



Iterative procedure for a nonlinear circuit  
by Marcia M Peterson

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of  
MASTER OF SCIENCE in Mathematics  
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Abstract:

This paper is a constructive proof of the existence and uniqueness of a periodic solution of the nonlinear differential equation (i)  $dx/dt + f(x) = p(t)$  where  $p(t)$  is continuous,  $p(t) = p(t + 2\pi)$ ,  $f(x)$  is continuous, and  $0 < a < b$ . This differential equation has a physical origin in the consideration of an electrical circuit which contains a linear inductance and a nonlinear resistance connected in series with a periodic external electromotive force. In the problem considered here the resistance is assumed to have an associated potential drop  $f(x)$  such that  $f(x)$  has a continuous derivative with respect to current flow,  $x$ , and  $f'(x)$  is bounded between positive constants.

The solution of the equation is approximated by a sequence of functions determined by the following iteration scheme. A linear term  $bx$  is first added to both members of (i) and it is rewritten in the form (ii)  $dx/dt + bx = p(t) + bx - f(x)$ .

The first iterate is taken to be the periodic solution of the linear equation resulting when the expression  $bx - f(x)$  is deleted from (ii). This solution is then substituted into (ii) and the second iterate taken to be the periodic solution of the resulting linear equation and so on. It is then proved that the sequence of functions so obtained converges to a periodic function which satisfies the differential equation (i) and that this solution is unique.

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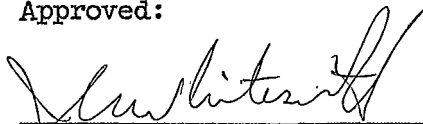
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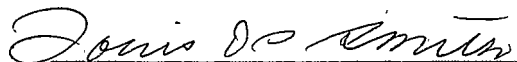
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## Abstract

This paper is a constructive proof of the existence and uniqueness of a periodic solution of the nonlinear differential equation

$$(i) \frac{dx}{dt} + f(x) = p(t)$$

where  $p(t)$  is continuous,  $p(t) = p(t + 2\pi)$ ,  $f'(x)$  is continuous, and  $0 < a < f'(x) < b$ .

This differential equation has a physical origin in the consideration of an electrical circuit which contains a linear inductance and a nonlinear resistance connected in series with a periodic external electromotive force. In the problem considered here the resistance is assumed to have an associated potential drop  $f(x)$  such that  $f(x)$  has a continuous derivative with respect to current flow,  $x$ , and  $f'(x)$  is bounded between positive constants.

The solution of the equation is approximated by a sequence of functions determined by the following iteration scheme. A linear term  $bx$  is first added to both members of (i) and it is rewritten in the form

$$(ii) \frac{dx}{dt} + bx = p(t) + bx - f(x).$$

The first iterate is taken to be the periodic solution of the linear equation resulting when the expression  $bx - f(x)$  is deleted from (ii). This solution is then substituted into (ii) and the second iterate taken to be the periodic solution of the resulting linear equation and so on.

It is then proved that the sequence of functions so obtained converges to a periodic function which satisfies the differential equation (i) and that this solution is unique.

## INTRODUCTION

Ohm's Law states that the potential drop across a conductor is a constant (called the resistance) multiplied by the current flowing through the conductor. It is well known that the state of current flow in a circuit containing Ohmic resistors, linear inductance coils, and linear capacitors connected in series with an external electromotive force can be determined by elementary means as the solution of a linear differential equation. If, however, the potential drop across the resistor is a nonlinear function of the current, the differential equation to be considered is no longer linear and other methods of solution must be sought.

A class of nonlinear resistors which have important engineering applications has been termed quasi-linear by Duffin<sup>1</sup> [1]. These materials have the property that the derivative of the potential drop with respect to current is bounded between positive constants. That is, if the current flowing through the resistor is  $x$  and  $g(x)$  is the potential drop across the resistor, then  $0 < \alpha < g'(x) < \beta$ . Among the materials exhibiting this behavior are selenium, copper oxide ( $\text{Cu}_2\text{O}$ ), and silicon carbide (thyrite) [2].

In this thesis the problem considered is the existence of a periodic solution of a nonlinear differential equation resulting when a circuit contains a quasi-linear resistor. Circuit and network problems involving nonlinear resistors have been treated extensively by Duffin [2] and by Swartz [4]. Duffin established the existence of a unique periodic

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1. Single numbers in brackets will indicate references listed in the Literature Cited.

solution and Swartz used an iteration procedure to actually construct the solution. In [4] Swartz obtained the approximating functions or iterates as Fourier series.

The role of this paper is to give (in the special case of a simple circuit containing no capacitor) a constructive proof of the existence of a periodic solution which is different from the proofs of Duffin and Swartz.

It will be assumed throughout that the electromotive force is a periodic function of time of period  $2\pi$ . Furthermore it will be assumed that the derivative of potential drop with respect to current is not only bounded but also continuous.

DESCRIPTION OF THE PROBLEM  
AND THE ITERATION TECHNIQUE

The circuit chosen for consideration in this thesis consists of a nonlinear resistor and a linear inductance coil of inductance  $L$  connected in series with a periodic external electromotive force,  $E$ .

It is well known that if the resistor is Ohmic with resistance  $R$ , the flow of current,  $I$ , in this circuit satisfies the linear differential equation

$$L \frac{dI}{dt} + RI = E(t).$$

To be considered instead is the nonlinear problem arising when the linear resistor is replaced by a nonlinear resistor having an associated potential drop,  $g(I)$ , where  $g'(I)$  is continuous and bounded between positive constants. For convenience the following notation will be adopted and used throughout: let  $x$  denote current; let  $\frac{g(x)}{L} = f(x)$ ; let  $\frac{E(t)}{L} = p(t)$ .

