



Subsystems of and interlacing theorems for interface Sturm-Liouville systems
by Dennis Numan Winslow

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in
Mathematics

Montana State University

© Copyright by Dennis Numan Winslow (1983)

Abstract:

This thesis deals with several important problems concerned with interface Sturm-Liouville systems. One of these problems is to determine how separation or interlacing theorems may be applied to the characteristic equations or auxiliary functions of such systems. Our attempt to solve this problem requires the introduction of the concept of a subsystem of a Sturm-Liouville interface system. Another problem is to discover if some results known to hold for ordinary Sturm—Liouville systems also hold for interface Sturm-Liouville systems. The first of these problems is solved by developing a theorem which asserts that the zeros of the auxiliary function of the entire system interlace the zeros of the products of the auxiliary functions of certain subsystems. This interlacing is used to solve the last problem by providing the basis for induction proofs which show that the set of characteristic values of any Sturm-Liouville interface system is countably infinite, contains a smallest element, is unbounded above, and has no finite limit point. Although information is also obtained about auxiliary functions of Sturm-Liouville interface systems and subsystems,, the main conclusions of this thesis are that the interlacing theorem is available for use in investigating characteristic values of Sturm-Liouville interface systems and that some important properties of characteristic values of ordinary Sturm-Liouville systems are also properties of characteristic values of interface Sturm-Liouville systems.

**SUBSYSTEMS OF AND INTERLACING THEOREMS
FOR INTERFACE STURM-LIOUVILLE SYSTEMS**

by

Dennis Numan Winslow

**A thesis submitted in partial fulfillment
of the requirements for the degree**

of

Doctor of Philosophy

in

Mathematics

**MONTANA STATE UNIVERSITY
Bozeman, Montana**

July 1983

D378
W732
cop.2

ii

APPROVAL

of a thesis submitted by

Dennis Numan Winslow

This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

June 14, 1983
Date

Louis C. Barrett
Chairperson, Graduate Committee

Approved for the Major Department

July 26 1983
Date

K. J. Tinkert
Head, Major Department

Approved for the College of Graduate Studies

7-27-83
Date

Michael Malone
Graduate Dean

Date Dec 26, 1985

ACKNOWLEDGMENTS

The author would like to thank his graduate adviser Dr. Louis C. Barrett. Dr. Barrett's help in guiding the research for this thesis and his assistance in writing it were invaluable in completing this work.

The author would also like to extend his gratitude to the members of his graduate committee. Dr. Gary Bogar, Dr. John Lund, and Dr. David Wend are due special thanks for carefully reading this thesis and offering suggestions for its improvement.

Lynda Mikkola's expert typing, as well as her patience with the many corrections made in the wording and form of this thesis as it was being typed, are greatly appreciated.

TABLE OF CONTENTS

	<u>Page</u>
I. Introduction	1
II. Interface Sturm-Liouville Systems	7
1. Statement of the Problem and Related Theorems	7
2. A Fundamental Interlacing Theorem	10
3. Characteristic Equations and Auxiliary Functions of Interface Systems	11
III. An Interface Theorem	13
1. Special Representations of the Auxiliary Function	13
2. Subsystems of Interface Sturm-Liouville Systems	16
3. Interlacing Properties of the Characteristic Numbers of Certain Subsystems	18
IV. Characteristic Numbers and Auxiliary Functions	30
1. A Subsystem Method for Locating Characteristic Numbers of Interface Systems	30
2. Further Properties of Characteristic Values of Interface Systems	32
3. Determinant Forms of Auxiliary Functions	39
4. Characteristic Determinants of Subsystems	41
5. Characteristic Determinants Derived from Different Bases	44
V. Illustrative Examples	50
1. A Specific Example of an Interface Sturm-Liouville System	50
2. Nonnegative Characteristic Values	51
3. Auxiliary Functions and Their Derivations	53
4. Interlacing Properties	55
5. Characteristic Functions and Their Orthogonality	63
6. Intervals Containing Characteristic Values of an Interface System	68
VI. Summary and Topics for Further Research	71
VII. Bibliography	73

LIST OF TABLES

<u>Table</u>		<u>Page</u>
1	Auxiliary functions of system (1 - 1;3) and its subsystems	56
2	Characteristic values of subsystems of system (1 - 1;3)	57
3	Characteristic functions of the subsystems of system (1 - 1;3)	64

LIST OF FIGURES

<u>Figure</u>		<u>Page</u>
1	Interlacing by the auxiliary function of the entire system	60
2	Interlacing by auxiliary functions of systems defined for $-2 < x < 1$	61
3	Interlacing by auxiliary functions of systems defined for $-1 < x < 2$	62
4	Demonstration of the result of Theorem 1, Sec. 1, Chap. 3	70

ABSTRACT

This thesis deals with several important problems concerned with interface Sturm-Liouville systems. One of these problems is to determine how separation or interlacing theorems may be applied to the characteristic equations or auxiliary functions of such systems. Our attempt to solve this problem requires the introduction of the concept of a subsystem of a Sturm-Liouville interface system. Another problem is to discover if some results known to hold for ordinary Sturm-Liouville systems also hold for interface Sturm-Liouville systems. The first of these problems is solved by developing a theorem which asserts that the zeros of the auxiliary function of the entire system interlace the zeros of the products of the auxiliary functions of certain subsystems. This interlacing is used to solve the last problem by providing the basis for induction proofs which show that the set of characteristic values of any Sturm-Liouville interface system is countably infinite, contains a smallest element, is unbounded above, and has no finite limit point. Although information is also obtained about auxiliary functions of Sturm-Liouville interface systems and subsystems, the main conclusions of this thesis are that the interlacing theorem is available for use in investigating characteristic values of Sturm-Liouville interface systems and that some important properties of characteristic values of ordinary Sturm-Liouville systems are also properties of characteristic values of interface Sturm-Liouville systems.

INTRODUCTION

Shortly after the advent of the differential calculus, nearly 320 years ago, various engineering, mathematical, and scientific problems gave rise to boundary value problems of one kind or another. Those involving differential equations in the real domain, with one or two point boundary conditions, were the first to receive serious attention. During the past century and a half, the mathematical literature concerning such systems has become very extensive and includes both practical and theoretical results of great importance.

It was not until 1897, however, that Nicoletti [18]⁺ introduced boundary conditions at more than two points into a differential system. He studied a single n th order equation together with boundary conditions at k points of an interval, under the restriction $k \leq n$. By 1908, Picone [19] had considered boundary conditions involving an infinite number of points along with integral boundary conditions. Wilder [32,33] subsequently introduced parameters into the differential equations. Independent of the earlier work of Nicoletti, Whyburn [29], after the manner of Carathéodory [10], developed a fundamental existence and uniqueness theorem for a quite general system involving n first-order ordinary differential equations with boundary conditions at $k \leq n$ points. With these points confined to an interval of definite length, and under a suitable Lipschitz condition, the solutions of the

⁺ Numbers within brackets refer to corresponding references of the bibliography.

system were shown to be continuous functions of the boundary points and values.

The studies of these individuals have been greatly expanded, and many new features of boundary value problems investigated, by other mathematicians. A variety of concepts and techniques such as Green's functions [15,29,30,34], Stieltje's integrals [6,25], matrix methods [1,2,7], and the calculus of variations [12,17,24,27] have been employed in the process. Much of the significant work done in this area, up until the year 1942, has already been reviewed in the literature [29], as have other major developments through the year 1955 [31]. Among the studies cited in these reviews, four are of particular interest to us because they have to do with interface characteristic value problems, as does this thesis.

Interface characteristic value problems involve differential equations on several adjoining open intervals together with a set of auxiliary conditions which apply at the ends of these intervals. At each point between neighboring intervals the relevant interface conditions relate left-hand and right-hand limiting values of a solution and its derivatives. The differential equations contain a parameter which may, or may not, appear in the boundary conditions.

In 1935, Reid [21] encountered such problems as a by-product while searching for functions which minimize the distance between several specified points when the functions are allowed to have discontinuities in their derivatives at the given points. Similar connections between the calculus of variations and certain interface problems were also

explored by Mansfield [17] who, in 1941, found conditions for self-adjointness of interface characteristic value problems.

Perhaps, the first extensive study of Sturm-Liouville interface boundary value problems was undertaken by Sangren [22]. The left and right most subintervals of his system could extend to $-\infty$ and ∞ , respectively. He showed that the related characteristic numbers were real and expansion theorems for the system were established. Applications of his results to problems in heat flow, potential and vibration theory, and nuclear reactor development were also pointed out.

Stallard [26], in 1955, studied matrix differential equations $Y' + PY = \lambda RY$ in an unknown matrix Y , with coefficient matrices P and R and a scalar parameter λ . The associated interface conditions were parameter free. He defined what is meant by a solution of such a system and gave existence and uniqueness theorems for solutions. His results also included conditions for self-adjointness, the development of Green's matrix, and expansion problems. He also gave conditions for self-adjointness of an interface system having $y'' + p_1 y' + p_2 y = \lambda r y$ as its differential equation, where p_1 , p_2 , and r are Lebesgue integrable functions.

During the past 25 years, boundary value problems have continued to provide a fruitful area of mathematical research.

By 1959, Barrett and Wylie [6] had used Stieltje's integrals to extend a number of results pertaining to classical Sturm-Liouville systems to systems whose boundary conditions contain the parameter. Applications of the theory to diffusion, heat flow, and vibration problems were also noted. That same year Barrett and Thorne [5]

investigated Sturm-Liouville interface boundary value problems of the kind considered in this thesis. The following year, 1960, Sangren [23] solved some two dimensional membrane and heat flow problems which, upon separation of variables, resulted in interface characteristic value problems to which his earlier work was applied. During 1964 other boundary value problems involving interface systems were studied by Barrett [3].

A difficulty that is often encountered in connection with characteristic value problems is that of finding the related characteristic values. Frequently, it is either impractical or impossible to find their exact values. One must then resort to numerical methods for their computation. Separation and interlacing theorems which have proven to be useful in this regard were given by Barrett and Bendixen [4] in 1965.

An interface characteristic value problem of still another sort is that of Jerome [16], published in 1970. He defined linear operators L and L^* and then considered a differential equation of the form $L^*Ly = \lambda y$ together with a mixed set of parameter free boundary conditions of either ordinary or interface types. Under suitable restrictions on the operators involved, he proceeded to establish positiveness and multiplicity properties of the characteristic numbers, determined orthogonality conditions for the characteristic functions, and gave a formula for the solution of an equation of the form $L^*Ly = g(x)$.

A quite recent history updating some of the work done on interface systems is set forth in two 1975 publications of Brown [8,9]. These

two articles have little to do with interface Sturm-Liouville systems, however.

Although a 1982 text by Wylie and Barrett [34] now discusses boundary value problems which entail interface Sturm-Liouville systems, the theory of such interface systems is still in an active state of development. The purpose of this thesis is to carry these investigations forward a little.

For the sake of clarity and easy reference, Chapter 1 begins with a characterization of just what is meant by an interface Sturm-Liouville system. After defining what is meant by characteristic values and characteristic functions of such systems, we state several well known theorems concerning these numbers and functions, and conditions under which they hold. Then characteristic equations and auxiliary functions of an interface system are derived. In Chapter 2 we introduce the notion of a subsystem and use a fundamental interlacing theorem to prove an interface interlacing theorem of considerable generality.

In Chapter 3, we apply our interface interlacing theorem to devise a subsystem method for locating characteristic numbers of an interface system within known intervals. In addition, we show that several well known properties of the characteristic numbers of ordinary Sturm-Liouville systems carry over to the characteristic numbers of interface Sturm-Liouville systems.

Ways of finding characteristic equations of both interface systems and their subsystems are discussed. This is done differently in

Chapters 2 and 3. A relation between auxiliary functions derived from different bases of the differential equations involved is also found.

Chapter 4 contains examples which serve to illustrate the theory of the first three chapters. Special attention is given to the interlacing phenomena of sets of characteristic numbers and the suborthogonality of sets of characteristic functions of the differential equations involved in various subsystems.

CHAPTER 1

INTERFACE STURM-LIOUVILLE SYSTEMS

1. Statement of the Problem and Related Theorems

Many problems in applied mathematics give rise to boundary value problems which, upon separation of variables, lead to differential systems of the type

$$(1) \quad \frac{d}{dx}[r(x)y'_k] + [\lambda p_k(x) + q_k(x)]y_k = 0, \quad x_{k-1} < x < x_k, \quad 1 \leq k \leq K$$

$$(0;1) \quad U_0(y_1) \equiv a_1 y'_1(a) + b_1 y_1(a) = 0$$

...

$$(k,1) \quad U_{k1}(y_k) + V_{k1}(y_{k+1}) \equiv [a_{k1} y'_k(x_k) + b_{k1} y_k(x_k)] \\ + [c_{k1} y'_{k+1}(x_k) + d_{k1} y_{k+1}(x_k)] = 0 \\ 1 \leq k \leq K-1$$

$$(k,2) \quad U_{k2}(y_k) + V_{k2}(y_{k+1}) \equiv [a_{k2} y'_k(x_k) + b_{k2} y_k(x_k)] \\ + [c_{k2} y'_{k+1}(x_k) + d_{k2} y_{k+1}(x_k)] = 0$$

...

$$(1;K) \quad V_K(y_K) \equiv c_K y'_K(b) + d_K y_K(b) = 0$$

where $a = x_0$, $x_K = b$, and $2 \leq K$. In a variety of applications, it turns out that the coefficient function r admits of a single representation $r(x)$ over the closed interval $[a,b]$, where r is continuous and $r(x) > 0$, [5, p. 2]. On each subinterval $x_{k-1} < x < x_k$ of (a,b) , p_k is continuous, $p_k(x) > 0$, q_k is continuous, and each p_k and q_k has finite limits at x_{k-1} and x_k . Expressions such as $y_k(x_k)$ and $y'_k(x_k)$ are to be interpreted as the left-hand limits

$$(2a) \quad y_k(x_k) = \lim_{x \rightarrow x_k^-} y_k(x) \quad \text{and} \quad y'_k(x_k) = \lim_{x \rightarrow x_k^-} y'_k(x) \quad 1 \leq k \leq K.$$

Similarly, $y_{k+1}(x_k)$ and $y'_{k+1}(x_k)$ stand for the right-hand limits

$$(2b) \quad y_{k+1}(x_k) = \lim_{x \rightarrow x_k^+} y_{k+1}(x) \quad \text{and} \quad y'_{k+1}(x_k) = \lim_{x \rightarrow x_k^+} y'_{k+1}(x) \quad 0 \leq k \leq K-1.$$

All of the constants $a_1, b_1, \dots, a_{k1}, \dots, d_{k2}, \dots, c_K, d_K$ are real.

Differential systems of this kind, involving boundary conditions of the type (0;1) and (1;K), and interface conditions of the sort (k,1) and (k,2) are referred to as interface Sturm-Liouville systems or Sturm-Liouville systems with interface conditions. Note that the semicolons in (0;1) and (1;K) serve to identify the boundary conditions, whereas commas are used in (k,1) and (k,2) to distinguish interface conditions. For brevity, we shall refer to the system just described as system (1 - 1;K).

A function y such that

$$(3) \quad y(x) = y_k(x), \quad x_{k-1} < x < x_k, \quad 1 \leq k \leq K$$

is called a solution of (1) on the fundamental interval (a,b) if, for $1 \leq k \leq K$, both y_k and y'_k are continuous and y_k satisfies (1) on the interval (x_{k-1}, x_k) . A solution y of (1) is said to be strictly nontrivial if there is no value of k for which $y_k(x) \equiv 0$. Interface systems having $y_k(x) \equiv 0$ for one or more values of k may be handled by considering two or more separate systems. Thus, from now on, only strictly nontrivial solutions for which each of the limits $y_k(x_k)$, $y'_k(x_k)$, $y_{k+1}(x_k)$, and $y'_{k+1}(x_k)$ exists, for the relevant values of k , will be called solutions of (1).

To make sure that the boundary conditions (0;1) and (1;K) are both present, the inequalities

$$(4) \quad (a) \quad a_1^2 + b_1^2 \neq 0 \quad (b) \quad c_K^2 + d_K^2 \neq 0$$

will be assumed to hold. With regard to the $2(K-1)$ interface conditions, we assume that

$$(5) \quad (a) \quad |AB|_k |CD|_k \neq 0 \quad (b) \quad R_k = |CD|_k / |AB|_k > 0 \quad 1 \leq k \leq K-1$$

where $|AB|_k$ and $|CD|_k$ are the determinants

$$(6) \quad (a) \quad |AB|_k = \begin{vmatrix} a_{k1} & b_{k1} \\ a_{k2} & b_{k2} \end{vmatrix} \quad (b) \quad |CD|_k = \begin{vmatrix} c_{k1} & d_{k1} \\ c_{k2} & d_{k2} \end{vmatrix} \quad 1 \leq k \leq K-1.$$

In this thesis the only interface systems we shall consider will be those in which (4) and (5) are satisfied.

Under hypotheses (4) and (5), it is known that there are values of λ , called characteristic values or characteristic numbers, for which the interface system has strictly nontrivial solutions. These solutions are called characteristic functions. The characteristic values and characteristic functions have the following important properties [5, pp. 3-13].

Theorem 1 The characteristic values are real.

Theorem 2 The characteristic values are simple roots of the characteristic equation.

Theorem 3 The characteristic functions are real.

Theorem 4 The characteristic functions are orthogonal over the fundamental interval (a, b) with respect to the weight function p defined by

$$p(x) = \left(\prod_{j=0}^{k-1} R_j \right) p_k(x) \quad x_{k-1} < x < x_k \quad 1 \leq k \leq K$$

where $R_0 = 1$.

2. A Fundamental Interlacing Theorem

If $F(\lambda) = 0$ is a characteristic equation of a Sturm-Liouville system, we shall call the function $F(\lambda)$ an auxiliary function of that system.

A convenient way of isolating one characteristic number from all the others of a Sturm-Liouville system containing just one pair of interface conditions, is to use the fact that the zeros of an auxiliary function of the system interlace the zeros of appropriately chosen products of simpler auxiliary functions of ordinary Sturm-Liouville systems which have no interface conditions [4, pp. 75-78]. In saying that the zeros of one function interlace the zeros of another, we mean the following:

Definition_1 The zeros of a function $f(\lambda)$ interlace the zeros of a function $g(\lambda)$ on an interval I if, and only if, on this interval:

1. no simple zero of $g(\lambda)$ is a zero of $f(\lambda)$
2. each double zero of $g(\lambda)$ is a zero of $f(\lambda)$
3. all zeros of $f(\lambda)$ are simple whereas the zeros of $g(\lambda)$ are either simple or double
4. between any two consecutive zeros of $g(\lambda)$, whether simple or double, there lies exactly one zero of $f(\lambda)$.

