

ABSTRACT TILING ACTIONS, EXPANSIVENESS AND LOCAL STRUCTURE

by

Hannah Faith Sobek Bergren

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ABSTRACT

A significant amount of literature is devoted to the study of the dynamical properties of the translation actions associated to self-affine tilings or Delone sets. A natural step is to axiomatize these essential properties among all $\mathbb{R}^d$ -actions on compact metric spaces. We propose a set of dynamical axioms of such an action which yields a topological conjugacy between the $\mathbb{R}^d$ -action and the translation action associated to a self-affine repetitive aperiodic tiling. In particular, we show that these axioms admit an expanding metric on the local cross-section of the phase space, which implies the existence of a local cross-section that is a Cantor set. We also investigate an interesting example of a tiling space that contains non-FLC tilings, which exhibits an unusually complicated local structure.

INTRODUCTION

The main goal of this work is to give an intrinsic characterization of continuous $\mathbb{R}^d$ -actions on a compact metric space that are topologically conjugate to the translation action on the tiling space associated to a self-affine repetitive aperiodic tiling of $\mathbb{R}^d$ with finite local complexity (FLC). This type of tiling space has been widely studied, mostly focusing on the dynamical properties that are invariant under topological conjugacies [13, 33, 37, 39]. The repetitivity hypothesis is a form of almost periodicity stipulating that any finite patch repeats so that it can be found within a sufficiently large window placed anywhere over the tiling. FLC requires that, up to translation, there are finitely many different patches of any given maximum size. We define both of these terms with more rigor in section 2.

In the current state of tiling theory, in order to understand the translation action on a tiling space one must first define a concrete set of prototiles and a set of rules (or procedures) prescribing how tiles may abut, which can be rather cumbersome. Our characterization aspires to create a more abstract and economical definition of tiling actions which will bring tiling theory into the field of study of topological dynamics of higher rank actions. Kwapisz proposed 4 axioms to describe an abstract self-affine tiling action [20]. Under the assumption of FLC, it was shown that there exists a topological conjugacy between the abstract self-affine tiling action with FLC and the action associated with an aperiodic repetitive self-affine tiling with FLC. We aim to show that the FLC hypothesis on the abstract tiling action is redundant, which is equivalent to showing that a compact metric space that admits an abstract self-affine tiling action is locally the product of $\mathbb{R}^d$ and a Cantor set. This equivalence is not obvious and hinges on the way Markov partitions in the proof of Kwapisz's theorem

are constructed. The tiles in the abstract space are obtained by intersecting orbits of the $\mathbb{R}^d$ action with atoms of the Markov partitions.

The secondary goal of this thesis is to better understand the local structure of tiling spaces of non-FLC tilings. Tilings that do not meet the FLC condition have been studied in [7, 11, 12, 17, 18, 36] and the common feature is the presence of fault lines. We define the fault line precisely in section 5, but roughly it is an infinite line of tile edges along which tiles may abut in an infinite variety of ways. Frank and Sadun introduce an example of a non-FLC tiling of $\mathbb{R}^2$ in which the effect of a horizontal fault line is described by two 1-dimensional substitution tilings [12, 37]. Here the tiling above the fault line is completely unrelated to the tiling below the fault line. This feature has a huge effect on the topology of the tiling space because it introduces a new degree of freedom. Frank and Sadun informally describe the tiling space as an odd mixture of 2- and 3-dimensional pieces, where the 3-dimensional pieces are dense. We formalize their discussion and give a more rigorous description of the tiling space.

Paradigm and $\mathbb{Z}^d$ -Actions

Let $T = (T^t)_{t \in \mathbb{Z}^d}$ be an $\mathbb{Z}^d$ -action on a metric space $(\mathbf{X}, d)$. The result of applying T^t to $\mathbf{x} \in \mathbf{X}$ is denoted by $T^t \mathbf{x}$. T is *expansive* if and only if any two orbits separate at some time by a definite distance; i.e., $\exists_{\Delta_0 > 0} \forall_{\mathbf{x}, \mathbf{y} \in \mathbf{X}}$ if $d(T^n \mathbf{x}, T^n \mathbf{y}) \leq \Delta_0$ for all $n \in \mathbb{Z}^d$, then $\mathbf{x} = \mathbf{y}$. A homeomorphism of $\mathbf{X}$ is expansive if and only if the $\mathbb{Z}$ -action generated by it is expansive. In 1979, Mañé showed that if there exists an expansive homeomorphism of a compact metric space $\mathbf{X}$ that is minimal (i.e. every orbit is dense), then $\mathbf{X}$ has topological dimension zero (i.e. $\mathbf{X}$ is totally disconnected) [25]. Since tiling actions of aperiodic repetitive self-affine tilings with FLC are both expansive and minimal and their underlying tiling spaces are locally

the product of $\mathbb{R}^d$ and a Cantor set, it was natural to hope that one could generalize Mañé's result to $\mathbb{Z}^d$ and $\mathbb{R}^d$ actions for $d \geq 2$. However, we were unable to utilize Mañé's constructions because they heavily relied on the stable and unstable sets of the homeomorphism, which do not have good analogs for higher rank actions. So we turned our attention to Fathi, who proved (Theorem 3.1) that if a compact metric space admits an expansive homeomorphism with zero topological entropy, then it has zero topological dimension [10]. Fathi's main tool was the existence of an expanding metric; i.e., a metric that is expanded in a finite time by a definite multiplicative factor. We were able to generalize Fathi's theorem. In particular, for expansive $\mathbb{Z}^d$ -actions we obtained the following result, proved in section 3.

Theorem 1.1. *For $d \in \mathbb{N}$ and a norm $|\cdot|$ on $\mathbb{R}^d$ there is $\tilde{b} > 0$ such that if T is an expansive $\mathbb{Z}^d$ -action on a compact metric space $\mathbf{X}$, then there exists a metric $\tilde{d}$ on $\mathbf{X}$ defining its topology and numbers $\tilde{k} > 1$ and $\tilde{\epsilon} > 0$ so that, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$,*

$$\max_{n \in \mathbb{Z}^d, |n| < \tilde{b}} \left\{ \tilde{d}(T^n \mathbf{x}, T^n \mathbf{y}) \right\} \geq \min \left\{ \tilde{k} \tilde{d}(\mathbf{x}, \mathbf{y}), \tilde{\epsilon} \right\}.$$

Even though the $\mathbb{Z}^d$ results do not play a direct role in our results about tilings, we have chosen to include them in this paper because we could not find any generalization of Fathi's theorem to higher rank actions in literature. Our main task is to generalize this construction to the appropriate class of $\mathbb{R}^d$ -actions, for which matters are complicated by the fact that $\mathbb{R}^d$ is connected (unlike $\mathbb{Z}^d$).

Results for $\mathbb{R}^d$ -Actions

The appropriate class of $\mathbb{R}^d$ -actions is one in which the dynamical properties exhibited by tiling actions are codified in purely topological terms. We now state the axioms of such actions, first proposed by Kwapisz in [20]. Let $T = (T^t)_{t \in \mathbb{R}^d}$ be an

$\mathbb{R}^d$ -action on a compact metric space $\mathbf{X}$ with the metric d . For the ball of radius $R \geq 0$ centered at $t = 0$ we simply write B_R . We say a linear map Λ is *expanding* if and only if all of its eigenvalues λ satisfy $|\lambda| > 1$. Consider the following axioms for $r_0, l_0, \Delta_0 > 0$:

- (I) **(LF)** $_{2r_0}$: $\forall_{\mathbf{x} \in \mathbf{X}, t \in \mathbb{R}^d}$ If $\mathbf{x} + t = \mathbf{x}$ and $|t| < 2r_0$ then $t = 0$;
- (II) **(WE)** $_{r_0, \Delta_0}$: $\forall_{\mathbf{x}, \mathbf{y} \in \mathbf{X}}$ If $d(\mathbf{x} + t, \mathbf{y} + t) \leq \Delta_0$ for all $t \in \mathbb{R}^d$, then $\mathbf{x} = \mathbf{y} + s$ for some $s \in B_{r_0}$;
- (III) **(PS)** $_{l_0, r_0, \Delta_0}$: $\forall_{0 < \delta < \Delta_0} \exists_{\delta_1 > 0} \forall_{\mathbf{x}, \mathbf{y} \in \mathbf{X}, U \subset \mathbb{R}^d \text{ connected}}$ If $\inf_{|s| \leq l_0} d(\mathbf{x} + t, \mathbf{y} + s + t) < \delta_1$ for all $t \in U$, then $\exists s \in B_{r_0}$ such that $d(\mathbf{x} + t, \mathbf{y} + s + t) < \delta$ for all $t \in U$;
- (IV) **(EXP)**: There exists a homeomorphism $\Phi : \mathbf{X} \rightarrow \mathbf{X}$ and an expanding linear map $\Lambda : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $T^{\Lambda t} \circ \Phi = \Phi \circ T^t$ for all $t \in \mathbb{R}^d$.

In the abbreviations above, (LF) stands for “Local Freedom,” (WE) is short for “Weak Expansivity,” (PS) is “Phase Stability” and (EXP) represents the “Self-affinity” of the action. These axioms, particularly (WE) and (PS), require some explanation, which we provide in Chapter 2.

Definition 1.2 (Kwapisz [20]). *An action of $\mathbb{R}^d$ on a compact metric space $\mathbf{X}$ is considered an abstract tiling action iff there exist positive constants r_0, Δ_0, l_0 with $l_0 < r_0$ so that axioms (I-III) hold.*

Definition 1.3 (Kwapisz [20]). *An abstract tiling action also satisfying (IV) is called an abstract self-affine tiling action. If additionally $\mathbf{X}$ is locally the product of a Cantor set and $\mathbb{R}^d$, the action is FLC.*

We should note that, in [20], an abstract action T on $\mathbf{X}$ is defined to be FLC iff $\mathbf{X}$ is locally the product of a Cantor set and $\mathbb{R}^d$, which is not to be confused with

the standard notion of FLC as applied to concrete tilings. It is only on the force of the following theorem that the use of the FLC phrasing for abstract tiling actions is justified.

Theorem 1.4 (Kwapisz [20]). *For any minimal abstract self-affine tiling action T (acting on X), there is a repetitive self-affine tiling with FLC such that its tiling space Y maps to X via a finite-to-one continuous map $\pi : Y \rightarrow X$ which factors the translation action on Y to T . The factor map $\pi : Y \rightarrow X$ is 1-1 on the complement of a nowhere dense set of measure zero. Moreover, if the abstract tiling action T is additionally FLC, then π is a homeomorphism conjugating T to the translating action on Y .*

We note that the measure referenced in the theorem is the unique action invariant probability measure (see Theorem 9.1 in [20] and [39]). Kwapisz utilizes the local product structure afforded by the abstract tiling action and the existence of Markov partitions to make a connection with concrete tilings. Our main goal in this paper is to remove the FLC condition in the “moreover” statement above. It is convenient to introduce two additional conditions on T that guarantee the existence of a well behaved cross-section $\mathbf{X}_0 \subseteq \mathbf{X}$. We define the Delone set in Chapter 2, for now think of it as a closed subset of $\mathbb{R}^d$ that is uniformly discrete and relative dense.

(DEL): $\exists_{R_D, r_D > 0}$ such that $\forall_{\mathbf{x} \in \mathbf{X}} D_{\mathbf{x}} := \{t : \mathbf{x} + t \in \mathbf{X}_0\}$ is an (R_D, r_D) -Delone set.

(TRA): $\forall_{\epsilon > 0} \exists_{\delta_{\epsilon} \in (0, \epsilon)}$ such that $\forall_{\mathbf{x} \in \mathbf{X}}$ if $d(\mathbf{x}, \mathbf{X}_0) < \delta_{\epsilon}$, then $\exists \tau \in B_{\epsilon} \cap D_{\mathbf{x}}$.

Definition 1.5. *An action T on $\mathbf{X}$ is said to have good cross-section $\mathbf{X}_0$ iff $\mathbf{X}_0$ is a closed subset of $\mathbf{X}$ satisfying (DEL) and (TRA).*

Now we state our main result.

Theorem 1.6. *Any minimal abstract self-affine tiling action T on a compact metric space $\mathbf{X}$ with a good cross-section $\mathbf{X}_0$ is topologically conjugate to the translation action on the tiling space of a repetitive self-affine tiling with FLC.*

The majority of the work necessary to remove the assumption of FLC lies in establishing the existence of a Cantor component in the local structure of $\mathbf{X}$. Indeed, by the arguments in [20], Theorem 1.6 readily follows from Theorem 1.7:

Theorem 1.7. *If T is a minimal abstract self-affine tiling action on a compact metric space $\mathbf{X}$ with a good cross-section $\mathbf{X}_0$, then $\mathbf{X}$ is locally homeomorphic to the product of $\mathbb{R}^d$ and a Cantor set.*

The key technical ingredient in the proof of 1.7 is a generalization of Theorem 1.1 for $\mathbb{Z}^d$ -actions to $\mathbb{R}^d$ -actions asserting existence of an expanding metric on a cross-section to the action. To understand the metric on the cross-section we need to isolate the “times” (i.e. the translations) in which an orbit intersects the cross-section. Since the orbits of two points may not intersect the cross-section at the same “time”, the following definition describes the situation when the orbits of two points return about the same “time”.

Definition 1.8. *We say $t, t' \in \mathbb{R}^d$ are $(\Delta_2, R, \mathbf{x}, \mathbf{z})$ -corresponding if and only if $t \in D_{\mathbf{x}}, t' \in D_{\mathbf{z}}, |t - t'| < \Delta_2$, and $\min\{|t|, |t'|\} < R + \Delta_2$.*

Theorem 1.9. *Let T be an abstract tiling action on a compact metric space $\mathbf{X}$ with a good cross-section $\mathbf{X}_0$. Fix $\Delta_2 \in (0, r_D/2)$. Then there exists a metric d_{exp} on $\mathbf{X}_0$ defining its topology and numbers $b > 0$, $k > 1$, and $\epsilon > 0$ so that for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}_0$, if $d_{exp}(\mathbf{x}, \mathbf{y}) < \epsilon$, then*

$$\sup \left\{ d_{exp} \left(T^e \mathbf{x}, T^{e'} \mathbf{y} \right) : e, e' \text{ } (\Delta_2, b, \mathbf{x}, \mathbf{y})\text{-corresponding} \right\} \geq k d_{exp}(\mathbf{x}, \mathbf{y}).$$

Remark 1.10. *In Theorem 1.9, one can take $\epsilon > 0$ as small as they wish without changing d_{exp} , b , k , and Δ_2 . Moreover, given any $\tilde{\Delta}_2 > 0$, the implication in the theorem holds with Δ_2 replaced by $\tilde{\Delta}_2$ provided ϵ is sufficiently small. This is because, as we shall prove in Proposition 2.27, $D_{\mathbf{x}}$ depends continuously on $\mathbf{x} \in \mathbf{X}$.*

In this generalization, axiom (WE) replaces the expansiveness requirement in Theorem 1.1. (PS) serves to overcome the “slippage” or “shearing” problem that may occur for $\mathbb{R}^d$ -actions, which is absent for actions of discrete groups like $\mathbb{Z}^d$. In Chapter 2, we show that property (DEL) is automatically satisfied for abstract self-affine tiling actions.

Overview

Chapter 2 contains prerequisite material of definitions and examples. In particular, we discuss the details of each axiom while providing examples as well as some important implications, including a proof of (DEL). We also provide brief introductions to group actions, tiling spaces, dimensional entropy, and Hausdorff dimension.

In chapter 3 we establish the existence of an expanding metric for expansive $\mathbb{Z}^d$ actions (Theorem 1.1). We actually provide two versions of this result, the first is a bit weaker in that the expanding radius b depends on the action. The proofs provide context and a springboard for the more elaborate arguments used in proving Theorem 1.9 in Chapter 4.

As already mentioned, Chapter 4 contains the proof of the existence of an expanding metric on the good cross-section of an abstract tiling action (Theorem 1.9). Using this expanding metric we are able to show that the Hausdorff dimension of the local cross-section is less than or equal to a constant times the one-dimensional entropy of the action. We then show that the one-dimensional entropy of an abstract

self-affine tiling action is zero. This implies that the Hausdorff dimension of the local cross-section is zero. Hence it is totally disconnected. Lastly, we prove that a space that admits an abstract self-affine tiling action and a good cross-section is locally homeomorphic to the product of a Cantor set and $\mathbb{R}^d$, our main result.

The remainder of this thesis is an analysis of a specific self-affine tiling without finite local complexity. This example was first discussed by Frank and Sadun [12, 37] (throughout this paper we will refer to it as the Frank-Sadun example). It illustrates how failure of the (WE) axiom can lead to violation of FLC and generate a very complex local structure. We give precise statements and complete proofs of the results that describe the local product structure of this tiling space. It turns out to be even more complicated than what Frank and Sadun suggest in their rather informal exposition. The main result about the local topological structure of the tiling space is Theorem 5.3, and Chapter 5 is essentially entirely devoted to the proof.

Finally, in the appendix we establish the relevant topology on closed subsets of $\mathbb{R}^d$ and show its equivalence to the topology induced by the Hausdorff distance between closed subsets of $\mathbb{R}^d$ with respect to an appropriate bounded metric.

Table 1.1: Some Key Notations.

T	the action on a space ($\mathbb{Z}^d$, $\mathbb{R}^d$, or a tiling action)
$\mathbf{X}$	a compact metric space (arbitrary or a tiling space)
$\mathbf{X}_0$	a closed subset of $\mathbf{X}$, usually a good cross-section for T
B_r	an open ball of radius r centered at the origin in $\mathbb{R}^d$
Λ	a linear expanding map on $\mathbb{R}^d$
Φ	the homeomorphism on $\mathbf{X}$ in the self-affinity definition,
Φ	also the inflation-substitution rule on tilings dimension ≥ 2
$D_{\mathbf{x}}$	the set of returns for $\mathbf{x}$ to the set $\mathbf{X}_0$
$\mathcal{P}$	a patch in a tiling $\mathbf{x}$
$\mathcal{X}$	the space of all tilings given a prototile set
$\mathcal{A}$	a finite set of symbols (an alphabet)
$\mathcal{A}^{\mathbb{Z}}$	the set of all bi-infinite pointed words
$\mathcal{A}^*$	the set of all finite words
ϕ	a symbolic substitution on $\mathcal{A}$, which extends to a map on $\mathcal{A}^*$
A	a substitution matrix
$\mathcal{T}_{\phi}$	the tiling space associated to the substitution ϕ
$B_{\delta}(\mathbf{X}_0)$	the δ neighborhood of $\mathbf{X}_0$
$W_{\epsilon}^s(\mathbf{x}, \Phi)$	the ϵ -local stable set of $\mathbf{x}$ with respect to Φ
$W_{\epsilon}^u(\mathbf{x}, \Phi)$	the ϵ -local unstable set of $\mathbf{x}$ with respect to Φ
$A_{a,b}$	the annulus in $\mathbb{R}^d$ centered at the origin with inner radius a and outer radius b
C^R	the intersection of C with the closed ball of radius R in $\mathbb{R}^d$
$B_{\epsilon}(A)$	the ϵ neighborhood of the set A
$\mathbf{U}_{\epsilon}^R(A)$	a basic set in the topology on the closed subsets in $\mathbb{R}^d$
$\eta_n(T)$	the n -dimensional entropy of T , a $\mathbb{Z}^d$ -action
$\tilde{\eta}_n(T)$	the n -dimensional entropy of T , a $\mathbb{R}^d$ -action
$\dim(X)$	the topological dimension of the space X
$\text{HD}_d(X)$	the Hausdorff dimension of X with respect to the metric d

PRELIMINARIES

Group Actions

Definition 2.1. *If G is an additive group with identity element 0 and X is a set, then a group action of G on X is a function $T : G \times X \rightarrow X : (g, x) \mapsto T(g, x)$ satisfying the following axioms for all $x \in X$:*

$$(i) \quad T(0, x) = x \text{ and}$$

$$(ii) \quad T(g_2, T(g_1, x)) = T(g_1 + g_2, x) \quad (\forall g_1, g_2 \in G).$$

We restrict to the case when X is a topological space (usually compact), G is a topological group $\mathbb{R}^d$ or $\mathbb{Z}^d$ for $d \geq 1$, and T is continuous. We shall write T^t for $T(t, \cdot)$ and the set $\{T^t(x) \mid t \in G\}$ is referred to as the *orbit of x under T* . For more details on group actions see [5, 8, 9, 16, 38].

Example 2.2 (Translation). $\mathbb{R}^d$ acts on itself by translations: for $t, w \in \mathbb{R}^d$, let $T^t : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be defined by $T^t(w) = w + t$. The axioms for a group action follow from properties of vector addition: $T^0(w) = w$ and $T^{t_1}(T^{t_2}(w)) = T^{t_1+t_2}(w)$.

Example 2.3 (Rotation). Given $\alpha, \beta \in \mathbb{R}$, $\mathbb{Z}^2$ acts on the circle $S^1 := \mathbb{R}/\mathbb{Z}$ by rotations: for $(n, m) \in \mathbb{Z}^2$, let $T^{(n,m)} : S^1 \rightarrow S^1$ be defined by $T^{(n,m)}x = x + n\alpha + m\beta$.

Example 2.4 (Kronecker Action [16]). $\mathbb{R}^d$ acts on $\mathbb{T}^d = S^1 \times \cdots \times S^1 = [0, 1]^d / \sim$ by translation: for fixed vectors $w_1, \dots, w_d \in \mathbb{R}^d$ and $x \in \mathbb{T}^d$ define $T^t(x) = x + t_1 w_1 + \cdots + t_d w_d$ where $t = (t_1, \dots, t_d) \in \mathbb{R}^d$.

The three actions above are by isometries.

Example 2.5 (Denjoy Example [16]). A homeomorphism $f : S^1 \rightarrow S^1$ is called a Denjoy map iff $\exists \Lambda \subseteq S^1$ where Λ is a Cantor set, $f(\Lambda) = \Lambda$ and the rotation number of f is irrational. The Denjoy action is the $\mathbb{Z}$ -action on Λ given by $T^t := f|_{\Lambda}^t$.

The four actions above are minimal (i.e., the orbit of every point x is dense in X) iff $(\alpha, \beta) \notin \mathbb{Q}^2$ in Example 2.3 and iff $w_1, \dots, w_d$ are independent over $\mathbb{Q}$ in Example 2.4. The following example will be used to illustrate axioms (WE) and (PS) in section 2.

Example 2.6 (Rotation). $\mathbb{R}$ acts on the annulus $A \subset \mathbb{R}^2$ centered at the origin by rotation on its concentric circles with the same linear speed: given polar coordinates $(r, \theta) \in A$, let $T^t(r, \theta) = (r, \theta + \frac{t}{r})$.

Example 2.7 (d-Dimensional Shift Action [5]). For a finite alphabet $\mathcal{A}$, $\mathbb{Z}^d$ acts on $X = \mathcal{A}^{\mathbb{Z}^d}$ by shifts (or translations): for $x = (x(n))_{n \in \mathbb{Z}^d} \in X$ and $k \in \mathbb{Z}^d$ we take $(T^k x)(n) = x(n + k) \quad (n \in \mathbb{Z}^d)$.

A subset $Y \subseteq X$ is called invariant iff the orbit of any $x \in Y$ is contained in Y . The d -dimensional shift has many invariant closed subsets. Restricting the maps T^t to such a subset produces a new action $T|_Y$. The following is an example of such an action.

Example 2.8 (Ledrappier Example [5]). $\mathbb{Z}^2$ acts on Y , a subset of $X = \{0, 1\}^{\mathbb{Z}^2}$, defined by the condition

$$x(i, j) + x(i + 1, j) + x(i, j + 1) \equiv 0 \pmod{2}.$$

We note that the above Y is treated as an infinite cartesian product of $\mathbb{Z}/2\mathbb{Z}$. Ledrappier showed that this example is mixing but not mixing of all orders [9].

Tiling Spaces

There is a great deal of literature on tiling spaces and their actions, for a good introduction to this theory see [13, 33, 37, 39]. We give a brief account.

Definition 2.9. A tiling $\mathbf{x}$ is a set of tiles that have pairwise disjoint interiors, covering all of $\mathbb{R}^d$. Each tile is a translate of an element from a fixed finite set $\{p_1, p_2, \dots, p_n\}$, called a *prototile set*. We require that each prototile p_i is the closure of its interior and is compact.

Two tiles are equivalent if both are translations of the the same prototile. A *patch* of a tiling $\mathbf{x}$ is a finite subset $\mathcal{P}$ of $\mathbf{x}$. The union of the tiles in the patch is a subset of $\mathbb{R}^d$, denoted $|\mathcal{P}|$. We define the *diameter of $\mathcal{P}$* as $\text{diam}(\mathcal{P}) := \max\{|p - q| : p, q \text{ points in } |\mathcal{P}|\}$. We call a patch with diameter not exceeding $R > 0$ an R -patch. Two patches $\mathcal{P}$ and $\mathcal{Q}$ are equivalent (up to translation), denoted $\mathcal{P} \equiv \mathcal{Q}$, if and only if there exists a translation that matches up equivalent tiles; i.e., $\exists t \in \mathbb{R}^d$ such that $\mathcal{P} = \mathcal{Q} + t$. A tiling $\mathbf{x}$ is *aperiodic* if and only if $\mathbf{x} + t = \mathbf{x}$ implies $t = 0$. A tiling $\mathbf{x}$ has *finite local complexity* (FLC) if and only if for all $R > 0$ there exists in $\mathbf{x}$ only finitely many equivalence classes of R -patches. That is, for all $R > 0$ there exists $n \in \mathbb{N}$ and patches $\mathcal{P}_1, \dots, \mathcal{P}_n$ such that if $\mathcal{P}$ is an R -patch in x then there exists $j \in \{1, \dots, n\}$ such that $\mathcal{P} \equiv \mathcal{P}_j$.

Let $\mathcal{X}$ denote the space of all tilings of $\mathbb{R}^d$ with a given prototile set. $\mathbb{R}^d$ acts on $\mathcal{X}$ by translation, $T^t(\mathbf{x}) = \mathbf{x} + t$.

Definition 2.10. Given a tiling $\mathbf{x}_0 \in \mathcal{X}$, the tiling space of $\mathbf{x}_0$ is the orbit closure $\mathbf{X} := \text{cl}\{\mathbf{x}_0 + t : t \in \mathbb{R}^d\}$ where the closure is taken with an appropriate metrizable topology in $\mathcal{X}$.

In the standard topology, two tilings in $\mathbf{X}$ are close if they can be made to agree on a large patch centered at the origin by a single small translation (a large patch is a patch whose underlying set contains a large ball, for a precise definition see [33]). In this topology one must require that x_0 has FLC in order to guarantee the compactness of $\mathbf{X}$ (essentially [34]). Furthermore, with the FLC condition $\mathbf{X}$ is

locally a product of $\mathbb{R}^d$ and a totally disconnected set (which is Cantor if every tiling in $\mathbf{X}$ is aperiodic) [2]. Without the stipulation of FLC we must consider a weaker topology, the *compact-open* topology [21], which can be described as follows. Pick a point q_i in every prototile p_i . To a translated prototile $p_i + t$ associate the point $q_i + t$. Call it a *control point* of the tile. Then any tiling of $\mathbb{R}^d$ can be recorded as an n -tuple (n = number of prototiles) of subsets of $\mathbb{R}^d$, (Λ_i) , each collecting the control points of the tiles of type p_i (i.e. translates of p_i). The sets Λ_i are closed subsets of $\mathbb{R}^d$ and, after using the stereographic projection, we can use the Hausdorff topology on such subsets viewed as subsets of the d -dimensional sphere. That is, the space of all tilings with given prototiles can be viewed as embedded into the space of all closed subsets of the n -fold cartesian product of the d -dimensional sphere. This topology is the third topology described (informally) by Frank and Sadun [12, 37].

If FLC is satisfied these two topologies on $\mathbf{X}$ are equivalent [21]. In the second topology, a tiling space $\mathbf{X}$ is always compact, which can be shown by using a Cantor diagonalization argument. Note that FLC is not topologically invariant in the sense that there are two tilings, one with FLC and one without, such that the translation actions on their respective tiling spaces (obtained by taking the closure with respect to the second topology) are topologically conjugate [30]. $\mathbf{X}$ is sometimes referred to as the *hull* of $\mathbf{x}_0$. We now give a few simple examples of hulls, which are illustrated in Figure 2.1.

Example 2.11. Suppose $\mathbf{x}_0$ is a 1-dimensional periodic tiling with one prototile of unit length. Then $\mathbf{x}_0 + t = \mathbf{x}_0$ for all $t \in \mathbb{Z}$. Thus $\mathbf{X} \cong \mathbb{R}/\mathbb{Z} \cong S^1$.

Example 2.12. Similarly, if $\mathbf{x}_0$ is a 2-dimensional periodic tiling with one unit square prototile, then $\mathbf{X} \cong \mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{T}^2$.

Example 2.13. Suppose $\mathbf{x}_0$ is a 1-dimensional tiling with two prototiles, say $a = [0, 1]$ and $b = [0, 2]$, and there is only one appearance of a b in $\mathbf{x}_0$. Then $\mathbf{X}$ includes all tilings with one b tile and tilings with no b tile. Thus $\mathbf{X}$ has two path-components: a circle as in example 2.11 and a line that asymptotically approaches that circle with both ends.

Example 2.14. Using the same prototiles from the example before, suppose $\mathbf{x}_0$ is a 1-dimensional tiling with one half consisting of a tiles and the other half consisting of b tiles. Then $\mathbf{X}$ has three path-components: a circle made of the tilings with all a tiles, a circle made of the tilings with all b tiles and a line asymptotically approaching both circles made of the tilings where half of $\mathbb{R}$ is tiled by a tiles and the other half is tiled by b tiles.

A tiling $\mathbf{x}$ is said to be *repetitive* if for any R -patch $\mathcal{P}$ in $\mathbf{x}$ there exists $R_{\mathcal{P}} > 0$ such that every open ball of radius $R_{\mathcal{P}}$ in $\mathbf{x}$ contains an R -patch equivalent to $\mathcal{P}$. This property is also known as the local isomorphism property [39]. Informally, this means that each local picture of $\mathbf{x}$ occurs everywhere in $\mathbf{x}$. A tiling is repetitive if and only if the tiling action is minimal (i.e., every orbit is dense) [31].

A popular class of tilings to study is that of self-affine tilings. A tiling $\mathbf{x}$ is *self-affine* iff there is a homeomorphism $\Phi : \mathbf{X} \rightarrow \mathbf{X}$ and an expanding linear map $\Lambda : \mathbb{R}^d \mapsto \mathbb{R}^d$ such that

$$\Phi(\mathbf{x}) = \mathbf{x} \text{ and } T^{\Lambda t} \circ \Phi = \Phi \circ T^t \text{ for all } t \in \mathbb{R}^d.$$

This is an abstract definition given by Kwapisz [20]. Traditionally, self-affinity of $\mathbf{x}$ is expressed via a so called inflation-substitution process whereby Λ inflates a tile p to $\Lambda(p)$ which is then subdivided into tiles from $\mathbf{x}$ according to some fixed rule. We will formalize this rule shortly. It is easiest to do this for one-dimensional tilings. Indeed,

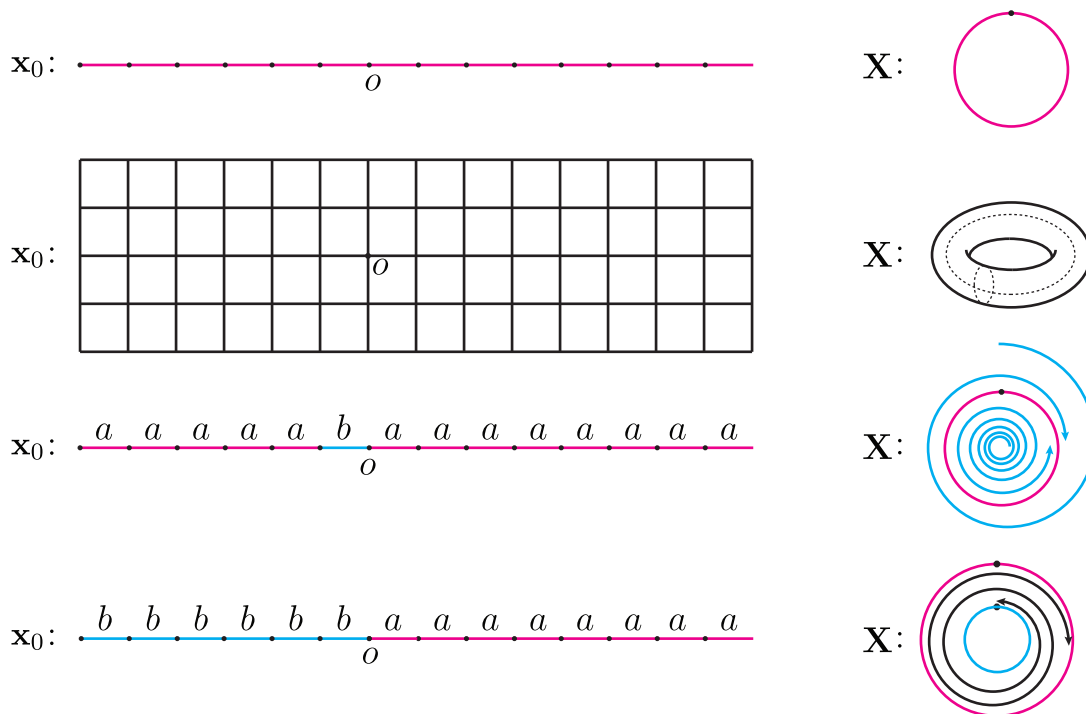


Figure 2.1: Examples of tilings and their hulls

when the prototiles are line segments, one-dimensional self-affine tilings coincide with so called substitution tilings, which we introduce now.

Substitution tilings are closely related to self-affine tilings and the field of symbolic dynamics [13,33,37]. Many of the examples in this paper will be substitution tilings. While we refer the reader to [1] for details, let us give a summary of this development. As before we fix a finite set $\mathcal{A}$. We think of $\mathcal{A}$ as an alphabet, the elements of $\mathcal{A}$ as letters, and a sequence of symbols (or juxtaposition of letters) as a word. For instance, the word 1101 corresponds to a sequence $(a_k)_{k=1}^4$ with $a_1 = 1$, $a_2 = 1$, $a_3 = 0$, and $a_4 = 1$. Often it is convenient to use sequences indexed by other sequences of $\mathbb{Z}$. To indicate the 0th element of a sequence we use a pointed word where a period is placed between the 0th and the 1st letter. For instance, $(a_k)_{k=-2}^3$

with $a_k = 0$ for $k \leq 0$ and $a_k = 1$ for $k > 0$ is represented by the pointed word 000.111. The set of all bi-infinite pointed words is synonymous with $\mathcal{A}^{\mathbb{Z}}$. Recall from Example 2.7 that $\mathbb{Z}$ acts on $\mathcal{A}^{\mathbb{Z}}$ by the shift map. Here the action $T^k x$ amounts to moving the period in x k places to the left. Denote the set of all finite words of letters from $\mathcal{A}$ by $\mathcal{A}^*$. A *symbolic substitution* on $\mathcal{A}$ is a map $\phi : \mathcal{A} \rightarrow \mathcal{A}^*$ associating a word to each letter. This map can be extended to $\mathcal{A}^*$ by replacing each letter by the associated word.

The substitution matrix associated to ϕ is given by $A = (a_{ij})_{i,j=1}^d$ where d is the cardinality of $\mathcal{A}$ and a_{ij} records the number of occurrences of the i th letter in $\phi(j)$. We are particularly interested in ϕ with a substitution matrix that is *primitive*; i.e. there exists a positive $n \in \mathbb{Z}$ such that every entry of A^n is positive. In such case, there exists a Perron-Frobenius eigenvalue $\lambda_1 > 1$ (the largest positive real eigenvalue) with multiplicity of one (Theorem 3.3.1 [6]).

A bi-infinite word w is called *ϕ -admissible* if each finite sub-word of w can be found as a sub-word in $\phi^n(a)$ for some $a \in \mathcal{A}$ and $n \geq 0$. By assigning a prototile to each letter in $\mathcal{A}$ we can consider the *substitution tiling* $\mathcal{T}_\phi$, the set of all tilings whose sequence of tiles correspond to a ϕ -admissible word. Here the prototiles (intervals in $\mathbb{R}$) are distinguishable by their labels (or color), but not necessarily by their lengths. Given a primitive substitution matrix with Perron-Frobenius eigenvalue λ_1 with corresponding left-eigenvalue $L = (L_1, \dots, L_k)$, the prototiles $p_1, \dots, p_k$ have lengths $L_1, \dots, L_k$. ϕ acts on $\mathcal{T}_\phi$ by stretching each tile a by a factor of λ_1 and replacing it by a sequence of prototiles corresponding to the associated the word $\phi(a)$. If A is primitive and $\mathbf{x}_0 \in \mathcal{T}_\phi$, then $\mathbf{x}_0$ is repetitive and $\mathcal{T}_\phi = \text{cl}\{\mathbf{x}_0 + t \mid t \in \mathbb{R}\}$ [37].

Another way to construct a one-dimensional tiling from a ϕ -admissible word is to project a strand (a broken line of unit horizontal and vertical segments) onto the eigenspace associated to λ_1 [1]. We utilize this technique in section 5 and we defer

the construction until then. We will now provide several examples of substitution tilings. The first two examples are of 1-dimensional tilings and thus can be described combinatorially, while the remaining 2-dimensional examples must also include a geometric description.

Example 2.15 (Thue-Morse). Suppose $\mathcal{A} = \{a, b\}$ and the substitution $\phi : \mathcal{A} \rightarrow \mathcal{A}^*$ is defined by the rules $\phi(a) = ab$ and $\phi(b) = ba$. This substitution has matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, Perron-Frobenius eigenvalue $\lambda_1 = 2$ and corresponding left-eigenvector $L = (1, 1)$. Let $w := \lim_{n \rightarrow \infty} \phi^n(b.a)$. Note that $\phi^2(w) = w$. Let $\mathbf{x}_0$ be the 1-dimensional tiling of a and b tiles, both of unit length but of distinct color, corresponding to the word w . Then $\mathcal{T}_\phi = \text{cl}\{\mathbf{x}_0 + t \mid t \in \mathbb{R}\}$.

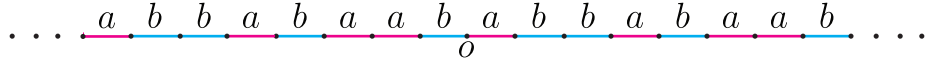


Figure 2.2: Thue-Morse substitution tiling

Example 2.16 (Fibonacci). Suppose $\mathcal{A} = \{a, b\}$ and the substitution $\phi : \mathcal{A} \rightarrow \mathcal{A}^*$ is defined by the rules $\phi(a) = b$ and $\phi(b) = ab$. This substitution has matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, Perron-Frobenius eigenvalue $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and corresponding left-eigenvector $L = (1, \lambda_1)$. Again let $w := \lim_{n \rightarrow \infty} \phi^n(b.a)$ and see that $\phi^2(w) = w$. Let $\mathbf{x}_0$ be the 1-dimensional tiling of a and b tiles with lengths one and λ_1 , respectively, corresponding to the word w . Again $\mathcal{T}_\phi = \text{cl}\{\mathbf{x}_0 + t \mid t \in \mathbb{R}\}$. Note that the asymptotic ratio of the number of a tiles to the number of b tiles in any $\mathbf{x} \in \mathcal{T}_\phi$ is irrational. Thus there are no periodic tilings in $\mathcal{T}_\phi$.

We now formalize self-affinity and the inflation-substitution rule as promised earlier.