The differential equation governing the current flow then becomes

$$(2.1) \quad \frac{dx}{dt} + f(x) = p(t)$$

where the following assumptions are made:

- 1)  $p(t)$  is continuous
- 2)  $p(t) = p(t + 2\pi)$
- 3)  $f'(x)$  is continuous
- 4)  $0 < a < f'(x) < b$ .

The periodic solution of equation (2.1) is approximated by a sequence of functions  $\{x_n(t)\}$  determined by the iteration scheme herein described. The linear term,  $bx$ , is first added to both members of (2.1)

and the equation is rearranged to

$$(2.2) \quad \frac{dx}{dt} + b(x) = p(t) + F(x)$$

where  $F(x) = bx - f(x)$ .

The first iterate,  $x_0(t)$ , is taken as the periodic solution of the linear equation obtained by deletion of the nonlinear term,  $F(x)$ , in (2.2). This solution is then substituted into (2.2) and  $x_1(t)$  is taken to be the periodic solution of the resulting linear equation,

$$\frac{dx}{dt} + bx = p(t) + F(x_0(t)), \text{ etc.}$$

In general, the  $(n + 1)^{\text{st}}$  iterate will be the periodic solution of

$$\frac{dx}{dt} + bx = p(t) + F(x_n(t)).$$

It remains, then, to show that the sequence of approximating functions so obtained converges to a periodic function and that this function is a unique periodic solution of (2.1). The method of proof that will be used here is an interesting analogy to the Cauchy-Picard method of proof of existence and uniqueness for the initial-value problem associated with a first-order differential equation.

## PRELIMINARIES

### A. Periodic Solution of the Linear Equation

To obtain the first iterate, the periodic solution of the linear equation formed by deletion of the nonlinear term in (2.2) will be found; i.e. the first iterate will be the periodic solution of

$$(3.1) \quad \frac{dx}{dt} + bx = p(t)$$

where  $b$  is a positive constant and  $p(t)$  is continuous and periodic.

$e^{bt}$  is easily seen to be an integrating factor for this equation.

That is to say, if  $x(t)$  satisfies (3.1) then

$$(3.2) \quad \frac{d}{dt} [e^{bt} bx(t)] = e^{bt} p(t).$$

Whereupon integration gives

$$(3.3) \quad x(t) = e^{-bt} \left[ \int e^{bt} p(t) dt + C \right] \quad \text{or}$$

$$(3.4) \quad x(t) = e^{-bt} \left\{ \int_{\alpha}^t e^{bs} p(s) ds \right\}$$

where  $C$  and  $\alpha$  are parameters. Equations (3.3) and (3.4) are entirely equivalent (in the sense that each gives the general solution of (3.1))

for the following reason: since  $p$  is continuous,

$$\frac{d}{dt} \int_{\alpha}^t e^{bs} p(s) ds = p(t) e^{bt}.$$

Thus the expressions multiplying  $e^{-bt}$  in (3.3) and in (3.4) have exactly the same derivative and therefore may differ only by a constant; i.e., the right member of (3.4) may always be obtained by specializing the constant  $C$  in (3.3). Hence, the general solution of (3.1) is

$$x(t) = \int_{\alpha}^t e^{b(s-t)} p(s) ds.$$

It now remains to choose  $\alpha$  so that the periodicity requirement is satisfied. That is,  $\alpha$  must be chosen so that  $x(t) = x(t + 2\pi)$  or

$$\int_{\alpha}^t e^{b(s-t)} p(s) ds = \int_{\alpha}^{t+2\pi} e^{b(s-t-2\pi)} p(s) ds.$$

This is done by making the substitution  $\sigma = s - 2\pi$  in the right member and recalling that  $p(\sigma) = p(\sigma + 2\pi)$  to give

$$\begin{aligned} \int_{\alpha}^t e^{b(s-t)} p(s) ds &= \int_{\alpha-2\pi}^t e^{b(\sigma-t)} p(\sigma) d\sigma \\ &= \int_{\alpha-2\pi}^{\alpha} e^{b(\sigma-t)} p(\sigma) d\sigma + \int_{\alpha}^t e^{b(\sigma-t)} p(\sigma) d\sigma. \end{aligned}$$

So it is seen that periodicity of  $x(t)$  requires that

$$\int_{\alpha-2\pi}^{\alpha} e^{b(\sigma-t)} p(\sigma) d\sigma = e^{-bt} \int_{\alpha-2\pi}^{\alpha} e^{b\sigma} p(\sigma) d\sigma = 0$$

or, since  $e^{-bt} \neq 0$ , (and changing the integration dummy)

$$(3.5) \int_{\alpha-2\pi}^{\alpha} e^{bs} p(s) ds = 0.$$

That this condition is satisfied for  $\alpha = -\infty$  will be established by showing that  $\lim_{\alpha \rightarrow -\infty} \int_{\alpha-2\pi}^{\alpha} e^{bs} p(s) ds = 0$  for  $b > 0$ .

(The existence of the improper integral,  $\int_{-\infty}^t e^{bs} p(s) ds$ , will be noted

shortly.)

Since  $p(s)$  is continuous the Mean-Value Theorem is applicable so

$$\int_{\alpha-2\pi}^{\alpha} e^{bs} p(s) ds = e^{b\xi} p(\xi) 2\pi \quad (\alpha - 2\pi < \xi < \alpha).$$

Now let  $\alpha \rightarrow -\infty$ . Then  $\xi \rightarrow -\infty$  and  $\lim_{\alpha \rightarrow -\infty} \int_{\alpha-2\pi}^{\alpha} e^{bs} p(s) ds = \lim_{\xi \rightarrow -\infty} e^{b\xi} p(\xi) 2\pi = 0,$

for  $p(\xi)$  is bounded (being continuous and periodic) and  $b$  is positive.