Now let $F(\lambda)$ be any function which can be expressed in the form

$$(1) \quad F(\lambda) = f_1(\lambda)f_4(\lambda) - f_2(\lambda)f_3(\lambda)$$

where f_1, f_2, f_3, f_4 are continuously differentiable functions over an interval I , and define W_{ij} to be the wronskian

$$(2) \quad W_{ij}(\lambda) = \begin{vmatrix} f_i(\lambda) & f_j(\lambda) \\ f_i'(\lambda) & f_j'(\lambda) \end{vmatrix}$$

of f_i and f_j , $1 \leq i, j \leq 4$. Also suppose

$$(3) \quad W_{12}(\lambda)W_{34}(\lambda) < 0 \quad \text{on } I.$$

Then, as a fundamental interlacing theorem we have [4, pp. 71-75]:

Theorem 1 The zeros of $F(\lambda)$ interlace the zeros of either of the product functions $f_1(\lambda)f_3(\lambda)$ or $f_2(\lambda)f_4(\lambda)$ on I .

In Chapter 2 we shall show how this theorem can be applied to the characteristic equation $F(\lambda) = 0$ of a general interface Sturm-Liouville system, or to $F(\lambda)$ itself, so as to isolate the characteristic numbers of the system from one another. Further properties of the characteristic numbers will also be established in Chapter 3.

3. Characteristic Equations and Auxiliary Functions of Interface Systems

Since, for $1 \leq k \leq K$, the same parameter λ occurs in each of the equations

$$(1) \quad \frac{d}{dx}[r(x)y'_k] + [\lambda p_k(x) + q_k(x)]y_k = 0 \quad x_{k-1} < x < x_k$$

solutions of these equations will not depend on x alone, as (3), Sec. 1, suggests, but on both λ and x . However, as in (1), a prime on y still denotes differentiation with respect to x . If $u_k(x, \lambda)$ and $v_k(x, \lambda)$ are linearly independent solutions of (1) on (x_{k-1}, x_k) , a complete solution of (1) on (a, b) is given by

$$(2) \quad y(x, \lambda) = A_k u_k(x, \lambda) + B_k v_k(x, \lambda) \quad x_{k-1} < x < x_k \quad 1 \leq k \leq K$$

where each A_k and B_k may depend on λ , but not on x .

Once a complete solution $y(x, \lambda)$ of (1) on (a, b) is known, a characteristic equation for the interface system $(1 - 1; K)$, Sec. 1, can be found by requiring $y(x, \lambda)$ to satisfy the auxiliary conditions

(0;1) through (1;K). When all of these conditions, except (1;K), are imposed on $y(x,\lambda)$, a system of $2K - 1$ homogeneous linear algebraic equations is obtained in the $2K$ parameters A_k and B_k . Since such a system always has nontrivial solutions, there are solutions of (1) on (a,b) which satisfy all of the auxiliary conditions, except (1;K). In fact, if $u(x,\lambda)$ is any one of these, they all have the form [5, pp. 9-11].

$$(3) \quad y(x,\lambda) = Au(x,\lambda) \quad A \neq 0$$

where A may depend on λ , but not on x or k .

Imposing (1;K) on (3), we get

$$(4) \quad V_K(y) \equiv c_K y'(b,\lambda) + d_K y(b,\lambda) = 0$$

as a characteristic equation, and

$$(5) \quad V_K(y) \equiv c_K y'(b,\lambda) + d_K y(b,\lambda)$$

as an auxiliary function of our interface Sturm-Liouville system.

Since $V_K(y) = AV_K(u)$, (4) is simply the equation

$$(6) \quad V_K(u) \equiv c_K u'(b,\lambda) + d_K u(b,\lambda) = 0$$

multiplied through by $A \neq 0$. Hence (4) and (6) are equivalent, and so the function

$$(7) \quad F(\lambda) \equiv V_K(u) \equiv c_K u'(b,\lambda) + d_K u(b,\lambda)$$

has the same zeros as (5). The solutions of (6), i.e. the zeros of (7), are the characteristic numbers of system $(1 - 1;K)$. Because these numbers remain unchanged, no matter what particular function of the family (3) is identified as $u(x,\lambda)$, it is customary to refer to (6) as the characteristic equation, and to (7) as the auxiliary function of the interface system, even though (6) and (7) may change by a nonzero multiple with each different choice of u .

CHAPTER 2

AN INTERFACE INTERLACING THEOREM

1. Special Representations of the Auxiliary Function

In this chapter we shall show that the fundamental interlacing theorem, Sec. 2, Chap. 1, can be applied to the auxiliary function $F(\lambda)$ of an interface Sturm-Liouville system for the purpose of isolating characteristic numbers of the system within intervals. We shall derive our result by proving that $F(\lambda)$ can be written in the form

$$(1) \quad F(\lambda) = f_1(\lambda)f_4(\lambda) - f_2(\lambda)f_3(\lambda)$$

required by the fundamental interlacing theorem, where for all real values of λ , each $f_i(\lambda)$, $1 \leq i \leq 4$, is continuously differentiable and $W_{12}(\lambda)W_{34}(\lambda) < 0$.

A number of distinct representations of F , each having the structure (1) may be found through the use of determinants (as is done in Sec. 3, Chap. 3) or as follows. With k fixed, and $1 \leq k < K$, let $u(x, \lambda)$ now be any specific solution of (1), Sec. 1, Chap. 1, on (a, x_k) which satisfies all auxiliary conditions of the interface system involving values of x less than x_k , and let

$$(2) \quad u(x, \lambda) = u_j(x, \lambda) \quad x_{j-1} < x < x_j \quad 1 \leq j \leq k$$

be a representation of u . By analogy with (3), Sec. 3, Chap. 1, all solutions of this kind have the form $Au(x, \lambda)$. Similarly, suppose $v(x, \lambda)$ is any particular solution of (1), Sec. 1, Chap. 1, on (x_k, b) which satisfies all auxiliary conditions involving values of x greater

than x_k , and let

$$(3) \quad v(x, \lambda) = v_j(x, \lambda) \quad x_{j-1} < x < x_j \quad k+1 \leq j \leq K$$

be a representation of v . All such solutions have the form $Bv(x, \lambda)$. Thus, every solution of the differential equations of system (1 - 1; K) on (a, b) which satisfies all of the auxiliary conditions, except the two interface conditions at x_k , has a representation in terms of $u(x, \lambda)$ and $v(x, \lambda)$ of the form

$$(4) \quad y(x, \lambda) = \begin{cases} Au(x, \lambda) & \text{on } (a, x_k) & A \neq 0 \\ Bv(x, \lambda) & \text{on } (x_k, b) & B \neq 0 \end{cases}$$

where A and B may depend on λ , but not on x or k . Imposing the interface conditions $(k, 1)$ and $(k, 2)$ on (4) we have

$$(5) \quad \begin{bmatrix} U_{k1}(u) & V_{k1}(v) \\ U_{k2}(u) & V_{k2}(v) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which, for each k such that $1 \leq k < K$, gives

$$(6) \quad F(\lambda) \equiv U_{k1}(u)V_{k2}(v) - U_{k2}(u)V_{k1}(v)$$

as a representation of the auxiliary function F . That these $K - 1$ representations all have the structure (1) is obvious, for we may set

$$f_1(\lambda) = U_{k1}(u), \quad f_2(\lambda) = U_{k2}(u), \quad f_3(\lambda) = V_{k1}(v), \quad f_4(\lambda) = V_{k2}(v).$$

For each fixed k , the function u as presently defined plays much the same role as did the function u appearing in (3), Sec. 3, Chap. 1. Indeed, if $U_{k1}(y_k) = 0$, instead of $V_K(y_K) = 0$, is taken as a right-hand boundary condition, then V_K gets replaced by U_{k1} in (6) and (7) of Sec. 3, Chap. 1. Thus, we can immediately interpret $U_{k1}(u)$ as the auxiliary function of the Sturm-Liouville system

$$(1;k) \quad \frac{d}{dx}[r(x)y_j'] + [\lambda p_j(x) + q_j(x)]y_j = 0, \quad x_{j-1} < x < x_j, \quad 1 \leq j \leq k$$

$$(0;1) \quad U_0(y_1) = 0$$

...

$$(j,1) \quad U_{j1}(y_j) + V_{j1}(y_{j+1}) = 0$$

$$1 \leq j \leq k-1$$

$$(j,2) \quad U_{j2}(y_j) + V_{j2}(y_{j+1}) = 0$$

...

$$(1;k) \quad U_{k1}(y_k) = 0$$

where, if $k = 1$, there are no interface conditions. If, in this system, the right-hand boundary condition $(1;k)$ is replaced by $U_{k2}(y_k) = 0$, the resulting system has $U_{k2}(u)$ as its auxiliary function.

Recalling the meaning (3) of v , and noting the effect of taking $V_{k1}(y_{k+1}) = 0$ as the relevant left-hand boundary condition, we similarly find that $V_{k1}(v)$ represents the auxiliary function of the Sturm-Liouville system

$$(k+1;K) \quad \frac{d}{dx}[r(x)y_j'] + [\lambda p_j(x) + q_j(x)]y_j = 0, \quad x_{j-1} < x < x_j, \quad k+1 \leq j \leq K$$

$$(k;1) \quad V_{k1}(y_{k+1}) = 0$$

...

$$(j,1) \quad U_{j1}(y_j) + V_{j1}(y_{j+1}) = 0$$

$$k+1 \leq j \leq K-1$$

$$(j,2) \quad U_{j2}(y_j) + V_{j2}(y_{j+1}) = 0$$

...

$$(1;K) \quad V_K(y_K) = 0$$

where there are no interface conditions in the system if $k = K-1$. If, in this system, the left-hand boundary condition $(k;1)$ is replaced by $V_{k2}(y_{k+1}) = 0$, the resulting system has $V_{k2}(v)$ as its auxiliary function.

2. Subsystems of Interface Sturm-Liouville Systems

Clearly, the differential systems of Sec. 1 having $U_{k1}(u)$, $U_{k2}(u)$, $V_{k1}(v)$, and $V_{k2}(v)$ as auxiliary functions, all involve differential equations and auxiliary conditions which are closely related to, if not included among, those of system $(1 - 1; K)$, Sec. 1, Chap. 1. It is therefore natural to refer to the first four of these five systems as subsystems of the last one. To describe this connection precisely, let us introduce the Kronecker delta

$$(1) \quad \delta_{jk} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases} \quad j, k = 0, 1, 2, 3, \dots$$

and for each fixed set of integers $\{i, h, m, n\}$, define a subsystem of system $(1 - 1; K)$ to be a differential system of the type:

$$(i+1:h) \quad \frac{d}{dx}[r(x)y'_j] + [\lambda p_j(x) + q_j(x)]y_j = 0, \quad x_{j-1} < x < x_j, \quad i+1 \leq j \leq h$$

$$(i;m) \quad \delta_{0i}U_0(y_1) + (1 - \delta_{0i})[\delta_{1m}V_{i1}(y_{i+1}) + \delta_{2m}V_{i2}(y_{i+1})] = 0$$

...

$$(j,1) \quad U_{j1}(y_j) + V_{j1}(y_{j+1}) = 0 \quad i+1 \leq j \leq h-1$$

$$(j,2) \quad U_{j2}(y_j) + V_{j2}(y_{j+1}) = 0$$

...

$$(n;h) \quad \delta_{hK}V_K(y_K) + (1 - \delta_{hK})[\delta_{n1}U_{h1}(y_h) + \delta_{n2}U_{h2}(y_h)] = 0$$

where $0 \leq i \leq h-1$, $1 \leq h \leq K$, and $m, n = 1, 2$. We shall identify these subsystems by the notation $(i; m - n; h)$, rather than by the notation $(i+1:h - n; h)$ which fails to specify the relevant value of m when $0 < i$.

As is evident from $(i+1:h)$, if $i < h-1$, the differential equations belonging to a subsystem of an interface system whose fundamental

interval is (a,b) must be solved on two or more of the successive subintervals originally identified with (a,b) . For $i = 0$, the left-hand boundary condition $(i;m)$ is $U_0(y_1) = 0$, as in system $(0;1-1;k)$, Sec. 1, and the value of m is irrelevant. If $0 < i$, condition $(i;m)$ becomes $V_{i1}(y_{i+1}) = 0$, or $V_{i2}(y_{i+1}) = 0$, according as $m = 1$, or $m = 2$, and, for either value of m , it follows from (5) and (6b), Sec. 1, Chap. 1, that $c_{im}^2 + d_{im}^2 \neq 0$. For $h = K$, the right-hand boundary condition $(n;h)$ is $V_K(y_K) = 0$, as in system $(k;1-1;K)$, Sec. 1, and the value of n is irrelevant. If $h < K$, condition $(n;h)$ becomes $U_{h1}(y_h) = 0$, or $U_{h2}(y_h) = 0$, according as $n = 1$, or $n = 2$, and, in either case, (5) and (6a), Sec. 1, Chap. 1, imply $a_{hn}^2 + b_{hn}^2 \neq 0$. Finally, the interface conditions $(j,1)$ and $(j,2)$ are contained among those of the original interface system. Thus, we see that every subsystem $(i;m-n;h)$ of $(1-1;K)$, Sec. 1, Chap. 1, for which $i < h-1$, is itself an interface Sturm-Liouville system which has (x_i, x_h) as its fundamental interval. To all such subsystems, Theorems 1 through 4, Sec. 1, Chap. 1, apply. In particular, Theorem 4 yields an important suborthogonality property of the characteristic functions of each subsystem whose fundamental interval is (x_i, x_h) : it is that these characteristic functions are orthogonal over (x_i, x_h) with respect to the weight function p given by

$$(2) \quad p(x) = \left(\prod_{s=i}^{j-1} R_s \right) p_j(x) \quad x_{j-1} < x < x_j \quad i+1 \leq j \leq h$$

where $R_i = 1$ and, for $i+1 \leq s \leq h-1$, R_s is defined by (5b) and (6), Sec. 1, Chap. 1.

A subsystem, for which $i = h-1$, is an ordinary Sturm-Liouville system with no interface conditions. On the other hand, if $i = 0$ and $h = K$, the subsystem $(0; m - n; K)$ and the original interface system $(1 - 1; K)$ are the same. In other words, every interface system is a subsystem of itself. A proper subsystem of an interface system is a subsystem which is distinct from the interface system.

3. Interlacing Properties of the Characteristic Numbers of Certain Subsystems

Having now defined what is meant by a subsystem of an interface system, let us henceforth denote system $(1 - 1; K)$, Sec. 1, Chap. 1, by $(1 - 1; K)$, and its subsystem $(i; m - n; h)$, Sec. 2, by $(i; m - n; h)$, and simply remember the sections where these concepts are defined.

In Sec. 1, we found that, for each fixed k such that $1 \leq k < K$, the functions

$$(1) \quad f_1(\lambda) = U_{k1}(u) \equiv a_{k1}u'(x_k, \lambda) + b_{k1}u(x_k, \lambda)$$

$$(2) \quad f_2(\lambda) = U_{k2}(u) \equiv a_{k2}u'(x_k, \lambda) + b_{k2}u(x_k, \lambda)$$

$$(3) \quad f_3(\lambda) = V_{k1}(v) \equiv c_{k1}v'(x_k, \lambda) + d_{k1}v(x_k, \lambda)$$

$$(4) \quad f_4(\lambda) = V_{k2}(v) \equiv c_{k2}v'(x_k, \lambda) + d_{k2}v(x_k, \lambda)$$

could be interpreted as auxiliary functions of the respective subsystems $(0; m - 1; k)$, $(0; m - 2; k)$, $(k; 1 - n; K)$, and $(k; 2 - n; K)$. We also saw that the auxiliary function of $(1 - 1; K)$ could be written as

$$(5) \quad F(\lambda) = f_1(\lambda)f_4(\lambda) - f_2(\lambda)f_3(\lambda)$$

which has the structure required by the fundamental interlacing theorem, Sec. 2, Chap. 1. As has been pointed out, each of the functions f_i , $1 \leq i \leq 4$, is an auxiliary function of either an

ordinary, or else an interface, Sturm-Liouville subsystem of $(1 - 1; K)$. Any one of the f_i which is itself the auxiliary function of an interface subsystem can, in the same way that (5) was derived, be represented by a difference of products of auxiliary functions of subsystems of that interface subsystem.

By applying the process just mentioned to the auxiliary function of each new interface subsystem evolved, it is possible to express $F(\lambda)$ solely in terms of a finite number of sums and differences of products of auxiliary functions of ordinary Sturm-Liouville systems. But, as is well known, the auxiliary functions of all these Sturm-Liouville systems are analytic for all real values of their argument, [14, pp. 388-389]. Hence, $F(\lambda)$ is analytic on $(-\infty, \infty)$. Moreover, the functions $f_i(\lambda)$ of (5) are also analytic and, therefore, continuously differentiable, as required by Theorem 1, Sec. 2, Chap. 1, on $(-\infty, \infty)$. Summarizing these comments, we have:

Lemma 1 For all real values of its argument, the auxiliary function of an interface Sturm-Liouville system is analytic.

Lemma 2 For all real values of λ , and for each fixed value of k such that $1 \leq k < K$, the functions $f_i(\lambda)$, defined by Eqs. (1) through (4) have continuous derivatives of all orders.

Having shown that $F(\lambda)$ has the form (5) and that each $f_i(\lambda)$, as given by (1) through (4), is continuously differentiable, we must still prove that the remaining hypothesis of the fundamental interlacing theorem holds before that theorem can be applied to $F(\lambda)$ and $f_1(\lambda)f_3(\lambda)$, or to $F(\lambda)$ and $f_2(\lambda)f_4(\lambda)$. That is, we must show that, for all real values of λ

$$(6) \quad W_{12}(\lambda)W_{34}(\lambda) = \begin{vmatrix} f_1(\lambda) & f_2(\lambda) \\ f_1'(\lambda) & f_2'(\lambda) \end{vmatrix} \begin{vmatrix} f_3(\lambda) & f_4(\lambda) \\ f_3'(\lambda) & f_4'(\lambda) \end{vmatrix} < 0.$$

Of course, the primes here signify differentiation with respect to λ .

Once (6) has been verified, the following interface interlacing theorem will be established.

Theorem 1 For each k such that $1 \leq k < K$, let the corresponding functions $f_i(\lambda)$, $1 \leq i \leq 4$, be those defined by Eqs. (1) through (4). For each set $\{f_1, f_2, f_3, f_4\}$ thus defined, the zeros of the auxiliary function $F(\lambda)$ of system $(1 - 1; K)$ interlace the zeros of either of the product functions $f_1(\lambda)f_3(\lambda)$ or $f_2(\lambda)f_4(\lambda)$ on $-\infty < \lambda < \infty$.