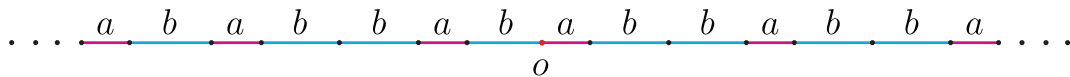


Figure 2.3: Fibonacci substitution tiling

Definition 2.17. A tiling $\mathbf{x}$ is Λ -self-affine iff there is an expanding linear map $\Lambda : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that

- (a) Λp is a patch of $\mathbf{x}$ for every tile p of $\mathbf{x}$, and
- (b) if p and p' are two such tiles that are equivalent (by a translation) then the patches Λp and $\Lambda p'$ are equivalent as well.

In other words, $\mathbf{x}$ is unchanged if we first inflate it by applying Λ and then substitute each inflated tile Λp by a collection of tiles that only depends on the type of p . This is equivalent to $\mathbf{x}$ being Λ -subdividing in [39].

It is natural to also talk about an inflation-substitution rule Φ (with inflation Λ) in absence of a specific tiling $\mathbf{x}$; i.e., only given a set of prototiles $\{p_1, \dots, p_k\}$. Such a rule is an assignment that associates to every prototile p_i a collection of tiles, denoted $\Phi(p_i)$, that tiles the inflated tile Λp_i (i.e. its elements have disjoint interior and union to Λp_i). Such a rule Φ can then be applied to translates of prototiles via $\Phi(p_i + t) := \Phi(p_i) + \Lambda t$, and also to tilings (or patches of tilings) tile-by-tile. In this framework, a tiling $\mathbf{x}$ is self-affine if there is a inflation-substitution rule Φ such that $\Phi(\mathbf{x}) = \mathbf{x}$. In practice, one constructs such a tiling $\mathbf{x}$ as limits $\Phi^n(p)$ as $n \rightarrow \infty$ where p is a suitably chosen seed tile (or seed patch). This is illustrated in following examples.

Example 2.18 (Chair). The set of four prototiles for the “chair” substitution is given in figure 2.4. The inflation-subdividing map Φ inflates each tile by a factor of 2 and

subdivides it into four tiles. Tilings in $\mathcal{T}_\Phi$ are aperiodic, repetitive, self-affine tilings with FLC [13, 23, 37].

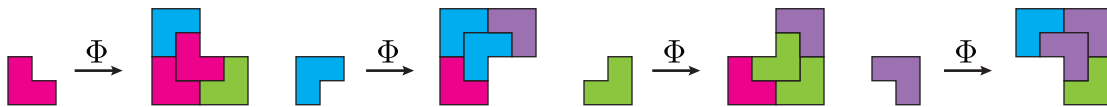


Figure 2.4: Chair prototiles and inflation-substitution rule

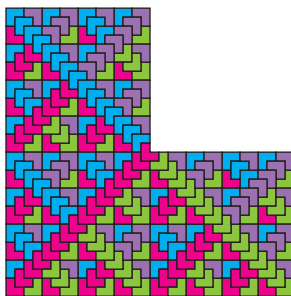


Figure 2.5: Chair tiling

Example 2.19 (Penrose [32]). The Penrose substitution tiling is one of the most famous non-periodic tilings and it has several variants. Consider the variant arising from two prototiles, a half-kite and a half-dart seen in figure 2.6. The inflation-subdividing map Φ inflates each tile by a factor of $\lambda = \frac{1+\sqrt{5}}{2}$ and subdivides according to figure 2.6. Tilings in $\mathcal{T}_\Phi$ are aperiodic, repetitive, self-affine tilings with FLC.

Example 2.20 (Pinwheel [29]). For this example we relax tile equivalence between tiles and prototiles by allowing not only translations but also rotations. We begin with two right triangular prototiles of side lengths 1, 2, and $\sqrt{5}$ (see figure 2.8). The

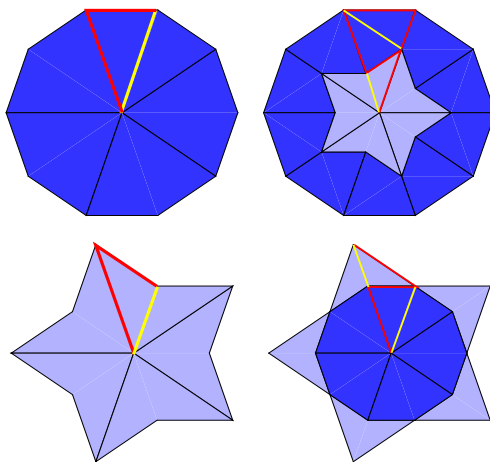


Figure 2.6: Penrose prototiles and inflation-substitution rule

Source: Jaroslaw Kwapisz

inflation-substitution map Φ inflates each tile by a factor of $\sqrt{5}$ and then subdivides it into 5 triangles as shown in figure 2.8. Because the triangle prototiles have angles that are irrational multiples of π they appear in the tiling in infinitely many orientations. Thus the Pinwheel tiling does not have FLC. Furthermore, this tiling is locally the product of an arc (an open subset of the circle of all possible orientations), a disk and a Cantor set [2, 37, 41].

Example 2.21 (Frank-Sadun [12, 37]). Let a and b be horizontal rectangular prototiles with height 1 and widths $\lambda_1 = \frac{1+\sqrt{13}}{2}$ and 3, respectively. The map Φ inflates tiles by a factor of λ_1 and subdivides as depicted in figure 2.10. In chapter 5 we will show that the tilings in $\mathcal{T}_\Phi$ do not have FLC. This will be due to the fault line phenomenon. Along a fault line the vertical edges of tiles below the fault line can be offset from the vertical edges of tiles above by an arbitrary amount. Thus there are infinitely many ways two tiles can abut in the tiling. We will also show that the local structure of $\mathcal{T}_\Phi$ is more complicated than the usual product of $\mathbb{R}^d$ and a Cantor set.

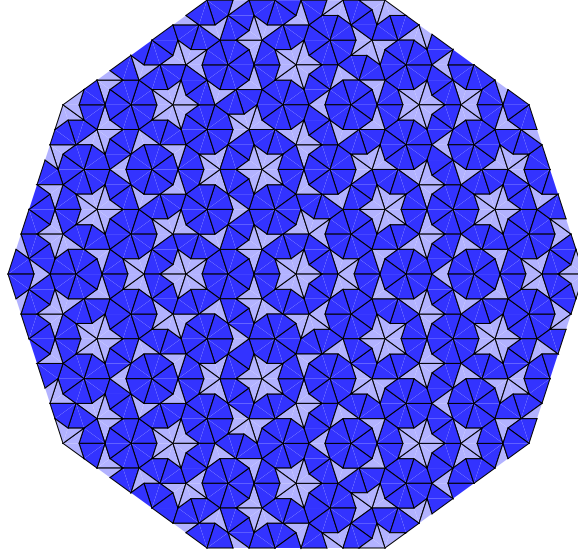


Figure 2.7: Penrose tiling

Source: Jaroslaw Kwapisz



Figure 2.8: Pinwheel prototiles and inflation-substitution rule

Because there is an obvious parallel between substitution dynamical systems and self-affine tiling dynamical systems, we utilize a slightly abusive notation. Throughout this paper, especially in Chapter 5, we use ϕ to denote both a symbolic substitution and sometimes to denote an inflation-substitution rule for a tiling of $\mathbb{R}$. Again Φ is used to represent an inflation-substitution rule on a higher dimensional tiling as well as the map in the abstract formulation of self-affinity (EXP). The intended use should be clear from the context.

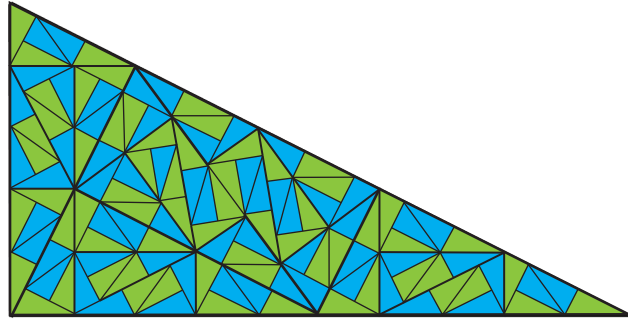


Figure 2.9: Pinwheel tiling



Figure 2.10: Frank-Sadun substitution rule

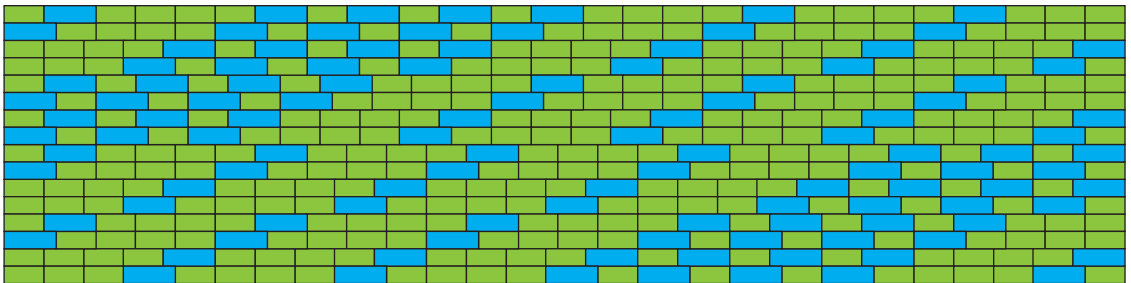


Figure 2.11: Frank-Sadun substitution tiling

Axioms

Recall that our goal in this paper is to find a class of $\mathbb{R}^d$ actions on a compact metric space that are topologically conjugate to the tiling action on an aperiodic repetitive self-affine tiling with FLC. Kwapisz proposed four axioms for this class of $\mathbb{R}^d$ actions [20]. In this section we will examine each of the four axioms individually, provide examples, and introduce two additional conditions stipulating the existence of a well behaved cross-section to the action: (DEL) and (TRA). In particular, we will show that (DEL) is consequence of axioms (I-IV), however, (TRA) does not automatically follow from Kwapisz's axioms. We state (DEL) and (TRA) separately to highlight the important aspects of the action used to show the existence of an expanding metric in section 4.

Let $T = (T^t)_{t \in \mathbb{R}^d}$ be an $\mathbb{R}^d$ -action on a compact metric space $\mathbf{X}$ with the metric d and $\mathbf{X}_0$ be a closed subset of $\mathbf{X}$. To denote the ball of radius $R \geq 0$ centered at $t = 0$ we simply write B_R . A discrete set D is said to be (R_D, r_D) -Delone in $\mathbb{R}^d$ if every open ball of radius R_D in $\mathbb{R}^d$ contains at least one element of D and if each open ball of radius r_D in $\mathbb{R}^d$ contains at most one element of D [2, 22]. The axioms for an abstract self-affine tiling action are listed below for some positive constant r_0 , l_0 and Δ_0 .

- (I) **(LF)** $_{2r_0}$: $\forall_{\mathbf{x} \in \mathbf{X}, t \in \mathbb{R}^d}$ if $\mathbf{x} + t = \mathbf{x}$ and $|t| < 2r_0$ then $t = 0$.
- (II) **(WE)** $_{r_0, \Delta_0}$: $\forall_{\mathbf{x}, \mathbf{y} \in \mathbf{X}}$ if $d(\mathbf{x} + t, \mathbf{y} + t) \leq \Delta_0$ for all $t \in \mathbb{R}^d$, then $\mathbf{x} = \mathbf{y} + s$ for some $s \in B_{r_0}$.
- (III) **(PS)** $_{l_0, r_0, \Delta_0}$: $\forall_{0 < \delta < \Delta_0} \exists_{\delta_1 > 0} \forall_{\mathbf{x}, \mathbf{y} \in \mathbf{X}, U \subset \mathbb{R}^d \text{ connected}}$ if $\inf_{|s| \leq l_0} d(\mathbf{x} + t, \mathbf{y} + s + t) < \delta_1$ for all $t \in U$, then there is $s \in B_{r_0}$ such that $d(\mathbf{x} + t, \mathbf{y} + s + t) < \delta$ for all $t \in U$.

(IV) **(EXP)**: there exists a continuous $\Phi : \mathbf{X} \rightarrow \mathbf{X}$ and an expanding linear map $\Lambda : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $T^{\Lambda t} \circ \Phi = \Phi \circ T^t$ for all $t \in \mathbb{R}^d$.

(V) **(DEL)**: $\exists_{R_D, r_D > 0}$ such that $\forall_{\mathbf{x} \in \mathbf{X}} \quad D_{\mathbf{x}} := \{t : \mathbf{x} + t \in \mathbf{X}_0\}$ is an (R_D, r_D) -Delone set.

(VI) **(TRA)**: $\forall_{\epsilon > 0} \exists_{\delta_\epsilon \in (0, \epsilon)}$ such that $\forall_{\mathbf{x} \in \mathbf{X}}$ if $d(\mathbf{x}, \mathbf{X}_0) < \delta_\epsilon$, then $\exists \tau \in B_\epsilon \cap D_{\mathbf{x}}$.

Local Freedom

$$\mathbf{(LF)}_{2r_0} : \quad \forall_{\mathbf{x} \in \mathbf{X}, t \in \mathbb{R}^d} \text{ if } \mathbf{x} + t = \mathbf{x} \text{ and } |t| < 2r_0 \text{ then } t = 0 \quad (2.1)$$

An action is called free if only the identity element fixes any point. The acronym LF refers to the local freedom of the action where $2r_0$ is the scale of freedom. Essentially it says that the action does not contain an orbit of “small” period. Note that Kwapisz states (LF) for r_0 [20]. Inconsequentially we use $2r_0$ so that our Temporal Alignment Lemma 4.2 holds. It is quite clear that any tiling action on an aperiodic tiling (e.g. Thue-Morse, Penrose, Pinwheel, etc.) satisfies (LF) since $r_0 = \infty$. Moreover (LF) holds for any tiling action, even if the tiling space has periodic tilings, as long as $2r_0$ is less than the minimum inner radius of a prototile (the minimum distance from the center of the prototile to an edge). Actions that (LF) excludes are actions that have points that are fixed under the action of $\mathbb{R}^d$ or its connected subgroup. For example, the action of $\mathbb{R}$ on $\mathbb{R} \cup \{\infty\}$ given by $T^t x = e^{-t} x$ has $x = 0$ and $x = \infty$ fixed. Another example that fails (LF) is the $\mathbb{R}$ -action by rotation on a disk centered at the origin, which is a fixed point. (LF) is a requisite to (DEL), which, in return, is necessary to show that the local cross-section is a Cantor set.

Weak Expansivity

$$\begin{aligned}
 (\mathbf{WE})_{r_0, \Delta_0} : \quad & \forall_{\mathbf{x}, \mathbf{y} \in \mathbf{X}} \text{ if } d(\mathbf{x} + t, \mathbf{y} + t) \leq \Delta_0 \text{ for all } t \in \mathbb{R}^d, \text{ then} \\
 & \mathbf{x} = \mathbf{y} + s \text{ for some } s \in B_{r_0}.
 \end{aligned} \tag{2.2}$$

The (WE) axiom is a weak form of expansivity where Δ_0 is the expansivity constant. It was also considered in [4, 19, 27]. In the context of a tiling action, (WE) basically says that if two tilings stay close for all time, then one must be a small translation of the other. In other words, we can distinguish any two orbits if we observe them with resolution Δ_0 or better. As desired, the tiling action on tilings with FLC satisfies (WE). Moreover, the tiling action on some non-FLC tilings also satisfies (WE), such as the tiling action on the Pinwheel tiling described in example 2.20. One instructive example that satisfies (WE) is the flow on an annulus that rotates its concentric circles with the same linear speed (Example 2.6). Here (WE) is satisfied due to the shearing action between different circles.

As stated earlier, we want our axioms to exclude tiling actions like that in the Frank-Sadun example because it does not have the proper local structure. The (WE) axiom does this for us and it excludes any action by isometries. In section 5 we prove that the Frank-Sadun example fails (WE).

The following lemma gives a more quantitative version of (WE), which we will utilize in the proof of Lemma 2.29.

Lemma 2.22. *T satisfies (WE) if and only if for all $\epsilon > 0$ there exists $R > 0$ such that*

$$\forall_{\mathbf{x}, \mathbf{y} \in \mathbf{X}} \forall_{t \in B_R} \quad d(\mathbf{x} + t, \mathbf{y} + t) \leq \Delta_0 \implies \exists s \in \bar{B}_{r_0} \text{ such that } d(\mathbf{x}, \mathbf{y} + s) < \epsilon. \tag{2.3}$$

Proof: ($\Leftarrow$) Suppose $d(\mathbf{x} + t, \mathbf{y} + t) \leq \Delta_0$ for all $t \in \mathbb{R}^d$. Take $\epsilon_n = \frac{1}{n}$ and $R_n > 0$ so that (2.3) holds. Then (2.3) reads

$$\forall_{t \in B_n} \quad d(\mathbf{x} + t, \mathbf{y} + t) \leq \Delta_0 \quad \implies \quad \exists s_n \in \bar{B}_{r_0} \quad d(\mathbf{x}, \mathbf{y} + s_n) < \frac{1}{n}.$$

Passing to a subsequential limit yields $s \in \bar{B}_{r_0}$ such that $d(\mathbf{x}, \mathbf{y} + s) = 0$; i.e., $\mathbf{x} = \mathbf{y} + s$.

($\Rightarrow$) Suppose the implication (2.3) does not hold. Then there exists $\epsilon > 0$ so that, for all $n > 0$, there exist $\mathbf{x}_n, \mathbf{y}_n \in \mathbf{X}$ such that

$$\forall_{t \in B_n} \quad d(\mathbf{x}_n + t, \mathbf{y}_n + t) \leq \Delta_0 \quad \text{yet} \quad \inf_{s \in B_{r_0}} d(\mathbf{x}_n, \mathbf{y}_n + s) \geq \epsilon.$$

By compactness there exists a sequence of indices $n_k \rightarrow \infty$ such that $\mathbf{x}_{n_k} \rightarrow \mathbf{x}$ and $\mathbf{y}_{n_k} \rightarrow \mathbf{y}$ with $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. Then we have

$$\forall_{t \in \mathbb{R}^d} \quad d(\mathbf{x} + t, \mathbf{y} + t) \leq \Delta_0 \quad \text{and} \quad \inf_{s \in B_{r_0}} d(\mathbf{x}, \mathbf{y} + s) \geq \epsilon.$$

But (WE) guarantees $s \in B_{r_0}$ such that $\mathbf{x} = \mathbf{y} + s$, a contradiction. $\square$

Phase Stability

$$\begin{aligned} (\mathbf{PS})_{l_0, r_0, \Delta_0} : \quad & \forall_{0 < \delta < \Delta_0} \exists_{\delta_1 > 0} \forall_{\mathbf{x}, \mathbf{y} \in \mathbf{X}, U \subset \mathbb{R}^d \text{ connected}} \text{ if } \inf_{|s| \leq l_0} d(\mathbf{x} + t, \mathbf{y} + s + t) < \delta_1 \\ & \text{for all } t \in U, \text{ then } \exists s \in B_{r_0} \text{ such that } d(\mathbf{x} + t, \mathbf{y} + s + t) < \delta \\ & \text{for all } t \in U. \end{aligned} \tag{2.4}$$

The (PS) axiom is an attempt to express a type of phase stability whereby, if two large orbit pieces are close to each other, then any two close points remain near

each other under translation along their respective orbits. In the context of tilings, (PS) says that if anywhere $\mathbf{x}$ looks very much like a small translation of $\mathbf{y}$, then there exists a single translation to make $\mathbf{y}$ look pretty much the same as $\mathbf{x}$. The l_0 constant acts as an upper bound on the size of the “harmless shear” between $\mathbf{x}$ and $\mathbf{y}$. In that sense it is reasonable to require that l_0 does not exceed the local freedom scale. Note that (PS) allows for arbitrarily large and connected U . If a bound were placed on the diameter of U , then (PS) would simply follow from continuity of T . For our proofs, we only need U to be solid ellipsoids in $\mathbb{R}^d$.

It is noted in [20] that actions satisfying (PS) include actions by isometries, the tiling action on some tilings with FLC (e.g. Penrose and Thue-Morse), and even on some non-FLC tilings (e.g. the Frank-Sadun tiling). However, (PS) excludes the tiling action on some non-FLC tilings, like the Pinwheel tiling. In this case, when two tilings are close by small rotation near the origin, their large orbit pieces may not be close. Roughly speaking, the orbit pieces look like a pair of scissors; near the origin they are close, but away from the origin they are far apart. An important example is the action of rotation on the annulus with constant linear speed (Example 2.6), which fails (PS) since it contains arbitrarily large shears. Observe that this action satisfies (LF) and (WE) yet its local cross-section is an interval of $\mathbb{R}$ rather than a Cantor set. So by requiring actions to satisfy (PS) we remove problematic examples such as this.

Abstract Self-Affinity

(EXP) : there exists a homeomorphism $\Phi : \mathbf{X} \rightarrow \mathbf{X}$ and an expanding linear map

$$\Lambda : \mathbb{R}^d \rightarrow \mathbb{R}^d \text{ such that } T^{\Lambda t} \circ \Phi = \Phi \circ T^t \text{ for all } t \in \mathbb{R}^d. \quad (2.5)$$

(EXP) is the abstract formulation of the self-affinity property. The requirement that Φ is invertible is often automatically satisfied when Φ is just continuous due to the recognizability theorem for aperiodic tiling spaces [39]. Observe that (LF) and (EXP) force aperiodicity: applying Φ^{-n} to $\mathbf{x} + t = \mathbf{x}$ yields $\Phi^{-n}\mathbf{x} + \Lambda^{-n}t = \Phi^{-n}\mathbf{x}$ where $|\Lambda^{-n}t| \rightarrow 0$ as $n \rightarrow \infty$ by the expanding property of Λ and (LF) implies there exists n so that $\Lambda^{-n}t = 0$. Clearly actions satisfying (EXP) include the tiling action on self-affine tilings. We show that any $\mathbb{R}^d$ action satisfying (EXP) has zero one-dimensional entropy (Theorem 4.16). Recall from the introduction that an *abstract self-affine tiling action* is an action of $\mathbb{R}^d$ that satisfies axioms (I-IV).

Delone Returns and Transversality

Recall that an action T on $\mathbf{X}$ has a *good cross-section* $\mathbf{X}_0$ iff $\mathbf{X}_0$ is a closed subset of $\mathbf{X}$ satisfying (DEL) and (TRA).

(DEL): $\exists_{R_D, r_D > 0}$ such that $\forall_{\mathbf{x} \in \mathbf{X}} \quad D_{\mathbf{x}} := \{t : \mathbf{x} + t \in \mathbf{X}_0\}$ is an (R_D, r_D) -Delone set.

(TRA): $\forall_{\epsilon > 0} \exists_{\delta_\epsilon \in (0, \epsilon)}$ such that $\forall_{\mathbf{x} \in \mathbf{X}}$ if $d(\mathbf{x}, \mathbf{X}_0) < \delta_\epsilon$, then $\exists \tau \in B_\epsilon \cap D_{\mathbf{x}}$.

(DEL) asserts that there is a closed subset whose set of return times is uniformly discrete and relatively dense. Specifically, for a fixed $\mathbf{x} \in \mathbf{X}_0$ each open ball of radius r_D in $\mathbb{R}^d$ contains at most one point from $D_{\mathbf{x}}$ and each open ball of radius R_D in $\mathbb{R}^d$ contains at least one point from $D_{\mathbf{x}}$. For more information about Delone sets see [2, 22]. The sixth axiom (TRA) conveys the transversality of the orbits of the action T to the cross-section $\mathbf{X}_0$. In this section, we show that (DEL) is automatically satisfied for an abstract self-affine tiling action. Unfortunately, (TRA) does not follow from self-affinity because points near the ends of $\mathbf{X}_0$ can take a long time to intersect $\mathbf{X}_0$. This is illustrated in Figure 2.12, where τ is the “first return time” of $\mathbf{x}$ to $\mathbf{X}_0$ and

τ is large. Also in this section, we define a topology on the set of all closed subsets of $\mathbb{R}^d$ that is equivalent to the one induced by the Hausdorff distance. Using this topology, we show that if T has a good cross-section $\mathbf{X}_0$, then $\mathbf{x} \mapsto D_{\mathbf{x}}$ is continuous.

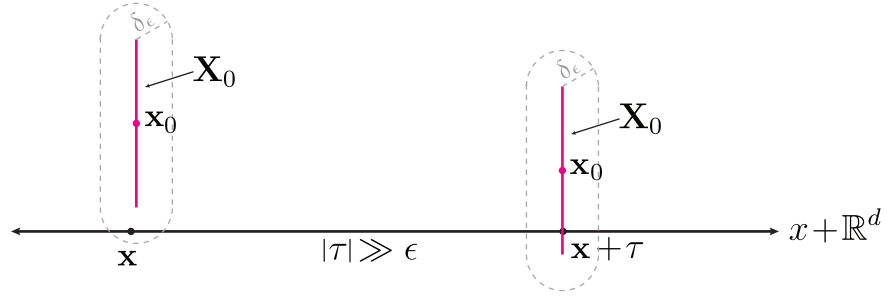


Figure 2.12: Failure of (TRA)

As stated in the introduction, the tiling space of concrete aperiodic tilings with FLC is locally the product of $\mathbb{R}^d$ and a Cantor set. Specifically, the $\mathbb{R}^d$ factor is the local unstable set of a tiling under the action of Φ and consists of all small translations of the tiling. The Cantor factor is the stable set of a tiling and consists of all tilings that agree with it exactly on a very large central patch. We denote the ϵ -local stable and unstable set of $\mathbf{x} \in \mathbf{X}$ with respect to Φ as

$$W_{\epsilon}^s(\mathbf{x}, \Phi) := \{\mathbf{y} \in \mathbf{X} \mid d(\Phi^n \mathbf{x}, \Phi^n \mathbf{y}) < \epsilon \text{ for all } n \geq 0\} \text{ and} \quad (2.6)$$

$$W_{\epsilon}^u(\mathbf{x}, \Phi) := \{\mathbf{y} \in \mathbf{X} \mid d(\Phi^{-n} \mathbf{x}, \Phi^{-n} \mathbf{y}) < \epsilon \text{ for all } n \geq 0\}. \quad (2.7)$$

Recall that an abstract self-affine tiling action comes equipped with positive constants r_0 , l_0 , and Δ_0 satisfying axioms (I-IV), which we will use in the following statements

and proofs. For $\Delta \in (0, \Delta_0)$, $E \subset \mathbb{R}^d$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, we write

$$\mathbf{x}|_E^{(\Delta)} \stackrel{s}{=} \mathbf{y}|_E^{(\Delta)} \text{ iff } d(\Phi^m(\mathbf{x} + t), \Phi^m(\mathbf{y} + t)) < \Delta \quad \forall_{t \in E, m \geq 0}. \quad (2.8)$$

This relation abstracts the instance when $\mathbf{x}$ and $\mathbf{y}$ have the same patches over E without adjusting by a small translation.

Kwapisz showed that an abstract self-affine tiling action has a similar local product structure. This product structure is critical for verifying (DEL) and (TRA). It hinges on the following lemma, which is an abstract analog of the fact that two sufficiently close tilings can be made to agree on a large central patch after a small translation (Axiom (PS) played a critical role here).

Lemma 2.23 (Kwapisz [20]). *Given $\epsilon > 0$, $\mathbf{x} \in \mathbf{X}$, and any abstract self-affine tiling action T on $\mathbf{X}$, the set $W_\epsilon^s(\mathbf{x}, \Phi)$ (where Φ is from (EXP)) is a local cross-section in the sense that there is $\epsilon_1 > 0$ (independent of $\mathbf{x}$) so that if $d(\mathbf{y}, \mathbf{x}) < \epsilon_1$, then there exists $\mathbf{y}_* \in W_\epsilon^s(\mathbf{x}, \Phi)$ with $\mathbf{y} = \mathbf{y}_* + t_*$ for some $t_* \in B_{r_0}$. Moreover, given $R > 0$, $\epsilon_1 > 0$ can be taken small enough (only depending on Φ) so that*

$$\mathbf{x} + \tau|_{B_R}^{(\Delta_0)} \stackrel{s}{=} \mathbf{y}_* + \tau|_{B_R}^{(\Delta_0)} \text{ for all } \tau \in B_{r_0}. \quad (2.9)$$

Remark 2.24 (Kwapisz [20]). *If we take ϵ small enough, for all $\mathbf{x} \in \mathbf{X}$, we have $\mathbf{y}_* + \tau \in W_\epsilon^s(\mathbf{x} + \tau, \Phi)$ for all $\tau \in B_{r_0}$ and $\mathbf{y}_*$ is the unique point of $(\mathbf{y} + B_{r_0}) \cap W_\epsilon^s(\mathbf{x}, \Phi)$.*

In such case, the traditional notation is $\mathbf{y}_* = [\mathbf{x}, \mathbf{y}]_\Phi$. Taking $\epsilon > 0$ small enough Lemma 2.23 gives a *local product structure* near every $\mathbf{x} \in \mathbf{X}$; that is, there is a homeomorphic identification between a neighborhood of $\mathbf{x}$ and $(\mathbf{x} + B_{r_0}) \times W_\epsilon^s(\mathbf{x}, \Phi)$ given by $\mathbf{y} \leftrightarrow (\mathbf{x} + t, \mathbf{y}_*)$. We are now ready to establish (DEL).

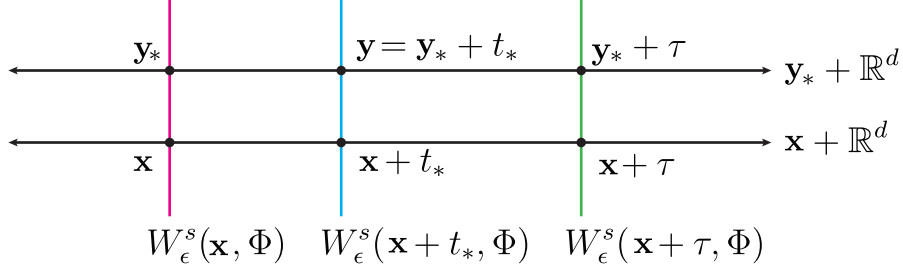


Figure 2.13: Local product structure

Proposition 2.25. *(DEL) holds for an abstract self-affine action T on a compact metric space $\mathbf{X}$.*

Proof: Suppose $\epsilon > 0$ is as in Remark 2.24. Fix $\mathbf{x}_0 \in \mathbf{X}$. Define $\mathbf{X}_0 := W_{\epsilon_1/2}^s(\mathbf{x}_0, \Phi)$ where ϵ_1 is provided by Lemma 2.23. Suppose $\mathbf{x} \in \mathbf{X}_0$, $\mathbf{x} + s \in \mathbf{X}_0$ and $|s| < r_0$. Then $d(\mathbf{x}, \mathbf{x} + s) < \epsilon_1$ by the definition of $W_{\epsilon_1/2}^s$. Lemma 2.23 and Remark 2.24 imply $s = 0$ since $\mathbf{x} \in (\mathbf{x} + B_{r_0}) \cap W_{\epsilon_1/2}^s(\mathbf{x}_0, \Phi)$ and $\mathbf{x}$ is unique. Thus taking $r_D := r_0$, every ball of radius r_0 in $\mathbb{R}^d$ contains at most one point of $D_{\mathbf{x}}$, since $(\mathbf{x} + B_{r_0}) \cap W_{\epsilon_1/2}^s(\mathbf{x}_0, \Phi)$ is a single point. For all open sets $U \subset \mathbf{X}$ and for all $\mathbf{x} \in U$ there exists $R > 0$ such that $\{t \in \mathbb{R}^d : \mathbf{x} + t \in U\}$ is R -dense by the minimality of T . Applying this to the open set $U = B_{\epsilon_1}(\mathbf{x}_0)$ and combining with the fact that, for all $\mathbf{y} \in B_{\epsilon_1}(\mathbf{x}_0)$, there exists $s \in B_{r_0}$ such that $\mathbf{y} + s \in \mathbf{X}_0$, from Lemma 2.23, we see that $R_D = R + r_0$ works; i.e., $D_{\mathbf{x}}$ is R_D -dense. $\square$

We now define a topology on closed subsets of $\mathbb{R}^d$ that we will use for the Delone sets $D_{\mathbf{x}}$ in Theorem 1.9. For any closed set in $\mathbb{R}^d$ we define $C^R := C \cap \overline{B}_R(0)$, which is a compact subset of $\mathbb{R}^d$, and $B_{\epsilon}(A) := \{t \in \mathbb{R}^d : |t - a| < \epsilon, \forall a \in A\}$. Given a closed

subset $A \subseteq \mathbb{R}^d$ we define a family of closed subsets of $\mathbb{R}^d$ by

$$\mathbf{U}_\epsilon^R(A) := \left\{ C \underset{\text{closed}}{\subseteq} \mathbb{R}^d : C^R \subseteq B_\epsilon(A) \quad \text{and} \quad A^R \subseteq B_\epsilon(C) \right\}$$

for $R > 0$ and $\epsilon > 0$.

Proposition 2.26. *The collection of closed sets*

$$\mathcal{B} = \left\{ \mathbf{U}_\epsilon^R(A) : A \underset{\text{closed}}{\subseteq} \mathbb{R}^d, \epsilon > 0, R > 0 \right\} \quad (2.10)$$

forms a basis of a topology on the set of all closed subsets of $\mathbb{R}^d$.

The topology given by $\mathcal{B}$ is equivalent to the one induced by using the Hausdorff distance between closed subsets of $\mathbb{R}^d$ with respect to a desirable bounded metric. We relegate this basic fact and Proposition 2.26 to the appendix.

Proposition 2.27. *For any abstract self-affine tiling action T on $\mathbf{X}$ with a good cross-section $\mathbf{X}_0$, the map $\mathbf{x} \mapsto D_{\mathbf{x}}$ is continuous; i.e.,*

$$\forall_{\epsilon, R > 0} \quad \exists_{\delta > 0} \quad \forall_{\mathbf{x}, \mathbf{y} \in \mathbf{X}} \quad d(\mathbf{x}, \mathbf{y}) < \delta \implies D_{\mathbf{y}} \in \mathbf{U}_\epsilon^R(D_{\mathbf{x}}). \quad (2.11)$$

Proof: It suffices to prove that $d(\mathbf{x}, \mathbf{y}) < \delta \implies D_{\mathbf{y}}^R \subseteq B_\epsilon(D_{\mathbf{x}})$ since we get the other inclusion simply by swapping $\mathbf{x}$ and $\mathbf{y}$. Suppose that the implication fails for $R > 0$ and $\epsilon > 0$. That is, for each $\delta = \frac{1}{n}$ there exist $\mathbf{x}_n, \mathbf{y}_n \in \mathbf{X}$ with $d(\mathbf{x}_n, \mathbf{y}_n) < \frac{1}{n}$ and $D_{\mathbf{y}_n}^R \not\subseteq B_\epsilon(D_{\mathbf{x}_n})$. Thus for each n there exists $t_n \in \mathbb{R}^d$ with $\mathbf{y}_n + t_n \in \mathbf{X}_0$ and $|t_n| \leq R$, yet

$$\forall \tau \in B_\epsilon(t_n) \quad \mathbf{x}_n + \tau \notin \mathbf{X}_0. \quad (2.12)$$

Upon passing to a subsequence, we may assume that $t_n \rightarrow t$ for some t with $|t| \leq R$ and $\mathbf{x}_n \rightarrow \mathbf{x}$ for some $\mathbf{x}$. Because $(T^t)_{|t| \leq R}$ is equicontinuous we have $d(\mathbf{x}_n + t_n, \mathbf{y}_n +$

$t_n) \rightarrow 0$ and so

$$d(\mathbf{x}_n + t_n, \mathbf{X}_0) \leq d(\mathbf{y}_n + t_n, \mathbf{X}_0) + d(\mathbf{x}_n + t_n, \mathbf{y}_n + t_n) \rightarrow 0. \quad (2.13)$$

On the other hand, we see that for the $\delta_\epsilon > 0$ supplied by (TRA) it must be that $d(\mathbf{x}_n + t_n, \mathbf{X}_0) \geq \delta_\epsilon$ for each n , otherwise 2.12 would be violated. This last inequality contradicts (2.13). $\square$

Weak Expansivity on the Cross-Section

We assume in this section that a good cross-section exists and we express (WE) entirely in terms of intersections of $\mathbb{R}^d$ -orbits with $\mathbf{X}_0$. To be weakly expansive on the cross-section roughly means that if the orbits of two points intersect the cross-section at about the same time and near each other, then the original points must have been very close to begin with. In other words, we can distinguish any two orbits whose return times are within Δ_2 of each other by observing them with resolution Δ_1 or better. This is made precise by the following definition.

Definition 2.28. *For $\Delta_1 > 0$ and $\Delta_2 \in (0, r_0/2)$, T satisfies $(WE_0)_{\Delta_1, \Delta_2}$ on $\mathbf{X}_0$ if for all $\epsilon_1 > 0$ there exists $R > 0$ so that for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}_0$ with $D_{\mathbf{y}} \in \mathbf{U}_{\Delta_2}^R(D_{\mathbf{x}})$*

$$\forall_{t, t' \text{ } (\Delta_2, R, \mathbf{x}, \mathbf{y})\text{-corresponding}} \quad d(\mathbf{x} + t, \mathbf{y} + t') < \Delta_1 \implies d(\mathbf{x}, \mathbf{y}) < \epsilon_1. \quad (2.14)$$

Note that the meaning of $t, t' \text{ } (\Delta_2, R, \mathbf{x}, \mathbf{y})\text{-corresponding}$ is given in Definition 1.8.

Lemma 2.29. *If T satisfies (DEL) and $(WE)_{r_0, \Delta_0}$ for $r_0 \in [0, r_D]$ and $\Delta_0 > 0$, then T satisfies $(WE_0)_{\Delta_1, \Delta_2}$ on $\mathbf{X}_0$ for some $\Delta_1 > 0$ and all $\Delta_2 \in (0, r_D/2)$.*

Proof: Suppose T satisfies $(WE)_{r_0, \Delta_0}$ for some $r_0 \in [0, r_D]$ and $\Delta_0 > 0$. Fix $\epsilon_1 > 0$ arbitrarily. By equicontinuity of $(T^t)_{|t| < R_D}$ there exists $\Delta_1 > 0$ so that

$$\forall_{\mathbf{x}', \mathbf{y}' \in \mathbf{X}} \quad d(\mathbf{x}', \mathbf{y}') < 2\Delta_1 \Rightarrow \forall_{t' \in B_{R_D}} \quad d(\mathbf{x}' + t', \mathbf{y}' + t') < \Delta_0. \quad (2.15)$$

It follows that, for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}_0$

$$\forall_{\tilde{t} \in D_{\mathbf{x}}^{R+R_D}} \quad d(\mathbf{x} + \tilde{t}, \mathbf{y} + \tilde{t}) < 2\Delta_1 \implies \forall_{t \in B_R} \quad d(\mathbf{x} + t, \mathbf{y} + t) < \Delta_0. \quad (2.16)$$

Indeed, for any $t \in B_R$ there exists $\tilde{t} \in D_{\mathbf{x}}^{R+R_D}$ with $|t - \tilde{t}| \leq R_D$ (by the Delone property of $D_{\mathbf{x}}$) and so $d(\mathbf{x} + \tilde{t}, \mathbf{y} + \tilde{t}) < 2\Delta_1$ (by the hypothesis of (2.16)). Defining $\mathbf{x}' := \mathbf{x} + \tilde{t}$, $\mathbf{y}' := \mathbf{y} + \tilde{t}$ and $t' := t - \tilde{t}$, we get $d(\mathbf{x} + t, \mathbf{y} + t) < \Delta_0$ by (2.15).

