The condition for an instability to set in is given by

$$1 + \frac{\tau_\eta}{\tau_K} + \frac{\tau_\eta}{\tau_\zeta} + \frac{\tau_\eta}{\tau_{MF}} + \frac{\tau_\eta}{\tau_{GR}} = 0. \quad (9.18)$$

As already discussed in Chapter 7, below the superfluid-transition temperature of roughly 10^9K the ratios τ_η/τ_K and τ_η/τ_ζ are much less than one and can be ignored. Furthermore Ipser and Lindblom (1991) have investigated the $l = m$ f -mode oscillations of Newtonian polytropes. (An f -mode is a p -mode in which there is no node in the radial eigenfunction.) Using their results the ratio τ_η/τ_{GR} can be written as

$$\left[\frac{\tau_\eta}{\tau_{GR}} \right]_m = \frac{1}{[\gamma_m(\Omega)]^{2m+1}} \left[\frac{\tau_\eta(0)}{\tau_{GR}(0)} \right]_m \left[\alpha_m(\Omega) - \frac{m\Omega}{\omega_m(0)} \right]^{2m+1}, \quad (9.19)$$

where a 0 refers to the zero angular velocity case and $\gamma_m \approx 1$ and $\alpha_m \approx 1$ are very slowly varying functions of the angular velocity Ω . Also, the results in Chapter 7 show that the ratio τ_η/τ_{MF} is bounded by

$$\frac{\tau_\eta}{\tau_{MF}} > 10^5 \left(\frac{\lambda}{10^6 \text{cm}} \right)^2 \left(\frac{\Omega}{10^3 \text{s}^{-1}} \right) \left(\frac{T}{10^9 \text{K}} \right)^2. \quad (9.20)$$

Equation (9.20) should give a good estimate of the results for an $l = m$ f -mode when the wavelength λ is related to the neutron star radius R by $\lambda \approx 2\pi R/m$.

Making this substitution in Eq. (9.20) and combining the result with Eq. (9.19) gives an approximate expression for τ_η/τ , which is given by

$$\begin{aligned} \frac{\tau_\eta}{\tau} \approx 1 + \frac{\tau_\eta}{\tau_{\text{MF}}} + \frac{\tau_\eta}{\tau_{\text{GR}}} \approx 10^5 \left(\frac{4\pi^2}{m^2} \right) \left(\frac{R}{10^6 \text{cm}} \right)^2 \left(\frac{\Omega}{10^3 \text{s}^{-1}} \right) \left(\frac{T}{10^9 \text{K}} \right)^2 \\ + \frac{1}{[\gamma_m(\Omega)]^{2m+1}} \left[\frac{\tau_\eta(0)}{\tau_{\text{GR}}(0)} \right]_m \left[\alpha_m(\Omega) - \frac{m\Omega}{\omega_m(0)} \right]^{2l+1}. \end{aligned} \quad (9.21)$$

Setting the right-hand side of this equation equal to zero, the result can be used to search for approximate solutions to the equation $1/\tau(\Omega_{\text{CR}}) = 0$. Since an underestimate has been used for $\tau_\eta/\tau_{\text{MF}}$, this would also result in an underestimate for Ω_{CR} . However, using the numerical results in Ipser and Lindblom (1991) for the various quantities which appear in Eq. (9.21) it is found that *no* solution to $1/\tau(\Omega_{\text{CR}}) = 0$ exist. Thus, mutual friction completely suppresses the gravitational-radiation instability for temperatures below the superfluid-transition temperature.

Ipser and Lindblom also calculate Ω_{CR} for nonsuperfluid neutron stars and for temperatures above the superfluid-transition temperature. (At higher temperatures bulk viscosity dominates the damping.) Their results for nonsuperfluid matter are given in Figure 8. For comparison, the expected results for a superfluid neutron star are given in Figure 9, assuming a superfluid-transition temperature of approximately 10^9K .