It is possible, however, that  $p(s)$  may be such that

$\int_{\alpha-2\pi}^{\alpha} e^{bs} p(s) ds$  vanishes for finite  $\alpha$ . In view of this, it will be shown

that then  $\int_{-\infty}^{\alpha} e^{bs} p(s) ds$  also vanishes.

To this end write

$$(3.6) \quad \int_{-\infty}^{\alpha} e^{bs} p(s) ds = \sum_{n=0}^{\infty} \int_{\alpha-(n+1)2\pi}^{\alpha-2n\pi} e^{bs} p(s) ds$$

and make the change of variable  $s = \eta - 2n\pi$  in the right member. Taking account of the periodicity of  $p$ , (3.6) becomes

$$\begin{aligned} \int_{-\infty}^{\alpha} e^{bs} p(s) ds &= \sum_{n=0}^{\infty} \int_{\alpha-2\pi}^{\alpha} e^{b(\eta-2n\pi)} p(\eta - 2n\pi) d\eta \\ &= \sum_{n=0}^{\infty} e^{-2n\pi b} \int_{\alpha-2\pi}^{\alpha} e^{b\eta} p(\eta) d\eta \\ &= \left( \int_{\alpha-2\pi}^{\alpha} e^{b\eta} p(\eta) d\eta \right) \sum_{n=0}^{\infty} (e^{-2\pi b})^n \end{aligned}$$

Since  $b$  is positive,  $0 < e^{-2\pi b} < 1$ , and the series,

$$\sum_{n=0}^{\infty} (e^{-2\pi b})^n, \text{ converges to a positive value, say } L.$$

Therefore, if (3.5) holds for finite  $\alpha$ ,

$$\int_{-\infty}^{\alpha} e^{bs} p(s) ds = L \int_{\alpha-2\pi}^{\alpha} e^{bs} p(s) ds = 0 \text{ and, hence}$$

$$\begin{aligned} e^{-bt} \int_{-\infty}^t e^{bs} p(s) ds &= e^{-bt} \int_{-\infty}^{\alpha} e^{bs} p(s) ds + e^{-bt} \int_{\alpha}^t e^{bs} p(s) ds \\ &= e^{-bt} \int_{\alpha}^t e^{bs} p(s) ds. \end{aligned}$$

Thus, in any event, periodicity of  $x(t)$  implies that

$$x(t) = \int_{-\infty}^t e^{b(s-t)} p(s) ds.$$

#### B. Existence of the First Iterate

In this section it will be shown that the integral,

$$\int_{-\infty}^t e^{b(s-t)} p(s) ds, \text{ converges.}$$

As previously noted, the continuity and periodicity of  $p(s)$  imply its boundedness for all  $s$ . That is, there exists a positive constant  $M$  such that  $|p(s)| \leq M$  for all  $s$ . Using this condition,

$$\int_{-\infty}^t \left| e^{b(s-t)} p(s) \right| ds \leq e^{-bt} M \int_{-\infty}^t e^{bs} ds = \frac{M}{b}.$$

Thus  $\int_{-\infty}^t e^{b(s-t)} p(s) ds$  converges absolutely, hence it converges, and

the existence of the first iterate,

$$(3.7) \quad x(t) = \int_{-\infty}^t e^{b(s-t)} p(s) ds, \text{ is assured.}$$

### C. Uniqueness

It is observed at this point that the uniqueness of the first iterate is implicit in the foregoing.

Every function,  $x(t)$ , which satisfies (3.1) must also, of course, satisfy (3.2). Thus all solutions of (3.1) are given by (3.3) or its equivalent (3.4), the equivalence of these equations having been established. In other words, all solutions of the linear equation, (3.1), may be obtained by specializing the constant  $\alpha$  in (3.4).

But periodicity of  $x(t)$  requires the verity of (3.5) and, as shown, this condition will be satisfied by  $\alpha = -\infty$  even if (3.5) holds for finite  $\alpha$ . Thus any other periodic solution of (3.1) must necessarily be given by (3.7); i.e., (3.7) must be a unique periodic solution of the linear equation.

This solution will henceforth be call  $x_0(t)$ .

## PROOF OF CONVERGENCE

### A. Existence of the Successive Iterates

The existence of the successive iterates,  $x_n(t)$ , obtained by the procedure described on page 4 will now be proved by induction.

The existence of the first iterate,

$$x_0(t) = \int_{-\infty}^t e^{b(s-t)} p(s) ds,$$

has been proved in section B page 8.

Now assume the  $k^{\text{th}}$  iterate

$$x_{k-1}(t) = \int_{-\infty}^t e^{b(s-t)} [p(s) + F(x_{k-2}(s))] ds$$

exists and is periodic.

The  $(k+1)^{\text{st}}$  iterate is taken to be the periodic solution of the linear equation

$$(4.1) \quad \frac{dx}{dt} + bx = p(t) + F(x_{k-1}(t))$$

where  $b$  is positive,  $p(t)$  is continuous,  $p(t) = p(t + 2\pi)$ , and  $F(x)$  is continuous.

By hypothesis,  $x_{k-1}(t)$  exists and therefore is continuous. Since  $x_{k-1}(t)$  is also assumed to be periodic,  $F(x_{k-1}(t))$  is continuous and periodic. From this and the conditions on  $p(t)$ , it follows that

$[p(t) + F(x_{k-1}(t))]$  is continuous and periodic. Thus repetition of the arguments used to establish (3.7) result in the following periodic solution of (4.1):

$$(4.2) \quad x_k(t) = \int_{-\infty}^t e^{b(s-t)} [p(s) + F(x_{k-1}(s))] ds.$$

Therefore  $x_n(t)$  ( $n = 0, 1, 2, \dots$ ) exists and is periodic.

### B. Convergence of the Sequence of Iterates

It will first be shown that for functions  $y(t)$  and  $z(t)$  defined for all  $t$ ,  $F(x(t))$  satisfies the Lipschitz condition

$$(4.3) \quad |F(y(t)) - F(z(t))| \leq K |y(t) - z(t)|$$

for all  $t$  and some  $K > 0$ .