Choose $\Delta_2 \in (0, r_D/2)$ so that $\sup_{|s| < \Delta_2, \mathbf{y} \in \mathbf{X}_0} d(\mathbf{y} + s, \mathbf{y}) \leq \Delta_1$. We claim that there exists $\epsilon > 0$ such that for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}_0$

$$\inf_{|s| < r_0} d(\mathbf{x}, \mathbf{y} + s) < \epsilon \implies d(\mathbf{x}, \mathbf{y}) < \epsilon_1. \quad (2.17)$$

Otherwise, there exist sequences $\mathbf{x}_n, \mathbf{y}_n \in \mathbf{X}_0$ and $s_n \in \mathbb{R}$ with $|s_n| \leq r_0$ so that $d(\mathbf{x}_n, \mathbf{y}_n + s_n) < \frac{1}{n}$ yet $d(\mathbf{x}_n, \mathbf{y}_n) \geq \epsilon_1$. By compactness, there exists a subsequence $(n_k)_{n_k \rightarrow \infty}$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}_0$, s with $|s| \leq r_0$ such that $\mathbf{x}_{n_k} \rightarrow \mathbf{x}$, $\mathbf{y}_{n_k} \rightarrow \mathbf{y}$, and $s_{n_k} \rightarrow s$. Then $d(\mathbf{x}, \mathbf{y} + s) = 0$ yet $d(\mathbf{x}, \mathbf{y}) \geq \epsilon_1 > 0$. In particular, $\mathbf{y} + s \in \mathbf{X}_0$ (since $\mathbf{x} \in \mathbf{X}_0$), which means that $s \in D_{\mathbf{y}}$. Since $|s| \leq r_0 \leq r_D$ (and $0 \in D_{\mathbf{y}}$) it must be that $s = 0$, a contradiction.

To prove that T is $(WE_0)_{\Delta_1, \Delta_2}$ on $\mathbf{X}_0$, take $\epsilon > 0$ as in the claim and let $R > 0$ be as provided by Lemma 2.22 for this ϵ . Take $\tilde{R} := R + R_D$. Consider arbitrary $\mathbf{x}, \mathbf{y} \in \mathbf{X}_0$ with $D_{\mathbf{y}} \in \mathbf{U}_{\Delta_2}^{\tilde{R}}(D_{\mathbf{x}})$ and $d(\mathbf{x} + t, \mathbf{y} + t') < \Delta_1$ for all t, t' ($\Delta_2, \tilde{R}, \mathbf{x}, \mathbf{y}$)-

corresponding. Then for $t \in D_{\mathbf{x}}^{\tilde{R}}$ we get

$$d(\mathbf{x} + t, \mathbf{y} + t) \leq d(\mathbf{x} + t, \mathbf{y} + t') + d(\mathbf{y} + t', \mathbf{y} + t) \leq \Delta_1 + \Delta_1 = 2\Delta_1 \quad (2.18)$$

where $t' \in D_{\mathbf{y}}$ is chosen as the point of $D_{\mathbf{y}}$ closest to t . See that $d(\mathbf{x} + t, \mathbf{y} + t') \leq \Delta_1$ in (2.18) because t, t' are $(\Delta_2, \tilde{R}, \mathbf{x}, \mathbf{y})$ -corresponding (here we used that $D_{\mathbf{y}} \in \mathbf{U}_{\Delta_2}^{\tilde{R}}(D_{\mathbf{x}})$ to get $|t - t'| < \Delta_2$) and $d(\mathbf{y} + t', \mathbf{y} + t) \leq \Delta_1$ in (2.18) is by the choice of Δ_2 .

From (2.16) we get $d(\mathbf{x} + t, \mathbf{y} + t) \leq \Delta_0$ for all $t \in B_R$. Now, Lemma 2.22 supplies $s \in B_{r_0}$ so that $d(\mathbf{x}, \mathbf{y} + s) < \epsilon$. Finally, (2.17) gives us $d(\mathbf{x}, \mathbf{y}) < \epsilon_1$, completing the proof of $(\text{WE}_0)_{\Delta_1, \Delta_2}$. $\square$

Dimensional Entropy

Following Milnor [26], we define n -dimensional entropy of a $\mathbb{Z}^d$ -action for $0 \leq n \leq d$. Let X be a compact metric space with metric d and T be a $\mathbb{Z}^d$ -action on X . For all bounded sets $S \subseteq \mathbb{R}^d$, define the metric

$$d_S^T(x, y) := \max_{e \in S \cap \mathbb{Z}^d} \{d(T^e x, T^e y)\}. \quad (2.19)$$

We say that $E \subseteq X$ is a (T, S, ϵ) -separated set if and only if

$$\forall x, y \in E, \quad x \neq y \Rightarrow d_S^T(x, y) \geq \epsilon. \quad (2.20)$$

Define

$$\text{sep}(T, S, \epsilon) := \max\{\#E : E \text{ is a } (T, S, \epsilon) \text{ - separated set}\} \quad (2.21)$$

and

$$H(T, S, \epsilon) := \ln \text{sep}(T, S, \epsilon). \quad (2.22)$$

Definition 2.30. For $Q := [-1, 1]^d$, $R > 0$, and $\epsilon > 0$, we define

$$\eta_n(T, \epsilon) := \limsup_{R \rightarrow \infty} \frac{H(T, RQ, \epsilon)}{R^n} \quad \text{for } 0 \leq n \leq d. \quad (2.23)$$

Then the n -dimensional entropy of T is

$$\eta_n(T) := \lim_{\epsilon \rightarrow 0^+} \eta_n(T, \epsilon) \quad (2.24)$$

where the limit exists by monotonicity.

We remark that Milnor defines $\eta_n(T, \epsilon)$ for any compact set $Q \subseteq \mathbb{R}^d$ and $R > 0$ by

$$\eta_n(T, \epsilon) := \sup_B \limsup_{R \rightarrow \infty} \frac{H(T, RQ + B, \epsilon)}{R^n} \quad \text{for } 0 \leq n \leq d, \quad (2.25)$$

where the supremum is taken over all bounded $B \subseteq \mathbb{R}^d$ [26]. When Q is a cube, this yields the same value as (2.23). In fact, (2.23) and (2.25) agree for any compact set $Q \subseteq \mathbb{R}^d$ with the property

$$\exists \lambda > 1 \quad \bar{Q} \subseteq \text{int } \lambda Q. \quad (2.26)$$

Remark 2.31. If T is an $\mathbb{R}^d$ -action, we make analogous definitions using the distance function

$$\tilde{d}_{RQ}^T(x, y) := \max_{e \in RQ} \{d(T^e x, T^e y)\} \quad (2.27)$$

which yields the corresponding quantities $\widetilde{\text{sep}}(T, RQ, \epsilon)$, $\tilde{H}(T, RQ, \epsilon)$, and $\tilde{\eta}_n(T)$. By restricting the $\mathbb{R}^d$ -action T to $\mathbb{Z}^d$, we get $\eta_n(T) = \tilde{\eta}_n(T)$.

Proof: Clearly, for all $x, y \in X$, $d_{RQ}^T(x, y) \leq \tilde{d}_{RQ}^T(x, y)$. So $\text{sep}(T, RQ, \epsilon) \leq \widetilde{\text{sep}}(T, RQ, \epsilon)$ and $\eta_n(T) \leq \tilde{\eta}_n(T)$. Furthermore, by the continuity of T , for any

$\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that

$$\forall_{x,y \in \mathbf{X}} \quad d(x,y) \leq \delta \Rightarrow d(T^t x, T^t y) \leq \epsilon \quad (\forall_{t \in B_{\sqrt{d}}(0)}). \quad (2.28)$$

It follows that

$$d_{RQ}^T(x,y) \leq \delta \Rightarrow \tilde{d}_{RQ}^T(x,y) \leq \epsilon \quad (\forall_{x,y \in \mathbf{X}}) \quad (2.29)$$

and so every (T, RQ, ϵ) -sep set is also a (T, RQ, δ) -sep set. Therefore, $\text{sep}(T, RQ, \delta) \geq \widetilde{\text{sep}}(T, RQ, \epsilon)$ and taking $\epsilon \searrow 0$ we get $\eta_n(T) \geq \tilde{\eta}_n(T)$. $\square$

Example 2.32. Let $X = \{0,1\}^{\mathbb{Z}^2}$ and T be the shift action, so that for $k \in \mathbb{Z}^2$, $T^k x = x(n+k)$ (as in Example 2.7). For sufficiently small $\epsilon > 0$, we have $C(\epsilon) > 0$ so that $2^{(2R+1)^2} \leq \text{sep}(T, RQ, \epsilon) \leq 2^{(2R+C(\epsilon))^2}$ for all $R \in \mathbb{N}$. That is, $\text{sep}(T, RQ, \epsilon)$ is of order $2^{(2R)^2}$ and

$$\eta_1(T) = \limsup_{R \rightarrow \infty} \frac{\ln(2^{(2R)^2})}{R^1} = \limsup_{R \rightarrow \infty} 4R \ln 2 = \infty \quad (2.30)$$

$$\eta_2(T) = \limsup_{R \rightarrow \infty} \frac{\ln(2^{(2R)^2})}{R^2} = 4 \ln 2 \quad (2.31)$$

$$\eta_3(T) = \limsup_{R \rightarrow \infty} \frac{\ln(2^{(2R)^2})}{R^3} = \limsup_{R \rightarrow \infty} \frac{4 \ln 2}{R} = 0. \quad (2.32)$$

Example 2.33. Let T be the rotation action from Example 2.3. Note that this action is not expansive. Considering that the length of S^1 is $L = 1$, $\text{sep}(T, RQ, \epsilon) \sim L/\epsilon$ and $\eta_1(T, \epsilon) = \limsup_{R \rightarrow \infty} \frac{\ln(L/\epsilon)}{R} = 0$.

Example 2.34. Let $X = \{0,1\}^{\mathbb{Z}} \times S^1$ and for a fixed α let $T^{(k,l)}(x,y) = (x(n+k), y + l\alpha)$; i.e., the action is a product of the shift and rotation. For sufficiently small $\epsilon > 0$, $\text{sep}(T, RQ, \epsilon)$ grows like 2^{2R} and $\eta_1(T) = \limsup_{R \rightarrow \infty} \frac{2R \ln 2}{R} = 2 \ln 2$.

Hausdorff Dimension

In this section we define topological and Hausdorff dimension and show that the Hausdorff dimension is at least as big as the topological dimension. This is standard and we use [15] as our main source for the following definitions, theorem and lemmas. All proofs in this section are attributed to Hurewicz and Wallman and are only provided for completeness. We start by inductively defining what it means for a separable metric space to have topological dimension $\leq n$. In it $\dim(X) > n$ stands for the negation of $\dim(X) \leq n$.

Definition 2.35. *The empty set and only the empty set has topological dimension -1 , $\dim(\emptyset) = -1$.*

(i) *A space X has dimension $\leq n$ ($n \geq 0$) at a point p if p has arbitrarily small neighborhoods whose boundaries have dimension $\leq n - 1$.*

(ii) *X has dimension $\leq n$, $\dim(X) \leq n$, if X has dimension $\leq n$ at each of its points.*

(iii) *X has dimension n at a point p if X has dimension $\leq n$ at a point p and X has dimension $> n - 1$ at p .*

(iv) *X has dimension n if $\dim(X) \leq n$ and $\dim(X) > n - 1$.*

(v) *X has dimension ∞ if $\dim(X) > n$ for each n .*

As shown in [15], if a set has topological dimension zero, then it is totally disconnected.

Definition 2.36. *Let X be a metric space and p an arbitrary real number, $0 \leq p < \infty$. For any $A \subset X$ denote the diameter of A by $\text{diam}(A)$. Given $\epsilon > 0$, let*

$$m_p^\epsilon(X) = \inf \sum_{i=1}^{\infty} \text{diam}(A_i)^p \tag{2.33}$$

where $X = A_1 \cup A_2 \cup \cdots$ is any decomposition of X in a countable number of subsets of diameter less than ϵ . Then the p -measure of X is given by

$$m_p(X) = \sup_{\epsilon > 0} m_p^\epsilon(X). \quad (2.34)$$

Definition 2.37. Given an arbitrary metric space X with metric d , the Hausdorff dimension of X is defined by

$$HD_d(X) = \sup\{p \in \mathbb{R} : m_p(X) > 0\}. \quad (2.35)$$

Theorem 2.38. Let X be a metric space with metric d . Then

$$HD_d(X) \geq \dim(X). \quad (2.36)$$

Lemma 2.39. Let X be a space of dimension n , $0 \leq n < \infty$. Then

$$m_n(X) > 0. \quad (2.37)$$

Lemma 2.40. Let X be a space such that $m_{n+1}(X) = 0$ for some n with $0 \leq n < \infty$. Then

$$\dim(X) \leq n. \quad (2.38)$$

First we will prove Theorem 2.38 from Lemmas 2.39 and 2.40, then we prove the lemmas.

Proof of Theorem 2.38: Suppose $\dim(X) = n$, $0 \leq n < \infty$. Lemma 2.39 implies $m_n(X) > 0$. So $\sup\{p : m_p(X) > 0\} \geq n$. That is $HD_d(X) \geq n$ $\square$

To prove Lemma 2.39 we will show that Lemma 2.39 is equivalent to Lemma 2.40 and prove Lemma 2.40.

Claim 2.41. *Lemma 2.39 is equivalent to Lemma 2.40.*

Proof of Claim 2.41: ($\Rightarrow$) Suppose Lemma 2.40 fails, i.e., $k := \dim(X) \geq n + 1$ and $m_{n+1}(X) = 0$. By Lemma 2.39, $m_k(X) > 0$. Since $p \mapsto m_p(X)$ is non-increasing $m_{n+1}(X) > 0$, a contradiction.

($\Leftarrow$) Suppose Lemma 2.39 fails, i.e., $\dim(X) = n$ and $m_n(X) = 0$. Lemma 2.40 gives $\dim(X) \leq n - 1$, a contradiction. $\square$

Proof of Lemma 2.40: Let $x_0 \in X$. For $r > 0$, let $S(r) := \{x \in X : d(x, x_0) = r\}$.

Claim 2.42. *If X is a space such that $m_{p+1}(X) = 0$, $0 \leq p < \infty$, then for almost every r , $m_p(S(r)) = 0$.*

Before proving Claim 2.42, we show that Lemma 2.40 follows by induction on n . We use that $\partial B(r) \subseteq S(r)$ here. We start with $n = 0$: Suppose $m_1(X) = 0$. Claim 2.42 implies $m_0(S(r)) = 0$ for a.e. r . That is, for all x_0 , $S(r) = \emptyset$ for a.e. r . Hence $\dim(X) \leq 0$ by Definition 2.35. We assume the induction hypothesis, $m_{n+1}(X) = 0 \Rightarrow \dim(X) \leq n$, and suppose $m_{n+2}(X) = 0$. Claim 2.42 implies $m_{n+1}(S(r)) = 0$ for a.e. r . The induction hypothesis gives $\dim(S(r)) \leq n$ for a.e. r . Since r can be arbitrarily small, $\dim$ of X at $x_0 \leq n + 1$. Since x_0 was arbitrary, $\dim(X) \leq n + 1$. This proves Lemma 2.40.

Proof of Claim 2.42: Consider first an arbitrary set $E \subseteq X$. Let

$$r_1 = \inf_{x \in E} d(x, x_0), \quad r_2 = \sup_{x \in E} d(x, x_0). \quad (2.39)$$

Then

$$r_2 - r_1 \leq \text{diam}(E). \quad (2.40)$$

From (2.39) and (2.40) it follows that

$$\begin{aligned} \int_0^\infty \text{diam}(S(r) \cap E)^p dr &= \int_{r_1}^{r_2} \text{diam}(S(r) \cap E)^p dr \\ &\leq \text{diam}(E)^p \int_{r_1}^{r_2} dr \\ &\leq \text{diam}(E)^{p+1}. \end{aligned} \quad (2.41)$$

From Definition 2.37 and the hypothesis $m_{p+1}(X) = 0$ there exists a sequence of decompositions of X , $X = A_1^n \cup A_2^n \cup \dots$, such that $\epsilon_n := \sup_k \text{diam}(A_k^n) < \infty$ and

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} \text{diam}(A_i^n)^{p+1} = 0. \quad (2.42)$$

From (2.41) and (2.42) we get

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} \int_0^\infty \text{diam}(S(r) \cap A_i^n)^p dr = 0. \quad (2.43)$$

Letting $f_{n,i}(r) := \text{diam}(S(r) \cap A_i^n)^p \geq 0$, equation (2.43) says

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} \|f_{n,i}\|_1 = 0, \quad (2.44)$$

where $\|\cdot\|_1$ is the norm on $L^1([0, \infty))$. Then by the triangle inequality we have

$$\lim_{n \rightarrow \infty} \left\| \sum_{i=1}^{\infty} f_{n,i} \right\|_1 \leq \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} \|f_{n,i}\|_1 = 0 \quad (2.45)$$

That is

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} \text{diam}(S(r) \cap A_i^n)^p = 0, \quad (2.46)$$

where the limit is taken in the mean. By Corollary 2.3 in [40] (a well-known corollary to the Riesz-Fischer Theorem) there exists a subsequence n_k with $n_k \rightarrow \infty$ such that

$$\lim_{k \rightarrow \infty} \sum_{i=1}^{\infty} \text{diam}(S(r) \cap A_i^{n_k})^p = 0 \quad \text{for a.e. } r. \quad (2.47)$$

This implies that $m_p(S(r)) = 0$ for a.e. r . This proves Claim 2.42, which then proves Lemma 2.40. □

RESULTS FOR $\mathbb{Z}^d$ -ACTIONSExistence of an Expanding Metric (Dependent on the Action)

As stated in the introduction of this thesis, this chapter serves as a jumping-off point for the arguments made in the next chapter. However, some might find the results in this chapter of independent interest because we present two theorems which generalize Fathi's theorem to a higher dimensional action. The first theorem guarantees a metric that is expanded by some factor $k > 1$ within finite amount of time, but that amount of time depends on the action. The second theorem will remove this dependence.

First, recall that a homeomorphism h on a metric space (X, d) is said to be *expansive* if there exists $c > 0$ such that for any two distinct point $x, y \in X$, there exists $n \in \mathbb{Z}$ such that $d(h^n(x), h^n(y)) \geq c$. In such case, $c > 0$ is called an *expansiveness constant* for h [6].

Theorem 3.1 (Fathi [10]). *If $h : X \rightarrow X$ is an expansive homeomorphism of a compact metric space X , then there exists a metric d on X defining its topology and numbers $k > 1, \epsilon > 0$ such that*

$$\forall_{x,y \in X} \quad \max\{d(h(x), h(y)), d(h^{-1}(x), h^{-1}(y))\} \geq \min\{kd(x, y), \epsilon\}. \quad (3.1)$$

Fathi's approach, which we emulate in our generalizations, relies on Frink's metrization theorem, where the usual triangle inequality is replaced by a generalized triangle inequality.

Theorem 3.2 (Frink [14]). *Suppose there exists a function ρ on a topological space X such that, for every two points $x, y \in X$, ρ satisfies the following conditions:*

I. $\rho(x, y) \geq 0$;

II. $\rho(x, y) = 0$ if and only if $x = y$;

III. $\rho(x, y) = \rho(y, x)$, (symmetry);

IV. If $\rho(x, y) < \epsilon$ and $\rho(z, y) < \epsilon$, then $\rho(x, z) < 2\epsilon$, (generalized triangle property).

Then X is metrizable. In particular, there exists a metric d on X such that, for every two points $x, y \in X$,

$$d(x, y) \leq \rho(x, y) \leq 4d(x, y)$$

.

We are now ready to state our first generalization of Fathi's theorem to an expansive $\mathbb{Z}^d$ -action.

Theorem 3.3. *Let T be an expansive $\mathbb{Z}^d$ -action on a compact metric space $\mathbf{X}$. Then there exists a metric d on $\mathbf{X}$ defining its topology and numbers $b > 0$, $k > 1$, and $\epsilon > 0$ so that, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$,*

$$\max_{e \in \mathbb{Z}^d, |e| < b} d(T^e \mathbf{x}, T^e \mathbf{y}) \geq \min \{kd(\mathbf{x}, \mathbf{y}), \epsilon\}. \quad (3.2)$$

b is called the expanding radius for T .

Proof: Choose some metric δ on $\mathbf{X}$ defining its topology. Let $c > 0$ be an expansiveness constant for T , i.e.,

$$\forall_{\mathbf{x}, \mathbf{y} \in \mathbf{X}} \quad \mathbf{x} \neq \mathbf{y} \implies \sup_{n \in \mathbb{Z}^d} \{\delta(T^n \mathbf{x}, T^n \mathbf{y})\} > c.$$

For any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, define $n : \mathbf{X} \times \mathbf{X} \rightarrow \mathbb{R}_+$ by

$$n(\mathbf{x}, \mathbf{y}) := \begin{cases} \infty & \text{if } \mathbf{x} = \mathbf{y}, \\ \inf \{ |t| : t \in \mathbb{Z}^d, \delta(T^t \mathbf{x}, T^t \mathbf{y}) > c \} & \text{if } \mathbf{x} \neq \mathbf{y}, \end{cases} \quad (3.3)$$

where $|\cdot|$ is the Euclidean norm. Observe that $n(\mathbf{x}, \mathbf{y})$ is upper semi-continuous as a function of $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. To see this take $(\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{X}$. Now suppose $(\mathbf{x}_n, \mathbf{y}_n) \rightarrow (\mathbf{x}, \mathbf{y})$ as $n \rightarrow \infty$. We must show that $\limsup_{n \rightarrow \infty} n(\mathbf{x}_n, \mathbf{y}_n) \leq n(\mathbf{x}, \mathbf{y})$. This is clear when $\mathbf{x} = \mathbf{y}$ and $n(\mathbf{x}, \mathbf{y}) = \infty$. For $\mathbf{x} \neq \mathbf{y}$, take $t \in \mathbb{Z}^d$ such that $\delta(T^t \mathbf{x}, T^t \mathbf{y}) > c$ and $|t| \leq n(\mathbf{x}, \mathbf{y})$. By continuity of T^t , there exists $n_0 \in \mathbb{N}$ such that $\delta(T^t \mathbf{x}_n, T^t \mathbf{y}_n) > c$ for $n \geq n_0$. So $n(\mathbf{x}_n, \mathbf{y}_n) \leq |t| \leq n(\mathbf{x}, \mathbf{y})$ for all $n \geq n_0$ and the inequality follows.

By a simple argument using compactness of $\mathbf{X}$, one can pick $m > 0$ so that

$$\forall \mathbf{a}, \mathbf{b} \in \mathbf{X} \quad \delta(\mathbf{a}, \mathbf{b}) > \frac{c}{2} \implies n(\mathbf{a}, \mathbf{b}) \leq m. \quad (3.4)$$

To show this suppose that for each $m > 0$ there exist $\mathbf{a}_m, \mathbf{b}_m \in \mathbf{X}$ such that $\delta(\mathbf{a}_m, \mathbf{b}_m) > \frac{c}{2}$ and $n(\mathbf{a}_m, \mathbf{b}_m) > m$. By compactness of $\mathbf{X}$, there exists a sequence $m_k \rightarrow \infty$ such that $(\mathbf{a}_{m_k}, \mathbf{b}_{m_k}) \rightarrow (\mathbf{a}, \mathbf{b})$ for some $\mathbf{a}, \mathbf{b} \in \mathbf{X}$. By Upper semi-continuity of $n(\mathbf{x}, \mathbf{y})$

$$\infty = \limsup_{m_k \rightarrow \infty} m_k \leq \limsup_{m_k \rightarrow \infty} n(\mathbf{a}_{m_k}, \mathbf{b}_{m_k}) \leq n(\mathbf{a}, \mathbf{b}).$$

From $n(\mathbf{a}, \mathbf{b}) = \infty$ we have $\delta(T^t \mathbf{a}, T^t \mathbf{b}) < c$ for all t . So $\mathbf{a} = \mathbf{b}$ by expansiveness.

This contradicts $\delta(\mathbf{a}, \mathbf{b}) = \lim_{m_k \rightarrow \infty} \delta(\mathbf{a}_{m_k}, \mathbf{b}_{m_k}) > \frac{c}{2}$. Choose $\alpha > 1$ so that $\alpha^m \leq 2$.

Now we define $\rho : \mathbf{X} \times \mathbf{X} \rightarrow \mathbb{R}_+$ by

$$\rho(\mathbf{x}, \mathbf{y}) := \alpha^{-n(\mathbf{x}, \mathbf{y})}. \quad (3.5)$$

It is easy to see that ρ is lower semi-continuous since $n(\mathbf{x}, \mathbf{y})$ is upper semi-continuous. Although ρ typically violates the triangle inequality and thus is not a metric, it satisfies the properties (i) $\rho(\mathbf{x}, \mathbf{y}) = \rho(\mathbf{y}, \mathbf{x})$ and (ii) $\rho(\mathbf{x}, \mathbf{y}) = 0$ iff $\mathbf{x} = \mathbf{y}$ (for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$).

Claim 3.4. ρ defines the topology on $\mathbf{X}$.

Proof of claim: To show that ρ defines the topology on $\mathbf{X}$, we must show that

1. for all $r > 0$ there exists $\epsilon > 0$ such that for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ $\rho(\mathbf{x}, \mathbf{y}) < \epsilon \Rightarrow \delta(\mathbf{x}, \mathbf{y}) < r$ and
2. for all $\epsilon > 0$ there exists $r > 0$ such that for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ $\delta(\mathbf{x}, \mathbf{y}) < r \Rightarrow \rho(\mathbf{x}, \mathbf{y}) < \epsilon$.

To prove part 1 it suffices to show for all $r > 0$ there exists $m > 0$ such that $n(\mathbf{x}, \mathbf{y}) > m \Rightarrow \delta(\mathbf{x}, \mathbf{y}) \leq r$. This is proved using the argument for (3.4) with $\frac{c}{2}$ replaced by r and taking $\epsilon := \alpha^{-m}$.

Suppose part 2 does not hold; i.e., suppose there exists $\epsilon > 0$ such that for all $r > 0$ there exist $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ with $\delta(\mathbf{x}, \mathbf{y}) < r$ and $\rho(\mathbf{x}, \mathbf{y}) \geq \epsilon$. Let $N := \left\lceil \frac{-\ln \epsilon}{\ln \alpha} \right\rceil$. By equicontinuity of $(T^t)_{|t| \leq N}$ there exists $r > 0$ such that $\delta(\mathbf{x}, \mathbf{y}) < r \Rightarrow \delta(T^t \mathbf{x}, T^t \mathbf{y}) < c$ for all $|t| \leq N$. Hence $n(\mathbf{x}, \mathbf{y}) > N$ and so $\rho(\mathbf{x}, \mathbf{y}) < \epsilon$ by our choice of N . This contradicts $\rho(\mathbf{x}, \mathbf{y}) \geq \epsilon$. So part 2 holds. $\square$

Claim 3.5. For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$, $\rho(\mathbf{x}, \mathbf{y}) \leq 2 \max \{\rho(\mathbf{z}, \mathbf{y}), \rho(\mathbf{z}, \mathbf{x})\}$.

Proof of claim: Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$. Take $n \in \mathbb{Z}^d$ with $|n| \leq n(\mathbf{x}, \mathbf{y})$ such that $\delta(T^n \mathbf{x}, T^n \mathbf{y}) > c$. By the triangle inequality, $\delta(T^n \mathbf{z}, T^n \mathbf{x}) > \frac{c}{2}$ or $\delta(T^n \mathbf{z}, T^n \mathbf{y}) > \frac{c}{2}$. In the first case, when $\delta(T^n \mathbf{z}, T^n \mathbf{x}) > \frac{c}{2}$, (3.4) supplies $k \in \mathbb{R}^d$ with $|k| \leq m$ such

that $\delta(T^{n+k}\mathbf{z}, T^{n+k}\mathbf{x}) > c$, so $n(\mathbf{z}, \mathbf{x}) \leq |n| + |k| \leq n(\mathbf{x}, \mathbf{y}) + m$. This gives

$$\rho(\mathbf{x}, \mathbf{y}) \leq 2\alpha^{-m}\rho(\mathbf{x}, \mathbf{y}) = 2\alpha^{-(n(\mathbf{x}, \mathbf{y})+m)} \leq 2\alpha^{-n(\mathbf{z}, \mathbf{x})} = 2\rho(\mathbf{z}, \mathbf{x}),$$

where we used the choice of α for the first inequality. In the second case, when $\delta(T^n\mathbf{z}, T^n\mathbf{y}) > \frac{c}{2}$, $\rho(\mathbf{x}, \mathbf{y}) \leq 2\rho(\mathbf{z}, \mathbf{y})$ by the same argument with $\mathbf{x}$ and $\mathbf{y}$ swapped.

□

Now that we have (i), (ii), and Claim 3.5, we apply Frink's Metrization Theorem 3.2 to ρ to get a metric d on $\mathbf{X}$ such that

$$d(\mathbf{x}, \mathbf{y}) \leq \rho(\mathbf{x}, \mathbf{y}) \leq 4d(\mathbf{x}, \mathbf{y}) \quad (\mathbf{x}, \mathbf{y} \in \mathbf{X}). \quad (3.6)$$

It directly follows from this inequality that d defines the topology on $\mathbf{X}$, since ρ defined the topology on $\mathbf{X}$. It remains to show that d satisfies inequality (3.2) in the theorem.

We start by picking $s > 0$ so that

$$k := \frac{\alpha^{s-\sqrt{d}}}{4} > 1. \quad (3.7)$$

By using continuity of T , we find $\epsilon > 0$ such that

$$d(\mathbf{x}, \mathbf{y}) < \epsilon \implies n(\mathbf{x}, \mathbf{y}) > s \quad (\mathbf{x}, \mathbf{y} \in \mathbf{X}). \quad (3.8)$$

Observe that (3.2) holds trivially when $\mathbf{x} = \mathbf{y}$ or $d(\mathbf{x}, \mathbf{y}) \geq \epsilon$. Hence, we argue for arbitrary $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ with $\mathbf{x} \neq \mathbf{y}$ and $d(\mathbf{x}, \mathbf{y}) < \epsilon$. We fix such $\mathbf{x}, \mathbf{y}$ and let $t \in \mathbb{Z}^d$ be such that $\delta(T^t\mathbf{x}, T^t\mathbf{y}) > c$ and $n(\mathbf{x}, \mathbf{y}) = |t|$; in particular, $\rho(\mathbf{x}, \mathbf{y}) = \alpha^{-|t|}$.

By the choice of ϵ , we have $0 < s < |t|$, which guarantees existence of $e, n \in \mathbb{Z}^d$ such that

$$|e| \leq b := s + \sqrt{d}, \quad |n| \leq |t| - s + \sqrt{d}, \quad \text{and} \quad t = e + n. \quad (3.9)$$

(Indeed, there clearly are $\bar{e}, \bar{n} \in \mathbb{R}^d$ with $|\bar{e}| < s$, $|\bar{n}| < |t| - s$, and $t = \bar{e} + \bar{n}$ (see Figure 3.1). So taking $e \in \mathbb{Z}^d$ with $|e - \bar{e}| < \sqrt{d}$ — facilitated by $\sqrt{d}$ -density of $\mathbb{Z}^d$ in $\mathbb{R}^d$ — and $n := t - e$ does the job). In view of (3.9) and the hypothesis on t ,

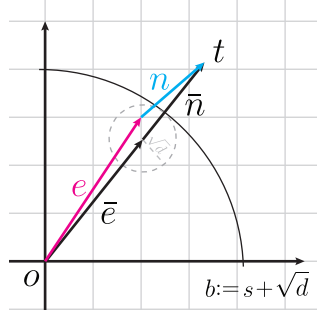


Figure 3.1: Expanding radius b dependent on T , a $\mathbb{Z}^d$ -action

$\delta(T^n T^e \mathbf{x}, T^n T^e \mathbf{y}) = \delta(T^t \mathbf{x}, T^t \mathbf{y}) > c$. So

$$n(T^e \mathbf{x}, T^e \mathbf{y}) \leq |n| \leq |t| - s + \sqrt{d}. \quad (3.10)$$

Since $\rho(T^e \mathbf{x}, T^e \mathbf{y}) := \alpha^{-n(T^e \mathbf{x}, T^e \mathbf{y})}$, (3.10) gives

$$\max_{e \in \mathbb{Z}^d, |e| < b} \{\rho(T^e \mathbf{x}, T^e \mathbf{y})\} \geq \alpha^{-(|t| - s + \sqrt{d})} = \alpha^{s - \sqrt{d}} \rho(\mathbf{x}, \mathbf{y}).$$

This combines with (3.6) to yield

$$\max_{e \in \mathbb{Z}^d, |e| < b} d(T^e \mathbf{x}, T^e \mathbf{y}) \geq \frac{\alpha^{s - \sqrt{d}}}{4} d(\mathbf{x}, \mathbf{y}) = k d(\mathbf{x}, \mathbf{y}),$$

which establishes (3.2). □

Existence of an Expanding Metric (Independent of the Action)

Theorem 1.1 is our second generalization of Fathi's theorem and it removes the dependence of the expanding radius on the action and we restate it now.

Theorem 3.6. *For $d \in \mathbb{N}$ and a norm $|\cdot|$ on $\mathbb{R}^d$ there is $\tilde{b} > 0$ such that if T is an expansive $\mathbb{Z}^d$ -action on a compact metric space $\mathbf{X}$, then there exists a metric $\tilde{d}$ on $\mathbf{X}$ defining its topology and numbers $\tilde{k} > 1$ and $\tilde{\epsilon} > 0$ so that, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$,*

$$\max_{e \in \mathbb{Z}^d, |n| < \tilde{b}} \left\{ \tilde{d}(T^e \mathbf{x}, T^e \mathbf{y}) \right\} \geq \min \left\{ \tilde{k} \tilde{d}(\mathbf{x}, \mathbf{y}), \tilde{\epsilon} \right\}. \quad (3.11)$$

$\tilde{b}$ is called the expanding radius.

Proof: Fix $d \in \mathbb{N}$. Let η be such that $\mathbb{Z}^d$ is η -dense in $\mathbb{R}^d$ with respect to the norm $|\cdot|$, i.e. for every $p \in \mathbb{R}^d$ there is $n \in \mathbb{Z}^d$ so that $|p - n| \leq \eta$. Choose

$$\tilde{a} \geq 2\eta, \quad \tilde{b} = \tilde{a} + 2\eta, \quad \text{and} \quad \tilde{c} \geq \frac{\tilde{a} + \tilde{b}}{2}. \quad (3.12)$$

Consider an expansive $\mathbb{Z}^d$ -action T . Let the metric d , and numbers k , b , and ϵ be as in Theorem 3.3. Pick $\beta \in (1, \infty)$ so that $\beta^b < k$.

Define

$$\tilde{d}(\mathbf{x}, \mathbf{y}) := \max_{e \in \mathbb{Z}^d, |e| \leq b + \tilde{c}} \frac{d(T^e \mathbf{x}, T^e \mathbf{y})}{\beta^{|e|}} \quad (\mathbf{x}, \mathbf{y} \in \mathbf{X}). \quad (3.13)$$

It is straightforward to check that $\tilde{d}$ is a metric on $\mathbf{X}$ that defines its topology. Let $K_\epsilon = \{(\mathbf{x}, \mathbf{y}) \in \mathbf{X} : d(\mathbf{x}, \mathbf{y}) \geq \epsilon\}$. Fix $t \in \mathbb{Z}^d$ with $\tilde{a} < |t| < \tilde{b}$. Define

$$\tilde{\epsilon}(t, \epsilon) := \inf_{(\mathbf{x}, \mathbf{y}) \in K_\epsilon} \tilde{d}(T^t \mathbf{x}, T^t \mathbf{y}). \quad (3.14)$$

Since $\tilde{\epsilon}(t, \epsilon) > 0$ by the continuity of T , we have

$$\tilde{\epsilon} = \tilde{\epsilon}(\epsilon) := \min_{t \in \mathbb{Z}^d, \tilde{a} < |t| < \tilde{b}} \tilde{\epsilon}(t, \epsilon) > 0. \quad (3.15)$$

We are now ready to prove (3.11). Take $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. Suppose $d(\mathbf{x}, \mathbf{y}) \geq \epsilon$. Then (3.11) holds because the construction of $\tilde{\epsilon}$ gives

$$\max_{e \in \mathbb{Z}^d, \tilde{a} < |e| < \tilde{b}} \left\{ \tilde{d}(T^e \mathbf{x}, T^e \mathbf{y}) \right\} \geq \tilde{\epsilon}. \quad (3.16)$$

Suppose $d(x, y) < \epsilon$. Then by Theorem 3.3 we have

$$\max_{e \in \mathbb{Z}^d, |e| < b} \{d(T^e \mathbf{x}, T^e \mathbf{y})\} \geq kd(\mathbf{x}, \mathbf{y}), \quad (3.17)$$

where b is the expanding radius from Theorem 3.3. Choose $e_0 \in \mathbb{Z}^d$, $|e_0| < b + \tilde{c}$, at which the maximum in (3.13) is attained.

Claim 3.7. $|e_0| > \frac{\tilde{a} + \tilde{b}}{2}$.

Proof: Let $B_b := \{e \in \mathbb{R}^d \mid 0 \leq |e| \leq b\}$. Suppose $|e_0| \leq \frac{\tilde{a} + \tilde{b}}{2}$.

