Equiripple approximation based on a rotating unit vector
by Donald Allen Rudberg

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Electrical Engineering
Montana State University

Abstract:
A transformation, first presented by Bennett in 1952, leading to equiripple lowpass magnitude squared functions is re-examined. It is cast in the form of a complex rotating unit vector. Bandpass and high-pass generating functions leading to magnitude squared functions follow immediately when considered as rotating vectors. Pole locations are arbitrary. Modifications provide general equiripple rational, function with and without symmetry about the origin. The method is shown to be well adapted to formation of equiripple passband and, stopband functions by iterative procedures. Any lengths of passband and stopband can be accomodated. Limitations of the method are pointed out. : Numerous examples are included, both single band functions with arbitrary poles and equiripple passband and stopband functions. The method is recommended to students,and practicing engineers as either, a complement to or substitute for the elliptic function approach.

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221

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ABSTRACT

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Numerous examples are included, both single band functions with arbitrary poles and equiripple passband and stopband functions. The method is recommended to students and practicing engineers as either a complement to or substitute for the elliptic function approach.

CHAPTER 1

INTRODUCTION

1.1 A DISCUSSION OF FILTER FUNCTIONS

When speaking of transfer function approximation as applied to lumped, linear and finite networks, one implies that the approximant under consideration fulfills two requirements: (1) It describes properties of a physically realizable network and (2) it satisfies certain error criteria. The approximants to be discussed in this presentation are those describing amplitude-frequency relations.

Physical realizability requires that the ratio of the Laplace transform of a response or output variable to the Laplace transform of an excitation or input variable be a rational function. This rational function must have real coefficients (Its poles and zeros appear in conjugate pairs), it must have no right half plane poles and any j axis poles must be simple.

Let the rational function (transfer function) so defined be called $T(p)$. The magnitude squared of $T(p)$ for real frequency, i.e. $p = j\omega$, is

$$|T(j\omega)|^2 = \frac{1}{1 + \varepsilon^2 H(j\omega)} \tag{1.1}$$

$H(j\omega)$ is called a filter function and ε is the ripple factor. Usually $0 \leq H(j\omega) \leq 1$ in the passband so that $\frac{1}{1 + \varepsilon^2} \leq |T(j\omega)|^2 \leq 1$ in that interval. Given a ripple specification, one can easily determine ε. The desired transfer function, $T(p)$, is found by substituting $-p^2 = \omega^2$ into $|T(j\omega)|^2$ and factoring $|T(j\omega)|^2_{\omega^2 = -p^2}$ into left half plane and right half plane poles, using the j axis as a sorting boundary. Left half plane poles are retained along with half of those on the j axis.

Any zero configuration can be chosen such that the numerator of $T(p)$ has real coefficients. A discussion of the effects of various zero choices is an extensive subject in its own right and will not be treated. We will be concerned with the formation of $H(j\omega)$ only.

Ideally, $H(j\omega) = 0$ in the passband and is infinitely large outside the passband. A response of this nature is unrealizable (5) and a rational function approximation must be made. There are numerous criteria as to goodness of fit for the rational function. A few criteria are:

1. Maximal flatness. At a prescribed point in the passband, the maximum possible number of successive derivatives of the approximant are equated to the corresponding derivatives of the ideal function.
2. Bounding the maximum error. A very simple criterion but sometimes all that is required.
3. Minimization of average error. This can be misleading in near zero average error does not necessarily involve a small maximum error.
4. Minimization of root-mean-square error. Effect of random errors is minimized. It can involve great computational labor for high degree functions. Resulting $|T(j\omega)|^2$ resemble Chebyshev approximation when $\varepsilon = 1$.
5. Equal maxima and minima of error (equiripple error). This is also called approximation in the Chebyshev sense. $H(j\omega)$ exhibits maxima of unity and minima of zero in the passband.

In what follows, exclusively equiripple $H(j\omega)$ will be dealt with. They will be written as $H(p)$ where $p = \sigma + j\omega$ is a general complex variable. In some instances the filter function may appear as $R^2(x)$ where x is a real variable. Examples are the Chebyshev rational functions of Section 4.4 and the general equiripple passband and stopband functions of Section 4.5. Then

$$|T(x)|^2 = \frac{1}{1 + \varepsilon^2 R^2(x)} \qquad (1.2)$$

would be the corresponding magnitude squared transfer function. Substitution of $\omega^2 = x^2$ will give the usual $|T(j\omega)|^2$.

1.2 A BRIEF REVIEW OF PROGRESS

It appears to the author that somewhere around the year 1960, a partition between philosophies of approach to equiripple functions can be made. Prior to that time, work centered about generation of functions in a non-iterative fashion. Closed form trigonometric functions, elliptic expressions and series expansions were popular. This was necessary because of the great number of computations involved in iterative processes, coupled with the lack of complete computer facilities and programming languages.

The first equiripple functions encountered by most engineers were and are the classical Chebyshev polynomials. They are defined and their properties investigated in standard texts on network synthesis (6,8,19,21). They are generated in a closed form trigonometric fashion. Darlington (7) introduced a procedure using a Chebyshev polynomial series to obtain near equiripple character of rational functions. At about the same time Bennett published his Ph.D. dissertation (2) and subsequently produced an article (3) based on that thesis. The pertinent concepts are a set of transformations providing functions of an equiripple nature over a prescribed lowband interval. Sharpe (15) also proposed an equiripple generating function. It was not general in that poles were restricted to real and imaginary axes only. Helman (11) described a procedure operating in much the same manner as that of Bennett. His method involves a rotating radius vector in the proof of equiripple properties. Only angle information was used since the radius vector followed a general elliptical path. In contrast, the author's work will use a strictly circular path for a radius vector. Both angle and magnitude are utilized. Filters having equiripple approximation of zero in lowband and equiripple approximation of infinity in highband were generated by elliptic functions (1, 5) in a non-iterative fashion. Bennett (4) proposed an iterative process for elliptic function filters, giving an example. An excellent review of the state of the approximation problem in late 1954 was given by Winkler (20). With the exception of Bennett's work (4), all of the above procedures had the feature of being expressed in non-iterative terms, usually involving elliptic or trigonometric functions.

About 1960 widespread use of digital computers and development of easily understood programming languages, e.g., Fortran, brought about renewed interest in implementation of iterative procedures. Engineers began to take different approaches to the equiripple problem. One of the new approaches was that of McGee (13), based upon equalization of extrema in an iterative fashion. General equiripple passband and stopband functions were solved by extending the procedure. Temes (17) and Smith (16) employed much the same method in generating filters to fit arbitrary specifications as to passband and stopband shape. Watanabe (18) achieved equiripple behavior in one or two passbands but not in the stopbands simultaneously. A very interesting attack was made by Humphereys (12) in which points of maxima and minima in the passband are chosen explicitly. A presentation and review of both the elliptic function approach and computer usage was given by Calahan (5). An extensive biblography was also included.

1.3 PURPOSES OF THE THESIS

The thesis has three main purposes: (1) To cast the original transformation of Bennett (3) in a different light, providing a pragmatic and accurate principle for equiripple approximation; (2) to present easily used closed form lowpass, highpass and bandpass generating functions and (3) to indicate procedures based on such functions that lead to more complex filters. Some of the procedures employ the closed form functions in an iterative fashion; others are strictly closed form. It is pointed out that there exist limitations of the principle with regard to formation of functions with preassigned zeros outside the passband, herein called extrapassband zeros.

CHAPTER 2.

THE COMPLEX VECTOR PRINCIPLE

2.1 INTRODUCTION

In this chapter a method is presented to generate, in a closed form, rational functions of the complex variable $p = \sigma + j\omega$. They have the following properties: (1) They are magnitude squared functions for $p = j\omega$, (2) they exhibit maxima of +1 and minima of zero in the interval of interest, being equal to +1 at the interval boundaries, (3) poles lie in quadrantal symmetry outside the interval and are arbitrary and (4) all zeros of the functions lie in the interval and cannot be preselected, being located essentially by the poles. As a result, functions generated by this method have, in the interval of interest, the maximum number of maxima and minima possible for a given degree. Denominator degree will never exceed numerator degree unless the interval of interest extends to infinity, in which case a zero or pair of zeros may appear at infinity.

In what follows, the term "passband" will be used in place of the phrase "interval of interest". It is less cumbersome and also implies a passband in the electric filter sense, which is the primary application area of such functions in electrical engineering.

The equiripple generative procedure employed by Bennett in his thesis (2) and articles (3,4) is that of a function in an auxiliary z-plane being conformally mapped onto the p-plane. The entire real frequency axis ($j\omega$ axis) of such a z-plane is mapped onto a desired passband on the p-pland real frequency axis. Pole locations in the z-plane (prepositioned by an initial mapping from the plplane to the z-plane) are brought back to the p-plane in the same mapping. This is the same view taken by Helman (11) and Humphereys (12) in their assessment of the general utility of the functions.

Continued treatment of such functions as transformations has tended to obscure their full elegance and simplicity. The beginning of a new view appears in the proof of equiripple properties by Bennett (3) when he speaks of the z-plane function as $\varepsilon^{j\phi}$, i.e., a complex vector. An initial exposure to the functions in a modern network synthesis course presented by Dr. Bennett coupled with continued discussions afterward have led to a complete adoption of the complex vector principle. Transformations have been eliminated entirely. All operations from construction of the complex vector function to the final rational function will be carried out in the complex p-plane. Proofs and examples in the exposition of this chapter will be for the lowpass case. Modifications for highpass and bandpass cases encountered in Chapter 3 will be made as necessary.

2.2 PRESENTATION OF THE METHOD IN WORDS AND EQUATIONS

A stepwise outline of the method follows:

1. A mixed rational and irrational function of p is constructed. It is called F(p) and has the following properties:

   A. It acts as a rotating vector of unit magnitude in the passband.

   B. It always has a value of +1 or −1 at band edge.

   C. It has singularities at certain eventual pole locations.

Specifically, it has singularities at all eventual pole locations that are in the left half plane or the positive real frequency axis (the $j\omega$ axis). Poles located in the right half plane and on the negative real frequency axis are not manifested in F(p) as singularities. Figure 2-1 indicates the region for which poles are placed in F(p). The omitted poles are the negatives of those included. They will be caused to appear in the final rational function as a result of a later step. It is assumed that all pole locations are exclusive of the passband.

2. Unity is added to or subtracted from F(p), addition or subtraction being dependent on the value of F(p) at band edge. For $F(p)|_{\text{Band Edge}} = 1$, addition is indicated; for $F(p)|_{\text{Band Edge}} = -1$, subtraction is indicated. This sum or difference is divided by 2. The resulting function, called G(p), has the following properties:

   A. Its magnitude oscillates between 0 and 1 in the passband.

   B. It always has unity magnitude at band edge.

   C. Its numerator is rational.

   D. Its denominator is that of F(p). The remarks made about singularities in F(p) hold for G(p).

   E. It is nonzero at the pole locations not manifested as singularities, i.e., at those points in the right half plane and on the negative real frequency axis at which

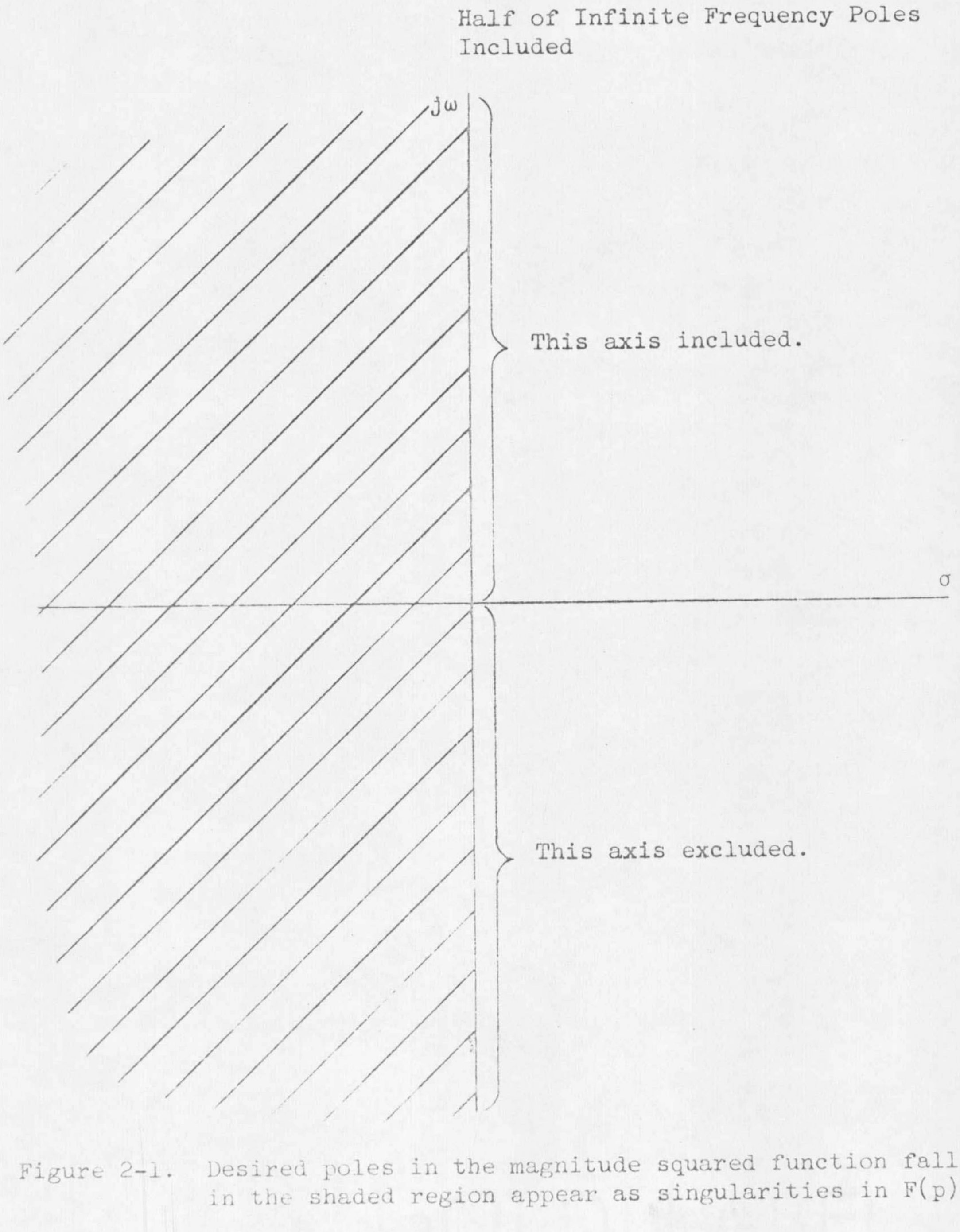


Figure 2-1. Desired poles in the magnitude squared function falling in the shaded region appear as singularities in F(p).

poles are desired.

3. A function called "G(p) conjugate" and denoted by $\bar{G}(p)$ is constructed. $\bar{G}(p)$ has the following properties:

   A. It is the conjugate of G(p) for p in the passband.

   B. It has a rational numerator that is either the same as that of G(p) or is its negative.

   C. It has singularities at those pole locations that were not singularities of G(p), i.e., its singularities occur at pole locations in the nonshaded region of Figure 2-1, including the negative j-axis.

   D. It is nonzero at the singularities of G(p).

4. The product G(p) $\bar{G}(p)$ is formed and called H(p). H(p) has the following properties:

   A. It is a rational function.

   B. It contains all the desired pole locations in both left half plane, right half plane and the $j\omega$-axis. The locations are in quadrantal symmetry.

   C. It oscillates between 0 and 1 in the passband.

   D. It is equal to 1 at band edges.

To illustrate in a few equations how this would work, assume that the mixed rational and irrational function for F(p) has been found. Its properties are expressed in equations.

$$F(p)\Big|_{p=j\omega} = e^{j\phi(\omega)} \tag{2.1}$$

$$F(p)\big|_{p=\text{Band Edge}} = \pm 1 \tag{2.2}$$

$$F(p)\big|_{p \to p_i} \to \infty \tag{2.3}$$

for prescribed p_i in the left half plane or the positive $j\omega$-axis.

Now add or subtract unity and divide by 2 to form $G(p)$.

$$G(p)\big|_{p=j\omega} = \frac{e^{j\phi(\omega)} \pm 1}{2} \tag{2.4}$$

and

$$0 \leq |G(j\omega)| \leq 1 \tag{2.5}$$

in the passband.

$$|G(p)|_{p=\text{Band Edge}} = 1 \tag{2.6}$$

The numerator of $G(p)$ is rational. Also

$$G(p)\big|_{p \to p_i} \to \infty \tag{2.7}$$

for prescribed p_i in the left half plane or on the positive $j\omega$-axis. Finally

$$G(p)\big|_{p=-p_i} \neq 0. \tag{2.8}$$

$\bar{G}(p)$, the conjugate of $G(p)$ has these properties:

$$\bar{G}(p)\Big|_{p=j\omega} = \frac{e^{-j\phi(\omega)} \pm 1}{2} \tag{2.9}$$

$$[\text{Numerator } \bar{G}(p)] = \pm[\text{Numerator } G(p)] \tag{2.10}$$

$$\bar{G}(p)\Big|_{p\to -p_i} \to \infty \tag{2.11}$$

$$\bar{G}(p)\Big|_{p=p_i} \neq 0 \tag{2.12}$$

Multiply $G(p)$ by $\bar{G}(p)$ to form $H(p)$. $H(p)$ is a purely rational function with the following properties:

$$H(p)\Big|_{p\to \pm p_i} \to \infty \tag{2.13}$$

for all p_i in $F(p)$. In the passband, $H(p)$ is expressed as

$$H(p)\Big|_{p=j\omega} = \left|\frac{e^{j\phi(\omega)} \pm 1}{2}\right|^2$$

$$= \left|\frac{\cos\phi(\omega) \pm 1 + j\ \sin\phi(\omega)}{2}\right|^2$$

$$= \frac{\cos^2\phi(\omega) + 1 \pm 2\cos\phi(\omega) + \sin^2\phi(\omega)}{4}$$

$$= \frac{1 \pm \cos\phi(\omega)}{2} \tag{2.14}$$

so that $0 \leq H(j\omega) \leq 1$ in the passband. Also

$$H(p)\Big|_{p=\text{Band Edge}} = 1. \tag{2.15}$$

It is assumed that the desired poles of $H(p)$ are in quadrantal symmetry so that $H(p)\big|_{p=j\omega}$ is a real function. For $H(p)$ to be a magnitude squared function it is necessary that it satisfy one more requirement: All j-axis critical frequencies must be of even multiplicity. The zeros of $H(p)$ will be so because it is positive on both sides of the zeros. Poles of $H(p)$ on the j-axis must be chosen of even multiplicity.

The foregoing remarks do not specify precise function forms and manipulations. They do make evident that an approach using the concept of a rotating vector provides a new insight into equiripple functions. Rather than a result of transformations or of maxima and minima choices in the passband, they are immediate and lucid. The particular form to be presented initially produces an approximation to 1/2 with a ripple value of 1/2. Later modifications will allow approximations to other functions.

2.3 THE LOWPASS FUNCTIONS

2.3.1 Lowpass F(p)

The initial $F(p)$ for the lowpass case is defined as[1]

[1]The F(p) functions for lowpass, highpass and bandpass cases were introduced to the author in private communication with Dr. B. J. Bennett.

$$F(p) \triangleq \prod_{i=1}^{n} \frac{m_i p + \sqrt{p^2 + \omega_H^2}}{-m_i p + \sqrt{p^2 + \omega_H^2}} \tag{2.16}$$

where ω_H is the upper band edge frequency. For simplicity, ω_H will be normalized to unity in all subsequent expressions. Presence in $F(p)$ of the parameters m_i and the index n indicate that $F(p)$ might properly be written as $F(p, m_i, n)$ at an added expense of notational complexity. The expense will not be borne. It will be understood in the following that $F(p)$ implicitly involves the parameters m_i and index n. Parameters m_i are used to fix the singularities of $F(p)$. Index n refers to the number of zeros in the passband (from $-\omega_H$ to ω_H in lowpass).

Let us look at an $F(p)$ with a single factor in order to ascertain its properties. We have

$$F(p) = \frac{m_i p + \sqrt{p^2 + 1}}{-m_i p + \sqrt{p^2 + 1}} \, . \tag{2.17}$$

Assume for the moment that m_i is real and positive. For $p = j\omega$

$$F(p)\Big|_{p=j\omega} = \frac{jm_i\omega + \sqrt{1 - \omega^2}}{-jm_i\omega + \sqrt{1 - \omega^2}} \, . \tag{2.18}$$

Two regions must be considered: (1) ω in the passband, i.e., $|\omega| \leq 1$ and (2) ω outside the passband, i.e., $|\omega| > 1$.

In the range $|\omega| \leq 1$, $F(p)$ has the property that its numerator is composed of an imaginary part $(jm_i\omega)$ and a real part $(\sqrt{1 - \omega^2})$, while the denominator consists of the negative imaginary part and the same real part. One might write this as

$$F(p)\big|_{p=j\omega} = \frac{j\beta(\omega) + \alpha(\omega)}{-j\beta(\omega) + \alpha(\omega)}, \quad |\omega| \leqslant 1$$

$$= e^{j\phi(\omega)} \tag{2.19}$$

where

$$\phi(\omega) = 2\tan^{-1}\frac{m_i\omega}{\sqrt{1-\omega}} \, . \tag{2.20}$$

As ω increases from 0 to 1, ϕ increases continuously from 0 to π, passing through $\pi/2$. $F(p)$ is indeed a rotating unit vector in this region, rotation being counter-clockwise. Figure 2-2 depicts $F(p)$ in the passband.

In the range $|\omega| > 1$, the numerator of $F(p)$ becomes purely imaginary as does the denominator, making the overall $F(p)$ strictly real. However, the numerator is the sum of two terms and the denominator is the difference of two terms.

$$F(p)\big|_{p=j\omega} = \frac{m_i\omega + \sqrt{\omega^2 - 1}}{-m_i\omega + \sqrt{\omega^2 - 1}}, \quad 1 < |\omega| \tag{2.21}$$

The possibility of the denominator becoming zero and therefore producing an infinite value of $F(p)$ presents itself. This would occur whenever

$$m_i\omega = \sqrt{\omega^2 - 1} \tag{2.22}$$

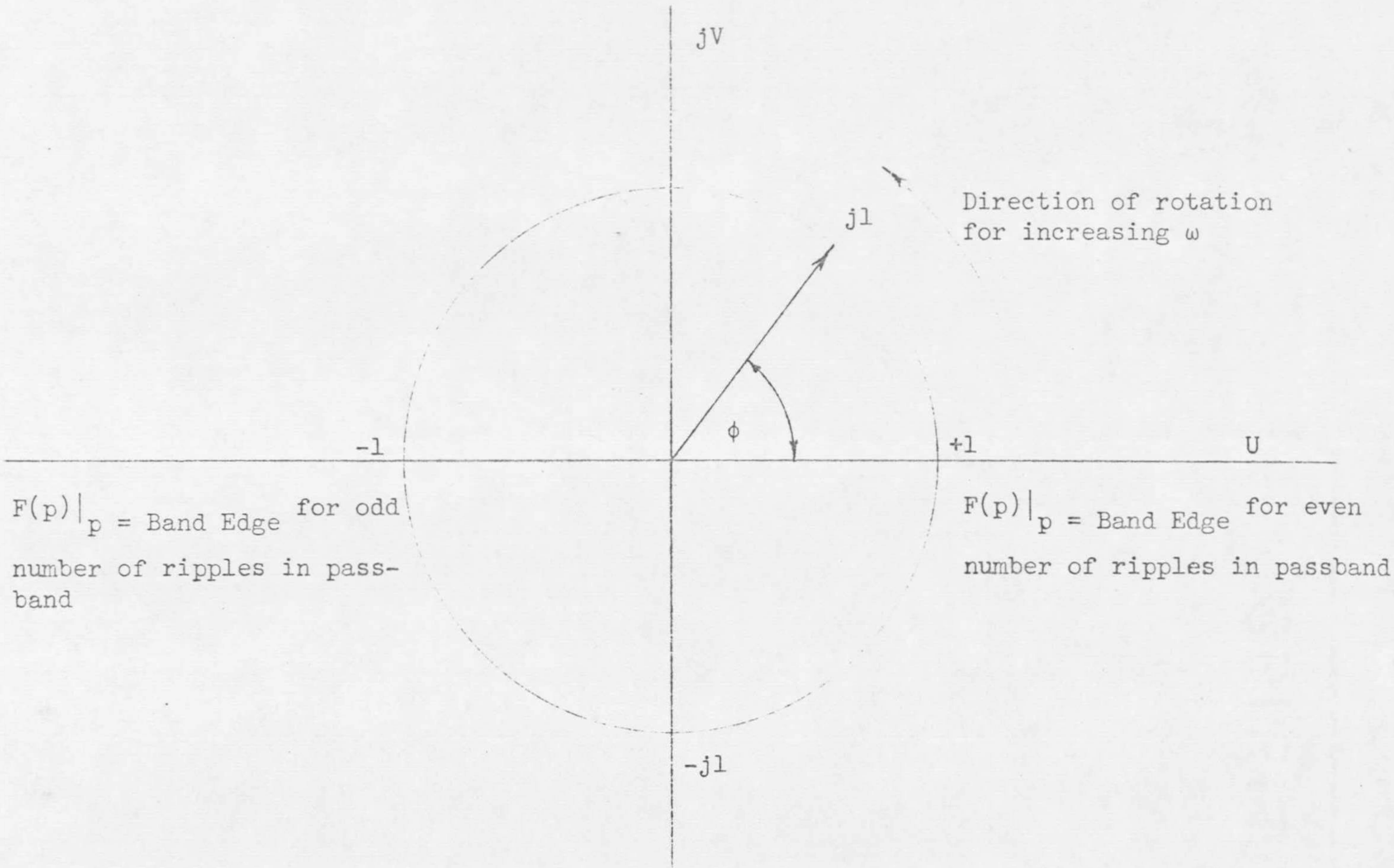


Figure 2-2. Action of $F(p) = U + jV$ for p in the passband.

or

$$\omega = \frac{1}{\sqrt{1 - m_i^2}} \tag{2.23}$$

which can be satisfied for a particular value of $\omega > 1$ if $0 < m_i < 1$. The fact that no negative value of ω will satisfy the equation is consistent with the remark that F(p) has its singularities only in the left half plane and on the positive j-axis. Singularities on the negative j-axis will be produced by $\bar{G}(p)$. $F(j\omega)$ for ω outside the passband is shown in Figure 2-3. In actual use m_i would be determined from the desired pole location, $p_i = j\omega_i$, by

$$m_i = \left.\frac{\sqrt{p^2 + 1}}{p}\right|_{p=j\omega_i} \tag{2.24}$$

which will yield a value of m_1 lying in the interval (0,1) for $\omega_1 > 1$. If an infinite frequency singularity is desired, it can be achieved by setting

$$m_i = \lim_{p\to\infty} \frac{\sqrt{p^2 + 1}}{p}$$

$$= 1. \tag{2.25}$$

So it is clear that the single factor F(p) does those things required of it.

Extension of these concepts to a multiple factor F(p) is straight-

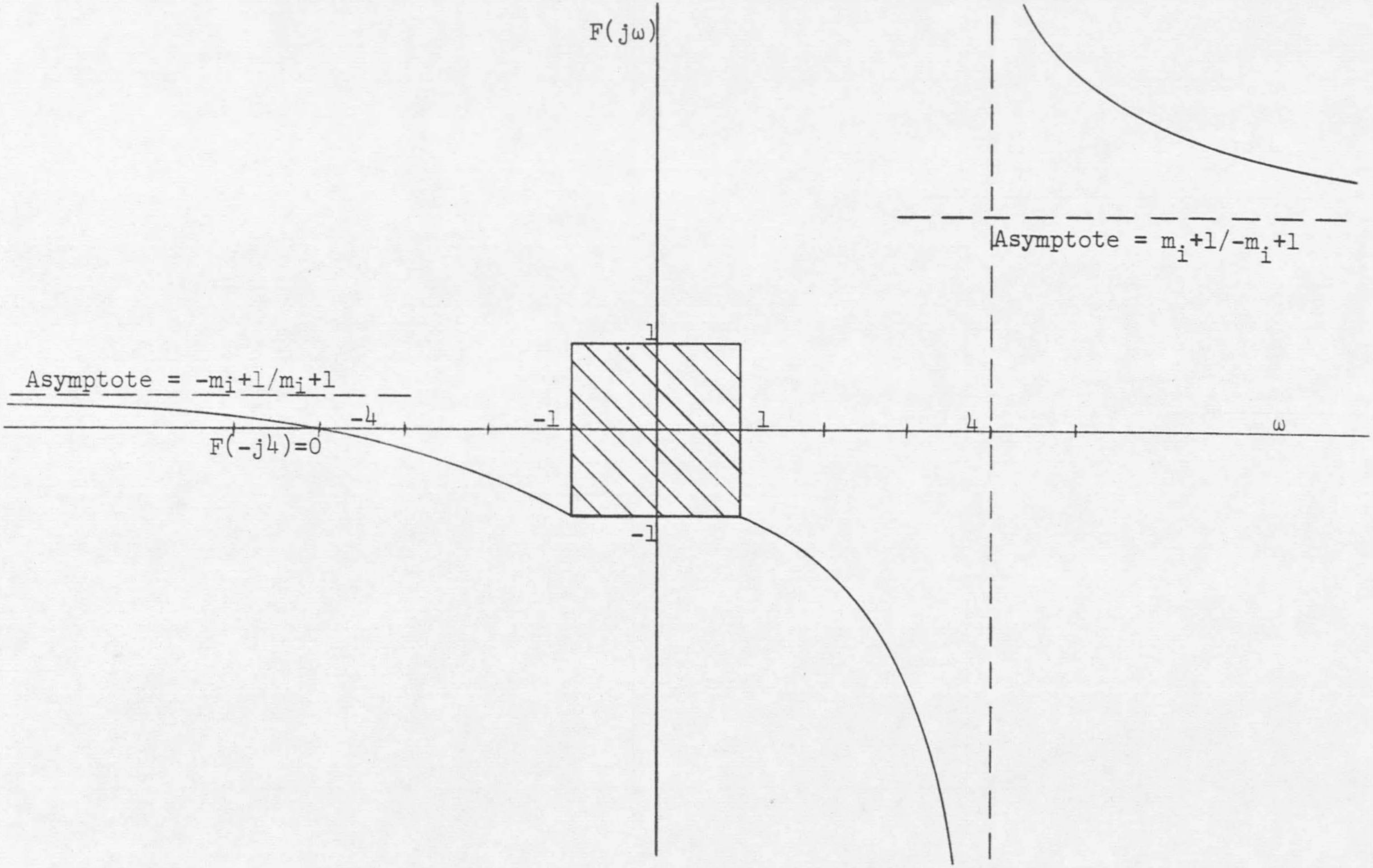


Figure 2-3. $F(j\omega)$ outside the passband. Singularity at $\omega = 4$.

forward. Writing

$$F(p) = \prod_{i=1}^{n} \frac{m_i p + \sqrt{p^2 + 1}}{-m_i p + \sqrt{p^2 + 1}} = \prod_{i=1}^{n} F_i(p) \tag{2.26}$$

and again assuming all m_i to be real and positive, we see that since each individual factor, $F_i(p)$, is of exactly the nature described above, the overall $F(p)$ has like properties.

$$F_i(p)\big|_{p=j\omega} = e^{j\phi_i(\omega)}\ , \quad |\omega| \leqslant 1 \tag{2.27}$$

where

$$\phi_i(\omega) = 2\tan^{-1} \frac{m_i \omega}{\sqrt{1 - \omega^2}} \tag{2.28}$$

so that $0 \leqslant \phi_i(\omega) \leqslant \pi$ as $0 \leqslant \omega \leqslant 1$, rotation being in a counter-clockwise direction. Then

$$\begin{aligned} F(p)\big|_{p=j\omega} &= \prod_{i=1}^{n} F_i(p)\big|_{p=j\omega} \\ &= \prod_{i=1}^{n} e^{j\phi_i(\omega)} \\ &= e^{j\Sigma\phi_i(\omega)} \\ &= e^{j\Phi(\omega)} \end{aligned} \tag{2.29}$$

where

$$\Phi(\omega) = \sum_{i=1}^{n} \phi_i(\omega) \tag{2.30}$$

so that $0 \leqslant \Phi(\omega) \leqslant n\pi$ as $0 \leqslant \omega \leqslant 1$, rotation being in a counter-clockwise direction. Again, in the region $1 < \omega$, each $F_i(p)$ can have a singularity at some ω_i, manifested in the equation by choosing

$$m_i = \left.\frac{\sqrt{p^2 + 1}}{p}\right|_{p=j\omega_i} \tag{2.31}$$

and if a singularity at infinity is desired

$$m_i = \lim_{p\to\infty} \frac{\sqrt{p^2 + 1}}{p}$$

$$= 1. \tag{2.32}$$

Each $F_i(p)$ will produce a preselected singularity so that the overall $F(p)$ will have singularities at ω_i, $i = 1, 2, \cdots, n$. It is seen from the above that each $F_i(p)$ will be zero at $-\omega_i$ and $F(p)$ will be zero at $\omega = -\omega_i$, $i - 1, 2, \cdots, n$, as shown in Figure 2-4.