$F(x) = bx - f(x)$  where  $f'(x)$  is continuous and  $0 < a < f'(x) < b$ . So  $F'(x) = b - f'(x)$  is certainly continuous. Now rewriting the above inequalities gives

$$-b < a - b < f'(x) - b < 0$$

or  $b > b - a > b - f'(x) > 0$ .

So it is seen that  $0 < F'(x) < b - a$  for all  $x$ . Then let  $b - a = K$ .

Since  $F'(x)$  is continuous it follows from the Law of the Mean that

$$F(y(t)) - F(z(t)) = F'(\xi)(y(t) - z(t))$$

for some  $\xi$  between  $y(t)$  and  $z(t)$ , or

$$|F(y(t)) - F(z(t))| = |F'(\xi)| |y(t) - z(t)|.$$

But  $0 < F'(\xi) < K$  giving

$$|F(y(t)) - F(z(t))| \leq K |y(t) - z(t)| \text{ for all } t.$$

By induction it will now be proved that for all  $t$  and  $M$  a positive constant

$$(4.4) \quad |x_n(t) - x_{n-1}(t)| \leq \frac{K^{n-1} M}{b^n} \quad (n = 1, 2, \dots).$$

$F(x_0(t))$  is continuous and periodic in  $t$  and therefore is bounded for all  $t$ . So there exists a positive constant  $M$  such that

$|F(x_0(t))| \leq M$  for all  $t$ . Now

$$\begin{aligned} |x_1(t) - x_0(t)| &= \left| \int_{-\infty}^t e^{b(s-t)} [p(s) + F(x_0(s)) - p(s)] ds \right| \\ &\leq e^{-bt} \int_{-\infty}^t e^{bs} |F(x_0(s))| ds. \end{aligned}$$

Then, by the boundedness of  $F(x_0(s))$ , (4.4) is verified for  $n = 1$ ; i.e.,

$$|x_1(t) - x_0(t)| \leq e^{-bt} M \int_{-\infty}^t e^{bs} ds = \frac{M}{b}.$$

Assume  $|x_k(t) - x_{k-1}(t)| \leq \frac{K^{k-1} M}{b^k}$  for all  $t$ .

For  $n = k + 1$

$$\begin{aligned} |x_{k+1}(t) - x_k(t)| &= \left| \int_{-\infty}^t e^{b(s-t)} [F(x_k(s)) - F(x_{k-1}(s))] ds \right| \\ &\leq e^{-bt} \int_{-\infty}^t e^{bs} |F(x_k(s)) - F(x_{k-1}(s))| ds. \end{aligned}$$

Since  $F(x(t))$  satisfies the Lipschitz condition for all  $t$ , this becomes

$$|x_{k+1}(t) - x_k(t)| \leq K e^{-bt} \int_{-\infty}^t e^{bs} |x_k(s) - x_{k-1}(s)| ds.$$

But, by hypothesis,  $|x_k(s) - x_{k-1}(s)| \leq \frac{K^{k-1} M}{b^k}$  for all  $t$ , so

$$|x_{k+1}(t) - x_k(t)| \leq \frac{K e^{-bt} K^{k-1} M}{b^k} \int_{-\infty}^t e^{bs} ds = \frac{K^k M}{b^{k+1}}.$$

Thus (4.4) is established.

Now consider the sequence of iterates  $\{x_n(t)\}$ , and note that

$$x_n(t) = x_0(t) + \sum_{k=1}^n (x_k(t) - x_{k-1}(t));$$

i.e.,  $x_n(t)$  is the  $n^{\text{th}}$  partial sum of the series

$$(4.5) \quad x_0(t) + \sum_{k=1}^{\infty} (x_k(t) - x_{k-1}(t)).$$

From (4.4)  $|x_n(t) - x_{n-1}(t)| \leq \frac{K^{n-1} M}{b^n} \quad (n = 1, 2, \dots).$

for all  $t$  ( $M > 0$  and  $K > 0$ ).

Since  $K < b$  the series  $\sum_{n=1}^{\infty} \frac{K^{n-1} M}{b^n} = \frac{M}{b} \sum_{n=1}^{\infty} \left(\frac{K}{b}\right)^{n-1}$  converges.

Thus (4.5) is majorized by a convergent series of positive constants and, according to the Weierstrass M Test, must converge absolutely and uniformly for all  $t$ . Hence, the sequence  $\{x_n(t)\}$  also converges uniformly to a limit function,  $x(t)$ , for all  $t$ .

Two remarks pertinent to the limit function: 1. If  $\lim_{n \rightarrow \infty} x_n(t) = x(t)$ , then  $x(t)$  is continuous since the  $x_n(t)$  ( $n = 0, 1, 2, \dots$ ) are continuous and the convergence is uniform. 2.  $x(t)$  is periodic as the following theorem shows.

Theorem 1.

If (1)  $\{x_n(t)\}$  converges to  $x(t)$  on an interval  $I$ ;

(2)  $x_n(t + 2\pi) = x_n(t)$  ( $n = 0, 1, 2, \dots$ ) ( $t, t + 2\pi \in I$ ),

then  $x(t + 2\pi) = x(t)$  ( $t, t + 2\pi \in I$ ).

Proof:  $|x(t + 2\pi) - x(t)| = \left| [x(t + 2\pi) - x_n(t)] + [x_n(t) - x(t)] \right|$

$$\leq |x(t + 2\pi) - x_n(t)| + |x_n(t) - x(t)|$$

$$= |x_n(t + 2\pi) - x(t + 2\pi)| + |x_n(t) - x(t)|$$

where the last equality follows from hypothesis (2).