Then $e_0 + B_b \subseteq B_{b+\tilde{c}}$ due to our choice of $\tilde{c}$. Hence, for $j \in B_b$, $j + e_0 \in B_{b+\tilde{c}}$ and we can estimate

$$\frac{d(T^{e_0} \mathbf{x}, T^{e_0} \mathbf{y})}{\beta^{|e_0|}} \geq \max_{e \in B_{b+\tilde{c}}} \frac{d(T^e \mathbf{x}, T^e \mathbf{y})}{\beta^{|e|}} \quad (3.18)$$

$$\geq \max_{j \in B_b \setminus \{0\}} \frac{d(T^{j+e_0} \mathbf{x}, T^{j+e_0} \mathbf{y})}{\beta^{|j+e_0|}} \quad (3.19)$$

$$\geq \max_{j \in B_b \setminus \{0\}} \frac{d(T^j(T^{e_0} \mathbf{x}), T^j(T^{e_0} \mathbf{y}))}{\beta^{|e_0|+b}} \quad (3.20)$$

$$\geq \frac{k}{\beta^b} \frac{d(T^{e_0} \mathbf{x}, T^{e_0} \mathbf{y})}{\beta^{|e_0|}} \quad (3.21)$$

where inequality (3.20) used the fact that $|j + e_0| \leq |e_0| + b$ and inequality (3.21) used (3.17). But the resulting inequality (3.21) cannot be true since we chose $\beta^b < k$. $\square$

Claim 3.8. *There exists $n \in \mathbb{Z}^d$ with $\tilde{a} < |n| < \tilde{b}$ such that*

$$|n| + |e_0 - n| \leq |e_0| + 2\eta. \quad (3.22)$$

Proof: Let $p := \frac{e_0}{|e_0|} \left(\frac{\tilde{a} + \tilde{b}}{2} \right) \in A_{\tilde{a}, \tilde{b}} := \{m \in \mathbb{R}^d \mid \tilde{a} < |m| < \tilde{b}\}$. Since $\mathbb{Z}^d$ is η dense in $\mathbb{R}^d$ and $B_\eta(p) \subseteq A_{\tilde{a}, \tilde{b}}$ (because $\frac{\tilde{b} - \tilde{a}}{2} = \eta$ by (3.12)), there exists $n \in \mathbb{Z}^d \cap B_\eta(p) \cap A_{\tilde{a}, \tilde{b}}$. Since $|e_0| > \frac{\tilde{a} + \tilde{b}}{2}$ from Claim 3.7 we have $|e_0 - p| = |e_0| - |p|$ (see Figure 3.2). By the triangle inequality

$$|e_0 - n| \leq |n - p| + |e_0 - p| \leq \eta + (|e_0| - |p|) \quad (3.23)$$

and

$$|n| \leq |n - p| + |p| \leq \eta + |p|. \quad (3.24)$$

Putting these together we get

$$|n| + |e_0 - n| \leq \eta + |p| + \eta + (|e_0| - |p|) \leq |e_0| + 2\eta, \quad (3.25)$$

which is (3.22). $\square$

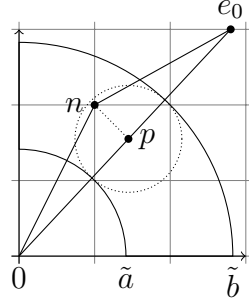


Figure 3.2: Expanding radius $\tilde{b}$ independent of T , a $\mathbb{Z}^d$ -action

From (3.22) we get $|e_0 - n| \leq |e_0| + 2\eta - |n| \leq b + \tilde{c} + 2\eta - a \leq b + \tilde{c}$. By taking $e = e_0 - n$ under the maximum below, we can estimate

$$\tilde{d}(T^n \mathbf{x}, T^n \mathbf{y}) = \max_{e \in \mathbb{Z}^d, |e| < b + \tilde{c}} \frac{d(T^{n+e} \mathbf{x}, T^{n+e} \mathbf{y})}{\beta^{|e|}} \quad (3.26)$$

$$\geq \frac{d(T^{n+e_0-n} \mathbf{x}, T^{n+e_0-n} \mathbf{y})}{\beta^{|e_0-n|}} \quad (3.27)$$

$$= \frac{d(T^{e_0} \mathbf{x}, T^{e_0} \mathbf{y})}{\beta^{|e_0|}} \cdot \beta^{|e_0|-|e_0-n|} \quad (3.28)$$

$$\geq \tilde{d}(\mathbf{x}, \mathbf{y}) \beta^{|n|-2\eta} \quad (3.29)$$

$$\geq \tilde{d}(\mathbf{x}, \mathbf{y}) \beta^{a-2\eta} \quad (3.30)$$

where we used (3.22) and $|n| > \tilde{a}$, in that order. By our choice of $\tilde{a}$ we have $\beta^{\tilde{a}-2\eta} > 1$.

Taking $\tilde{k} = \beta^{\tilde{a}-2\eta}$, inequality (3.30) becomes

$$\tilde{d}(T^n \mathbf{x}, T^n \mathbf{y}) \geq \tilde{k} \tilde{d}(\mathbf{x}, \mathbf{y}). \quad (3.31)$$

Finally, (3.11) follows from (3.16) and (3.31). $\square$

Remark 3.9. In dimension $d = 1$, (3.22) holds with η set to 0 and equations (3.26-3.31) work with $\tilde{a} = 0$ and $\tilde{b} = \tilde{a} + 1$, recovering Fathi's result.

RESULTS FOR $\mathbb{R}^d$ -ACTIONSTemporal Alignment Lemma

In this chapter, we will be concerned with abstract tiling actions with a good cross-section $\mathbf{X}_0$. We start with a convenient notation for the condition on proximity of two orbits that already appeared in the definition of (WE_0) (Definition 2.28) and Lemma 2.29.

Definition 4.1. *For any $\Delta_1, \Delta_2, R > 0$ and $\mathbf{x}, \mathbf{z} \in \mathbf{X}_0$, we write*

$$\mathbf{x}|_R \stackrel{\Delta_1}{\underset{\Delta_2}{\equiv}} \mathbf{z}|_R$$

iff $D_{\mathbf{z}} \in \mathbf{U}_{\Delta_2}^R(D_{\mathbf{x}})$, and for all t, t' that are $(\Delta_2, R, \mathbf{x}, \mathbf{z})$ -corresponding, $d(\mathbf{x}+t, \mathbf{z}+t') < \Delta_1$.

Note that the relation is symmetric, $\mathbf{x}|_R \stackrel{\Delta_1}{\underset{\Delta_2}{\equiv}} \mathbf{z}|_R$ iff $\mathbf{z}|_R \stackrel{\Delta_1}{\underset{\Delta_2}{\equiv}} \mathbf{x}|_R$. We express it in words by saying that $\mathbf{x}$ and $\mathbf{z}$ are R -proximate with *maximum separation* Δ_1 and *temporal alignment* to within Δ_2 . In the context of tiling actions, this is roughly equivalent to saying that the two tilings $\mathbf{x}$ and $\mathbf{z}$ agree, up to Δ_2 -small translation, on their central R -patches. (Actually, on $R + r(\Delta_2)$ -patches where $r(\Delta_2) \rightarrow \infty$ as $\Delta_2 \searrow 0$.)

The following technical lemma about improving the temporal alignment (at the cost of reducing R) heavily relies on the axiom (PS) and will play a key role in the proof of Theorem 1.9.

Lemma 4.2. *Let T be an abstract tiling action on a compact metric space $\mathbf{X}$ with a good cross-section $\mathbf{X}_0$. Then for all $\Delta_1 > 0, \Delta_2 \in (0, \min\{l_0, 10\Delta_0\})$ there exists*

$\hat{\Delta}_1 > 0$ such that

$$\forall_{R > R_D + 1} \forall_{\mathbf{x}, \mathbf{z} \in \mathbf{X}_0} \quad \mathbf{x}|_R \stackrel{\hat{\Delta}_1}{\Delta_2} \mathbf{z}|_R \implies \mathbf{x}|_{R-R_D-1} \stackrel{\hat{\Delta}_1}{\Delta_2/2} \mathbf{z}|_{R-R_D-1} \quad (4.1)$$

Proof: Fix $\Delta_1 > 0$, $l_0 > 0$ and $R > R_D + 1$. Take any $\Delta_2 \in (0, \min\{l_0, 10\Delta_0\})$.

Claim 4.3. *For all $\epsilon > 0$ there exists $\delta > 0$ such that for all $\mathbf{x}, \mathbf{z} \in \mathbf{X}_0$, for all $(\Delta_2, R, \mathbf{x}, \mathbf{z})$ -corresponding t, t' , and for all $|s| \leq r_0$ we have*

$$d(\mathbf{x} + t, \mathbf{z} + t') < \delta \quad \text{and} \quad d(\mathbf{x} + t, \mathbf{z} + t + s) < \delta \implies |t + s - t'| < \epsilon. \quad (4.2)$$

Proof of Claim: Suppose Claim 4.3 fails. Then there would exist $\epsilon > 0$ such that for all $n > 0$ there exist $\mathbf{x}_n, \mathbf{z}_n \in \mathbf{X}_0$, $(\Delta_2, R, \mathbf{x}, \mathbf{z})$ -corresponding t_n, t'_n and s_n with $|s_n| \leq r_0$ such that

$$d(\mathbf{x}_n + t_n, \mathbf{z}_n + t'_n) < \frac{1}{n} \quad \text{and} \quad d(\mathbf{x}_n + t_n, \mathbf{z}_n + t_n + s_n) < \frac{1}{n} \quad (4.3)$$

yet

$$|t_n + s_n - t'_n| \geq \epsilon. \quad (4.4)$$

Using compactness, there exists subsequence (n_k) , $n_k \rightarrow \infty$, such that $\mathbf{x}_{n_k} \rightarrow \mathbf{x}$, $\mathbf{z}_{n_k} \rightarrow \mathbf{z}$, $t_{n_k} \rightarrow t$, $t'_{n_k} \rightarrow t'$, $s_{n_k} \rightarrow s$ with $\mathbf{x}, \mathbf{z} \in \mathbf{X}_0$, $t \in D_{\mathbf{x}}, t' \in D_{\mathbf{z}}$ and $|s| \leq r_0$.

Then

$$\lim_{n_k \rightarrow \infty} d(\mathbf{x}_{n_k} + t_{n_k}, \mathbf{z}_{n_k} + t'_{n_k}) = d(\mathbf{x} + t, \mathbf{z} + t') = 0$$

and

$$\lim_{n_k \rightarrow \infty} d(\mathbf{x}_{n_k} + t_{n_k}, \mathbf{z}_{n_k} + t_{n_k} + s_{n_k}) = d(\mathbf{x} + t, \mathbf{z} + t + s) = 0.$$

Since t_{n_k}, t'_{n_k} are corresponding and thus satisfy $|t_{n_k} - t'_{n_k}| \leq \Delta_2$, we also have

$$\lim_{n_k \rightarrow \infty} |t_{n_k} + s_{n_k} - t'_{n_k}| = |t + s - t'| \leq \Delta_2 + r_0. \quad (4.5)$$

Thus $\mathbf{z} + t' = \mathbf{z} + t + s$; i.e. $\mathbf{z} + \tau = \mathbf{z}$ for $\tau := t + s - t'$. So $|\tau| \geq 2r_0$ by (LF) $_{2r_0}$. However (4.5) and (4.4) imply $\epsilon \leq |\tau| \leq \Delta_2 + r_0$. Since $\Delta_2 + r_0 < 2r_0$ we get a contradiction. This proves Claim 4.3. $\square$

Now take $\epsilon \in (0, \Delta_2/4)$ and $\delta \in (0, \Delta_2/10)$ as in Claim 4.3. Apply (PS) $_{l_0, r_0, \Delta_0}$ for this δ to get $\delta_1 > 0$.

Claim 4.4. *There exists $\hat{\Delta}_1 \in (0, \Delta_1)$ such that for all $R > R_D + 1$,*

$$\forall_{\mathbf{x}, \mathbf{z} \in \mathbf{X}_0} \quad \forall_{|\tilde{t}| < R - R_D + 1} \quad \mathbf{x} \Big|_R \stackrel{\hat{\Delta}_1}{\Delta_2} \mathbf{z} \Big|_R \implies \inf_{|s| \leq l_0} d(\mathbf{x} + \tilde{t}, \mathbf{z} + \tilde{t} + s) < \delta_1. \quad (4.6)$$

Proof of Claim: Fix $\mathbf{x}, \mathbf{z} \in \mathbf{X}_0$ and $R > R_D + 1$. By continuity of the action there exists $\hat{\Delta}_1 > 0$ (depending on R_D but not R) such that $|\tilde{t} - t| \leq R_D$ guarantees that

$$d(\mathbf{x} + t, \mathbf{z} + t') < \hat{\Delta}_1 \implies d(\mathbf{x} + \tilde{t}, \mathbf{z} + \tilde{t} + t' - t) = d(\mathbf{x} + t + \tilde{t} - t, \mathbf{z} + t' + \tilde{t} - t) < \delta_1.$$

Suppose

$$\mathbf{x} \Big|_R \stackrel{\hat{\Delta}_1}{\Delta_2} \mathbf{z} \Big|_R \quad (4.7)$$

and $|\tilde{t}| < R - R_D - 1$. Then there exists $t \in D_{\mathbf{x}}$ with $|\tilde{t} - t| \leq R_D$ (by the Delone property of $D_{\mathbf{x}}$). Since

$$|t| \leq |\tilde{t}| + |\tilde{t} - t| \leq R - R_D - 1 + R_D = R - 1 < R,$$

(4.7) ensures that there exists a corresponding $t' \in D_{\mathbf{z}}$ such that $|t - t'| < \Delta_2$.

Since $|t' - t| < \Delta_2 < l_0$ we get

$$\inf_{|s| < l_0} d(\mathbf{x} + \tilde{t}, \mathbf{z} + \tilde{t} + s) < \delta_1.$$

Finally, by the arbitrariness of $\tilde{t}$ with $|\tilde{t}| < R - R_D - 1$ Claim 4.4 is shown. $\square$

So far we have fixed the values $l_0 > 0$, $\Delta_1, \Delta_2 \in (0, \min\{l_0, 10\Delta_0\})$, $R > R_D + 1$, $\epsilon \in (0, \Delta_2/4)$, and $\delta \in (0, \Delta_2/10)$. Now we take $\hat{\Delta}_1 \in (0, \delta)$ satisfying Claim 4.4 and fix $\mathbf{x}, \mathbf{z} \in \mathbf{X}_0$ such that

$$\mathbf{x}|_R \stackrel{\hat{\Delta}_1}{\Delta_2} \mathbf{z}|_R. \quad (4.8)$$

Let $R' := R - R_D - 1$ and $U := B_{R'+1}$. Then by Claim 4.4 we have

$$\inf_{|s| < l_0} d(\mathbf{x} + \tilde{t}, \mathbf{z} + \tilde{t} + s) < \delta_1 \quad (\forall \tilde{t} \in U).$$

Invoking $(\text{PS})_{l_0, r_0, \Delta_0}$ we get $s \in B_{r_0}$ independent of t so that

$$d(\mathbf{x} + \tilde{t}, \mathbf{z} + \tilde{t} + s) < \delta \quad (\forall \tilde{t} \in U). \quad (4.9)$$

For any t, t' $(\Delta_2/2, R', \mathbf{x}, \mathbf{z})$ -corresponding, t, t' are also $(\Delta_2, R, \mathbf{x}, \mathbf{z})$ -corresponding. So

$$d(\mathbf{x} + t, \mathbf{z} + t') < \hat{\Delta}_1. \quad (4.10)$$

Applying (4.9) with $\tilde{t} := t$ we get

$$d(\mathbf{x} + t, \mathbf{z} + t + s) < \delta. \quad (4.11)$$

Since $\hat{\Delta}_1 < \delta$ we can invoke Claim 4.3 to get $|t + s - t'| < \epsilon$. Using a similar argument with $t = t' = 0$ we also get $|s| < \epsilon$. Therefore

$$|t - t'| \leq |t + s - t'| + |s| \leq 2\epsilon < 2(\Delta_2/4) = \Delta_2/2.$$

Hence, by the arbitrariness of t and t' and (4.10), we get

$$\mathbf{x}|_{R'} \stackrel{\hat{\Delta}_1}{\underset{\Delta_2/2}{\equiv}} \mathbf{z}|_{R'}$$

as desired. □

Proof of the Expanding Metric Theorem 1.9

Theorem 1.9 is a generalization of Theorem 3.3 for $\mathbb{Z}^d$ -actions to $\mathbb{R}^d$ -actions. The expanding metric is defined on the closed subset $\mathbf{X}_0 \subset \mathbf{X}$ provided by (DEL). The arguments made in this proof parallel those made in the proof of Theorem 3.3, with some extra work due to the dependence of the set of return times $D_{\mathbf{x}}$ on $\mathbf{x}$.

Theorem 4.5 (1.9). *Let T be an abstract tiling action on a compact metric space $\mathbf{X}$ with a good cross-section $\mathbf{X}_0$. Fix $\Delta_2 \in (0, r_D/2)$. Then there exists a metric d_{exp} on $\mathbf{X}_0$ defining its topology and numbers $b > 0$, $k > 1$, and $\epsilon > 0$ so that for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}_0$, if $d_{exp}(\mathbf{x}, \mathbf{y}) < \epsilon$, then*

$$\sup \left\{ d_{exp} \left(T^e \mathbf{x}, T^{e'} \mathbf{y} \right) : e, e' (\Delta_2, b, \mathbf{x}, \mathbf{y})\text{-corresponding} \right\} \geq k d_{exp}(\mathbf{x}, \mathbf{y}). \quad (4.12)$$

We say that b is an expanding radius for T on $\mathbf{X}_0$.

Proof: Let T be an $\mathbb{R}^d$ -action on $\mathbf{X}$ with a good cross-section $\mathbf{X}_0$ satisfying $(LF)_{2r_0}$, $(WE)_{r_0, \Delta_0}$, $(PS)_{l_0, r_0, \Delta_0}$. Choose some metric d on $\mathbf{X}$ defining its topology. Let

$\Delta_1, \Delta_2 > 0$ be the constants provided by Lemma 2.29, i.e.,

$$\forall_{\epsilon > 0} \quad \exists_{R > 0} \quad \forall_{\mathbf{x}, \mathbf{y} \in \mathbf{X}_0} \quad \mathbf{x}|_R \stackrel{\Delta_1}{\Delta_2} \mathbf{y}|_R \implies d(\mathbf{x}, \mathbf{y}) < \epsilon. \quad (4.13)$$

Fix $\hat{\Delta}_1$ as in Lemma 4.2. For any $\mathbf{x}, \mathbf{y} \in \mathbf{X}_0$, define $n : \mathbf{X}_0 \times \mathbf{X}_0 \rightarrow \mathbb{R}_+$ by

$$n(\mathbf{x}, \mathbf{y}) := \sup\{R > 0 : \mathbf{x}|_R \stackrel{\hat{\Delta}_1}{\Delta_2} \mathbf{y}|_R\}. \quad (4.14)$$

Note that $n(\mathbf{x}, \mathbf{y}) \geq 0$ because for $R = 0$, $D_{\mathbf{x}}^R = D_{\mathbf{y}}^R = \{0\}$ so $\mathbf{x}|_R \stackrel{\hat{\Delta}_1}{\Delta_2} \mathbf{y}|_R$ holds trivially. In (4.13) take $\epsilon = \frac{\hat{\Delta}_1}{2}$ to get $L > 0$ such that

$$\mathbf{x}|_L \stackrel{\hat{\Delta}_1}{\Delta_2} \mathbf{y}|_L \implies d(\mathbf{x}, \mathbf{y}) < \frac{\hat{\Delta}_1}{2} \quad (\mathbf{x}, \mathbf{y} \in \mathbf{X}_0). \quad (4.15)$$

Choose $\alpha > 1$ so that $\alpha^{L+R_D+1} \leq 2$. Now define $\rho : \mathbf{X}_0 \times \mathbf{X}_0 \rightarrow \mathbb{R}_+$ by

$$\rho(\mathbf{x}, \mathbf{y}) := \alpha^{-n(\mathbf{x}, \mathbf{y})}. \quad (4.16)$$

Although ρ typically violates the triangle inequality and thus is not a metric, it does satisfy the properties (i) $\rho(\mathbf{x}, \mathbf{y}) = \rho(\mathbf{y}, \mathbf{x})$ and (ii) $\rho(\mathbf{x}, \mathbf{y}) = 0$ iff $\mathbf{x} = \mathbf{y}$ (for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}_0$). Once we have established the following claims, we will use Frink's Metrization Theorem to get a true metric.

Claim 4.6. ρ defines the topology on $\mathbf{X}_0$.

Proof of claim: To show that ρ defines the topology on $\mathbf{X}_0$, we must show that

- (a) for all $r > 0$ there exists $\epsilon > 0$ such that $\rho(\mathbf{x}, \mathbf{y}) < \epsilon \implies d(\mathbf{x}, \mathbf{y}) < r$ (for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}_0$) and

(b) for all $\epsilon > 0$ there exists $r > 0$ such that $d(\mathbf{x}, \mathbf{y}) < r \Rightarrow \rho(\mathbf{x}, \mathbf{y}) < \epsilon$ (for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}_0$).

To prove part (a) it suffices to show that, for all $r > 0$, there exists $m > 0$ such that $n(\mathbf{x}, \mathbf{y}) > m \Rightarrow d(\mathbf{x}, \mathbf{y}) \leq r$. Fix $r > 0$. Then (WE₀) provides an $R > 0$ such that

$$\mathbf{x}|_R \stackrel{\hat{\Delta}_1}{\Delta_2} \mathbf{y}|_R \implies d(\mathbf{x}, \mathbf{y}) < r.$$

Choose $m := R$. Then

$$n(\mathbf{x}, \mathbf{y}) > R \implies \mathbf{x}|_R \stackrel{\hat{\Delta}_1}{\Delta_2} \mathbf{y}|_R \implies d(\mathbf{x}, \mathbf{y}) < r.$$

To prove part (b) consider $\epsilon > 0$ and take $R > 0$ so that $\alpha^{-R} < \epsilon$. For all $\Delta_2 \in (0, r_0/2]$, Proposition 2.27 provides $r_1 > 0$ such that

$$\forall \mathbf{x}, \mathbf{y} \in \mathbf{X}_0 \quad d(\mathbf{x}, \mathbf{y}) < r_1 \implies D_{\mathbf{y}} \in \mathbf{U}_{\Delta_2}^R(D_{\mathbf{x}}).$$

Also, by equicontinuity of $(T^t)_{|t| \leq R}$, there exists $r_2 > 0$ such that for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}_0$

$$d(\mathbf{x}, \mathbf{y}) < r_2 \implies d(T^t \mathbf{x}, T^t \mathbf{y}) \leq \hat{\Delta}_1/2 \quad \forall |t| \leq R.$$

Now shrink Δ_2 so that, for all $\mathbf{y} \in \mathbf{X}_0$, $d(T^t \mathbf{y}, T^{t'} \mathbf{y}) \leq \hat{\Delta}_1/2$ whenever $|t - t'| < \Delta_2$.

Thus for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}_0$ with $d(\mathbf{x}, \mathbf{y}) < r_2$ we have

$$d(T^t \mathbf{x}, T^{t'} \mathbf{y}) \leq d(T^t \mathbf{x}, T^t \mathbf{y}) + d(T^t \mathbf{y}, T^{t'} \mathbf{y}) \leq \hat{\Delta}_1/2 + \hat{\Delta}_1/2 \leq \hat{\Delta}_1$$

(for all t, t' $(\Delta_2, R, \mathbf{x}, \mathbf{y})$ -corresponding) by the triangle inequality. Set $r = \min\{r_1, r_2\}$ to get

$$d(\mathbf{x}, \mathbf{y}) < r \implies \mathbf{x}|_R \stackrel{\hat{\Delta}_1}{\underset{\Delta_2}{=}} \mathbf{y}|_R \implies n(\mathbf{x}, \mathbf{y}) \geq R.$$

So $\rho(\mathbf{x}, \mathbf{y}) = \alpha^{-n(\mathbf{x}, \mathbf{y})} \leq \alpha^{-R} < \epsilon$ by our choice of R . $\square$

Claim 4.7. *For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}_0$, $\rho(\mathbf{x}, \mathbf{z}) \leq 2 \max\{\rho(\mathbf{x}, \mathbf{y}), \rho(\mathbf{y}, \mathbf{z})\}$.*

Proof of claim: Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}_0$ and consider $R \leq \min\{n(\mathbf{x}, \mathbf{y}), n(\mathbf{y}, \mathbf{z})\}$ so that

$$\mathbf{x}|_R \stackrel{\hat{\Delta}_1}{\underset{\Delta_2}{=}} \mathbf{y}|_R \quad \text{and} \quad \mathbf{y}|_R \stackrel{\hat{\Delta}_1}{\underset{\Delta_2}{=}} \mathbf{z}|_R.$$

Then

$$D_{\mathbf{y}} \in \mathbf{U}_{\Delta_2}^R(D_{\mathbf{x}}) \quad \text{and} \quad D_{\mathbf{z}} \in \mathbf{U}_{\Delta_2}^R(D_{\mathbf{y}}).$$

Putting those together gives us

$$D_{\mathbf{z}} \in \mathbf{U}_{2\Delta_2}^R(D_{\mathbf{x}}) \tag{4.17}$$

By the choice of L given in (4.15) we have

$$\mathbf{x}|_{R-L} \stackrel{\hat{\Delta}_1/2}{\underset{\Delta_2}{=}} \mathbf{y}|_{R-L} \quad \text{and} \quad \mathbf{y}|_{R-L} \stackrel{\hat{\Delta}_1/2}{\underset{\Delta_2}{=}} \mathbf{z}|_{R-L}.$$

Lemma 4.2 implies that

$$\mathbf{x}|_{R-L-R_D-1} \stackrel{\hat{\Delta}_1/2}{\underset{\Delta_2/2}{=}} \mathbf{y}|_{R-L-R_D-1} \quad \text{and} \quad \mathbf{z}|_{R-L-R_D-1} \stackrel{\hat{\Delta}_1/2}{\underset{\Delta_2/2}{=}} \mathbf{y}|_{R-L-R_D-1}.$$

Now we apply the triangle inequality and (4.17) to get

$$\mathbf{x}|_{R-L-R_D-1} \stackrel{\hat{\Delta}_1}{\underset{\Delta_2}{=}} \mathbf{z}|_{R-L-R_D-1}. \tag{4.18}$$

That is

$$\rho(\mathbf{x}, \mathbf{z}) \leq \alpha^{-(R-L-R_D-1)} \leq \alpha^{L+R_D+1} \alpha^{-R}. \quad (4.19)$$

By the arbitrariness of R , we get

$$\rho(\mathbf{x}, \mathbf{z}) \leq \alpha^L \alpha^{-\min\{n(\mathbf{x}, \mathbf{y}), n(\mathbf{y}, \mathbf{z})\}} \leq 2 \max\{\rho(\mathbf{x}, \mathbf{y}), \rho(\mathbf{y}, \mathbf{z})\}, \quad (4.20)$$

as desired. This proves Claim 4.7. $\square$

Now that we have (i), (ii), and Claim 4.7, we apply Frink's Metrization Theorem 3.2 to ρ to get a metric d_{exp} on $\mathbf{X}_0$ such that

$$d_{\text{exp}}(\mathbf{x}, \mathbf{y}) \leq \rho(\mathbf{x}, \mathbf{y}) \leq 4d_{\text{exp}}(\mathbf{x}, \mathbf{y}) \quad (\mathbf{x}, \mathbf{y} \in \mathbf{X}_0). \quad (4.21)$$

It directly follows from this inequality that d_{exp} defines the topology on $\mathbf{X}$, since ρ defined the topology on $\mathbf{X}_0$. It remains to show that d_{exp} satisfies inequality (4.12) in the theorem.

We start by picking $s > 0$ so that

$$k := \frac{\alpha^{s-2(R_D+\Delta_2)}}{4} > 1, \quad (4.22)$$

where R_D is the Delone constant for $D_{\mathbf{x}}$. Let $R := R_D + \Delta_2$. From the second inequality in (4.21), there exists $\epsilon > 0$ such that

$$d_{\text{exp}}(\mathbf{x}, \mathbf{y}) < \epsilon \implies n(\mathbf{x}, \mathbf{y}) > s + R \quad (\mathbf{x}, \mathbf{y} \in \mathbf{X}_0). \quad (4.23)$$

To prove (4.12) we fix $\mathbf{x}, \mathbf{y} \in \mathbf{X}_0$ with $d_{\text{exp}}(\mathbf{x}, \mathbf{y}) < \epsilon$. Then $D_{\mathbf{y}} \in \mathbf{U}_{\Delta_2}^{s+R}(D_{\mathbf{x}})$ by (4.23). Consider an arbitrary $A > 0$ such that

$$A \leq n(\mathbf{x} - e, \mathbf{y} - e') \quad \text{for all } e, e' (\Delta_2, s + R, \mathbf{x}, \mathbf{y})\text{-corresponding.} \quad (4.24)$$

This means that

$$\mathbf{x} - e|_A \stackrel{\hat{\Delta}_1}{=} \mathbf{y} - e'|_A \quad \text{for all } e, e' (\Delta_2, s + R, \mathbf{x}, \mathbf{y})\text{-corresponding.} \quad (4.25)$$

Fix $t, t' (\Delta_2, A - 2R + s, \mathbf{x}, \mathbf{y})$ -corresponding with $|t|, |t'| \leq A - 2R + s$. Now split t and t' into collinear parts $t = \bar{e} + \bar{n}$ and $t' = \bar{e}' + \bar{n}'$ so that $|\bar{e}| = s$, $|\bar{n}| = |t| - s$, $|\bar{e}'| = s$, and $|\bar{n}'| = |t'| - s$. Take e, e' to be $(\Delta_2, s + R, \mathbf{x}, \mathbf{y})$ -corresponding with $|e - \bar{e}| < R_D$ and $|e' - \bar{e}'| < R$.

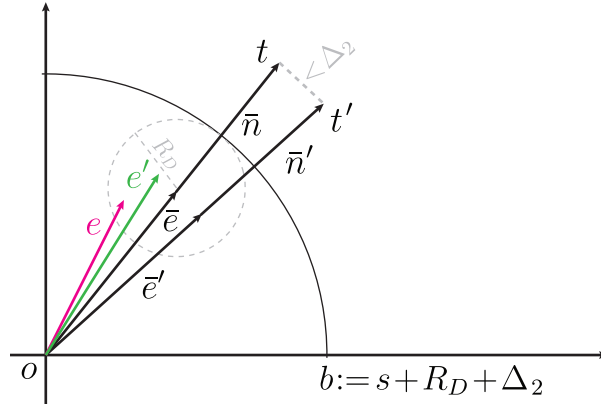


Figure 4.1: Expanding radius b for T , an abstract tiling action

Set $n := t - e$ and $n' := t' - e'$. Using the triangle inequality we get

$$|n| \leq |\bar{n}| + |n - \bar{n}| \leq (A - 2R) + (2R) \leq A \quad \text{and}$$

$$|n'| \leq |\bar{n}'| + |n' - \bar{n}'| \leq (A - 2R) + (2R) \leq A.$$

Obviously $n \in D_{\mathbf{x}-e}$ and $n' \in D_{\mathbf{y}-e'}$. From (4.25) we get $D_{\mathbf{y}-e'} \in \mathbf{U}_{\Delta_2}^A(D_{\mathbf{x}-e})$. So $|n - n'| < \Delta_2$. That is, n, n' are $(\Delta_2, A, \mathbf{x} - e, \mathbf{y} - e')$ -corresponding. Using this and (4.25), we get

$$d(\mathbf{x} - t, \mathbf{y} - t') = d(\mathbf{x} - e - n, \mathbf{y} - e' - n') < \hat{\Delta}_1.$$

By the arbitrariness of t and t' , we get $\mathbf{x}|_{A-2R+s} \stackrel{\hat{\Delta}_1}{\Delta_2} \mathbf{y}|_{A-2R+s}$ and so

$$n(\mathbf{x}, \mathbf{y}) \geq A - 2R + s.$$

That is $A \leq n(\mathbf{x}, \mathbf{y}) - s + 2R$. By the arbitrariness of A , we get $n(\mathbf{x} - e, \mathbf{y} - e') \leq n(\mathbf{x}, \mathbf{y}) - s + 2R$ for all $e, e' (\Delta_2, s + R, \mathbf{x}, \mathbf{y})$ -corresponding. So

$$\inf\{n(\mathbf{x} - e, \mathbf{y} - e') : e, e' (\Delta_2, s + R, \mathbf{x}, \mathbf{y})\text{-corresponding}\} \leq n(\mathbf{x}, \mathbf{y}) - s + 2R.$$

That is

$$\sup\{\rho(\mathbf{x} - e, \mathbf{y} - e') : e, e' (\Delta_2, s + R, \mathbf{x}, \mathbf{y})\text{-corresponding}\} \geq \alpha^{s-2R} \rho(\mathbf{x}, \mathbf{y}).$$

Applying inequality (4.21) we get

$$\begin{aligned} \sup\{d_{\text{exp}}(\mathbf{x} - e, \mathbf{y} - e') : e, e' (\Delta_2, s + R, \mathbf{x}, \mathbf{y})\text{-corresponding}\} &\geq \frac{\alpha^{s-2R}}{4} d_{\text{exp}}(\mathbf{x}, \mathbf{y}) \\ &= k d_{\text{exp}}(\mathbf{x}, \mathbf{y}), \end{aligned}$$

where the equality comes from (4.22). Letting $b := s + R$, we get inequality (4.12) as desired. $\square$

Dimension Estimation Theorem ($\mathbb{Z}^d$)

The reader who is only concerned about our main results may skip this section as it only pertains to $\mathbb{Z}^d$ -actions. This theorem generalizes the bound used by Fathi along analogous lines [10].

Theorem 4.8. *Let T be an expansive $\mathbb{Z}^d$ -action on a compact metric space $\mathbf{X}$, with metric $\tilde{d}$ and constants $\tilde{k}$ and $\tilde{b}$ as in Theorem 1.1. Then*

$$HD_{\tilde{d}}(\mathbf{X}) \leq \tilde{b} \cdot \frac{\eta_1(T)}{\ln \tilde{k}} \quad (4.26)$$

Proof: Fix $\epsilon > 0$. Let $E \subset \mathbf{X}$ be a (T, RQ, ϵ) -separated set of maximal cardinality. Then E is also a (T, RQ, ϵ) -spanning set, i.e.

$$\forall \mathbf{x} \in \mathbf{X}, \exists \mathbf{y} \in E \quad \text{such that} \quad d_{RQ}^T(\mathbf{x}, \mathbf{y}) < \epsilon. \quad (4.27)$$

Here we use the metric d_{RQ}^T defined in (2.19) with $d = \tilde{d}$. Then the collection of ϵ -balls in the d_{RQ}^T metric given by $\mathcal{U} = \{B_{\epsilon}^{d_{RQ}^T}(\mathbf{x}) : \mathbf{x} \in E\}$ is a covering of X .

Claim 4.9. *For $U \in \mathcal{U}$, $\text{diam}(U) \leq 2\epsilon \tilde{k}^{-l+1}$, where $l := \lfloor R/\tilde{b} \rfloor$.*

Proof of Claim: Fix $\mathbf{x} \in E$. Let $U = B_\epsilon^{d_{RQ}}(\mathbf{x}) \in \mathcal{U}$. Take $\mathbf{y} \in U$. Then there exists $k_1 \in A_{\tilde{a}, \tilde{b}} \cap \mathbb{Z}^d$ (recall $\tilde{a}$ comes from the proof of Theorem 3.6) such that

$$\tilde{d}(\mathbf{T}^{k_1} \mathbf{x}, \mathbf{T}^{k_1} \mathbf{y}) \geq \tilde{k} \tilde{d}(\mathbf{x}, \mathbf{y}) \quad (4.28)$$

by Theorem 1.1. Applying Theorem 1.1 again we find there exists $k_2 \in \mathbb{Z}^d$ with $k_2 - k_1 \in A_{\tilde{a}, \tilde{b}}$ such that

$$\tilde{d}(\mathbf{T}^{k_2} \mathbf{x}, \mathbf{T}^{k_2} \mathbf{y}) \geq \tilde{k} \tilde{d}(\mathbf{T}^{k_1} \mathbf{x}, \mathbf{T}^{k_1} \mathbf{y}) \geq \tilde{k}^2 \tilde{d}(\mathbf{x}, \mathbf{y}). \quad (4.29)$$

This can be repeated $l := \lfloor R/\tilde{b} \rfloor$ times to get $k_l \in \mathbb{Z}^d$ with $k_l - k_{l-1} \in A_{\tilde{a}, \tilde{b}}$ such that

$$\tilde{d}(\mathbf{T}^{k_l} \mathbf{x}, \mathbf{T}^{k_l} \mathbf{y}) \geq \tilde{k} \tilde{d}(\mathbf{T}^{k_{l-1}} \mathbf{x}, \mathbf{T}^{k_{l-1}} \mathbf{y}) \geq \tilde{k}^l \tilde{d}(\mathbf{x}, \mathbf{y}). \quad (4.30)$$

Thus

$$\tilde{d}(\mathbf{x}, \mathbf{y}) \leq \tilde{k}^{-l} \tilde{d}(\mathbf{T}^{k_l} \mathbf{x}, \mathbf{T}^{k_l} \mathbf{y}) \leq \tilde{k}^{-l} \cdot \epsilon \quad (4.31)$$

where the last inequality is due to the definition of U and the fact that $k_l \in lB_{\tilde{b}}(0) \subseteq B_{\frac{R}{\tilde{b}}, \tilde{b}}(0) = RQ$. By the arbitrariness of $\mathbf{x}, \mathbf{y} \in U$ we have $\text{diam}(U) \leq 2\epsilon \tilde{k}^{-l+1}$. $\square$

For $s > 0$, Claim 4.9 gives

$$\sum_{U \in \mathcal{U}} \text{diam}(U)^s \leq \text{sep}(\mathbf{T}, RQ, \epsilon) \left(2\epsilon \tilde{k}^{-l+1} \right)^s \quad (4.32)$$

$$\leq \text{sep}(\mathbf{T}, RQ, \epsilon) \cdot e^{s \ln(2\epsilon) + s(-l+1) \ln \tilde{k}}. \quad (4.33)$$

By definition we have

$$\eta_n(T, \epsilon) = \limsup_{R \rightarrow \infty} \frac{\ln \text{sep}(\mathbf{T}, RQ, \epsilon)}{R^n} \quad \text{for } 0 \leq n \leq d. \quad (4.34)$$

Fix $\delta > 0$ and $n = 1$. Then there exists R_0 such that for all $R \geq R_0$,

$$\frac{\ln \text{sep}(T, RQ, \epsilon)}{R} \leq \eta_1(T, \epsilon) + \delta. \quad (4.35)$$

So

$$\text{sep}(T, RQ, \epsilon) \leq e^{R(\eta_1(T, \epsilon) + \delta)}. \quad (4.36)$$

Hence

$$\sum_{U \in \mathcal{U}} \text{diam}(U)^s \leq e^{R(\eta_1(T, \epsilon) + \delta + s \frac{-l+1}{R} \ln \tilde{k} + \frac{s \ln(2\epsilon)}{R})} \quad (4.37)$$

$$= e^{R(\eta_1(T, \epsilon) + \delta - s \frac{\lfloor R/\tilde{b} \rfloor - 1}{R} \ln \tilde{k} + \frac{s \ln(2\epsilon)}{R})}. \quad (4.38)$$

To guarantee convergence for the right hand side of (4.38) as $R \rightarrow \infty$ it suffices to require $\eta_1(T, \epsilon) + \delta - \frac{s \ln \tilde{k}}{\tilde{b}} < 0$. Indeed then $\eta_1(T, \epsilon) + \delta - s \frac{\lfloor R/\tilde{b} \rfloor - 1}{R} \ln \tilde{k} + \frac{s \ln(2\epsilon)}{R} < 0$ for sufficiently large R because it differs from $\eta_1(T, \epsilon) + \delta - \frac{s \ln \tilde{k}}{\tilde{b}}$ by a term of the order of a constant times R^{-1} . So any exponent $s > \frac{\tilde{b}(\eta_1(T, \epsilon) + \delta)}{\ln \tilde{k}}$ guarantees that the sum (4.37) is bounded as $R \rightarrow \infty$. Thus

$$\text{HD}_{\tilde{d}}(X) \leq \tilde{b} \cdot \frac{\eta_1(T, \epsilon) + \delta}{\ln \tilde{k}}. \quad (4.39)$$

Now letting $\delta \rightarrow 0$ and then $\epsilon \rightarrow 0$ we get

$$\text{HD}_{\tilde{d}}(X) \leq \tilde{b} \cdot \frac{\eta_1(T)}{\ln \tilde{k}} \quad (4.40)$$

as desired. □

Dimension Estimation Theorem ($\mathbb{R}^d$)

Theorem 4.10. *Let T be an abstract tiling action on a compact metric space $\mathbf{X}$ with a good cross-section $\mathbf{X}_0$. Taking $b > 0$, $k > 1$, $\Delta_2 > 0$, $\epsilon > 0$ and d_{exp} as in Theorem 1.9, we have*

$$HD_{d_{\text{exp}}}(\mathbf{X}_0) \leq b \cdot \frac{\eta_1(T)}{\ln k} \quad (4.41)$$

Proof: Let d be a metric on $\mathbf{X}$ defining its topology and consider the metric $\tilde{d}_{RQ}^T$ defined in (2.27).