2.3.2 F(p) For Complex Poles

The assumption that m_i is real restricts possible singularities to the real and imaginary axes only, leaving out potential locations in the complex plane. This restriction will now be removed.

Let us assume that a fourth degree equiripple lowpass function

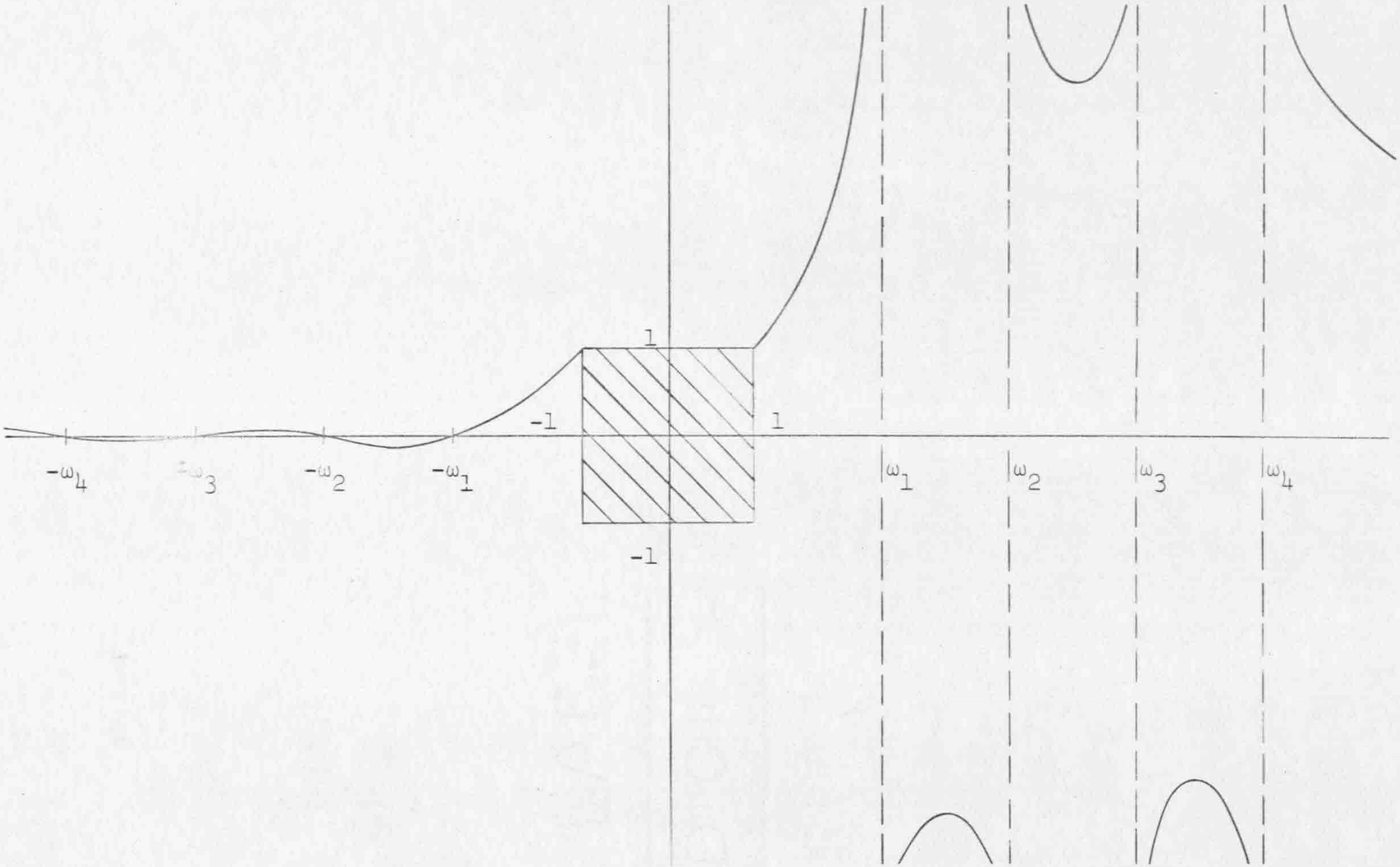


Figure 2-4. $F(p)$ constructed from four individual factors. Singularities at $\omega = \omega_1, \omega_2, \omega_3, \omega_4$.

with a pole configuration shown in Figure 2-5 is desired, viz., poles at $p = \pm\sigma_i \pm j\omega_i$. For such a function $n = 2$ and

$$F(p) = \frac{m_1 p + \sqrt{p^2+1}}{-m_1 p + \sqrt{p^2+1}} \cdot \frac{m_2 p + \sqrt{p^2+1}}{-m_2 p + \sqrt{p^2+1}}$$

$$= \frac{(m_1 m_2 + 1)p^2 + 1 + (m_1 + m_2)p\sqrt{p^2+1}}{(m_1 m_2 + 1)p^2 + 1 - (m_1 + m_2)p\sqrt{p^2+1}} \quad (2.33)$$

The pole set to be considered in formation of $F(p)$ is the left half plane set, $p = -\sigma_i \pm j\omega_i$. Graphical representations of various calculations involved are shown in Figures 2-6 and 2-7. Let the desired pole location be $p_i = r_i e^{j\theta_i}$, where r_i is the radius vector to p_i and θ_i is the angle measured counter-clockwise from the positive real axis. Express m_i as

$$m_i = \frac{\sqrt{r_i^2 e^{j2\theta_i} + 1}}{r_i e^{j\theta_i}} \quad (2.34)$$

The term inside the radical is rewritten as

$$r_i^2 e^{j2\theta_i} + 1 = R_i^2 e^{j2\psi_i} \quad (2.35)$$

Since $\pi/2 < \theta_i < \pi$, then $\pi > \psi_i > \theta_i$ and ψ_i has the same sign as θ_i. The expression for m_i becomes

$$m_i = \frac{\sqrt{R_i^2 e^{j2\psi_i}}}{r_i e^{j\theta_i}}$$

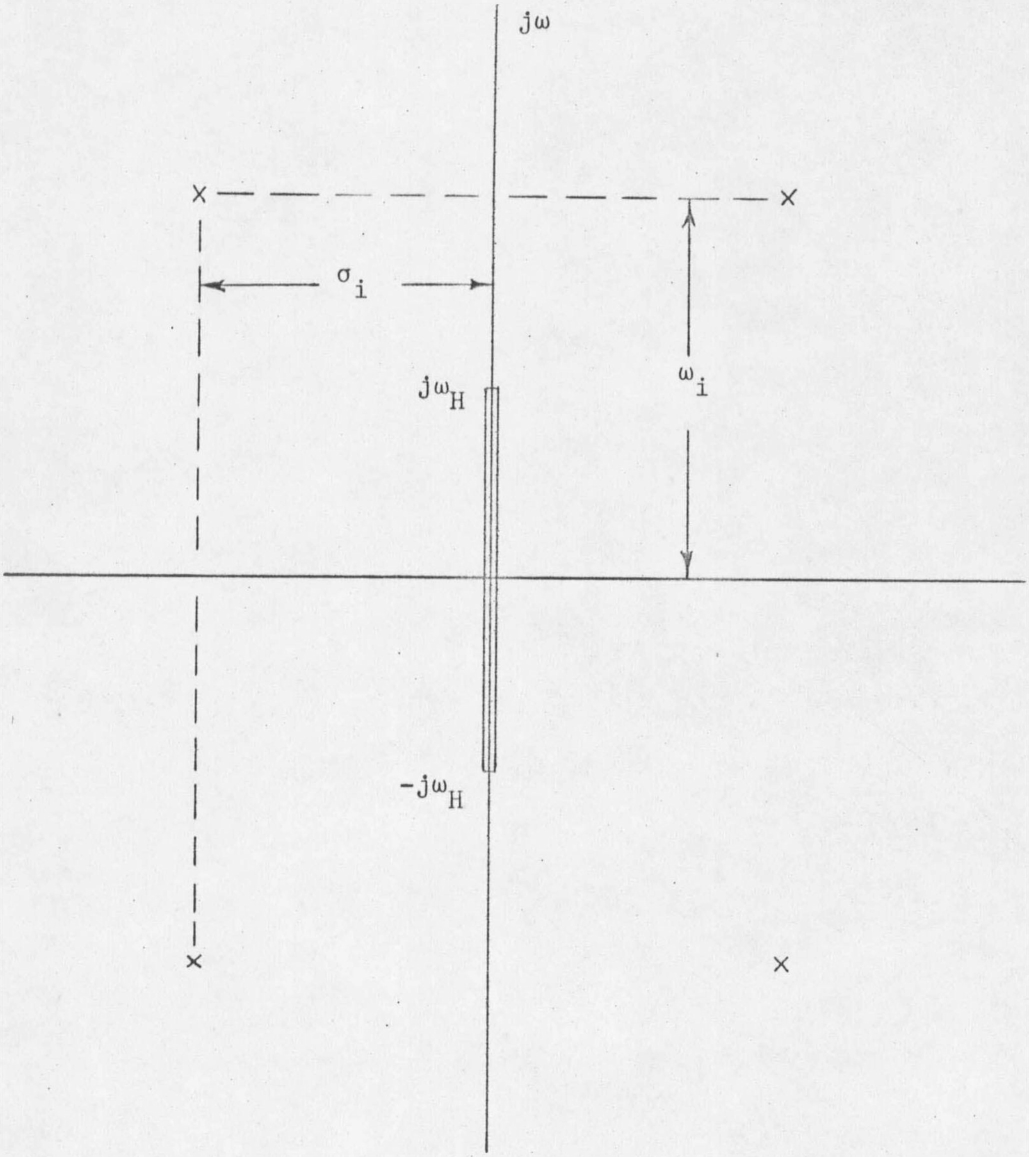


Figure 2-5. P-plane plot of the lowpass 4^{th} degree function with poles at $p = \pm\sigma_i \pm j\omega_i$. Passband shown as double lined axis.

$$= \frac{R_i}{r_i} e^{j(\psi_i - \theta_i)} \tag{2.36}$$

which is a vector in the first quadrant as in Figure 2-7(b).

Since m_i is always analytic at p_i and real for p real, Schwarz's Reflection Principle holds, showing that the m associated with $\overline{p_i}$ is $\overline{m_i}$. Let $m_i = \alpha_i + j\beta_i$ so that

$$m_i\overline{m_i} = \alpha_i^2 + \beta_i^2 \tag{2.37}$$

and

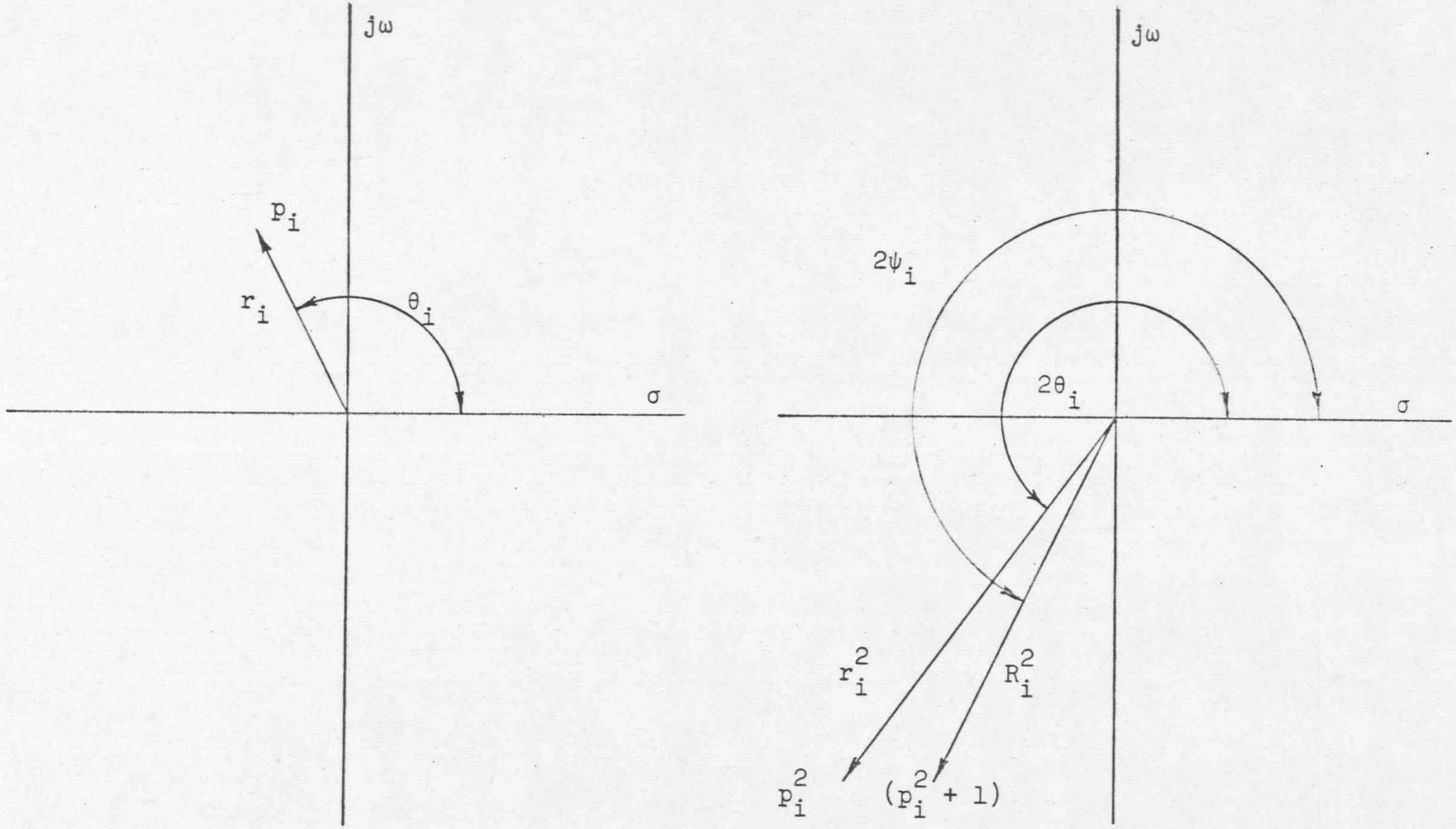
$$m_i + \overline{m_i} = 2\alpha_i. \tag{2.38}$$

Both expressions are strictly real and their substitution into F(p) yields

$$F(p) = \frac{(\alpha_i^2 + \beta_i^2 + 1)p^2 + 1 + 2\alpha_i p\sqrt{p^2 + 1}}{(\alpha_i^2 + \beta_i^2 + 1)p^2 + 1 - 2\alpha_i p\sqrt{p^2 + 1}} \tag{2.39}$$

so that the rotating vector concept again holds when conjugate poles are considered.

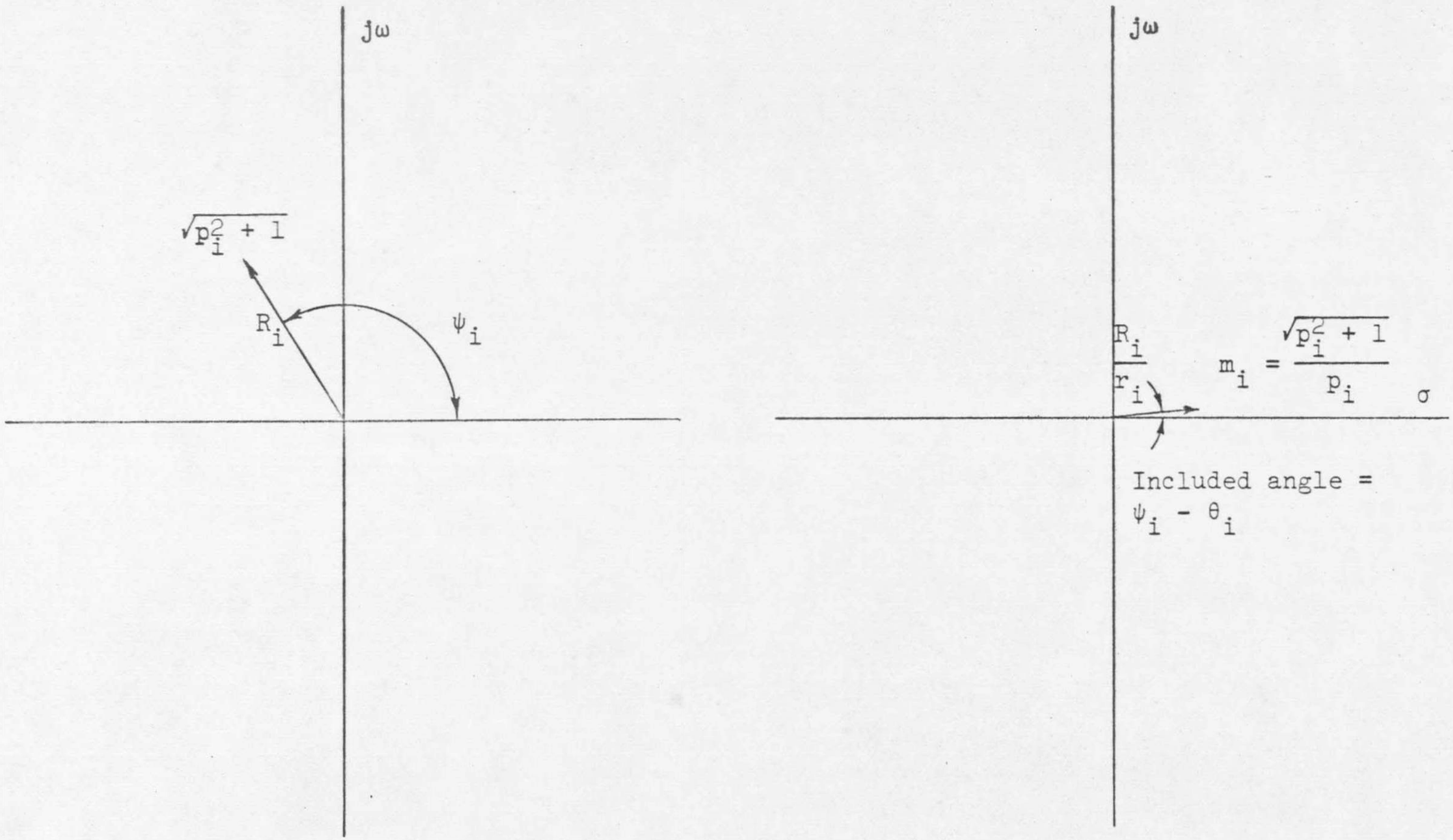
It is seen that singularities exist in F(p) only at p_i and $\overline{p_i}$. Singularities will be produced in the right half plane, i.e., at $-p_i$ and $-\overline{p_i}$ through multiplying the denominator by $[(\alpha_i^2 + \beta_i^2 + 1)p^2 + 1 +$



(a) Vector representation of $p = -\sigma_i + j\omega_i$

(b) Vector representation of p_i^2 and $(p_i^2 + 1)$

Figure 2-6. Formation of m_i for $p = p_i$. Sheet 1.



(a) Vector representation of $\sqrt{p_i^2 + 1}$

(b) Vector representation of $m_i = \frac{\sqrt{p_i^2 + 1}}{p_i}$

Figure 2-7. Formation of m_i for $p = p_i$. Sheet 2.

$2\alpha_i p\sqrt{p^2 \mp 1}$] which is [Denominator F(-p)]. Reasons for choosing such an expression will be given in the section concerning formation of $\overline{G(p)}$. Figure 2-8 shows the general nature of m_i for p_i in various regions.

2.3.3 Lowpass G(p)

G(p) is formed from F(p) by the expression

$$G(p) = \frac{F(p) \pm 1}{2} \quad (2.40)$$

In the passband

$$G(p)\Big|_{p=j\omega} = \frac{e^{j\phi(\omega)}}{2} \pm \frac{1}{2}, \quad |\omega| \leq 1 \quad (2.41)$$

which is a rotating vector of one-half unit length, centered at $(\pm\frac{1}{2}, 0)$. A representation is shown in Figure 2-9, (a) and (b). Its magnitude varies between limits of 0 and 1. Choice of addition or subtraction of 1 to F(p) is governed by requirements at band edge. For the usual case, an equiripple function is required to have a normalized magnitude of unity at band edges. Reference to equations (2.29) and (2.30) shows that

$$G(p)\Big|_{p=j\omega} = \frac{e^{jn\pi} \pm 1}{2}$$

$$= \begin{cases} \dfrac{1 \pm 1}{2} & n \text{ even} \\[2ex] \dfrac{-1 \pm 1}{2} & n \text{ odd} \end{cases} \quad (2.42)$$

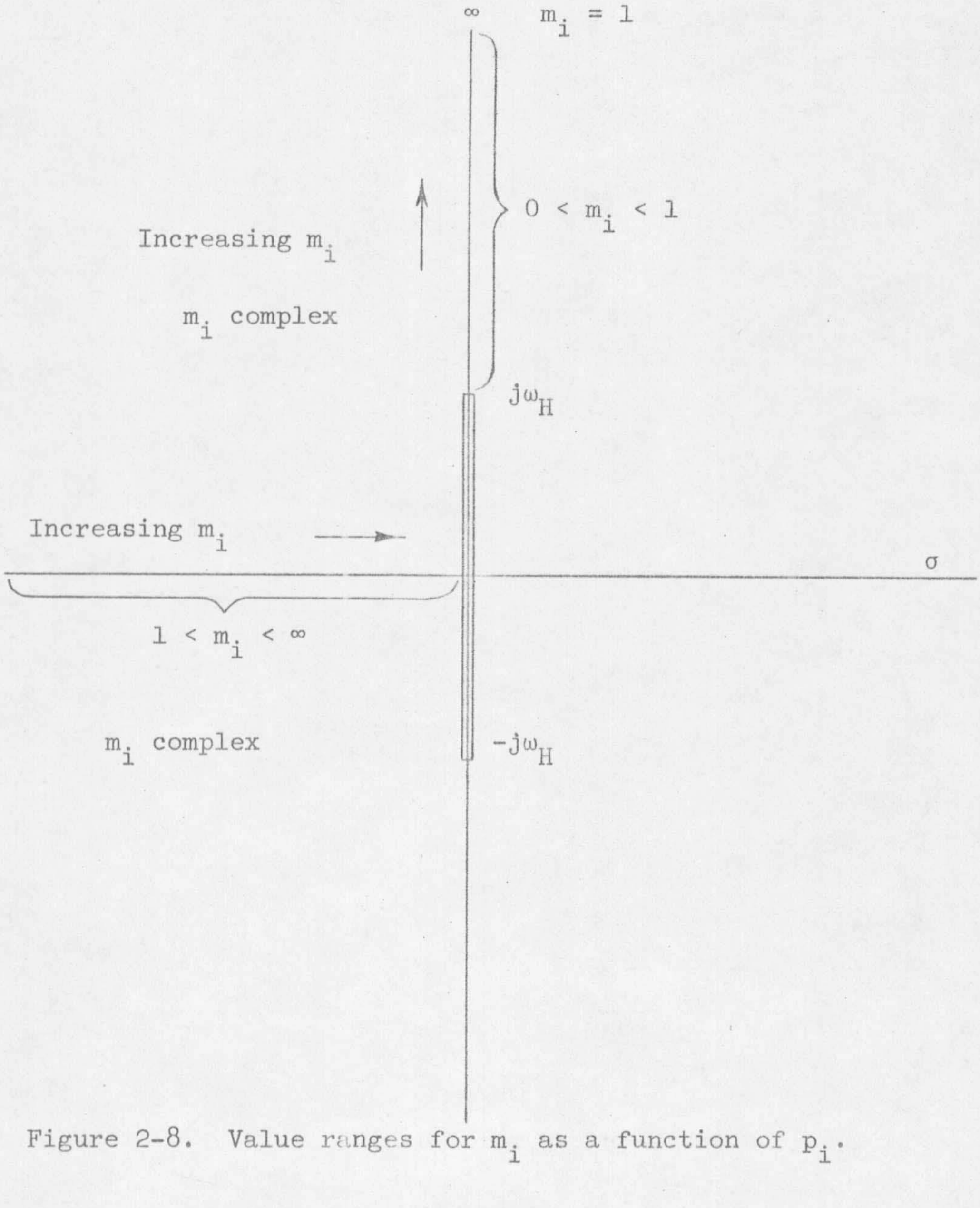
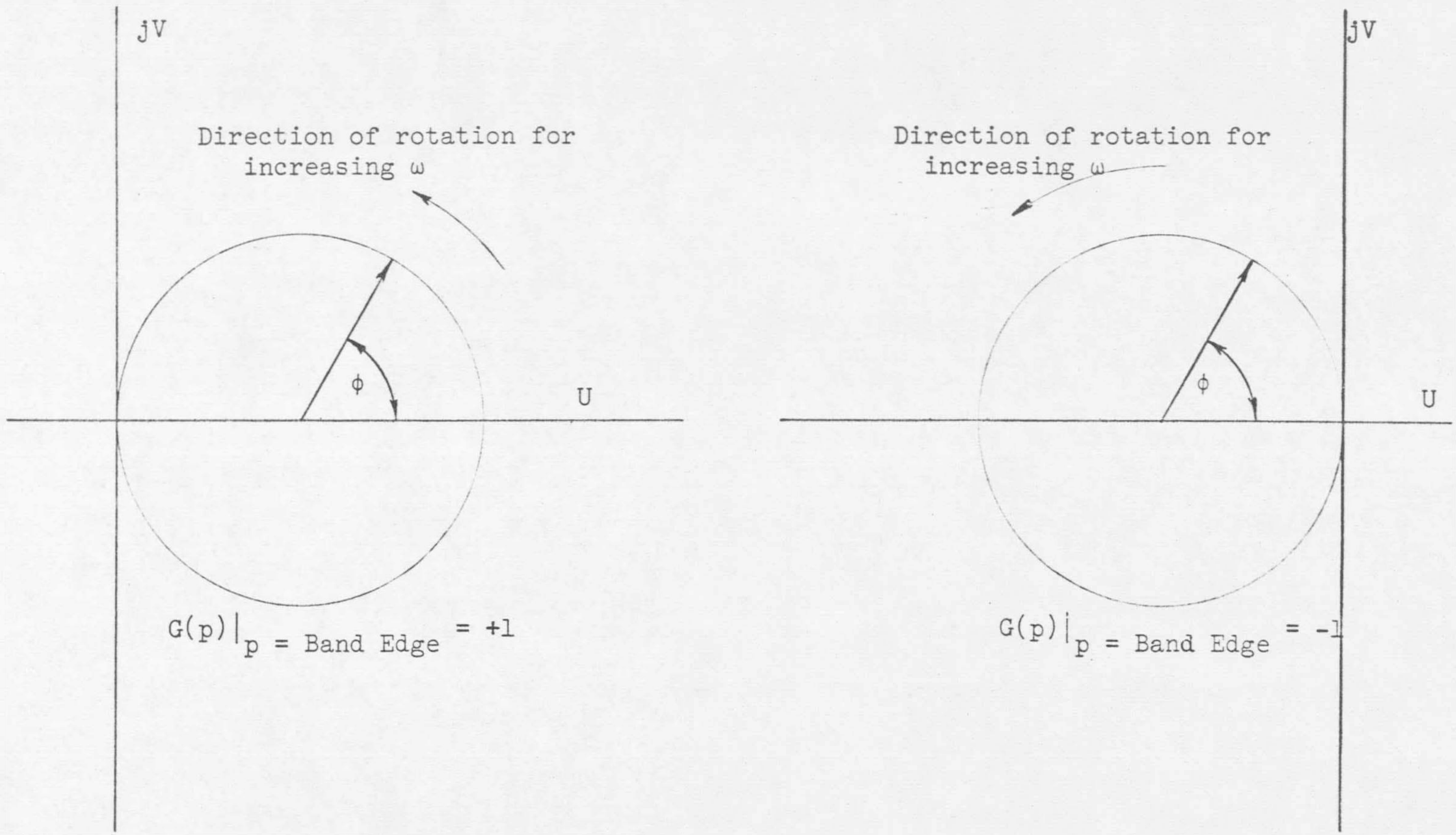


Figure 2-8. Value ranges for m_i as a function of p_i.



(a) Even number of ripples

(b) Odd number of ripples

Figure 2-9. Action of $G(p) = U + jV$ for p in the passband.

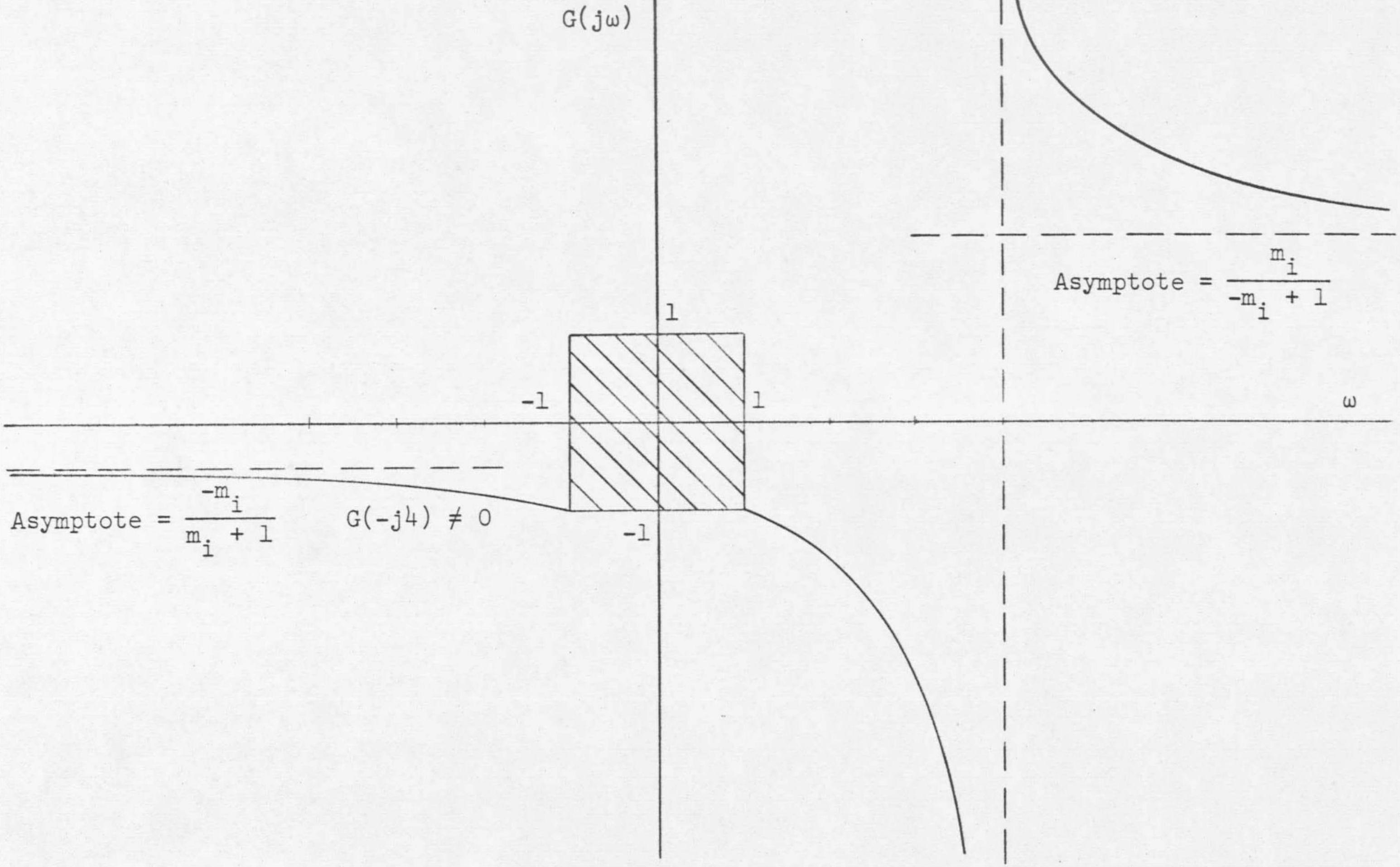


Figure 2-10. $G(j\omega)$ outside the passband. Singularity at $\omega = 4$.

at band edge. Clearly we need the positive sign for n even and the negative sign for n odd. Concisely,

$$G(p) = \frac{F(p) + (-1)^n}{2} \quad . \tag{2.43}$$

G(p) has the same denominator as does F(p), giving it the same singularities. Since $|F(j\omega)| \neq 1$ for $|\omega| > 1$, there will be no zeros of G(p) outside the passband and exactly n zeros in the passband.