Proof that $W_{12}(\lambda)W_{34}(\lambda) < 0$ First, the structure of $W_{12}(\lambda)W_{34}(\lambda)$ will be investigated. Making use of Eqs. (1) and (2), and familiar properties of determinants, we obtain

$$\begin{aligned} (7) \quad W_{12}(\lambda) &= \begin{vmatrix} f_1(\lambda) & f_2(\lambda) \\ f_1'(\lambda) & f_2'(\lambda) \end{vmatrix} \\ &= \begin{vmatrix} a_{k1}u'(x_k, \lambda) + b_{k1}u(x_k, \lambda) & a_{k2}u'(x_k, \lambda) + b_{k2}u(x_k, \lambda) \\ a_{k1} \frac{d}{d\lambda}u'(x_k, \lambda) + b_{k1} \frac{d}{d\lambda}u(x_k, \lambda) & a_{k2} \frac{d}{d\lambda}u'(x_k, \lambda) + b_{k2} \frac{d}{d\lambda}u(x_k, \lambda) \end{vmatrix} \\ &= \begin{vmatrix} u'(x_k, \lambda) & u(x_k, \lambda) \\ \frac{d}{d\lambda}u'(x_k, \lambda) & \frac{d}{d\lambda}u(x_k, \lambda) \end{vmatrix} \begin{vmatrix} a_{k1} & a_{k2} \\ b_{k1} & b_{k2} \end{vmatrix} \\ &= W[u'(x_k, \lambda), u(x_k, \lambda)] |AB|_k \end{aligned}$$

where $W[u'(x_k, \lambda), u(x_k, \lambda)]$ is the wronskian of $u'(x_k, \lambda)$ and $u(x_k, \lambda)$ with respect to the independent variable λ .

Similarly, through the use of Eqs. (3) and (4), we find

$$(8) \quad W_{34}(\lambda) = W[v'(x_k, \lambda), v(x_k, \lambda)] |CD|_k.$$

From (7) and (8), there follows

$$(9) \quad W_{12}(\lambda)W_{34}(\lambda) = W[u'(x_k, \lambda), u(x_k, \lambda)]W[v'(x_k, \lambda), v(x_k, \lambda)]|AB|_k|CD|_k.$$

But, as (5b), Sec. 1, Chap. 1, clearly implies

$$(10) \quad |AB|_k|CD|_k > 0.$$

Therefore, for λ real, (6) and the inequality

$$(11) \quad W[u'(x_k, \lambda), u(x_k, \lambda)]W[v'(x_k, \lambda), v(x_k, \lambda)] < 0$$

are logically equivalent.

To establish (11), we shall show, in turn, that, for $-\infty < \lambda < \infty$,

$$(12) \quad W[u'(x_k, \lambda), u(x_k, \lambda)] > 0$$

and that

$$(13) \quad W[v'(x_k, \lambda), v(x_k, \lambda)] < 0.$$

Recalling (2) of Sec. 1, we have

$$(14) \quad [r(x)u_j'(x, \lambda)]' + [\lambda p_j(x) + q_j(x)]u_j(x, \lambda) = 0.$$

For any λ_0 distinct from λ , this becomes

$$(15) \quad [r(x)u_j'(x, \lambda_0)]' + [\lambda_0 p_j(x) + q_j(x)]u_j(x, \lambda_0) = 0.$$

Both (14) and (15) hold on the intervals $x_{j-1} < x < x_j$, $1 \leq j \leq k$.

Multiplying (14) by $u_j(x, \lambda_0)$, and (15) by $u_j(x, \lambda)$, and subtracting the first equation thus obtained from the second, we get

$$(16) \quad u_j(x, \lambda)[ru_j'(x, \lambda_0)]' - u_j(x, \lambda_0)[ru_j'(x, \lambda)]' + (\lambda_0 - \lambda)p_j u_j(x, \lambda)u_j(x, \lambda_0) = 0$$

which can be written

$$(17) \quad (\lambda_0 - \lambda)p_j(x)u_j(x, \lambda)u_j(x, \lambda_0) \\ = \frac{d}{dx}\{r(x)[u_j(x, \lambda_0)u_j'(x, \lambda) - u_j(x, \lambda)u_j'(x, \lambda_0)]\}.$$

We shall denote the wronskian of $u_j(x, \lambda_0)$ and $u_j(x, \lambda)$, relative to the independent variable x , by a lower case w , thus,

$$(18) \quad w[u_j(x, \lambda_0), u_j(x, \lambda)] = u_j(x, \lambda_0)u_j'(x, \lambda) - u_j(x, \lambda)u_j'(x, \lambda_0).$$

Using this notation, (17) becomes

$$(19) \quad (\lambda_0 - \lambda) p_j(x) u_j(x, \lambda) u_j(x, \lambda_0) = \frac{d}{dx} \{ r(x) w[u_j(x, \lambda_0), u_j(x, \lambda)] \}.$$

Multiplying both members of this equation by $\prod_{s=0}^{j-1} R_s$, then integrating from x_{j-1} to x_j , and subsequently summing from 1 to k over j , we obtain

$$(20) \quad (\lambda_0 - \lambda) \sum_{j=1}^k \int_{x_{j-1}}^{x_j} \left(\prod_{s=0}^{j-1} R_s \right) p_j(x) u_j(x, \lambda) u_j(x, \lambda_0) dx \\ = \sum_{j=1}^k \left(\prod_{s=0}^{j-1} R_s \right) r(x) w[u_j(x, \lambda_0), u_j(x, \lambda)] \Big|_{x_{j-1}}^{x_j}.$$

If $k = 1$, then, for all real values of λ , $u(x, \lambda) = u_1(x, \lambda)$, with x confined to the interval (a, x_1) . Thus, no interface conditions apply. On the other hand, if $k > 1$, at least two interface conditions must be satisfied by u . Let us consider these two cases separately.

In case $k = 1$, the second member of (20) reduces to

$$(21) \quad r(x_1) w[u(x_1, \lambda_0), u(x_1, \lambda)] - r(a) w[u(a, \lambda_0), u(a, \lambda)].$$

But $u(x, \lambda)$ and $u(x, \lambda_0)$ are functions such that

$$a_1 u'(a, \lambda) + b_1 u(a, \lambda) = 0$$

$$a_1 u'(a, \lambda_0) + b_1 u(a, \lambda_0) = 0$$

where a_1 and b_1 are not both zero. This implies that

$$(22) \quad w[u(a, \lambda_0), u(a, \lambda)] = 0.$$

So, in this case, (20) simplifies to

$$(23) \quad (\lambda_0 - \lambda) \int_a^{x_1} p_1(x) u(x, \lambda) u(x, \lambda_0) dx = r(x_1) w[u(x_1, \lambda_0), u(x_1, \lambda)]$$

which we write as

$$(24) \quad \int_a^{x_1} p_1(x) u(x, \lambda) u(x, \lambda_0) dx = r(x_1) w[u(x_1, \lambda_0), u(x_1, \lambda)] / (\lambda_0 - \lambda).$$

Since the limit of $p_1(x)$ is finite as x tends to either a or x_1 , p_1 is bounded on (a, x_1) . For all real values of λ , the limits $u(a, \lambda)$

and $u(x_1, \lambda)$ are also finite. Moreover, for any $\delta > 0$, u is bounded and uniformly continuous throughout the rectangular domain

$$D = \{(x, \lambda) | a < x < x_1, |\lambda_0 - \lambda| \leq \delta\},$$

[15, p. 218]. This implies

$$(25) \quad \lim_{\lambda_0 \rightarrow \lambda} \int_a^{x_1} p_1(x) u(x, \lambda) u(x, \lambda_0) dx = \int_a^{x_1} p_1(x) u^2(x, \lambda) dx.$$

Now, the right-hand member of (24) is equal to

$$(26) \quad r(x_1) \left[u'(x_1, \lambda) \frac{u(x_1, \lambda_0) - u(x_1, \lambda)}{\lambda_0 - \lambda} - u(x_1, \lambda) \frac{u'(x_1, \lambda_0) - u'(x_1, \lambda)}{\lambda_0 - \lambda} \right].$$

Equating the limit of this expression, as $\lambda_0 \rightarrow \lambda$, to the right-hand member of (25), we obtain

$$(27) \quad \int_a^{x_1} p_1(x) u^2(x, \lambda) dx = r(x_1) W[u'(x_1, \lambda), u(x_1, \lambda)].$$

This shows that (12) must hold when $k = 1$, because $u(x, \lambda)$ is nontrivial and $p_1(x) > 0$ on (a, x_1) ; furthermore, $r(x_1) > 0$.

If $k > 1$, $u(x, \lambda)$ not only satisfies condition (0;1) of system (1 - 1;K), but also all of the interface conditions (j,1) and (j,2), $1 \leq j \leq k-1$. Hence,

$$(28) \quad \begin{aligned} a_{j1} u'_j(x_j, \lambda) + b_{j1} u_j(x_j, \lambda) &= -[c_{j1} u'_{j+1}(x_j, \lambda) + d_{j1} u_{j+1}(x_j, \lambda)] \\ a_{j2} u'_j(x_j, \lambda) + b_{j2} u_j(x_j, \lambda) &= -[c_{j2} u'_{j+1}(x_j, \lambda) + d_{j2} u_{j+1}(x_j, \lambda)]. \end{aligned}$$

When solved for $u'_j(x_j, \lambda)$ and $u_j(x_j, \lambda)$, by Cramer's Rule, the solutions to these pairs of equations give

$$(29) \quad \begin{aligned} |AB|_j u'_j(x_j, \lambda) &= |BC|_j u'_{j+1}(x_j, \lambda) + |BD|_j u_{j+1}(x_j, \lambda) \\ |AB|_j u_j(x_j, \lambda) &= -|AC|_j u'_{j+1}(x_j, \lambda) - |AD|_j u_{j+1}(x_j, \lambda) \end{aligned} \quad 1 \leq j \leq k-1$$

where the meanings of $|BC|_j$, $|BD|_j$, $|AC|_j$, and $|AD|_j$ are evident by analogy with (6a) and (6b), Sec. 1, Chap. 1.

From (29) it follows that, for any λ_0 distinct from λ ,

$$(30) \quad \begin{aligned} |AB|_j u'_j(x_j, \lambda_0) &= |BC|_j u'_{j+1}(x_j, \lambda_0) + |BD|_j u_{j+1}(x_j, \lambda_0) \\ |AB|_j u_j(x_j, \lambda_0) &= -|AC|_j u'_{j+1}(x_j, \lambda_0) - |AD|_j u_{j+1}(x_j, \lambda_0) \end{aligned} \quad 1 \leq j \leq k-1.$$

Combining equations (29) and (30), with (18) in mind, we find

$$(31) \quad \begin{aligned} |AB|_j^2 w[u_j(x_j, \lambda_0), u_j(x_j, \lambda)] \\ = [|BD|_j |AC|_j - |BC|_j |AD|_j] w[u_{j+1}(x_j, \lambda_0), u_{j+1}(x_j, \lambda)] \\ = |AB|_j |CD|_j w[u_{j+1}(x_j, \lambda_0), u_{j+1}(x_j, \lambda)] \end{aligned}$$

which, in conjunction with (5b), Sec. 1, Chap. 1, gives

$$(32) \quad w[u_j(x_j, \lambda_0), u_j(x_j, \lambda)] = R_j w[u_{j+1}(x_j, \lambda_0), u_{j+1}(x_j, \lambda)] \quad 1 \leq j \leq k-1.$$

Since (22) still holds when $k > 1$, the second member of (20) now equals

$$\begin{aligned} r(x_1) w[u_1(x_1, \lambda_0), u_1(x_1, \lambda)] + \\ \sum_{j=2}^k \left(\prod_{s=0}^{j-1} R_s \right) \{ r(x_j) w[u_j(x_j, \lambda_0), u_j(x_j, \lambda)] - r(x_{j-1}) w[u_j(x_{j-1}, \lambda_0), u_j(x_{j-1}, \lambda)] \}. \end{aligned}$$

By rearranging terms, this can be written

$$\begin{aligned} \sum_{j=1}^{k-1} \left(\prod_{s=0}^{j-1} R_s \right) r(x_j) \{ w[u_j(x_j, \lambda_0), u_j(x_j, \lambda)] - R_j w[u_{j+1}(x_j, \lambda_0), u_{j+1}(x_j, \lambda)] \} + \\ \left(\prod_{s=0}^{k-1} R_s \right) r(x_k) w[u_k(x_k, \lambda_0), u_k(x_k, \lambda)]. \end{aligned}$$

But, in virtue of (32), every term in the preceding sum over j is zero.

In the final term u_k may be replaced by u , because, $u(x, \lambda) = u_k(x, \lambda)$, on (x_{k-1}, x_k) . Hence, when $k > 1$, (20) simplifies to

$$(33) \quad \begin{aligned} (\lambda - \lambda_0) \sum_{j=1}^k \int_{x_{j-1}}^{x_j} \left(\prod_{s=0}^{j-1} R_s \right) p_j(x) u_j(x, \lambda) u_j(x, \lambda_0) dx \\ = \left(\prod_{s=0}^{k-1} R_s \right) r(x_k) w[u(x_k, \lambda_0), u(x_k, \lambda)] \end{aligned}$$

or, dividing through by $\lambda - \lambda_0$,

$$(34) \quad \sum_{j=1}^k \int_{x_{j-1}}^{x_j} \left(\prod_{s=0}^{j-1} R_s \right) p_j(x) u_j(x, \lambda) u_j(x, \lambda_0) dx \\ = \left(\prod_{s=0}^{k-1} R_s \right) r(x_k) w[u(x_k, \lambda_0), u(x_k, \lambda)] / (\lambda - \lambda_0).$$

Now, for $1 \leq j \leq k$, the limit of $p_j(x)$ is finite as x tends to either x_{j-1} or x_j , and so p_j is bounded on (x_{j-1}, x_j) . For all real values of λ , the limits $u_j(x_{j-1}, \lambda)$ and $u_j(x_j, \lambda)$ are also finite. In fact, for any $\delta > 0$, $u_j(x, \lambda)$ is bounded and uniformly continuous throughout the rectangular domain

$$D_j = \{(x, \lambda) | x_{j-1} < x < x_j, |\lambda_0 - \lambda| \leq \delta\},$$

[15, p. 218]. The limit of the first member of (34), as λ_0 tends to λ , is therefore equal to

$$\sum_{j=1}^k \int_{x_{j-1}}^{x_j} \lim_{\lambda_0 \rightarrow \lambda} \left(\prod_{s=0}^{j-1} R_s \right) p_j(x) u_j(x, \lambda) u_j(x, \lambda_0) dx \\ = \sum_{j=1}^k \int_{x_{j-1}}^{x_j} \left(\prod_{s=0}^{j-1} R_s \right) p_j(x) u_j^2(x, \lambda) dx.$$

The second member of (34) equals

$$\left(\prod_{s=0}^{k-1} R_s \right) r(x_k) \left[u'(x_k, \lambda) \frac{u(x_k, \lambda_0) - u(x_k, \lambda)}{\lambda_0 - \lambda} - u(x_k, \lambda) \frac{u'(x_k, \lambda_0) - u'(x_k, \lambda)}{\lambda_0 - \lambda} \right].$$

Equating the limit of this expression, as $\lambda_0 \rightarrow \lambda$, to the second member of the preceding equation, we get

$$(35) \quad \sum_{j=1}^k \int_{x_{j-1}}^{x_j} \left(\prod_{s=0}^{j-1} R_s \right) p_j(x) u_j^2(x, \lambda) dx = \left(\prod_{s=0}^{k-1} R_s \right) r(x_k) w[u'(x_k, \lambda), u(x_k, \lambda)].$$

If $k = 1$, this result reduces to (27), in which case we have already seen that (12) holds. Of course, (35) also shows that (12) must hold if $k > 1$ because, for $1 \leq j \leq k$, each $u_j(x, \lambda)$ is nontrivial and each $p_j(x) > 0$ on (x_{j-1}, x_j) ; moreover, $r(x_k) > 0$ and, for $0 \leq s \leq k-1$, $R_s > 0$.

Having verified (12), let us next work with (3), Sec. 1, and proceed to establish (13). Instead of (14) and (15), we now have

$$(36) \quad [r(x)v_j'(x, \lambda)]' + [\lambda p_j(x) + q_j(x)]v_j(x, \lambda) = 0$$

$$(37) \quad [r(x)v_j'(x, \lambda_0)]' + [\lambda_0 p_j(x) + q_j(x)]v_j(x, \lambda_0) = 0$$

where $\lambda_0 \neq \lambda$, as before. These equations hold on $x_{j-1} < x < x_j$ for $k+1 \leq j \leq K$. From (36) and (37) we find, in the same way (19) was derived from (14) and (15), that

$$(38) \quad (\lambda_0 - \lambda)p_j(x)v_j(x, \lambda)v_j(x, \lambda_0) = \frac{d}{dx}\{r(x)w[v_j(x, \lambda_0), v_j(x, \lambda)]\}.$$

Multiplying both members of this result by $\prod_{s=k}^{j-1} R_s$, then integrating from

x_{j-1} to x_j , and subsequently summing from $k+1$ to K over j , we obtain

$$(39) \quad (\lambda_0 - \lambda) \sum_{j=k+1}^K \int_{x_{j-1}}^{x_j} \left(\prod_{s=k}^{j-1} R_s \right) p_j(x) v_j(x, \lambda) v_j(x, \lambda_0) dx \\ = \sum_{j=k+1}^K \left(\prod_{s=k}^{j-1} R_s \right) r(x) w[v_j(x, \lambda_0), v_j(x, \lambda)] \Big|_{x_{j-1}}^{x_j}.$$

If $k = K-1$, no interface conditions apply to v . But, if $k < K-1$, v must satisfy at least two such conditions. In either case

$$(40) \quad v(x, \lambda) = v_K(x, \lambda) \quad x_{K-1} < x < b \quad \lambda \text{ real.}$$

Furthermore

$$c_K v'(b, \lambda) + d_K v(b, \lambda) = 0$$

$$c_K v'(b, \lambda_0) + d_K v(b, \lambda_0) = 0.$$

Since c_K and d_K are not both zero,

$$(41) \quad w[v(b, \lambda_0), v(b, \lambda)] = 0.$$

With this result in mind, we now proceed by cases.

In case $k = K-1$, (39) reduces to

$$(42) \quad \int_{x_{K-1}}^b p_K(x) v(x, \lambda) v(x, \lambda_0) dx \\ = -r(x_{K-1}) w[v(x_{K-1}, \lambda_0), v(x_{K-1}, \lambda)] / (\lambda_0 - \lambda).$$

The same procedure and underlying reasoning that led from (24) to (27) also yields

$$(43) \quad -\int_{x_{K-1}}^b p_K(x) v^2(x, \lambda) dx = r(x_{K-1}) W[v'(x_{K-1}, \lambda), v(x_{K-1}, \lambda)]$$

as a consequence of (42). On (x_{K-1}, b) , $v(x, \lambda)$ is nontrivial and $p_K(x) > 0$; moreover, $r(x_{K-1}) > 0$. Hence, it follows from (43) that (13) holds for the case $k = K-1$.