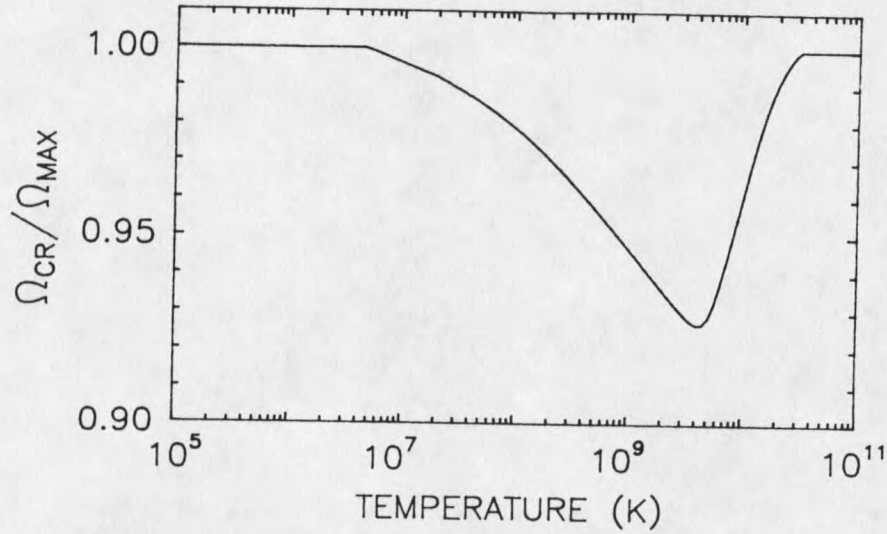


Figure 8. Critical angular velocity versus temperature without superfluidity. The results shown are for a $1.5M_{\odot}$ neutron star. The equation of state is approximated by a polytrope of index $n = 1$. (After Ipser and Lindblom 1991.)

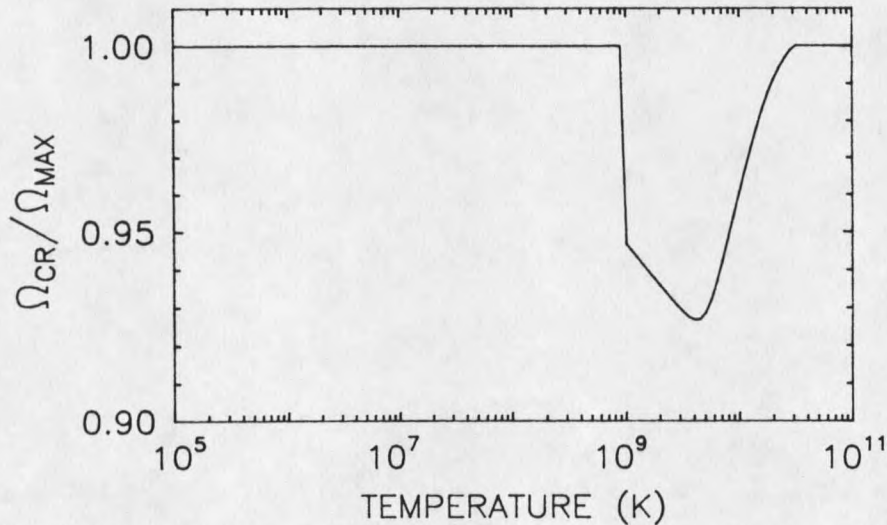


Figure 9. Critical angular velocity versus temperature with superfluidity. A superfluid-transition temperature of approximately 10^9 K is assumed. The figure shows that the gravitational-radiation instability is completely suppressed below the transition temperature to superfluidity as explained in the text.

CHAPTER 10

SUMMARY

Superfluid hydrodynamics in neutron stars has been developed extensively in this thesis. A more detailed summary of the important results from each chapter is given below.

In Chapter 2 a slightly new derivation of the Landau equations is given. This provides a more convenient starting point for the generalization of these equations to superfluid mixtures, which is given in Chapter 3. The most important result of Chapter 3 is that the derivation given here shows that a new class of vorticity-preserving forces can be included in the equations for a mixture of N superfluids. These forces have a profound influence on the coupling of the charged species of the mixture to the electromagnetic field. For a specific value of these forces it is shown that the superfluids respond to an appropriate Lorentz force that is far more natural than that proposed in the previous studies of Vardanyan and Sedrakyan (1981) or Holm and Kupershmidt (1987). Also in Chapter 3 the method Bekarevich and Khalatnikov (1961) use to account for the vortex energies is generalized to the case of charged fluids. The results in Chapter 3 have been published in Mendell and Lindblom (1991).