A more compact expression for G(p) can be formed directly from F(p) by writing

$$G(p) = \frac{\text{Rational Part of } F(p) \text{ Numerator}}{F(p) \text{ Denominator}} \quad . \tag{2.44}$$

Validity of this remark is shown by a consideration of the form of F(p). For pairs of conjugate poles, there are in F(p) factors of the form

$$\left[\frac{(1 + m_i\overline{m_i})p^2 + 1 + (m_i + \overline{m_i})p\sqrt{p^2 + 1}}{(1 + m_i\overline{m_i})p^2 + 1 - (m_i + \overline{m_i})p\sqrt{p^2 + 1}}\right] \quad . \tag{2.45}$$

For single poles on the real axis or at infinity, there are in F(p) factors of the form

$$\frac{m_i p + \sqrt{p^2 + 1}}{-m_i p + \sqrt{p^2 + 1}} \quad . \tag{2.46}$$

In both numerator and denominator of the conjugate pole F(p), the factors are made up of two types of terms: an even term that is rational and an odd term that is irrational; and for the single real pole function, the factors are made up of an odd term that is rational and an even term that is irrational. Expansion of the products of such F(p) shows an important feature: For n even, the <u>even</u> terms are all rational and the odd terms are all irrational and for n odd, the <u>odd</u> terms are all rational and the even terms are all irrational. In fact the even and odd terms of both numerator and denominator are identical in form, differing only in sign. In the numerator are all positive terms while in the denominator the even terms are all positive and the odd terms are all negative. The form of F(p) is $\frac{M + N}{M - N}$ where M and N are the even and odd parts respectively of the numerator and denominator. From this

$$G(p) = \frac{\frac{M + N}{M - N} + (-1)^n}{2}$$

$$= \frac{M + (-1)^n M + N - (-1)^n N}{2(M - N)} . \tag{2.47}$$

For n even, M is the rational part of the numerator of F(p) and

$$G(p) = \frac{M + (+1)M + N - (+1)N}{2(M - N)}$$

$$= \frac{M}{M - N} . \tag{2.48}$$

For n odd, N is the rational part of the numerator of F(p) and

$$G(p) = \frac{M + (-1)M + N - (-1)N}{2(M - N)}$$

$$= \frac{N}{M - N} \quad . \tag{2.49}$$

So the move from F(p) to G(p) is immediate and uncomplicated. G(p) always has a rational numerator, all of whose zeros are in the passband, and has the mixed rational and irrational denominator of F(p).

2.3.4 Lowpass $\bar{G}(p)$

In accord with the outlined procedure, a function is sought with the property that it is the conjugate of G(p) in the passband. Although it is not strictly the conjugate of G(p) for all p, nomenclature will be corrupted and such a function will be called "G(p) conjugate" and denoted by $\bar{G}(p)$. $\bar{G}(p)$ has the single necessary and sufficient property that

$$\bar{G}(p) = \overline{\left[\frac{e^{j\phi(\omega)} + (-1)^n}{2}\right]} = \left[\frac{e^{-j\phi(\omega)} + (-1)^n}{2}\right] . \tag{2.50}$$

G(p) itself can be written as

$$G(p) = \begin{cases} \dfrac{M}{M - N} & n \text{ even} \\ \dfrac{N}{M - N} & n \text{ odd} \end{cases} . \tag{2.51}$$

Take first the case for n even and $|\omega| \leq 1$. M is an even polynomial such that it is real for $p = j\omega$ so it is its own conjugate. M - N is a mixed function whose conjugate is M + N. The desired $\bar{G}(p)$ is $\frac{M}{M + N}$.

For n odd, N is an odd polynomial for $p = j\omega$ such that $-N$ is its conjugate. $M - N$ is again a mixed function with $M + N$ as its conjugate. So $\bar{G}(p) = \frac{-N}{M + N}$. Generally,

$$\bar{G}(p) = \begin{cases} \frac{M}{M + N} & n \text{ even} \\ \frac{-N}{M + N} & n \text{ odd} \end{cases} . \tag{2.52}$$

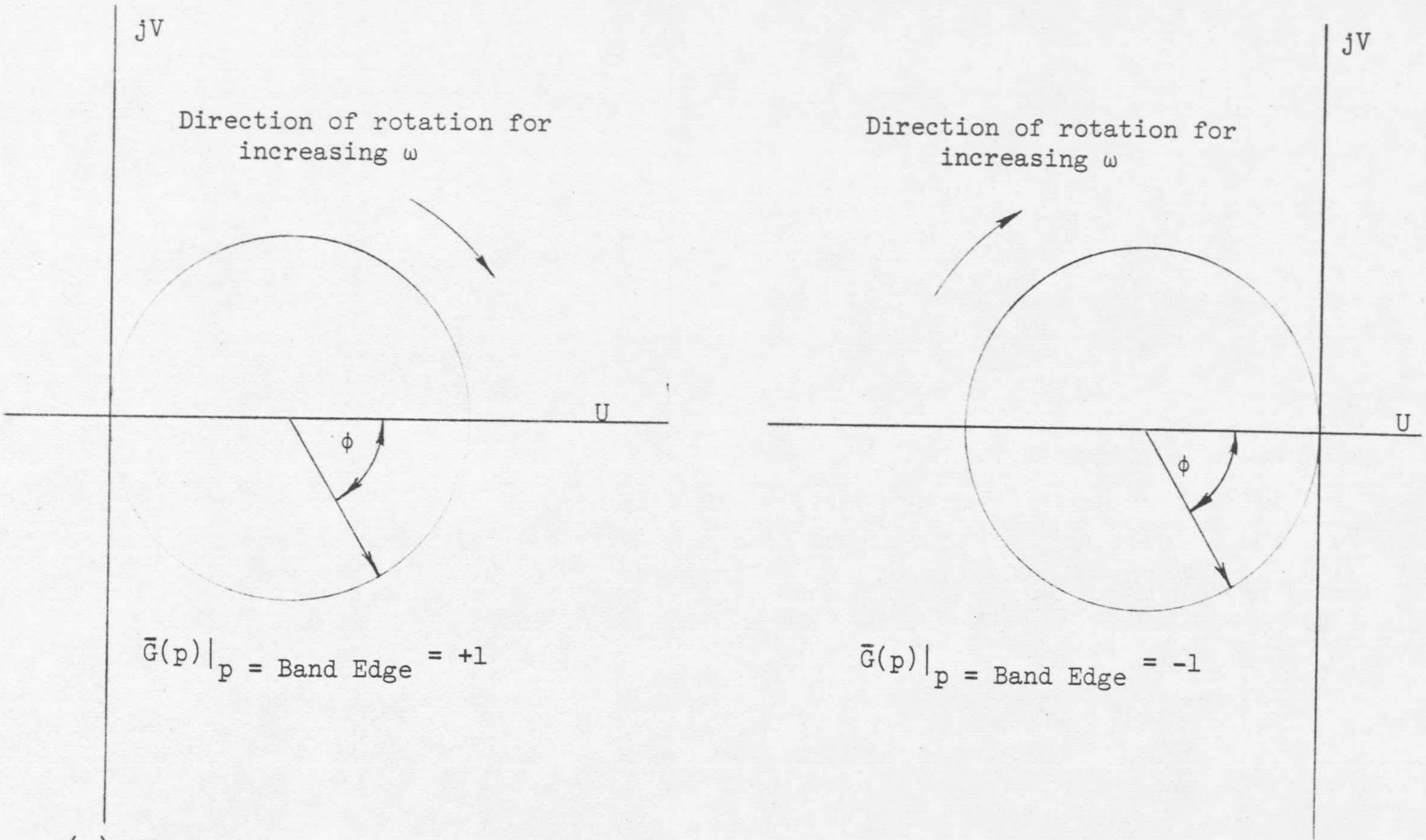
For the lowpass case, $M + N$ is zero at points negative of those for which $M - N$ is zero. So for $G(p)$ with singularities at p_i, $(i = 1, 2, \cdots, n)$ $\bar{G}(p)$ has singularities at $-p_i$, $(i = 1, 2, \cdots, n)$. Compare Figures 2-10 and 2-12; also Figures 2-9 and 2-11.

2.3.5 <u>Formation of H(P)</u>

H(p), the final rational function, is

$$H(p) = G(p)\bar{G}(p) = \begin{cases} \frac{M^2}{M^2 - N^2} & n \text{ even} \\ \frac{-N^2}{M^2 - N^2} & n \text{ odd} \end{cases} . \tag{2.53}$$

H(p) is guaranteed rational because M is a polynomial and N is a polynomial multiplying the radical $\sqrt{p^2 + 1}$. Hence their squares are rational.



(a) Even number of ripples

(b) Odd number of ripples

Figure 2-11. Action of $\bar{G}(p) = U + jV$ for p in the passband

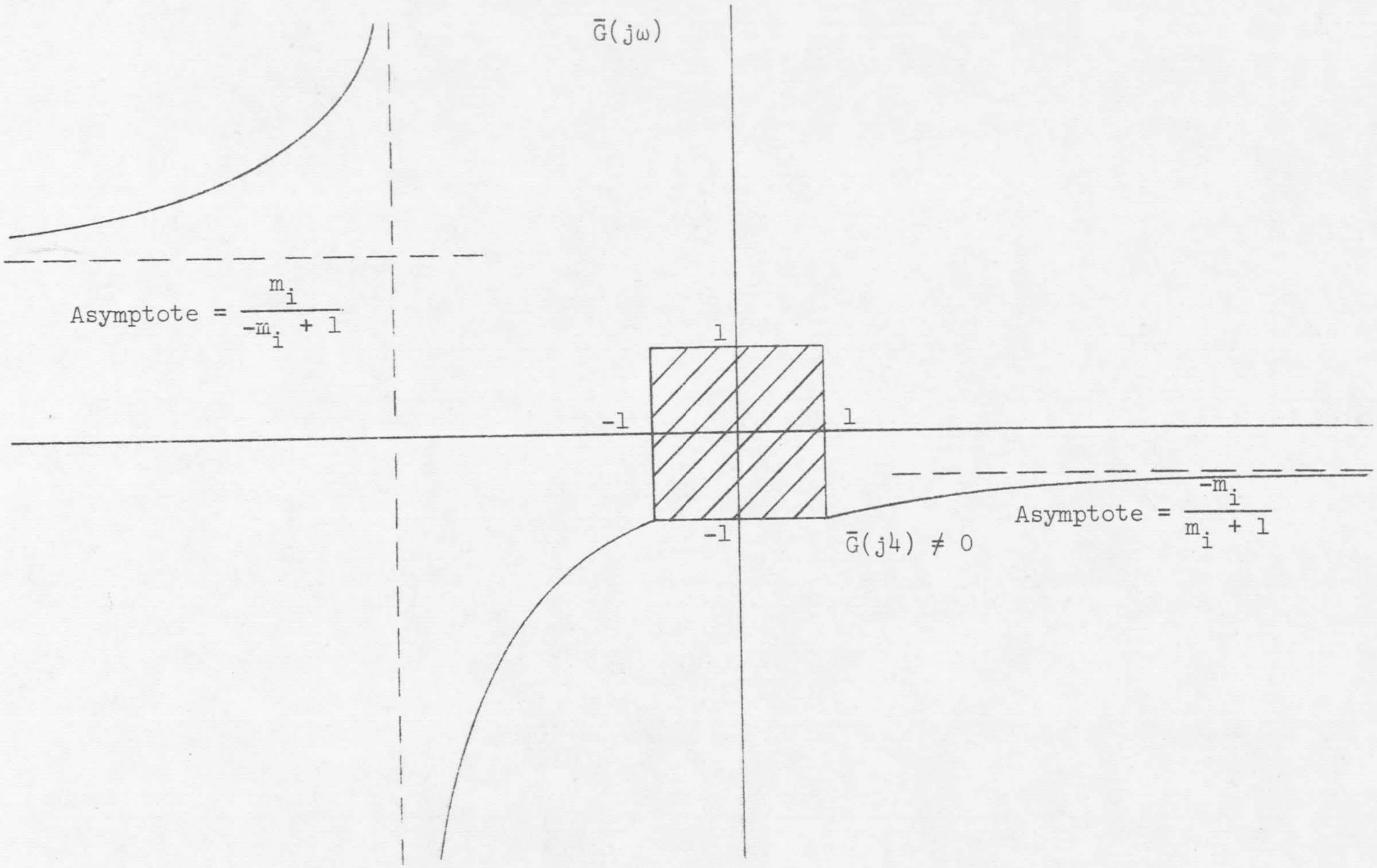


Figure 2-12. $\bar{G}(j\omega)$ outside the passband. Singularity at $\omega = -4$.

CHAPTER 3

THREE FUNDAMENTAL FUNCTIONS

3.1 INTRODUCTION

In this chapter three fundamental types of equiripple functions are presented. They are: (1) Lowpass, (2) highpass and (3) bandpass. Modifications of the lowpass case presentation of Chapter 2 pertinent to highpass and bandpass cases are made as required. Such changes are concerned with the manner in which singularities appear in F(p), G(p) and $\tilde{G}(p)$, in addition to necessary band edge changes. The fundamental rotating vector principle remains invariant. It is shown in examples that not all steps of the method as presented in Chapter 2 must be included to arrive at numerical coefficient values. It is also shown that lowpass functions can be considered as a subclass of bandpass functions for computational purposes.

3.2 LOWPASS FUNCTIONS

Most widely known among equiripple lowpass functions are the classical Chebyshev polynomials. They approximate zero with ripple maxima of +1 and minima of -1 and have all poles at infinity. The procedure presented thus far yields lowpass magnitude squared functions only so it is expected that generation of equiripple functions with poles at infinity and passband from zero to one will yield the Chebyshev polynomials squared. This it indeed does.

For such functions with all poles at infinity, all m_i are unity. For C_n^2, the nth Chebyshev polynomial squared, F(p) becomes

$$F(p) = \left[\frac{p + \sqrt{p^2 + 1}}{-p + \sqrt{p^2 + 1}}\right]^n . \tag{3.1}$$

If one were to require, for example, the second Chebyshev polynomial squared, Figure 3-1, the following steps provide it:

1. Set up $F(p) = \dfrac{2p^2 + 1 + 2p\sqrt{p^2 + 1}}{2p^2 + 1 - 2p\sqrt{p^2 + 1}}$. (3.2)

2. Form $G(p) = \dfrac{2p^2 + 1}{2p^2 + 1 - 2p\sqrt{p^2 + 1}}$. (3.3)

3. Form the conjugate of G(p) over the passband.

$$\bar{G}(p) = \frac{2p^2 + 1}{2p^2 + 1 + 2p\sqrt{p^2 + 1}} \tag{3.4}$$

4. Multiply G(p) by $\overline{G(p)}$ to form H(p).

$$H(p) = G(p)\ \overline{G(p)}$$

$$= \frac{4p^4 + 4p^2 + 1}{4p^4 + 4p^2 + 1 - (4p^4 + 4p^2)}$$

$$= 4p^4 + 4p^2 + 1 \tag{3.5}$$

which for $p = j\omega$ is the second Chebyshev polynomial squared. As pointed out in Chapter 2, H(p) is a rational function with double order zeros in the passband. It has poles at those locations inserted in F(p) plus the negative of those locations. In this case there were two locations at infinity in F(p). These two poles plus two more at minus infinity appear in H(p). With rational functions, infinity

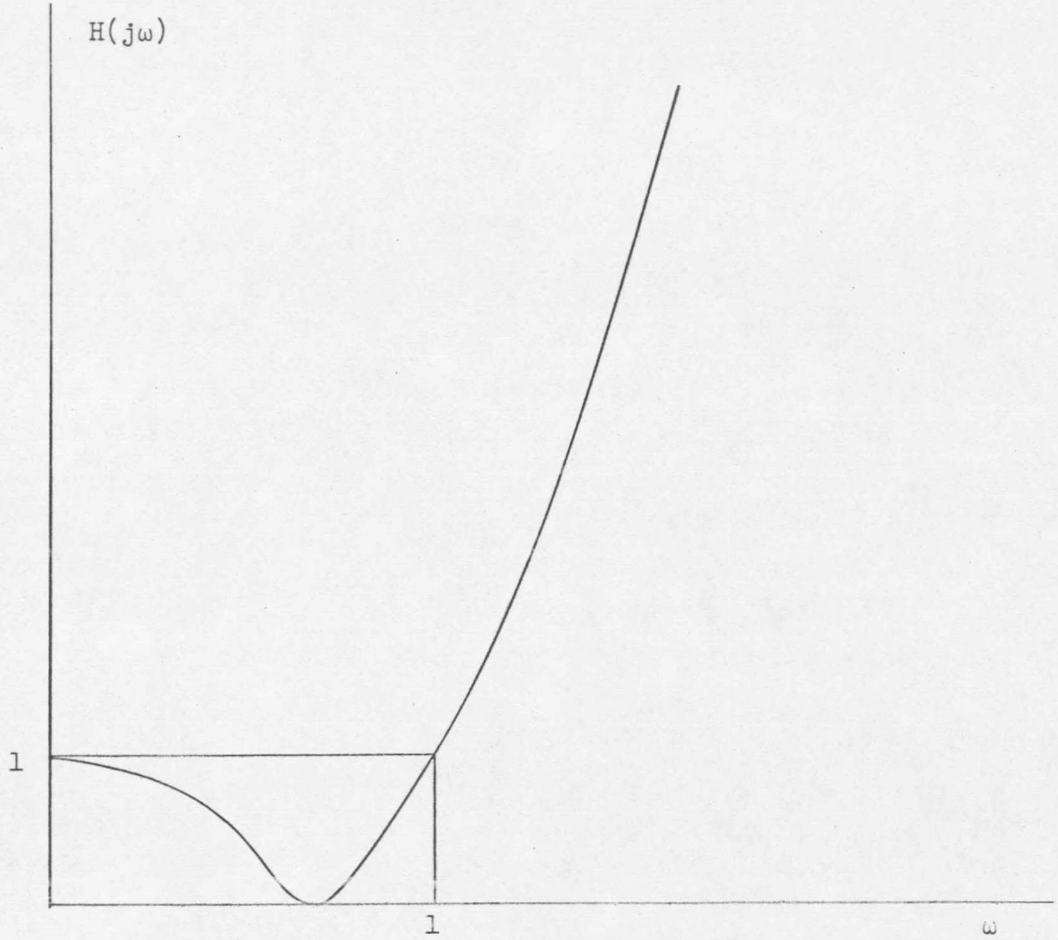


Figure 3-1. Second Chebyshev polynomial squared.

approached from any direction can be considered as a point (21) and we say $H(p)$ has four poles at infinity. As another example, consider the requirement of an $H(p)$ function with a double order zero in the passband $(0 \leq \omega \leq 1)$ and finite poles at $p = \pm j \frac{2}{\sqrt{3}}$. Further assume that $H(p)$ is to be a magnitude squared function so that the poles must be double. Figure 3-2 depicts the function. The result is that

$$F(p) = \prod_{i=1}^{2} \frac{m_i p + \sqrt{p^2 + 1}}{-m_i p + \sqrt{p^2 + 1}} \tag{3.6}$$

where

$$m_1 = m_2 = \left. \frac{\sqrt{p^2 + 1}}{p} \right|_{p = +j \frac{2}{\sqrt{3}}} . \tag{3.7}$$

The usual steps follow:

1. $$F(p) = \frac{(m_1 m_2 + 1)p^2 + 1 + (m_1 + m_2)p\sqrt{p^2 + 1}}{(m_1 m_2 + 1)p^2 + 1 - (m_1 + m_2)p\sqrt{p^2 + 1}}$$
$$= \frac{\frac{5}{4} p^2 + 1 + p\sqrt{p^2 + 1}}{\frac{5}{4} p^2 + 1 - p\sqrt{p^2 + 1}} \tag{3.8}$$

2. $$G(p) = \frac{\frac{5}{4} p^2 + 1}{\frac{5}{4} p^2 + 1 - p\sqrt{p^2 + 1}} \tag{3.9}$$

3. $$\overline{G(p)} = \frac{\frac{5}{4} p^2 + 1}{\frac{5}{4} p^2 + 1 + p\sqrt{p^2 + 1}} \tag{3.10}$$

4. $$H(p) = \frac{\frac{25}{16} p^4 + \frac{5}{2} p^2 + 1}{\frac{25}{16} p^4 + \frac{5}{2} p^2 + 1 - p^4 - p^2}$$

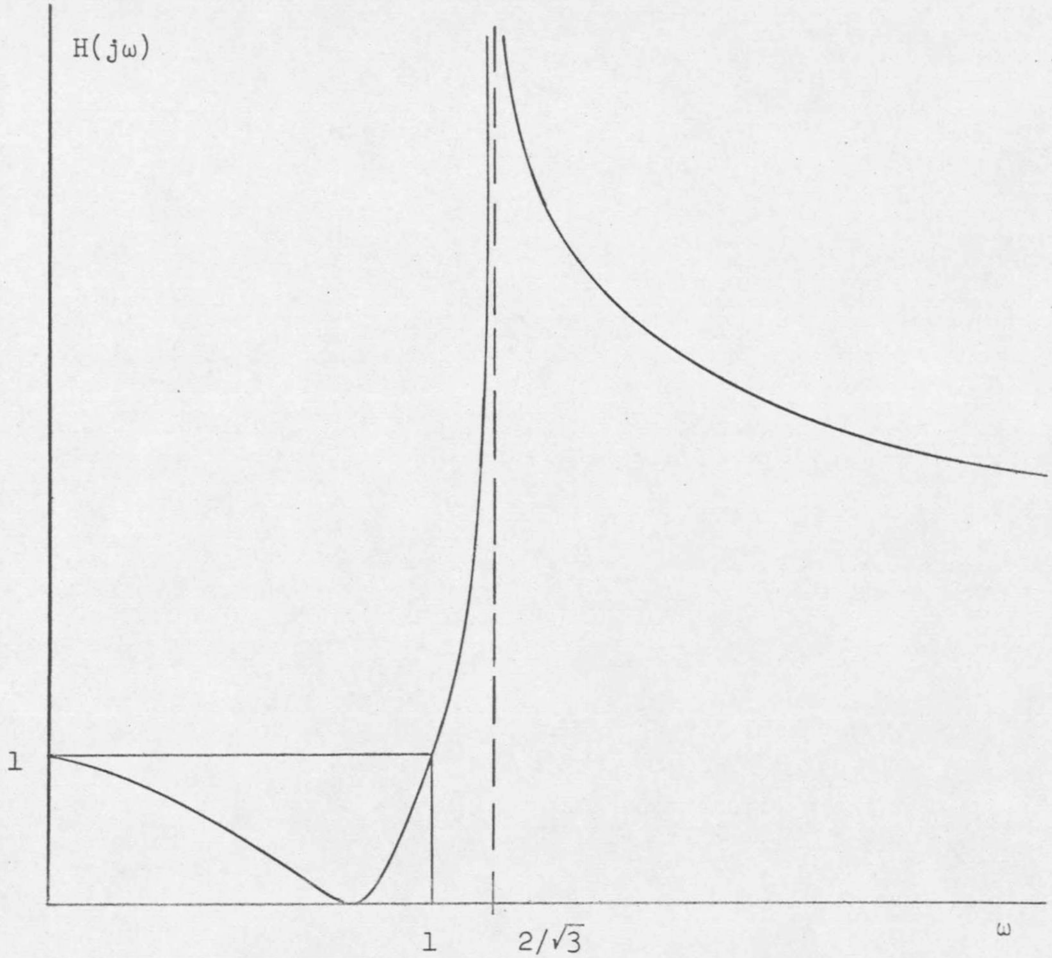


Figure 3-2. Magnitude squared function of degree 4. Poles at $p = j\,2/\sqrt{3}$

$$= \frac{\frac{25}{16} p^4 + \frac{5}{2} p^2 + 1}{\frac{9}{16} p^4 + \frac{3}{2} p^2 + 1} \tag{3.11}$$

As expected, the denominator factors into $(\frac{3}{4} p^2 + 1)^2$ which exhibits the desired pole locations.

The foregoing examples have a common redundancy: They actually include computation of the denominator polynomial. Since the poles are specified ab initio, this polynomial is already known to within a constant multiplier. The first example is to have poles at infinity and its denominator must be only a constant. The second example requires poles at $p = \pm j \frac{2}{\sqrt{3}}$ and must be $(p + \frac{4}{3})^2$ times a constant multiplier. Both denominators were carried in the examples only to show that the method indeed produces the proper denominator. A new computational recipe that omits such redundancy is

1. Compute {Rational part $\prod_{i=1}^{n} (m_i p + \sqrt{p^2 + 1})$}.

2. Multiply this by its conjugate over the passband.

3. Form $K \prod_{i=1}^{q} (p + p_i)$ as the denominator of $H(p)$ where K is the constant multiplier to be computed below, and p_i is one of the total number, 1, of finite pole locations. Infinite frequency poles do not explicitly appear, being produced by numerator degree exceeding denominator degree.

4. Knowing the desired ripple value and the value of the numerator function at band edge, compute

$$K = (\text{Numerator}|_{\text{Band Edge}})/(\text{Ripple value}). \tag{3.12}$$

3.3 HIGH PASS FUNCTIONS

High pass functions are generable by performing a transformation of variable, $p = \frac{\omega_c}{p^*}$, on the lowpass F(p) to give a highpass function in the variable p*; or by a direct formation of a highpass F(p) based on the rotating vector principle. We will adopt the latter view because of its greater generality and potential for providing insight. The band will extend over the interval $[\omega_L, \infty)$. Considering the form of the lowpass F(p), some expression involving a radical that undergoes a change from real value to imaginary value at the critical point $\omega = \omega_L$ would seem reasonable as a start. F(p) should have the form

$$F(p) = \frac{\alpha + j\beta}{\alpha - j\beta}$$

$$= e^{j\phi(\omega)} \tag{3.13}$$

for $|\omega| \geq \omega_L$ and be strictly real for $|\omega| < \omega_L$. Some suggestions are:

1. $$F(p) = \prod_{i=1}^{n} \frac{m_i p + \sqrt{-(p^2 + \omega_L^2)}}{-m_i p + \sqrt{-(p^2 + \omega_L^2)}} \tag{3.14}$$

2. $$F(p) = \prod_{i=1}^{n} \frac{m_i/p + \sqrt{\omega_L^2/p^2 + 1}}{-m_i/p + \sqrt{\omega_L^2/p^2 + 1}} \tag{3.15}$$

3. $$F(p) = \prod_{i=1}^{n} \frac{m_i + \sqrt{p^2 + \omega_L^2}}{-m_i + \sqrt{p^2 + \omega_L^2}} \tag{3.16}$$

All of these are suitable since they have a common feature of being (1) real outside the passband and (2) a rotating vector in the passband. The final choice is entirely left to the user. We will adopt (3.16) because of its simplicity.

Properties of this F(p) will be investigated. For $n = 1$, $p = j\omega$ and m_1 real and positive.

$$F(p)\Big|_{p = j\omega} = \frac{m_1 + \sqrt{\omega_L^2 - \omega^2}}{-m_1 + \sqrt{\omega_L^2 - \omega^2}} \tag{3.17}$$

In the passband, $|\omega| \geqslant \omega_L$

$$F(p)\Big|_{p = j} = \frac{m_1 + j\sqrt{\omega^2 - \omega_L^2}}{-m_1 + j\sqrt{\omega^2 - \omega_L^2}}$$

$$= e^{j\phi(\omega)} \tag{3.18}$$

where

$$\phi = -\pi + 2\tan^{-1} \frac{\sqrt{\omega^2 - \omega_L^2}}{m_1} \tag{3.19}$$

and as $\omega_L \leq \omega < \infty$, $-\pi \leq \phi < 0$; rotation being in a counter-clockwise direction. Outside the passband, i.e., for $|\omega| < \omega_L$,

$$F(p)\Big|_{p = j\omega} = \frac{m_1 + \sqrt{\omega_L^2 - \omega^2}}{-m_1 + \sqrt{\omega_L^2 - \omega^2}} \quad . \tag{3.20}$$

The function is real and has singularities at both p_1 and $-p_1$, determined by setting $m_1 = \sqrt{\omega_L^2 - p_1^2}$. See Figure 3-3. This is different from the lowpass case, which exhibited a singularity at p_1, but a zero value at $-p_1$. One can see from Figure 3-3 that at no point outside the passband will $F(p)$ be less than unity in magnitude, insuring that $G(p)$ will have no zeros outside the passband. The same sort of even function behavior will be exhibited by bandpass functions.