In case $k < K-1$, $v(x, \lambda)$ must satisfy the interface conditions $(j, 1)$ and $(j, 2)$, $k+1 \leq j \leq K-1$, as well as condition $(1; K)$ of system $(1 - 1; K)$. This implies that, for $k+1 \leq j \leq K-1$, Eqs. (28) through (32) hold with u_j and u_{j+1} replaced by v_j and v_{j+1} , respectively. Specifically, the analog of (32) is

$$(44) \quad w[v_j(x_j, \lambda_0), v_j(x_j, \lambda)] = R_j w[v_{j+1}(x_j, \lambda_0), v_{j+1}(x_j, \lambda)], \quad k+1 \leq j \leq K-1.$$

Returning to the second member of (39), we substitute in the limits of integration, remembering that (41) holds for $k < K-1$, to obtain

$$\begin{aligned}
& \sum_{j=k+1}^{K-1} \sum_{s=k}^{j-1} (\prod R_s) \{ r(x_j) w[v_j(x_j, \lambda_0), v_j(x_j, \lambda)] - \\
& \quad r(x_{j-1}) w[v_j(x_{j-1}, \lambda_0), v_j(x_{j-1}, \lambda)] \} - \\
& \quad \sum_{s=k}^{K-1} (\prod R_s) r(x_{K-1}) w[v_K(x_{K-1}, \lambda_0), v_K(x_{K-1}, \lambda)]
\end{aligned}$$

which, by rearranging terms, can be written

$$\begin{aligned}
& \sum_{j=k+1}^{K-1} \sum_{s=k}^{j-1} (\prod R_s) r(x_j) \{ w[v_j(x_j, \lambda_0), v_j(x_j, \lambda)] - R_j w[v_{j+1}(x_j, \lambda_0), v_{j+1}(x_j, \lambda)] \} \\
& \quad - R_k r(x_k) w[v_{k+1}(x_k, \lambda_0), v_{k+1}(x_k, \lambda)].
\end{aligned}$$

From this, (44) and the fact that $v(x, \lambda) = v_{k+1}(x, \lambda)$ on (x_k, x_{k+1}) , we see that the second member of (39) reduces to

$$-R_k r(x_k) w[v(x_k, \lambda_0), v(x_k, \lambda)].$$

Thus, when divided through by $\lambda_0 - \lambda$, (39) becomes simply

$$\begin{aligned}
(45) \quad & \sum_{j=k+1}^K \int_{x_{j-1}}^{x_j} \sum_{s=k}^{j-1} (\prod R_s) p_j(x) v_j(x, \lambda) v_j(x, \lambda_0) dx \\
& = -R_k r(x_k) w[v(x_k, \lambda_0), v(x_k, \lambda)] / (\lambda_0 - \lambda).
\end{aligned}$$

The procedure and reasoning used to derive (35) from (34) can be readily adapted to (45). This yields

$$(46) \quad - \sum_{j=k+1}^K \int_{x_{j-1}}^{x_j} \sum_{s=k}^{j-1} (\prod R_s) p_j(x) v_j^2(x, \lambda) dx = R_k r(x_k) W[v'(x_k, \lambda), v(x_k, \lambda)]$$

as a limiting form of (45) as $\lambda_0 \rightarrow \lambda$.

If $k = K-1$, (46) reduces to (43) in which case we have already seen that (13) holds. Further inspection of (46) shows that (13) also holds if $k < K-1$ because, for $k+1 \leq j \leq K$, each $v_j(x, \lambda)$ is nontrivial and $p_j(x) > 0$ on (x_{j-1}, x_j) ; furthermore, $r(x_k) > 0$ and, for $k \leq s \leq K-1$, $R_s > 0$.

In proving (12) and (13), we have not only established (11), but have verified (6), as well. As a matter of fact, we have proven that, for each k such that $1 \leq k < K$, all hypotheses of the fundamental interlacing theorem are satisfied on $(-\infty, \infty)$ by the functions $F(\lambda)$ and $f_i(\lambda)$, $1 \leq i \leq 4$, defined by Eqs. (1) through (5). Thus our proof of Theorem 1 is complete.

Of course, with appropriate modifications, Theorem 1 may be applied to any interface subsystem $(i; m - n; h)$ of system $(1 - 1; K)$. For, with i , h , m , and n given and $i < h - 1$, it is possible to choose k such that $i < k < h$. Then functions u and v can be defined on (x_1, x_k) and on (x_k, x_h) in the same way these functions were defined on (a, x_k) and on (x_k, b) in dealing with system $(1 - 1; K)$. Once this has been done, Eqs. (1) through (4) define functions $f_i(\lambda)$, $1 \leq i \leq 4$, such that, for each fixed k , the zeros of the auxiliary function of the given subsystem interlace the zeros of either of the product functions $f_1 f_3$ or $f_2 f_4$ on $-\infty < \lambda < \infty$.

CHAPTER 3

CHARACTERISTIC NUMBERS AND AUXILIARY FUNCTIONS

1. A Subsystem Method for Locating Characteristic Numbers of Interface Systems

From Theorem 1, Sec. 3, Chap. 2, we know that for each fixed k , $1 \leq k < K$, the characteristic numbers of an interface Sturm-Liouville system $(1 - 1; K)$, Sec. 1, Chap. 1, interlace both the union of the characteristic values of the subsystems $(0; m - 1; k)$ and $(k; 1 - n; K)$, and the union of the characteristic values of the subsystems $(0; m - 2; k)$ and $(k; 2 - n; K)$. Clearly, all of these subsystems involve a boundary condition of the original interface system.

A connection between the characteristic numbers of an interface system and the characteristic numbers of its proper subsystems, whether the subsystems contain a boundary condition of the original system or not, is described by the next theorem.

Theorem 1 There is at least one characteristic value of an interface system between any three consecutive characteristic values of any one of its proper subsystems.

Proof: For a subsystem $(i; m - n; h)$ to be a proper subsystem of system $(1 - 1; K)$, either $i \neq 0$ or $h \neq K$. If $K > 2$, the proper subsystems of $(1 - 1; K)$ may be separated into three different categories according as $i = 0$, $h = K$, or $0 < i < h < K$.

If $0 = i$, then $h < K$, and the value of m is irrelevant. In this case, it follows from our interface interlacing theorem, namely, Theorem 1, Sec. 3, Chap. 2, that the zeros of the auxiliary function $F(\lambda)$ of system $(1 - 1; K)$ interlace the zeros of either of the product functions $f_1(\lambda)f_3(\lambda)$, or $f_2(\lambda)f_4(\lambda)$, where f_1, f_2, f_3, f_4 are auxiliary functions of the respective subsystems $(0; m - 1; h)$, $(0; m - 2; h)$, $(h; 1 - n; K)$, $(h; 2 - n; K)$. This implies, in particular, that there is at least one zero of $F(\lambda)$ between any two consecutive zeros of either $f_1(\lambda)$ or $f_2(\lambda)$. In other words, if $0 = i$ and $n = 1$ or 2 , there is at least one characteristic value of $(1 - 1; K)$ between any two, and thus between any three, consecutive characteristic values of a proper subsystem $(i; m - n; h)$.

If $h = K$, then $0 < i$, and the value of n is irrelevant. In this case the interface interlacing theorem implies that the zeros of $F(\lambda)$ interlace the zeros of either $f_1(\lambda)f_3(\lambda)$, or $f_2(\lambda)f_4(\lambda)$, where f_1, f_2, f_3, f_4 now stand for auxiliary functions of the respective subsystems $(0; m - 1; i)$, $(0; m - 2; i)$, $(i; 1 - n; K)$, $(i; 2 - n; K)$. As a particular consequence of this, it follows that there is at least one zero of $F(\lambda)$ between any two consecutive zeros of either $f_3(\lambda)$, or $f_4(\lambda)$. In other words, if $h = K$ and $m = 1$ or 2 , there is at least one characteristic value of $(1 - 1; K)$ between any two, and hence between any three, consecutive characteristic values of a proper subset $(i; m - n; h)$.

This proves the theorem for proper subsystems which contain a boundary condition of the original interface system, as is always the case if $K = 2$.

If $K > 2$, the theorem must still be established when m and n are specified, and $0 < i < h < K$. Under these conditions, the interface interlacing theorem guarantees the existence of at least one characteristic value of system $(1 - 1; K)$ between any two consecutive characteristic values of the subsystem $(0; m - n; h)$, and the existence of at least one characteristic value of $(0; m - n; h)$ between any two consecutive characteristic values of $(i; m - n; h)$. Thus, between any three consecutive characteristic values of the subsystem $(i; m - n; h)$ there must be at least two characteristic values of $(0; m - n; h)$ and therefore at least one characteristic value of system $(1 - 1; K)$. The proof is now complete.

In general, the application of Theorem 1 to an interface system and one of its subsystems will produce very little useful information concerning the exact location of the characteristic values of the interface system. Those subsystems whose characteristic values are close together will give the best results. For every interval determined by the first and third of three consecutive characteristic numbers of such a subsystem will be relatively small and contain at least one characteristic value of the interface system itself. These ideas are illustrated in Sec. 6, Chap. 4.

2. Further Properties of Characteristic Values of Interface Systems

In this section repeated use will be made of the following well known theorem [15, p. 235] pertaining to an ordinary Sturm-Liouville system consisting of a differential equation of the type

$$(1) \quad [r(x)y']' + [\lambda p(x) + q(x)]y = 0 \quad a \leq x \leq b$$

and boundary conditions

$$(2) \quad a_1 y'(a) + b_1 y(a) = 0 \quad c_1 y'(b) + d_1 y(b) = 0$$

where the functions r , p , and q are real and continuous, and $r(x)$ and $p(x)$ are positive on $[a, b]$. The coefficients a_1 , b_1 , c_1 , and d_1 , as well as the functions r , p , and q , are independent of λ .

Theorem 1 An ordinary Sturm-Liouville system has a countably infinite set $\{\lambda_n\}$ of characteristic values. This set has a smallest member, is unbounded above, and has no finite limit point.

Because the characteristic numbers of an ordinary Sturm-Liouville system have no finite limit point and are unbounded above, they can be ordered so that

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots < \lambda_n < \dots$$

where λ_n increases without bound as n continues to increase.

The next three lemmas show that an interface, as well as an ordinary, Sturm-Liouville system has a countably infinite set of characteristic numbers which contains a smallest member, has no finite limit point, and is unbounded above.

Lemma 1 The set $\{\lambda_n\}$ of characteristic numbers of an interface Sturm-Liouville system is a countably infinite set which is unbounded above.

Proof The proof proceeds by induction. Since we are dealing with an interface system $(1 - 1; K)$, we first consider the case $K = 2$.

All four proper subsystems of an interface system $(1 - 1; 2)$ are ordinary Sturm-Liouville systems. By Theorem 1, the auxiliary function corresponding to each of these subsystems has a countable infinity of zeros which are unbounded above. Since this is true, in particular, of the auxiliary functions f_1 and f_3 of subsystems $(0; m - 1; 1)$ and

$(1; 1 - n; 2)$, it is certainly true of the product function $f_1 f_3$. But, according to the interface interlacing theorem, between any two consecutive zeros of $f_1(\lambda) f_3(\lambda)$ there is exactly one zero of the auxiliary function $F(\lambda)$ of the interface system $(1 - 1; 2)$. Moreover, every double zero of $f_1(\lambda) f_3(\lambda)$ is a simple zero of $F(\lambda)$. Hence, $F(\lambda)$ has a countably infinite set of zeros which is unbounded above; that is to say, the lemma is true if $K = 2$.

Suppose next that the lemma holds for all interface Sturm-Liouville systems $(1 - 1; k)$ involving k successive intervals (x_{j-1}, x_j) , $1 \leq j \leq k$, where k is an integer greater than or equal to 2. Then, the procedure of Sec. 1, Chap. 2, can be used to express the auxiliary function $F(\lambda)$ of any given interface system $(1 - 1; k+1)$ over $k+1$ successive intervals in the form

$$F(\lambda) = f_1(\lambda) f_4(\lambda) - f_2(\lambda) f_3(\lambda)$$

where, to make this representation of F specific, we may take $f_1(\lambda)$ and $f_2(\lambda)$ to be auxiliary functions of the ordinary Sturm-Liouville systems $(0; m - 1; 1)$ and $(0; m - 2; 1)$, respectively, each of which is now a subsystem of $(1 - 1; k+1)$. The functions $f_3(\lambda)$ and $f_4(\lambda)$ then become auxiliary functions of the respective subsystems of $(1 - 1; k+1)$ identified by $(1; 1 - n; k+1)$ and $(1; 2 - n; k+1)$. Both of these subsystems are interface systems over k successive intervals. Therefore, each f_i has a countable infinity of zeros which are unbounded above. But, the zeros of F interlace those of either $f_1 f_3$ or $f_2 f_4$. Hence, F too has an infinite number of zeros which are unbounded

above. By induction, the lemma is thus true for every interface Sturm-Liouville system $(1 - 1; K)$ for which $2 \leq K$.

In proving the next lemma, we shall make use of a wronskian of the type

$$(3) \quad w_{ij}(\lambda) = \begin{vmatrix} F(\lambda) & f_i(\lambda)f_j(\lambda) \\ F'(\lambda) & [f_i(\lambda)f_j(\lambda)]' \end{vmatrix} \quad 1 \leq i, j \leq 4.$$

Lemma 2 Every interface Sturm-Liouville system has a smallest characteristic number.

Proof Proceeding by induction, we first consider an interface system $(1 - 1; K)$ for which $K = 2$. All four proper subsystems of an interface system $(1 - 1; 2)$ over two successive intervals are ordinary Sturm-Liouville systems. In virtue of Theorem 1, the auxiliary function $f_i(\lambda)$ corresponding to each of these subsystems must therefore have a smallest zero Λ_i , $1 \leq i \leq 4$. For instance, in this notation, Λ_1 and Λ_3 are in turn the smallest zeros of the auxiliary functions $f_1(\lambda)$ and $f_3(\lambda)$ of the respective subsystems $(0; m - 1; 1)$ and $(1; 1 - n; 2)$. Replacing the auxiliary function $F(\lambda)$ of $(1 - 1; 2)$ by $f_1(\lambda)f_4(\lambda) - f_2(\lambda)f_3(\lambda)$ in the determinant $w_{13}(\lambda)$, defined by (3), and utilizing (2), Sec. 2, Chap. 1, we find that

$$(4) \quad w_{13}(\lambda) = f_3^2(\lambda)W_{12}(\lambda) - f_1^2(\lambda)W_{34}(\lambda).$$

From (6), Sec. 3, Chap. 2, we know that $W_{12}(\lambda)W_{34}(\lambda) < 0$. Now either $\Lambda_1 = \Lambda_3$, or else $\Lambda_1 \neq \Lambda_3$, and in either case we may let $\Lambda = \min \{\Lambda_1, \Lambda_3\}$. Then neither $f_1(\lambda)$ nor $f_3(\lambda)$ can be zero on $-\infty < \lambda < \Lambda$, and it follows from (4) that $w_{13}(\lambda) \neq 0$ there. This implies the zeros of F and f_1f_3 separate each other on $(-\infty, \Lambda)$ [4, p. 70]. Hence, if two distinct zeros α and β of $F(\lambda)$ were to exist, both less than Λ , there would be a

number γ such that $f_1(\gamma)f_3(\gamma) = 0$, where $\alpha < \gamma < \beta$. But this is impossible since neither $f_1(\lambda)$ nor $f_3(\lambda)$ can have a zero less than Λ . Thus, if $F(\lambda)$ has a zero less than Λ , it has only one, and that zero is the smallest characteristic value of the system $(1 - 1; 2)$.

Now, for all values of λ , the zeros of F interlace the zeros of f_1f_3 . Thus, if $F(\lambda) \neq 0$ on $(-\infty, \Lambda)$ and $\Lambda_1 = \Lambda_3$, then Λ is the smallest characteristic value of system $(1 - 1; 2)$. On the other hand, if $\Lambda_1 \neq \Lambda_3$, the unique zero of $F(\lambda)$, between Λ and the next larger zero of f_1f_3 , is the smallest characteristic value of $(1 - 1; 2)$.

Clearly, F either does, or does not, have a zero less than Λ , and either $\Lambda_1 = \Lambda_3$, or else $\Lambda_1 \neq \Lambda_3$. In each of these cases we have shown that $F(\lambda)$ has a smallest zero. This proves the lemma for interface Sturm-Liouville systems $(1 - 1; K)$ when $K = 2$.

The induction hypothesis is now made that every interface system $(1 - 1; k)$, over k successive intervals, has a smallest characteristic number. If $F(\lambda)$ is the auxiliary function of any interface system $(1 - 1; k+1)$ over $k+1$ successive intervals, F has a representation

$$F(\lambda) = f_1(\lambda)f_4(\lambda) - f_2(\lambda)f_3(\lambda)$$

in which f_1 and f_2 are auxiliary functions of the respective ordinary Sturm-Liouville systems $(0; m - 1; 1)$ and $(0; m - 2; 1)$, each of which is now a subsystem of $(1 - 1; k+1)$. The functions f_3 and f_4 are auxiliary functions of the interface subsystems of $(1 - 1; k+1)$ identified by $(1; 1 - n; k+1)$ and $(1; 2 - n; k+1)$, both of which involve k successive intervals. From Theorem 1, and our hypothesis on k , each $f_i(\lambda)$, $1 \leq i \leq 4$, has a smallest zero. This is true, in particular, of $f_1(\lambda)$ and $f_3(\lambda)$. Let us denote their smallest zeros by Λ_1 and Λ_3 . Then, the

same reasoning which established this lemma for $K = 2$ can be applied to F , f_1 , and f_3 , as presently defined. This leads to the conclusion that the lemma holds for every interface system $(1 - 1; k+1)$, if it holds for every interface system over k successive intervals. Thus, by induction, every interface Sturm-Liouville system $(1 - 1; K)$, with $2 \leq K$, has a smallest characteristic number.

Our next lemma states, in effect, that the set of characteristic numbers of an interface system has $+\infty$ as its only limit point. This result is essential to the ordering of arbitrary infinite subsets of the set of all characteristic numbers of such a system.

Lemma 3 The set of characteristic values of an interface Sturm-Liouville system has no finite limit point.