Let  $\epsilon > 0$  be given. By the convergence of  $\{x_n(t)\}$  there exist integers  $N_1$  and  $N_2$  such that for  $n > N_1$ , then  $|x_n(t) - x(t)| < \epsilon/2$  and for  $n > N_2$ , then  $|x_n(t + 2\pi) - x(t + 2\pi)| < \epsilon/2$ . Let  $N$  be the maximum of  $N_1$  and  $N_2$ . Then for  $n > N$ ,

$$\begin{aligned} |x(t + 2\pi) - x(t)| &\leq |x_n(t + 2\pi) - x(t + 2\pi)| + |x_n(t) - x(t)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

But the left member of the above inequality is independent of  $n$ . Hence the left member is zero and the conclusion follows.

## VERIFICATION OF THE SOLUTION

For convenience, some essential conclusions will be stated and proved in the form of more general theorems before proceeding with the verification of the solution.

### Theorem 2.

- If (1)  $f_n(s)$  is continuous ( $n = 0, 1, 2, \dots$ ),  
 (2)  $f_n(s + 2\pi) = f_n(s)$  ( $n = 0, 1, 2, \dots$ ),  
 (3)  $\{f_n(s)\}$  converges uniformly to  $f(s)$  on  $I[-\infty, t_0]$ ,  
 (4)  $b > 0$ ;

then  $\int_{-\infty}^t e^{bs} f_n(s) ds$  ( $t \leq t_0$ ) converges uniformly with respect to  $n$ .

Proof: It must be shown that corresponding to every positive number  $\epsilon$  there is a number  $T$  ( $T < t_0$ ) such that

$$\left| \int_{\alpha_1}^{\alpha_2} e^{bs} f_n(s) ds \right| < \epsilon \text{ for } \alpha_1 < \alpha_2 < T \text{ and all positive integers } n \quad [3].$$

By hypotheses (1) and (2) the  $f_n(s)$  ( $n = 0, 1, 2, \dots$ ) are bounded; i.e., there exist positive numbers  $M_n$  such that

$$|f_n(s)| \leq M_n \quad (n = 0, 1, 2, \dots) \text{ for all } s \in I[-\infty, t_0]. \text{ Also}$$

$$|f_n(s)| = |f_n(s) - f(s) + f(s)| \leq |f_n(s) - f(s)| + |f(s)|.$$

According to hypothesis (3), to every  $\epsilon' > 0$  there corresponds an integer  $N$  such that  $|f_n(s) - f(s)| < \epsilon'$  for all  $n > N$  and  $s \in I[-\infty, t_0]$ .  $f(s)$  is continuous for it is defined by a uniformly convergent sequence of continuous functions. Hence,  $f(s)$  is bounded and there exists a positive number  $M'$  such that  $|f(s)| \leq M'$  ( $s \in I[-\infty, t_0]$ ).

Thus  $|f_n(s)| < \epsilon' + M'$  for all  $n > N$ . Now let  $M$  be the maximum of  $\epsilon' + M'$  and  $M_n$  ( $n = 0, 1, 2, \dots$ ). Then  $|f_n(s)| \leq M$  ( $n = 0, 1, 2, \dots$ ) for all  $s \in I [-\infty, t_0]$ .

It then follows that

$$\left| \int_{\alpha_1}^{\alpha_2} e^{bs} f_n(s) ds \right| \leq \int_{\alpha_1}^{\alpha_2} e^{bs} |f_n(s)| ds$$

$$\leq M \int_{\alpha_1}^{\alpha_2} e^{bs} ds = \frac{M}{b} (e^{b\alpha_2} - e^{b\alpha_1}).$$

For a given  $\epsilon > 0$  the choice of  $T = \frac{1}{b} \ln \frac{b\epsilon}{M}$  suffices for the conclusion of the theorem. For then since  $b$  is positive, if  $\alpha_2 < T$ ,  $e^{b\alpha_2} < \frac{b\epsilon}{M}$ . Then also  $e^{b\alpha_2} - e^{b\alpha_1} < \frac{b\epsilon}{M}$  or

$$\frac{M}{b} (e^{b\alpha_2} - e^{b\alpha_1}) < \epsilon.$$

Theorem 3.

If (1)  $f_n(t)$  ( $n = 0, 1, 2, \dots$ ) is continuous on  $I[-\infty, t_0]$ ,

(2)  $\{f_n(t)\}$  converges uniformly to  $f(t)$  on  $I[-\infty, t_0]$ ,

(3)  $\int_{-\infty}^t f_n(s) ds$  ( $t \leq t_0$ ) converges uniformly with respect to  $n$ ,

(4)  $\int_{-\infty}^t f(s) ds$  ( $t \leq t_0$ ) exists;

then  $\lim_{n \rightarrow \infty} \int_{-\infty}^t f_n(s) ds = \int_{-\infty}^t f(s) ds$  ( $t \leq t_0$ ).

Proof: Let  $\epsilon > 0$  be given. Then for  $R < t \leq t_0$ ,

$$\left| \int_{-\infty}^t f_n(s) ds - \int_{-\infty}^t f(s) ds \right| \leq$$

$$\left| \int_{-\infty}^R f_n(s) ds \right| + \left| \int_R^t f_n(s) ds - \int_R^t f(s) ds \right| + \left| \int_{-\infty}^R f(s) ds \right| \leq$$

$$\left| \int_{-\infty}^R f_n(s) ds \right| + \int_R^t |f_n(s) - f(s)| ds + \left| \int_{-\infty}^R f(s) ds \right|.$$

Now the conditions of hypotheses (3) and (4) allow one to choose  $R$  small enough so that

$$\left| \int_{-\infty}^R f_n(s) ds \right| < \frac{\epsilon}{3} \text{ for } n = 0, 1, 2, \dots \text{ and } \left| \int_{-\infty}^R f(s) ds \right| < \frac{\epsilon}{3}.$$

Also, by hypothesis (2), there is an integer  $N$  such that

$$|f_n(s) - f(s)| < \frac{\epsilon}{3(t_0 - R)} \text{ for all } n > N \text{ and all } s \in I[-\infty, t_0].$$

Thus for  $n > N$

$$\left| \int_{-\infty}^t f_n(s) ds - \int_{-\infty}^t f(s) ds \right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \int_R^t \frac{\epsilon ds}{3(t_0 - R)} < \epsilon, \text{ and the conclu-}$$

sion is evident.