Another difference is in $\bar{G}(p)$. When $\bar{G}(p)$ was formed in lowpass functions, it contained singularities at $-p_1$ if $G(p)$ contained them at p_1. In highpass, $\bar{G}(p)$ contains no singularities whatsoever, all desired pole locations of $H(p)$ having appeared in $G(p)$. When $H(p) = G(p)\,\dfrac{1}{\bar{G}(p)}$ is formed, the singularities then belong to a magnitude squared function and may properly be called poles. They will automatically appear at p_i and $-p_i$. For a general highpass function,

$$F(p) = \prod_{i=1}^{n} \frac{m_i + \sqrt{p^2 + \omega_L^2}}{-m_i + \sqrt{p^2 + \omega_L^2}} \tag{3.21}$$

where n is one half the desired degree of $H(p)$. For $p = j\omega$ and all m_i real and positive

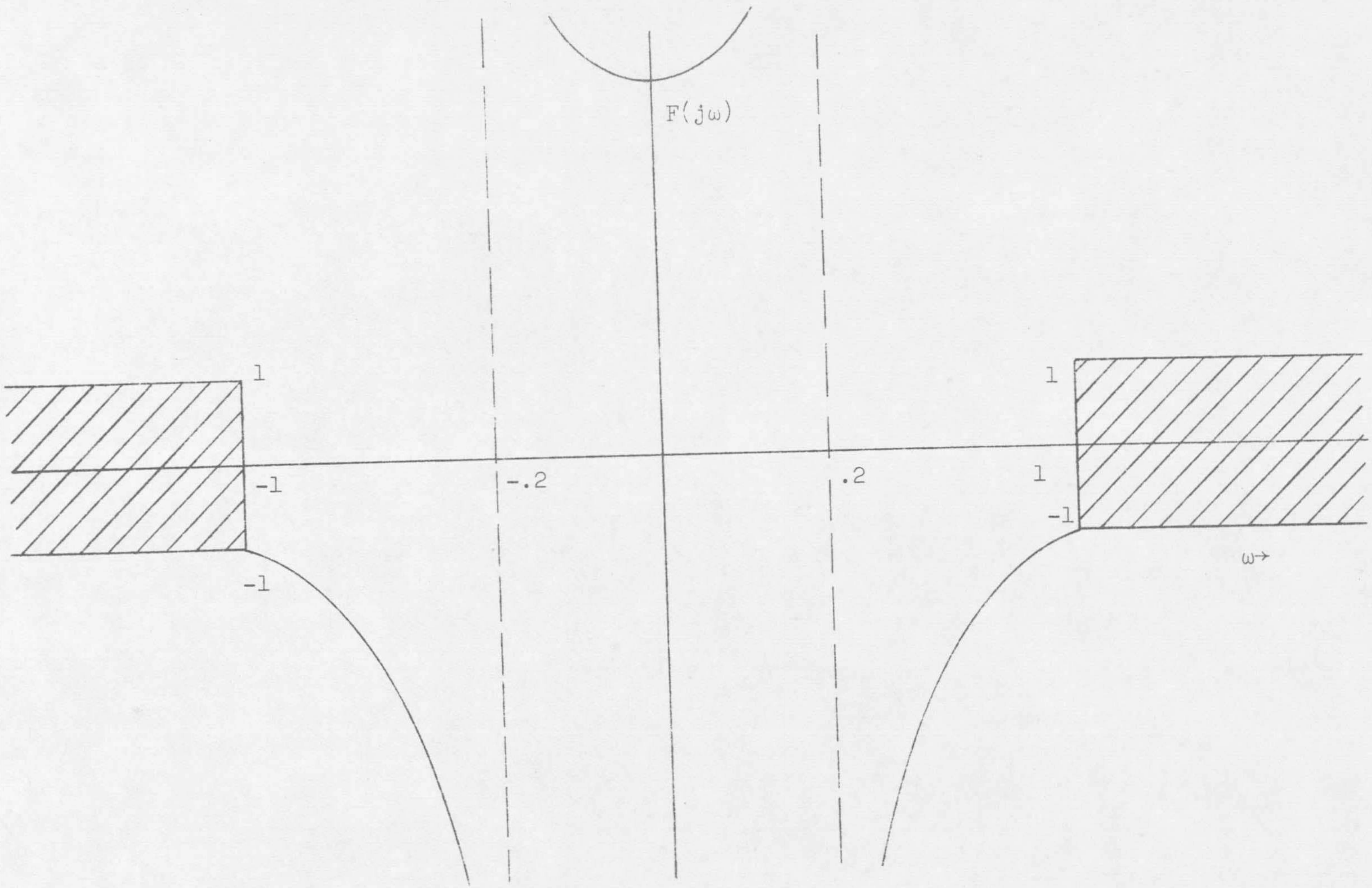


Figure 3-3. Highpass F(p) outside the passband. Poles at $p = \pm j.2$.

$$F(p)\Big|_{p = j\omega} = \prod_{i=1}^{n} \frac{m_i + \sqrt{\omega_L^2 - \omega^2}}{-m_i + \sqrt{\omega_L^2 - \omega^2}} \; , \tag{3.22}$$

In the passband, $|\omega| \geq \omega_L$

$$F(p)\Big|_{p = j\omega} = \prod_{i=1}^{n} \frac{m_i + j\sqrt{\omega^2 - \omega_L^2}}{-m_i + j\sqrt{\omega^2 - \omega_L^2}}$$

$$= e^{j\Phi(\omega)} \tag{3.23}$$

where

$$\Phi = -n\pi + 2\sum_{i=1}^{n} \tan^{-1} \frac{\sqrt{\omega^2 - \omega_L^2}}{m_i} \tag{3.24}$$

and as $\omega_L \leq \omega < \infty$, $-n\pi \leq \Phi < 0$; rotation being in a counter-clockwise direction. Again, the function is real for ω outside the passband

$$F(p)\Big|_{p = j\omega} = \prod_{i=1}^{n} \frac{m_i + \sqrt{\omega_L^2 - \omega^2}}{-m_i + \sqrt{\omega_L^2 - \omega^2}} \tag{3.25}$$

and has singularities at $p = p_i$ and $p = -p_i$, determined by setting $m_i = \sqrt{p^2 + \omega_L^2}\Big|_{p = p_i}$; $i = 1, 2, 3, \cdots, n$. Conjugate singularities are handled in a similar manner as in lowpass. Invoking the Schwarz reflection principle again will show that, for $p = p_i$ leading to m_i, $p = \overline{p_i}$ will lead to $\overline{m_i}$. The two term product in $F(p)$ is

$$F(p) = \frac{p^2 + (\omega_L^2 + m_i\overline{m_i}) + (m_i + \overline{m_i})\sqrt{p^2 + \omega_L^2}}{p^2 + (\omega_L^2 + m_i\overline{m_i}) - (m_i + \overline{m_i})\sqrt{p^2 + \omega_L^2}} \qquad (3.26)$$

in which all coefficients are real. It is also desired that the vector rotate in a counter-clockwise direction in the passband, requiring in addition that all numerator coefficients be positive. Counter-clockwise rotation is necessary since all vectors formed from real axis and imaginary axis poles rotate counter-clockwise. Multiplication by a vector rotating clockwise will cause effective cancellation of rotation, scattering zeros of $G(p)$, and consequently $H(p)$, throughout the complex plane. As a result, a general rule has been established: All m_i will be chosen such that they have positive real parts. An instance of application of this rule appears in the example of this section. It might be noted that choice of all m_i with negative real part would do just as well. Vectors would rotate clockwise in such a circumstance. Consistency of choice is all that is required.

With the stated rule applied, the highpass vector rotates from -2π to 0 in a counter-clockwise direction as ω goes from ω_L to infinity. It has singularities at $p = p_1$, $-p_1$ and also at $p = \overline{p_1}$, $\overline{-p_1}$, i.e., for the entire quadruplet of pole locations. Upon multiplication by its conjugate denominator, the entire quadruplet will appear as poles of a magnitude squared function.

Formation of $G(p)$ follows the usual rules. Since $F(p)\big|_{p = j\omega_L} = (-1)^n$, $G(p)$ is

$$G(p) = \frac{F(p) + (-1)^n}{2} \tag{3.27}$$

which gives G(p) a magnitude of unity at band edge. Following a similar precedure to that in lowpass, it is easily proved that a direct formulation of G(p) is

$$G(p) = \frac{\text{Rational Part of Numerator of } F(p)}{\text{Denominator of } F(p)} , \tag{3.28}$$

Since the proof is so simple and like that of lowpass, it will be omitted. The magnitude squared function, H(p), is formed as $H(p) = G(p)\,\overline{\bar{G}(p)}$ with $\bar{G}(p)$ having the same property as in the lowpass case, i.e., $\bar{G}(p)$ is the conjugate function of G(p) for $p = j\omega$ and $\omega_L \leq \omega < \infty$. $\bar{G}(p)$ will have the form

$$\bar{G}(p) = \begin{cases} \dfrac{M}{M + N} & n \text{ even} \\ \dfrac{-N}{M + N} & n \text{ odd} \end{cases} \tag{3.29}$$

so that $\bar{G}(p)$ has no singularities whatsoever. H(p) is therefore rational and has the required singularities, which can now be called poles because they belong to a magnitude squared function. As a numerical example of the highpass procedure, a magnitude function having a passband beginning at $\omega_L = 1$, and with one and one half ripples in the positive passband (a 6^{th} degree function overall) is generated. Poles are to be located at $p = \pm 1, \pm 1 \pm j1$. The presence of complex poles will help to make clear this part of the procedure. A plot of

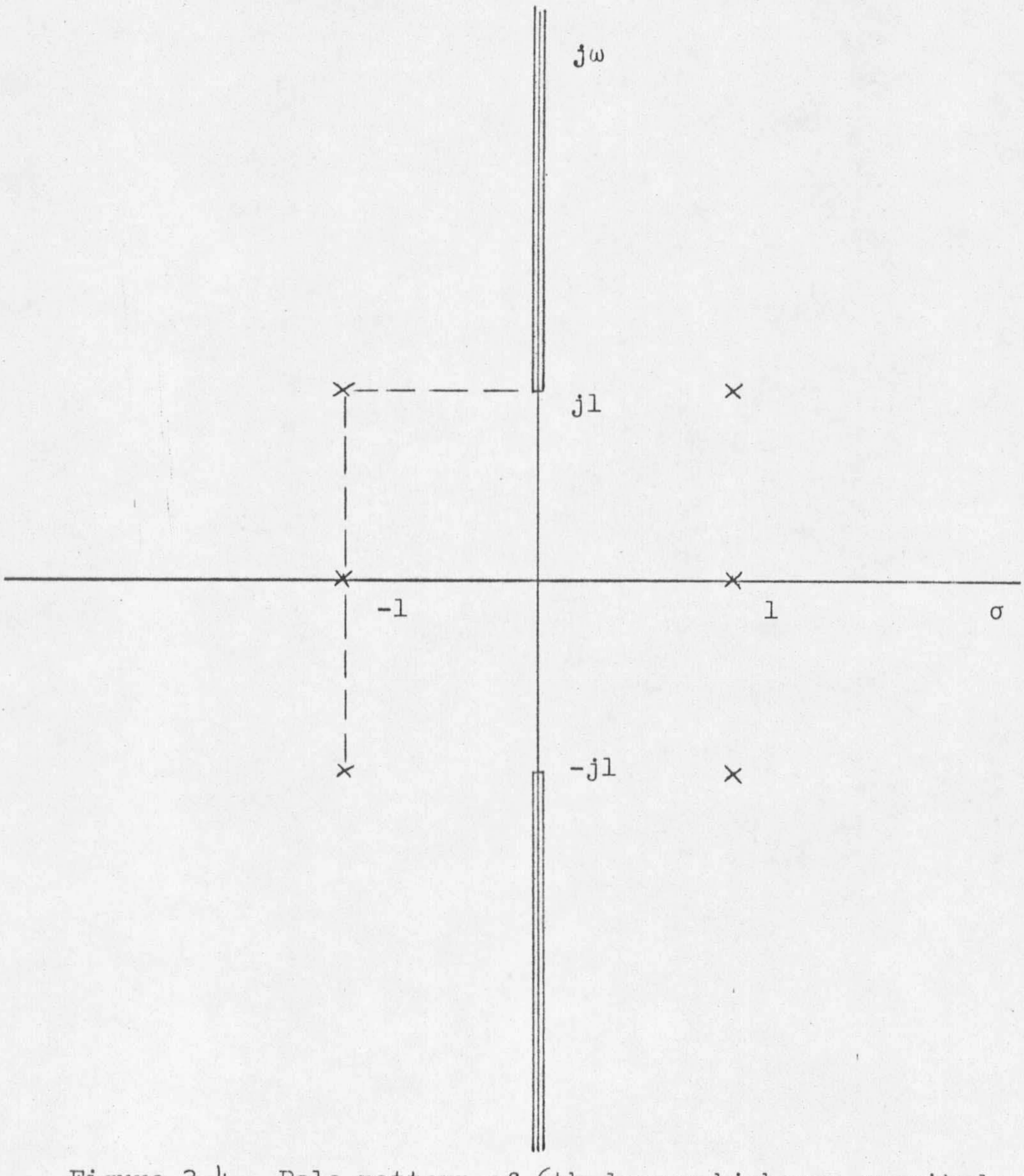


Figure 3-4. Pole pattern of 6th degree highpass magnitude squared function. Passband shown as double lined axis.

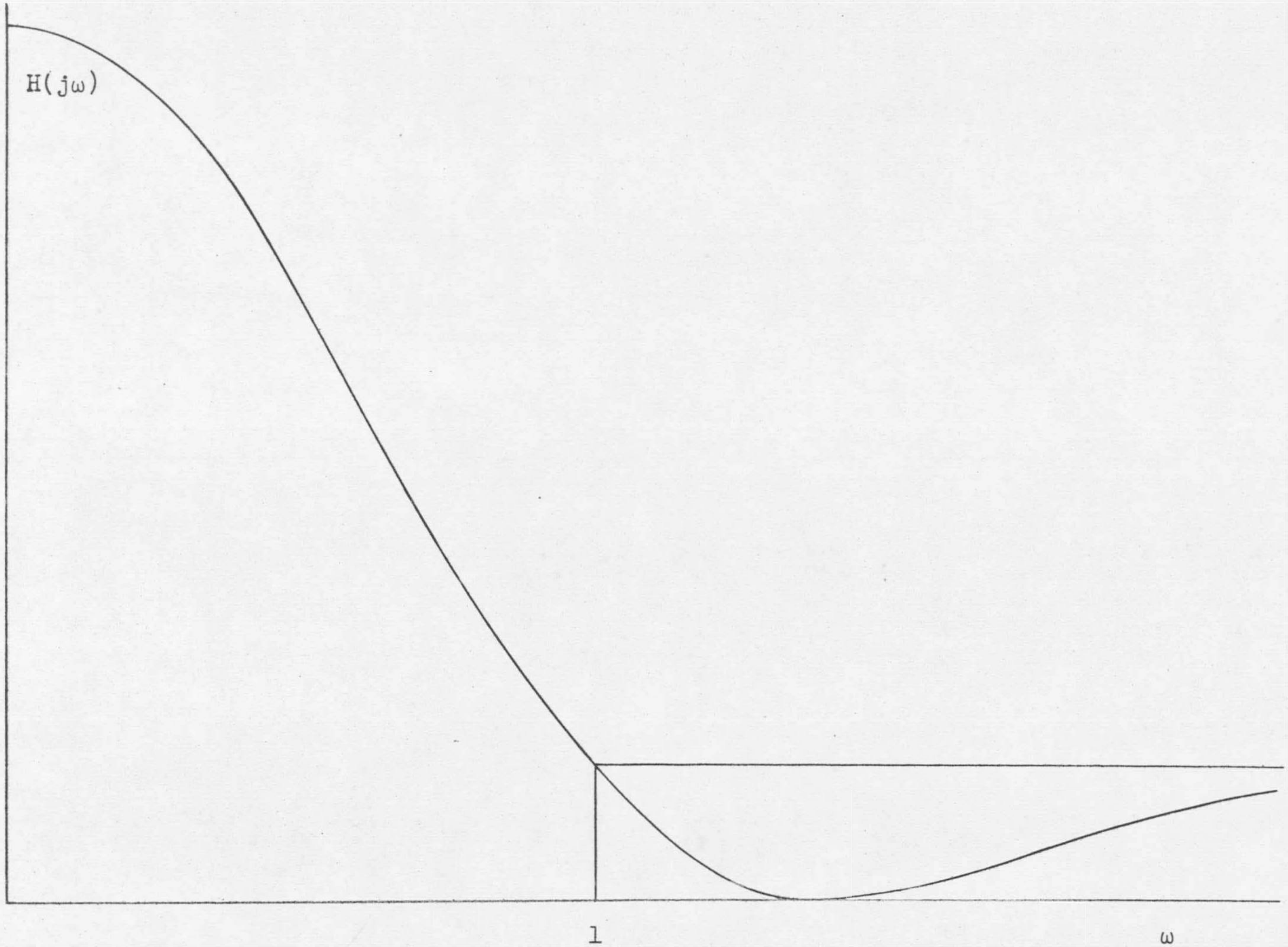


Figure 3-5. $H(j\omega)$ versus ω for function with pole pattern of Figure 3-4.

the function is shown in Figures 3-4 and 3-5. The steps are:

1. Determine m_i

$$m_1 = \sqrt{p^2 + 1}\Big|_{p = -1} = \sqrt{2} \tag{3.30}$$

$$m_2 = \sqrt{p^2 + 1}\Big|_{p = -1+j1} = \sqrt[4]{5}\ \underline{/1/2 \tan^{-1} - 2} \tag{3.31}$$

$$m_2 = 1.49535e^{j \pm 2.5880188}$$

$$= \mp 1.27203 \pm j.786122 \tag{3.33}$$

An application of the rule that m_i have always positive real part requires that we choose

$$m_2 = 1.27203 - j.786122 \tag{3.34}$$

and

$$m_3 = \bar{m}_2 = 1.27203 - j.786122 \tag{3.35}$$

2. Construct F(p)

$$F(p) = \prod_{i=1}^{3} \frac{m_i + \sqrt{p^2 + 1}}{-m_i + \sqrt{p^2 + 1}}$$

$$= \frac{m_1 + \sqrt{p^2+1}}{-m_1 + \sqrt{p^2+1}} \cdot \frac{p^2 + (m_2\overline{m}_2 + 1) + (m_2 + \overline{m}_2)\sqrt{p^2+1}}{p^2 + (m_2\overline{m}_2 + 1) - (m_2 + \overline{m}_2)\sqrt{p^2+1}}$$

$$= \frac{2 + \sqrt{p^2+1}}{-2 + \sqrt{p^2+1}} \cdot \frac{p^2 + 3.23606 + (2.54406)\sqrt{p^2+1}}{p^2 + 3.23606 - (2.54406)\sqrt{p^2+1}}$$

$$= \frac{(3.95827p^2 + 7.12054) + (p^2 + 6.83389)\sqrt{p^2+1}}{(3.95827p^2 + 7.12054) + (p^2 + 6.83389)\sqrt{p^2+1}} \qquad (3.36)$$

3. Form G(p)

$$G(p) = \frac{(3.15827p^2 + 7.12054)}{(3.95827p^2 + 7.12054) - (p^2 + 6.83389)\, p^2 + 1} \qquad (3.37)$$

4. Multiply G(p) by $\overline{G}(p)$ to form H(p)

$$H(p) = G(p)\,\overline{G}(p)$$

$$= \frac{(3.95827p^2 + 7.12054)}{(3.95827p^2 + 7.12054)^2 - (p^2 + 6.83389)^2(p^2 + 1)}$$

$$= \frac{15.6679p^4 + 56.3700p^2 + 50.7021}{-p^6 + 1.00010p^4 - 4.00010p^2 + 4.00000} \qquad (3.38)$$

The denominator is, ideally

$$(p^2 + 2p + 2)(p^2 - 2p + 2)(1 + p)(1 - p) = -p^6 + p^4 - 4p^2 + 4 \qquad (3.39)$$

Considering the example has only 6 places carried throughout, H(p) is quite accurate.

Again redundancy exists in the example. There is no need to carry the denominator along. It will be that one specified ab initio and can be written directly in $H(p)$. It was computed solely for illustrative purposes. There is need for only the rational part of the numerator of $F(p)$. A new computational procedure would be:

1. Compute the m_i.

2. Compute the rational part of the numerator of $F(p)$.

3. Multiply such polynomial by its conjugate.

4. Place this over the desired denominator to form $H(p)$.

5. Set the constant multiplier to achieve desired ripple.

3.4 BAND PASS FUNCTIONS

These functions will be of an equiripple nature over the range $\omega_L \leq |\omega| \leq \omega_H$. The $F(p)$ used for generating is

$$F(p) = \prod_{i=1}^{n} \frac{m_i\sqrt{p^2 + \omega_L^2} + \sqrt{p^2 + \omega_H^2}}{-m_i\sqrt{p^2 + \omega_L^2} + \sqrt{p^2 + \omega_H^2}} \qquad (3.40)$$

where ω_L denotes lower band edge and ω_H denotes higher band edge. Assuming m_1 to be real and positive, the simplest $F(p)$ is

$$F(p) = \frac{m_1\sqrt{p^2+\omega_L^2}+\sqrt{p^2+\omega_H^2}}{-m_1\sqrt{p^2+\omega_L^2}+\sqrt{p^2+\omega_H^2}} \; . \tag{3.41}$$

It has the following properties for $p = j\omega$:

$$F(p)\Big|_{p=j\omega} = \frac{m_1\sqrt{\omega_L^2-\omega^2}+\sqrt{\omega_H^2-\omega^2}}{-m_1\sqrt{\omega_L^2-\omega^2}+\sqrt{\omega_H^2-\omega^2}}$$

$$= \begin{cases} \dfrac{\alpha(\omega)+\beta(\omega)}{-\alpha(\omega)+\beta(\omega)} & \text{for } |\omega| < \omega_L \\[2ex] \dfrac{j\alpha(\omega)+\beta(\omega)}{-j\alpha(\omega)+\beta(\omega)} & \text{for } \omega_L \leqslant |\omega| \leqslant \omega_H \; . \\[2ex] \dfrac{j\alpha(\omega)+j\beta(\omega)}{-j\alpha(\omega)+j\beta(\omega)} = \dfrac{\alpha(\omega)+\beta(\omega)}{-\alpha(\omega)+\beta(\omega)} & \text{for } |\omega| > \omega_H \end{cases} \tag{3.42}$$

In the passband, $F(p)\big|_{p=j\omega}$ is a rotating unit vector

$$F(p)\Big|_{p=j\omega} = e^{j\phi(\omega)} \; , \tag{3.43}$$

where

$$\phi = 2\tan^{-1}\frac{\sqrt{\omega^2-\omega_L^2}}{\sqrt{\omega_H^2-\omega^2}} \tag{3.44}$$

and as $\omega_L \leqslant \omega \leqslant \omega_H$, $0 \leqslant \phi \leqslant \pi$; rotation being in a counter-clockwise direction. Outside the passband, i.e., for $|\omega| < \omega_L$ and $|\omega| > \omega_H$

$$F(p)\Big|_{p=j\omega} = \frac{m_1\sqrt{\omega_L^2 - \omega^2} + \sqrt{\omega_H^2 - \omega^2}}{-m_1\sqrt{\omega_L^2 - \omega^2} + \sqrt{\omega_H^2 - \omega^2}} \qquad (3.45)$$

$F(p)$ is real and has singularities at $p = p_1$ and $p = -p_1$ according to the value of $m_1 = \sqrt{\dfrac{p_1^2 + \omega_H^2}{p_1^2 + \omega_L^2}}$.

For n/2 ripples in the passband (an overall 2n degree function)

$$F(p) = \prod_{i=1}^{n} \frac{m_i\sqrt{p^2 + \omega_L^2} + \sqrt{p^2 + \omega_H^2}}{-m_i\sqrt{p^2 + \omega_L^2} + \sqrt{p^2 + \omega_H^2}} \qquad (3.46)$$

In the passband,

$$F(p)\Big|_{p=j\omega} = e^{j\Phi(\omega)} \qquad (3.47)$$

where

$$\Phi = 2\sum_{i=1}^{n} \tan^{-1} \frac{\sqrt{\omega^2 - \omega_L^2}}{\sqrt{\omega_H^2 - \omega^2}} \qquad (3.48)$$

and as $\omega_L \leq \omega \leq \omega_H$, $0 \leq \Phi \leq n\pi$; rotation being in a counter-clockwise direction. Outside the passband $F(p)$ is real with singularities at $p = p_i$ and $p = -p_i$, $i = 1, 2, \cdots, n$. As in highpass, all singularities appear in $F(p)$ and $G(p)$, $\bar{G}(p)$ having none at all. The usual dichotomy regarding pole locations results in m_i computed from left half plane and positive j-axis poles. The origin is actually on the boundary of the region for which poles are to be used in computing m_i. Since computation of m_i for poles at $p = p_i$ automatically inserts a pole at

$p = -p_i$ into $F(p)$, $G(p)$ and $H(p)$, computation of m_i for one half of the desired $H(p)$ poles at the origin is indicated. There will always be an even number of poles at the origin because $H(p)$ is an even function and one half of their number will be an integer.

A computational procedure for bandpass functions follows:

1. Compute m_i's according to

$$m_i = \sqrt{\frac{p_i^2 + \omega_L^2}{p_i^2 + \omega_H^2}}. \tag{3.49}$$

There will always be an even number of m_i to compute because there must be an integer number of ripples in the passband, making n even.

2. Construct F(p)

$$F(p) = \prod_{i=1}^{n} \frac{m_i\sqrt{p^2 + \omega_L^2} + \sqrt{p^2 + \omega_H^2}}{-m_i\sqrt{p^2 + \omega_L^2} + \sqrt{p^2 + \omega_H^2}}, \quad n \text{ even}. \tag{3.50}$$

3. Form G(p) by

$$G(p) = \frac{F(p) + (-1)^n}{2}$$

$$= \frac{F(p) + 1}{2} \tag{3.51}$$

or

$$G(p) = \frac{\text{Rational part of Numerator of } F(p)}{\text{Denominator of } F(p)} . \qquad (3.52)$$

Again, it is not difficult to show that a rational numerator exists for $G(p)$. The proof follows the same lines as the lowpass case and will not be presented.

4. Form $\bar{G}(p)$. $\bar{G}(p)$ is formable directly from $G(p)$ by noting that in the passband, $G(p)$ has the form

$$G(p) = \frac{(\text{Even Real Polynomial})}{(\text{Even Real Polynomial}) + (\text{Irrational Imaginary Expression})} \qquad (3.53)$$

from which $\bar{G}(p)$ takes the form

$$\bar{G}(p) = \frac{(\text{Even Real Polynomial})}{(\text{Even Real Polynomial}) - (\text{Irrational Imaginary Expression})} .$$

5. Form $H(p)$ as

$$H(p) = G(p)\,\bar{G}(p) . \qquad (3.54)$$

$H(p)$ is a rational function.

As an example, suppose one constructs a bandpass function with 4 ripples in the band extending from $\omega_L = 1$ to $\omega_H = 2$. There will be 8 zeros per band for a total of 16 zeros or a 16^{th} degree magnitude squared function. Poles are to be placed in a quadrantal set at $p = \pm 1 \pm j1$, on the real axis at $p = \pm 1$ and in imaginary axis pairs at zero, $\pm j4$ and infinity for a total of 16 poles. See Figures 3-6 and 3-7.

Computations for this function were carried out on an IBM 1620 II computer using a Fortran program written by the author. A simplified computational scheme using the rational part of the numerator of F(p) and separate denominator computation and normalization was employed. It is similar in all respects to the simplified highpass procedure given at the end of Section 3.3. Table 3-1 lists the input specifications and final coefficient values plus pertinent values of the m_i. The conjugate product refers to $m_i\overline{m_i}$ and the conjugate sum refers to $m_i + \overline{m_i}$. A plot of the function vs. omega is shown in Figure 3-6. The program was made of a completely general nature for bandpass functions. Although the IBM 1620 is not a particularly fast machine as modern digital computers go, output for up to 6^{th} degree functions was limited essentially by print rate of a 240 line per minute printer. For 16^{th} degree functions, computation of numerator coefficients required about 2 seconds. In general, computational operations and time required increase in proportion to the square of the desired degree.

As a concluding remark in covering the three fundamental functions, it is clear that for computational purposes, one can consider lowpass as a special case of the bandpass. With $\omega_L = 0$, all values of m_i are identical to those resulting from strict lowpass operations.

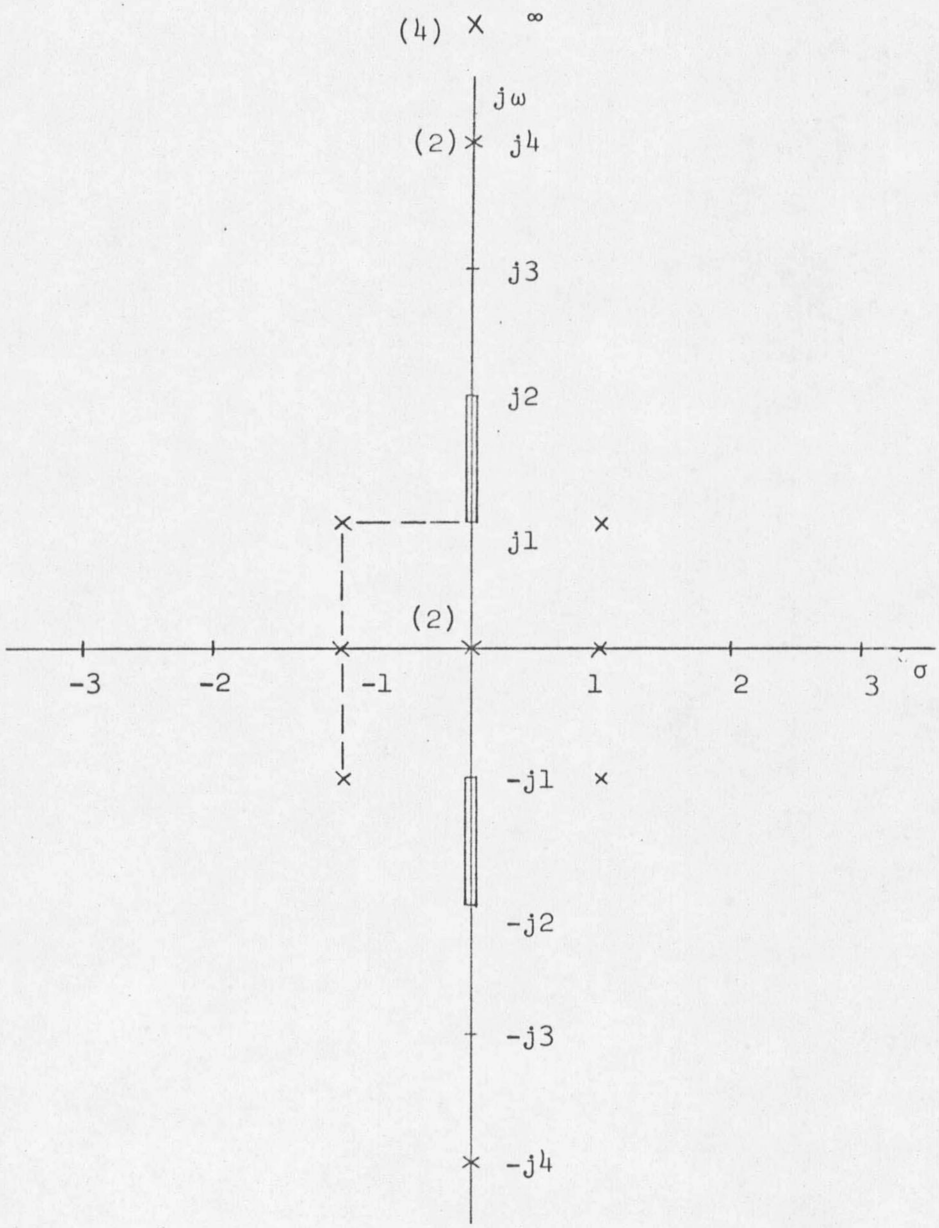


Figure 3-6. Pole pattern for 16th degree magnitude squared function. Passband shown as double lined axis.