Claim 4.11. *There exists $\tilde{\epsilon} > 0$ so that*

$$\begin{aligned} \forall_{\mathbf{x}, \mathbf{y} \in \mathbf{X}_0} \quad \forall_{R > 0} \quad \tilde{d}_{RQ}^T(\mathbf{x}, \mathbf{y}) \leq \tilde{\epsilon} \Rightarrow D_{\mathbf{y}} \subseteq \mathbf{U}_{\Delta_2}^R(D_{\mathbf{x}}) \text{ and } \forall_{t, t' \text{ } (\Delta_2, R, \mathbf{x}, \mathbf{y})\text{-corresponding}} \\ d_{\text{exp}}(T^t \mathbf{x}, T^{t'} \mathbf{y}) < \epsilon. \end{aligned} \quad (4.42)$$

Proof of Claim: This follows from the continuity of $\mathbf{x} \mapsto D_{\mathbf{x}}$ and the fact that d_{exp} and d induce the same topology on $\mathbf{X}_0$. To be precise, take $\tilde{\epsilon} > 0$. Consider $\mathbf{x}, \mathbf{y} \in \mathbf{X}_0$ and $R > 0$ so that $\tilde{d}_{RQ}^T(\mathbf{x}, \mathbf{y}) \leq \tilde{\epsilon}$. Whenever $t \in D_{\mathbf{x}} \cap RQ$, we have $d(T^t \mathbf{x}, T^t \mathbf{y}) \leq \tilde{\epsilon}$ (by the definition of $\tilde{d}_{RQ}^T$). Here $T^t \mathbf{y}$ need not be in $\mathbf{X}_0$, but continuity of $\mathbf{z} \mapsto D_{\mathbf{z}}$ ensures that, if $\tilde{\epsilon}$ is small enough there is t' so that $T^{t'} \mathbf{y} \in \mathbf{X}_0$ and $|t - t'| < \Delta_2$. In fact, upon shrinking $\tilde{\epsilon}$ more if necessary, $|t - t'|$ will be so small that $d(T^t \mathbf{x}, T^{t'} \mathbf{y}) \leq d(T^t \mathbf{x}, T^t \mathbf{y}) + d(T^{t-t'} \mathbf{y}, \mathbf{y})$ can be made as small as we like. In particular, we can achieve $d_{\text{exp}}(T^t \mathbf{x}, T^{t'} \mathbf{y}) < \epsilon$. This means that for every $t \in D_{\mathbf{x}}^R \subseteq RQ$ there exists t' so that t, t' are $(\Delta_2, R, \mathbf{x}, \mathbf{y})$ -corresponding, and vice versa for $t' \in D_{\mathbf{y}}^R$. Moreover, $d_{\text{exp}}(T^t \mathbf{x}, T^{t'} \mathbf{y}) < \epsilon$ for such t, t' , as desired. $\square$

Fix $\tilde{\epsilon} > 0$ as in Claim 4.11. Let $E \subset \mathbf{X}$ be a $(T, RQ, \epsilon) - \widetilde{\text{sep}}$ set of maximal cardinality. Then E is also a $(T, RQ, \epsilon) - \widetilde{\text{span}}$ set and so the collection of ϵ -balls in

the $\tilde{d}_{RQ}^\Gamma$ metric given by $\mathcal{U} = \{B_\epsilon^{\tilde{d}_{RQ}^\Gamma}(\mathbf{x}) : \mathbf{x} \in E\}$ is a covering of $\mathbf{X}$. In particular, the set $\mathcal{V} = \{U \cap \mathbf{X}_0 : U \in \mathcal{U}\}$ is a covering of $\mathbf{X}_0$.

Claim 4.12. *For all $V \in \mathcal{V}$ and $R > 0$, $\text{diam}_{d_{\text{exp}}}(V) \leq 2\epsilon k^{-R/b+1}$.*

Proof of Claim: Fix $\mathbf{x} \in E$ and $R > 0$. Let $V = B_\epsilon^{\tilde{d}_{RQ}^\Gamma}(\mathbf{x}) \cap \mathbf{X}_0 \in \mathcal{V}$. Take any $\mathbf{y} \in V$. By Claim 4.11, we have $d_{\text{exp}}(\mathbf{x}, \mathbf{y}) \leq \epsilon$. Application of Theorem 1.9 gives k_1, k'_1 $(\Delta_2, b, \mathbf{x}, \mathbf{y})$ -corresponding such that

$$d_{\text{exp}}(T^{k_1}\mathbf{x}, T^{k'_1}\mathbf{y}) \geq k d_{\text{exp}}(\mathbf{x}, \mathbf{y}). \quad (4.43)$$

As long as $b < R$, we still have $d_{\text{exp}}(T^{k_1}\mathbf{x}, T^{k'_1}\mathbf{y}) \leq \epsilon$ (by Claim 4.11) and we can apply Theorem 1.9 again to get k_2, k'_2 with $k_2 - k_1, k'_2 - k'_1$ $(\Delta_2, b, T^{k_1}\mathbf{x}, T^{k'_1}\mathbf{y})$ -corresponding and $k_2 - k_1 \in B_b(0)$ such that

$$d_{\text{exp}}(T^{k_2}\mathbf{x}, T^{k'_2}\mathbf{y}) \geq k d_{\text{exp}}(T^{k_1}\mathbf{x}, T^{k'_1}\mathbf{y}) \geq k^2 d_{\text{exp}}(\mathbf{x}, \mathbf{y}). \quad (4.44)$$

This can be repeated $l := \lfloor R/b \rfloor$ times to get k_l, k'_l that are $(\Delta_2, b, \mathbf{x}, \mathbf{y})$ -corresponding with $d_{\text{exp}}(T^{k_l}\mathbf{x}, T^{k'_l}\mathbf{y}) \leq \epsilon$. Hence

$$d_{\text{exp}}(\mathbf{x}, \mathbf{y}) \leq k^{-1} d_{\text{exp}}(T^{k_1}\mathbf{x}, T^{k'_1}\mathbf{y}) \leq \dots \leq k^{-l} d_{\text{exp}}(T^{k_l}\mathbf{x}, T^{k'_l}\mathbf{y}) \leq k^{-l} \cdot \epsilon. \quad (4.45)$$

By the arbitrariness of $\mathbf{y} \in V$ we have $\text{diam}_{d_{\text{exp}}}(V) \leq 2\epsilon k^{-R/b+1}$. $\square$

For $s > 0$, Claim 4.12 gives

$$\sum_{U \in \mathcal{U}} \text{diam}(U)^s \leq \widetilde{\text{sep}}(\mathbf{T}, RQ, \tilde{\epsilon}) (2\epsilon k^{-R/b+1})^s \quad (4.46)$$

$$\leq \widetilde{\text{sep}}(\mathbf{T}, RQ, \tilde{\epsilon}) \cdot e^{s \ln(2\epsilon) + s(-R/b+1) \ln k}. \quad (4.47)$$

By definition we have

$$\tilde{\eta}_n(T, \tilde{\epsilon}) = \limsup_{R \rightarrow \infty} \frac{\ln \widetilde{\text{sep}}(\mathbf{T}, RQ, \tilde{\epsilon})}{R^n} \quad \text{for } 0 \leq n \leq d. \quad (4.48)$$

Fix $\delta > 0$ and $n = 1$. Then there exists R_0 such that for all $R \geq R_0$,

$$\frac{\ln \widetilde{\text{sep}}(\mathbf{T}, RQ, \tilde{\epsilon})}{R} \leq \tilde{\eta}_1(T, \tilde{\epsilon}) + \delta. \quad (4.49)$$

So

$$\widetilde{\text{sep}}(\mathbf{T}, RQ, \tilde{\epsilon}) \leq e^{R(\tilde{\eta}_1(T, \tilde{\epsilon}) + \delta)}. \quad (4.50)$$

Hence

$$\sum_{U \in \mathcal{U}} \text{diam}(U)^s \leq e^{R(\tilde{\eta}_1(T, \tilde{\epsilon}) + \delta + s \frac{-R/b+1}{R} \ln k + \frac{s \ln(2\epsilon)}{R})}. \quad (4.51)$$

To guarantee convergence for the right hand side of (4.51) as $R \rightarrow \infty$ it suffices to require that $\tilde{\eta}_1(T, \tilde{\epsilon}) + \delta - \frac{s \ln k}{b} < 0$. Indeed then $\tilde{\eta}_1(T, \tilde{\epsilon}) + \delta + s \frac{-R/b+1}{R} \ln k + \frac{s \ln(2\epsilon)}{R} < 0$ for sufficiently large R because it differs from $\tilde{\eta}_1(T, \tilde{\epsilon}) + \delta - \frac{s \ln k}{b}$ by a term of the order of a constant times R^{-1} . So any exponent $s > \frac{b(\tilde{\eta}_1(T, \tilde{\epsilon}) + \delta)}{\ln k}$ guarantees that the sum (4.51) is bounded as $R \rightarrow \infty$. Thus

$$\text{HD}_{d_{\text{exp}}}(\mathbf{X}) \leq b \cdot \frac{\tilde{\eta}_1(T, \tilde{\epsilon}) + \delta}{\ln k}. \quad (4.52)$$

Now letting $\delta \rightarrow 0$ and then $\tilde{\epsilon} \rightarrow 0$ we get

$$\text{HD}_{d_{\text{exp}}}(\mathbf{X}) \leq b \cdot \frac{\tilde{\eta}_1(\mathbf{T})}{\ln k} \quad (4.53)$$

as desired. □

Self-Affine Actions Have Zero Entropy

In this section, let T be an $\mathbb{R}^d$ -action (or $\mathbb{Z}^d$ -action) on a compact metric space $(\mathbf{X}, d_{\mathbf{X}})$ and S be an $\mathbb{R}^d$ -action (or $\mathbb{Z}^d$ -action) on a compact metric space $(\mathbf{Y}, d_{\mathbf{Y}})$. The dynamical system $(\mathbf{Y}, S)$ is a *factor* of $(\mathbf{X}, T)$ iff there exists a continuous surjective map $h : \mathbf{X} \rightarrow \mathbf{Y}$ such that $h \circ T^t = S^t \circ h$ for all $t \in \mathbb{R}^d$. If h is invertible, then the dynamical systems $(\mathbf{X}, T)$ and $(\mathbf{Y}, S)$ are called *conjugate* [6]. Recall from Remark 2.31, that the entropy of an $\mathbb{R}^d$ -action can be determined by calculating the entropy of its associated $\mathbb{Z}^d$ -action obtained by restricting to times $t \in \mathbb{Z}^d$. In particular, in the proofs of items 4.13, 4.14, and 4.15, all times are integers in $\mathbb{Z}^d$.

Proposition 4.13. *If $(\mathbf{Y}, S)$ is a factor of $(\mathbf{X}, T)$, then*

$$\eta_1(T) \geq \eta_1(S). \quad (4.54)$$

Proof: This is standard. Suppose $(\mathbf{Y}, S)$ is a factor of $(\mathbf{X}, T)$, i.e. there exists a continuous surjective map $h : \mathbf{X} \rightarrow \mathbf{Y}$ such that

$$h \circ T^t = S^t \circ h \quad \forall_{t \in \mathbb{R}^d}. \quad (4.55)$$

By compactness, h is uniformly continuous. So for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$d_{\mathbf{X}}(\mathbf{x}_1, \mathbf{x}_2) < \delta(\epsilon) \Rightarrow d_{\mathbf{Y}}(h(\mathbf{x}_1), h(\mathbf{x}_2)) < \epsilon. \quad (4.56)$$

Let $E \subset \mathbf{Y}$ be (S, RQ, ϵ) -separated. Let $F \subset \mathbf{X}$ be obtained by choosing one pre-image for every $\mathbf{y} \in E$. Take $\mathbf{x}, \mathbf{y} \in F$, $\mathbf{x} \neq \mathbf{y}$. Then $h(\mathbf{x}), h(\mathbf{y}) \in E$ and there exists $e \in RQ \cap \mathbb{Z}^d$ such that $d_{\mathbf{Y}}(S^e h(\mathbf{x}), S^e h(\mathbf{y})) > \epsilon$. Then (4.56) implies that

$d_{\mathbf{X}}(T^e \mathbf{x}, T^e \mathbf{y}) > \delta(\epsilon)$. This shows that F is a $(T, RQ, \delta(\epsilon))$ -separated set. Thus

$$\text{sep}(T, RQ, \delta(\epsilon)) \geq \text{sep}(S, RQ, \epsilon). \quad (4.57)$$

So

$$\frac{1}{R} \ln \text{sep}(T, RQ, \delta(\epsilon)) \geq \frac{1}{R} \ln \text{sep}(S, RQ, \epsilon), \quad (4.58)$$

and letting $R \rightarrow \infty$ yields

$$\eta_1(T, \delta(\epsilon)) \geq \eta_1(S, \epsilon). \quad (4.59)$$

Finally, letting $\epsilon \rightarrow 0$ gives

$$\eta_1(T) \geq \eta_1(S). \quad (4.60)$$

□

Corollary 4.14. *If S is topologically conjugate to T , then*

$$\eta_1(T) = \eta_1(S). \quad (4.61)$$

Proof: If S is topologically conjugate to T , then S is a factor of T and T is a factor of S . So applying Proposition 4.13 both ways yields (4.61). □

Proposition 4.15. *If $S^t = T^{\Lambda t}$, where $\Lambda : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is an injective linear map with $\lambda Q \subseteq \Lambda Q$, then*

$$\eta_1(S) \geq \lambda \eta_1(T). \quad (4.62)$$

Proof: First, observe that there exists $r_\Lambda \in \mathbb{R}$ such that for all $R > 0$ and $t \in \Lambda RQ \cap \mathbb{Z}^d$ there exists $t' \in \Lambda RQ \cap \Lambda \mathbb{Z}^d$ such that $|t - t'| < r_\Lambda$, where $|\cdot|$ is some norm on $\mathbb{Z}^d$.

By equicontinuity of $(T^e)_{|e| \leq r_\Lambda}$, for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$d(x, y) < \delta(\epsilon) \Rightarrow d(T^e x, T^e y) < \epsilon \quad \forall e \in \mathbb{Z}^d \cap B_{r_\Lambda}(0). \quad (4.63)$$

If $d(T^t x, T^t y) \geq \epsilon$ for some $t \in \Lambda RQ \cap \mathbb{Z}^d$, then $d(T^{t'} x, T^{t'} y) \geq \delta(\epsilon)$ for some $t' \in \Lambda RQ \cap \Lambda \mathbb{Z}^d$ with $|t - t'| < r_\Lambda$ by our observation. From the definition of S $d(S^{t''} x, S^{t''} y) \geq \delta(\epsilon)$ for the $t'' \in RQ \cap \mathbb{Z}^d$ that satisfies $\Lambda t'' = t'$. It follows that

$$\text{sep}(S, RQ, \delta(\epsilon)) \geq \text{sep}(T, \Lambda RQ, \epsilon). \quad (4.64)$$

Since $\lambda RQ \subseteq \Lambda RQ$, we have

$$\text{sep}(T, \Lambda RQ, \epsilon) \geq \text{sep}(T, \lambda RQ, \epsilon). \quad (4.65)$$

So

$$\frac{1}{R} \log \text{sep}(S, RQ, \delta(\epsilon)) \geq \lambda \frac{1}{\lambda R} \log \text{sep}(T, (\lambda R)Q, \epsilon). \quad (4.66)$$

Letting $R \rightarrow \infty$, we get

$$\eta_1(S, \delta(\epsilon)) \geq \lambda \eta_1(T, \epsilon). \quad (4.67)$$

Now let $\epsilon \rightarrow 0$ to get

$$\eta_1(S) \geq \lambda \eta_1(T) \quad (4.68)$$

as desired. □

Theorem 4.16. *Suppose T is a self-affine $\mathbb{R}^d$ -action (or $\mathbb{Z}^d$ -action) on a compact metric space $\mathbf{X}$. If $\eta_1(T) < \infty$, then*

$$\eta_1(T) = 0. \quad (4.69)$$

Proof: Let T be a self-affine $\mathbb{R}^d$ -action. By definition there exists an injective, linear, expanding map $\Lambda : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and a homeomorphism $\Phi : X \rightarrow X$ such that

$$\forall t \in \mathbb{R}^d \text{ (or } \mathbb{Z}^d), \quad \Phi \circ T^t = T^{\Lambda t} \circ \Phi. \quad (4.70)$$

Because Λ is expanding, there is $\lambda > 1$ and $n \in \mathbb{N}$ such that $\lambda Q \subseteq \Lambda^n Q$. Let $S^t := T^{\Lambda^n t}$. By (4.70) and Corollary 4.14 we get

$$\eta_1(T) = \eta_1(S). \quad (4.71)$$

We apply Proposition 4.15 to get

$$\eta_1(S) \geq \lambda \eta_1(T) \quad (4.72)$$

That is

$$\eta_1(T) \geq \lambda \eta_1(T) \quad (4.73)$$

with $\lambda > 1$. Since $\eta_1(T) < \infty$, we conclude that

$$\eta_1(T) = 0. \quad (4.74)$$

□ One needs to work quite a bit more to see that $\eta_1(T) < \infty$ for self-affine $\mathbb{R}^d$ -action T .

Theorem 4.17. *If T is a self-affine abstract tiling action, then $\eta_1(T) = 0$.*

Proof: By passing to an iterate of Φ if necessary, we can assume that there exists $\lambda > 1$ so that for all $t \in \mathbb{R}^d$ $|\Lambda t| \geq \lambda |t|$. Fix $\Delta \in (0, \Delta_0)$, where Δ_0 is the expansivity

constant from (WE). Let $\epsilon > 0$ be such that for all $\mathbf{y}_*, \mathbf{x} \in \mathbf{X}$

$$\mathbf{y}_* \in W_\epsilon^s(\mathbf{x}, \Phi) \implies \mathbf{y}_*|_{B_1}^{(\Delta/2)} \equiv \mathbf{x}|_{B_1}^{(\Delta/2)}. \quad (4.75)$$

That is, for all $m \in \mathbb{N}_0$ $d(\Phi^m \mathbf{y}_* + \Lambda^m t, \Phi^m \mathbf{x} + \Lambda^m t) < \Delta/2$ for all $|t| < 1$. Let $\epsilon_1 > 0$ be as provided by Lemma 2.3 in [20]. Let $\delta_1 > 0$ be so that

$$|t| < \delta_1 \implies \forall_{\mathbf{x} \in \mathbf{X}} d(\mathbf{x}, T^t \mathbf{x}) < \min\{\epsilon_1/2, \Delta/2\}. \quad (4.76)$$

Upon fixing $\mathbf{z}_0 \in \mathbf{X}$ and taking $R_0 > 0$ large enough, the orbit piece $\Sigma := \mathbf{z}_0 + B_{R_0}$ is so dense in $\mathbf{X}$ that it intersects $W_\epsilon^s(\mathbf{x}, \Phi) \cap B_{\epsilon_1/2}(\mathbf{x})$ for every $\mathbf{x} \in \mathbf{X}$. This follows by minimality and the local product structure at every $\mathbf{x}$.

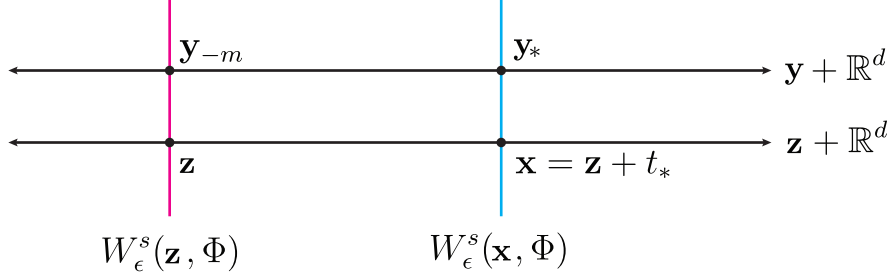
Consider any $R > 1$. Take $m := \left\lceil \frac{\ln R}{\ln \lambda} \right\rceil \in \mathbb{N}$. Select $D \subseteq B_{R_0} \subseteq \mathbb{R}^d$ that is $\|\Lambda\|^{-m} \delta_1$ -dense and of cardinality $\#D \leq C_0 \|\Lambda\|^m$ where $C_0 = C_0(\delta_1, R_0)$. Set $E := \mathbf{z}_0 + D$.

Claim 4.18. $\Phi^m(E)$ is $(T, \frac{R}{\sqrt{d}}Q, \Delta) - \widetilde{\text{spanning}}$.

Proof of Claim: Consider any $\mathbf{y} \in \mathbf{X}$. Set $\mathbf{y}_{-m} := \Phi^{-m}(\mathbf{y})$. Take $\mathbf{z} \in B_{\epsilon_1/2}(\mathbf{y}_{-m}) \cap W_\epsilon^s(\mathbf{y}_{-m}) \cap \Sigma$. Take t_* with $|t_*| < \|\Lambda\|^{-m} \delta_1$ so that $\mathbf{x} := \mathbf{z} + t_* \in E$. Since $|t_*| < \delta_1$, then $d(\mathbf{x}, \mathbf{z}) < \epsilon_1/2$. Because $d(\mathbf{y}_{-m}, \mathbf{x}) \leq d(\mathbf{y}_{-m}, \mathbf{z}) + d(\mathbf{z}, \mathbf{x}) < \epsilon_1/2 + \epsilon_1/2 < \epsilon_1$ we can use Lemma 2.3 in [20] to secure $\mathbf{y}_* := [\mathbf{x}, \mathbf{y}_{-m}]$; i.e., $\{y_*\} = W_\epsilon^s(\mathbf{x}, \Phi) \cap \mathbf{y}_{-m} + B_{r_0}$. Note that $\mathbf{y}_* = \mathbf{y}_{-m} + t_*$ (see Figure 4.2). This used Remark 2.4 in [20].

By choice of ϵ and $\mathbf{y}_* \in W_\epsilon^s(\mathbf{x}, \Phi)$ gives

$$d(\Phi^m \mathbf{y}_* + \Lambda^m t, \Phi^m \mathbf{x} + \Lambda^m t) < \Delta/2 \quad \forall_{|t| < 1}. \quad (4.77)$$

Figure 4.2: $\mathbf{y}_*$ from the local product structure

Also

$$\begin{aligned}
 d(\Phi^m \mathbf{y}_* + \Lambda^m t, \Phi^m \mathbf{y}_{-m} + \Lambda^m t) &= d(\Phi^m \mathbf{y}_{-m} + \Lambda^m t_* + \Lambda^m t, \mathbf{y} + \Lambda^m t) \\
 &= d(\mathbf{y} + \Lambda^m t + \Lambda^m t_*, \mathbf{y} + \Lambda^m t) \\
 &\leq \Delta/2
 \end{aligned} \tag{4.78}$$

because $|\Lambda^m t_*| < \|\Lambda^m\| \|\Lambda\|^{-m} \delta_1 = \delta_1$ and by the choice of δ_1 . Putting together (4.77) and (4.78) gives

$$d(\mathbf{y} + \Lambda^m t, \Phi^m \mathbf{x} + \Lambda^m t) < \Delta/2 + \Delta/2 = \Delta \quad \forall |t| < 1. \tag{4.79}$$

Thus $d(\mathbf{y} + \tau, \Phi^m \mathbf{x} + \tau) < \Delta$ for all $|\tau| < \lambda^m$. Since $\frac{1}{\sqrt{d}} R Q \subseteq B_R \subseteq B_{\lambda^m}$ we get $\tilde{d}_{\frac{R}{\sqrt{d}} Q}^T(\Phi^m \mathbf{x}, \mathbf{y}) < \Delta$. By arbitrariness of $\mathbf{y}$, $\Phi^m(E)$ is $(T, \frac{R}{\sqrt{d}} Q, \Delta) - \widetilde{\text{spanning}}$. $\square$

From Claim 4.18 we have, for all sufficiently small $\Delta > 0$ and all $R > 0$,

$$\begin{aligned}
 \widetilde{\text{span}}(T, \frac{R}{\sqrt{d}} Q, \Delta) &\leq \sharp \Phi^m(E) = \sharp E = \sharp D \leq C_0 \|\Lambda\|^m \\
 &\leq C_0 e^{\frac{\ln \|\Lambda\|}{\ln \lambda} \cdot m \ln \lambda} \\
 &= C_0 (\lambda^m)^{\frac{\ln \|\Lambda\|}{\ln \lambda}} \leq C_0 R^\nu
 \end{aligned}$$

where $\nu := \frac{\ln \|\Lambda\|}{\ln \lambda} \in [1, \infty)$. Thus

$$\lim_{R \rightarrow \infty} \frac{1}{R} \ln \widetilde{\text{span}}(\mathbf{T}, \frac{R}{\sqrt{d}}Q, \Delta) \leq \lim_{R \rightarrow \infty} \frac{1}{R} \ln C_0(\sqrt{d}R)^\nu = 0. \quad (4.80)$$

By the standard relationship between separated and spanning sets we have

$$\lim_{R \rightarrow \infty} \frac{1}{R} \ln \widetilde{\text{sep}}(\mathbf{T}, \frac{R}{\sqrt{d}}Q, \Delta) = 0 \text{ as well. Taking } \Delta \searrow 0 \text{ gives } \eta_1(\mathbf{T}) = 0. \quad \square$$

Remark 4.19. *We have shown a much stronger statement than $\eta_1(\mathbf{T}) = 0$. The growth of $\widetilde{\text{span}}(\mathbf{T}, \frac{R}{\sqrt{d}}Q, \Delta)$ and $\widetilde{\text{sep}}(\mathbf{T}, \frac{R}{\sqrt{d}}Q, \Delta)$ is at worst the power function R^ν for any given Δ . This is a well known fact for tiling spaces of concrete self-affine tilings [22, 28].*

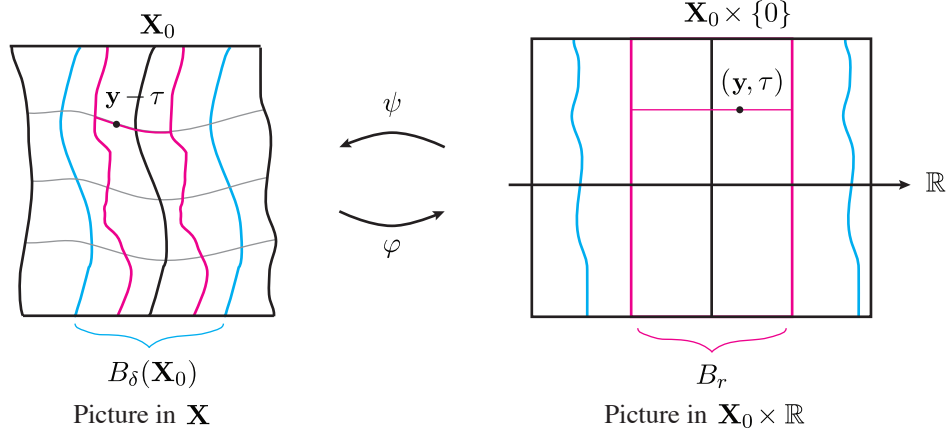
Proof of the Local Structure Theorem

Theorem 4.20 (Theorem 1.7). *If $\mathbf{T}$ is a minimal abstract self-affine tiling action on a compact metric space $\mathbf{X}$ with a good cross-section $\mathbf{X}_0$, then $\mathbf{X}$ is locally homeomorphic to the product of $\mathbb{R}^d$ and a Cantor set.*

Proof. Define a continuous function $\psi : \mathbf{X}_0 \times \mathbb{R} \rightarrow \mathbf{X}$ by $\psi(\mathbf{y}, \tau) := \mathbf{y} - \tau$. (TRA) provides $\delta > 0$ so that

$$\mathbf{x} \in B_\delta(\mathbf{X}_0) \implies D_{\mathbf{x}} \cap B_{r_D} = \{\tau = \tau(\mathbf{x})\}. \quad (4.81)$$

The map $\varphi : B_\delta(\mathbf{X}_0) \rightarrow \mathbf{X}_0 \times B_{r_D}$ given by $\varphi(\mathbf{x}) = (\mathbf{x} + \tau(\mathbf{x}), \tau(\mathbf{x}))$ is continuous since the map $\mathbf{x} \mapsto D_{\mathbf{x}}$ is continuous by Proposition 2.27. By uniform continuity, taking $r > 0$ small, $\psi(\mathbf{X}_0 \times B_r) \subseteq B_\delta(\mathbf{X}_0)$. By construction, $\psi \circ \varphi = id_{B_\delta(\mathbf{X}_0)}$ and $\varphi \circ \psi = id_{\mathbf{X}_0 \times B_r}$. It follows that the sets $U := \mathbf{X}_0 \times B_r$ and $V := \psi(U)$ are open and $\psi|_U : U \rightarrow V$ is a homeomorphism.

Figure 4.3: Local homeomorphism on $\mathbf{X}$

It remains to show that $\mathbf{X}_0$ is a Cantor set; i.e., compact, totally disconnected and perfect. $\mathbf{X}_0$ is compact as a closed subset of a compact set. By Theorem 4.16 $\eta_1(\mathbf{T}) = 0$. By Theorem 4.10 $\text{HD}_{d_{\text{exp}}}(\mathbf{X}_0) \leq b \cdot \frac{\eta_1(\mathbf{T})}{\ln k} = 0$. Thus $\mathbf{X}_0$ is totally disconnected. To show that $\mathbf{X}_0$ is perfect take $x \in \mathbf{X}_0$. By minimality, there exists a sequence $(t_k)_{k \in \mathbb{N}} \subset \mathbb{R}^d$ so that $\lim_{k \rightarrow \infty} \mathbf{x} + t_k = \mathbf{x}$. Let $\mathbf{x}_k := \mathbf{x} + t_k \in \mathbf{X}$. Again, (TRA) provides $\delta > 0$ so that

$$\mathbf{x}_k \in B_\delta(\mathbf{X}_0) \implies D_{\mathbf{x}} \cap B_{r_D} = \{\tau_k = \tau(\mathbf{x}_k)\}. \quad (4.82)$$

So for this $\delta > 0$ there exists $N \in \mathbb{N}$ such that $\mathbf{x}_k \in B_\delta(\mathbf{X}_0)$ for all $k > N$. Since $\mathbf{x} \mapsto D_{\mathbf{x}}$ is continuous and $\tau(\mathbf{x}) = 0$, letting $\mathbf{x}'_k := \mathbf{x}_k + \tau_k \in \mathbf{X}_0$ for all $k > N$, we have $\lim_{k \rightarrow \infty} \mathbf{x}'_k = \mathbf{x}$. By aperiodicity $\mathbf{x}'_k \neq \mathbf{x}$ (for all k). Thus $\mathbf{x}$ is an accumulation point in $\mathbf{X}_0$. $\square$

A NON-FLC SELF-AFFINE TILING OF $\mathbb{R}^2$ Introduction

We now turn our attention to a 2-dimensional substitution tiling with horizontal fault lines, that was first considered in [12]. We are interested in this example because the local structure is more complicated than the usual product of $\mathbb{R}^2$ and a Cantor set. Our main goal is to flesh out all the details necessary to complement the somewhat informal exposition in [12]. Tilings in this space will have essentially the same rows, up to horizontal translation, which is governed by the 1-dimensional substitution tiling of the row and its coding.

Consider the rectangular prototiles a and b , where both have height 1 and widths $\lambda_1 = \frac{1 + \sqrt{13}}{2}$ and 3, respectively. Now consider the self-affine substitution Φ described in Figure 5.1.



Figure 5.1: Frank-Sadun substitution rule

For any $n \in \mathbb{N}$, $\Phi^n(a)$ and $\Phi^n(b)$ are called *n th-order super-prototiles*. A tiling of $\mathbb{R}^2$ is *allowed* by the substitution Φ if for any finite patch P in the tiling, there exists $n \in \mathbb{N}$ such that a translated copy of P is contained in some n th-order super-prototile. Let $\mathcal{T}$ be the set of all tilings of $\mathbb{R}^2$ with prototiles a and b . The *substitution tiling space*, $\mathcal{T}_\Phi$, is the smallest closed set containing all allowed tilings. Consider the open-compact topology defined in Chapter 2 in which, roughly speaking, two tilings are ϵ -close if they contain the same tiles out to a distance of $1/\epsilon$ and if each tile in the first tiling is within a translation of size $\leq \epsilon$ of the corresponding tile in the second

tiling [12, 21]. Note that we allow the translation to depend on the tile. In the case of tilings with FLC (roughly, finitely many local pictures) one can require that there is a single ϵ -translation for all the tiles out to a distance $1/\epsilon$. Under this topology, $\mathcal{T}_\Phi$ is compact and the $\mathbb{R}^2$ -action on $\mathcal{T}_\Phi$ induced by translating tilings is minimal [12, 31, 34].

It is shown that the FLC hypothesis implies that the tiling action satisfies (WE). In addition to describing the local structure, we show that tilings associated to this particular substitution fail to have FLC and that the action is not weakly expansive as stated in the following two propositions.

Proposition 5.1. *Tilings in $\mathcal{T}_\Phi$ do not have FLC.*

Proposition 5.2. *The tiling action on $\mathcal{T}_\Phi$ does not satisfy (WE).*

The culprit will be a phenomenon of the fault line. Following [12], we say that a tiling $x \in \mathcal{T}_\Phi$ exhibits a *fault line* if and only if there is a line, ℓ , that is a union of edges of tiles in $\mathbf{x}$ with direction vector $v \in \mathbb{R}^2$ satisfying the following property. For every $\epsilon > 0$ there exists a tiling $\mathbf{x}' \in \mathcal{T}_\Phi$ such that:

- (1) on one half of the plane with boundary ℓ , $\mathbf{x}' = \mathbf{x}$, and
- (2) there exists $t \in (0, \epsilon]$ such that on the other half of the plane with boundary ℓ ,
 $\mathbf{x} = \mathbf{x}' \pm tv$.

In our case, ℓ will be a horizontal line and so $v = (1, 0)$.

We remark that if some tiling $\mathbf{x}_0 \in \mathcal{T}_\Phi$ contains a fault line, then FLC fails for every $\mathbf{x} \in \mathcal{T}_\Phi$. To see this consider any tiling $\mathbf{x} \in \mathcal{T}_\Phi$ and assume it has FLC. Since $\mathbf{x}$ has FLC, then $\mathcal{T}_\Phi := cl\{\mathbf{x}_0 + t : t \in \mathbb{R}^2\}$ is compact if the closure is taken with the standard topology described in Chapter 2. Since $\mathbf{x}_0$ contains a fault line, for some $R > 0$ and any $\epsilon > 0$ there exists a patch of diameter less than R containing two tiles abutting along ℓ with edges offset by $t \in (0, \epsilon]$ (see Figure 5.2). Then such a patch

also appears in $\mathbf{x}$ up to translation since $\mathcal{T}_\Phi$ by minimality of the translation action. Because ϵ is arbitrary, $\mathbf{x}$ contains infinitely many distinct patches with diameter less than R , thus violating the FLC hypothesis for $\mathbf{x}$.

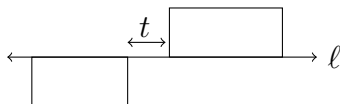


Figure 5.2: Arbitrary offset of tiles along the fault line

In pursuance of the local characterization of $\mathcal{T}_\Phi$, we introduce some preliminary terms and notation. We say a tiling $\mathbf{x}$ is *aligned* if the x -axis is made of tile edges. We consider a *row* r in any tiling $\mathbf{x}$ to be the collection of tiles in $\mathbf{x}$ whose union of interiors contains a horizontal line. We index the rows according to its vertical position where r_0 denotes the row whose union of interiors and bottom edges contain the x -axis. By a vertical translation (which induces a homeomorphism of $\mathcal{T}_\Phi$) any tiling in $\mathcal{T}_\Phi$ can be placed in the subset of aligned tilings denoted

$$\mathcal{T}_a := \{\mathbf{x} \in \mathcal{T}_\Phi : \mathbf{x} \text{ aligned} \}.$$

Because small neighborhoods in $\mathcal{T}_\Phi$ of a point $\mathbf{x} \in \mathcal{T}_a$ are homeomorphic to the Cartesian product $\mathbb{R} \times U$ where U is a small neighborhood of $\mathbf{x}$ in $\mathcal{T}_a$, we investigate the local structure of $\mathcal{T}_a$. Note that $\mathcal{T}_a$ comes with the $\mathbb{R}$ -action induced by the horizontal translations.

Our main tool will be a coding procedure associating to any $\mathbf{x} \in \mathcal{T}_a$ an element $c(\mathbf{x}) \in \mathcal{G} := \{0, 1\}^{\mathbb{N}}$ (essentially the group of 2-adic integers). Note that we will write the elements of $\mathcal{G}$ as infinite words, i.e., infinite concatenations of 0's and 1's (with

no intervening commas). The subset $\mathcal{G}_0^+$ consisting of words that terminate with $0^\infty := 0000\dots$ is a dense subgroup. Identifying its element $d_1d_2\dots d_N0^\infty$ with the natural number $d_1+d_22+\dots+d_N2^{N-1}$ establishes a group isomorphism $\mathcal{G}_0 \rightarrow \mathbb{N}_0$. We shall use $f : \mathcal{G} \rightarrow \mathcal{G}$ for the translation of $\mathcal{G}$ by $\mathbf{1} := 10^\infty$ with carry to the right, which is often referred to as the *adding machine*. (For example, if $c = 1100\dots \in \mathcal{G}$, then $f(c) = c + \mathbf{1} = 1100\dots + 100\dots = 00100\dots$). $\mathcal{G}_0^+$ is the forward orbit of 0^∞ under the action of f , $G_0^+ = \{f^k(0^\infty) : k \in \mathbb{N}_0\}$. The backward orbit is $G_0^- = \{f^k(0^\infty) : -k \in \mathbb{N}\}$, where $k = 0$ is excluded. It consists of all the sequences that terminate in 1^∞ and thus is also dense in $\mathcal{G}$. For instance $f^{-1}(0^\infty) = 1^\infty$ because $1^\infty + 10^\infty = 0^\infty$. Denote by $\mathcal{G}_0$ the bi-infinite orbit of 0^∞ under f , $\mathcal{G}_0 := \mathcal{G}_0^- \cup \mathcal{G}_0^+ = \{f^k(0^\infty) : k \in \mathbb{Z}\}$. The mapping $\mathbb{Z} \ni k \mapsto f^k(0^\infty) \in \mathcal{G}_0$ is an isomorphism of groups. We postpone the details of the construction of $c(\mathbf{x})$ as the code of the 0th row of $\mathbf{x}$ and only note that it depends continuously on $\mathbf{x}$.