Proof As we have repeatedly observed, if $K = 2$, the auxiliary function $F(\lambda)$ of system $(1 - 1; 2)$ can be written

$$F(\lambda) = f_1(\lambda)f_4(\lambda) - f_2(\lambda)f_3(\lambda)$$

where each $f_i(\lambda)$, $1 \leq i \leq 4$, is the auxiliary function of an ordinary Sturm-Liouville system. From Lemma 2, and its proof, we know that F has a smallest zero λ_1 and that, if λ_1 is less than the smallest zero of f_1f_3 , it is the only zero of F with this property. According to Theorem 1, neither the zeros of f_1 , nor those of f_3 , have a finite limit point. Thus, for any number c greater than λ_1 , the interval $[\lambda_1, c]$ contains only a finite number of zeros of f_1f_3 , say m . Since the zeros of F interlace the zeros of f_1f_3 , there is at most one zero of F at each zero of f_1f_3 , and between successive zeros of f_1f_3 there is exactly one zero of F . Hence, allowing for the possibility that λ_1

is less than the smallest zero of $f_1 f_3$, and that F has a zero less than or equal to c , but larger than the largest zero of $f_1 f_3$ in $[\lambda_1, c]$, we see that $[\lambda_1, c]$ can contain at most $2m + 1$ zeros of F . This proves the lemma for interface systems over two successive intervals.

Now suppose that, for every interface system $(1 - 1; k)$ over k successive intervals, the corresponding set of characteristic values has no finite limit point. The auxiliary function F of an arbitrary interface system $(1 - 1; k+1)$ over $k + 1$ successive intervals has a representation

$$F(\lambda) = f_1(\lambda)f_4(\lambda) - f_2(\lambda)f_3(\lambda)$$

where f_1 and f_2 are again auxiliary functions of ordinary Sturm-Liouville systems, while both f_3 and f_4 are auxiliary functions of interface systems over k successive intervals. It follows, in particular, that neither the zeros of f_1 , nor those of f_3 , have a finite limit point. Reasoning with F , f_1 and f_3 , as presently defined, just as in the case $K = 2$, we find that if λ_1 is the smallest zero of F , and c is any number larger than λ_1 , then only a finite number of zeros of F can belong to the interval $[\lambda_1, c]$. Thus, the zeros of F have no finite limit point, and the lemma is true by induction.

By combining the results of Lemmas 1 through 3, we obtain the following analog of Theorem 1 for interface systems.

Theorem 2 Every interface Sturm-Liouville system has a countably infinite set of characteristic values. Each such set has a smallest member, is unbounded above, and has no finite limit point.

Since every subsystem of an interface Sturm-Liouville system is either an ordinary, or else an interface, Sturm-Liouville system,

Theorems 1 and 2 may also be combined. This gives:

Theorem 3 The set of all characteristic values of any subsystem of an interface Sturm-Liouville system is countably infinite, has a smallest member, is unbounded above, and has no finite limit point.

Here is another theorem which we state without proof [5, p. 13]. It provides sufficient conditions for the characteristic numbers of an interface system to be nonnegative. The determinants $|AC|_k$, $|BD|_k$, $|BC|_k$, and $|AD|_k$ appearing in the theorem are like those we encountered in Sec. 3, Chap. 2.

Theorem 4 If λ is a characteristic number of an interface Sturm-Liouville system (1-1;K) in which $a_1b_1 \leq 0$, $c_Kd_K \geq 0$, $q_k(x) \leq 0$ on (x_{k-1}, x_k) , and for $1 \leq k \leq K-1$, $|AC|_k|BD|_k = 0$, $|AC|_k|BC|_k \geq 0$, and $|AD|_k|BD|_k \geq 0$, then $\lambda \geq 0$.

As a special case of Theorem 4 we have the classical result:

Corollary 1 If λ is a characteristic number of an ordinary Sturm-Liouville system (1-2), and if $a_1b_1 \leq 0$, $c_1d_1 \geq 0$, and $q(x) \leq 0$ on $[a,b]$, then $\lambda \geq 0$.

We shall have occasion to utilize several of the foregoing theorems in the examples of Chap. 4.

3. Determinant Forms of Auxiliary Functions

In this section an auxiliary function of system (1-1;K), Sec. 1, Chap. 1, is derived by imposing all of the boundary and interface conditions on a complete solution

(1) $y(x, \lambda) = A_k u_k(x, \lambda) + B_k v_k(x, \lambda) \quad x_{k-1} < x < x_k \quad 1 \leq k \leq K$
on (a,b) of the differential equations (1) of the system. In this way,

$2K$ linear homogeneous algebraic equations are obtained in the $2K$ unknowns A_k, B_k . An auxiliary function of the interface system is then provided by the determinant of the coefficients of the unknowns. Determinants which represent auxiliary functions of Sturm-Liouville systems are called characteristic determinants. Specifically, we have:

Theorem 1 An auxiliary function $F(\lambda)$ of system $(1 - 1; K)$ is given by the characteristic determinant

$$(2) \quad \begin{vmatrix} U_0(u_1) & U_0(v_1) & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ U_{11}(u_1) & U_{11}(v_1) & V_{11}(u_2) & V_{11}(v_2) & 0 & 0 & \dots & 0 & 0 \\ U_{12}(u_1) & U_{12}(v_1) & V_{12}(u_2) & V_{12}(v_2) & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & U_{21}(u_2) & U_{21}(v_2) & V_{21}(u_3) & V_{21}(v_3) & \dots & 0 & 0 \\ 0 & 0 & U_{22}(u_2) & U_{22}(v_2) & V_{22}(u_3) & V_{22}(v_3) & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & V_K(u_K) & V_K(v_K) \end{vmatrix}.$$

Proof If the boundary and interface conditions of system $(1 - 1; K)$ are imposed on (1), and coefficients subsequently collected on the A_k 's and B_k 's in the resulting equations, the following linear algebraic system is obtained:

$$(0;1) \quad U_0(u_1)A_1 + U_0(v_1)B_1 = 0$$

...

$$(k,1) \quad U_{k1}(u_k)A_k + U_{k1}(v_k)B_k + V_{k1}(u_{k+1})A_{k+1} + V_{k1}(v_{k+1})B_{k+1} = 0$$

$$1 \leq k \leq K-1$$

$$(k,2) \quad U_{k2}(u_k)A_k + U_{k2}(v_k)B_k + V_{k2}(u_{k+1})A_{k+1} + V_{k2}(v_{k+1})B_{k+1} = 0$$

...

$$(1;K) \quad V_K(u_K)A_K + V_K(v_K)B_K = 0.$$

Setting the determinant of the coefficients of the A_k 's and B_k 's equal to zero yields the characteristic equation, and the determinant (2) itself is an auxiliary function, of system $(1 - 1; K)$.

In Sec. 1, Chap. 2, one method of expressing the auxiliary function of an interface system in the form

$$(3) \quad F(\lambda) = f_1(\lambda)f_4(\lambda) - f_2(\lambda)f_3(\lambda)$$

was explained in detail, and we have used this result many times since. Here is a theorem which provides an alternative way of accomplishing the same thing.

Theorem 2 It is possible to obtain $K - 1$ representations (3) of the auxiliary function $F(\lambda)$ of system $(1 - 1; K)$ as follows. For each fixed k , $1 \leq k < K$, take $f_1(\lambda)$ to be the principal minor of the determinant (2) whose elements are common to the first $2k$ rows and columns, and take $f_4(\lambda)$ to be the principal minor whose elements are common to the last $2K - 2k$ rows and columns, of (2). To obtain $f_2(\lambda)$, simply replace $U_{k1}(u_k)$ and $U_{k1}(v_k)$ in $f_1(\lambda)$ by $U_{k2}(u_k)$ and $U_{k2}(v_k)$, respectively. To obtain $f_3(\lambda)$ replace $V_{k2}(u_{k+1})$ and $V_{k2}(v_{k+1})$ in $f_4(\lambda)$ by $V_{k1}(u_{k+1})$ and $V_{k1}(v_{k+1})$, respectively.

Proof The theorem comes as an immediate result upon expanding the representation of $F(\lambda)$ given by (2), using determinants of order $2k$ from the first $2k$ columns of (2) in accordance with Laplace's generalized expansion theorem for determinants [13, pp. 268-270].

4. Characteristic Determinants of Subsystems

In this section we examine how a determinant defining an auxiliary function for any proper subsystem $(i; m - n; h)$ of system $(1 - 1; K)$ can

be extracted from the characteristic determinant (2), Sec. 3, of system $(1 - 1; K)$. To see how this can be done, we first impose the auxiliary conditions of $(i; m - n; h)$ upon a complete solution of its differential equations over (x_1, x_h) . The system of linear algebraic equations thus obtained can be written

$$(i; m) \delta_{0i} [U_0(u_1)A_1 + U_0(v_1)B_1] + (1 - \delta_{0i}) [\delta_{1m} \{V_{i1}(u_{i+1})A_{i+1} + V_{i1}(v_{i+1})B_{i+1}\}] \\ + (1 - \delta_{0i}) [\delta_{2m} \{V_{i2}(u_{i+1})A_{i+1} + V_{i2}(v_{i+1})B_{i+1}\}] = 0$$

...

$$(j, 1) U_{j1}(u_j)A_j + U_{j1}(v_j)B_j + V_{j1}(u_{j+1})A_{j+1} + V_{j1}(v_{j+1})B_{j+1} = 0$$

$i+1 \leq j \leq h-1$

$$(j, 2) U_{j2}(u_j)A_j + U_{j2}(v_j)B_j + V_{j2}(u_{j+1})A_{j+1} + V_{j2}(v_{j+1})B_{j+1} = 0$$

...

$$(n; h) \delta_{hK} [V_K(u_K)A_K + V_K(v_K)B_K] +$$

$$(1 - \delta_{hK}) [\delta_{n1} \{U_{h1}(u_h)A_h + U_{h1}(v_h)B_h\} + \delta_{n2} \{U_{h2}(u_h)A_h + U_{h2}(v_h)B_h\}] = 0.$$

A characteristic determinant of $(i; m - n; k)$ is, of course, defined by the determinant of the coefficients of the A's and B's appearing in these equations.

Since we are assuming that $(i; m - n; h)$ is a proper subsystem of $(1 - 1; K)$, this coefficient determinant must be a minor of the determinant (2), Sec. 3. The following theorem describes the precise nature of this minor.

Theorem 1 If $0 < i < h < K$, an auxiliary function of a subsystem $(i; m - n; h)$ of system $(1 - 1; K)$ is defined by the characteristic determinant

$$(1) \begin{vmatrix} V_{im}(u_{i+1}) & V_{im}(v_{i+1}) & 0 & 0 & 0 & \dots & 0 & 0 \\ U_{i+11}(u_{i+1}) & U_{i+11}(v_{i+1}) & V_{i+11}(u_{i+2}) & V_{i+11}(v_{i+2}) & 0 & \dots & 0 & 0 \\ U_{i+12}(u_{i+1}) & U_{i+12}(v_{i+1}) & V_{i+12}(u_{i+2}) & V_{i+12}(v_{i+2}) & 0 & \dots & 0 & 0 \\ 0 & 0 & U_{i+21}(u_{i+2}) & U_{i+21}(v_{i+2}) & \dots & \dots & \dots & \dots \\ 0 & 0 & U_{i+22}(u_{i+2}) & U_{i+22}(v_{i+2}) & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & U_{hn}(u_h) & U_{hn}(v_h) \end{vmatrix}$$

If $i = 0$, the characteristic determinant of $(0; m - n; h)$ is obtained by replacing the elements $V_{im}(u_{i+1})$ and $V_{im}(v_{i+1})$ in the first row of (1) by $U_0(u_1)$ and $U_0(v_1)$, respectively. If $h = K$, the characteristic determinant of $(i; m - n; K)$ is found by replacing the elements $U_{hn}(u_h)$ and $U_{hn}(v_h)$ in the last row of (1) by $V_K(u_K)$ and $V_K(v_K)$, respectively.

As a special case of Theorem 1, we note that if $0 < i = h-1 < K-1$, the subsystem $(i; m - n; h)$ is an ordinary Sturm-Liouville system whose characteristic determinant is

$$\begin{vmatrix} V_{h-1m}(u_h) & V_{h-1m}(v_h) \\ U_{hn}(u_h) & U_{hn}(v_h) \end{vmatrix}.$$

To understand how the characteristic determinant (1) of a subsystem $(i; m - n; h)$ is extracted from the characteristic determinant (2), Sec. 3, observe that the columns of (1) come from columns $2i+1$ through $2h$ of (2) and that when $0 < i < h-1$, the rows of (1), excluding the first and last, come from rows $2i+2$ through $2h-1$ of (2). These rows and columns of (1) are fixed as soon as i and h are specified.

Thus, only the first and last rows of (1) depend on the values of m and n .

If $i = 0$, the first row of (1) comes from the first row of (2). If $0 < i$, the first row of (1) comes from row $2i$, or row $2i+1$, of (2) according as $m = 1$, or $m = 2$.

If $h = K$, the last row of (1) comes from the last row of (2). If $h < K$, the last row of (1) comes from row $2h$, or row $2h+1$, of (2) according as $n = 1$, or $n = 2$.

5. Characteristic Determinants Derived from Different Bases

Eq. (7), Sec. 3, Chap. 1, provided us with an auxiliary function

$$(1) \quad F(\lambda) = V_K(u) \equiv c_K u'(b, \lambda) + d_K u(b, \lambda)$$

of the interface Sturm-Liouville system $(1 - 1; K)$. Other auxiliary functions of the system were also found to exist; however, we saw that to each such function, say G , there corresponded a function $A(\lambda)$ such that

$$(2) \quad G(\lambda) = A(\lambda)F(\lambda) \quad A(\lambda) \neq 0 \quad \lambda \text{ real.}$$

Of course, the same kind of relation holds for auxiliary functions of ordinary Sturm-Liouville systems. Summarizing these remarks as a theorem, we have:

Theorem 1 If $F(\lambda)$ is an auxiliary function of either an ordinary or an interface Sturm-Liouville system, all other auxiliary functions of the system are of the form $A(\lambda)F(\lambda)$ where $A(\lambda)$ is never zero.

This theorem ensures that the characteristic numbers of any given Sturm-Liouville system are unique.

Using linearly independent solutions u_k and v_k to the k th differential equation of system $(1 - 1; K)$, $1 \leq k \leq K$, we obtained a corresponding auxiliary function $F(\lambda)$ given by the characteristic determinant (2), Sec. 3. Suppose μ_k and ν_k provide another basis for all solutions of the k th differential equation of $(1 - 1; K)$ on (x_{k-1}, x_k) . The same process applied to the μ_k and ν_k , yields a characteristic determinant $G(\lambda)$ which also defines an auxiliary function of the interface system. We shall now use matrices and determinants to give an inductive proof that the characteristic determinants $G(\lambda)$ and $F(\lambda)$ are related as stated by Theorem 1.

Proof Since u_k and v_k are linearly independent solutions of the same differential equation that the linearly independent functions μ_k and ν_k satisfy, u_k and v_k must be linear combinations of μ_k and ν_k , and conversely. Thus,

$$(3) \quad (a) \quad \begin{bmatrix} u_k \\ v_k \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{bmatrix} \begin{bmatrix} \mu_k \\ \nu_k \end{bmatrix} \quad (b) \quad \begin{bmatrix} \mu_k \\ \nu_k \end{bmatrix} = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix} \begin{bmatrix} u_k \\ v_k \end{bmatrix}$$

where $\alpha_k, \beta_k, \gamma_k, \delta_k, a_k, b_k, c_k$, and d_k may depend on λ but not on x .

Substituting the right-hand member of (3b) into the right-hand member of (3a) for $\begin{bmatrix} \mu_k \\ \nu_k \end{bmatrix}$, we have

$$(4) \quad \begin{bmatrix} u_k \\ v_k \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{bmatrix} \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix} \begin{bmatrix} u_k \\ v_k \end{bmatrix}.$$

Let us define

$$(5) \quad (a) \quad \Delta_k(\lambda) = \begin{vmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{vmatrix} \quad (b) \quad D_k(\lambda) = \begin{vmatrix} a_k & b_k \\ c_k & d_k \end{vmatrix} \quad 1 \leq k.$$

Then, for all real values of λ ,

$$(6) \quad \Delta_k(\lambda) D_k(\lambda) = 1$$

because the linear independence of u_k and v_k implies the product of the two square matrices in (4) is just the 2×2 identity matrix. From (6) we see that neither $\Delta_k(\lambda)$ nor $D_k(\lambda)$ is ever zero.

It is readily verified that U_0 has the linearity property

$$(7) \quad U_0(au + bv) = aU_0(u) + bU_0(v).$$

Of course V_k and each U_{k1} , U_{k2} , V_{k1} , and V_{k2} also has this property. With this in mind, consider the case in which $G(\lambda)$ and $F(\lambda)$ are characteristic determinants of an ordinary Sturm-Liouville system.

Then

$$\begin{aligned} G(\lambda) &= \begin{vmatrix} U_0(\mu_1) & U_0(v_1) \\ V_1(\mu_1) & V_1(v_1) \end{vmatrix} = \begin{vmatrix} U_0(a_1 u_1 + b_1 v_1) & U_0(c_1 u_1 + d_1 v_1) \\ V_1(a_1 u_1 + b_1 v_1) & V_1(c_1 u_1 + d_1 v_1) \end{vmatrix} \\ &= \begin{vmatrix} a_1 U_0(u_1) + b_1 U_0(v_1) & c_1 U_0(u_1) + d_1 U_0(v_1) \\ a_1 V_1(u_1) + b_1 V_1(v_1) & c_1 V_1(u_1) + d_1 V_1(v_1) \end{vmatrix} = \begin{vmatrix} U_0(u_1) & U_0(v_1) \\ V_1(u_1) & V_1(v_1) \end{vmatrix} \begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix} \\ &= D_1(\lambda) F(\lambda). \end{aligned}$$

With $A(\lambda) = D_1(\lambda)$, this proves the theorem for ordinary Sturm-Liouville systems.