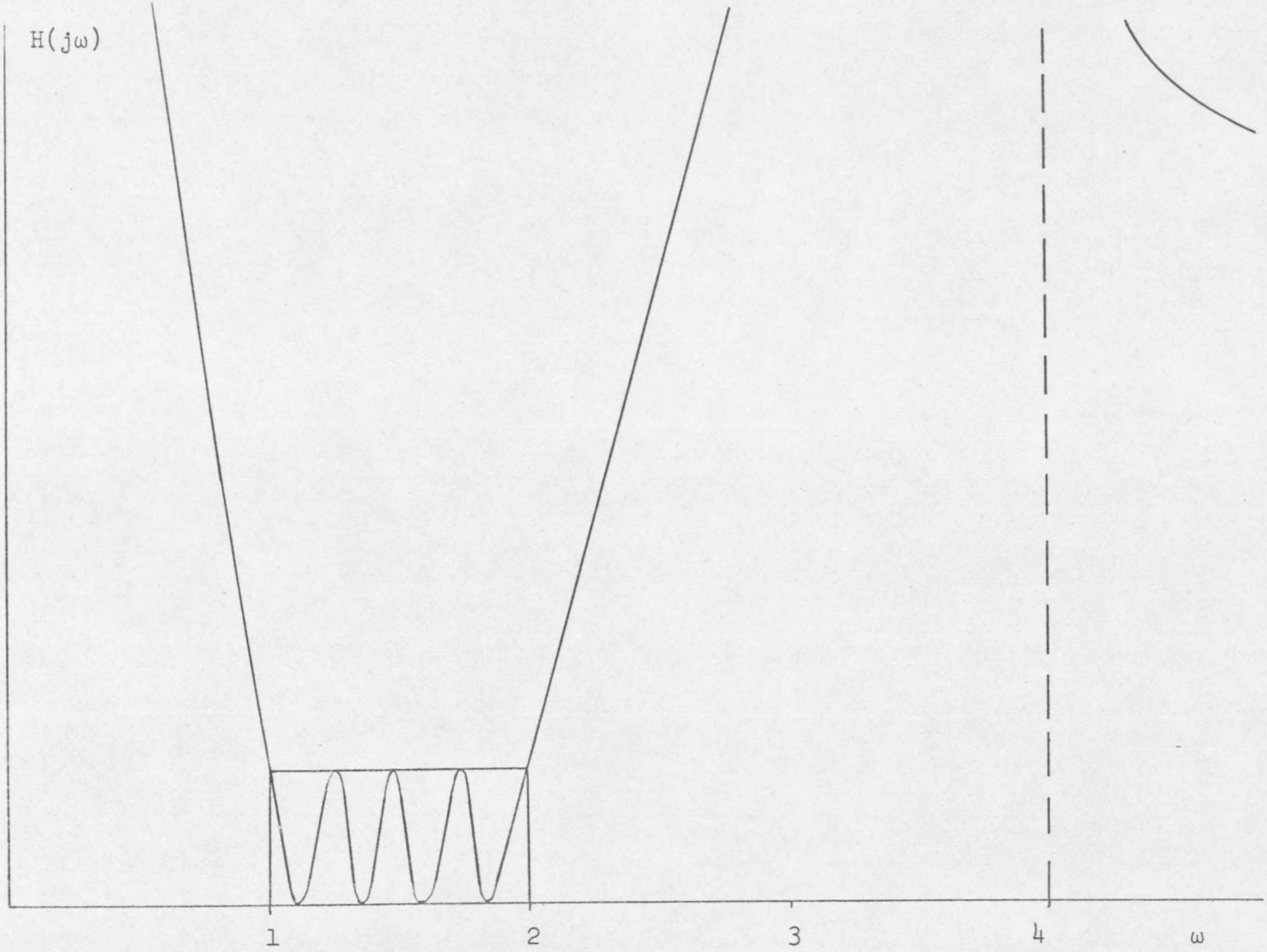


Figure 3-7. $H(j\omega)$ versus ω for function with pole pattern of Figure 3-4.

TABLE 3-1. Pertinent Values in the Computation of Figure 3-7.

Magnitude squared function of degree 16

Passband extends from 1.0 to 2.0.

Pole locations	Corresponding m_i
±1.0 ±j1.0	Conjugate product = 2.0000000
(4 poles total)	Conjugate sum = 2.6832816
±j4.0 (4 poles total)	.89442719
± 1.0 (2 poles total)	1.5811388
∞ (4 poles total)	1.0000000
0.0 (2 poles total)	2.0000000

$$H(p)=\frac{73861.049p^{16}+1400691.7p^{14}+11288710p^{12}+50410944p^{10}-136211290p^{8}+227754900p^{6}+229963580p^{4}+128165900p^{2}+30201551}{11.158108p^{12}+345.90135p^{10}+2544.0486p^{8}-1472.8703p^{6}+9997.6648p^{4}-11425.903p^{2}}$$

Passband zeros at

$\omega = \pm 1.0391112, \pm 1.3084135, \pm 1.6866165, \pm 1.9610111$

CHAPTER 4

VARIATIONS AND APPLICATIONS

4.1 INTRODUCTION

Variations of the three fundamental functions to produce rational functions with and without symmetry about the origin begin the chapter. The variations are of a more tactical nature than strategic but do require an understanding of Chapters 2 and 3. Applications include generation of the Chebyshev rational function (CRF), having equiripple passband approximation of zero from $\omega = 0$ to $\omega = 1$ and equiripple stopband approximation of infinity beginning at $\omega = \omega_c$ and extending to $\omega = \infty$. More general equiripple functions having passbands and stopbands of finite length are presented. A rapidly convergent iterative procedure is used in both.

A limitation of the power of the rotating vector principle is encountered when one desires functions with a known denominator and certain preassigned zeros, i.e., when functions without the maximum number of passband ripples possible for their degree are required. It is shown that the usual procedure of constructing $F(p)$ from individual factors for each pole is not in general possible. A solution for such requirements is shown; however it is not particularly satisfying, requiring the solution of nonlinear simultaneous equations for determination of passband zero locations. This result blocks development of a rapid iterative procedure for multiband equiripple filter functions. It is felt by the author that the procedure due to McGee (13) is still the most efficient for such filters.

A limited solution to the multiple passband equiripple filter is presented. It is limited in the sense that: (1) All passbands have the same number of ripples, (2) all passbnads have the same amplitude of ripple and (3) the poles are not independently specifiable.

The chapter concludes with a short inquiry into equiripple functions formed by adding arbitrary constants to F(p) in formation of G(p).

4.2 NON-MAGNITUDE-SQUARED RATIONAL FUNCTIONS

All functions produced so far satisfy the three requirements for magnitude squared functions: (1) All poles and zeros are in quadrantal symmetry, (2) all j-axis poles and zeros are of even multiplicity and (3) H(p) is non-negative for all ω. In addition they are symmetrical about zero, i.e., they are strictly even or odd functions in p^2. Such a feature is not necessary to magnitude functions in general but is characteristic of these.

Suppose now that one wanted to produce not magnitude squared functions but simply rational functions; meaning that requirements (2) and (3) are removed, leaving as the single criterion that all poles and zeros be in quadrantal symmetry. Such functions have simple zeros in the passband and approximate zero with ripple maxima of +1 and minima of −1. Examples are the entire set of Chebyshev polynomials and the set of Chebyshev rational functions. Notationwise they will be called R(x), rational functions of the real variable x. Even or odd nature of the resulting functions will be retained. See Figure 4-1 for a general sketch. It is assumed that one knows the degree of the desired rational function and its denominator in factored form; i.e., pole locations are explicit.

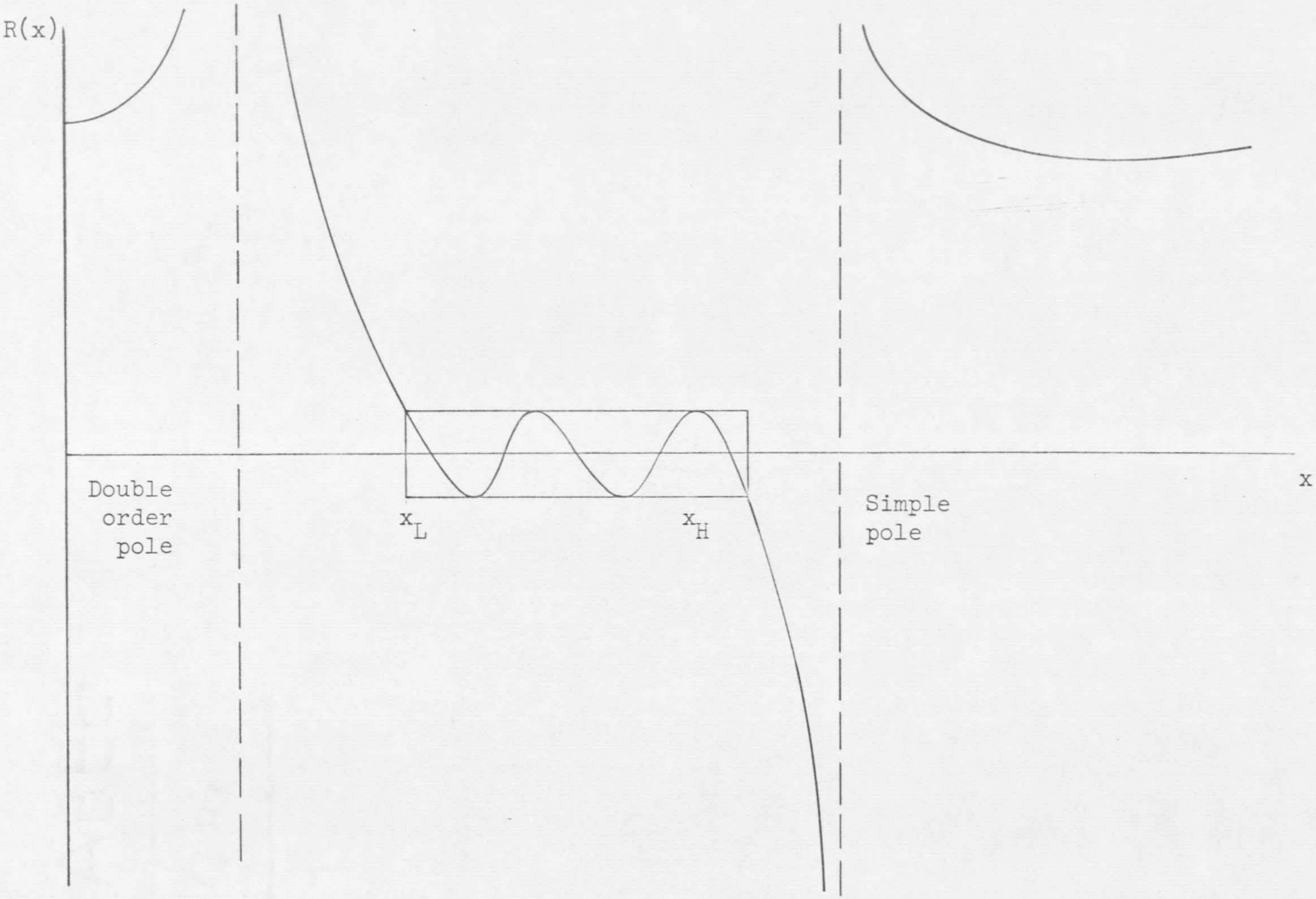


Figure 4-1. General form of an R(x).

The approach to such $R(x)$ is based on beginning as though its square, $R^2(x)$, were to be constructed. An $F(x)$ consistent with $R^2(x)$ is constructed and a corresponding $G(x)$ is formed. Here the procedure is truncated. The numerator of $G(x)$ is the proper numerator of $R(x)$. $R(x)$ denominator is known so that only constant multiplier adjustment is required.

The procedure begins by constructing a modified $F(x)$ suitable for x a real variable and n the degree of $R(x)$

$$F(x) = \prod_{i=1}^{n} \frac{m_i\sqrt{x^2 - x_L^2} + \sqrt{x^2 - x_H^2}}{-m_i\sqrt{x^2 - x_L^2} + \sqrt{x^2 - x_H^2}} \tag{4.1}$$

for bandpass and lowpass, and

$$F(x) = \prod_{i=1}^{n} \frac{m_i + \sqrt{x^2 - x_H^2}}{-m_i + \sqrt{x^2 - x_H^2}} \tag{4.2}$$

for highpass. $F(x)$ is constructed so that if operations were carried through in the magnitude squared sense, all poles of $R(x)$ would appear twice, i.e., $F(x)$ is consistent with the formation of $R^2(x)$. It is seen that the rotating vector property exists in the passband for x a real variable. Passbands are symmetrically placed about the origin. Outside the passband $F(x)$ is real and can be made to exhibit desired poles by computing m_i as

$$m_i = \sqrt{\frac{x_i^2 - x_H^2}{x_i^2 - x_L^2}} \tag{4.3}$$

in bandpass and lowpass and

$$m_i = \sqrt{x_i^2 - x_L^2} \tag{4.4}$$

in highpass. Again

$$G(x) = \frac{F(x) + (-1)^n}{2} \tag{4.5}$$

and has a rational numerator exhibiting simple zeros in the passband. By carrying through in a parallel manner as before, one can form $R^2(x)$ as the product of $G(x)$ and $\bar{G}(x)$. $R(x)$ is formed by truncating the process at $G(x)$, using $G(x)$ numerator over the desired denominator and setting levels with a constant multiplier.

Computationally the procedure is:

1. Square the required denominator. It is assumed that the denominator is in factored form so this amounts to doubling of each factor.

2. Construct $F(x)$ as though this squared denominator were to be the final one of $R^2(x)$.

3. Set the rational part of $F(x)$ numerator over the required denominator. The function formed has all desired pole and zero locations and is correct to within a constant

multiplier.

4. Adjust the value at band edge to whatever ripple is desired through a constant multiplier.

As an example, consider the set of all Chebyshev polynomials. They are lowpass, extending from 0 to 1, and have all poles at infinity. F(x) for the nth polynomial is

$$F(x) = \prod_{i=1}^{n} \frac{x + \sqrt{x^2 - 1}}{-x + \sqrt{x^2 - 1}}$$

$$= \left(\frac{x + \sqrt{x^2 - 1}}{-x + \sqrt{x^2 - 1}} \right)^n \tag{4.6}$$

G(x) is formed as

$$G(x) = \frac{\text{Rational Part } (x + \sqrt{x^2 - 1})^n}{(-x + \sqrt{x^2 - 1})^n} \tag{4.7}$$

R(x) is formed by setting the numerator of G(x) over the desired denominator. For the Chebyshev polynomials the denominator is a constant; in fact unity. Then

$$R(x) = T_n(x)$$

$$= \text{Rational part } (x + \sqrt{x^2 - 1})^n$$

$$= \text{Real part } (x + j\sqrt{1 - x^2})^n \qquad (4.8)$$

which is recognized as one of the common defining equations for $T_n(x)$ (19). Other examples are shown in Figures 4-2, 4-3 and 4-4. Tables 4-1, 4-2 and 4-3 list pertinent computational values.

4.3 RATIONAL FUNCTIONS WITHOUT EVEN OR ODD SYMMETRY[1]

Generation of functions with even or odd symmetry is not an intrinsic property of the rotating vector principle. It is easily adapted to generate functions, $R(x)$, exhibiting equiripple behavior over any finite or infinite interval. Procedure is much the same as in generation of rational functions of Section 4.2. It is assumed that the degree of the function and its denominator are known. $F(x)$ is constructed as though $R^2(x)$ were to be the end result. The rational part of $F(x)$ numerator will then be the desired numerator. The denominator is already known so that only a constant multiplier is required to adjust ripple value.

For n the desired function degree, modified $F(x)$ are

$$F(x) = \prod_{i=1}^{n} \left[\frac{m_i\sqrt{x - x_L} + \sqrt{x - x_H}}{-m_i\sqrt{x - x_L} + \sqrt{x - x_H}} \right]^2 \qquad (4.9)$$

for passband over the interval $x_L \leqslant x \leqslant x_H$,

[1] The material in this section does not generally lead to useful filter functions. It is intended only as a mathematical sidelight to the generation of filter functions and can be omitted with no loss of continuity.

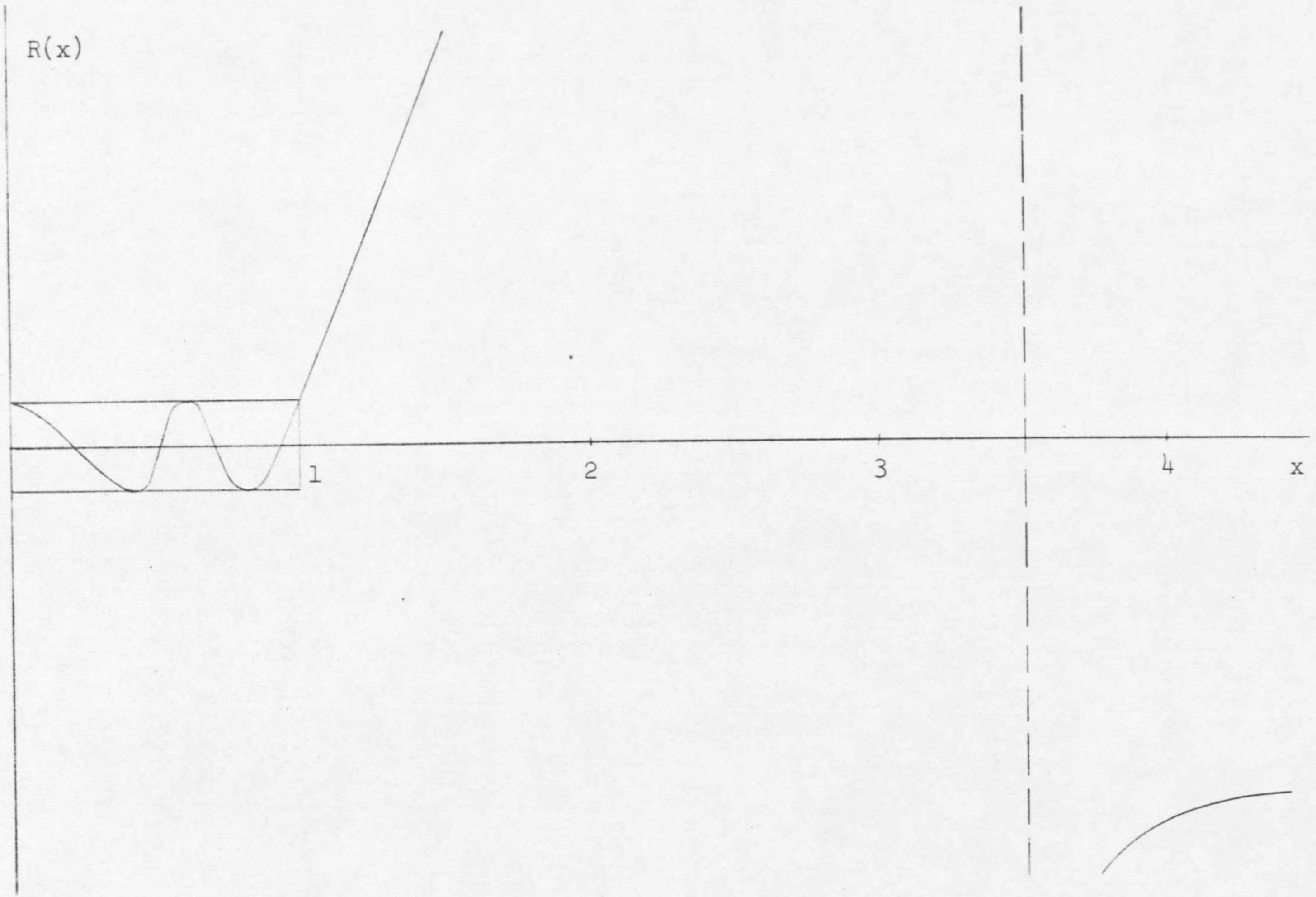


Figure 4-2. Lowpass R(x).

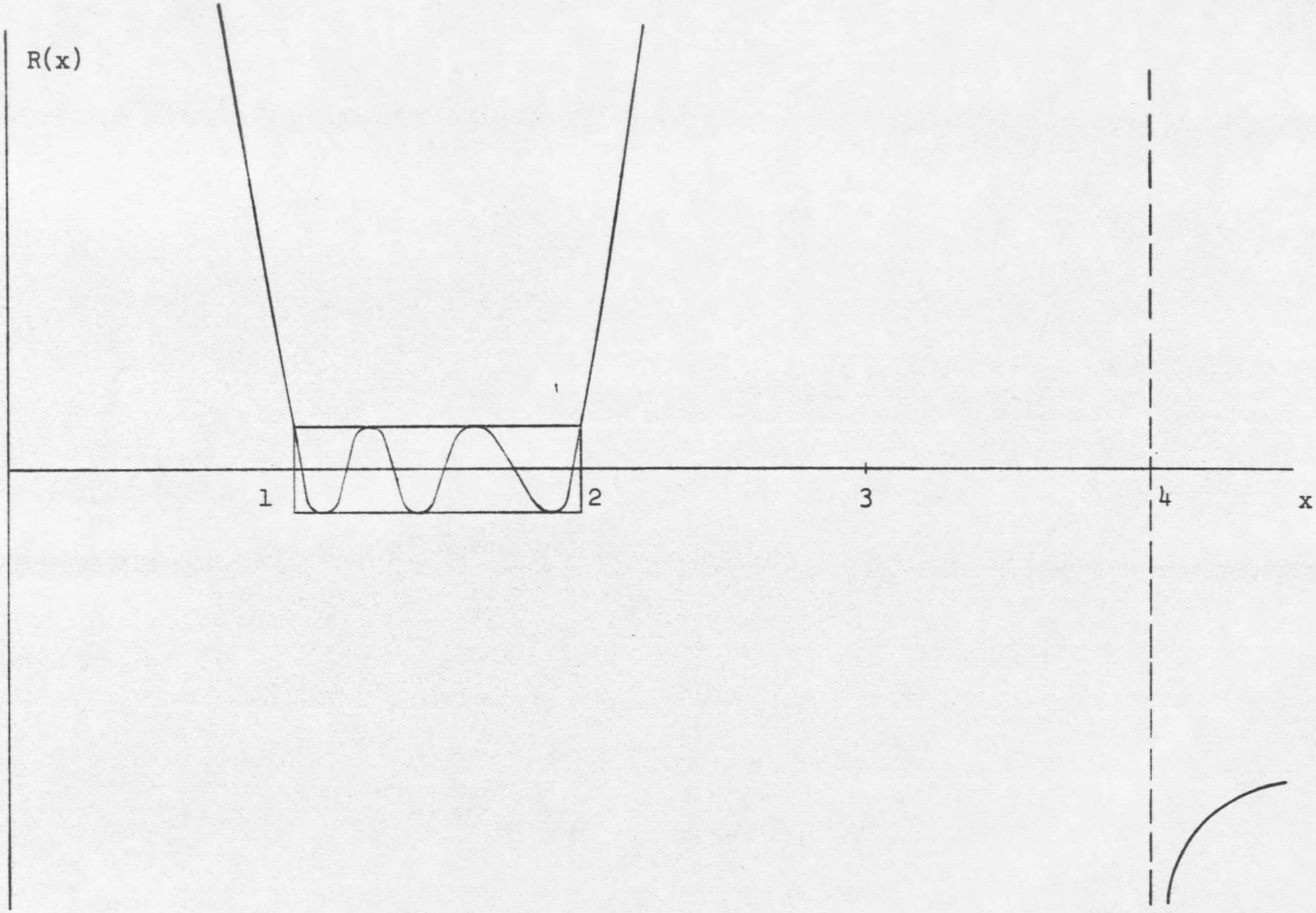


Figure 4-3. Bandpass R(x).

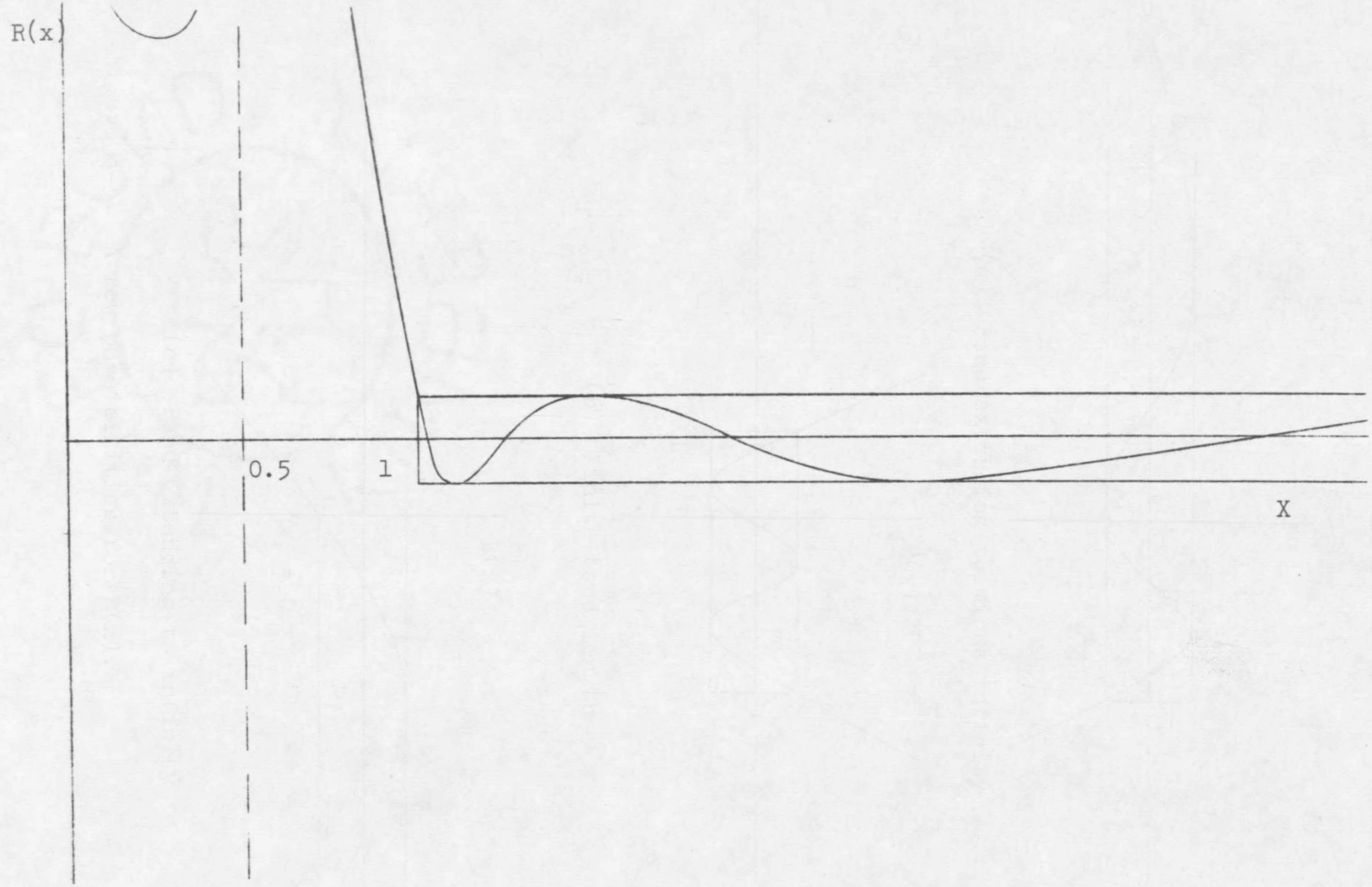


Figure 4-4. Highpass R(x).

TABLE 4-1. Pertinent Values in the Computation of Figure 4-2.

Rational function of degree 8.

Passband extends from 0.0 to 1.0.

Pole locations		
±3.0 : 1.5	Conjugate product =	.94933375
(4 poles total)	Conjugate sum =	1.9473060
±3.5 (2 poled total)		.95831485
∞ (2 poles total)		1.0000000

$$R(x) = \frac{116.45967x^8 - 235.68310x^6 + 149.73715x^4 - 30.696048x^2 + 1.0000000}{-.00064498 74x^6 + .01660871 8x^4 - .18829932x^2 + 1.0000000}$$

Passband zeros at

$x = \pm .19961558,\ \pm .56469322,\ \pm .83743802,\ \pm .98164067$

TABLE 4-2. Pertinent Values in the Computation of Figure 4-3.

Rational function of degree 12.

Passband extends from 1.0 to 2.0.

Pole locations	Corresponding m_i
±1.0 ±j1.0	Conjugate product = 2.0000000
(4 poles total)	Conjugate sum = 2.6832816
±4.0 (2 poles total)	.89442719
±j1.0 (2 poles total)	1.5811388
∞ (2 poles total)	1.0000000

$$R(x) = \frac{\begin{array}{c}13901.108x^{12} - 190695.48x^{10} + 1043585.4x^{8} \\ - 2908371.0x^{6} + 4346962.1x^{4} - 3304097.0x^{2} \\ + 999443.87\end{array}}{\begin{array}{c}-34.714285x^{10} + 520.71427x^{8} + 416.57142x^{6} \\ +2082.8571x^{4} + 2221.7142x^{2}\end{array}}$$

Passband zeros at

x = ± 1.0138523, ± 1.1212712, ± 1.3185613,

± 1.5712704, ± 1.819223, ± 1.9782853.

TABLE 4-3. Pertinent Values in the Computation of Figure 4-4.

Rational function of degree 9.

Passband extends from 1.0 to infinity.

Pole Locations	Corresponding m_i
$\pm 1.0 \quad j1.0$	Conjugate Product = 2.2360680
(4 poles total)	Conjugate Sum = 2.5440393
± 0.5 (2 poles total)	.86602540
$\pm j1.0$ (2 poles total)	1.4142136
0.0 (1 pole total)	1.0000000

$$R(x) = \frac{10.648557x^8 - 190.18398x^6 + 855.76559x^4 - 1346.7748x^2 + 678.04462}{1.0000000x^9 + .75000000x^7 + 3.7500000x^5 + 3.0000000x^3 - 1.0000000x}$$

Passband zeros at

$x = \pm\ 1.0254304,\ \pm\ 1.24600004,\ \pm\ 1.8023073,\ \pm\ 3.4652179$

$$F(x) = \prod_{i=1}^{n} \left[\frac{m_i + \sqrt{x_L - x}}{-m_i + \sqrt{x_L - x}} \right]^2 \tag{4.10}$$

for passband over the interval $x_L \leqslant x < \infty$; and

$$F(x) = \prod_{i=1}^{n} \left[\frac{m_i + \sqrt{x - x_H}}{-m_i + \sqrt{x - x_H}} \right]^2 \tag{4.11}$$

for passband over the interval $-\infty < x \leqslant x_H$. It will be explained shortly why $F(x)$ appears as a product of squared rather than simple terms. All three functions are rotating vectors in their respective passbands and have singularities at $x = x_i$ in accordance with

$$m_i = \sqrt{\frac{x_i - x_H}{x_i - x_L}}, \tag{4.12}$$

$$m_i = \sqrt{x_L - x_i} \tag{4.13}$$

and

$$m_i = \sqrt{x_i - x_H} \tag{4.14}$$

respectively.

Resultant $R(x)$ functions have, in general, no symmetry whatsoever. Correspondingly, concepts of lowpass, bandpass and highpass have no meaning. One speaks only of a passband interval. For example, $R(x)$ for passband over the interval $x_L \leqslant x \leqslant x_H$ will be equiripple over that range only. Action of $R(x)$ for $-x_H \leq x \leq -x_L$ is not necessarily constrained. In fact it is possible for these ranges to overlap. Figure 4-5 shows various possibilities for $R(x)$.

An important point must be observed when using these $F(x)$. The lack of symmetry means that if a singularity is placed in $F(x)$ at $x = x_i$, its negative would not appear in $R^2(x)$. Rather there will appear a simple pole of $R^2(x)$ at $x = x_i$. Consequently, each individual pole of $R^2(x)$ must be explicitly placed in $F(x)$. In terms of $R(x)$, each individual pole must be placed twice in $F(x)$. For $R(x)$ with n zeros, there are n poles and n squared factors in $F(x)$, precisely enough to allow placing each pole of $R(x)$ twice.