One way to explain the local structure of $\mathcal{T}_a$ is to first embed it into the big product space

$$\mathcal{Y} := \mathcal{G} \times \prod_{k \in \mathbb{Z}} \mathcal{T}_\phi.$$

The embedding is simply given by $\mathbf{x} \mapsto (c(\mathbf{x}), (r_k)_{k \in \mathbb{Z}})$ where r_k is the k -th row of $\mathbf{x}$ thought of as a tiling of the real line. (This map is obviously a homeomorphism onto its image because already $\mathbf{x} \mapsto (\mathbf{x}_k)_{k \in \mathbb{Z}}$ is such and $c(\cdot)$ is continuous. From now on consider $\mathcal{T}_\Phi$ as identified with the image of this embedding. In particular, in the rest of this section, when we refer to a metric on $\mathcal{T}_\Phi$ it will be the restriction of the following metric on $\mathcal{Y}$:

$$d_{\mathcal{Y}}((c, (r_k)), (c', (r'_k))) := \max \left\{ d_{\mathcal{G}}(c, c'), \max_{k \in \mathbb{Z}} d_{\mathcal{T}_\phi}(r_k, r'_k) / 2^{|k|} \right\}.$$

Now, $\mathcal{Y}$ is a huge space and $\mathcal{T}_\Phi$ sits in it only as a small subspace and will be captured by examining its projections onto at most three factors of in the infinite cartesian product defining $\mathcal{Y}$. Let then $\pi_{\mathcal{G}} : \mathcal{Y} \rightarrow \mathcal{G}$ and $\pi_k : \mathcal{Y} \rightarrow \mathcal{T}_\phi$ be the projections $\pi_{\mathcal{G}}(c, (r_k)) := c$ and $\pi_k(c, (r_k)) := r_k$. The main role will be played by the projection $\pi_{\mathcal{G},0}(c, (r_k)) := (c, r_0)$ and the fibers of its restriction to $\mathcal{T}_\Phi$, for which we reserve the notation

$$Z(c, r) := \pi_{\mathcal{G},0}^{-1}(c, r) \cap \mathcal{T}_a \quad ((c, r) \in \mathcal{G} \times \mathcal{T}_\phi).$$

We write $\mathcal{T}_a$ as a disjoint union $\mathcal{T}_a = \mathcal{T}_g \cup \mathcal{T}_b$ where

$$\mathcal{T}_g := \pi_{\mathcal{G},0}^{-1}(\mathcal{G} \setminus \mathcal{G}_0 \times \mathcal{T}_\phi) \quad \text{and} \quad \mathcal{T}_b := \pi_{\mathcal{G},0}^{-1}(\mathcal{G}_0 \times \mathcal{T}_\phi).$$

Theorem 5.3. *In the disjoint union $\mathcal{T}_a = \mathcal{T}_g \cup \mathcal{T}_b$, both $\mathcal{T}_g$ and $\mathcal{T}_b$ are dense. Moreover:*

- (i) $Z(c, r)$ is a point for $(c, r) \in \mathcal{G} \setminus \mathcal{G}_0 \times \mathcal{T}_\phi$, and the restriction $\pi_{\mathcal{G},0}|_{\mathcal{T}_g} : \mathcal{T}_g \rightarrow \mathcal{G} \setminus \mathcal{G}_0 \times \mathcal{T}_\phi$ is a homeomorphism;
- (ii) $Z(c, r)$ is homeomorphic to $\mathcal{T}_\phi$ for $(c, r) \in \mathcal{G}_0 \times \mathcal{T}_\phi$;
- (iii) if $(c^{(n)}, r^{(n)}) \subset \mathcal{G}_0 \times \mathcal{T}_\phi$ and $(c^{(n)}, r^{(n)}) \rightarrow (c, r) \in \mathcal{G} \setminus \mathcal{G}_0 \times \mathcal{T}_\phi$ then $Z(c^{(n)}, r^{(n)}) \rightarrow Z(c, r)$; (Note that $Z(c, r)$ is just a point here.)
- (iv) if $(c^{(n)}) \subset \mathcal{G}_0$ and $c^{(n)} \rightarrow c \in \mathcal{G}$ with $c^{(n)} \neq c$ then the fibers $Z(c, r)$ shrink to a point, i.e., $\lim_{n \rightarrow \infty} \sup_{r \in \mathcal{T}_\phi} \text{diam}(Z(c^{(n)}, r^{(n)})) = 0$;
- (v) $\forall_{x \in \mathcal{T}_b} \exists_{(c^{(n)}, r^{(n)}) \subset \mathcal{G}_0 \times \mathcal{T}_\phi} Z(c^{(n)}, r^{(n)}) \rightarrow \{x\}$;
- (vi) $\forall_{x \in \mathcal{T}_b} \exists_{(c^{(n)}, r^{(n)}) \subset \mathcal{G} \setminus \mathcal{G}_0 \times \mathcal{T}_\phi} Z(c^{(n)}, r^{(n)}) \rightarrow \{x\}$;

Corollary 5.4. $\mathcal{T}_g$ is locally homeomorphic to $\mathbb{R} \times V$ where V is the Cantor set with countably many points removed. (In fact, $\mathbb{R} \times \text{Cantor} \times U$ where U is an open subset of $\mathcal{G} \setminus \mathcal{G}_0$.) Inside any non-empty open set in $\mathcal{T}_a$ there sits an embedded copy of $\mathcal{T}_\phi$.

Corollary 5.5. There is no open subset of $\mathcal{T}_a$ that is homeomorphic to $\mathbb{R} \times \text{Cantor}$.

Analysis of a Pair of 1-Dimensional Tilings

To begin our analysis of the 2-dimensional tiling space defined above, we first analyze the associated pair of 1-dimensional substitution tilings. We can do this because what happens on either side of the fault line can be described by a pair of 1-dimensional substitutions. Consider the substitutions $\phi_1(a) = ba$, $\phi_1(b) = aaa$ and $\phi_2(a) = ab$, $\phi_2(b) = aaa$. Both substitutions have the abelianization matrix $A = \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}$ with Perron-Frobenius eigenvalue $\lambda_1 = \frac{1+\sqrt{13}}{2} \approx 2.3$ and the remaining eigenvalue $\lambda_2 = \frac{1-\sqrt{13}}{2} \approx -1.3$. A bi-infinite word $w = \dots w_{-2}w_{-1}.w_0w_1\dots$ is called admissible if for each finite sub-word w_k there exists $n \in \mathbb{N}$ such that w_k is contained in $\phi_i^n(a)$. Let a and b be 1-dimensional prototiles with lengths λ_1 and 3, respectively. The tiling space $\mathcal{T}_{\phi_i}$ consists of all tilings of $\mathbb{R}$ by translates of the prototiles a and b with the property that any finite connected patch of tiles has the associated word that is admissible.

One-dimensional tilings spaces can be modeled and analyzed in the “strand space,” which is introduced in [1]. We will use this convenient realization to prove Lemmas 5.8 and 5.9, so we recall it now.

Let ω_i denote the eigenvector for eigenvalue λ_i of the substitution matrix A . Set $E_i = \text{lin}(\omega_i)$ so that $\mathbb{R}^2 = E_1 \oplus E_2$. The projection of a point p onto E_1 in the direction of E_2 is given by $pr_1(p) := \langle p, \omega_1^* \rangle \omega_1$, where ω_1^* is the Perron eigenvector of the adjoint A^T normalized so that $\langle \omega_1, \omega_1^* \rangle = 1$. Then the length of the projection onto E_1 is

given by $|p|_1 = \langle p, \omega_1 \rangle$. Define $pr_2(p)$ similarly and the length of the projection onto E_2 is given by $|p|_2 = \langle p, \omega_2 \rangle$.

Let e_i be the i th unit basis vector in $\mathbb{R}^2$ and $I_i := \{te_i : t \in [0, 1]\}$ be the i th unit line segment on the corresponding axis. $I_1 = \{te_1 : t \in [0, 1]\}$ will be associated to a tile of type a and $I_2 = \{te_2 : t \in [0, 1]\}$ will be associated to a tile of type b . The term *strand* refers to an oriented broken line $\gamma = \{I_{i_k} + x_k\}$, $x_k \in \mathbb{R}^2$, consisting of translated copies of the basic segments meeting tip-to-tail in a connected union. For example, a strand associated to the word $aaaba$ is pictured in figure 5.3.

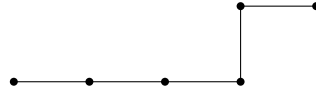


Figure 5.3: A strand associated to $aaaba$

The space of all bi-infinite strands in $\mathbb{R}^2$ is denoted by $\mathcal{F}$. The substitution ϕ_i induces a map ϕ_i that takes edges to strands. More precisely, the image of the edge $x + I_i$ is given by $\phi_i(x + I_i) = Ax + I_j$. The map on edges ϕ_i naturally extends to an inflation-substitution map on strands edge by edge; i.e., $\phi_i(\gamma)$ is the strand obtained by concatenating the strands $\phi_i(e)$ where e is an edge of γ .

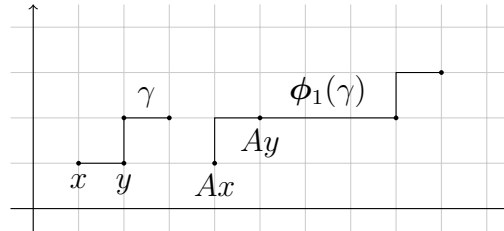


Figure 5.4: ϕ_1 applied to the strand γ

Strands up to translation along E_2 are synonymous with tilings via the projection pr_1 onto E_1 (see figure 5.5). We define the space of all tilings with prototiles a and b

as $\mathcal{T} := \{\gamma \in \mathcal{F} : 0 \in \gamma\} \equiv \mathcal{F}/E_1$ with the tiling topology. As expected, $\mathcal{T}$ is compact and ϕ induces a self-map on $\mathcal{T}$, which we also denote by ϕ .

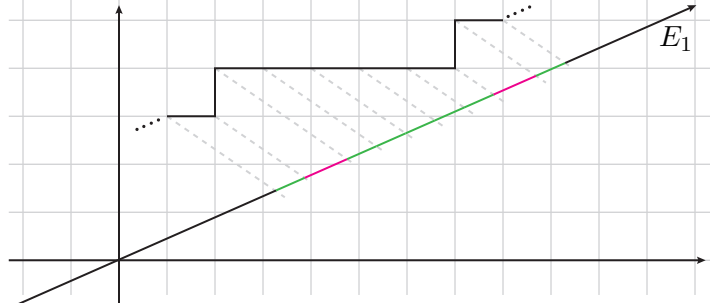


Figure 5.5: A tiling produced by projecting a strand onto E_1

Given a substitution ϕ , we define the big tiling space associated to ϕ as the global attractor of $\phi : \mathcal{T} \mapsto \mathcal{T}$:

$$\mathcal{T}_\phi^{\text{big}} := \bigcap_{n \in \mathbb{N}} \phi^n(\mathcal{T}). \quad (5.1)$$

This means that $\gamma \in \mathcal{T}_\phi^{\text{big}}$ iff $\forall n \in \mathbb{N} \exists \gamma_{-n} \in \mathcal{T}$ with $\gamma = \phi^n(\gamma_{-n})$. This space is actually larger than the $\mathcal{T}_\phi$ defined at the beginning of this section because it may include a finite number of additional orbits (the so called wandering orbits of the *welded strands* described in [BK]). When we eliminate these orbits from $\mathcal{T}_\phi^{\text{big}}$, we get the subset $\mathcal{T}_\phi$, which is the unique minimal set under translation and is equal to the original tiling space associated to the substitution ϕ . For the tiling we are considering $\mathcal{T}_{\phi_i}^{\text{big}} = \mathcal{T}_{\phi_i}$ because any central substrand of length 2 in $\mathcal{F}_{\phi_i}$ have words aa , ab , ba , all of which are subwords of $\phi_i^2(a)$, so there are no welded strands.

Through the next three lemmas we will see that the tiling spaces $\mathcal{T}_{\phi_1}$ and $\mathcal{T}_{\phi_2}$ are the same space. First, consider the ϕ_1 and ϕ_2 on the level of words.

Lemma 5.6. *For any word w of finite length, $\phi_2(w)$ is a cyclic permutation of $\phi_1(w)$ obtained by removing a from the end of the word $\phi_1(w)$ and replacing it at the beginning.*

Proof: Let $\hat{\phi}_1$ be a map on words of finite length obtained by first applying ϕ_1 to the word and then removing a from the right end and placing it at the left end. For example, $\hat{\phi}_1(a) = \hat{b}a = ab$. Then Lemma 5.6 says that for any word w , $\phi_2(w) = \hat{\phi}_1(w)$. We will prove this lemma by induction on the length of w .

Suppose w is a word of length 1. Then either $w = a$ or $w = b$. If $w = a$, then $\hat{\phi}_1(w) = \hat{b}a = ab = \phi_2(w)$. If $w = b$, then $\hat{\phi}_1(w) = a\hat{a}a = aaa = \phi_2(w)$.

Assume for any word w of length n , we have $\hat{\phi}_1(w) = \phi_2(w)$. Let w be a word of length $n + 1$. Then there exists a word $\tilde{w}$ of length n such that $w = a\tilde{w}$ or $w = b\tilde{w}$. If $w = a\tilde{w}$, then $\phi_2(w) = \phi_2(a)\phi_2(\tilde{w}) = ab\hat{\phi}_1(\tilde{w}) = \hat{\phi}_1(a)\hat{\phi}_1(\tilde{w}) = \hat{\phi}_1(a\tilde{w}) = \hat{\phi}_1(w)$. If $w = b\tilde{w}$, then $\phi_2(w) = \phi_2(b)\phi_2(\tilde{w}) = aaa\hat{\phi}_1(\tilde{w}) = \hat{\phi}_1(b)\hat{\phi}_1(\tilde{w}) = \hat{\phi}_1(b\tilde{w}) = \hat{\phi}_1(w)$. Thus for any word w , $\phi_2(w) = \hat{\phi}_1(w)$. $\square$

Lemma 5.7. *If w is a bi-infinite word, the bi-infinite word $\phi_2(w)$ is the right shift of the word $\phi_1(w)$ with a being the first letter after the period in $\phi_2(w)$. Interpreting w as a tiling $\mathbf{w}$, the tiling $\phi_2(\mathbf{w})$ associated to the word $\phi_2(w)$ is obtained by translating the tiling $\phi_1(\mathbf{w})$ associated to the word $\phi_1(w)$ to the right by the distance equal to the length of an a tile, $|a| = \lambda_1$.*

Proof: Let $w = (w_k)_{-\infty}^{\infty}$, $w_k \in \mathcal{A}$, be an arbitrary bi-infinite word. Denote the finite subword $\phi_i(w_k)$ with the last a removed by $\phi_i(w_k) \setminus a$. Then

$$\begin{aligned}
\phi_2(w) &= (\phi_2(w_k))_{-\infty}^{\infty} = \left(\widehat{\phi_1(w_k)} \right)_{-\infty}^{\infty} \\
&= \dots \widehat{\phi_1(w_{-2})} \widehat{\phi_1(w_{-1})} \widehat{\phi_1(w_0)} \widehat{\phi_1(w_1)} \dots \\
&= \dots a(\phi_1(w_{-2}) \setminus a) a(\phi_1(w_{-1}) \setminus a) \dots a(\phi_1(w_0) \setminus a) a(\phi_1(w_1) \setminus a) \dots \\
&= \dots a\phi_1(w_{-2})(\phi_1(w_{-1}) \setminus a) \dots a\phi_1(w_0)\phi_1(w_1) \dots \\
&= \phi_1(w) \quad \text{shifted once to the right.}
\end{aligned}$$

In terms of the 1-dimensional tilings, we have $\phi_2(\mathbf{w}) = \phi_1(\mathbf{w}) + \lambda_1$. □

Lemma 5.8. *The tiling spaces $\mathcal{T}_{\phi_1}$ and $\mathcal{T}_{\phi_2}$ are the same space.*

Proof: Define the bi-infinite word $w_1 := \lim_{n \rightarrow \infty} \phi_1^{(2n)}(a.a)$. Then $\phi_1^2(w_1) = w_1$. From Lemma 5.7, on the level of tilings we have $\phi_2(\mathbf{w}_1) = \phi_1(\mathbf{w}_1) + |a|$. Let γ_1 be the bi-infinite strand associated the word w_1 , i.e. $\gamma_1 = \lim_{n \rightarrow \infty} \phi_1^{2n}(I_1 \cup I_1)$, where $(I_1 \cup I_1)$ is centered at the origin. Then $\phi_2(\gamma_1) = \phi_1(\gamma_1) + |a|$. Set $t_0 := \frac{|a|}{1-\lambda_1} \in \mathbb{R}$. Then

$$\begin{aligned}
\phi_2^2(\gamma_1 + t_0) &= \phi_2(\phi_2(\gamma_1 + t_0)) \\
&= \phi_2(\phi_1(\gamma_1) + \lambda_1 t_0 + |a|) \\
&= \phi_1^2(\gamma_1) + \lambda_1^2 t_0 + \lambda_1 |a| + |a| \\
&= \gamma_1 + \lambda_1^2 t_0 + \lambda_1 |a| + |a| \\
&= \gamma_1 + \frac{|a|}{1 - \lambda_1} \\
&= \gamma_1 + t_0
\end{aligned}$$

In other words, $\gamma_1 + t_0$ is fixed by ϕ_2^2 . Therefore $\gamma_1 + t_0 \in \mathcal{T}_{\phi_2}^{\text{big}} = \mathcal{T}_{\phi_2}$. By translation invariance of $\mathcal{T}_{\phi_2}$, we get $\gamma_1 \in \mathcal{T}_{\phi_2}$. Finally, $\mathcal{T}_{\phi_2} = \mathcal{T}_{\phi_1}$ by minimality, since $\gamma_1 \in \mathcal{T}_{\phi_1}$ and $\gamma_1 \in \mathcal{T}_{\phi_2}$. We have just shown the tiling spaces defined by substitutions ϕ_1 and ϕ_2 are the same space. $\square$

Starting with a pair of a tiles both beginning at 0 with one above the other, $\begin{pmatrix} a \\ a \end{pmatrix}$, and applying ϕ_1 on the lower edges and ϕ_2 on the upper edges successively, we get patches of tiles described by the following pairs of words:

$$\alpha_1 = \begin{pmatrix} ba \\ ab \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} aaaba \\ abaaa \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} bababaaaaba \\ abaaaababab \end{pmatrix}, \quad \dots \quad \alpha_n = \begin{pmatrix} \phi_1^n(a) \\ \phi_2^n(a) \end{pmatrix}.$$

Notice at any given point along the horizontal fault line, there is a discrepancy between the number of a tiles in the top row and the number of a tiles in the bottom row. We define the discrepancy of a tiles between the top and bottom rows of the n th iterate up to the point $\tau \in \mathbb{R}$ by

$$\begin{aligned} a^{(n)}(\tau) := & (\text{the number of } a \text{ tiles in the top row of } \alpha_n \text{ entirely contained in } [0, \tau]) \\ & - (\text{the number of } a \text{ tiles in the bottom row of } \alpha_n \text{ entirely contained in } [0, \tau]). \end{aligned} \tag{5.2}$$

Note that $a^{n+2}(\cdot)$ is an extension of $a^n(\cdot)$, since the pair of words α_n is contained in the pair of words α_{2n} . That means $a^{2n}(\tau) = a^n(\tau)$ for $\tau \in \mathbb{R}$, provided both are defined.

Lemma 5.9. *The discrepancy $a^{(n)}(\tau)$ is unbounded; i.e. $\sup_{n, \tau} a^{(n)}(\tau) = +\infty$.*

Proof: Again we will use strands to prove this lemma. Denote the strand corresponding to the word $\phi_i^{(n)}(a)$ by $\gamma_i^{(n)}$. Define the maximum distance between

strands $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$ as

$$D_n := \max \left\{ |p_1^{(n)} - p_2^{(n)}|_2 : p_i^{(n)} \in \gamma_i^{(n)} \text{ and } p_1^{(n)} - p_2^{(n)} \in E_2 \right\}. \quad (5.3)$$

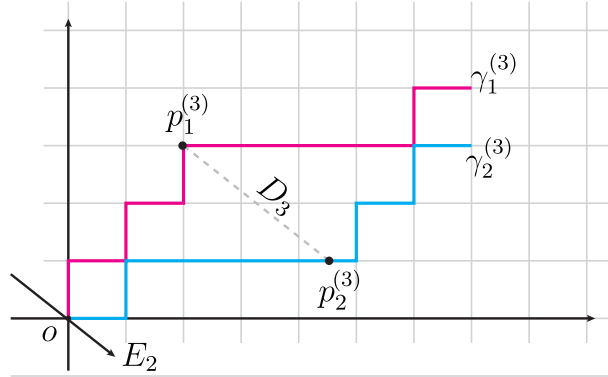


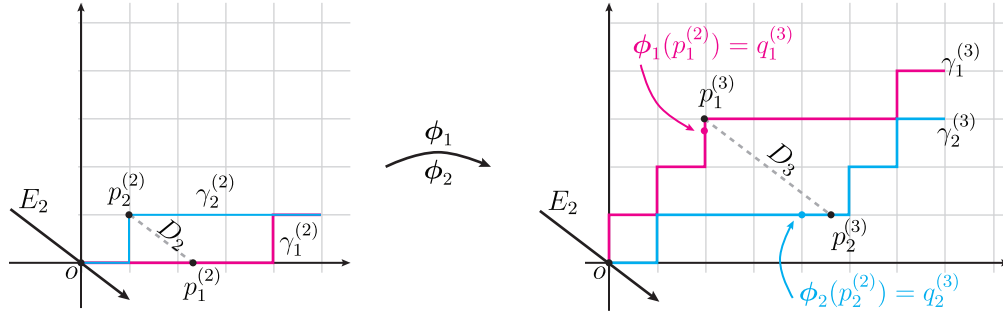
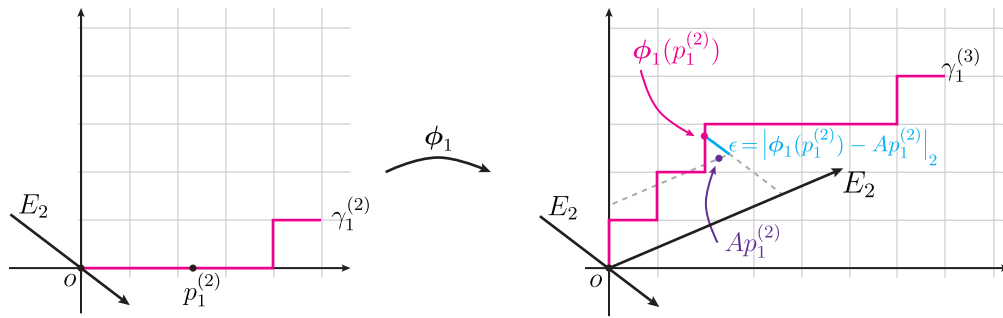
Figure 5.6: The distance D_3 between strands $\gamma_1^{(3)}$ and $\gamma_2^{(3)}$

Recall that the map on edges ϕ_i naturally extends to an inflation-substitution map on strands edge by edge. Let us denote the resulting strand of ϕ_i applied to $\gamma_i^{(n)}$ by $(\phi_i)_{\gamma_i^{(n)}}$ for $i = 1, 2$. Since ϕ_i of points $p_i^{(n)}$ on strands $\gamma_i^{(n)}$ may not be the same points at which the maximum D_{n+1} is attained, we will denote them by $q_i^{(n+1)}$ (see Figure 5.7).

Using a computational aid we estimate that the maximum distance between strands $\gamma_1^{(7)}$ and $\gamma_2^{(7)}$ is $D_7 \approx 9.853 \geq 9.85$. As long as $D_n \geq 9.85$, we have $D_{n+1} \geq 1.01D_n$. To see this fact, let

$$\epsilon_i := \sup \{ |(\phi_i)_\eta(p) - Ap|_2 : \eta \text{ is a strand in } \mathbb{R}^2 \text{ and } p \text{ is a point on } \eta \}$$

(see Figure 5.8 for an example of the distance $|(\phi_i)_\eta(p) - Ap|_2$). Again using a

Figure 5.7: ϕ_i applied to $\gamma_i^{(2)}$ Figure 5.8: The distance $|\phi_1(p_1^{(2)}) - Ap_1^{(2)}|_2$

computational aid we estimate that $\epsilon_i \leq 1.366$. Then

$$\begin{aligned}
D_{n+1} &= |p_1^{(n+1)} - p_2^{(n+1)}|_2 \geq |q_1^{(n+1)} - q_2^{(n+1)}|_2 \\
&\geq |Ap_1^{(n)} - Ap_2^{(n)}|_2 - \epsilon_1 - \epsilon_2 \\
&\geq |Ap_1^{(n)} - Ap_2^{(n)}|_2 - 2.736 \\
&\geq |\lambda_2| |p_1^{(n)} - p_2^{(n)}|_2 - 2.736 \\
&\geq |\lambda_2| D_n - 2.2736 \\
&\geq 1.01 [D_n + (|\lambda_2| - 1.01) D_n - 2.736] \\
&\geq 1.01 D_n
\end{aligned}$$

where we use that $(|\lambda_2| - 1.01) D_n - 2.736 \geq (1.3)(9.85) - 2.736 \geq 0$. Hence $D_{n+k} \geq (1.01)^k D_n$ for $n \geq 7$ and $k \geq 0$. So for any $k \geq 0$, $D_{7+k} \geq (1.01)^k D_7 \geq (1.01)^k (9.85)$ and $\lim_{k \rightarrow \infty} (1.01)^k (9.85) = \infty$, i.e. D_{7+k} diverges.

To see that $a^{(n)}(\tau)$ also diverges as $n \rightarrow \infty$, let $p_i = (x_i, y_i)$ ($i = 1, 2$) be the points on $\gamma_i^{(n)}$ satisfying $p_1 - p_2 \in E_2$ and $\text{pr}_2(p_i) = \tau$, for some fixed $\tau \in \mathbb{R}$ and $n \in \mathbb{N}$ (see Figure 5.9). Then $a^{(n)}(\tau) = \lfloor x_1 \rfloor - \lfloor x_2 \rfloor$. If α is the angle between E_2 and

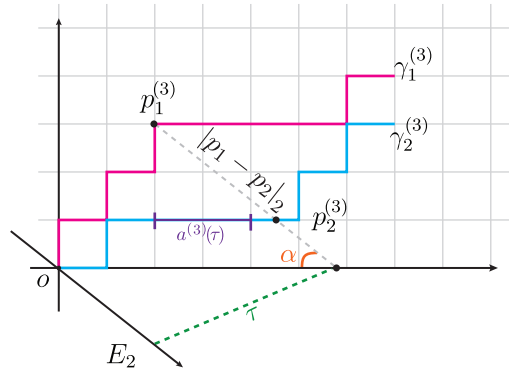


Figure 5.9: Geometric estimation of the discrepancy $a^{(n)}(\tau)$

the x -axis, then $a^{(n)}(\tau) \approx |p_1 - p_2|_2 \cos \alpha$. Because $\cos \alpha \neq 0$ and $\max\{|p_1 - p_2|_2 : p_1 - p_2 \in E_2\}$ is unbounded, then $a^{(n)}(\tau)$ is also unbounded. $\square$

Analysis of the 2-Dimensional Tiling

Recall the substitution and its associated self-affine tiling space $\mathcal{T}_\Phi$ described in the introduction of this chapter. Note that the inflation property of Φ is given by the



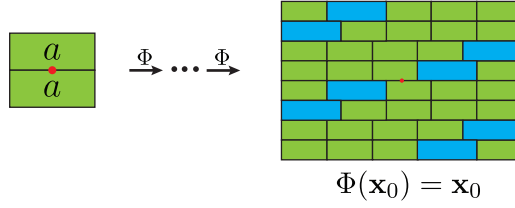
Figure 5.10: Frank-Sadun substitution rule

matrix $\Lambda := \begin{bmatrix} \lambda_1 & 0 \\ 0 & 2 \end{bmatrix}$ where λ_1 is the Perron-Frobenius eigenvalue of A , the substitution matrix given in the previous section.

In [39], Solomyak proved recognizability for all aperiodic tilings with FLC. In this section we will explore the 2-dimensional substitution tiling in detail, which will produce several necessary results, the main result being recognizability (Lemma 5.29). Our development follows the ideas of Frank-Sadun and relies heavily on the coding assigned to each horizontal row of tiles in a tiling. We will first describe the coding for a particular tiling $\mathbf{x}_0$. Then we will extend that coding to translates of $\mathbf{x}_0$ and finally to any tiling in the space $\mathcal{T}_\Phi$.

Coding Rows in Tiling $\mathbf{x}_0$

Let $\mathbf{x}_0 \in \mathcal{T}_\Phi$ be the tiling of $\mathbb{R}^2$ grown from the two tile patch $\begin{pmatrix} a \\ a \end{pmatrix}$ centered at the origin under Φ ; i.e., $\mathbf{x}_0 := \lim_{n \rightarrow \infty} \Phi^n \begin{pmatrix} a \\ a \end{pmatrix}$ (see figure 5.11). Clearly $\Phi(\mathbf{x}_0) = \mathbf{x}_0$ and for any $t \in \mathbb{R}^2$ we have $\Phi(\mathbf{x}_0 + t) = \Phi(\mathbf{x}_0) + \Lambda t = \mathbf{x}_0 + \Lambda t$. By minimality of the $\mathbb{R}^2$ -action on $\mathcal{T}_\Phi$ we see that $\mathcal{T}_\Phi = \text{cl}\{\mathbf{x}_0 + t : t \in \mathbb{R}^2\}$.

Figure 5.11: Growing $\mathbf{x}_0$ under Φ

Since every patch of tiles in $\mathbf{x}_0$ is contained in an image of an a -tile we have a natural hierarchy of n super-prototiles (or simply n -supertiles) that will allow us to generate binary codes for the horizontal rows (of tiles) in $\mathbf{x}_0$. Recall that a row in any tiling $\mathbf{x} \in \mathcal{T}_\Phi$ is the collection of tiles in $\mathbf{x}$ whose union of interiors intersects a horizontal line. It is sometimes useful to view a row in $\mathbf{x} \in \mathcal{T}_\Phi$ as an element of the 1-dimensional tiling space $\mathcal{T}_\phi := \mathcal{T}_{\phi_1} = \mathcal{T}_{\phi_2}$ (Lemma 5.8) as projections onto a horizontal line (figure 5.5).

Definition 5.10. *The code of a row r in $\mathbf{x}_0$ is a one-sided sequence $c \in \mathcal{G}$ where the n th digit in the sequence is 0 if the row r intersects the lower half of an n -supertile and 1 if the row r intersects the upper half of an n -supertile.*

We mention that which n -supertile is used is immaterial. Observe that r_0 , the row of tiles whose bottom edges make up the x -axis, will have code $c_0 = 0^\infty$ because it intersects the lower half of an n -supertile for all n . Similarly, r_{-1} , the row of tiles whose upper edges make up the x -axis, has code $c_{-1} = 1^\infty$ because it intersects the upper half of an n -supertile for all n . The first row above r_0 , denoted r_1 , will have code $c_1 = 1000\dots$ because it intersects the upper half of an n -supertile and the lower half of an n -supertile for all $n > 1$. The codes of adjacent rows are related via the diadic adding machine $f : \mathcal{G} \rightarrow \mathcal{G}$ described in the introduction to Chapter 5. Precisely,

taking c_k to denote the code of the k th row r_k in the tiling $\mathbf{x}_0$, we have the following lemma:

Lemma 5.11. *For all $k \in \mathbb{Z}$, $c_{k+1} = f(c_k)$.*

Proof: Fix $k \in \mathbb{Z}$ and consider the following three cases:

1. Simple Case: $c_k = 0w$, where $w \in \mathcal{G}$
2. Generic Case: $c_k = 111 \dots 10w$, where $w \in \mathcal{G}$
3. Critical Case: $c_k = 1^\infty$

Simple Case: Suppose $c_k = 0w$. Since the first digit is 0, r_k must intersect the lower half of a 1-supertile. The row directly above r_k necessarily intersects the upper half of that same 1-supertile see Figure 5.12. That means the first digit of c_{k+1} must be 1. Note that for all $n > 1$, r_k and r_{k+1} intersect the same half of an n -supertile. So the remaining digits of c_k and c_{k+1} are the same, as given by w . Now, observe that adding 1 to the code of r_k by using binary addition with carry yields the appropriate code for r_{k+1} : $f(c_k) = c_k + \mathbf{1} = 0w + 100 \dots = 1w = c_{k+1}$.

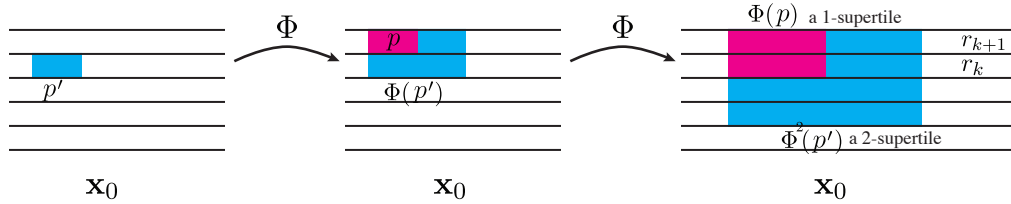


Figure 5.12: Simple case: row r_k intersects the lower half of a 1-supertile

Generic Case: Suppose the code for r_k begins with n ones, $c_k = 111 \dots 10w$ for some $w \in \{0, 1\}^{\mathbb{N}}$. The zero in the $(n + 1)$ th digit implies that r_k intersects the lower half

of an $(n + 1)$ -supertile. The first n ones in the sequence imply that r_k intersects the upper half an m -supertile for all $m \leq n$.

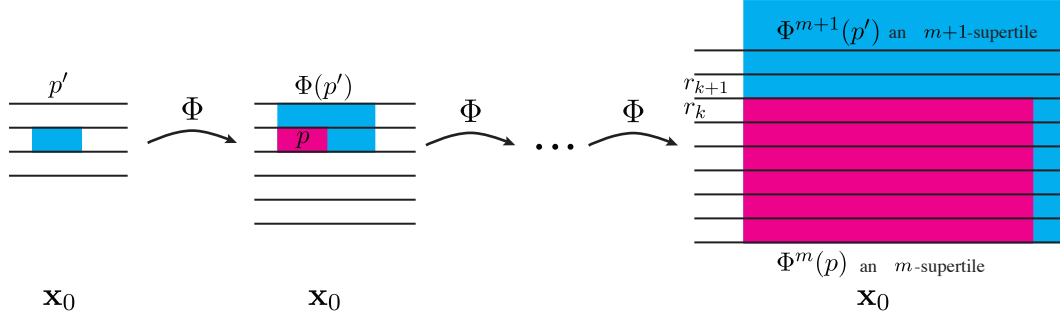


Figure 5.13: Generic case: row r_k intersects the upper half of a m -supertile

It follows that r_{k+1} intersects the upper half of the same the $(n + 1)$ -supertile as r_k , and r_k and r_{k+1} intersect the same half of an m -supertile for each $m > n + 1$. Because row r_{k+1} lies above row r_k and r_k intersects the upper half of all m -supertiles for $m \leq n$, r_{k+1} must intersect the lower half of a different set of supertiles. So the code for r_{k+1} will begin with n zeros, $c_{k+1} = 000 \dots 01w$. Now observe that by adding $\mathbf{1}$ (with carry) to the code of r_k we get the appropriate code for r_{k+1} : $f(c_k) = c_k + \mathbf{1} = 111 \dots 10w + 100 \dots = 000 \dots 01w = c_{k+1}$.

Critical Case: Suppose that $c_k = 1111 \dots = 1^\infty$. This means that r_k always intersects the top half of all supertiles. This is in fact row r_{-1} in the tiling $\mathbf{x}_0$. So the row above must forever intersects the lower half of all supertiles, albeit they must be images of different supertiles. So r_k and r_{k+1} never intersect the image of the same supertile. When we add $\mathbf{1}$ (with carry) to the code of r_k we will see that r_k and r_{k+1} never share a coordinate, thus satisfying the conditions above.

$$f(c_k) = c_k + \mathbf{1} = 111 \dots + 100 \dots = 000 \dots = c_{k+1}$$

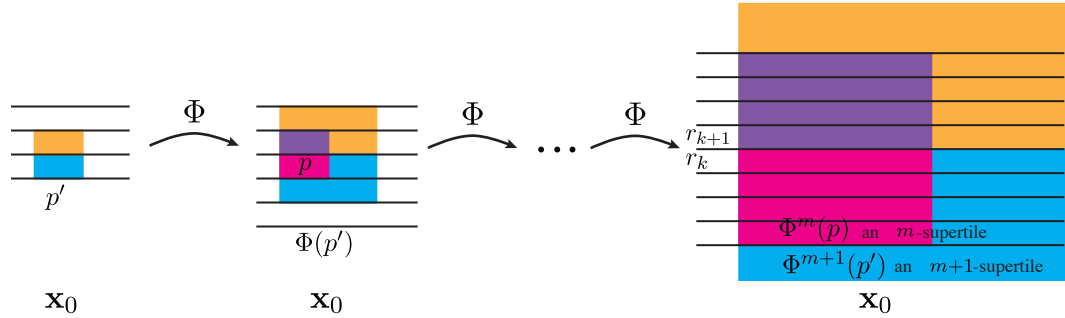


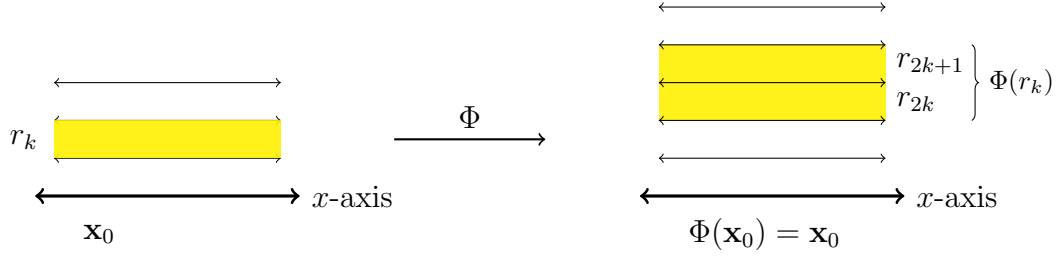
Figure 5.14: Critical case: row r_k is always intersects the upper half of supertiles

Since $k \in \mathbb{Z}$ was arbitrary, Lemma 5.11 holds for all $k \in \mathbb{Z}$. $\square$

Observe that all even numbered rows intersect the bottom half of 1-supertiles and thus have codes that begin with 0. Similarly, all odd numbered rows intersect the top half of 1-supertiles and thus have codes that begin with 1.

Lemma 5.12. *Let r_k be a row in the tiling $\mathbf{x}_0$ with code c_k . Then $c_{2k} = 0c_k$ and $c_{2k+1} = 1c_k$.*

Proof: Take a row r_k in $\mathbf{x}_0$ with code c_k . Under the inflation substitution of Φ every row gets inflated and replaced by two rows. In particular, r_k gets mapped to $r_{2k} \cup r_{2k+1}$ (see figure 5.15). This also means that every $(n+1)$ -supertile in $\Phi(\mathbf{x}_0)$ arises from an n -supertile in $\mathbf{x}_0$, and the tops and bottoms halves of the two correspond to each other via Φ . Hence the $(n+1)$ st digit in c_{2k} and c_{2k+1} is the same as the n th digit of c_k for $n \geq 1$. The first digit is dictated by the even and odd property: c_{2k} begins with 0 because $2k$ is even and c_{2k+1} begins with 1 because $2k+1$ is odd. Thus $c_{2k} = 0c_k$ and $c_{2k+1} = 0c_k + 1$. $\square$

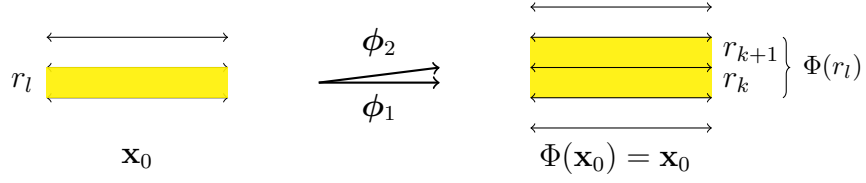
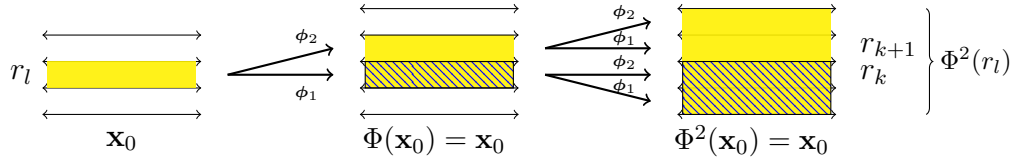
Figure 5.15: Correspondence of rows in $\mathbf{x}_0$ under Φ

The next proposition shows how individual rows can be obtained from the row below. Eventually we see that most tilings in $\mathcal{T}_\Phi$ are determined by one row and its code.