Theorem 4.

If (1)  $\{x_n(s)\}$  converges uniformly to  $x(s)$  on  $I[-\infty, t_0]$ ,

(2)  $|F(y(s)) - F(z(s))| \leq K |y(s) - z(s)|$  for some  $K > 0$  and all  $s \in I[-\infty, t_0]$ ; then  $\{F(x_n(s))\}$  converges uniformly to  $F(x(s))$  for all  $s \in I[-\infty, t_0]$ .

Proof: Let  $\epsilon > 0$  be given. According to hypothesis (1), to every positive number  $\epsilon'$  there corresponds an integer  $N$  such that

$|x_n(s) - x(s)| < \epsilon'$  for all  $n > N$  and all  $s \in I[-\infty, t_0]$ . The conclusion follows from the choice  $\epsilon' = \frac{\epsilon}{K}$ . For then

$|F(x_n(s)) - F(x(s))| \leq K |x_n(s) - x(s)| < K \epsilon' = \epsilon$  for all  $n > N$  and all  $s \in I[-\infty, t_0]$ .

Lemma.

If (1)  $\{F(x_n(s))\}$  converges uniformly to  $F(x(s))$  on  $I[-\infty, t_0]$ ,

(2)  $b > 0$ ;

then  $\{e^{bs} F(x_n(s))\}$  converges uniformly to  $e^{bs} F(x(s))$  on  $I[-\infty, t_0]$ .

Proof: By hypothesis, corresponding to any given  $\epsilon > 0$  there is an integer  $N$  such that  $|F(x_n(s)) - F(x(s))| < \epsilon e^{-bt_0}$  for all  $n > N$  and all  $s \in I[-\infty, t_0]$ . But then since  $b$  is positive, for all  $n > N$  and all  $s \in I[-\infty, t_0]$ ,  $e^{bs} |F(x_n(s)) - F(x(s))| < \epsilon$  or  $|e^{bs} F(x_n(s)) - e^{bs} F(x(s))| < \epsilon$ , and the conclusion follows.

In section B on page 11 it was established that the sequence of iterates  $\{x_n(t)\}$  converges uniformly to a periodic function,  $x(t)$ , for all  $t$ . It will now be proved that this limit function is a periodic solution of the nonlinear differential equation

$$(5.1) \quad \frac{dx}{dt} + bx = p(t) + F(x)$$

where  $F(x) = bx + f(x)$ ,  $f'(x)$  is continuous,  $0 < a < f'(x) < b$ ,  $p(t)$  is continuous, and  $p(t + 2\pi) = p(t)$ .

To this end it is first noted that if  $x(t)$  satisfies the integral

equation

$$(5.2) \quad x(t) = \int_{-\infty}^t e^{b(s-t)} [p(s) + F(x(s))] ds,$$

it also satisfies the differential equation (5.1). Consider (5.2). By removing the factor  $e^{-bt}$  from the integrand and then differentiating there results

$$\begin{aligned} \frac{dx(t)}{dt} &= -be^{-bt} \int_{-\infty}^t e^{bs} [p(s) + F(x(s))] ds + [p(t) + F(x(t))] \\ &= -bx(t) + p(t) + F(x(t)) \text{ or } \frac{dx(t)}{dt} + bx(t) = p(t) + F(x(t)). \end{aligned}$$

It is now to be verified that  $x(t) = \lim_{n \rightarrow \infty} x_n(t)$  is a periodic solution of (5.1) by proving that  $x(t)$  satisfies (5.2). In what follows, let  $t_0$  be any number.

It has been seen that  $\{x_n(s)\}$  converges uniformly for all  $s$  and that  $F(x_n(s))$  ( $n = 0, 1, 2, \dots$ ) satisfies the Lipschitz condition for all  $s$ . In particular these statements are true for  $s \in I[-\infty, t_0]$ . Thus the hypotheses of Theorem 4, and those of the Lemma, are satisfied so

$$(a) \quad \{e^{bs} F(x_n(s))\} \text{ converges uniformly to } e^{bs} F(x(s)) \text{ for } s \in I[-\infty, t_0].$$

The  $F(x_n(s))$  ( $n = 0, 1, 2, \dots$ ) then fulfill the conditions of Theorem 2 since, as has been previously observed,  $F(x_n(s))$  ( $n = 0, 1, 2, \dots$ ) is continuous and periodic for all  $s$  and, according to Theorem 4,

$$\{F(x_n(s))\} \text{ converges uniformly to } F(x(s)) \text{ for } s \in I[-\infty, t_0]. \text{ Thus}$$

$$(b) \quad \int_{-\infty}^t e^{bs} F(x_n(s)) ds \quad (t \leq t_0) \text{ converges uniformly with respect to } n.$$