For a given set of specifications in $R(x)$: (1) x_L and x_H, (2) degree of $R(x) = n$, (3) pole locations; $x = x_1, x_2, \cdots, x_n$ and (4) value of $R(x)$ at one band edge, a computational procedure for finite passband is:

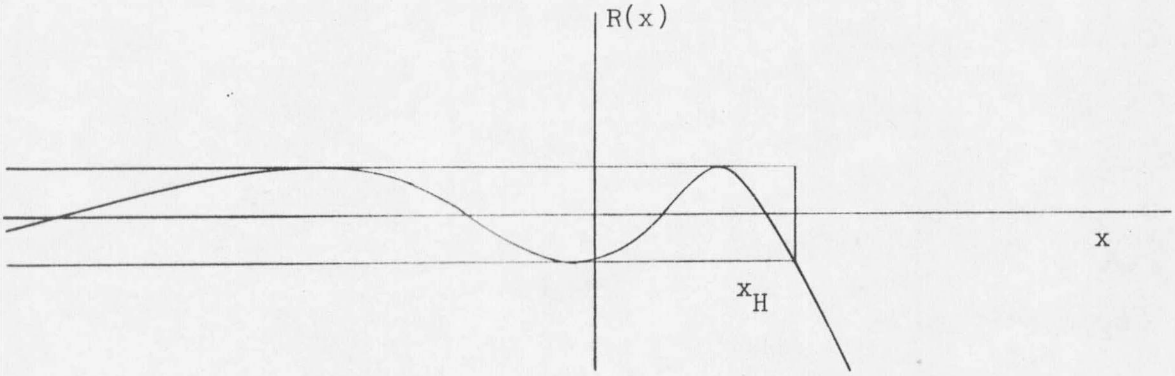
1. Construct $F(x)$. Equation (4.9) is used.

$$F(x) = \prod_{i=1}^{n} \left[\frac{m_i\sqrt{x - x_L} + \sqrt{x - x_H}}{-m_i\sqrt{x - x_L} + \sqrt{x - x_H}}\right]^2 \tag{4.9}$$

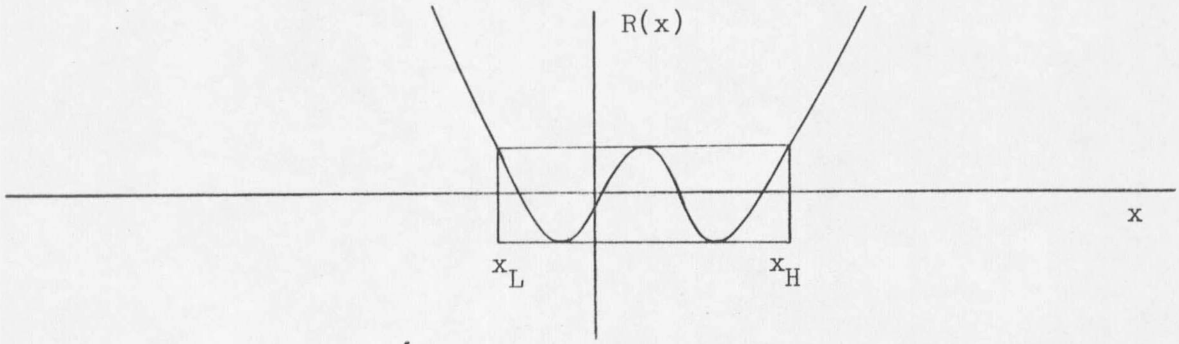
where, in accord with Equation (4.12)

$$m_i = \sqrt{\frac{x_i - x_H}{x_i - x_L}} \tag{4.12}$$

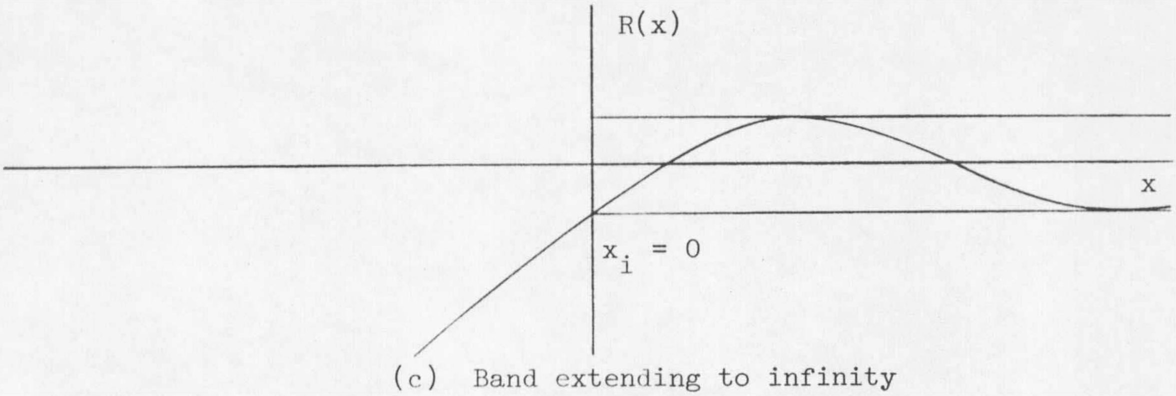
2. $G(x)$ will always be



(a) Band extending to minus infinity



(b) Finite band length



(c) Band extending to infinity

Figure 4-5. A few possibilities for R(x).

$$G(x) = \frac{F(x) + 1}{2} \tag{4.15}$$

because an even number of factors are always present in $F(x)$. Set the numerator of $G(x)$ over a denominator possessing the desired pole locations of $R(x)$. There results

$$R(x) = \frac{\text{Rational Part [Num } F(x)]}{K \text{ [Denominator]}} \tag{4.16}$$

3. Adjust K to give the desired value at specified band edge.

As a numerical example, consider a rational function that is to be equiripple from $x = x_L$ to $x = x_H$ but not for the range $x = -x_L$ to $x = -x_H$. There are to be 3 zeros in the interval and the denominator is to provide poles at $x = x_1$, x_2 and x_3. See Figure 4-6. An appropriate $F(x)$ would be

$$F(x) = \prod_{i=1}^{3} \left[\frac{m_i\sqrt{x - x_L} + \sqrt{x - x_H}}{-m_i\sqrt{x - x_L} + \sqrt{x - x_H}} \right]^2 \tag{4.17}$$

where

$$m_i = \sqrt{\frac{x_i - x_H}{x_i - x_L}}; \; i = 1, 2, 3 \tag{4.18}$$

For $x_L = 1$, $x_H = 2$ and all poles at infinity so that all $m_i = 1$, this

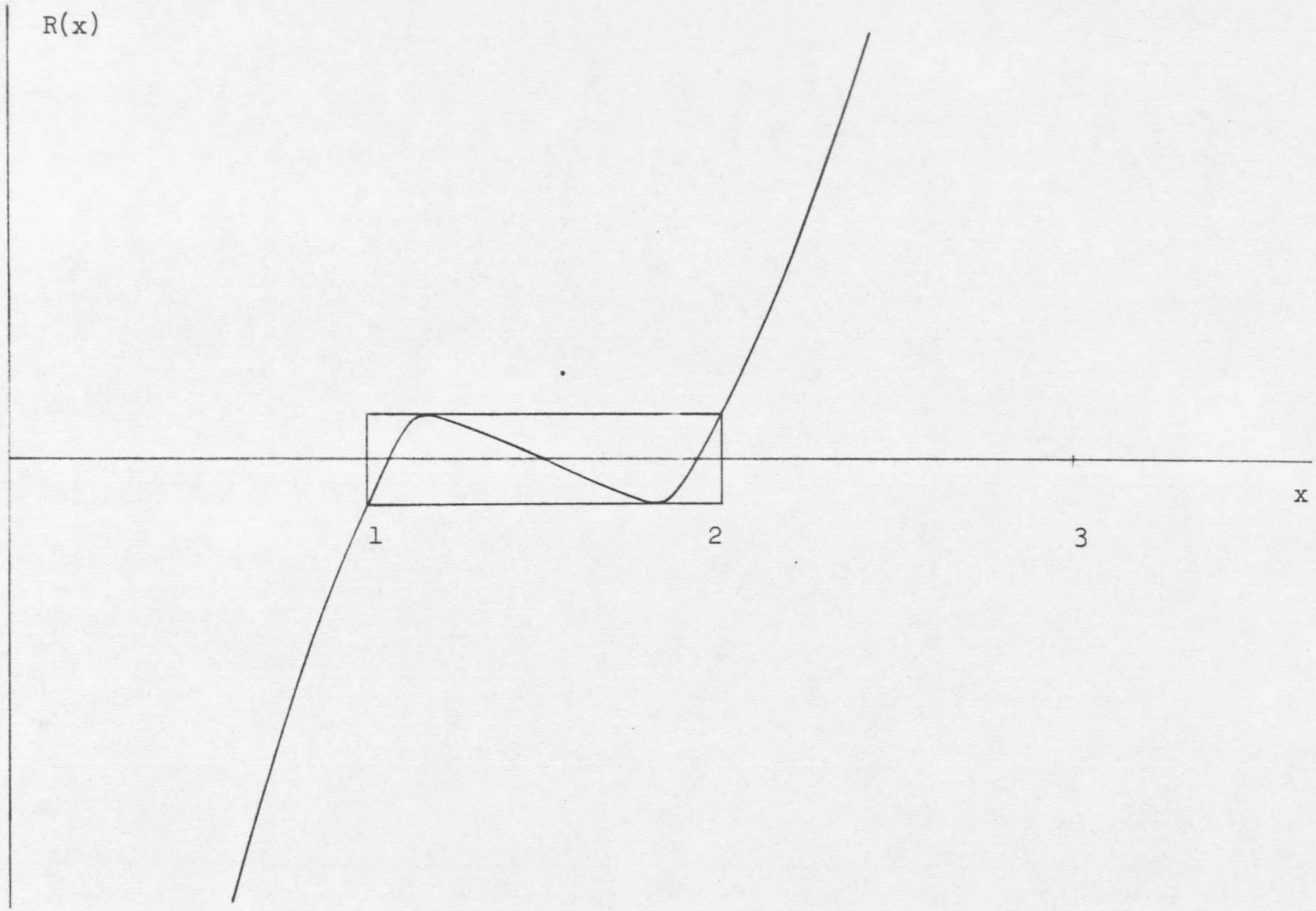


Figure 4-6. Plot of $R(x) = 32x^3 - 144x^2 + 210x - 99$.

becomes

$$F(x) = \left[\frac{\sqrt{x-1} + \sqrt{x-2}}{-\sqrt{x-1} + \sqrt{x-2}}\right]^6$$

$$= \frac{(32x^3 - 144x^2 + 210x - 99) + (32x^2 - 88x + 70)\sqrt{x-1}\sqrt{x-2}}{(32x^3 - 144x^2 + 210x - 99) - (32x^2 - 88x + 70)\sqrt{x-1}\sqrt{x-2}} \qquad (4.19)$$

Then

$$G(x) = \frac{32x^3 - 144x^2 + 210x - 99}{(32x^3 - 144x^2 + 210x - 99) - (32x^2 - 88x + 70)\sqrt{x-1}\sqrt{x-2}} \qquad (4.20)$$

We are interested in the numerator only, so that

$$R(x) = \frac{32x^3 - 144x^2 + 210x - 99}{K} \, . \qquad (4.21)$$

For $R(1) = \mp 1$ or, what is the same thing, $R(2) = \pm 1$ we find $K = \pm 1$.

4.4 THE CHEBYSHEV RATIONAL FUNCTION (CRF)[1]

This classic form is perhaps as familiar as the Chebyshev polynomials. There is no need to define or describe it here; it is well

[1] The term "Chebyshev rational function" has been used by some authors (4,11) to denote any rational function with equiripple behavior in the passband and arbitrary pole locations. The class of functions exhibiting equiripple lowpass approximation of zero and highpass approximation of infinity were refered to as complete elliptic filter functions. It is becoming more common (6, 19) to reserve "Chebyshev rational function" for this latter class only, while authors invent terms for the general class. This is the terminology adhered to in this presentation.

treated in most standard texts (5, 6, 8, 19, 21). An article by Papoulis (14) indicates an algebraic procedure for determining coefficients. Bennett (4) also proposed algebraic means for low degree functions based upon his original transformations. These are the same transformations that lead to the $F(p)$ functions discussed in Chapters 2 and 3. In that same article he also mentioned that a band to band iterative process which is rapidly convergent may be applied. Guillemin cast Bennett's iteration in different words and gave a low degree example (8). The procedure to be described in here is of an iterative nature and arrives at the CRF as a rational function of a real variable, as presented in Section 4.2.

In employing the functions of Chapter 3, two avenues of approach can be taken in producing the CRF. The first uses none of the already known properties of the CRF and is strictly a band to band iterative process, making first the lowband equiripple with respect to highband poles and finding the lowband zeros, then inverting the result and making the highband equiripple with respect to what are now lowband poles at the initial lowband zero locations. The second procedure initially appears a bit more sophisticated but produces identical results with respect to both speed and accuracy. It uses the property of geometric symmetry of poles and zeros of the CRF. For $x_L = 1$ this symmetry is about a center point, $\sqrt{x_c}$, where x_c is the beginning of the highband. For $\sqrt{x_c}$ the center frequency, x_{z_i} the location of the ith zero and x_{p_i} the location of the ith pole, then

$$x_{p_i} x_{z_i} = x_c ; \; i = 1, 2, \ldots, n \tag{4.22}$$

Stepwise, the first procedure is:

1. Given requirements of a CRF of degree n, lowband from 0 to 1 and highband from x_c to infinity, choose a set of highband pole locations to begin the iteration. Construct a lowpass $R_{L_1}(x)$ using these pole locations. This $R_{L_1}(x)$ is of course equiripple in lowpass.

2. Find the zeros of $R_{L_1}(x)$ and use them as poles for construction of a highpass $R_{H_1}(x)$ whose passband begins at x_c and runs to infinity. $R_{H_1}(x)$ is equiripple in highpass but is usually no longer equiripple in approximating infinity in lowpass. It will be equiripple in both bands only if the original guess for highband pole locations were exactly the final locations, a most unlikely prospect.

3. Find the zeros of $R_{H_1}(x)$ and use them as poles for the construction of a new lowpass function $R_{L_2}(x)$.

4. Continue the lowpass-highpass iteration until zero locations of $R_{L_k}(x)$ (or $R_{H_k}(x)$) differ from those of $R_{L_{(k-1)}}(x)$ (or

$R_{H_{(k-1)}}(x)$) by less than some preassigned measure.

The second procedure is:

1. Given requirements for a CRF of degree n, lowband from 0 to 1 and highband from x_c to infinity, choose a set of highpass pole locations to begin the iteration. Construct a lowpass $R_{L_1}(x)$ using these pole locations. This step is indentical to step (1) of the first method.

2. Find the zeros of $R_{L_1}(x)$. Let them be x_{z_1}, x_{z_2}, ..., x_{z_i}. Form $x_{p_1} = \frac{x_c}{x_{z_1}}$, $x_{p_2} = \frac{x_c}{x_{z_2}}$, ..., $x_{p_i} = \frac{x_c}{x_{z_i}}$ and use these as new pole locations to form $R_{L_2}(x)$.

3. Continue to form new $R_{L_k}(x)$ until zero locations of $R_{L_k}(x)$ differ from those of $R_{L_{(k-1)}}(x)$ by less than some preassigned measure.

Dependence of the second procedure upon geometric symmetry of the CRF is evident. Both methods produce results of identical occuracy in an identical number of iterations (Each new computation of an equiripple function is counted as one iteration, be the function lowpass or highpass.). This is not surprising in view of the fact that setting $x_{p_i} = \frac{x_c}{x_{z_i}}$ essentially constitutes a lowpass to highpass transformation. So the methods are equivalent. However, Procedure 2 results in a shorter computer program. Execution time is of course the same for both. Computationally a simplification is possible in both procedures. Rather than construct $R(x)$ completely in each iteration, only the zeros of the rational part of the numerator of $F(x)$ need be found. The programs written include this simplification.

Several runs on the computer provided the following information, some of which has been stated:

1. Both procedures gave identical results in the same number of iterations.

2. Although no convergence proof has been found, no example of failure to converge was encountered. Indeed, monotone convergence of zeros to their final location (accurate to 8 figures) was found in every case.

3. With the exception noted below, for a given x_c, the number of iterations was essentially independent of degree. For $x_c = 1.1$, about 13 iterations were required; for $x_c = 1.25$, about 9 iterations were required; for $x_c = 2.0$, about 7 iterations were required.

4. Initial choice of highband pole locations had almost no effect on the number of iterations required with one notable exception: For $n = 4, 8, 12, \cdots$, i.e., for $\frac{n}{2}$

even, when the initial poles were divided equally between half grouped at $x = x_c$ and half grouped at infinity, the number of iterations required decreased significantly. For $x_c = 1.1$, about 7 iterations were required; for $x_c = 1.25$, about 6 iterations were required; for $x_c = 2.0$, about 4 iterations were required. For $n = 2, 6, 10, \cdots$, i.e., for $\frac{n}{2}$ odd, the function will not allow an equal division of initial pole locations between band edge and infinity. By setting $\frac{n-1}{2}$ poles at ω_c and $\frac{n+1}{2}$ poles at infinity, a reduction of not more than one iteration was observed. 3, 5, 7, $\cdots$, an approximately equal division of initial highband poles between ω_c and infinity gave no reduction of iterations.

Figures 4-7 through 4-10 show migration of lowband zero locations for 4th degree and 8th degree functions; first for all highband poles chosen initially at infinity, then for equal division between x_c and infinity.

4.5 GENERAL EQUIRIPPLE PASS AND STOP FUNCTIONS.

These functions are a generalization of the CRF in that (1) the upper frequency band may extend over finite limits rather than to infinity, (2) the lower frequency band may begin at some finite, non-zero frequency or (3) both characteristics may be exhibited. See Figure 4-11 for an illustration of the general form. Since they do not in general exhibit the geometric symmetry characteristic of the CRF,

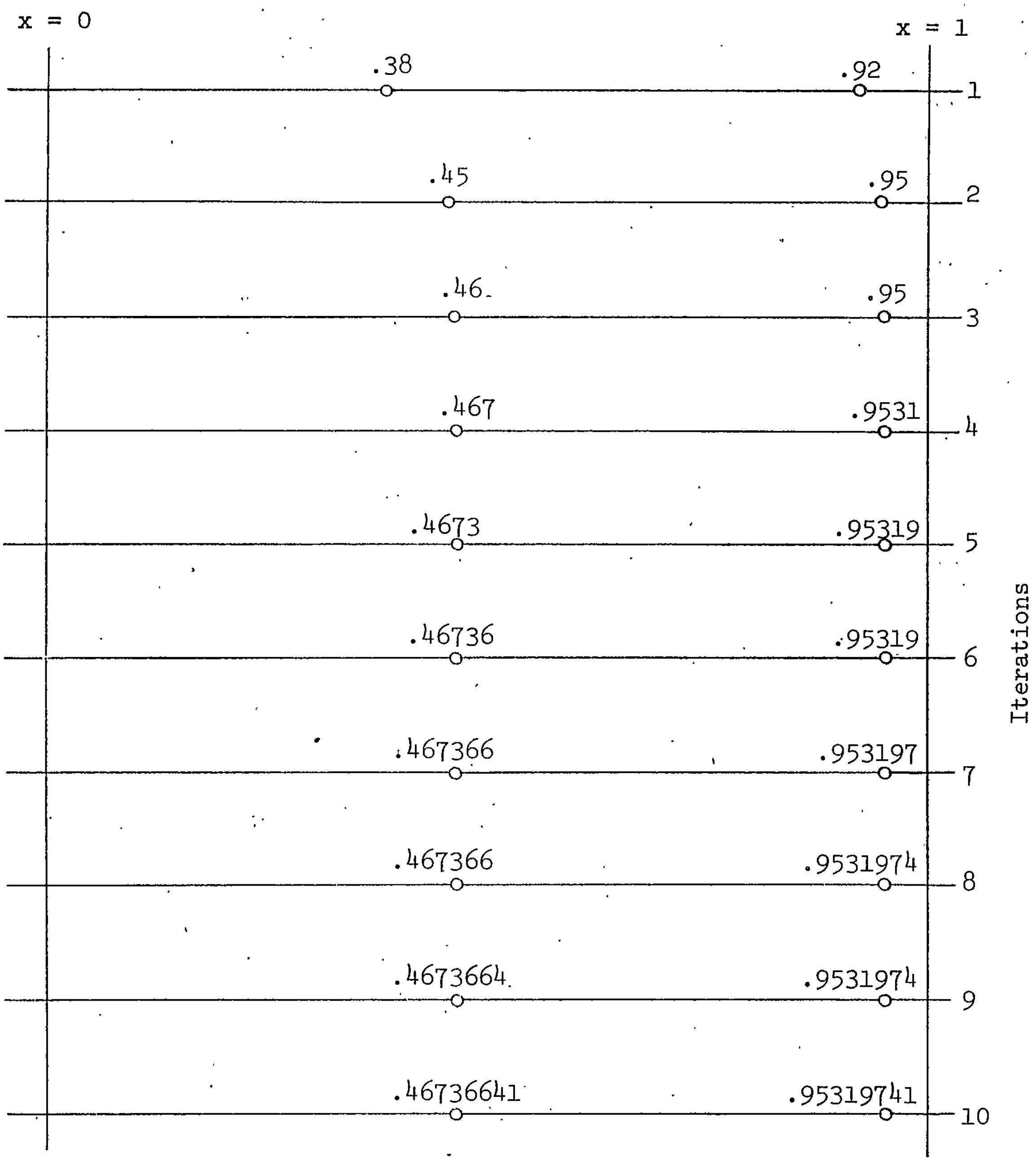


Figure 4-7. Zero migration for CRF of degree 4. $x_c = 1.25$.
All initial poles at infinity.

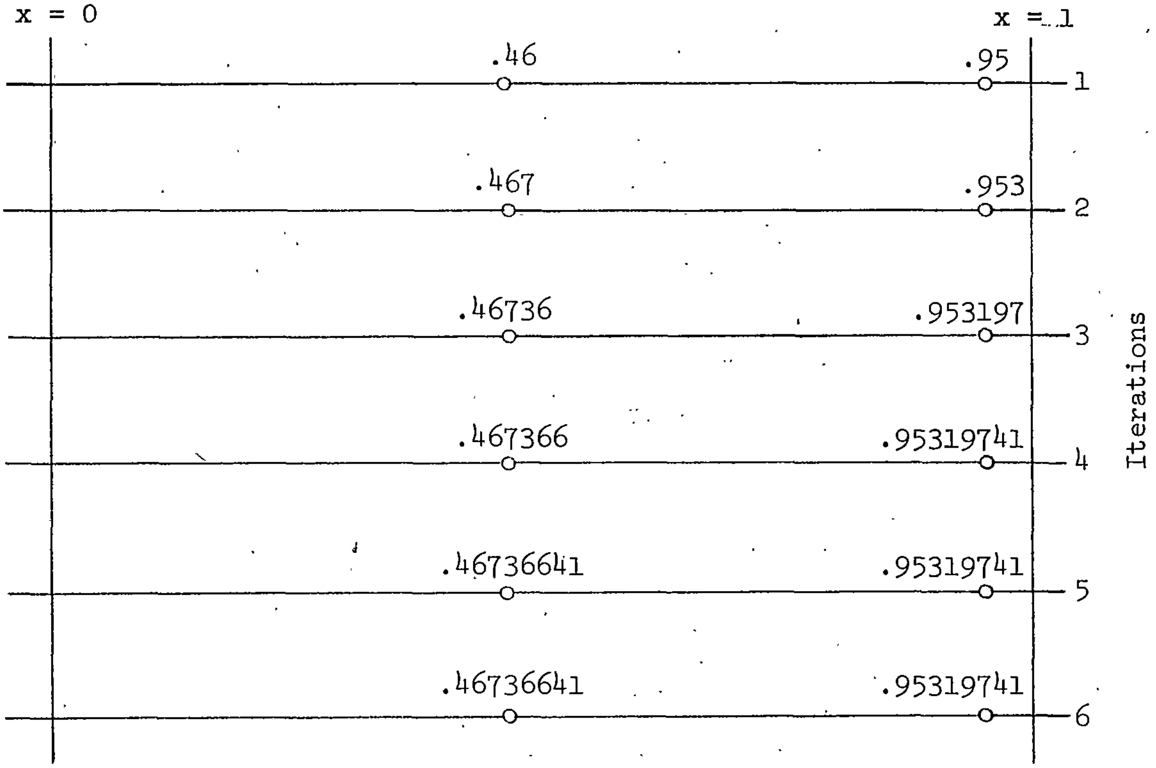


Figure 4-8. Zero migration for CRF of degree 4. $x_c = 1.25$. One initial pole location at x_c; one initial pole location at infinity.

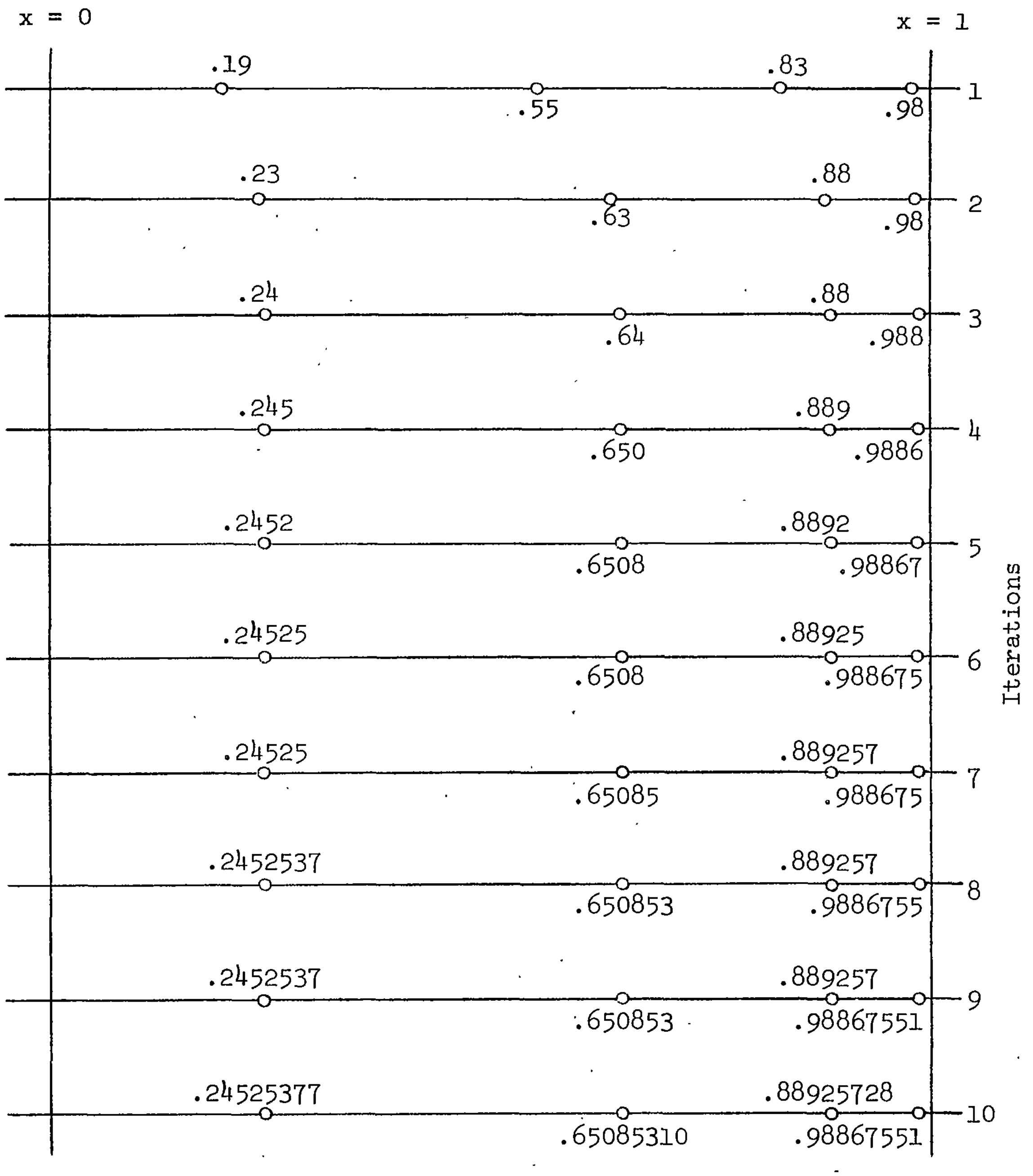


Figure 4-9. Zero migration for CRF of degree 8. $x_c = 1.25$. All initial pole locations at infinity.

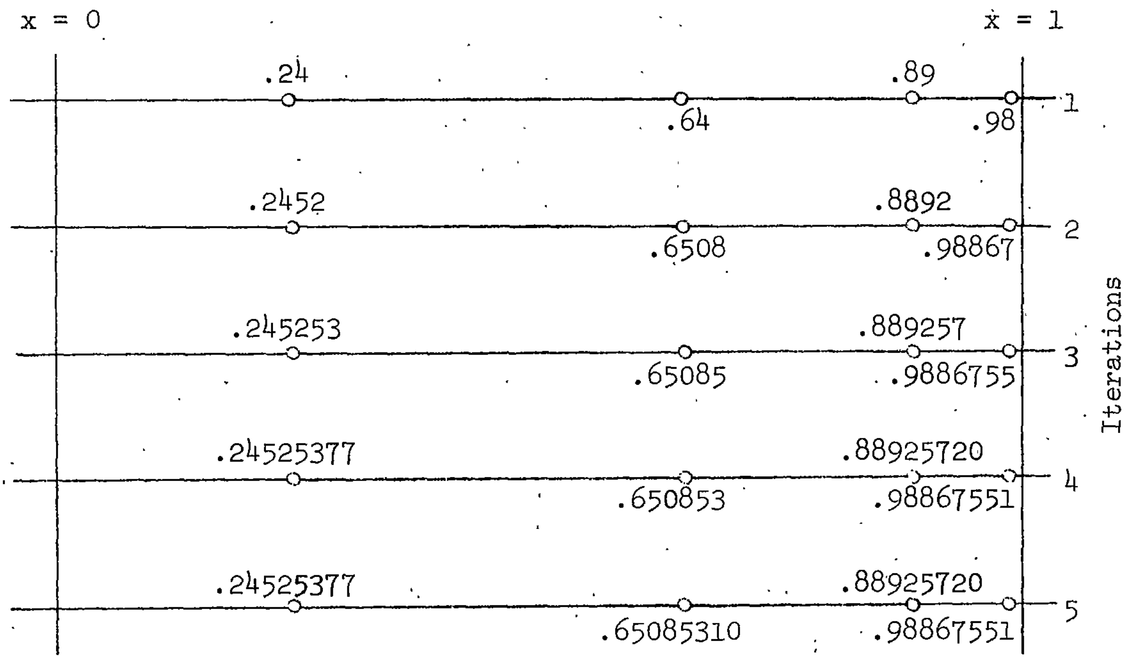


Figure 4-10. Zero migration for CRF of degree 8. $x_c = 1.25$. Two initial pole locations at x_c; two initial pole locations at infinity.