Lemma 5.13. *Let r_k be a row in $\mathbf{x}_0$ with coding of the form $c_k = 111 \dots 10w$ with m leading ones and $w \in \mathcal{G}$. Then the row r_{k+1} is obtained from row r_k by translation; namely $r_{k+1} = r_k + (\tau, 1)$ where $\tau := \lambda_1^{m+1} - \lambda_1^m - \dots - \lambda_1$.*

Proof: Let's begin with the simple case. Suppose row r_k has the coding $c_k = 0w$, where $w \in \mathcal{G}$. Then r_{k+1} has coding $c_{k+1} = 1w$ by Lemma 5.11. Observe that c_k and c_{k+1} have the same digits in place $n \geq 2$. This means that for $n \geq 2$, r_k and r_{k+1} intersect the same half of the same n -supertile. The first digits of c_k and c_{k+1} imply that there exists a 1-supertile such that r_k intersects the bottom half and r_{k+1} intersects the top half. Therefore there exists a row r_l such that $r_k \cup r_{k+1} \subseteq \Phi(r_l)$. Considering for a moment r_k and r_{k+1} as 1-dimensional tilings, we can write $r_k = \phi_1(r_l)$ and $r_{k+1} = \phi_2(r_l)$, according to the definition of Φ (see Figure 5.16). Lemma 5.7 gives us $\phi_2(r_l) = \phi_1(r_l) + \lambda_1$. That is $r_{k+1} = r_k + (\lambda_1, 1)$.

Now, suppose row r_k has coding $c_k = 10w$, where $w \in \mathcal{G}$. By the adding machine r_{k+1} has coding $c_{k+1} = 01w$. Note that rows r_k and r_{k+1} do not intersect the same 1-supertile because r_k intersects the upper half and r_{k+1} intersects the bottom half.

Figure 5.16: Case where $c_k = 0w$ Figure 5.17: Case where $c_k = 10w$

However, r_k and r_{k+1} both intersect the same n -supertile for $n \geq 2$ and they intersect the same half of the same n -supertile for $n \geq 3$. Furthermore, there exists a row r_l such that $r_k \cup r_{k+1} \subseteq \Phi^2(r_l)$ with r_k lying in the bottom half and r_{k+1} lying in the upper half of the image. Considering again r_k and r_{k+1} as 1-dimensional tilings, we can write $r_k = \phi_2(\phi_1(r_l))$ and $r_{k+1} = \phi_1(\phi_2(r_l))$ (see Figure 5.17). Using Lemma 5.7 and the property $\phi_i(r + t) = \phi_i(r) + \lambda_1 t$ for any 1-D tiling r and $t \in \mathbb{R}$, we get

$$\begin{aligned}
 \phi_1(\phi_2(r_l)) &= \phi_1(\phi_1(r_l) + \lambda_1) \\
 &= \phi_1(\phi_1(r_l)) + \lambda_1^2 \\
 &= \phi_2(\phi_1(r_l)) + \lambda_1^2 - \lambda_1
 \end{aligned}$$

Reverting back to considering r_k, r_{k+1} as rows of the 2-dimensional tiling, this means $r_{k+1} = r_k + (\lambda_1^2 - \lambda_1, 1)$.

Take an arbitrary $m \in \mathbb{N}$ and suppose that row r_k has code $c_k = 111 \dots 10w$ with m leading ones. By Lemma 5.11 we have $c_{k+1} = c_k + 1 = 000 \dots 01w$ with m leading

zeros. From these codings we determine that for all $n \leq m$, r_k and r_{k+1} do not intersect the same n -supertile because r_k intersects the upper half and r_{k+1} intersects the lower half. For $n > m + 1$ we see that c_k and c_{k+1} agree, so r_k and r_{k+1} intersect the same half of a n -supertile. From digit $m + 1$ we see that r_k intersects the lower half of a $(m + 1)$ -supertile and r_{k+1} intersects the upper half. This means that there exists a row r_l such that $r_k \cup r_{k+1} \subseteq \Phi^{m+1}(r_l)$. Expressing r_k and r_{k+1} as 1-dimensional tilings $\mathbf{r}_k$ and $\mathbf{r}_{k+1}$ we can write $\mathbf{r}_k = \phi_2^m(\phi_1(\mathbf{r}_l))$ and $\mathbf{r}_{k+1} = \phi_1^m(\phi_2(\mathbf{r}_l))$. The lemma follows then from the following claim.

Claim 5.14. $\phi_1^m(\phi_2(\mathbf{r}_l)) = \phi_2^m(\phi_1(\mathbf{r}_l)) + \lambda_1^{m+1} - \lambda_1^m - \dots - \lambda_1$

Proof of Claim: We will prove the claim by induction on m using Lemma 5.7 and the property $\phi_i(\mathbf{r} + t) = \phi_i(\mathbf{r}) + \lambda_1 t$ for any 1-D tiling $\mathbf{r}$ and $t \in \mathbb{R}$. We have already shown that the $m = 1$ case holds; i.e., $\phi_1(\phi_2(\mathbf{r}_l)) = \phi_2(\phi_1(\mathbf{r}_l)) + \lambda_1^2 - \lambda_1$. Suppose that the $m - 1$ case holds; i.e., $\phi_1^{m-1}(\phi_2(\mathbf{r}_l)) = \phi_2^{m-1}(\phi_1(\mathbf{r}_l)) + \lambda_1^m - \lambda_1^{m-1} - \dots - \lambda_1$. Then

$$\begin{aligned}
\phi_1^m(\phi_2(\mathbf{r}_l)) &= \phi_1(\phi_1^{m-1}(\phi_2(\mathbf{r}_l))) \\
&= \phi_1(\phi_2^{m-1}(\phi_1(\mathbf{r}_l)) + \lambda_1^m - \lambda_1^{m-1} - \dots - \lambda_1) \\
&= \phi_2(\phi_2^{m-1}(\phi_1(\mathbf{r}_l)) + \lambda_1^m - \lambda_1^{m-1} - \dots - \lambda_1) - \lambda_1 \\
&= \phi_2(\phi_2^{m-1}(\phi_1(\mathbf{r}_l))) + \lambda_1^{m+1} - \lambda_1^m - \dots - \lambda_1^2 - \lambda_1 \\
&= \phi_2^m(\phi_1(\mathbf{r}_l)) + \lambda_1^{m+1} - \lambda_1^m - \dots - \lambda_1^2 - \lambda_1
\end{aligned}$$

□

The next lemma will show the uniqueness of the translations found above.

Lemma 5.15. *The numbers $\lambda_1^{m+1} - \lambda_1^m - \dots - \lambda_1$ are distinct for each $m \in \mathbb{N}$.*

Proof: Suppose there exist $n \neq m$ such that

$$\lambda_1^{n+1} - \lambda_1^n - \dots - \lambda_1 = \lambda_1^{m+1} - \lambda_1^m - \dots - \lambda_1. \quad (5.4)$$

Without loss of generality, suppose $n < m$. Then equation (5.4) becomes

$$\begin{aligned} 0 &= \lambda_1^{m+1} - \lambda_1^m - \dots - 2\lambda_1^{n+1} = \lambda_1^{n+1}(\lambda_1^{m-n} - \lambda_1^{m-n-1} - \dots - \lambda_1 - 2) \text{ or} \\ 0 &= \lambda_1^{m-n} - \lambda_1^{m-n-1} - \dots - \lambda_1 - 2. \end{aligned} \quad (5.5)$$

Note that for $m - n = 1$, the right side of equation (5.5) becomes $\lambda_1 - 2 > 0$, a contradiction. To get a contradiction for all $k := m - n > 1$, we now show that $\lambda_1^k - \lambda_1^{k-1} - \dots - \lambda_1 \geq 3$ by using induction. For $k = 2$ we have $\lambda_1^2 - \lambda_1 = 3$ by the definition of λ_1 . Suppose we have

$$\lambda_1^k - \lambda_1^{k-1} - \dots - \lambda_1^2 - \lambda_1 > 3.$$

Then multiplying both sides by λ_1 and subtracting λ_1 we get

$$\lambda_1^{k+1} - \lambda_1^k - \dots - \lambda_1^2 - \lambda_1 > 3\lambda_1 - \lambda_1 = 2\lambda_1 > 3$$

completing the induction step. □

Coding via the Number of Leading Ones

Lemma 5.16 ahead shows that the recording of the number of leading ones in the codes of all rows uniquely determines the codes. For each $m \in \mathbb{N}$, set

$$B_m := \{c = \sigma_1\sigma_2\dots \in \mathcal{G} : \sigma_1 = \sigma_2 = \dots = \sigma_m = 1, \sigma_{m+1} = 0\}$$

and let $B_0 := \{c = \sigma_1\sigma_2 \dots \in \mathcal{G} : \sigma_1 = 0\}$. For $c \in \mathcal{G}$, we define the *itinerary* of c by

$$I(c) := (m_k)_{k \in \mathbb{N}_0} \in \{0, 1\}^{\mathbb{N}_0}$$

where $m_k \in \mathbb{N}_0$ satisfies $f^k(c) \in B_{m_k}$ for $k \in \mathbb{N}_0$. Notice that the k th entry of $I(c)$ records the number of leading ones in the k th iterate of c under f .

Lemma 5.16. *For any $c \in \mathcal{G}$, $I(c)$ uniquely determines c .*

The family $\{B_m\}_{m \in \mathbb{N}}$ is a partition of $\mathcal{G}$. We note that there exists an f -invariant metric d on $\mathcal{G}$ in which $\text{diam}(B_m) \searrow 0$ as $m \rightarrow \infty$. We also note that f is minimal; i.e., $\{f^n(c) : n \in \mathbb{N}\}$ is dense in $\mathcal{G}$ for every $c \in \mathcal{G}$.

Proof of Lemma 5.16: Fix $c \in \mathcal{G}$. Suppose the lemma fails; i.e., there exist $c, \hat{c} \in \mathcal{G}$, $\hat{c} \neq c$, with $I(\hat{c}) = (m_k)_{k \in \mathbb{N}_0} = I(c)$. Since $\hat{c} \neq c$ there exists $\epsilon > 0$ such that $d(c, \hat{c}) > \epsilon$. Take $m \in \mathbb{N}_0$ so that $\text{diam}(B_m) < \epsilon$. By minimality of f there exists infinitely many $k \in \mathbb{N}_0$ such that $f^k(c) \in B_m$. In particular, there is $k \in \mathbb{N}_0$ such that $f^k(c) \in B_m$. Because f is an isometry $d(f^k(c), f^k(\hat{c})) = d(c, \hat{c}) > \epsilon$. So $f^k(\hat{c}) \notin B_m$. By the definition of the itineraries $m_k = m$ and $m_k \neq m$, a contradiction. $\square$

Now that we know that the itinerary $(m_k)_k$ uniquely determines the code, we observe that it is not necessary to know the number of leading ones in every row to determine the codes for every row in our tiling (this is because the codes are related by the adding machine).

Corollary 5.17. *Let $k_0 \in \mathbb{Z}$ and let $c_k \in \mathcal{G}$ for each $k \in \mathbb{Z}$. If $(c_k)_{k \in \mathbb{Z}}$ is such that $c_{k+1} = f(c_k)$, then the sequence $(m_k)_{k > k_0}$, where $m_k :=$ the number of leading ones in c_k , determines $(c_k)_{k \in \mathbb{Z}}$ uniquely.*

Proof: Fix $k_0 \in \mathbb{Z}$. Take two arbitrary sequences $(c_k)_{k \in \mathbb{Z}}$ and $(\hat{c}_k)_{k \in \mathbb{Z}}$ from $\mathcal{G}$ satisfying $c_{k+1} = f(c_k)$ and $\hat{c}_{k+1} = f(\hat{c}_k)$ for all $k \in \mathbb{Z}$. Suppose $(c_k)_{k \in \mathbb{Z}}$ and $(\hat{c}_k)_{k \in \mathbb{Z}}$ have

$(m_k)_{k>k_0} = (\hat{m}_k)_{k>k_0}$. Note that $I(c_k) = (m_{k+l})_{l \in \mathbb{N}_0}$ because $f^l(c_k) = c_{k+l}$ has m_{k+l} leading ones. From $(m_k)_{k>k_0} = (\hat{m}_k)_{k>k_0}$ we get $m_{k_0+j} = \hat{m}_{k_0+j}$ for $j \in \mathbb{N}_0$. That is $I(c_{k_0}) = I(\hat{c}_{k_0})$. Then Lemma 5.16 implies $c_{k_0} = \hat{c}_{k_0}$. Using the adding machine on c_{k_0} we get $c_k = f^{k-k_0}(c_{k_0}) = f^{k-k_0}(\hat{c}_{k_0}) = \hat{c}_k$ for all $k \in \mathbb{Z}$. Thus $(c_k)_{k \in \mathbb{Z}} = (\hat{c}_k)_{k \in \mathbb{Z}}$. $\square$

Coding Rows for Translates of $\mathbf{x}_0$

Now that we have established binary coding of rows in $\mathbf{x}_0$ we can extend the coding to translates of $\mathbf{x}_0$. For any $t \in \mathbb{R}^2$, $\Phi(\mathbf{x}_0 + \Lambda^{-n}t) = \mathbf{x}_0 + \Lambda^{-n+1}t$. So we have the infinite backward orbit under Φ :

$$\dots \mapsto \mathbf{x}_0 + \Lambda^{-2}t \mapsto \mathbf{x}_0 + \Lambda^{-1}t \mapsto \mathbf{x}_0 + t \mapsto \mathbf{x}_0 + \Lambda t \quad (5.6)$$

and the associated infinite hierarchy of supertiles for $\mathbf{x}_0 + t$ whereby an n -supertile in $\mathbf{x}_0 + t$ is the image under Φ^n of a tile in $\mathbf{x}_0 + \Lambda^{-n}t$. This provides codes for the rows of $\mathbf{x}_0 + t$ whereby the n th digit of the code of a row is 0 or 1 depending on whether the row intersects the lower half or upper half of an n -supertile.

In subsequent considerations we will have to index the rows of any tiling $\mathbf{x} \in \mathcal{T}_\Phi$ by integers so let us attend to this simple task now. Recall that r_0 in $\mathbf{x}_0$ is the collection of all tiles in $\mathbf{x}_0$ whose bottom edges made up the x -axis. The tiling $\mathbf{x}_0 + t$ may not have a collection of tiles whose bottom edges make up the x -axis. So let us define the zero-th row of any tiling $\mathbf{x} \in \mathcal{T}_\Phi$ to be the collection of tiles in $\mathbf{x}$ whose union of interiors and bottom edges contain the x -axis. We then index the remaining rows in the natural way, increasing the index by one for each row above and decreasing by one for each row below.

Let us record how the indexing of a row changes under translation. Observe that if r is a row in $\mathbf{x}_0$ with code c , then $r + t$ is a row in $\mathbf{x}_0 + t$ and $r + t$ will have the

same code c . In particular, if $\mathbf{x}' := \mathbf{x}_0 + t$ with $t = (a, b) \in \mathbb{R}^2$ and r'_k is the k th row of $\mathbf{x}'$, then the correspondence between the rows of $\mathbf{x}_0$ and $\mathbf{x}'$ is given by $r'_k = r_{k-\lfloor b \rfloor} + t$. So row $r_{k-\lfloor b \rfloor}$ in $\mathbf{x}_0$ has the same code as row r'_k in $\mathbf{x}'$; i.e., $c'_k = c_{k-\lfloor b \rfloor}$. Using this property we can see that Lemma 5.11 and Lemma 5.13 immediately extend to $\mathbf{x}_0 + t$.

Lemma 5.18. *The code for row r'_{k+1} in tiling $\mathbf{x}' := \mathbf{x}_0 + t$ can be determined from the code of row r'_k by way of the adding machine: $c'_{k+1} = f(c'_k)$ for all $k \in \mathbb{Z}$.*

Lemma 5.19. *Let r'_k be a row in $\mathbf{x}' := \mathbf{x}_0 + t$ with coding of the form $c'_k = (111 \dots 10w)$ with m leading ones and $w \in \mathcal{G}$. Then the row r'_{k+1} is obtained from row r'_k by translation as follows: $r'_{k+1} = r'_k + (\tau, 1)$ where $\tau := \lambda_1^{m+1} - \lambda_1^m - \dots - \lambda_1$.*

Although these lemmas are quite clear we include their proofs for completeness.

Proof of Lemma 5.18: Let the code of row r'_k in $\mathbf{x}' := \mathbf{x}_0 + t$ be given by c'_k .

If $t = (a, b) \in \mathbb{R}^2$ then $c'_k = c_{k-\lfloor b \rfloor}$ where $c_{k-\lfloor b \rfloor}$ is the code for row $r_{k-\lfloor b \rfloor}$ in $\mathbf{x}_0$. Then Lemma 5.11 implies that row $r_{k-\lfloor b \rfloor+1}$ has code $c_{k-\lfloor b \rfloor+1} = f(c_{k-\lfloor b \rfloor})$. Since row $r_{k-\lfloor b \rfloor+1}$ in $\mathbf{x}_0$ corresponds to row r'_{k+1} in $\mathbf{x}'$ its code is given by $c'_{k+1} = c_{k-\lfloor b \rfloor+1} = f(c_{k-\lfloor b \rfloor}) = f(c'_k)$. $\square$

Proof of Lemma 5.19: Rows r'_k and r'_{k+1} in $\mathbf{x}'$ corresponds to rows $r_{k-\lfloor b \rfloor}$ and $r_{k-\lfloor b \rfloor+1}$ in $\mathbf{x}_0$. In particular, $r'_k = r_{k-\lfloor b \rfloor} + t$ and $r'_{k+1} = r_{k-\lfloor b \rfloor+1} + t$. Then

$$\begin{aligned} r'_{k+1} &= r_{k-\lfloor b \rfloor+1} + t \\ &= (r_{k-\lfloor b \rfloor} + (\tau, 1)) + t \quad \text{by Lemma 5.13} \\ &= (r_{k-\lfloor b \rfloor} + t) + (\tau, 1) \\ &= r'_k + (\tau, 1). \end{aligned}$$

$\square$

Extending Lemma 5.12 is not as straight forward as the previous lemmas because $\Phi(\mathbf{x}') \neq \mathbf{x}'$ for $t \neq 0$. So how do the codes of $\mathbf{x}'$ and $\Phi(\mathbf{x}')$ relate? The answer depends on how the x -axis intersects the tiling $\mathbf{x}'$.

Recall from the introduction to Chapter 5 that a tiling $\mathbf{x}$ is *aligned* if the x -axis is made of tile edges. Likewise, we say a tiling $\mathbf{x}$ is *half-aligned* if the x -axis lies exactly in the middle of a row of tiles; i.e., $\mathbf{x} + (0, 1/2)$ is aligned (see figures 5.18 and 5.19). Note that the tiling $\mathbf{x}_0$ is aligned; i.e., $\mathbf{x}_0 \in \mathcal{T}_a$. If $\Phi(\mathbf{x}) = \mathbf{x}' \in \mathcal{T}_a$, then $\mathbf{x}$ must be either aligned or half-aligned due to the vertical doubling property of Φ . The correct analogue of Lemma 5.12 is as follows.

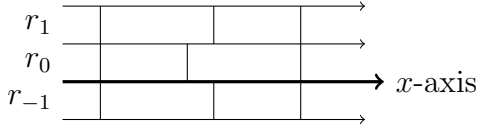


Figure 5.18: Aligned tiling

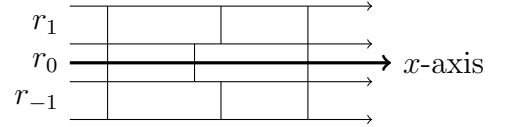
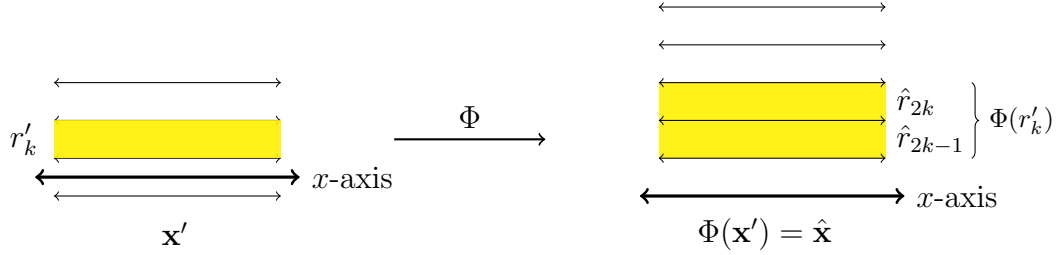


Figure 5.19: Half-aligned tiling

Lemma 5.20. *Let $\mathbf{x}' := \mathbf{x}_0 + t$ and $\hat{\mathbf{x}} := \Phi(\mathbf{x}') = \mathbf{x}_0 + \Lambda t$. If $\mathbf{x}' \in \mathcal{T}_a$, then $\hat{c}_{2k} = 0c'_k$ and $\hat{c}_{2k+1} = 1c'_k$ for all $k \in \mathbb{Z}$. If $\mathbf{x}'$ is half-aligned, then $\hat{c}_{2k-1} = 0c'_k$ and $\hat{c}_{2k} = 1c'_k$ for all $k \in \mathbb{Z}$.*

Proof: Suppose $\mathbf{x}' \in \mathcal{T}_a$. Using the definition of supertiles (following 5.6) we observe that the $(n+1)$ -supertiles in $\hat{\mathbf{x}}$ arise from n -supertiles in $\mathbf{x}'$ by application of Φ . As shown in the proof of Lemma 5.12, for any $k \in \mathbb{Z}$ r'_k gets mapped to $\hat{r}_{2k} \cup \hat{r}_{2k+1}$ (see figure 5.15). So the $n+1$ digits of $\hat{c}_{2k}$ and $\hat{c}_{2k+1}$ are the same as the n th digit of c'_k for $n \geq 1$. 1-supertiles in $\hat{\mathbf{x}}$ arise from the inflation of a tile in r'_k under Φ . The image of this tile will intersect both rows $\hat{r}_{2k}$ and $\hat{r}_{2k+1}$. So $\hat{c}_{2k}$ will begin with 0 because $\hat{r}_{2k}$ intersects the bottom half of the 1-supertile and $\hat{c}_{2k+1}$ will start with 1 because $\hat{r}_{2k+1}$

Figure 5.20: Correspondence of half-aligned rows under Φ

intersects the top half of the 1-supertile. Hence the codes are given by $\hat{c}_{2k} = 0c'_k$ and $\hat{c}_{2k+1} = 1c'_k$.

Suppose $\mathbf{x}'$ is half-aligned. Again we observe that the $(n+1)$ -supertiles in $\mathbf{x}'$ arise from n -supertiles in $\mathbf{x}$. Unlike the previous case, a row r'_k in $\mathbf{x}'$ gets mapped to $\hat{r}_{2k-1} \cup \hat{r}_{2k}$ in $\mathbf{x}'$ under Φ (see figure 5.20). This means that the $n+1$ digits of $\hat{r}_{2k-1}$ and $\hat{r}_{2k}$ are the same as the n th digit of r'_k for $n \geq 1$. Now the even numbered rows in $\hat{\mathbf{x}}$ have codes that begin with 1 because they intersect the upper half of the 1-supertiles and the odd numbered rows have codes that begin with 0 because they intersect the lower half of the 1-supertiles. Hence the codes of rows in $\mathbf{x}'$ are given by $c'_{2k-1} = 0c_k$ and $c'_{2k} = 1c_k$ for all $k \in \mathbb{Z}$. $\square$

Coding Rows in an Arbitrary Tiling $\mathbf{x} \in \mathcal{T}_\Phi$

Now we consider an arbitrary tiling $\mathbf{x} \in \mathcal{T}_\Phi$. Since the orbit of $\mathbf{x}_0$ under translation is dense in $\mathcal{T}_\Phi$, there exists a sequence $(t_n)_{n \in \mathbb{N}}$ with $t_n \in \mathbb{R}^2$ such that $\lim_{n \rightarrow \infty} \mathbf{x}_0 + t_n = \mathbf{x}$. Before we can extend our properties of codes from the previous two sections to codes of rows in $\mathbf{x}$, we need to know that the sequence of codes converge and are independent of the choice of $(t_n)_{n \in \mathbb{N}}$. We will restrict our attention to $x \in \mathcal{T}_a$ since the codes are preserved under translation.

Given $(t_n)_{n \in \mathbb{N}} \subseteq \mathbb{R} \times \mathbb{Z}$, we let $\mathbf{x}_n := \mathbf{x}_0 + t_n$ and denote the k th row in $\mathbf{x}_n$ by $r_k^{(n)}$, its code by $c_k^{(n)}$, and the number of leading ones in $c_k^{(n)}$ by $m_k^{(n)}$ (cf. Section 5). We are interested first in sequences $(t_n)_{n \in \mathbb{N}}$ satisfying the following hypothesis:

$$(H) \quad \mathbf{x} := \lim_{n \rightarrow \infty} \mathbf{x}_n \quad \text{and} \quad c_k := \lim_{n \rightarrow \infty} c_k^{(n)} \quad \text{exist for all } k \in \mathbb{Z}.$$

In the proof Corollary 5.23 we will see that such a sequence always exists for any $\mathbf{x} \in \mathcal{T}_\Phi$.

For each $k \in \mathbb{Z}$ we have a dichotomy about the sequence of leading ones; either

- (i) $\left(m_k^{(n)}\right)_{n \in \mathbb{N}}$ is bounded along a subsequence or
- (ii) $\left(m_k^{(n)}\right)_{n \in \mathbb{N}}$ tends to infinity.

Lemma 5.21. *Under hypothesis (H) (ii) can only hold for at most one k .*

Proof: Suppose not; i.e., there exist $k_1, k_2 \in \mathbb{Z}$ for which (ii) holds. Without loss of generality, let $k_2 > k_1$. Since $\left(m_{k_i}^{(n)}\right)_{n \in \mathbb{N}}$ tends to infinity there exists $N \in \mathbb{N}$ such that for $n > N$ $m_{k_1}^{(n)} \gg k_2 - k_1$. Because $c_{k_2+1}^{(n)}$ is obtained by applying the adding machine to $c_{k_2}^{(n)}$ it has $m_{k_2}^{(n)}$ leading zeros. $c_{k_2+1}^{(n)}$ can also be obtained by applying the adding machine $k_2 - k_1$ times to $c_{k_1+1}^{(n)}$. Because $m_{k_1}^{(n)} \gg k_2 - k_1$ this yields a one somewhere in the first $m_{k_2}^{(n)}$ positions in $c_{k_2+1}^{(n)}$, contradicting $c_{k_2+1}^{(n)}$ having $m_{k_2}^{(n)}$ leading zeros. $\square$

Lemma 5.22. *The limiting sequence $(c_k)_{k \in \mathbb{Z}}$ does not depend on the choice of $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R} \times \mathbb{Z}$; i.e., if (H) is satisfied for sequences $(t_n)_{n \in \mathbb{N}}$ and $(t'_n)_{n \in \mathbb{N}}$ with $\mathbf{x} = \mathbf{x}'$, then $(c_k)_{k \in \mathbb{Z}} = (c'_k)_{k \in \mathbb{Z}}$.*

Proof: From lemma 5.21 either $(t_n)_{n \in \mathbb{N}}$ satisfies (i) for all $k \in \mathbb{Z}$ or there is exactly one $k_* \in \mathbb{Z}$ for which (ii) holds. Similarly, either $(t'_n)_{n \in \mathbb{N}}$ satisfies (i) for all $k \in \mathbb{Z}$

or there is exactly one $k'_* \in \mathbb{Z}$ for which (ii) holds. If case (i) is satisfied for both sequences fix $k_0 \in \mathbb{Z}$ arbitrarily. If case (ii) holds for (t_n) or (t'_n) or both, let $k_0 = k_*$ or k'_* or $\max\{k_*, k'_*\}$, respectively. Fix $k > k_0$. Then (i) holds for k ; i.e., $(m_k^{(n)})_{n \in \mathbb{N}}$ is bounded along a subsequence. In particular, there is some m_k that is a limit of $m_k^{(n)}$ along a subsequence. Thus there exists an unbounded subset $N_k \subset \mathbb{N}$ so that $m_k^{(n)} = m_k$ for $n \in N_k$. Similarly, if m'_k is a limit of $m'^{(n)}_k$ along a subsequence, then there exists an unbounded subset $N'_k \subset \mathbb{N}$ so that $m'^{(n)}_k = m'_k$ for $n \in N'_k$.

By Lemma 5.13 rows $r_k^{(n)}, r_{k+1}^{(n)}$ are equivalent up to a horizontal translation by $\tau_k^{(n)} := \lambda^{m_k^{(n)}+1} - \lambda^{m_k^{(n)}} - \dots - \lambda$ for all $n \in \mathbb{N}$. Similarly, $r'^{(n)}_k, r'_{k+1}^{(n)}$ are equivalent up to a horizontal translation by $\tau'^{(n)}_k$. Note that, for each k , $\tau_k^{(n)}$ takes on finitely many values along the subsequence $n \in N_k$ because $(m_k^{(n)})_{n \in N_k}$ is bounded. Because $\mathbf{x}_n \rightarrow \mathbf{x}$ as $n \rightarrow \infty$ and by taking subsequential limits along N_k we get that the 1-dimensional tilings corresponding to rows r_k, r_{k+1} are equivalent up to some limiting translation τ_k . Similarly for $\mathbf{x}'$, the 1-dimensional tilings of rows r'_k, r'_{k+1} are equivalent up to a limiting translation τ'_k . Since $\mathbf{x} = \mathbf{x}'$, the tiling of r_{k+1} is preserved under translation by $\tau'_k - \tau_k$. By aperiodicity of $\mathcal{T}_{\phi_i}$ we have $\tau_k = \tau'_k$.

Therefore $m_k = m'_k$ for $k > k_0$, because the numbers $\lambda^{m+1} - \lambda^m - \dots - \lambda$ are distinct for each $m \in \mathbb{N}$ by Lemma 5.15. So $(c_k)_{k > k_0} = (c'_k)_{k > k_0}$ by Corollary 5.17. Finally, applying the adding machine yields $(c_k)_{k \in \mathbb{Z}} = (c'_k)_{k \in \mathbb{Z}}$, as desired. $\square$

Corollary 5.23. *Given any $\mathbf{x} \in \mathcal{T}_a$ and any sequence $(t_n)_{n \in \mathbb{N}} \subseteq \mathbb{R} \times \mathbb{Z}$ such that $\lim_{n \rightarrow \infty} \mathbf{x}_0 + t_n = \mathbf{x}$, for all $k \in \mathbb{Z}$, the limit $\lim_{n \rightarrow \infty} c_k^{(n)} = c_k$ exists and is independent of the choice of $(t_n)_{n \in \mathbb{N}}$.*

Proof: Let $\mathbf{x} = \lim_{n \rightarrow \infty} \mathbf{x}_0 + t_n$. $((c_k^{(n)})_{k \in \mathbb{Z}})_{n \in \mathbb{N}}$ is a sequence in the compact space $(\{0, 1\}^{\mathbb{Z}})^{\mathbb{N}}$. So it converges if and only if every convergent subsequence has the same limit. This is guaranteed by Lemma 5.22. $\square$

Given $\mathbf{x} \in \mathcal{T}_a$, there exists a sequence $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R} \times \mathbb{Z}$ such that $\lim_{n \rightarrow \infty} \mathbf{x}_0 + t_n = \mathbf{x}$ because the orbit of $\mathbf{x}_0$ is dense in $\mathcal{T}_\Phi$ under translation. So we define the *code of the k th row of $\mathbf{x}$* to be the limit c_k given by Corollary 5.23. Note that for $\mathbf{x} \in \mathcal{T}_\Phi$ not aligned, we can fix τ so that $\mathbf{x}'_0 := \mathbf{x} + (0, \tau)$ is aligned and let the codes of rows in $\mathbf{x}$ be defined as the codes of the corresponding rows in $\mathbf{x}'$.

Corollary 5.24. *The code c_0 depends continuously on $\mathbf{x} \in \mathcal{T}_a$.*

Proof: Suppose $\lim_{n \rightarrow \infty} \mathbf{x}^{(n)} = \mathbf{x}$ in $\mathcal{T}_a$. For all $n \in \mathbb{N}$ there exists $(t_{n,m})_{m=1}^\infty$ such that $\lim_{m \rightarrow \infty} \mathbf{x}_0 + t_{n,m} = \mathbf{x}^{(n)}$. Let $\mathbf{x}^{(n,m)} := \mathbf{x}_0 + t_{n,m}$. By Corollary 5.23

$$\lim_{m \rightarrow \infty} c_0^{(n,m)} = c_0^{(n)}. \quad (5.7)$$

Now choose a subsequence $\mathbf{x}^{(n,m_n)}$ such that $\lim_{n \rightarrow \infty} \mathbf{x}^{(n,m_n)} = \mathbf{x}$ and $\lim_{n \rightarrow \infty} c_0^{(n,m_n)} = c_0^{(n)}$. Then corollary 5.23 implies

$$\lim_{n \rightarrow \infty} c_0^{(n,m_n)} = c_0. \quad (5.8)$$

Finally, 5.7 and 5.8 show that $\lim_{n \rightarrow \infty} c_0^{(n)} = c_0$. □

It is easy to see that Lemma 5.18 and Lemma 5.19 extend to any tiling $\mathbf{x} \in \mathcal{T}_\Phi$ as follows.

Lemma 5.25. *For any tiling $\mathbf{x} \in \mathcal{T}_\Phi$, the code for row r_{k+1} can be determined from the code of row r_k by way of the adding machine: $c_{k+1} = f(c_k)$ for all $k \in \mathbb{Z}$.*

Lemma 5.26. *For any tiling $\mathbf{x} \in \mathcal{T}_\Phi$, let the code of row r_k be of the form $c_k = (111 \dots 10w)$, where $m \geq 0$ is the number of leading ones in c_k and $w \in \mathcal{G}$. Then the row r_{k+1} is obtained from row r_k by translation; namely $r_{k+1} = r_k + (\tau, 1)$ where $\tau := \lambda_1^{m+1} - \lambda_1^m - \dots - \lambda_1$.*

Remark 5.27. For every fixed $k \in \mathbb{Z}$, the translation τ in lemma 5.26 depends continuously on $\mathbf{x} \in \mathcal{T}_a$.

Proof. From Corollary 5.24 it suffices to show τ depends continuously on c_0 . Take $c_0^{(n)} \in \mathcal{G}$ such that $\lim_{n \rightarrow \infty} c_0^{(n)} = c_0$ where c_0 has m leading ones. For $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d_{\mathcal{G}}(c_0^{(n)}, c_0) < \epsilon$. Choose $M > m$ such that $\frac{1}{2^M} < \epsilon$. Then $c^{(n)}$ has m leading for $n > N$. That is $\tau^{(n)} = \tau$ for $n > N$. $\square$

Lemma 5.28. Suppose $\mathbf{x}, \mathbf{x}' \in \mathcal{T}_{\Phi}$ are such that $\Phi(\mathbf{x}) = \mathbf{x}'$.

If $\mathbf{x}$ is aligned, then the codes of $\mathbf{x}$ and $\mathbf{x}'$ satisfy

$$c'_{2k} = 0c_k \quad \text{and} \quad c'_{2k+1} = 1c_k \quad \text{for all } k \in \mathbb{Z}. \quad (5.9)$$

If $\mathbf{x}$ is half-aligned then the codes satisfy

$$c'_{2k} = 1c_k \quad \text{and} \quad c'_{2k+1} = 0c_{k+1} \quad \text{for all } k \in \mathbb{Z}. \quad (5.10)$$

Proof. Take $\mathbf{x}_n := \mathbf{x}_0 + t_n$ such that $\lim_{n \rightarrow \infty} \mathbf{x}_0 + t_n = \mathbf{x}$ and $\lim_{n \rightarrow \infty} c_k^{(n)} = c_k$ for all $k \in \mathbb{Z}$ by Corollary 5.23. Then $\mathbf{x}'_n := \Phi(\mathbf{x}_n) = \mathbf{x}_0 + \Lambda t_n$ converges to $\mathbf{x}' = \Phi(\mathbf{x})$ and $\lim_{n \rightarrow \infty} c_k'^{(n)} = c'_k$ for all k (by Corollary 5.23). Applying Lemma 5.12 to $\mathbf{x}'_n$ and $\mathbf{x}_n$ and passing to the limit with $n \rightarrow \infty$ yields gives (5.9) and (5.10). $\square$

Recognizability

In this section we establish that the substitution Φ is recognizable. In terms of the tiling dynamical system, this means that the inflation-substitution map $\Phi : \mathcal{T}_{\Phi} \rightarrow \mathcal{T}_{\Phi}$ is invertible. It is easy to see that Φ is surjective since any tiling can be approximated by $\mathbf{x}_0 + t$ and $\Phi(\mathbf{x}_0 + \Lambda^{-1}t) = \mathbf{x}_0 + t$. However, it is not clear a priori

that there exists only one preimage under Φ . We will use the codings of rows to show injectivity of Φ .

Lemma 5.29. *For any $\mathbf{x} \in \mathcal{T}_\Phi$, there exists a unique $\mathbf{x}' \in \mathcal{T}_\Phi$ such that $\Phi(\mathbf{x}') = \mathbf{x}$.*

Proof: Fix $\mathbf{x} \in \mathcal{T}_\Phi$. Suppose there exist tilings $\mathbf{x}', \mathbf{x}'' \in \mathcal{T}_\Phi$ such that $\Phi(\mathbf{x}') = \mathbf{x} = \Phi(\mathbf{x}'')$. Note that $\Phi(\mathbf{x}' + \Lambda^{-1}t) = \mathbf{x} + t = \Phi(\mathbf{x}'' + \Lambda^{-1}t)$ for all $t \in \mathbb{R}^2$. This means that if $\mathbf{x}$ is not aligned we can make a small vertical translation so that $\mathbf{x} + t$ is aligned. So we might as well assume that $\mathbf{x}$ is aligned. Let (c'_k) and (c''_k) be the codes of rows in $\mathbf{x}'$ and $\mathbf{x}''$ respectively. Because $\mathbf{x}$ is aligned, its preimages $\mathbf{x}'$ and $\mathbf{x}''$ must be either aligned or half-aligned. Suppose that $\mathbf{x}'$ is half-aligned and $\mathbf{x}''$ is aligned. Then Lemma 5.28 implies that for each $k \in \mathbb{Z}$ the codes satisfy $c_{2k} = 1c'_k = 0c''_k$ and $c_{2k+1} = 0c'_{k+1} = 1c''_k$, which cannot happen because the first digit cannot be both 1 and 0 simultaneously. Therefore we can conclude that $\mathbf{x}'$ and $\mathbf{x}''$ must have the same alignment. If $\mathbf{x}'$ and $\mathbf{x}''$ are both aligned, then Lemma 5.28 implies that the codes satisfy $c_{2k} = 0c'_k = 0c''_k$ and $c_{2k+1} = 1c'_k = 1c''_k$ ($k \in \mathbb{Z}$). If $\mathbf{x}'$ and $\mathbf{x}''$ are both half-aligned, then Lemma 5.28 implies that the codes satisfy $c_{2k} = 0c'_k + 1 = 0c''_k + 1$ and $c_{2k+1} = 0c'_{k+1} = 0c''_{k+1}$ ($k \in \mathbb{Z}$). In both cases we see that $c'_k = c''_k$ for each $k \in \mathbb{Z}$. In particular, the first symbols of c'_k and c''_k coincide, and our assumption that $\Phi(\mathbf{x}') = \mathbf{x} = \Phi(\mathbf{x}'')$ yields $\phi_1(r'_k) = r_{2k} = \phi_1(r''_k)$ and $\phi_2(r'_k) = r_{2k+1} = \phi_2(r''_k)$ ($k \in \mathbb{Z}$). Thus $r'_k = r''_k$ because ϕ_i is injective for $i = 1, 2$ (reference). In conclusion, we have shown that $\mathbf{x}'$ and $\mathbf{x}''$ have the same alignment and that their rows are equal, hence $\mathbf{x}' = \mathbf{x}''$. $\square$

Now that we have recognizability we can give the following definition of an n -supertile.