Recalling Theorem 1, Sec. 3, again, we observe that in terms of the linearly independent solutions μ_j and v_j of its j th differential equation, an interface system (1 - 1; 2) over two successive intervals (x_{j-1}, x_j) , $1 \leq j \leq 2$, has

$$G(\lambda) = \begin{vmatrix} U_0(\mu_1) & U_0(v_1) & 0 & 0 \\ U_{11}(\mu_1) & U_{11}(v_1) & V_{11}(\mu_2) & V_{11}(v_2) \\ U_{12}(\mu_1) & U_{12}(v_1) & V_{12}(\mu_2) & V_{12}(v_2) \\ 0 & 0 & V_2(\mu_2) & V_2(v_2) \end{vmatrix}$$

as a characteristic determinant. Using (3b), the linearity property of the U's and V's, and matrix multiplication, we easily find that the matrix

$$(8) \quad \begin{bmatrix} U_0(\mu_1) & U_0(v_1) & 0 & 0 \\ U_{11}(\mu_1) & U_{11}(v_1) & V_{11}(\mu_2) & V_{11}(v_2) \\ U_{12}(\mu_1) & V_{12}(v_1) & V_{12}(\mu_2) & V_{12}(v_2) \\ 0 & 0 & V_2(\mu_2) & V_2(v_2) \end{bmatrix}$$

equals the matrix product

$$(9) \quad \begin{bmatrix} U_0(u_1) & U_0(v_1) & 0 & 0 \\ U_{11}(u_1) & U_{11}(v_1) & V_{11}(u_2) & V_{11}(v_2) \\ U_{12}(u_1) & U_{12}(v_1) & V_{12}(u_2) & V_{12}(v_2) \\ 0 & 0 & V_2(u_2) & V_2(v_2) \end{bmatrix} \begin{bmatrix} a_1 & c_1 & 0 & 0 \\ b_1 & d_1 & 0 & 0 \\ 0 & 0 & a_2 & c_2 \\ 0 & 0 & b_2 & d_2 \end{bmatrix}.$$

The determinant of the first matrix in (9) is clearly a characteristic determinant of $(1 - 1; 2)$ expressed in terms of the linearly independent solutions u_j and v_j of the j th differential equation of the system, i.e., it is $F(\lambda)$. Laplace's generalized expansion shows that the determinant of the last matrix in (9) is equal to $D_1(\lambda)D_2(\lambda)$. Thus, equating determinants of (8) and (9), we obtain

$$(10) \quad G(\lambda) = F(\lambda)D_1(\lambda)D_2(\lambda) = A(\lambda)F(\lambda)$$

where we have set $A(\lambda) = D_1(\lambda)D_2(\lambda)$. Since neither $D_1(\lambda)$ nor $D_2(\lambda)$ is ever zero, the theorem is true for interface systems over two intervals.

Now assume the theorem holds for all interface Sturm-Liouville systems $(1 - 1; k)$ over k successive intervals (x_{j-1}, x_j) , where $1 \leq j \leq k$ and k is a fixed integer greater than or equal to 2. To each such system there corresponds an auxiliary function defined by a

characteristic determinant $g(\lambda)$ involving bases μ_k and v_k for the k differential equations of the system. Likewise, the system has an auxiliary function defined by a characteristic determinant $f(\lambda)$ involving linearly independent solutions u_k and v_k of its differential equations. We suppose that

$$(11) \quad \begin{bmatrix} \mu_j \\ v_j \end{bmatrix} = \begin{bmatrix} a_j & b_j \\ c_j & d_j \end{bmatrix} \begin{bmatrix} u_j \\ v_j \end{bmatrix} \quad 1 \leq j \leq k.$$

As (10) suggests, part of our assumption is

$$(12) \quad g(\lambda) = \left[\prod_{j=1}^k D_j(\lambda) \right] f(\lambda).$$

Let us pause here to observe that the determinants $D_j(\lambda)$ are uniquely determined by the coefficient matrices in (11) and, thus, are not affected by any change in the auxiliary conditions of any given interface system.

An interface system $(1 - 1; k+1)$ over $k+1$ successive intervals also has auxiliary functions defined by determinants $G(\lambda)$ and $F(\lambda)$ involving different sets of bases $\{\mu_j, v_j\}$ and $\{u_j, v_j\}$ for the $k+1$ differential equations of the system. As a special application of Theorem 2, Sec. 3, $G(\lambda)$ can be written

$$(13) \quad G(\lambda) = g_1(\lambda)g_4(\lambda) - g_2(\lambda)g_3(\lambda)$$

where $g_1(\lambda)$ and $g_2(\lambda)$ are characteristic determinants of the respective subsystems $(0; m - 1; k)$ and $(0; m - 2; k)$ in terms of the μ 's and v 's. Similarly, $g_3(\lambda)$ and $g_4(\lambda)$ are characteristic determinants of the respective subsystems $(k; 1 - 1; k+1)$ and $(k; 2 - 1; k+1)$. In terms of the u 's and v 's, these same subsystems have the respective characteristic determinants $f_1(\lambda)$, $f_2(\lambda)$, $f_3(\lambda)$, and $f_4(\lambda)$.

Both subsystems $(0; m-1; k)$ and $(0; m-2; k)$ are interface systems over k intervals, whereas $(k; 1-1; k+1)$ and $(k; 2-2; k+1)$ are ordinary Sturm-Liouville systems. By assumption, the theorem applies to characteristic determinants of interface systems over k successive intervals. Hence, recalling (12) and our remarks concerning the D_j 's, we have

$$(14) \quad (a) \quad g_1(\lambda) = \left[\prod_{j=1}^k D_j(\lambda) \right] f_1(\lambda) \quad \text{and} \quad (b) \quad g_2(\lambda) = \left[\prod_{j=1}^k D_j(\lambda) \right] f_2(\lambda).$$

We have proved the theorem holds for ordinary Sturm-Liouville systems.

Hence

$$(15) \quad (a) \quad g_3(\lambda) = D_{k+1}(\lambda) f_3(\lambda) \quad \text{and} \quad (b) \quad g_4(\lambda) = D_{k+1}(\lambda) f_4(\lambda).$$

Substituting from (14) and (15) into (13), we get

$$G(\lambda) = \left[\prod_{j=1}^{k+1} D_j(\lambda) \right] [f_1(\lambda) f_4(\lambda) - f_2(\lambda) f_3(\lambda)] = \left[\prod_{j=1}^{k+1} D_j(\lambda) \right] F(\lambda).$$

Finally, setting $A(\lambda) = \prod_{j=1}^k D_j(\lambda)$, we have

$$(16) \quad G(\lambda) = A(\lambda) F(\lambda) \quad \text{where } A(\lambda) \text{ is never zero}$$

as asserted by the theorem.

Not only have we proved the theorem for every interface system $(1-1; K)$, $K \geq 2$, and for every ordinary Sturm-Liouville system, but we have also established the relations

$$(17) \quad (a) \quad G(\lambda) = \left[\prod_{k=1}^K D_k(\lambda) \right] F(\lambda) \quad \text{and} \quad (b) \quad F(\lambda) = \left[\prod_{k=1}^K A_k(\lambda) \right] G(\lambda)$$

for characteristic determinants of Sturm-Liouville systems when $G(\lambda)$ and $F(\lambda)$ are derived from different bases.

CHAPTER 4

ILLUSTRATIVE EXAMPLES

1. A Specific Example of an Interface Sturm-Liouville System

In this section we consider an interface Sturm-Liouville system with the property that its characteristic values and those of its subsystems can be easily calculated. Working with this example we demonstrate how the results of Chapters 2 and 3 may be used to obtain information about interface Sturm-Liouville systems and their subsystems.

The interface system we shall study has

$$\begin{array}{ll}
 y_1'' + 16\pi^2\lambda y_1 = 0 & -2 < x < -1 \\
 (1) \quad y_2'' + \pi^2\lambda y_2 = 0 & -1 < x < 1 \\
 y_3'' + 4\pi^2\lambda y_3 = 0 & 1 < x < 2
 \end{array}$$

as its differential equations and

$$\begin{array}{ll}
 (0;1) & y_1(-2) = 0 \\
 (1,1) & y_1(-1) - y_2(-1) = 0 \\
 (1,2) & y_1'(-1) - 4y_2'(-1) = 0 \\
 (2,1) & y_2(1) - y_3(1) = 0 \\
 (2,2) & 2y_2'(1) - y_3'(1) = 0 \\
 (1;3) & y_3(2) = 0
 \end{array}$$

for its boundary and interface conditions. This system, which we shall identify as system $(1 - 1;3)$, is clearly a special case of system $(1 - 1;K)$ in which

$$-2 = a = x_0, \quad x_1 = -1, \quad x_2 = 1, \quad x_3 = b = 2, \quad K = 3$$

$$r(x) = 1, \quad -2 < x < 2$$

$$p_1(x) = 16\pi^2, \quad q_1(x) = 0, \quad -2 < x < -1$$

$$p_2(x) = \pi^2, \quad q_2(x) = 0, \quad -1 < x < 1$$

$$p_3(x) = 4\pi^2, \quad q_3(x) = 0, \quad 1 < x < 2.$$

Observe that the function r is continuous and $r(x) > 0$ on $[-2, 2]$.

Also note that, for $1 \leq k \leq 3$, p_k is continuous, $p_k(x) > 0$, and q_k is continuous on (x_{k-1}, x_k) ; moreover, p_k and q_k have finite limits at x_{k-1} and at x_k . Finally,

$$a_1^2 + b_1^2 = 0^2 + 1^2 = 1$$

$$c_3^2 + d_3^2 = 0^2 + 1^2 = 1$$

$$R_1 = \begin{vmatrix} 0 & -1 \\ -4 & 0 \end{vmatrix} \bigg/ \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 4$$

$$R_2 = \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} \bigg/ \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} = 1/2$$

and so we are dealing with an interface system that satisfies conditions (4) and (5), Sec. 1, Chap. 1.

2. Nonnegative Characteristic Values

The derivation of characteristic equations for an interface system and its subsystems is simpler if the characteristic numbers are all nonnegative. To test system (1 - 1;3) and its subsystems for this property, we observe that all boundary conditions of (1 - 1;3) and its subsystems involve only y , or else y' , as variables. Hence, the product of the coefficient of y and of y' in each of these boundary conditions is zero. For instance, $a_1 b_1 = 0$ and $c_3 d_3 = 0$ in the case of system (1 - 1;3). We have already pointed out that, for $1 \leq k \leq 3$, $q_k(x) = 0$ on (x_{k-1}, x_k) . The equations

$$|BD|_1 = |BD|_2 = \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} = 0 \quad |AC|_1 = \begin{vmatrix} 0 & 0 \\ 1 & -4 \end{vmatrix} = 0 \quad |AC|_2 = \begin{vmatrix} 0 & 0 \\ 2 & -1 \end{vmatrix} = 0$$

also hold. Therefore Theorem 4 or Corollary 1, Sec. 2, Chap. 3, may be applied and we find that neither system $(1 - 1; 3)$, nor any of its subsystems, has a negative characteristic number.

Let us next see if $\lambda = 0$ is a characteristic number of $(1 - 1; 3)$ or any of its subsystems. Were we to have $\lambda = 0$, a complete solution of the differential equations of $(1 - 1; 3)$ on $(-2, 2)$ would be

$$(1) \quad y_k(x) = A_k x + B_k \quad x_{k-1} < x < x_k \quad 1 \leq k \leq 3$$

where A_k, B_k may depend on λ but not on x . Imposing the relevant boundary and interface conditions on (1), we obtain

$$(0;1) \quad -2A_1 + B_1 = 0$$

$$(1,1) \quad -A_1 + B_1 + A_2 - B_2 = 0$$

$$(1,2) \quad A_1 - 4A_2 = 0$$

$$(2,1) \quad A_2 + B_2 - A_3 - B_3 = 0$$

$$(2,2) \quad 2A_2 - A_3 = 0$$

$$(1;3) \quad 2A_3 + B_3 = 0.$$

The determinant of the coefficients of this algebraic system is

$$(2) \quad \begin{vmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \end{vmatrix}.$$

Since the value of this determinant is -8 , no nontrivial solution of system $(1 - 1; 3)$ corresponds to $\lambda = 0$. Hence, $\lambda = 0$ is not a characteristic number of $(1 - 1; 3)$.

To see if $\lambda = 0$ is a characteristic number of some subsystem of $(1 - 1; 3)$, we make use of Theorem 1, Sec. 4, Chap. 3. This gives, for

instance,

$$\begin{vmatrix} -2 & 1 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ 1 & 0 & -4 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

as a characteristic determinant for the subsystem $(0; m - 1; 2)$. Since this determinant has a nonzero value, the subsystem $(0; m - 1; 2)$ cannot have $\lambda = 0$ as a characteristic value. Testing all other subsystems of $(1 - 1; 3)$ in the same way, we find that there is only one subsystem of $(1 - 1; 3)$ for which $\lambda = 0$ is a characteristic value. It is subsystem $(1; 2 - 2; 2)$ whose characteristic determinant

$$\begin{vmatrix} -4 & 0 \\ 2 & 0 \end{vmatrix}$$

has the value 0.

3. Auxiliary Functions and Their Derivations

Now that all of the characteristic values of system $(1 - 1; 3)$ and its subsystems are known to be nonnegative, we may replace the original parameter λ by λ^2 and, except in dealing with the subsystem $(1; 2 - 2; 2)$, require the new parameter λ to be positive. Under this replacement, the differential equations of $(1 - 1; 3)$ become

$$\begin{aligned} (1) \quad & y_1'' + 16\pi^2\lambda^2 y_1 = 0 & -2 < x < -1 \\ & y_2'' + \pi^2\lambda^2 y_2 = 0 & -1 < x < 1 \\ & y_3'' + 4\pi^2\lambda^2 y_3 = 0 & 1 < x < 2 \end{aligned}$$

while the related boundary and initial conditions remain unchanged. We shall now refer to this new system as system $(1 - 1; 3)$. For λ positive, a complete solution of Eqs. (1) over $(-2, 2)$ is

$$\begin{aligned}
 y_1(x) &= A_1 \cos 4\pi\lambda x + B_1 \sin 4\pi\lambda x & -2 < x < -1 \\
 (2) \quad y_2(x) &= A_2 \cos \pi\lambda x + B_2 \sin \pi\lambda x & -1 < x < 1 \\
 y_3(x) &= A_3 \cos 2\pi\lambda x + B_3 \sin 2\pi\lambda x & 1 < x < 2.
 \end{aligned}$$

Imposing the boundary and interface conditions on (2), we obtain

$$\begin{bmatrix} \cos 8\pi\lambda & -\sin 8\pi\lambda & 0 & 0 & 0 & 0 \\ \cos 4\pi\lambda & -\sin 4\pi\lambda & -\cos \pi\lambda & \sin \pi\lambda & 0 & 0 \\ \sin 4\pi\lambda & \cos 4\pi\lambda & -\sin \pi\lambda & -\cos \pi\lambda & 0 & 0 \\ 0 & 0 & \cos \pi\lambda & \sin \pi\lambda & -\cos 2\pi\lambda & -\sin 2\pi\lambda \\ 0 & 0 & -\sin \pi\lambda & \cos \pi\lambda & \sin 2\pi\lambda & -\cos 2\pi\lambda \\ 0 & 0 & 0 & 0 & \cos 4\pi\lambda & \sin 4\pi\lambda \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ A_3 \\ B_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

from which it is clear that a characteristic determinant of our new interface system is

$$(3) \quad \begin{vmatrix} \cos 8\pi\lambda & -\sin 8\pi\lambda & 0 & 0 & 0 & 0 \\ \cos 4\pi\lambda & -\sin 4\pi\lambda & -\cos \pi\lambda & \sin \pi\lambda & 0 & 0 \\ \sin 4\pi\lambda & \cos 4\pi\lambda & -\sin \pi\lambda & -\cos \pi\lambda & 0 & 0 \\ 0 & 0 & \cos \pi\lambda & \sin \pi\lambda & -\cos 2\pi\lambda & -\sin 2\pi\lambda \\ 0 & 0 & -\sin \pi\lambda & \cos \pi\lambda & \sin 2\pi\lambda & -\cos 2\pi\lambda \\ 0 & 0 & 0 & 0 & \cos 4\pi\lambda & \sin 4\pi\lambda \end{vmatrix}$$

Expanding (3), we obtain

$$(4) \quad F(\lambda) = \sin 8\pi\lambda$$

as an auxiliary function of system (1 - 1;3) as modified.

To extract a characteristic determinant for each proper subsystem of (1 - 1;3) from (3) we make use of Theorem 1, Sec. 4, Chap. 3. In this way we find that a characteristic determinant of the subsystem (0;m - 2;2) is

$$\begin{vmatrix} \cos 8\pi\lambda & -\sin 8\pi\lambda & 0 & 0 \\ \cos 4\pi\lambda & -\sin 4\pi\lambda & -\cos \pi\lambda & \sin \pi\lambda \\ \sin 4\pi\lambda & \cos 4\pi\lambda & -\sin \pi\lambda & -\cos \pi\lambda \\ 0 & 0 & -\sin \pi\lambda & \cos \pi\lambda \end{vmatrix}$$

which, when expanded, gives $\cos 6\pi\lambda$ as an auxiliary function.

A characteristic determinant of subsystem (1;2 - 2;2) is

$$\begin{vmatrix} -\sin \pi\lambda & -\cos \pi\lambda \\ -\sin \pi\lambda & \cos \pi\lambda \end{vmatrix}$$

which, when expanded and multiplied by -1, gives $\sin 2\pi\lambda$ as an auxiliary function. As was noted in Sec. 2, this is the only subsystem of (1 - 1;3) which has $\lambda = 0$ as a characteristic number. Since 0 is a zero of $\sin 2\pi\lambda$, this function provides an auxiliary function of (1;2 - 2;2) even though it was derived under the assumption $\lambda > 0$.

Auxiliary functions of (1 - 1;3) and of all its subsystems are listed and labeled in Table 1 where the auxiliary functions of the subsystems (0;m - 2;2) and (1;2 - 2;2), for instance, are denoted by $F_2(\lambda)$ and $f_6(\lambda)$, respectively. The domain of f_6 consists of all nonnegative numbers. All other auxiliary functions are defined only for positive values of the parameter λ .

4. Interlacing Properties

The elementary nature of the auxiliary functions of (1 - 1;3) and its subsystems makes it easy to find a formula for the n th characteristic number of each system. These formulas appear in column 2, Table 2. The last eight columns contain successive characteristic values of the subsystem of (1 - 1;3) whose auxiliary

Table 1. Auxiliary functions of system (1 - 1;3) and its subsystems.

System	Auxiliary Conditions	Auxiliary Functions
(0;m - 1;1)	$y_1(-2)=0$ and $y_1(-1)=0$	$f_1(\lambda)=\sin(4\pi\lambda)$
(0;m - 2;1)	$y_1(-2)=0$ and $y_1'(-1)=0$	$f_2(\lambda)=\cos(4\pi\lambda)$
(1;1 - 1;2)	$y_2(-1)=0$ and $y_2(1) = 0$	$f_3(\lambda)=\sin(2\pi\lambda)$
(1;2 - 1;2)	$y_2'(-1)=0$ and $y_2(1)=0$	$f_4(\lambda)=\cos(2\pi\lambda)$
(1;1 - 2;2)	$y_2(-1)=0$ and $y_2'(1)=0$	$f_5(\lambda)=\cos(2\pi\lambda)$
(1;2 - 2;2)	$y_2'(-1)=0$ and $y_2'(1)=0$	$f_6(\lambda)=\sin(2\pi\lambda)$
(2;1 - n;3)	$y_3(1)=0$ and $y_3(2)=0$	$f_7(\lambda)=\sin(2\pi\lambda)$
(2;2 - n;3)	$y_3'(1)=0$ and $y_3(2)=0$	$f_8(\lambda)=\cos(2\pi\lambda)$
	$y_1(-2)=0$	
(0;m - 1;2)	$y_1(-1)-y_2(-1)=0$	$F_1(\lambda)=\sin(6\pi\lambda)$
	$y_1'(-1)-4y_2'(-1)=0$	
	$y_2(1)=0$	
(0;m - 2;2)	Same as for $F_1(\lambda)$ except the condition at $x=1$ is $y_2'(1)=0$	$F_2(\lambda)=\cos(6\pi\lambda)$
	$y_2(-1)=0$	
(1;1 - n;3)	$y_2(1)-y_3(1)=0$	$F_3(\lambda)=\sin(4\pi\lambda)$
	$2y_2'(1)-y_3'(1)=0$	
	$y_3(2)=0$	
(1;2 - n;3)	Same as for $F_3(\lambda)$ except the condition at $x=-1$ is $y_2'(-1)=0$	$F_4(\lambda)=\cos(4\pi\lambda)$
(1 - 1;3)	(0;1) through (1;3)	$F(\lambda)=\sin(8\pi\lambda)$

Table 2. Characteristic values of subsystems of system (1 - 1;3).