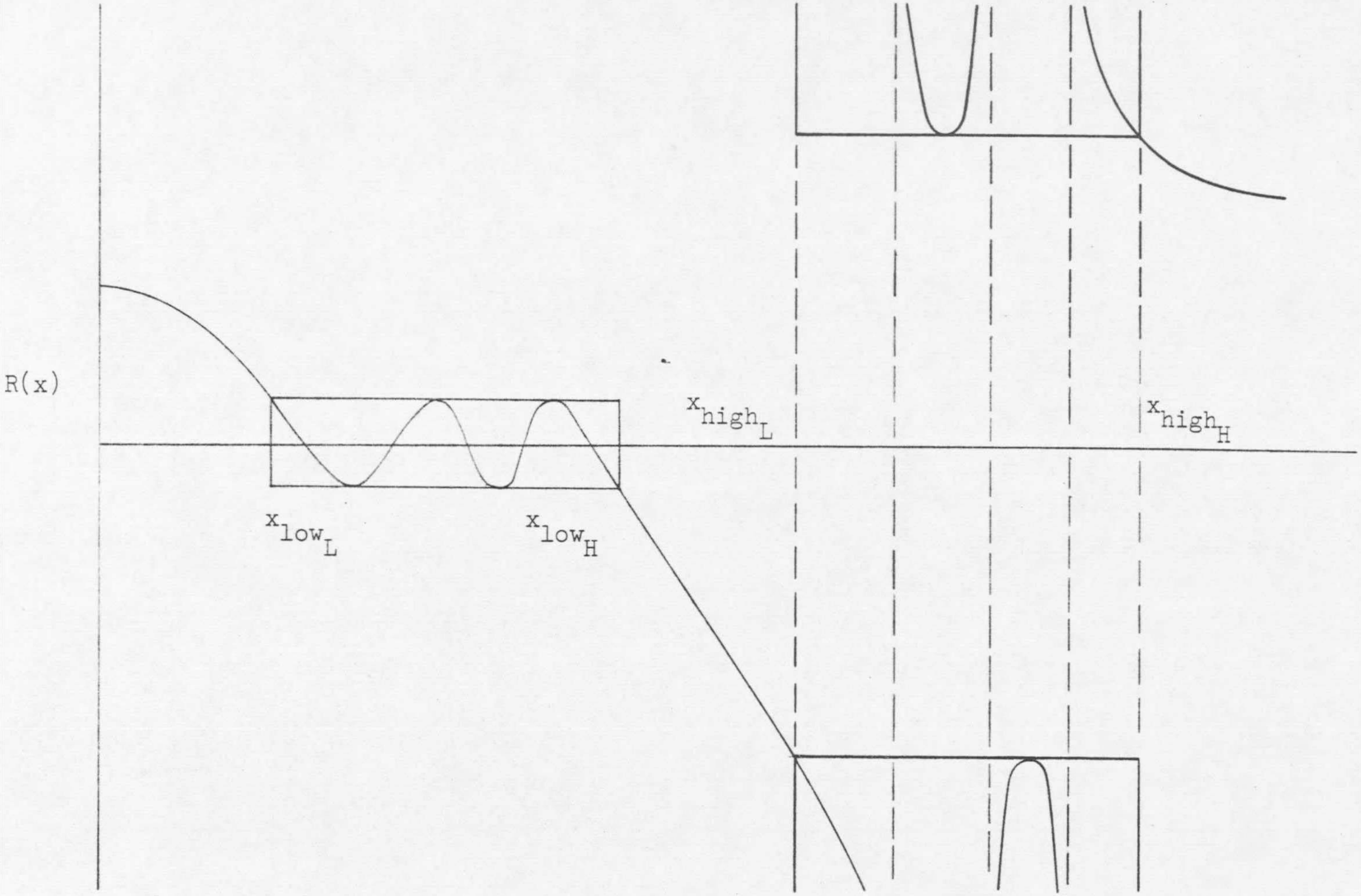


Figure 4-11. General equiripple passband and stopband function.

an iterative procedure like the first one mentioned in Section 4.4 is used. Specifically, it is:

1. Given lowband limits, $x_{low\,L}$ and $x_{low\,H}$, highband limits, $x_{high\,L}$ and $x_{high\,H}$ and desired degree n; choose a set of highband pole locations to begin the iteration. Construct a lowband $R_{L_1}(x)$ using these pole locations.

2. Find the zeros of $R_{L_1}(x)$ and use them as poles for construction of a highband $R_{H_1}(x)$.

3. Find the zeros of $R_{H_1}(x)$ and use them as poles for construction of a lowband $R_{L_2}(x)$.

4. Continue the iteration until zero locations of $R_{L_k}(x)$ (or $R_{H_k}(x)$) differ from those of $R_{L_{(k-1)}}(x)$ (or from those of $R_{H_{(k-1)}}(x)$) by less than some preassigned measure.

Again the computational simplification of finding only the zeros of G(x) numerator was used in writing a Fortran program. Execution of the program provided the following general information:

1. Monotone convergence of zeros and poles to their final locations (accurate to 8 figures) was again observed. This was true in every example although no convergence

proof has been found.

2. With the exception noted in (3), for given lowband and highband specifications, the number of iterations was essentially independent of degree. For lowband extending from 0 to 1.0, highband from 1.1 to 3.0, about 14 iterations were required; for lowband extending from 0 to 1.0, high-band from 1.5 to 3.0, about 9 iterations were required; for lowband extending from 0 to 1.0, highband from 2.0 to 3.0, about 7 iterations were required.

3. It was again found that for $\frac{n}{2}$ even, a choice of initial pole locations equally divided between the low and high limits of the highband produced convergence with significantly fewer iterations. Their number dropped to about 8, 6 and 5, respectively, for the bands listed in (2). Decrease in iterations for $\frac{n}{2}$ odd was not more than 1 with initial $\frac{n-1}{2}$ pole locations at low limit of the highband and $\frac{n+1}{2}$ at the high limit. For n odd and an approximately equal placement of initial poles at low and high limit of the highband, no change in the number of iterations was observed.

Figures 4-12 through 4-15 show migration of lowband zero locations and highband pole locations for 4th degree and 6th degree functions.

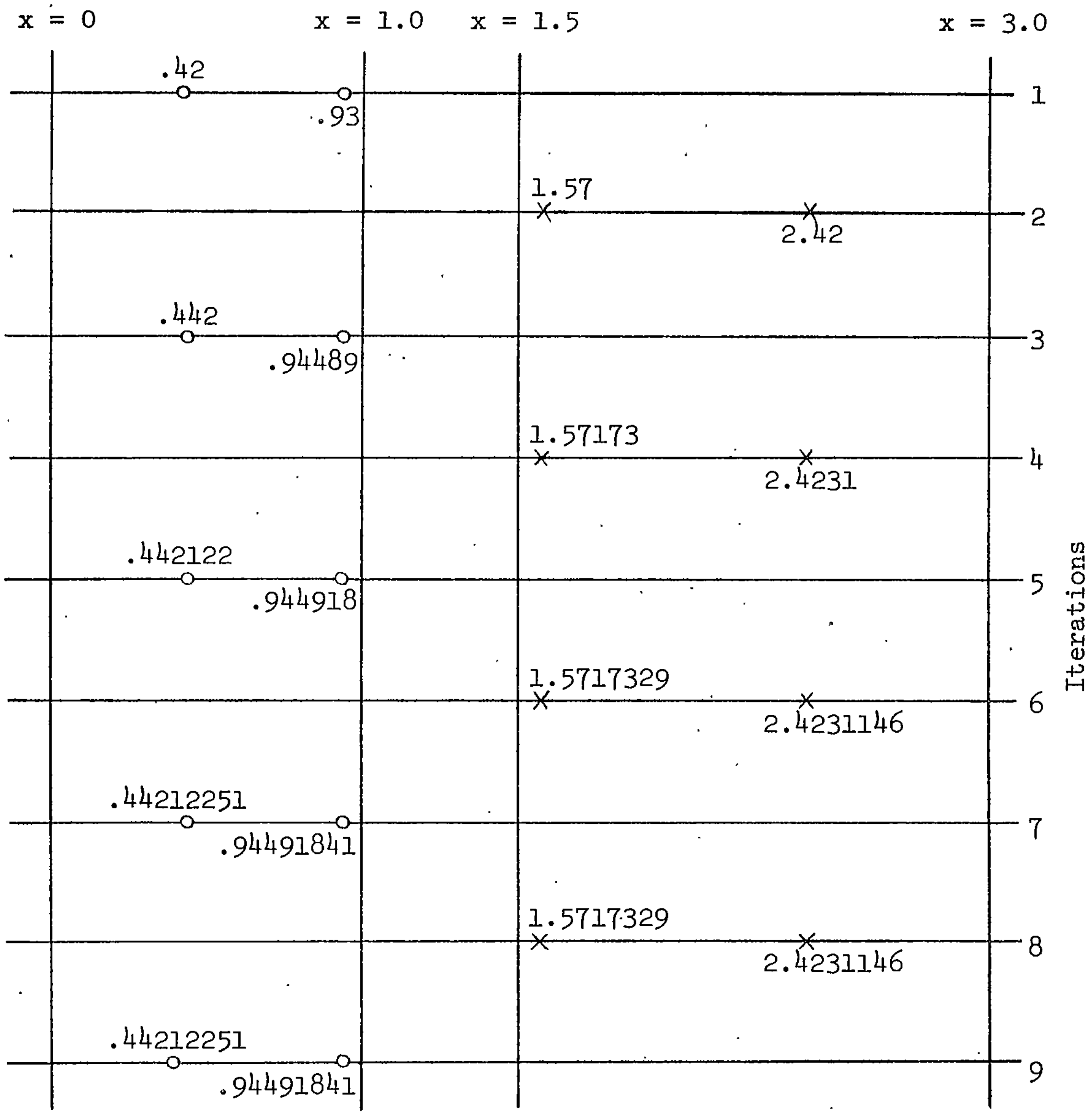


Figure 4-12. Zero and pole migration for equiripple passband and stopband function of degree 4. Band edges as shown. Initial pole locations at 2.0 and 2.5.

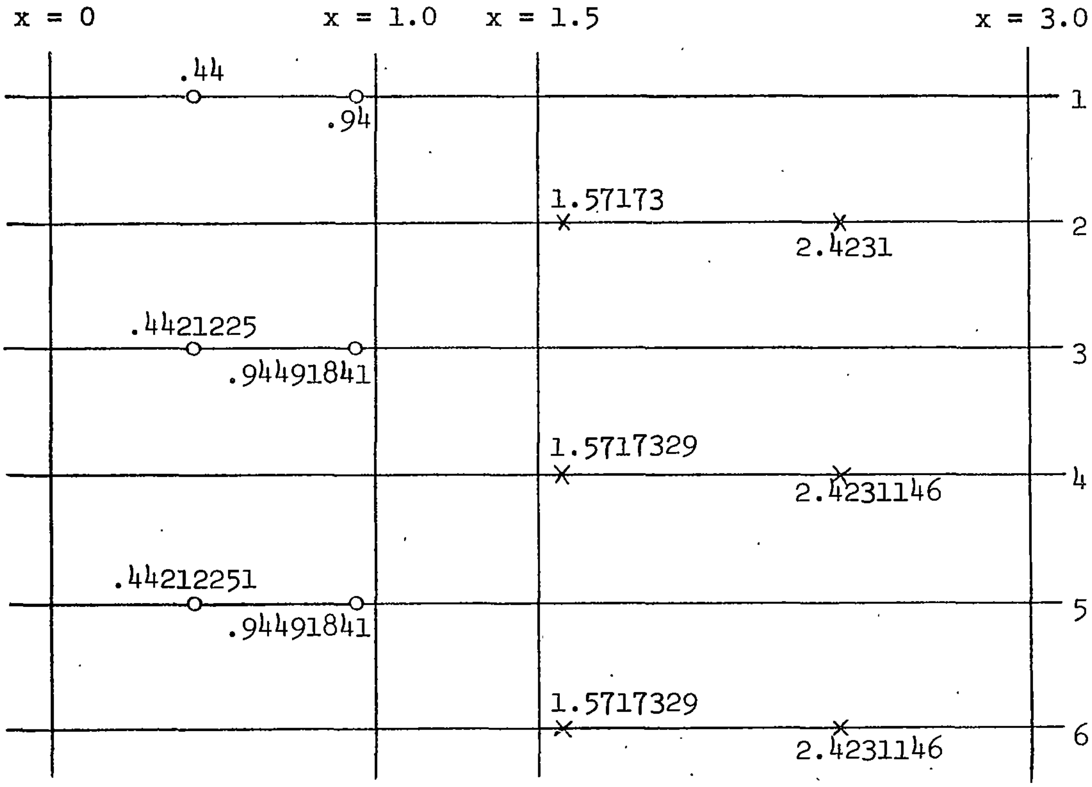


Figure 4-13. Zero and pole migration for equiripple passband and stopband function of degree 4. Band edges as shown. Initial pole locations at 1.5 and 3.0.

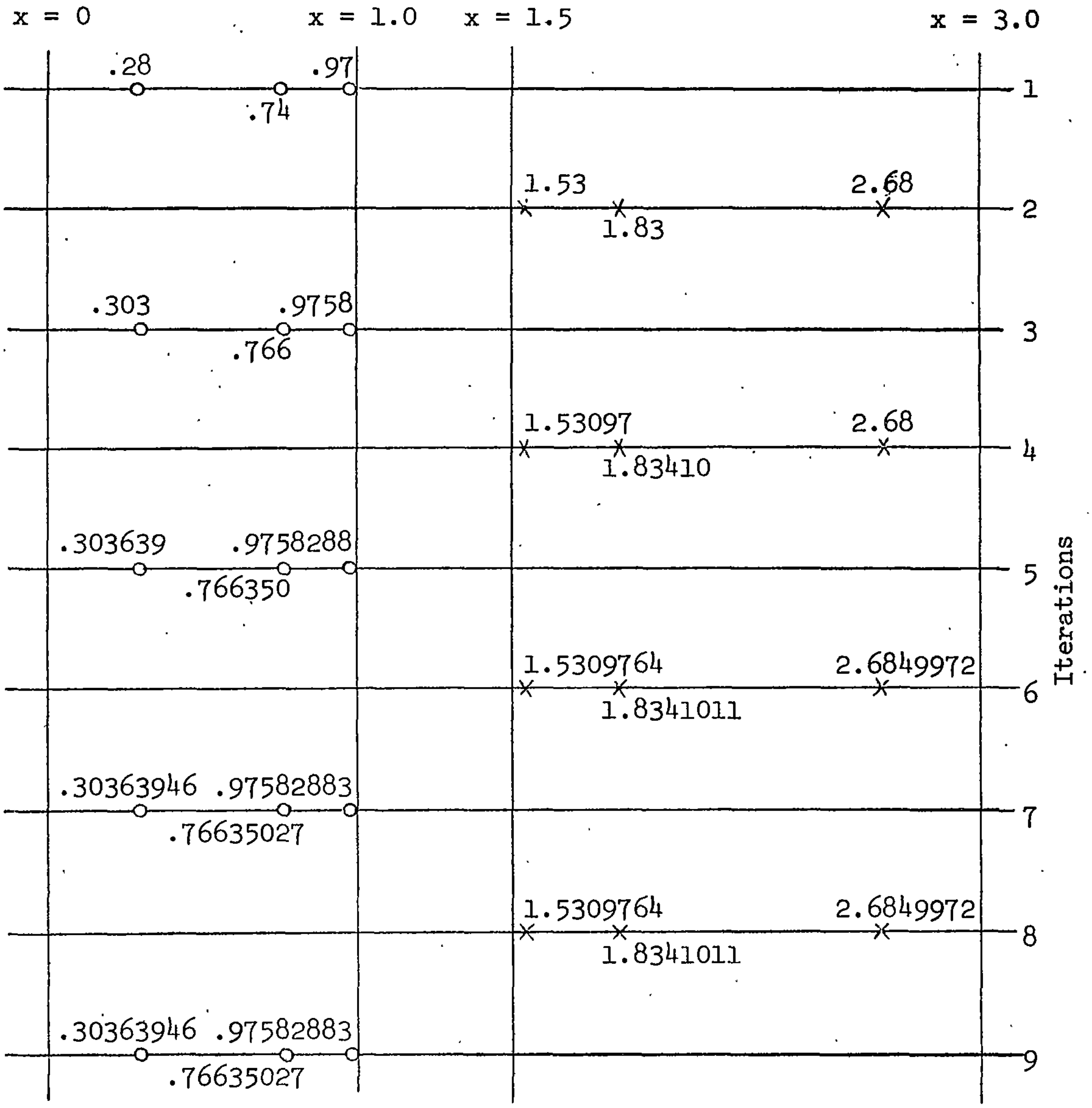


Figure 4-14. Zero and pole migration for equiripple passband and stopband function of degree 6. Band edges as shown. Initial pole locations at 1.875, 2.250 and 2.625.

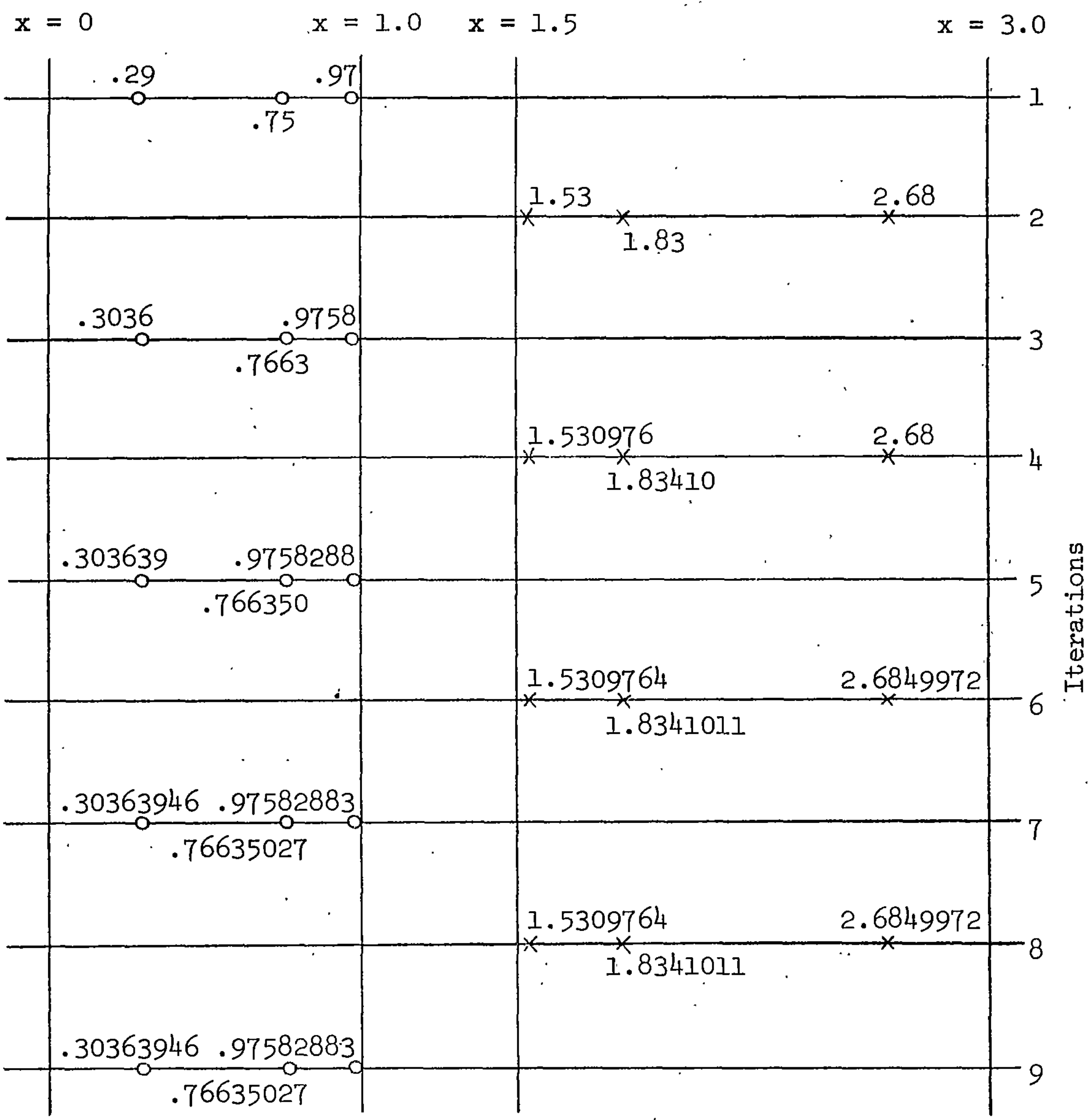


Figure 4-15. Zero and pole migration for equiripple passband and stopband function of degree 6. Band edges as shown. One initial pole location at 1.5; two initial pole locations at 3.0. Note no decrease in iterations.

All highband poles are initially chosen with about even distribution in highband, then placed soley at band edges.

4.6 EQUIRIPPLE FUNCTIONS WITH PREASSIGNED EXTRAPASSBAND ZEROS AND ARBITRARY POLES

The rotating vector principle as presented in Chapters 2 and 3 reaches a limitation when applied to the problem of functions with preassigned extrapassband zeros and arbitrary poles. Section 4.6.1 shows that factor by factor construction of $F(p)$ is not generally possible. Section 4.6.2 presents a complete, although not entirely satisfying, solution to the problem.

4.6.1 Problems Encountered in the Rotating Vector Principle

In order to clearly understand the nature of problems encountered when forming equiripple functions with prescribed extrapassband zeros, it is well to look at functions $F(p)$, $G(p)$ and $H(p)$ from another viewpoint. Heretofore attention has been paid only to the location of poles. Maxima and minima locations within the passband were not of express concern. The rotating vector principle implicitly set the passband zeros and accordingly the points at which $H(j\omega) = 1$. We will now indicate the mechanism by which these passband points are set.

Assume that $F(p)$ for some bandpass function of overall degree $2n$ has been constructed and $G(p)$ is now formed. We see that

$$G(p) = \frac{A(p)}{A(p) - B(p)\sqrt{p^2 + \omega_L^2}\sqrt{p^2 + \omega_H^2}} \tag{4.23}$$

where $A(p)$ is an even polynomial of degree n and $B(p)$ is an even polynomial of degree $n - 2$. From this

$$H(p) = \frac{A^2(p)}{A^2(p) - B^2(p)(p^2 + \omega_L^2)(p^2 + \omega_H^2)} \tag{4.24}$$

We now inquire into how the expression of $H(p)$ in this form displays explicitly the zeros and maxima in the passband.

The answers to such inquire are immediate. Zeros exist at the zeros of $A^2(p)$. All such zeros are in the passband and are double; therefore $H(p)$ is positive on both sides of the location. It is also evident that $H(p) = 1$ whenever $B^2(p)\ (p^2 + \omega_L^2)(p^2 + \omega_H^2) = 0$. All zeros of $B^2(p)$ are therefore in the passband, are double and interlace those of $A^2(p)$. Since they are double, $H(p)$ is less than unity on either side of them. The factors $(p^2 + \omega_L^2)$ and $(p^2 + \omega_H^2)$ provide single order zeros in the expression above. Hence $H(p)$ continues through such points without again turning downward.

In more complete terms one may write

$$H(p) = \frac{K^2 \prod_{i=1}^{n} (p^2 + z_i^2)^2}{K^2 \prod_{i=1}^{n} (p^2 + z_i^2)^2 - L^2 \left[\prod_{i=1}^{n-1} (p_i^2 + \omega_i^2)^2\right] (p^2 + \omega_L^2)(p^2 + \omega_H^2)} \tag{4.25}$$

The numerator and its identical denominator term indicate points at which $H(p) = 0$. The zeros are double. The right hand term of the denominator, viz., $-L^2 \left[\prod_{i=1}^{n-1} (p^2 + \omega_i^2)^2\right] (p^2 + \omega_L^2)(p^2 + \omega_H^2)$, indicates the points at which $H(p) = 1$. There are $n - 1$ double order internal points and the two band edges, both of which are single order points. Figure 4-16 shows the significance of these remarks. The rotating

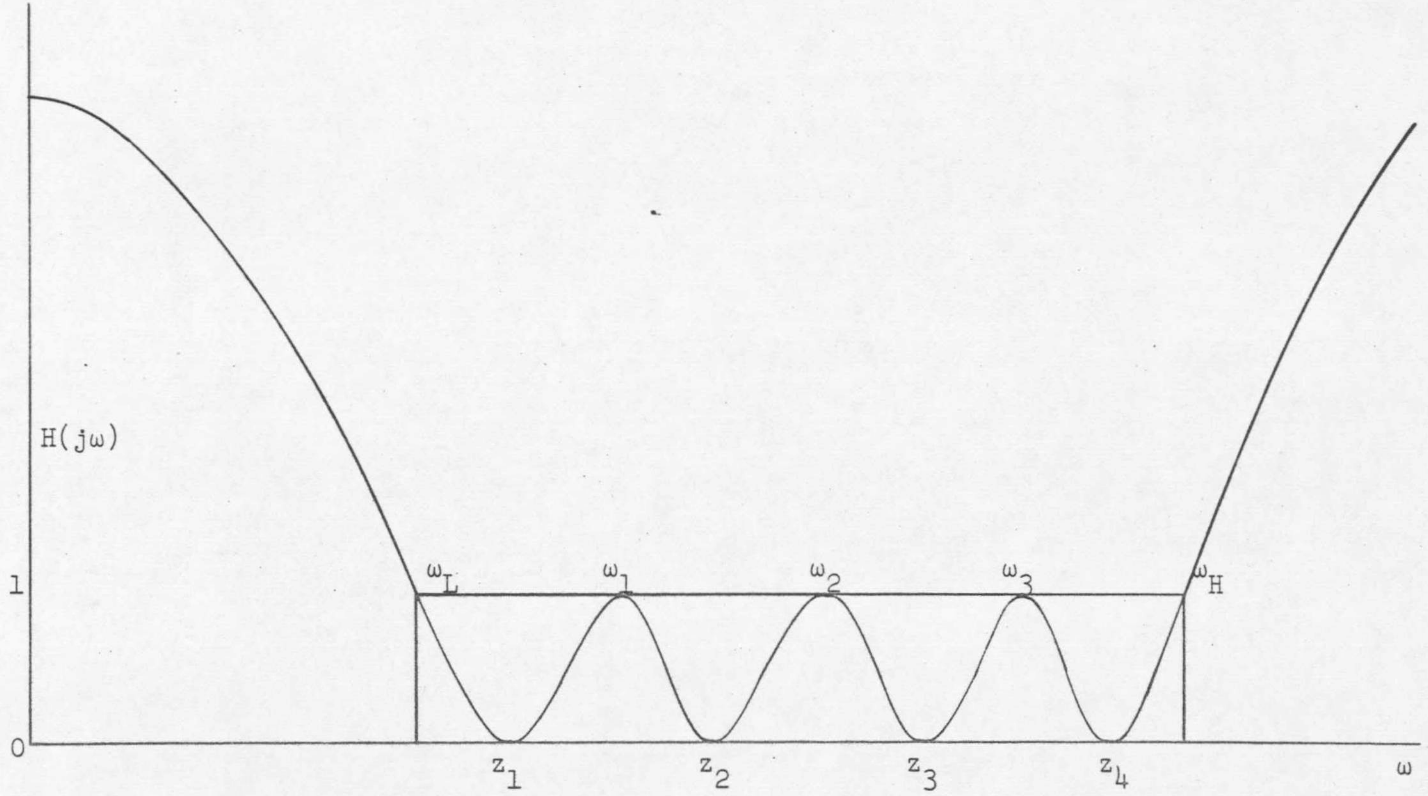


Figure 4-16. Passband of four ripple H(p).

vector principle used so far has not required explicit computation of either the z_i's or the ω_i's. They were generated automatically in the procedure with only a knowledge of desired pole locations and band edges.

Equation 4.25 for H(p) as a magnitude squared function is the same as that arrived at in another manner by Humphereys (9), showing that the procedures have a common point.

Now consider the requirement of a three ripple bandpass function with a double order zero outside the band at $\omega = z_{ext_1}$. All specified poles are assumed at large values of p to keep them from complicating the picture. Figure 4-17 shows the specified function. From Figure 4-17, the required H(p) is

$$H(p) = \frac{K^2\left[\prod_{i=1}^{3} (p^2 + z_i^2)^2\right](p^2 + z_{ext_1}^2)^2}{\{K^2\left[\prod_{i=1}^{3} (p^2 + z_i^2)^2\right](p^2 + z_{ext_1}^2)^2 - L^2\left[\prod_{i=1}^{2} (p^2 + \omega_i^2)^2\right](p^2 + \omega_L^2)(p^2 + \omega_H^2)(p^2 + \omega_{ext_1}^2)(p^2 + \omega_{ext_2}^2)\}} \tag{4.26}$$

For this to be formed in the usual fashion it would be necessary to have

$$G(p) = \frac{K\left[\prod_{i=1}^{3} (p^2 + z_i^2)\right](p^2 + z_{ext_1}^2)}{\{K\left[\prod_{i=1}^{3} (p^2 + z_i^2)\right](p^2 + z_{ext_1}^2) - L\left[\prod_{i=1}^{2} (p^2 + \omega_i^2)\right]\sqrt{p^2 + \omega_L^2}\,\sqrt{p^2 + \omega_H^2}\,\sqrt{p^2 + \omega_{ext_1}^2}\,\sqrt{p^2 + \omega_{ext_2}^2}\}} \tag{4.27}$$

The numerator is a polynomial, which has always been the case with G(p).

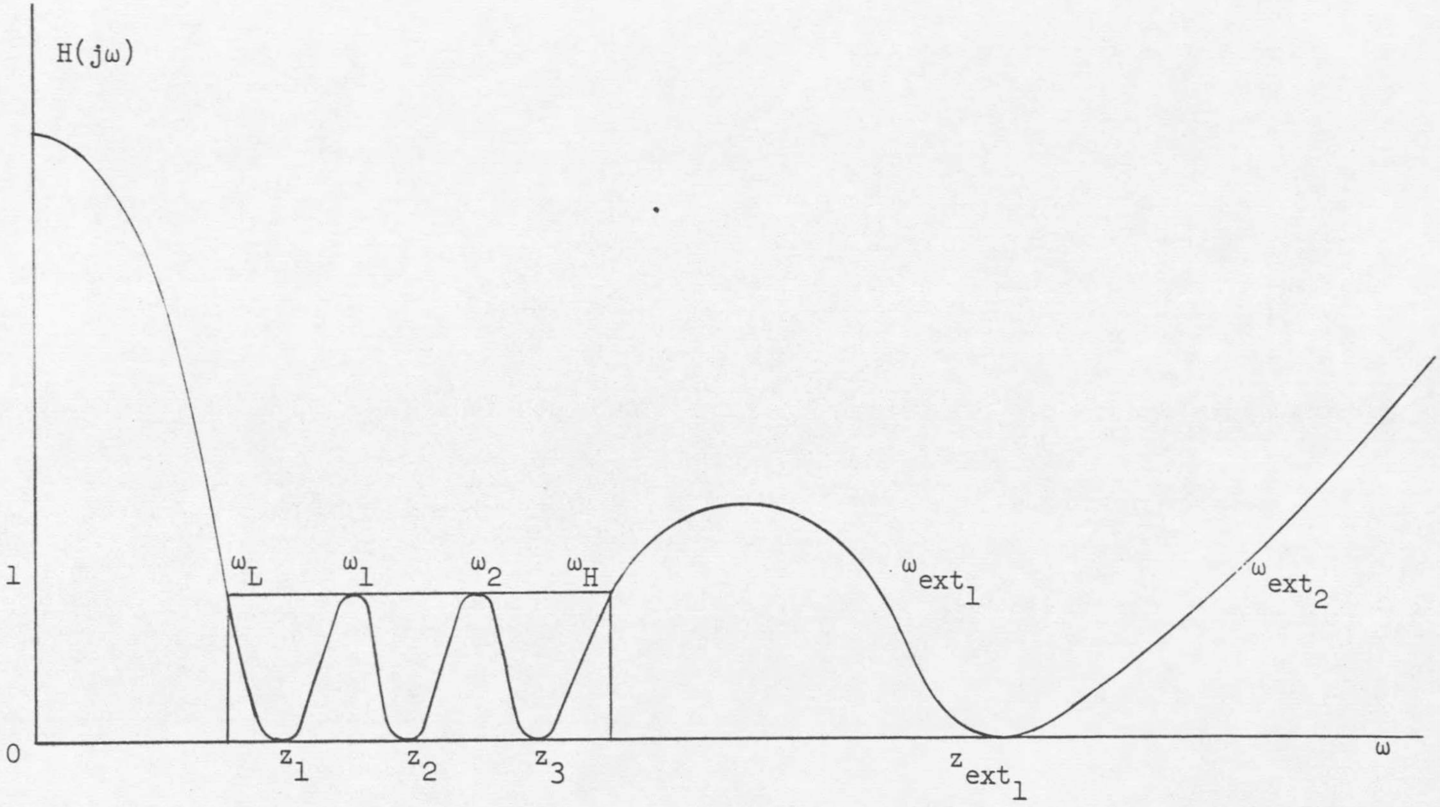


Figure 4-17. H(p) with extrapassband zero at z_{ext_1}.