Definition 5.30. Suppose $\mathbf{x}$ is a tiling in $\mathcal{T}_\Phi$. Let $\mathbf{x}_{-n} \in \mathcal{T}_\Phi$ be a tiling such that $\Phi^n(\mathbf{x}_{-n}) = \mathbf{x}$. Then a patch of $\mathbf{x}$ is called an n -supertile if and only if it equals $\Phi^n(T)$ for some tile T of $\mathbf{x}_{-n}$.

Proof of Non-FLC Tiling

Let $\bar{\mathbf{x}}_0$ be the tiling of the right half plane obtained by growing the patch $(\begin{smallmatrix} a \\ a \end{smallmatrix})$ under Φ and $\bar{r}_k$ be its rows ($k \in \mathbb{Z}$). Similar to $\mathbf{x}_0$, $\bar{\mathbf{x}}_0$ has a natural hierarchy of super tiles, and the codes of $\bar{r}_{-1}$ and $\bar{r}_0$ are $\bar{c}_{-1} = 1^\infty$ and $\bar{c}_0 = 0^\infty$. We observe that the tiles in $\bar{r}_{-1}$ spell the infinite word $w_{-1} := \phi_2^\infty(a)$ and the tiles in $\bar{r}_0$ spell the infinite word $w_0 := \phi_1^\infty(a)$. Let E be the set of the positions along the x -axis of all left edges of a -tiles in $\bar{r}_0$ and all right edges of a -tiles in $\bar{r}_{-1}$. Given $\tau \in E$, we define the *offset* $\mathcal{O}(\tau)$ between a -tiles in row $\bar{r}_0$ and a -tiles in row $\bar{r}_{-1}$ as follows. If τ is at the right edge of an a tile in $\bar{r}_{-1}$, then we define $\mathcal{O}(\tau) := \tau' - \tau \geq 0$, where τ' is the position of the left edge of the nearest a -tile of $\bar{r}_0$ entirely contained in $[\tau, \infty)$.

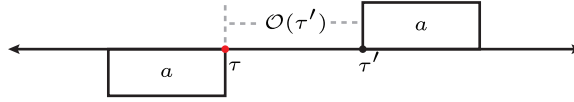
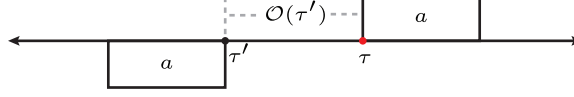


Figure 5.21: Offset defined using the tile edges from r_{-1}

If τ is at the left edge of an a -tile in $\bar{r}_0$, then we define $\mathcal{O}(\tau) := \tau - \tau' \geq 0$, where τ' is the position of the right edge of the nearest a -tile of $\bar{r}_{-1}$ entirely contained in $(0, \tau]$.

Note that our substitution Φ does not allow two horizontally adjacent b -tiles in any row of $\bar{\mathbf{x}}_0$. Because $|b| = 3$, we conclude that for all $\tau \in E$, $\mathcal{O}(\tau) \in [0, 3]$.

Figure 5.22: Offset defined using the tile edges from r_0

Additionally, $\mathcal{O}(\tau) \pmod{3}$ is determined by the number of a -tiles alone; precisely

$$\mathcal{O}(\tau) = \lambda_1 a(\tau) \pmod{3} \quad (5.11)$$

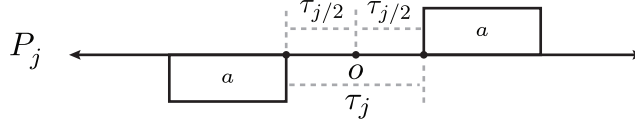
where $a(\tau) = a^{(\infty)}(\tau)$, as defined in (5.2). Let $A := \{\mathcal{O}(\tau) \pmod{3} \mid \tau \in E\}$, a subset of the circle $\mathbb{R}/3\mathbb{Z}$.

Lemma 5.31. *A is dense in $\mathbb{R}/3\mathbb{Z}$.*

Proof: Observe that $a(\tau)$ changes in steps of ± 1 as τ increases and recall that $a(\tau)$ is unbounded (by Lemma 5.9). This implies that, for any $m \in \mathbb{N}$, either $\{0, \lambda_1, \dots, m\lambda_1\} \pmod{3} \subset A$ or $\{0, -\lambda_1, \dots, -m\lambda_1\} \pmod{3} \subset A$. One of those possibilities takes place for infinitely many m , so A contains a forward orbit of the rotation by λ_1 or $-\lambda_1$. Hence A is a dense subset of $\mathbb{R}/3\mathbb{Z}$ because λ_1 is irrational. $\square$

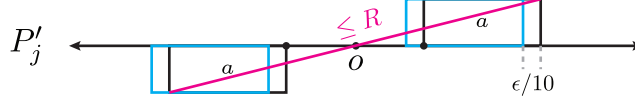
Lemma 5.32. *For any $s \in (0, 3)$, there exists a tiling $\mathbf{x} \in \mathcal{T}_a$ with a pair of a -tiles (one in row r_{-1} and one in row r_0) offset by s ; i.e., $\exists \tau \in E$ with $\mathcal{O}(\tau) = s$. Additionally, row r_{-1} will have code $c_{-1} = 1^\infty$.*

Proof: Fix an arbitrary $s \in (0, 3)$. By Lemma 5.31 there exists a sequence $(\tau_n)_{n=1}^\infty \subseteq E$ with $\lim_{n \rightarrow \infty} \mathcal{O}(\tau_n) = s$. Let $\mathbf{x}_n := \bar{\mathbf{x}}_0 - (\tau_n, 0)$, the tiling of the right half plane translated to the left by τ_n . Upon passing to a subsequence $n_k \rightarrow \infty$, by compactness there exists a tiling of the whole plane $\mathbf{x}_s := \lim_{k \rightarrow \infty} \mathbf{x}_{n_k}$ which contains a pair of tiles offset by s near the origin and r_{-1} has code $c_{-1} = 1^\infty$. $\square$

Figure 5.23: Patch P_j

Let us recall a few definitions pertinent to Proposition 5.1. A *patch* P of a tiling $\mathbf{x}$ is a connected subset of tiles in $\mathbf{x}$. The *diameter of a patch* P is defined as $\text{diam}(P) := \max\{|p - q| : p, q \text{ are points in tiles of } P\}$. We say that two patches are *equivalent* (up to translation), $P \equiv Q$, iff there exists $t \in \mathbb{R}^2$ such that $P = Q + t$. A tiling $\mathbf{x}$ has *finite local complexity* (FLC) iff for all $R > 0$ the set of equivalence classes of patches of diameter less than or equal to R is finite; i.e., $\forall_{R>0} \exists_{N \in \mathbb{N}} \exists_{P_1, P_2, \dots, P_N}$ such that if P is a patch of $\mathbf{x}$ with $\text{diam}(P) \leq R$ then $\exists_{j \in \{1, 2, \dots, N\}}$ such that $P \equiv P_j$.

Proof of Proposition 5.1: Take any $\mathbf{y} \in \mathcal{T}_\Phi$. Suppose $\mathbf{y}$ has FLC, so we can use the topology that requires only a single translation to match up two patches as described in Section Fix $N \in \mathbb{N}$. Pick $\tau_j \in (0, 3)$ for $j = 1, 2, \dots, N$ with $\tau_j \neq \tau_k$ for $j \neq k$. Let P_j be a patch containing two a -tiles offset by $\tau_j \in (0, 3)$ centered at the origin (see figure 5.23). Let $R = \max\{\text{diam}(P_j) + \sqrt{2}/5\}$. We will show that $\mathbf{y}$ has N inequivalent patches of diameter R , which shows that $\mathbf{y}$ fails to have FLC. For each τ_j we use Lemma 5.32 to get a tiling $\mathbf{x}_j \in \mathcal{T}_\Phi$ such that P_j is a patch of $\mathbf{x}_j$ centered at the origin. Let $\epsilon := \min_{j \neq k} (|\tau_j - \tau_k|)$ so that $0 < \epsilon < 1$. Since $cl\{\mathbf{y} + t : t \in \mathbb{R}^2\} = \mathcal{T}_\Phi$, for all $j \in \{1, 2, \dots, N\}$, there exists t_j such that $\mathbf{y} + t_j$ is so close to $\mathbf{x}_j$ that it contains a patch P'_j with two a -tiles offset by τ'_j with $|\tau'_j - \tau_j| < \epsilon/10$. Then by the triangle inequality we have $|\tau'_j - \tau'_k| \geq |\tau_j - \tau_k| - |\tau_j - \tau'_j| - |\tau_k - \tau'_k| \geq \epsilon - \epsilon/10 - \epsilon/10 = 8\epsilon/10 > 0$. That is $\tau'_j \neq \tau'_k$ for all $j \neq k$. Finally, any patch in $\mathbf{y} + t_j$ is also a patch in $\mathbf{y}$. Hence $\mathbf{y}$ has N different patches of diameter $R + \sqrt{5}/2$ (see figure 5.24). $\square$

Figure 5.24: Patch P'_j

The Critical Case of the Fault Line

Now that we have shown that the tiling space $\mathcal{T}_\Phi$ fails to have FLC, we attempt to describe its local structure in more detail. For the critical case where rows r and r' having codings $c = 1^\infty$ and $c' = 0^\infty$, we will see that the vertical edges of the tiles in r are horizontally offset by arbitrary distances to the vertical edges of tiles in r' . In fact, the tiles in r are unrelated to the tiles in r' . In the following results, we use the same notation for a 1-dimensional tiling r as we do for a row r of tiles in a 2-dimensional tiling $\mathbf{x}$. To get a row r corresponding to a 1-dimensional tiling r , simply inflate every tile in the 1-dimensional tiling r to have height one and place it in $\mathbb{R}^2$ so that the tiles saddling the origin now saddle the y -axis. We apologize for this abuse of notation, but we are confident that the context will provide the appropriate meaning.

Proposition 5.33. *For any two 1-dimensional tilings r_{-1} and r_0 obtained from our substitutions ϕ_1 and ϕ_2 , there exists a 2-dimensional tiling $\mathbf{x} \in \mathcal{T}_\Phi$ containing row r_0 directly above row r_{-1} with codes $c = 1^\infty$ and $c' = 0^\infty$.*

Proof: Take an arbitrary tiling r_{-1} from the tiling space determined by the substitution ϕ_2 and an arbitrary tiling r_0 from the tiling space determined by the substitution ϕ_1 . Denote the finite central patches of r_i by $\tilde{r}_i := r_i|_{-N}^N$. As shown in the 1-dimensional case, any finite part of r_{-1} is a sub-patch of $\phi_2^N(a)$ and any finite

part of r_0 is a sub-patch of $\phi_1^N(a)$ for all $N \in \mathbb{N}$ sufficiently large. So there exists $N \in \mathbb{N}$ and $s_N \in [0, 3]$ such that the image of the pair of a tiles offset by s_N under Φ^N contains the pair $(\tilde{r}_0, \tilde{r}_{-1})$. Better yet, from Lemma 5.32, there exists a tiling $\mathbf{x}_{s_N} \in \mathcal{T}_\Phi$ such that $\Phi^N(\mathbf{x}_{s_N}) \in \mathcal{T}_\Phi$ contains the central patches $(\tilde{r}_0, \tilde{r}_{-1})$. By compactness of $\mathcal{T}_\Phi$, there exists a convergent subsequence $\Phi^{N_k}(\mathbf{x}_{s_{N_k}})$ such that $\lim_{N_k \rightarrow \infty} \Phi^{N_k}(\mathbf{x}_{s_{N_k}}) = \mathbf{y} \in \mathcal{T}_\Phi$ and $\mathbf{y}$ has the desired pair of rows (r_0, r_{-1}) . $\square$

For completeness we include the following two lemmas, which give us an idea of size of $\mathcal{T}_\Phi$, mainly that it is at least as big as $\mathcal{G}$.

Lemma 5.34. *For any code $\sigma \in \mathcal{G}$ terminating with 0^∞ , i.e. $\sigma = \sigma_1\sigma_2\ldots\sigma_m000\ldots$ with $\sigma_i \in \mathcal{G}$, there exists a tiling $\mathbf{x} \in \mathcal{T}_\Phi$ with $c_0 = \sigma$.*

Proof: Suppose $\sigma = \sigma_1\sigma_2\ldots\sigma_m000\ldots$ with $\sigma_i \in \mathcal{G}$. Take $n := \sigma_1 + 2\sigma_2 + 2^2\sigma_3 + \ldots + 2^{(m-1)}\sigma_m$. Using Lemma 5.11, we see that the $(n+1)$ st row of $\mathbf{x}_0$ has the coding σ since $(((((000\ldots) + 1) + 1) + \cdots) + 1) = n$. So let $\mathbf{x} = \mathbf{x}_0 - (0, n)$, the tiling $\mathbf{x}_0$ translated down n units so that σ represents the first row of $\mathbf{x}$. $\square$

Lemma 5.35. *For any code $\sigma \in \mathcal{G}$, there exists a tiling $\mathbf{x}' \in \mathcal{T}_\Phi$ with code $c'_0 = \sigma$.*

Proof: Suppose $\sigma = \sigma_1\sigma_2\ldots\sigma_n\ldots$. Define $\sigma^{(n)} = \sigma_1\sigma_2\ldots\sigma_n000\ldots$. Let $\mathbf{x}^{(n)}$ be the tiling with first row coding of $\sigma^{(n)}$. By compactness of $\mathcal{T}_\Phi$, there exists a convergent subsequence $\mathbf{x}^{(n_k)} \in \mathcal{T}_\Phi$ such that $\lim_{n_k \rightarrow \infty} \mathbf{x}^{(n_k)} = \mathbf{x}' \in \mathcal{T}_\Phi$ and $\lim_{n_k \rightarrow \infty} \sigma^{(n_k)} = \sigma$ is the coding of the first row of $\mathbf{x}'$. $\square$

Proof that (WE) Fails

Proof of Proposition 5.2: Fix $r_0 > 0$ and $\Delta_0 > 0$.

Claim 5.36. : For any $\tau \leq \Delta_0$ there exists tilings $\mathbf{x}, \tilde{\mathbf{x}} \in \mathcal{T}_\Phi$ such that

$$r_k = \begin{cases} \tilde{r}_k & k < 0 \\ \tilde{r}_k + (\tau, 0) & k \geq 0 \end{cases}$$

i.e., the x -axis is a fault line.

If the claim is true, then $d(\mathbf{x} + t, \tilde{\mathbf{x}} + t) \leq \Delta_0$ for all $t \in \mathbb{R}^2$ yet for all $s \in B_{r_0}$ $\mathbf{x} \neq \tilde{\mathbf{x}} + s$. So (WE) fails.

Proof of claim: Arbitrarily fix $\tau \leq \Delta_0$. Take any 1-dimensional tilings $\tilde{r}_{-1}, \tilde{r}_0 \in \mathcal{T}_{\phi_1}$. Let $r_{-1} = \tilde{r}_{-1}$ and $r_0 = \tilde{r}_0 + \tau$. By Proposition 5.33, there exist tilings $\mathbf{x}, \tilde{\mathbf{x}} \in \mathcal{T}_\Phi$ with rows r_{-1}, r_0 and $\tilde{r}_{-1}, \tilde{r}_0$, respectively, with coding $c_{-1} = \tilde{c}_{-1} = 1^\infty$. Moreover, $r_k = \tilde{r}_k$ for $k < 0$ and $r_k = \tilde{r}_k + \tau$ for $k \geq 0$. $\square$

Proof of the Local Structure Theorem 5.3

$\mathcal{T}_g$ and $\mathcal{T}_b$ are both dense by minimality of the $\mathbb{R}^2$ -action on $\mathcal{T}_\Phi$. Now we prove statements (i-vi) individually.

(i) Fix $(c, r) \in \mathcal{G} \setminus \mathcal{G}_0 \times \mathcal{T}_\phi$. Then $Z(c, r)$ is a tiling $\mathbf{x} \in \mathcal{T}_a$ whose zeroth row r_0 corresponds to the tiling $r \in \mathcal{T}_\phi$ and has the code $c \in \mathcal{G} \setminus \mathcal{G}_0$; i.e., $\pi_{\mathcal{G},0}(x) = (c, r)$. To see that $\mathbf{x}$ is unique we invoke lemmas 5.25 and 5.26. Lemma 5.25 implies that the remaining codes $(c_k)_{k \neq 0}$ are determined by the orbit of c under the adding machine; i.e., $c_k = f^k(c)$. Now we apply Lemma 5.26 consecutively to each row using the codes $(c_k)_{k \in \mathbb{Z}}$ respectively. In other words, each row r_k in x is just a translation of r by the vector $\tau_k := \tau_k(c)$. Moreover, $\tau_k = \tau_k(c)$ depends continuously on c . Indeed, assuming $k > 0$ for specificity, $\tau_k(c)$ is given as a sum of translations between rows r_j and r_{j+1} where $j = 0, \dots, k-1$, and each such translation is a continuous function of c (Remark 5.27). It follows that the inverse of $\pi_{\mathcal{G},0}|_{\mathcal{T}_b}$ is continuous. That $\pi_{\mathcal{G},0}|_{\mathcal{T}_b}$

is continuous rests on the continuity of $c(x) = c_0$ as a function of $\mathbf{x}$ which is shown Corollary 5.24. $\square$

(ii) Fix $r_0 := r \in \mathcal{T}_\phi$ and $c_0 := c \in \mathcal{G}_0$. There exists $k_* \in \mathbb{Z}$ so that $c = f^{k_*}(1^\infty)$. For specificity assume that $k_* < 0$. Any tiling $\mathbf{x} \in Z(c, r)$ has a fault line between rows r_{k_*} and r_{k_*+1} which have codes 1^∞ and 0^∞ , respectively. The row r_{k_*} is on the other side of the fault line from r_0 and we will refer to it as the *transcritical row* of $\mathbf{x}$. (If $k_* \geq 0$ that would be the row r_{k_*+1} .) According to Proposition 5.33 rows r_{k_*} and r_{k_*+1} are completely unrelated. Therefore $\mathbf{x}$ is uniquely determined by the triple (c_0, r_0, r_{k_*}) . Precisely, any row r_k for $k \geq k_* + 1$ is a translations of r_0 by an amount determined by k and the code c (and thus by k and k_*), and any row r_k for $k \leq k_*$ is a translation of r_{k_*} by an amount determined by k and k_* .

Claim 5.37. *The dependence of $\mathbf{x}$ on (c_0, r_0, r_{k_*}) is continuous.*

In particular, the dependence of $\mathbf{x}$ on r_{k_*} is continuous. Thus the correspondence $r_{k_*} \mapsto x$ is a continuous bijection between compact spaces $\mathcal{T}_\phi$ and $Z(c, r)$, hence a homeomorphism.

Proof of Claim 5.37: Take a sequence $\mathbf{x}^{(n)} \subset Z(c, r)$ with the corresponding sequence of triples $(c_0^{(n)}, r_0^{(n)}, r_{k_*}^{(n)})_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow \infty} (c_0^{(n)}, r_0^{(n)}, r_{k_*}^{(n)}) = (c_0, r_0, r_{k_*})$ in the product topology. For $\epsilon > 0$ there exists $N_1, N_2, N_3 \in \mathbb{N}$ such that the following statements hold.

- (a) $d_{\mathcal{G}}(c_0^{(n)}, c_0) < \epsilon$ for $n > N_1$.
- (b) $d_{\mathcal{T}_\phi}(r_0^{(n)}, r_0) < \epsilon$ for $n > N_2$.
- (c) $d_{\mathcal{T}_\phi}(r_{k_*}^{(n)}, r_{k_*}) < \epsilon$ for $n > N_3$.

Furthermore, (b) implies that $d_{\mathcal{T}_\phi}(r_k^{(n)}, r_k) < \epsilon$ for $n > N_2$ and $k > k_*$ since τ_k depends continuously on c_0 . Similarly, (c) implies $d_{\mathcal{T}_\phi}(r_k^{(n)}, r_k) < \epsilon$ for $n > N_3$ and $k \leq k_*$. Thus $d_{\mathcal{T}_\phi}(r_k^{(n)}, r_k) < \epsilon$ for all $k \in \mathbb{Z}$ if $n > \max\{N_1, N_2, N_3\}$. That is $\lim_{n \rightarrow \infty} x^{(n)} = x$. $\square$

(iii) Take a sequence $(c^{(n)}, r^{(n)}) \subset \mathcal{G}_0 \times \mathcal{T}_\phi$ so that $\lim_{n \rightarrow \infty} (c^{(n)}, r^{(n)}) = (c, r) \in \mathcal{G} \setminus \mathcal{G}_0 \times \mathcal{T}_\phi$. Since $c^{(n)} \in \mathcal{G}_0$ we can write it as $c^{(n)} = f^{k_*^{(n)}}(0^\infty)$ and note that $k_*^{(n)} \rightarrow \infty$. Consider first a fixed $n \in \mathbb{N}$. By the description of the fiber in the proof of (ii) above, all the tilings in $Z(c^{(n)}, r^{(n)})$ agree on rows with indices in the range from $-|k_*^{(n)}|$ to $|k_*^{(n)}|$. (This is the range that does not include the fault line.) Moreover, the rows in this range depend continuously on the zeroth row and its code. Thus, if we fix any $N > 0$, if n is large enough that $|k_*^{(n)}| > N$, then the patches of any tiling in $Z(c^{(n)}, r^{(n)})$ consisting of the rows in the range $-N$ to N do not depend on the choice of the tiling, and they converge to the corresponding patch in the sole tiling making up $Z(c, r)$. $\square$

(iv) Write c_n as $c_n = f^{k_*^{(n)}}(0^\infty)$ and note that $k_*^{(n)} \rightarrow \infty$, as before. Use the correspondence $r_{k_*^{(n)}} \mapsto x$ between $Z(c^{(n)}, r^{(n)})$ and $\mathcal{T}_\phi$ from the proof of (ii) and the definition of the metric to see that $\text{diam}(Z(c^{(n)}, r^{(n)})) \leq 2^{-|k_*^{(n)}|} \rightarrow 0$. $\square$

(v) Fix $\mathbf{x} \in \mathcal{T}_b$. Take any tiling $\tilde{\mathbf{x}} \in \mathcal{T}_a$ with a fault line. Let k_* be the index of its transcritical row. By using minimality of the $\mathbb{R}^2$ -action on $\mathcal{T}_\phi$, one can find a sequence $(t_n) \subset \mathbb{R} \times \mathbb{Z}$ such that $\tilde{\mathbf{x}}^{(n)} := \tilde{\mathbf{x}} + t_n \rightarrow x$ and the second coordinate $t_{n,2}$ of t_n converges to infinity. (To be precise, minimality secures numbers $R_n \rightarrow \infty$ are such that $\mathbf{x}' + B_{R_n}(0)$ is $1/n$ -dense in $\mathcal{T}_\phi$ for any $\mathbf{x}'$. Select t_n from the ball $B_{R_n}((0, 2R_n))$ so that $\tilde{\mathbf{x}}^{(n)}$ is within distance $1/n$ from $\tilde{\mathbf{x}}$.) Note that $\{\tilde{\mathbf{x}}^{(n)}\} \subset Z(\tilde{c}^{(n)}, \tilde{r}^{(n)})$ where $\tilde{c}^{(n)} := c(\tilde{\mathbf{x}}^{(n)})$ and $\tilde{r}^{(n)}$ is the 0th row of $\tilde{\mathbf{x}}^{(n)}$. Moreover, $\tilde{\mathbf{x}}^{(n)}$ has its transcritical row at index $k_* + t_{n,2} \rightarrow \infty$, which implies (as in (iv)) that the diameters of $Z(\tilde{c}^{(n)}, \tilde{r}^{(n)})$ shrink to zero. Thus $\tilde{\mathbf{x}}^{(n)} \rightarrow \mathbf{x}$ guarantees $Z(\tilde{c}^{(n)}, \tilde{r}^{(n)}) \rightarrow \{\mathbf{x}\}$. $\square$

(vi) Fix $\mathbf{x} \in \mathcal{T}_b$. Take any tiling $\tilde{\mathbf{x}} \in \mathcal{T}_a$ without a fault line. By minimality of the $\mathbb{R}^2$ -action on $\mathcal{T}_\Phi$, there is a sequence $(t_n) \subset \mathbb{R} \times \mathbb{Z}$ such that $\tilde{\mathbf{x}}^{(n)} := \tilde{\mathbf{x}} + t_n \rightarrow \mathbf{x}$. Because $\tilde{\mathbf{x}}^{(n)}$ have no fault lines (just like $\tilde{\mathbf{x}}$), $\{\tilde{\mathbf{x}}^{(n)}\} = Z(\tilde{c}^{(n)}, \tilde{r}^{(n)})$ where $\tilde{c}^{(n)} := c(\tilde{x}^{(n)})$ and $\tilde{r}^{(n)}$ is the 0th row of $\tilde{\mathbf{x}}^{(n)}$. Thus $Z(\tilde{c}^{(n)}, \tilde{r}^{(n)}) \rightarrow \{\mathbf{x}\}$. $\square$

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APPENDIX ATOPOLOGY ON CLOSED UNBOUNDED SUBSETS OF $\mathbb{R}^D$

In this section, consider all sets to be closed unbounded subsets of $\mathbb{R}^d$ unless stated otherwise. Let $\mathbf{d}_S$ denote the Hausdorff distance between closed sets of $\mathbb{R}^d$ with respect to any metric d_S inducing the standard topology and with the property that distance between any two points near infinity is small; i.e., for all $\mu > 0$ there exists $R > 0$ such that

$$|t|, |t'| > R \implies d_S(t, t') < \mu. \quad (\text{A.1})$$

While the metric induced from the standard unit sphere via the stereographic projection onto $\mathbb{R}^d$ is a natural choice for d_S , we provide a more concrete construction.

Let $|\cdot|$ denote the Euclidean metric on $\mathbb{R}^d$. Fix a point $s \in \mathbb{R}^d$ and define $h(t) := \frac{1}{1 + |t - s|}$. Note that

$$\forall_{t, t'} \quad |h(t) - h(t')| \leq |t - t'| \quad (\text{A.2})$$

because $|t - s| - |t' - s| \leq |t - t'|$. Take

$$d_S(t, t') := \min\{|t - t'|, h(t) + h(t')\}. \quad (\text{A.3})$$

It is clear that $d_S(t, t') = d_S(t', t)$ and $d_S(t, t') = 0$ iff $t = t'$. To verify the triangle inequality we must verify the six possibilities in which the minima of

$$d_S(t, t'') \leq d_S(t, t') + d_S(t', t'') \quad (\text{A.4})$$

can be realized. We indicate the three first choices by (1,1,1). Here (A.4) becomes $|t - t''| \leq |t - t'| + |t' - t''|$, which is of course true. The situation of the three second choices, (2,2,2), is equally as clear: $h(t) + h(t'') \leq [h(t) + h(t')] + [h(t') + h(t'')]$. By the definition of “2” in (2,1,1), we have $h(t) + h(t'') \leq |t' - t''|$. So $h(t) + h(t'') \leq$

$|t - t'| + |t' - t''|$, which is (A.4) for this case. From the “1” in case (1,2,2), we have $|t - t''| \leq h(t) + h(t'') \leq h(t) + h(t') + h(t') + h(t'')$, which is (A.4) for this case. For case (1,1,2), we have $|t - t''| \leq h(t) + h(t'')$, $|t - t'| \leq h(t) + h(t')$, and $h(t') = h(t'') \leq |t' - t''|$. So

$$|t - t''| \leq h(t) - h(t') + h(t') + h(t'') \leq |t - t'| + h(t') + h(t'')$$

by (A.2), which is (A.4) for this case. For case (2,2,1), we have $h(t) + h(t'') \leq |t - t''|$, $h(t) + h(t') \leq |t - t'|$, and $|t' - t''| \leq h(t') + h(t'')$. So

$$h(t) + h(t'') = h(t) + h(t') + h(t'') - h(t') \leq h(t) + h(t') + |t' - t''|$$

by (A.2), which is again (A.4) for this case.

Our d_S is a simplified form of the metric from [24]. Another approach, borrowed from [3] (called “sphericalization” in [35]) is to use $d_S(t, t') := |t - t'|h(t)h(t')$; however, this is only a Frink metric.

In the following notation, we use bold to indicate that we are using the Hausdorff distance with respect to d_S defined in (A.3). For any set C , define

$$\overline{B}_\epsilon(C) := \{p \in \mathbb{R}^d : |p - q| \leq \epsilon \text{ for some } q \in C\}$$

and

$$\overline{B}_\epsilon^{d_S}(C) := \{p \in \mathbb{R}^d : d_S(p, q) \leq \epsilon \text{ for some } q \in C\}.$$

We will also define

$$\mathbf{B}_\mu(C) := \{\tilde{C} : \mathbf{d}_S(\tilde{C}, C) < \mu\}$$

and

$$\overline{\mathbf{B}}_\mu(C) := \{\tilde{C} : \mathbf{d}_S(\tilde{C}, C) \leq \mu\}.$$

Finally, recall our definition $C^R := C \cap \overline{B}_R(0)$.

We will need the upper semi-continuity property of C^R as a function of R , which is established in the following lemma.

Lemma A.1. *For all $\delta_1 > 0$, and $R > 0$ there exists $\delta_2 > 0$ such that $\forall C \underset{\text{closed}}{\subseteq} \mathbb{R}^d$*

$$C^{R+\delta_2} \subseteq B_{\delta_1}(C^R). \quad (\text{A.5})$$

Proof: Suppose the lemma fails. Then there exists $\delta_1 > 0$ and $R > 0$ such that for each $\delta_2 = \frac{1}{k}$ there is $C_k \underset{\text{closed}}{\subseteq} \mathbb{R}^d$ exists $p_k \in C_k^{R+1/k}$ with $p_k \notin B_{\delta_1}(C^R)$. Using compactness of the space of closed subsets of $\overline{B}_{R+1}(0)$ with respect to the Hausdorff topology on such sets, we can find a subsequence $k_n \rightarrow \infty$ such that $p_{k_n} \rightarrow p$ and $C_{k_n}^{R+1/k_n} \rightarrow C$ for some $p \in C$ with $C \underset{\text{closed}}{\subseteq} \overline{B}_{R+1}(0)$. Since $p_k \notin B_{\delta_1}(C^R)$ and $|p| \leq R$ we get $p \notin C$, a contradiction. $\square$

Recall our definition

$$\mathbf{U}_\epsilon^R(A) := \left\{ C \underset{\text{closed}}{\subseteq} \mathbb{R}^d : C^R \subseteq B_\epsilon(A^R) \quad \text{and} \quad A^R \subseteq B_\epsilon(C^R) \right\}.$$

Lemma A.2. *For all $A \subseteq \mathbb{R}^d$, $\epsilon > 0$, $R > 0$, $C \in \mathbf{U}_\epsilon^R(A)$ there exists $\mu > 0$ such that*

$$\mathbf{B}_\mu(C) \subseteq \mathbf{U}_\epsilon^R(A). \quad (\text{A.6})$$

Proof: Take $C \in \mathbf{U}_\epsilon^R(A)$. We need to show that there exists a $\mu > 0$ so that the following two inclusions

$$\tilde{C}^R \subseteq B_\epsilon(A) \quad \text{and} \quad A^R \subseteq B_\epsilon(\tilde{C}) \quad (\text{A.7})$$

hold whenever $\mathbf{d}_S(C, \tilde{C}) < \mu$. Since $C^R \subseteq B_\epsilon(A)$ and is compact there exists $\epsilon' \in (0, \epsilon)$ such that

$$C^R \subseteq B_{\epsilon'}(A). \quad (\text{A.8})$$

Set $\delta_1 := \frac{\epsilon - \epsilon'}{2}$. Take $\delta_2 \in (0, \delta_1)$ as in Lemma 4. Take $\mu_1 > 0$ small so that

$$\forall_{\tilde{C}} \quad \mathbf{d}_S(\tilde{C}, C) < \mu_1 \implies \tilde{C}^R \subseteq B_{\delta_2}(C^R) \quad (\text{A.9})$$

(the proof of existence for μ_1 is similar to the proof of Lemma 4). Consider $\tilde{C} \in \mathbf{B}_{\mu_1}(C)$ and $\tilde{c} \in \tilde{C}^R$. Take $c \in C^R$ with $|c - \tilde{c}| < \delta_2$ per (A.9). Then $|c| \leq R + \delta_2$. So

$$c \in C^{R+\delta_2} \underset{(A.5)}{\subseteq} B_{\delta_1}(C^R) \underset{(A.8)}{\subseteq} B_{\epsilon'+\delta_1}(A).$$

Hence $\tilde{c} \in B_{\epsilon'+\delta_1+\delta_2}(A) \subseteq B_\epsilon(A)$ since $\epsilon' + \delta_1 + \delta_2 < \epsilon$. So $\tilde{C} \subseteq B_\epsilon(A)$ for all $\tilde{C}$ with $\mathbf{d}_S(C, \tilde{C}) < \mu_1$.

Now we show the second inclusion in (A.7). Since $C \in \mathbf{U}_\epsilon^R(A)$ we have $A^R \subseteq B_\epsilon(C)$. Again, there exists $\epsilon'' \in (0, \epsilon)$ such that

$$A^R \subseteq B_{\epsilon''}(C). \quad (\text{A.10})$$

Set $\delta_3 := \frac{\epsilon - \epsilon''}{2}$. In the same way we secured μ_1 for (A.9) there is $\mu_2 > 0$ so that

$$\forall_{\tilde{C}} \quad \mathbf{d}_S(\tilde{C}, C) < \mu_2 \implies C^{R+\epsilon''} \subseteq B_{\delta_3}(\tilde{C}). \quad (\text{A.11})$$

Consider $\tilde{C} \in \mathbf{B}_{\mu_2}(C)$ and $a \in A^R$. Take $c \in C$ with $|a - c| < \epsilon''$, which we can do by (A.10). Since $|c| \leq R + \epsilon''$, $c \in C^{R+\epsilon''}$. Then there exists $\tilde{c} \in \tilde{C}$ such that $|c - \tilde{c}| < \delta_3$ by (A.11). Then $|a - \tilde{c}| \leq |a - c| < \epsilon'' + \delta_3 < \epsilon$. Thus $A^R \subseteq B_\epsilon(\tilde{C})$ for all $\tilde{C}$ with $\mathbf{d}_S(C, \tilde{C}) < \mu_2$.

Finally, let $\mu := \min\{\mu_1, \mu_2\}$. Then for all $\tilde{C} \in \mathbf{B}_\mu(C)$ we have

$$\tilde{C}^R \subseteq B_\epsilon(A) \quad \text{and} \quad A^R \subseteq B_\epsilon(\tilde{C}). \quad (\text{A.12})$$

That is $\tilde{C} \in \mathbf{U}_\epsilon^R(A)$. Hence $\mathbf{B}_\mu(C) \subseteq \mathbf{U}_\epsilon^R(A)$. We have shown Lemma A.2. $\square$

Corollary A.3. *If $C \in \mathbf{U}_{\epsilon_1}^{R_1}(A_1) \cap \mathbf{U}_{\epsilon_2}^{R_2}(A_2)$, then there exists $\mu > 0$ such that*

$$\mathbf{B}_\mu(C) \subseteq \mathbf{U}_{\epsilon_1}^{R_1}(A_1) \cap \mathbf{U}_{\epsilon_2}^{R_2}(A_2). \quad (\text{A.13})$$

Proof: Apply Lemma 5 twice while taking the minimum of the two μ 's. $\square$

Lemma A.4. *For all $C \subseteq \mathbb{R}^d$ and $\mu > 0$ there exists $R > 0$ and $\epsilon > 0$ such that*

$$\mathbf{U}_\epsilon^R(C) \subseteq \mathbf{B}_\mu(C). \quad (\text{A.14})$$

Proof: It suffices to prove

$$\mathbf{U}_\epsilon^R(C) \subseteq \overline{\mathbf{B}}_\mu(C) \quad (\text{A.15})$$

since $\overline{\mathbf{B}}_\mu \subseteq \mathbf{B}_{\mu'}$ if $\mu < \mu'$. Since d_S induces the standard topology on $\mathbb{R}^d$, for all $\mu > 0$ and $R > 0$ there exists $\epsilon > 0$ such that

$$|t|, |t'| < R + 10 \quad \text{and} \quad |t - t'| < \epsilon \implies d_S(t, t') < \mu. \quad (\text{A.16})$$

Take $\mu > 0$ as in Lemma A.2 and fix $R > 0$. From our property

$$|t|, |t'| > R - 1 \implies d_S(t, t') < \mu. \quad (\text{A.17})$$

Take $\epsilon \in (0, 1)$ as in (A.16) for our chosen μ and R . Consider $\tilde{C} \in \mathbf{U}_\epsilon^R(C)$. Then we have

$$(i) \quad \tilde{C}^R \subseteq B_\epsilon(C) \quad \text{and} \quad (ii) \quad C^R \subseteq B_\epsilon(\tilde{C}). \quad (\text{A.18})$$

Take $\tilde{c} \in \tilde{C}$. If $|\tilde{c}| > R - 1$, take any $c \in C$ with $|c| > R - 1$, which exists because we assumed C is unbounded. Then $d_S(c, \tilde{c}) < \mu$ by (A.17). If $|\tilde{c}| \leq R - 1$, then there exists $c \in C$ with $|c - \tilde{c}| < \epsilon$ by (i) in (A.18). Since $|c| \leq |\tilde{c}| + \epsilon < R - 1 + \epsilon \leq R + 10$, (A.16) yields $d_S(c, \tilde{c}) < \mu$. This shows that $\tilde{C} \subseteq \overline{B}_\mu^{ds}(C)$.

Now take $c \in C$. If $|c| > R - 1$, take any $\tilde{c} \in \tilde{C}$ with $|\tilde{c}| > R - 1$, which exists because we assumed $\tilde{C}$ is unbounded. Then $d_S(c, \tilde{c}) < \mu$ by (A.17). If $|c| \leq R - 1$, then there exists $\tilde{c} \in \tilde{C}$ with $|c - \tilde{c}| < \epsilon$ by (ii) in (A.18). Since $|\tilde{c}| \leq |c| + \epsilon < R - 1 + \epsilon \leq R + 10$, (A.16) yields $d_S(c, \tilde{c}) < \mu$. This shows that $C \subseteq \overline{B}_\mu^{ds}(\tilde{C})$. We have shown $\tilde{C} \subseteq \overline{B}_\mu^{ds}(C)$ and $C \subseteq \overline{B}_\mu^{ds}(\tilde{C})$, which means that $\mathbf{d}_S(C, \tilde{C}) \leq \mu$. So $\tilde{C} \in \overline{\mathbf{B}}_\mu(C)$. $\square$

We are ready to prove Proposition 2.26 from section 2, which we state now.

Proposition A.5. *The collection of closed sets*

$$\mathcal{B} = \left\{ \mathbf{U}_\epsilon^R(A) : A \underset{\text{closed}}{\subseteq} \mathbb{R}^d, \epsilon > 0, R > 0 \right\} \quad (\text{A.19})$$

forms a basis of topology on the set of all closed subsets of $\mathbb{R}^d$.

Proof of Proposition A.5: Apply Corollary A.3 and Lemma A.4. $\square$

Proposition A.6. *The topology provided by Proposition A.5 is equivalent to the one induced by $\mathbf{d}_S$ on closed unbounded subsets of $\mathbb{R}^d$.*

Proof: Apply Lemma A.2 and Lemma A.4. □