In Chapters 4 and 5 the equations in Chapter 3 are adapted to describe the superfluid interior region of a neutron star composed of superfluid neutrons, superconducting protons, and degenerate electrons and muons. The equations given in these chapters differ significantly from those derived previously in the

related studies of Easson and Pethick (1979), Epstein (1988), and those which describe an ordinary fluid used in most past studies of neutron star oscillations. The equations here generalize the MHD equations of Easson and Pethick (1979) to include dynamics for the superfluid neutrons and the effects of superconducting protons. They generalize the equations derived in Epstein (1988) to include the effects of gravitation, electromagnetism, rotation, and the vortices. They also correct an error in Epstein's work: the definitions in his paper of the kinetic energy density, the mass currents, and the mass density matrix are inconsistent in that they incorrectly imply that the mass currents are parallel to the superfluid velocities. The most important result of Chapter 4 is that a model of the energy density is constructed and used to relate certain dependent variables in the theory to the independent variables. A summary of the oscillation frequencies resulting from the forces which act on the fluids is also given in this Chapter. An important result shown in Chapter 5 is that a superconducting plasma will not support Alfvén waves, found in ordinary plasmas. Instead, a new mode with a frequency equal to the square root of the product of the cyclotron and vortex frequencies is shown to exist. This new mode may be observable in terrestrial superconductors. The results in Chapter 5 are applicable to studies of g-modes and p-modes in neutron stars, including the effects of rotation, electromagnetism, and the effects of the underlying vortices. The final set of equations presented at the end of Chapter 5 are especially suited to studies of p-modes in rapidly rotating neutron stars. The work presented in Chapters 4 and 5 is based on that given in Mendell (1991a).

In Chapters 6 and 7 the equations are generalized to include the effects of dissipation. The work presented in Chapters 6 and 7 is based on that given in Mendell (1991b). Most importantly, the effects of mutual friction are included for

the first time in the equations for a mixture of superfluids and in the equations which describe a superfluid neutron star. Mutual friction in neutron stars occurs because of the scattering of electrons off the quantized vortices which occur in the neutron and proton superfluids. At the end of Chapter 7 a convenient method is presented for computing the damping times of p-modes due to mutual friction and the other forms of dissipation. The important conclusion of Chapter 7 is mutual friction is the dominant internal damping mechanism of these modes in rapidly rotating superfluid neutron stars. This conclusion is confirmed by results for plane-wave solutions given in Chapter 8. Chapter 8 also investigates the propagation of sound through superfluid neutron-star matter. It is shown that the theory presented here results in two possible sound modes, as is true in terrestrial superfluids. These results also correct the earlier work of Epstein (1988).

In Chapter 9 it is argued that the damping (or growth) time of a mode due to gravitational-radiation reaction is independent of the presence of superfluidity. However, in general, dissipation is completely different in superfluid neutron stars compared with the case of no superfluidity. Especially important, as has been emphasized above, is mutual friction due to the scattering of electrons off the vortices. This form of dissipation occurs only in superfluids. The conclusion of Chapter 9 is that mutual friction will completely prevent the onset of the gravitational-radiation instability in neutron stars with temperatures below the superfluid-transition temperature. The work in Chapter 9 is summarized in Lindblom and Mendell (1991).

The conclusions reached in this thesis have been made using many justifiable simplifying assumptions. However, relativistic effects have been ignored for the following reasons. The fully relativistic treatment of the arbitrary oscillations of

rotating stars has yet to be accomplished, and further development of relativistic superfluid hydrodynamics is also needed. Studies using the nonrelativistic equations developed here can give qualitative results and should also indicate whether more complicated relativistic studies are justified. It can be expected that in typical neutron stars general relativity will introduce corrections to the Newtonian results of the order $GM/c^2 R \times 100\% \sim 10\% - 20\%$. For example, a recent study finds that general relativistic effects tend to enhance the gravitational-radiation instability, lowering the critical angular velocity from its Newtonian value by up to 8% (Cutler and Lindblom 1991). This is too small of a change to alter the conclusion of Chapter 9.