$F(x(s))$  is continuous for all  $s$  since the  $F(x_n(s))$  ( $n = 0, 1, 2, \dots$ )

are continuous and the convergence is uniform. Thus  $F(x(s))$  is also periodic in  $s$  (for  $x(s)$  is periodic). These conditions on  $F(x(s))$  establish its boundedness and therefore (by employing an argument similar to that of section B on page 8)

$$(c) \int_{-\infty}^t e^{bs} F(x(s)) ds \quad (t \leq t_0) \text{ exists.}$$

Obviously the functions  $e^{bs} F(x_n(s))$  ( $n = 0, 1, 2, \dots$ ) are continuous on  $I[-\infty, t_0]$ . But now the hypotheses of Theorem 3 are exactly this together with results (a), (b), and (c) above. This theorem in turn justifies termwise integration of the sequence  $\{e^{bs} F(x_n(s))\}$ . That is, for any number  $t \leq t_0$

$$\lim_{n \rightarrow \infty} \int_{-\infty}^t e^{bs} F(x_{n-1}(s)) ds = \int_{-\infty}^t \lim_{n \rightarrow \infty} e^{bs} F(x_{n-1}(s)) ds.$$

It readily follows that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^t e^{b(s-t)} F(x_{n-1}(s)) ds = \int_{-\infty}^t e^{b(s-t)} \lim_{n \rightarrow \infty} F(x_{n-1}(s)) ds$$

or, again invoking Theorem 4,

$$\lim_{n \rightarrow \infty} \int_{-\infty}^t e^{b(s-t)} F(x_{n-1}(s)) ds = \int_{-\infty}^t e^{b(s-t)} F(x(s)) ds.$$

$$\text{Since } \lim_{n \rightarrow \infty} \int_{-\infty}^t e^{b(s-t)} p(s) ds = \int_{-\infty}^t e^{b(s-t)} p(s) ds,$$

one obtains by addition

$$\lim_{n \rightarrow \infty} \int_{-\infty}^t e^{b(s-t)} [p(s) + F(x_{n-1}(s))] ds = \int_{-\infty}^t e^{b(s-t)} [p(s) + F(x(s))] ds.$$

$$\begin{aligned} \text{However, } x(t) &= \lim_{n \rightarrow \infty} x_n(t) \\ &= \lim_{n \rightarrow \infty} \int_{-\infty}^t e^{b(s-t)} [p(s) + F(x_{n-1}(s))] ds. \end{aligned}$$

Hence  $x(t) = \int_{-\infty}^t e^{b(s-t)} [p(s) + F(x(s))] ds$  for any  $t \leq t_0$ . But  $t_0$  was any

number so the integral equation (5.2) and, consequently, the differential equation (5.1) are satisfied by  $x(t)$  on any  $t$  interval.

## UNIQUENESS OF THE SOLUTION

It will first be demonstrated that if a periodic function satisfies the differential equation (5.1) it must also satisfy the integral equation (5.2).

Assume that  $x(t) = x(t + 2\pi)$  and  $\frac{dx(t)}{dt} + bx(t) = p(t) + F(x(t))$

where all conditions stated for (5.1) hold. Multiplication of both members of the differential equation by  $e^{bt}$  will make the left member exact, or  $\frac{d}{dt} [e^{bt} x(t)] = e^{bt} [p(t) + F(x(t))]$ . Since  $x(t)$  is differentiable,

it must be continuous; thus the right member is integrable so

$$e^{bt} x(t) = \int_{\alpha}^t e^{bs} [p(s) + F(x(s))] ds \text{ or } x(t) = e^{-bt} \int_{\alpha}^t e^{bs} [p(s) + F(x(s))] ds$$

where  $\alpha$  is a parameter.

Now  $x(t)$  is periodic as are both  $p(s)$  and  $F(x(s))$  (the last from continuity considerations). By taking account of this periodicity and proceeding with the argument on page 5, it is to be concluded that

$$e^{-bt} \int_{\alpha}^t e^{bs} [p(s) + F(x(s))] ds = e^{-bt} \int_{-\infty}^t e^{bs} [p(s) + F(x(s))] ds.$$

Thus  $x(t)$  satisfies (5.2) if it is a periodic solution of (5.1).

It remains now to prove that  $x(t)$ , the periodic solution of the nonlinear differential equation (5.1), is unique. This will be done by assuming that  $\bar{x}(t)$  is another periodic solution of (5.1) and then considering the difference  $\bar{x}(t) - x(t)$ .

Since any periodic solution of the differential equation in question also satisfies the integral equation (5.2), this difference may be

written

$$\bar{x}(t) - x(t) = \int_{-\infty}^t e^{b(s-t)} [F(\bar{x}(s)) - F(x(s))] ds.$$

So  $|\bar{x}(t) - x(t)| = \left| \int_{-\infty}^t e^{b(s-t)} [F(\bar{x}(s)) - F(x(s))] ds \right|$  or

$$(6.1) \quad |\bar{x}(t) - x(t)| \leq \int_{-\infty}^t e^{b(s-t)} |F(\bar{x}(s)) - F(x(s))| ds.$$

But  $F(x(s))$  satisfies the Lipschitz condition (4.3); i.e., for all  $s$  and

$$K = b - a > 0,$$

$$(6.2) \quad |F(\bar{x}(s)) - F(x(s))| \leq K |\bar{x}(s) - x(s)|.$$

Introduction of (6.2) into (6.1) results in

$$(6.3) \quad |\bar{x}(t) - x(t)| \leq K \int_{-\infty}^t e^{b(s-t)} |\bar{x}(s) - x(s)| ds.$$

From continuity and periodicity considerations,  $F(x(s))$  is evidently bounded for all  $s$ , and there is no loss in generality in taking the bound to be  $K$ . That is, there is no loss of generality in choosing  $b$  large enough so that  $|F(x(s))| \leq K$  for all  $s$ . Thus also

$$|F(\bar{x}(s)) - F(x(s))| \leq |F(\bar{x}(s))| + |F(x(s))| = 2K.$$

Substitution of this result into (6.1) will give the following inequality

$$|\bar{x}(t) - x(t)| \leq 2K \int_{-\infty}^t e^{b(s-t)} ds = \frac{2K}{b},$$

which, in turn, introduced into (6.3) gives  $|\bar{x}(t) - x(t)| \leq \frac{2K^2}{b}$ .