Auxiliary Function	λ_n	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
$f_1(\lambda)$	$n/4$	1/4 0.250	1/2 0.500	3/4 0.750	1 1.000	5/4 1.250	3/2 1.500	7/4 1.750	2 2.000
$f_2(\lambda)$	$(2n-1)/8$	1/8 0.125	3/8 0.375	5/8 0.625	7/8 0.875	9/8 1.125	11/8 1.375	13/8 1.625	15/8 1.875
$f_3(\lambda)$	$n/2$	1/2 0.500	1 1.000	3/2 1.500	2 2.000	5/2 2.500	3 3.000	7/2 3.500	4 4.000
$f_4(\lambda)$	$(2n-1)/4$	1/4 0.250	3/4 0.750	5/4 1.250	7/4 1.750	9/4 2.250	11/4 2.750	13/4 3.250	15/4 3.750
$f_5(\lambda)$	$(2n-1)/4$	1/4 0.250	3/4 0.750	5/4 1.250	7/4 1.750	9/4 2.250	11/4 2.750	13/4 3.250	15/4 3.750
$f_6(\lambda)$	$(n-1)/2$	0 0.000	1/2 0.500	1 1.000	3/2 1.500	2 2.000	5/2 2.500	3 3.000	7/2 3.500
$f_7(\lambda)$	$n/2$	1/2 0.500	1 1.000	3/2 1.500	2 2.000	5/2 2.500	3 3.000	7/2 3.500	4 4.000
$f_8(\lambda)$	$(2n-1)/4$	1/4 0.250	3/4 0.750	5/4 1.250	7/4 1.750	9/4 2.250	11/4 2.750	13/4 3.250	15/4 3.750
$F_1(\lambda)$	$n/6$	1/6 0.167	1/3 0.333	1/2 0.500	2/3 0.667	5/6 0.833	1 1.000	7/6 1.167	4/3 1.333
$F_2(\lambda)$	$(2n-1)/12$	1/12 0.083	1/4 0.250	5/12 0.417	7/12 0.583	3/4 0.750	11/12 0.917	13/12 1.083	5/4 1.250
$F_3(\lambda)$	$n/4$	1/4 0.250	1/2 0.500	3/4 0.750	1 1.000	5/4 1.250	3/2 1.500	7/4 1.750	2 2.000
$F_4(\lambda)$	$(2n-1)/8$	1/8 0.125	3/8 0.375	5/8 0.625	7/8 0.875	9/8 1.125	11/8 1.375	13/8 1.625	15/8 1.875
$F(\lambda)$	$n/8$	1/8 0.125	1/4 0.250	3/8 0.375	1/2 0.500	5/8 0.625	3/4 0.750	7/8 0.875	1 1.000

function appears in the corresponding row of column 1. In each case λ_1 denotes the smallest characteristic number. Both fractional and decimal values of the characteristic numbers are given.

From the interface interlacing theorem, Sec. 3, Chap. 2, and the fact that all characteristic values of $(1 - 1; 3)$ and its subsystems are nonnegative, we know that the zeros of $F(\lambda)$ must interlace the zeros of the product function $f_1(\lambda)F_3(\lambda)$ on $[0, \infty)$. A comparison of the characteristic numbers in the last row of Table 2, which are the successive zeros of $F(\lambda)$, with the union of the characteristic numbers in rows 1 and 11, confirms this property as far as our table goes. In this example, all zeros of $f_1(\lambda)$ and of $F_3(\lambda)$ happen to coincide, i.e., all zeros of $f_1(\lambda)F_3(\lambda)$ are double zeros and hence are zeros of $F(\lambda)$. These observations are portrayed geometrically in Fig. 1a where the zeros of $F(\lambda)$, $f_1(\lambda)$ and $F_3(\lambda)$ have been plotted along the positive λ axis.

Of course, the interface interlacing theorem also asserts that the zeros of $F(\lambda)$ interlace those of the product function $f_2(\lambda)F_4(\lambda)$, as well as those of $f_7(\lambda)F_1(\lambda)$ and of $f_8(\lambda)F_2(\lambda)$. Figs. 1b, 1c, and 1d illustrate this behavior of the zeros of $F(\lambda)$.

A useful summary of the preceding interlacing properties of $F(\lambda)$ in terms of subsystems is the following: For each integer k , $0 < k < 3$, the characteristic values of $(1 - 1; 3)$ interlace the union of the characteristic values of the two subsystems $(0; m - n_1; k)$ and $(k; m_1 - n; 3)$ provided $n_1 = m_1$, where $1 \leq m_1$, $n_1 \leq 2$.

Incidentally, Figs. 1a and 1b illustrate the fact that, in this example, the set of all zeros of $F(\lambda)$ is the union of the zeros of

$f_1(\lambda)F_3(\lambda)$ and those of $f_2(\lambda)F_4(\lambda)$. To see this note that all zeros of $f_1(\lambda)F_3(\lambda)$ and of $f_2(\lambda)F_4(\lambda)$ are double zeros. Thus, each zero of $f_1(\lambda)F_3(\lambda)$ is a zero of $F(\lambda)$, and each zero of $f_2(\lambda)F_4(\lambda)$ is a zero of $F(\lambda)$. Since the first zero of $f_2(\lambda)F_4(\lambda)$ comes before the first zero of $f_1(\lambda)F_3(\lambda)$, and since only the first zero of $F(\lambda)$ can possibly come before the first zero of $f_1(\lambda)F_3(\lambda)$, this first zero of $F(\lambda)$ must be the first zero of $f_2(\lambda)F_4(\lambda)$. The second zero of $F(\lambda)$ must then come between the first and second zeros of $f_2(\lambda)F_4(\lambda)$ and is then the first zero of $f_1(\lambda)F_3(\lambda)$. The third zero of $F(\lambda)$ is then the second zero of $f_2(\lambda)F_4(\lambda)$, and the fourth zero of $F(\lambda)$ is between the second and third zeros of $f_2(\lambda)F_4(\lambda)$. Thus it is the second zero of $f_1(\lambda)F_3(\lambda)$. Continuing in this way we see that the set of all zeros of $F(\lambda)$ is, indeed, the union of the zeros of $f_1(\lambda)F_3(\lambda)$ and of $f_2(\lambda)F_4(\lambda)$.

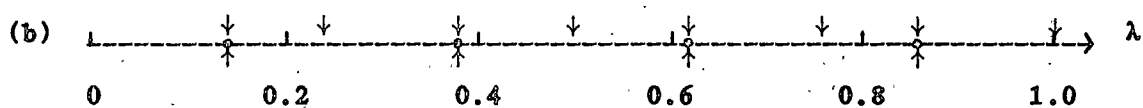
Had the zeros of $f_1(\lambda)F_3(\lambda)$ and of $f_2(\lambda)F_4(\lambda)$ failed to produce all of the zeros of $F(\lambda)$ then the zeros of $f_7(\lambda)F_1(\lambda)$ and of $f_8(\lambda)F_2(\lambda)$ might have been used to advantage.

As we know, a subsystem of $(1 - 1; 3)$ involving two adjoining intervals is itself a Sturm-Liouville interface system to which the interface interlacing theorem may be applied. Specifically, when applied to $F_1(\lambda)$ the theorem asserts that the zeros of $F_1(\lambda)$ interlace those of either $f_1(\lambda)f_3(\lambda)$ or $f_2(\lambda)f_4(\lambda)$. This behavior of the zeros of $F_1(\lambda)$ is portrayed in Figs. 2a and 2b. Similarly, the zeros of $F_2(\lambda)$ interlace the zeros of either $f_1(\lambda)f_5(\lambda)$ or $f_2(\lambda)f_6(\lambda)$. This is depicted by Figs. 2c and 2d.

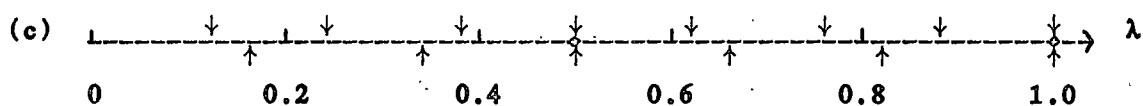
Figure 1. Interlacing by the auxiliary function of the entire system.



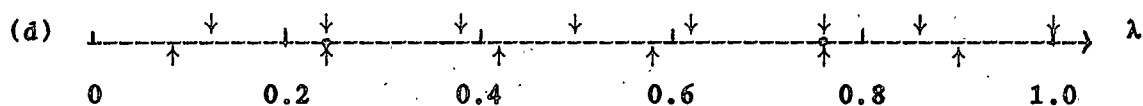
Consecutive zeros of $f_1(\lambda)$, $F_3(\lambda)$, and $F(\lambda)$ are indicated
by the symbols $\cdot$, $\uparrow$, and $\downarrow$, respectively.



Consecutive zeros of $f_2(\lambda)$, $F_4(\lambda)$, and $F(\lambda)$ are indicated
by the symbols $\cdot$, $\uparrow$, and $\downarrow$, respectively.



Consecutive zeros of $f_7(\lambda)$, $F_1(\lambda)$, and $F(\lambda)$ are indicated
by the symbols $\cdot$, $\uparrow$, and $\downarrow$, respectively.

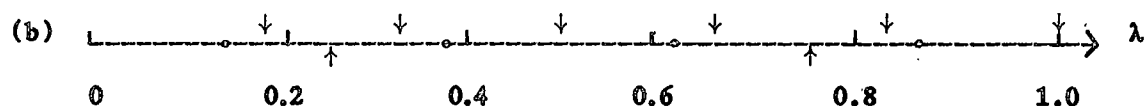


Consecutive zeros of $f_8(\lambda)$, $F_2(\lambda)$, and $F(\lambda)$ are indicated
by the symbols $\cdot$, $\uparrow$, and $\downarrow$, respectively.

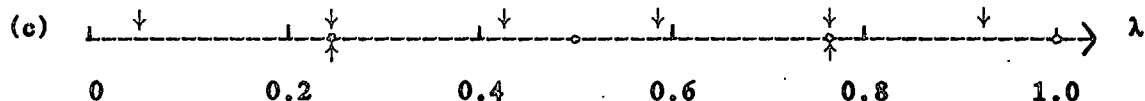
Figure 2. Interlacing by auxiliary functions of systems defined for $-2 < \kappa < 1$.



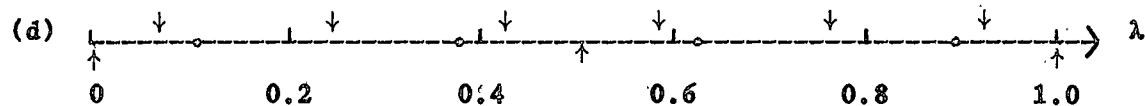
Consecutive zeros of $f_1(\lambda)$, $f_3(\lambda)$, and $F_1(\lambda)$ are indicated by the symbols $\cdot$, $\uparrow$, and $\downarrow$, respectively.



Consecutive zeros of $f_2(\lambda)$, $f_4(\lambda)$, and $F_1(\lambda)$ are indicated by the symbols $\cdot$, $\uparrow$, and $\downarrow$, respectively.

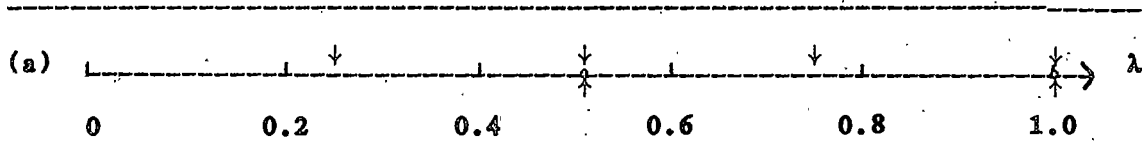


Consecutive zeros of $f_1(\lambda)$, $f_5(\lambda)$, and $F_2(\lambda)$ are indicated by the symbols $\cdot$, $\uparrow$, and $\downarrow$, respectively.

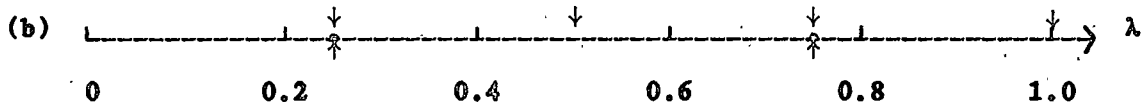


Consecutive zeros of $f_2(\lambda)$, $f_6(\lambda)$, and $F_2(\lambda)$ are indicated by the symbols $\cdot$, $\uparrow$, and $\downarrow$, respectively.

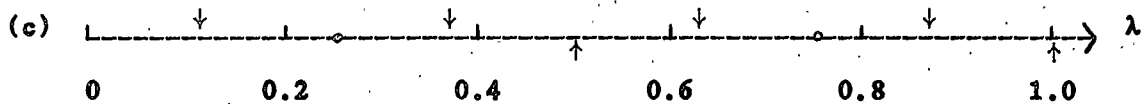
Figure 3. Interlacing by auxiliary functions of systems defined for $-1 < x < 2$.



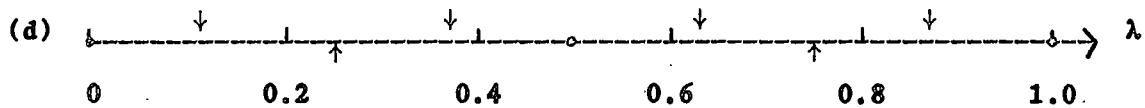
Consecutive zeros of $f_3(\lambda)$, $f_7(\lambda)$, and $F_3(\lambda)$ are indicated by the symbols $\cdot$, $\uparrow$, and $\downarrow$, respectively.



Consecutive zeros of $f_5(\lambda)$, $f_9(\lambda)$, and $F_3(\lambda)$ are indicated by the symbols $\cdot$, $\uparrow$, and $\downarrow$, respectively.



Consecutive zeros of $f_4(\lambda)$, $f_7(\lambda)$, and $F_4(\lambda)$ are indicated by the symbols $\cdot$, $\uparrow$, and $\downarrow$, respectively.



Consecutive zeros of $f_6(\lambda)$, $f_8(\lambda)$, and $F_4(\lambda)$ are indicated by the symbols $\cdot$, $\uparrow$, and $\downarrow$, respectively.

By the same token, the zeros of $F_3(\lambda)$ interlace the zeros of either $f_3(\lambda)f_7(\lambda)$ or $f_5(\lambda)f_8(\lambda)$ as shown in Figs. 3a and 3b. Finally the zeros of $F_4(\lambda)$ interlace the zeros of $f_4(\lambda)f_7(\lambda)$ or $f_6(\lambda)f_8(\lambda)$ as shown in Figs. 3c and 3d.

5. Characteristic Functions and Their Orthogonality

By utilizing the characteristic values of Table 2 to compute the characteristic functions of the subsystems of (1 - 1;3), we obtain Table 3. In listing the characteristic functions, all nonzero multiplying factors have been reduced to unity.

According to Theorem 4, Sec. 1, Chap. 1, the characteristic functions of system (1 - 1;3) are orthogonal on the fundamental interval $(-2,2)$ with respect to the weight function

$$p(x) = \left(\prod_{j=0}^2 R_j \right) p_k(x) \quad x_{k-1} < x < x_k \quad 1 \leq k \leq 3$$

where $R_0 = 1$ and, from Sec. 1, $R_1 = 4$ and $R_2 = 1/2$. We shall now verify this property directly. To do so, we suppose $m \neq n$ and consider

$$\begin{aligned} & \int_{-2}^{-1} 16\pi^2 \sin \frac{n\pi x}{2} \sin \frac{m\pi x}{2} dx \\ & + \int_{-1}^1 4\pi^2 \sin \frac{n\pi(x-3)}{8} \sin \frac{m\pi(x-3)}{8} dx \\ & + \int_1^2 8\pi^2 \sin \frac{n\pi(x-2)}{4} \sin \frac{m\pi(x-2)}{4} dx. \end{aligned}$$

Table 3. Characteristic functions of the subsystems of system (1 - 1;3).

System	Characteristic Functions	
(0;m - 1;1)	$y_1 = \sin[n\pi x]$	$-2 < x < -1$
(0;m - 2;1)	$y_1 = \sin[(2n-1)\pi x/2]$	$-2 < x < -1$
(1;1 - 1;2)	$y_2 = \sin[n\pi(x+1)/2]$	$-1 < x < 1$
(1;2 - 1;2)	$y_2 = \sin[(2n-1)\pi(x-1)/4]$	$-1 < x < 1$
(1;1 - 2;2)	$y_2 = \sin[(2n-1)\pi(x+1)/4]$	$-1 < x < 1$
(1;2 - 2;2)	$y_2 = \begin{cases} \sin[(n-1)\pi x/2] & n \text{ even} \\ \cos[(n-1)\pi x/2] & n \text{ odd} \end{cases}$	$-1 < x < 1$
(2;1 - n;3)	$y_3 = \sin[n\pi(x-2)]$	$1 < x < 2$
(2;2 - n;3)	$y_3 = \sin[(2n-1)\pi(x-2)/2]$	$1 < x < 2$
(0;m - 1;2)	$y_1 = \sin[2n\pi(x+2)/3]$	$-2 < x < -1$
	$y_2 = \sin[n\pi(x+5)/6]$	$-1 < x < 1$
(0;m - 2;2)	$y_1 = \sin[(2n-1)\pi(x+2)/3]$	$-2 < x < -1$
	$y_2 = \sin[(2n-1)\pi(x+5)/12]$	$-1 < x < 1$
(1;1 - n;3)	$y_2 = \sin[n\pi(x-3)/4]$	$-1 < x < 1$
	$y_3 = \sin[n\pi(x-2)/2]$	$1 < x < 2$
(1;2 - n;3)	$y_2 = \sin[(2n-1)\pi(x-3)/8]$	$-1 < x < 1$
	$y_3 = \sin[(2n-1)\pi(x-2)/4]$	$1 < x < 2$
(1 - 1;3)	$y_1 = \sin[n\pi x/2]$	$-2 < x < -1$
	$y_2 = \sin[n\pi(x-3)/8]$	$-1 < x < 1$
	$y_3 = \sin[n\pi(x-2)/4]$	$1 < x < 2$

The sum of these three integrals equals.

$$\begin{aligned}
 & 16\pi \left\{ \frac{\sin[(n-m)\pi x/2]}{(n-m)} - \frac{\sin[(n+m)\pi x/2]}{(n+m)} \right\} \Bigg|_{-2}^{-1} \\
 & + \left\{ \frac{\sin[(n-m)\pi(x-3)/8]}{(n-m)} - \frac{\sin[(n+m)\pi(x-3)/8]}{(n+m)} \right\} \Bigg|_{-1}^1 \\
 & + \left\{ \frac{\sin[(n-m)\pi(x-2)/4]}{(n-m)} - \frac{\sin[(n+m)\pi(x-2)/4]}{(n+m)} \right\} \Bigg|_1^2
 \end{aligned}$$

which upon substituting in the upper and lower limits, reduces to zero.