Another important area which needs further study is the interaction between neutron vortices and proton vortices. In general, the rotation axis will not be aligned with the magnetic field axis in a rotating magnetized neutron star. Thus, for sufficiently large-amplitude oscillations the neutron and proton vortices either will be forced to move through each other or they will become pinned onto or between each other. Sauls (1989) has used the neutron and proton energy gaps to estimate that the pinning energy due to an intersection between neutron and proton vortices is about $\epsilon_{\text{pin}} \sim .1\text{Mev} = 2 \times 10^{-7}\text{erg}$. The magnetic interaction between the vortices could be even more important, giving a pinning energy of

$$\epsilon_{\text{pin}} \sim (2/8\pi)(\Phi_n/\pi\Lambda^2)(\Phi_p/\pi\Lambda^2)(\pi\Lambda^3)\vec{\nu}_n \cdot \vec{\nu}_p \leq 5 \times 10^{-5}\text{erg}. \quad (10.1)$$

(See Chapter 4 for the definitions and magnitude estimates of the quantities in this equation.) The pinning force per unit length, $\vec{f}_{\text{pin}}$, is then roughly the pinning energy divided by both the distance between pinning sites along a vortex and the distance over which the pinning energy returns to zero as the vortices are

separated. For overlapping arrays of neutron and proton vortices the smallest of these distances is usually the distance between proton vortices,

$$d_p = [h/(\sqrt{3}m_p|\vec{\omega}_p + a_p\vec{B}|)]^{1/2} \approx 4.9 \times 10^{-10}(10^{12}\text{G}/B_o)^{1/2}\text{cm}. \quad (10.2)$$

Thus, a rough estimate of the magnitude of the pinning force per unit length is then

$$f_{\text{pin}} \sim \epsilon_{\text{pin}}/d_p^2 \sim 2 \times 10^{14}(B_o/10^{12}\text{G})\text{dyne/cm}. \quad (10.3)$$

Of course, the exact form of the pinning force will be complicated. For example, the pinning energy and distance between pinning sites will depend strongly on the value of the geometric factor $\vec{\nu}_n \cdot \vec{\nu}_p$, and this factor could be a complicated function of position in the star. But a comparison of the estimate of the pinning force with the drag force between the neutrons and protons, and with the Magnus force which acts on the vortices (see Chapter 6), shows that the pinning force could be important in comparison with these forces. If so, the results of this thesis remain valid in at least three important cases: (i) a rotating neutron star in which the interior magnetic field has been expelled, (ii) a magnetized neutron star which is not rotating, and (iii) a rotating magnetized neutron star in which the oscillation amplitudes are small compared with d_p . Further study of the neutron-vortex proton-vortex interaction is needed to determine whether the equations presented in this thesis are valid beyond these specific cases.

A third area which may be fruitful to study in the future is the spin triplet pairing of the neutrons. In this thesis the neutron vortices are treated as if in the spin singlet state. This is currently believed to be quite adequate on length scales much larger than the radius of the vortex cores, and thus it should be completely satisfactory in the smooth-averaged hydrodynamics presented in this

thesis. However, studies of vortices in ${}^3\text{He}$, which also form a superfluid with spin triplet pairing, have revealed complicated structures (e.g., see Salomaa and Volovick 1987). Further investigation of the neutron vortices could thus reveal unexpected results.

Finally, in Chapter 4 it was pointed out that the effective mass of the proton is uncertain, though probably bounded by $.3 \leq m_p^*/m_p < .7$ at nuclear density. It was also pointed out in Chapter 4 that there appears to be an inconsistency in the calculations of the effective mass and the mass density matrix. This needs to be straightened out in the future. It is not expected, however, that the outcome of such a study will significantly change the results of this thesis.

Of course the greatest uncertainty in the theory of neutron stars is the equation of state of nuclear matter. An accurate determination of the maximum angular velocity of a neutron star may provide a key to this problem. This in turn may help open the window to the discovery of other objects such as black holes or even quark stars. This thesis, hopefully, makes some small contribution to these and other quests for a deeper understanding of the universe. Whatever the case, neutron stars will continue to be remarkable objects which will fascinate the curious, perhaps forever.

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