This is then substituted into (6.2) and the process is repeated, obtaining

an improved bound on  $|\bar{x}(t) - x(t)|$  with each refinement.

By induction it is now shown that  $|\bar{x}(t) - x(t)| \leq \frac{2K^n}{b^n}$  for all positive integers  $n$ .

The verity of this statement has been established for  $n = 1$ . Now assume that after the  $k^{\text{th}}$  refinement one has obtained

$$|\bar{x}(t) - x(t)| \leq \frac{2K^k}{b^k}.$$

Then from (6.2),  $|F(\bar{x}(s)) - F(x(s))| \leq K |\bar{x}(s) - x(s)| = \frac{2K^{k+1}}{b^k}$ .

Then by introducing this result into (6.1) it is found that

$$|\bar{x}(t) - x(t)| \leq \frac{2K^{k+1}}{b^k} \int_{-\infty}^t e^{b(s-t)} ds = \frac{2K^{k+1}}{b^{k+1}}$$

and the conclusion follows.

Now consider the inequality

$$|\bar{x}(t) - x(t)| \leq 2 \left( \frac{K}{b} \right)^n.$$

Since  $K = b - a > 0$ ,  $b > K$  so  $\lim_{n \rightarrow \infty} \left( \frac{K}{b} \right)^n = 0$ .

But the inequality holds for all positive integers  $n$  and the left member is independent of  $n$ ; hence it must equal zero. Therefore,  $x(t)$  is the only periodic solution of the differential equation (5.1).

## SUMMARY

The results of this paper may be summarized by the following theorem.

Theorem 5. If the differential equation

$$(i) \quad \frac{dx}{dt} + f(x) = p(t)$$

where  $p(t)$  is continuous,  $p(t)$  is periodic,  $f'(x)$  is continuous, and  $0 < a < f'(x) < b$ , is written in the form

$$(ii) \quad \frac{dx}{dt} + bx = p(t) + [bx - f(x)]$$

and a sequence of functions  $x_n(t)$  is determined in the following fashion:

$x_0(t)$  is the periodic solution of the equation resulting when  $[bx - f(x)]$

is deleted from (ii);  $x_1(t)$  is the periodic solution of the equation

resulting when  $x_0(t)$  is substituted into the right member of (ii);

$x_2(t)$  is obtained by repetition of the process used to obtain  $x_1(t)$ ,

etc., then the sequence  $\{x_n(t)\}$  converges to a unique periodic solution

of (i).

## REMARKS

The first step in the solution was the addition of a linear term,  $bx$ , to both members of the equation. It is of interest to examine the consequences of failure to employ this tactic before performing the iteration.

If the term  $bx$  is not added to both members, the iteration evidently will merely consist of repeated quadratures. Furthermore, the equations solved to obtain the iterates will have no unique periodic solution. It is also clear that no constant terms may be allowed in the right member if the next iterate is to be periodic. A possible procedure then, would be to take all integration constants to be zero.

A simple linear example (in which the conditions of Theorem 5 are satisfied) will be used to illustrate this procedure if the linear term is not first added to both members.

$$\text{Consider } \frac{dx}{dt} + x = \sin t$$

$$\text{or } (8.1) \quad \frac{dx}{dt} = \sin t - x$$

and proceed with the iteration technique described in Theorem 5 setting all integration constants to zero.  $x_0(t)$  is the periodic solution of

$\frac{dx}{dt} = \sin t$  or  $x_0(t) = -\cos t$ . Then  $x_1(t)$  is the periodic solution of

$$\frac{dx}{dt} = \sin t + \cos t \quad \text{or}$$

$$x_1(t) = -\cos t + \sin t.$$

The next iterate is easily seen to be  $x_2(t) = \sin t$  and the procedure obviously fails for upon substitution into (8.1) one obtains  $\frac{dx}{dt} = 0$ .

However, as the following example indicates, in certain cases the

sequence of functions found by this method will converge to a periodic solution.

Consider the differential equation  $\frac{dx}{dt} = \sin t - \frac{x}{2}$  and note that, as is readily verified, the exact periodic solution is

$$x(t) = 0.4 \sin t - 0.8 \cos t.$$

Proceeding with the iteration as before,  $x_0(t)$  is the periodic solution of  $\frac{dx}{dt} = \sin t$  or  $x_0(t) = -\cos t$ . The successive iterates are then found to be as follows:

$$x_1(t) = 0.5 \sin t - \cos t$$

$$x_2(t) = 0.5 \sin t - 0.75 \cos t$$

$$x_3(t) = 0.375 \sin t - 0.75 \cos t$$

$$x_4(t) = 0.375 \sin t - 0.813 \cos t$$

$$x_5(t) = 0.406 \sin t - 0.813 \cos t$$

$$x_6(t) = 0.406 \sin t - 0.796 \cos t$$

etc.

It appears that in this example the sequence of iterates will converge to the periodic solution, as is indeed the case. In fact, this example illustrates the following theorem [5].

Theorem 6. Consider the differential equation

$$(i) \quad \frac{dx}{dt} = p(t) + \alpha f(x)$$

where  $p(t)$  is continuous,  $p(t + \pi) = -p(t)$ ,  $f(-x) = -f(x)$ , and

$0 < a < f'(x) < b$ . If a sequence of functions,  $x_n(t)$ , is determined in the following way:  $x_0(t)$  is found by deleting the term  $\alpha f(x)$  in (i), integrating and setting the constant of integration to zero;  $x_1(t)$  is

found by substituting  $x_0(t)$  into the right member of (i), integrating and setting the constant of integration to zero, etc., then the sequence  $\{x_n(t)\}$  converges to a unique periodic solution of (i) provided  $|a/b| < 1$ .

In view of the stringent conditions imposed by this theorem, the power of the approach used in adding the term  $bx$  to both members of the equation is apparent.

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