Hence, the stated orthogonality condition holds.

From (2), Sec. 2, Chap. 2, we also know suborthogonality conditions which apply to the characteristic functions of the various subsystems of (1 - 1;3). Let us verify, for instance, that the characteristic functions of (0;m - 1;2) are orthogonal on (-2,1) with respect to the weight function

$$p(x) = \left(\prod_{s=0}^1 R_s \right) p_j(x) \quad x_{j-1} < x < x_j \quad 1 \leq j \leq 2$$

where $R_0 = 1$ and $R_1 = 4$. With $m \neq n$, we evaluate the integrals of the sum

$$\begin{aligned}
 & \int_{-2}^{-1} 16\pi^2 \sin \frac{2n\pi(x+2)}{3} \sin \frac{2m\pi(x+2)}{3} dx \\
 & + \int_{-1}^1 4\pi^2 \sin \frac{n\pi(x+5)}{6} \sin \frac{m\pi(x+5)}{6} dx
 \end{aligned}$$

and get

$$\begin{aligned}
 & 16\pi \left\{ \frac{\sin[2(n-m)\pi(x+2)/3]}{(n-m)} - \frac{\sin[2(n+m)\pi(x+2)/3]}{(n+m)} \right\} \Bigg|_{-2}^{-1} \\
 & + 16\pi \left\{ \frac{\sin[(n-m)\pi(x+5)/6]}{(n-m)} - \frac{\sin[(n+m)\pi(x+5)/6]}{(n+m)} \right\} \Bigg|_{-1}^1.
 \end{aligned}$$

Substituting in the limit, we find that all of this reduces to zero. Hence, the suborthogonality of the characteristic functions of $(0; m-1; 2)$ is established.

For the subsystem $(1; 2-2; 2)$, which has 0 as a characteristic number, the corresponding suborthogonality condition asserts that the characteristic functions of the system are orthogonal on $(-1, 1)$ with respect to the weight function

$$(1) \quad p(x) = R_1 p_2(x) = \pi^2 \quad -1 < x < 1.$$

Since only one interval is involved here, we may clearly delete the nonzero factor (1) from the integral defining the suborthogonality of the characteristic functions in this case.

In our direct verification of the suborthogonality condition for $(1; 2-2; 2)$, we first assume n is even and m is odd. Since

$$\begin{aligned} \int_{-1}^1 \sin \frac{(n-1)\pi x}{2} \cos \frac{(m-1)\pi x}{2} dx &= \left[\frac{-\cos[(n-m)\pi x/2]}{(n-m)} - \frac{\cos[(n+m-2)\pi x/2]}{(n+m-2)} \right]_{-1}^1 \\ &= \left[\frac{-\cos[(n-m)\pi/2]}{\pi(n-m)} - \frac{\cos[(n+m-2)\pi/2]}{\pi(n+m-2)} \right] \\ &\quad + \left[\frac{\cos[(n-m)\pi/2]}{\pi(n-m)} - \frac{\cos[(n+m-2)\pi/2]}{\pi(n+m-2)} \right] \\ &= 0, \end{aligned}$$

the suborthogonality condition holds in this case.

If $n \neq m$ and if n and m are both even, we have

$$\begin{aligned}
 & \int_{-1}^1 \sin \frac{(n-1)\pi x}{2} \sin \frac{(m-1)\pi x}{2} dx \\
 &= \left[\frac{\sin[(n-m)\pi x/2]}{\pi(n-m)} - \frac{\sin[(n+m-2)\pi x/2]}{\pi(n+m-2)} \right]_{-1}^1 \\
 &= \left[\frac{\sin[(n-m)\pi/2]}{\pi(n-m)} - \frac{\sin[(n+m-2)\pi/2]}{\pi(n+m-2)} \right] \\
 &\quad + \left[\frac{\sin[(n-m)\pi/2]}{\pi(n-m)} - \frac{\sin[(n+m-2)\pi/2]}{\pi(n+m-2)} \right] \\
 &= 2 \left\{ \frac{\sin[(n-m)\pi/2]}{\pi(n-m)} - \frac{\sin[(n+m-2)\pi/2]}{\pi(n+m-2)} \right\} \\
 &= 0
 \end{aligned}$$

since both m and n are even.

For $n \neq m$ and for n and m both odd, the suborthogonality condition involves the integral

$$\begin{aligned}
 & \int_{-1}^1 \cos \frac{(n-1)\pi x}{2} \cos \frac{(m-1)\pi x}{2} dx \\
 &= \left[\frac{\sin[(n-m)\pi x/2]}{\pi(n-m)} + \frac{\sin[(n+m-2)\pi x/2]}{\pi(n+m-2)} \right]_{-1}^1 \\
 &= \left[\frac{\sin[(n-m)\pi/2]}{\pi(n-m)} + \frac{\sin[(n+m-2)\pi/2]}{\pi(n+m-2)} \right] \\
 &\quad + \left[\frac{\sin[(n-m)\pi/2]}{\pi(n-m)} + \frac{\sin[(n+m-2)\pi/2]}{\pi(n+m-2)} \right] \\
 &= 2 \left\{ \frac{\sin[(n-m)\pi/2]}{\pi(n-m)} + \frac{\sin[(n+m-2)\pi/2]}{\pi(n+m-2)} \right\} \\
 &= 0
 \end{aligned}$$

since both m and n are odd so that both $n-m$ and $n+m$ are even. Thus, the suborthogonality condition has been shown to hold in all cases for the characteristic functions of $(1; 2 - 2; 2)$.

In the same way the suborthogonality of the characteristic functions of the other subsystems of $(1 - 1; 3)$ can be verified.

6. Intervals Containing Characteristic Values of an Interface System

We have already mentioned that Theorem 1, Sec. 1, Chap. 3, may be useful if subsystem $(i; m - n; k)$ has characteristic values which are close together. Then the interval between the first and last of any three consecutive characteristic values of subsystem $(i; m - n; k)$ will be small. Moreover, each such interval will contain at least one characteristic value of $(1 - 1; K)$.

For example, consider the following system whose differential equations are

$$\begin{aligned}
 (1) \quad & y_1'' + \pi^2 \lambda^2 y_1 = 0 && \text{for } -2 < x < -1 \\
 & y_2'' + 100\pi^2 \lambda^2 y_2 = 0 && \text{for } -1 < x < 1 \\
 & y_3'' + 4\pi^2 \lambda^2 y_3 = 0 && \text{for } 1 < x < 2
 \end{aligned}$$

and where boundary and interface conditions are

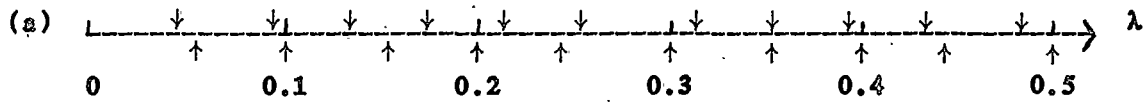
$$\begin{aligned}
 (0;1) \quad & y_1(-2) = 0 \\
 (1,1) \quad & y_1(-1) - y_2(-1) = 0 \\
 (1,2) \quad & 10y_1'(-1) - y_2'(-1) = 0 \\
 (2,1) \quad & y_2(1) - y_3(1) = 0 \\
 (2,2) \quad & y_2'(1) - 5y_3'(1) = 0 \\
 (1;3) \quad & y_3(2) = 0.
 \end{aligned}$$

The n th characteristic value of this interface system is given by $\lambda_n = n/23$. The n th characteristic values of the subsystems $(1; 1 - 1; 2)$, $(1; 2 - 1; 2)$, $(1; 1 - 2; 2)$, and $(1; 2 - 2; 2)$ are $\lambda_n = n/20$, $(2n-1)/40$, $(2n-1)/40$, and $n/20$, respectively. For each of these four

subsystems the line segment joining the n th and $(n+2)$ nd characteristic values is of length 0.1. Consequently, fairly small intervals are found which must contain at least one characteristic value of $(1 - 1; 3)$.

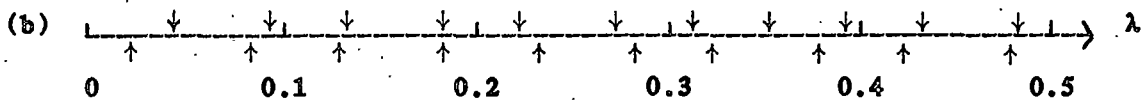
Figs. 4a through 4d demonstrate the fact that at least one characteristic value of system $(1 - 1; 3)$ lies between the smallest and largest of any three consecutive characteristic numbers of the four subsystems already named.

Figure 4. Demonstration of the result of Theorem 1, Sec. 1, Chap. 3.



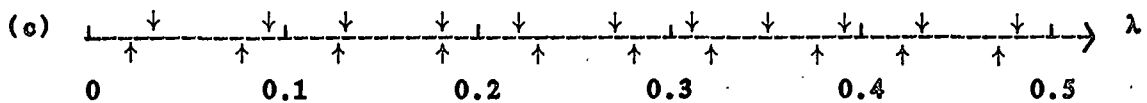
The characteristic values of $(1;1 - 1;2)$ and

$(1 - 1;3)$ are denoted by $\uparrow$ and $\downarrow$ respectively.



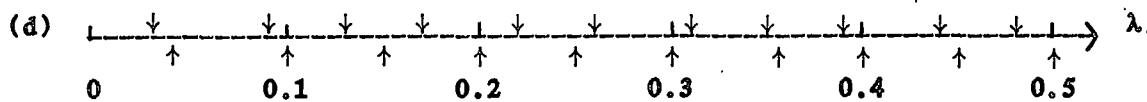
The characteristic values of $(1;2 - 1;2)$ and

$(1 - 1;3)$ are denoted by $\uparrow$ and $\downarrow$ respectively.



The characteristic values of $(1;1 - 2;2)$ and

$(1 - 1;3)$ are denoted by $\uparrow$ and $\downarrow$ respectively.



The characteristic values of $(1;2 - 2;2)$ and

$(1 - 1;3)$ are denoted by $\uparrow$ and $\downarrow$ respectively.

CHAPTER 5

SUMMARY AND TOPICS FOR FURTHER RESEARCH

In Sec. 2, Chap. 2 we introduced the concept of a subsystem of an interface Sturm-Liouville system. Then an interlacing theorem involving auxiliary functions of an interface Sturm-Liouville system and certain of its subsystems was established. In Chap. 3 some results concerning characteristic values of ordinary Sturm-Liouville systems were extended to characteristic values of interface Sturm-Liouville systems. We now list several topics for further research in areas related to those discussed in this thesis:

1. Obtain conditions yielding results similar to those of the fundamental interlacing theorem of Sec. 2, Chap. 1, when the $f_i(\lambda)$ for $1 \leq i \leq 4$ are allowed to have zeros of an order higher than one.
2. Seek to determine for an interface Sturm-Liouville system how the number of zeros of a characteristic function corresponding to the n th characteristic value may depend on n .
3. Find the asymptotic form as n approaches $+\infty$ of Sturm-Liouville interface characteristic functions which correspond to the n th characteristic number.
4. Seek useful families of functions for applying the Raleigh-Ritz Method to interface systems and conditions necessary for getting good results with this method.

5. Discover some variation of Sturm's Comparison Theorem which holds for interface systems.
6. Extend results obtained in this thesis to Sturm-Liouville interface systems involving the parameter λ in their boundary and interface conditions.
7. Study Sturm-Liouville interface systems for which (4) and (5) of Chap. 1 may not hold.
8. Investigate Sturm-Liouville interface systems of order higher than 2.

BIBLIOGRAPHY

1. H. F. Baker, Proceedings of the London Mathematical Society, Vol. 34, 1902, pp. 347-360.
2. H. F. Baker, Proceedings of the London Mathematical Society, Vol. 35, 1903, pp. 333-378.
3. Louis C. Barrett, Separation and Interlacing Theorems with Applications to the Characteristic Numbers of Two-Layer Sturm-Liouville Systems, NAVWEPS Report 8466, U.S. Naval Ordnance Test Station, China Lake, California, 1964.
4. Louis C. Barrett and Gerald Bendixen, Separation and Interlacing Theorems, Quarterly of Applied Mathematics, XXIII, No. 1, April, 1965, pp. 69-78.
5. Louis C. Barrett and Charles Thorne, Sturm-Liouville Systems with Generalized Boundary Conditions, NAVORD Report 6517, U.S. Naval Ordnance Test Station, China Lake, California, 1959.
6. Louis C. Barrett and C. Ray Wylie, Orthogonality Conditions in Systems with Both Continuous and Lumped Parameters, Revista de Mathematica y Fisica Teorica, Tucumán, Vol. XII, 1959, pp. 73-80.
7. G. D. Birkhoff and R. E. Langer, The Boundary Problems and Developments Associated with a System of Ordinary Linear Differential Equations of the First Order, Proceedings of the American Academy of Arts and Sciences, Vol. 57, 1922, pp. 51-128.
8. R. C. Brown, Duality Theory for nth Order Differential Operators Under Stieltjes Boundary Conditions, SIAM J. Math. Analysis, Vol. 6, 1975, pp. 882-900.
9. R. C. Brown, Adjoint Domains and Generalized Splines, Czech. Math. J., Vol. 25, 1975, pp. 134-147.
10. C. Carathéodory, Vorlesungen Über Reelle Funktionen, Leipzig, 1927, p. 665.
11. Earl A. Coddington and Norman Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, 1955.
12. Carl Herbert Denbow, A Generalized Form of the Problem of Bolza, Contributions to the Calculus of Variations 1933-1937, University of Chicago Press, 1937, pp. 449-484.

13. F. A. Ficken, Transformations and Matrices, Prentice Hall Inc., 1967, pp. 268-270.
14. Einar Hille, Lectures on Ordinary Differential Equations, Addison Wesley, pp. 388-389.
15. E. L. Ince, Ordinary Differential Equations, Dover, 1956.
16. J. W. Jerome, Linear Self-Adjoint Multipoint Boundary Value Problems and Related Approximation Schemes, Numerische Math., 15, 1970, pp. 433-449.
17. Ralph Mansfield, Differential Systems Involving k-Point Boundary Conditions, Contributions to the Calculus of Variations 1938-1941, University of Chicago Press, 1941, pp. 413-472.
18. O. Nicoletti, Sulle Condizioni Iniziali che Determinano gli Integrali delle Equazioni Differenziali Ordinarie, Atti della R. Accademia delle Scienze di Torino, Vol. 33, 1897, pp. 746 ff.
19. M. Picone, I Teoremi d'esistenza per gli Integrali di una Equazione Differenziale Lineare Ordinaria Soddisfacenti ad una Nuova Classe di Condizioni, Rendiconti della Accademia dei Lincei, Vol. 17, 1908, pp. 340-347.
20. T. J. Pignani and W. M. Whyburn, Differential Systems with Interface and General Boundary Conditions, Journal of the Elisha Mitchell Scientific Society, Vol. 72, No. 1, May 1956, pp. 1-14.
21. W. T. Reid, Discontinuous Solutions in the Non-parametric Problem of Mayer in the Calculus of Variations, Amer. J. Math., Vol. 57, 1935, pp. 69-93.
22. W. C. Sangren, Differential Equations with Discontinuous Coefficients, Oak Ridge National Laboratory Publication No. 1566 (Physics), 1953.
23. W. C. Sangren, Interfaces in Two Dimensions, SIAM Rev. 2, 1960, pp. 192-199.
24. Malcolm Finlay Smiley, Discontinuous Solutions for the Problem of Bolza in Parametric Form, Contributions to the Calculus of Variations, 1933-1937, University of Chicago Press, 1937, pp. 527-566.
25. A. Smorgorshevsky, Les Fonctions de Green des Systems Differentials Lineaires dans un Domaine a une Seule Dimension, Recueil Mathématique (n. s.), Vol. 7, 1940, pp. 179-196.

26. F. W. Stallard, Differential Systems with Interface Conditions, Oak Ridge National Laboratory Publication No. 1876 (Physics), 1955.
27. Robert Weinstock, Calculus of Variations, McGraw-Hill, 1952.
28. W. M. Whyburn, On the Fundamental Existence Theorems for Differential Systems, Annals of Math., Vol. 30, 1928, pp. 31-38.
29. W. M. Whyburn, Differential Equations with General Boundary Conditions, Bull. Amer. Math. Soc., Vol. 48, 1942, pp. 692-704.
30. W. M. Whyburn, Differential Systems with General Boundary Conditions, Seminar Reports in Mathematics, Univ. of California Publications in Mathematics, Vol 2, 1944, pp. 45-61.
31. W. M. Whyburn, Differential Systems with Boundary Conditions at More Than Two Points, Proceedings of the Conference on Differential Equations, University of Maryland, March 17, 18, and 19, 1955.
32. Charles E. Wilder, Expansion Problems of Ordinary Linear Differential Equations with Auxiliary Conditions at More Than Two Points, Trans. Amer. Math. Soc., Vol. 18, 1917, pp. 415-442.
33. Charles E. Wilder, Problems in the Theory of Ordinary Linear Differential Equations with Auxiliary Conditions at More Than Two Points, Trans. Amer. Math. Soc. Vol. 19, 1918, pp. 157-166.
34. C. Ray Wylie and Louis C. Barrett, Advanced Engineering Mathematics 5th Ed., McGraw-Hill, New York, 1982.

MONTANA STATE UNIVERSITY LIBRARIES
 stks D378.W732@Theses
 Subsystems of and interlacing theorems f RL

 3 1762 00185687 9

D378
 W732 Winslow, D. N.
 cop.2 Subsystems of and
 interlacing theorems
 for interface sturm-
 liouville systems

DATE	ISSUED TO
1982-3	

D 378
 W 732
 cop. 2