The right portion of G(p) denominator is not of the usual form. In contrast to the usual irrational part being only band edge information, it has radicals at two other points, $\omega = \omega_{ext_1}$, ω_{ext_2}. The associated F(p) is a rotating vector not only in the passband but also in the interval $\omega_{ext_1} \leq \omega \leq \omega_{ext_2}$.

More complex cases are shown in Figures 4-18, 4-19, and 4-20. In each, a three ripple bandpass function is shown, having double order zeros at $\omega = z_{ext_1}$, z_{ext_2}, the poles being at specified locations outside the portion shown. The numerator and left hand denominator expression of G(p) will again be polynomials, representing the locations of the double order zeros in H(p). The right hand denominator expression must have zeros at $\omega = \omega_1$, ω_2 and radicals with zeros at $\omega = \omega_L$, ω_H. In addition it must have radicals with zeros at $\omega = \omega_{ext_i}$. Figure 4-18 has simple radicals at $\omega = \omega_{ext_1}$, ω_{ext_2}, ω_{ext_3}, ω_{ext_4}; Figure 4-19 has simple radicals at $\omega = \omega_{ext_1}$, ω_{ext_3} and a polynomial factor at $\omega = \omega_{ext_2}$; Figure 4-20 has simple radicals with zeros at $\omega = \omega_{ext_1}$, ω_{ext_2} and a factor of the form $\sqrt{a_4 p^4 + a_2 p^2 + a_0}$ having zeros located in quadrantal symmetry with imaginary part somewhere between z_{ext_1} and z_{ext_2}.

In general, H(p) has the appearance

$$H(p) = \frac{A^2(p)C^2(p)}{A^2(p)C^2(p) - B^2(p)D(p)(p^2 + \omega_L^2)(p^2 + \omega_H^2)} \qquad (4.28)$$

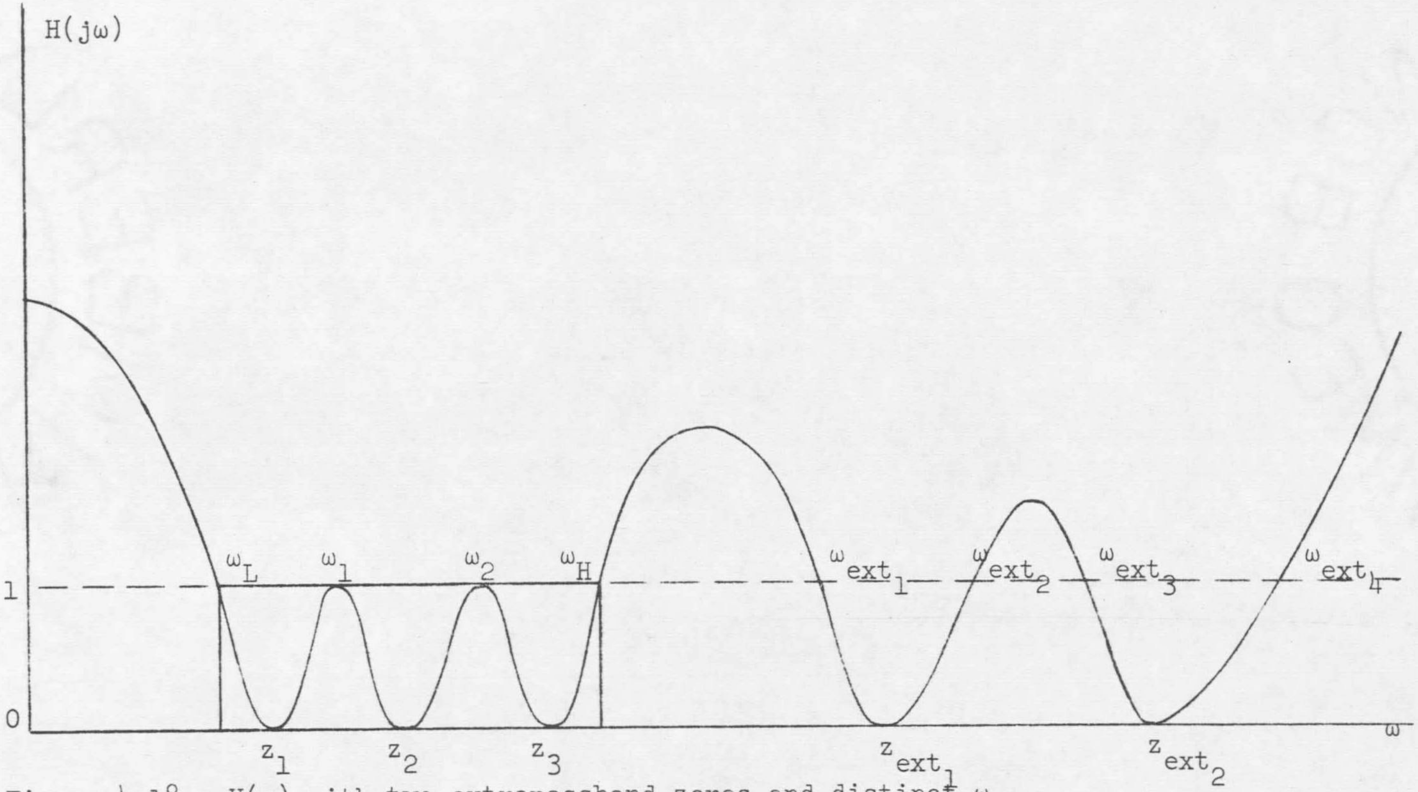


Figure 4-18. $H(p)$ with two extrapassband zeros and distinct ω_{ext_i}.

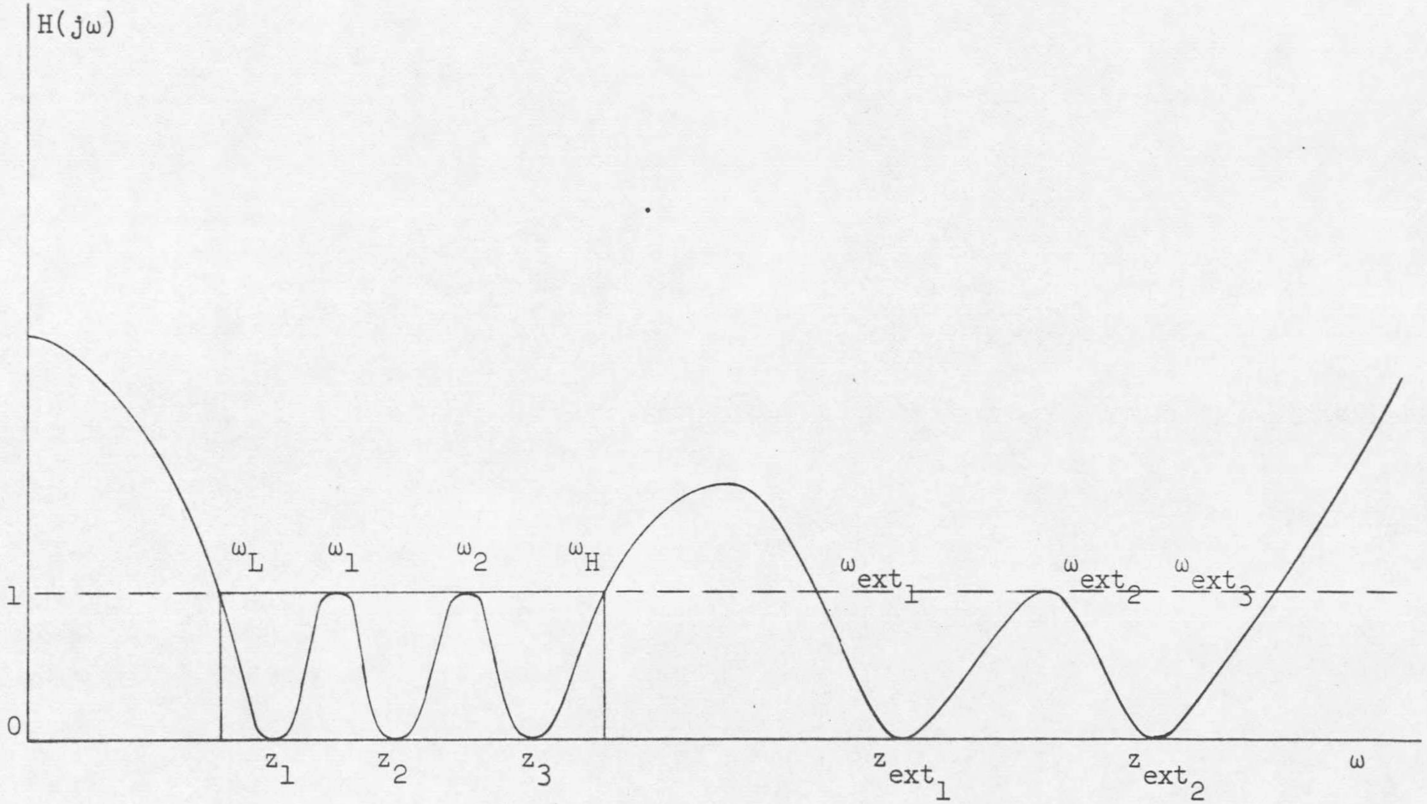


Figure 4-19. $H(p)$ with two extrapassband zeros and double order unity value location at ω_{ext_2}

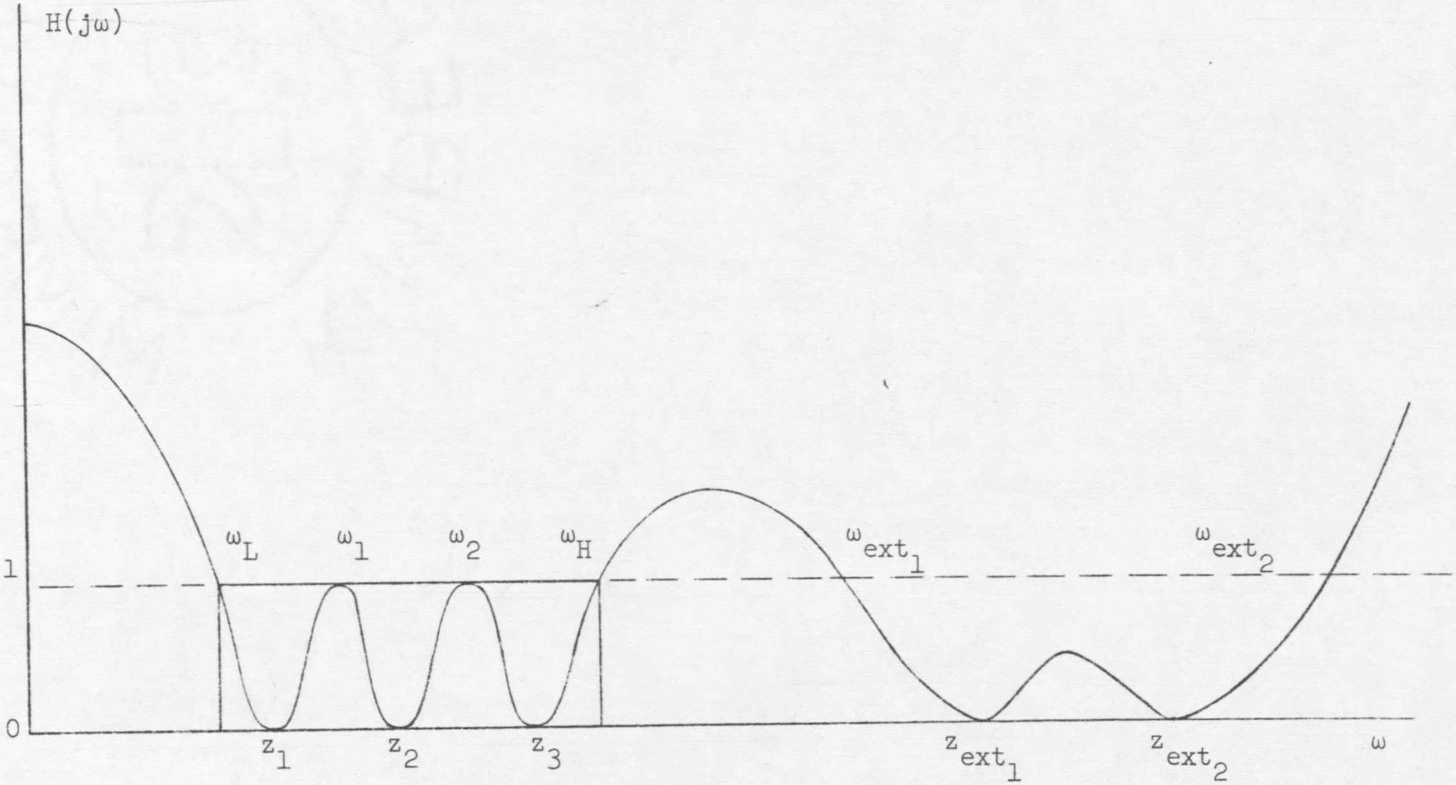


Figure 4-20. $H(p)$ with extrapassband zeros and unity value locations displaced off $j\omega$ axis.

$$A(p) = K \prod_{i=1}^{n} (p^2 + z_i^2) \tag{4.29}$$

indicates the passband zero locations,

$$B(p) = L \prod_{i=1}^{n-1} (p^2 + \omega_i)^2 \tag{4.30}$$

indicates the passband unity locations,

$$C(p) = \prod_{i=1}^{q} (p^2 + z^2_{ext_i}) \tag{4.31}$$

indicates extrapassband zeros and D(p) is a polynomial indicating extrapassband points of unity value, which may or may not be on the j-axis. From this

$$G(p) = \frac{A(p)C(p)}{A(p)C(p) - B(p)\sqrt{D(p)}\sqrt{p^2 + \omega_L^2}\sqrt{p^2 + \omega_H^2}} \tag{4.32}$$

and

$$F(p) = \frac{A(p)C(p) + B(p)\sqrt{D(p)}\sqrt{p^2 + \omega_L^2}\sqrt{p^2 + \omega_H^2}}{A(p)C(p) - B(p)\sqrt{D(p)}\sqrt{p^2 + \omega_L^2}\sqrt{p^2 + \omega_H^2}} \tag{4.33}$$

It is at this point that difficulties appear. It is shown in Appendix A that one cannot construct any general set of functions.

$$F_i(p) = \frac{C_i(p)\sqrt{p^2+\omega_L^2} + D_i(p)\sqrt{p^2+\omega_H^2}}{-C_i(p)\sqrt{p^2+\omega_L^2} + D_i(p)\sqrt{p^2+\omega_H^2}} \tag{4.34}$$

where $C_i(p)$ and $D_i(p)$ are expressions that are polynomials, radicals, or products of polynomials and radicals, such that their continued product will form the desired F(p), unless D(p) is known at the outset. Problems will appear in failure of radicals to disappear in cross products. Any resulting expressions will be cluttered with radicals which will not cancel. For a function with prescribed extrapassband zeros and prescribed denominator, the rotating vector principle will not yield an H(p) with rational numerator.

In essence, the problem arises from being unable to determine the set of ω_{ext_i} initially or to put it in other terms, from inability to determine over exactly what intervals outside the passband F(p) is to again become a rotating vector and through what angle it should rotate.

There remains another general class of functions for investigation. If it is not generally possible to determine an F(p) that is a rotating vector for all intervals in which $H(j\omega) \leqslant 1$, there are still those F(p) that are rotating vectors in the passband and strictly real outside the passband. It is shown in Appendix B that such functions will introduce undesired poles into H(p) and are not useful.

Such results do not mean that the rotating vector approach cannot produce H(p) with prescribed extrapassband zeros. What is meant is that the approach will not allow extrapassband zero placement with an arbitrarily prespecified denominator. One must accept the denominator that appears.

To illustrate this, let

$$F(p) = \frac{A(p) + B(p)\sqrt{p^2 + \omega_L^2}\sqrt{p^2 + \omega_H^2}}{A(p) - B(p)\sqrt{p^2 + \omega_L^2}\sqrt{p^2 + \omega_H^2}} \tag{4.35}$$

be formed in the usual way for certain prespecified pole locations. It is the usual bandpass $F(p)$. Let $C(p)$ be a polynomial representing the desired zero locations, the location of each eventual double order zero appearing once. Also require that $C(p)$ be real for all ω.

$$C(p) = \begin{cases} \prod\limits_{i=1} (p^2 + z_{ext_i}^2) & q \text{ even} \\ p/j \prod\limits_{i=1}^{q-1} (p^2 + z_{ext_i}) & q \text{ odd} \end{cases} \tag{4.36}$$

Insert $C(p)$ into $F(p)$ to obtain a new $F(p)$.

$$F(p) = \frac{A(p)C(p) + B(p)\sqrt{p^2 + \omega_L^2}\sqrt{p^2 + \omega_H^2}}{A(p)C(p) - B(p)\sqrt{p^2 + \omega_L^2}\sqrt{p^2 + \omega_H^2}} \tag{4.37}$$

The new $F(p)$ is still a rotating vector in the passband and will lead to

$$H(p) = \frac{A^2(p)C^2(p)}{A^2(p)C^2(p) - B^2(p)(p^2 + \omega_L^2)(p^2 + \omega_H^2)} \tag{4.38}$$

which exhibits the zeros of $C(p)$ in double order. The denominator is no longer that one originally specified.

4.6.2 A Solution to the Extrapassband Zero Problem

Although formation of an H(p) in the usual sense of factor by factor construction of F(p), formation of G(p) and finally H(p), has been shown in general not possible, the general form of H(p) offers a solution. The procedure is one of beginning with this general form and determining points where H(p) = 0 and H(p) = 1. It leads to a set of nonlinear simultaneous equations whose solutions are found by an iterative Newton Raphson process (9, 10).

We have as a desired H(p), Equation 4.28

$$H(p) = \frac{A^2(p)C^2(p)}{A^2(p)C^2(p) - B^2(p)D(p)(p^2 + \omega_L^2)(p^2 + \omega_H^2)} \tag{4.28}$$

where

$$A(p) \triangleq K \prod_{i=1}^{n} (p^2 + z_i^2) \tag{4.29}$$

$$B(p) \triangleq L \prod_{i=1}^{n-1} (p^2 + \omega_i^2) \tag{4.30}$$

$$C(p) \triangleq \prod_{k=1}^{q} (p^2 + z_{ext_k}^2) \tag{4.31}$$

and D(p) is an unknown even monic polynomial of degree 4q. Since D(p) is even, monic and of degree 4q, it has 2q coefficients that are unknown. K and L are unknown constants.

Known information about H(p) is:

1. The degree of all polynomials. Deg $(A(p)) = 2n$. Deg $(B(p)) = 2(n - 1)$. Deg $(C(p)) = 2q$. Deg $(D(p)) = 4q$.

2. Band edges, ω_L and ω_H.

3. The $2q$ extrapassband zero locations, $\pm z_{ext_k}$.

4. The overall denominator to within a constant multiplier. The zeros of the denominator are known.

The problem is to find all z_i. The unknowns are:

1. Passband zero locations, z_i. There are n of these.

2. Passband unity locations, ω_i. There are $n - 1$ of these.

3. Coefficients of $D(p)$. There are $2q$ of these.

4. Constants K and L.

A solution procedure is:

1. Expand $A^2(p)C^2(p) - B^2(p)D(p)(p^2 + \omega_L^2)(p^2 + \omega_H^2)^2$ into a polynomial. This polynomial will have $2(n = q) + 1$ unknown coefficients represented in terms of $2(n + q) + 1$

unknowns. The unknowns are: (1) n z_i locations, (2) n - 1 ω_i locations, (3) 2q coefficients of D(p) and (4) constants K and L.

2. From the desired pole locations, form the denominator. It need not be a monic polynomial but is most likely to be so by the process of factor multiplication. The denominator is even and of degree 4(n + q), having 2(n + q) + 1 coefficients.

3. Equate the 2(n + q) + 1 unknown coefficients of H(p) denominator to those of the desired denominator. A set of 2(n + q) + 1 nonlinear simultaneous equations in as many unknowns results.

4. Solve the set of 2(n + 1) + 1 nonlinear simultaneous equations. Out of this set of solutions one is interested in n z_i locations and the constant K.

An equation solving procedure is:

1. Let the equations be represented as $F_i = 0$; i = 1, 2, ..., 2(n + q) + 1.

2. Make an initial guess for the solution values, called x_i;

$i = 1, 2, \cdots, 2(n + 1) + 1.$

3. Numerically obtain the partial derivatives, $\partial F_j/\partial x_i$, as

$$\frac{\partial F_j}{\partial x_i} = \frac{F_j(x_i + \delta x_i) - F_j(x_i)}{\delta x_i} \tag{4.39}$$

where δ is a small constant.

4. Solve the resulting set of linear simultaneous equations (Newton-Raphson)

$$\sum_{i=1}^{2(n+q)+1} \frac{\partial F_j}{\partial x_i} \Delta x_i = -F_j; \; j = 1, \cdots, 2(n + q) + 1 \tag{4.40}$$

for the unknowns Δx_i.

5. Replace the initial values of x_i by $x_i + \Delta x_i$.

6. Repeat this process until the Δx_i are as small as desired.

A simple Fortran program was written, verifying this solution procedure. The method was not extensively pursued for the following reasons:

1. An initial guess quite close to the final solution is required for convergence.

2. It is inefficient. We are interested in n + 1 variables, yet must solve a set of 2(n + q) + 1 simultaneous nonlinear equations to arrive at them.

3. McGee (13) has presented a method for such operations that appears to be more efficient. It is also guaranteed to converge.

4.7 A CLOSED FORM SOLUTION FOR A CLASS OF MULTIPLE PASSBAND FILTERS

An extension of the rotating vector principle allows closed form construction of a class of multiple passband filter functions. Any number of passbands can be constructed immediately. Watanabe (18) produced functions with two equiripple passbands. McGee (13) required a doubly iterative procedure for solution, although his solution is general.

The class possesses the following characteristics:

1. All band are prespecified. They are $\omega_1 < \omega_2 < \cdots < \omega_{2k}$ where k is the number of passbands. See Figure 4-21.

2. All passbands have ripple between 0 and 1.

3. All passbands have the same number of zeros. Let this number be n so that $\frac{n}{2}$ is the total ripples per band.

4. Out of the total of 2nk pole locations; only n of them are independently specifiable.

Operations are based on construction of an F(p) that becomes a rotating vector for $\omega_\ell \leq \omega \leq \omega_{\ell+1}$, ℓ odd, and is a real function for other ω. From what has gone before it is reasonable to expect F(p) to be composed of products and sums of $\sqrt{p^2 + \omega_\ell^2}$ where ω_ℓ indicates a band edge.

Numerous F(p) can be constructed to provide the rotating vector feature. One suitable F(p) is

$$F(p) = \prod_{i=1}^{n} \frac{m_i\sqrt{p^2+\omega_1^2}\sqrt{p^2+\omega_3^2}\sqrt{p^2+\omega_5^2}\cdots + \sqrt{p^2+\omega_2^2}\sqrt{p^2+\omega_4^2}\sqrt{p^2+\omega_6^2}\cdots}{-m_i\sqrt{p^2+\omega_1^2}\sqrt{p^2+\omega_3^2}\sqrt{p^2+\omega_5^2}\cdots + \sqrt{p^2+\omega_2^2}\sqrt{p^2+\omega_4^2}\sqrt{p^2+\omega_6^2}\cdots} \tag{4.41}$$

Another F(p) is

$$F(p) = \prod_{i=1}^{n} \frac{m_i\sqrt{p^2+\omega_1^2}\sqrt{p^2+\omega_4^2}\sqrt{p^2+\omega_5^2}\cdots + \sqrt{p^2+\omega_2^2}\sqrt{p^2+\omega_3^2}\sqrt{p^2+\omega_6^2}\cdots}{-m_i\sqrt{p^2+\omega_1^2}\sqrt{p^2+\omega_4^2}\sqrt{p^2+\omega_5^2}\cdots + \sqrt{p^2+\omega_2^2}\sqrt{p^2+\omega_3^2}\sqrt{p^2+\omega_6^2}\cdots} \tag{4.42}$$

For $\omega_\ell \leq \omega \leq \omega_{\ell+1}$, ℓ odd, 4.41 and 4.42 are rotating vectors. In all other intervals they are real.

The difference between them comes in the band edge value of the individual factors. Write F(p) as

$$F(p) = \prod_{i=1}^{n} F_i(p) \tag{4.43}$$

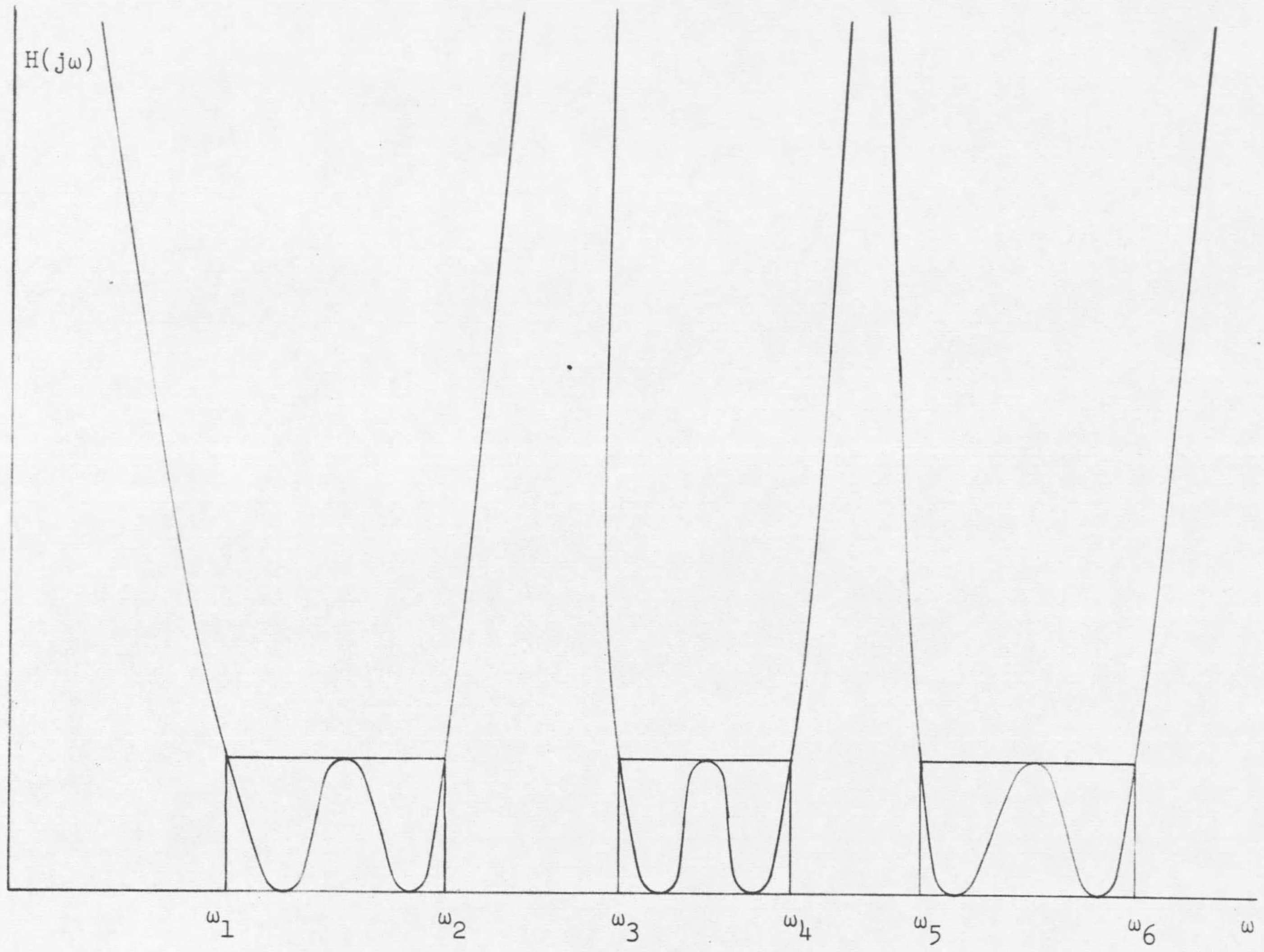


Figure 4-21. General multiple passband appearance.

where

$$F_i(p) = \frac{m_i\sqrt{p^2+\omega_1^2}\sqrt{p^2+\omega_3^2}\sqrt{p^2+\omega_5^2}\cdots + \sqrt{p^2+\omega_2^2}\sqrt{p^2+\omega_4^2}\sqrt{p^2+\omega_6^2}\cdots}{-m_i\sqrt{p^2+\omega_1^2}\sqrt{p^2+\omega_3^2}\sqrt{p^2+\omega_5^2}\cdots + \sqrt{p^2+\omega_2^2}\sqrt{p^2+\omega_4^2}\sqrt{p^2+\omega_6^2}\cdots} \tag{4.44}$$

for 4.41 and a corresponding expression for 4.42. In 4.44 $F_i(j\omega_\ell) = 1$ and $F_i(j\omega_{\ell+1}) = -1$, ℓ odd. $F_i(j\omega)$ alternates in sign for successive ω_ℓ. In the expression that corresponds to 4.42 $F_i(j\omega_1) = 1$, $F_i(j\omega_2) = F_i(j\omega_3) = -1$, $F_i(j\omega_4) = F_i(j\omega_5) = 1$, etc. $F_i(j\omega_\ell)$ alternates in sign for successive pairs of ω_ℓ, having the same sign at the end of one band as at the beginning of the next.

Bandpass filter functions always have n even. Both 4.41 and 4.42 give $F(j\omega_\ell) = 1$ for all ℓ and are suitable. For the special case of a filter function that is lowpass from $\omega_1 = 0$ to ω_2 and highpass from ω_3 to infinity, 4.41 becomes

$$F(p) = \prod_{i=1}^{n} \frac{m_i p\sqrt{p^2+\omega_3^2} + \sqrt{p^2+\omega_2^2}}{-m_i p\sqrt{p^2+\omega_3^2} + \sqrt{p^2+\omega_2^2}} \tag{4.45}$$

and 4.42 becomes

$$F(p) = \prod_{i=1}^{n} \frac{m_i p + \sqrt{p^2+\omega_2^2}\sqrt{p^2+\omega_3^2}}{-m_i p + \sqrt{p^2+\omega_2^2}\sqrt{p^2+\omega_3^2}} \tag{4.46}$$

For n odd, 4.45 gives $F(j\omega_2) = 1$ and $F(j\omega_3) = -1$. This would lead to $G(j\omega_2) = 0$ and $G(j\omega_3) = 1$. $G(j\omega)$ must be unity at both points. Proper values are provided by 4.46. For this reason only, 4.42 will be chosen for more examination. The discussion about pole sets will pertain equally to both 4.41 and 4.42 with obvious modifications.

Poles are placed in H(p) by setting

$$m_i = \sqrt{\frac{(p_i^2 + \omega_1^2)(p_i^2 + \omega_4^2)(p_i^2 + \omega_5^2)\cdots}{(p_i^2 + \omega_2^2)(p_i^2 + \omega_3^2)(p_i^2 + \omega_6^2)\cdots}} . \tag{4.47}$$

As remarked, each m_i value has associated with it k values of ω satisfying 4.47. Appendix C has a more detailed discussion of the interdependence of poles.

G(p) is formed in the usual way,

$$G(p) = \frac{F(p) + (-1)^n}{2} \tag{4.48}$$

as are $\overline{G}(p)$ and H(p). The author has written a plotting program using 4.42. Graphical results are shown in Figures 4-22 and 4-23.

4.8 FUNCTIONS GENERATED BY FORMING G(p) IN AN ALTERNATE WAY

Using an invariant F(p), it is interesting to construct G(p) in ways other than $G(p) = \frac{F(p) + (-1)^n}{2}$. Consider the addition of a constant, K, to F(p). Division of this by K + 1 will provide G(p) with value ±1 at band edge.

$$G(p) = \frac{F(p) + K(-1)^n}{K + 1} \tag{4.49}$$

In Figure 4-24 are shown G(p) in the passband for both K < 1 and K > 1.

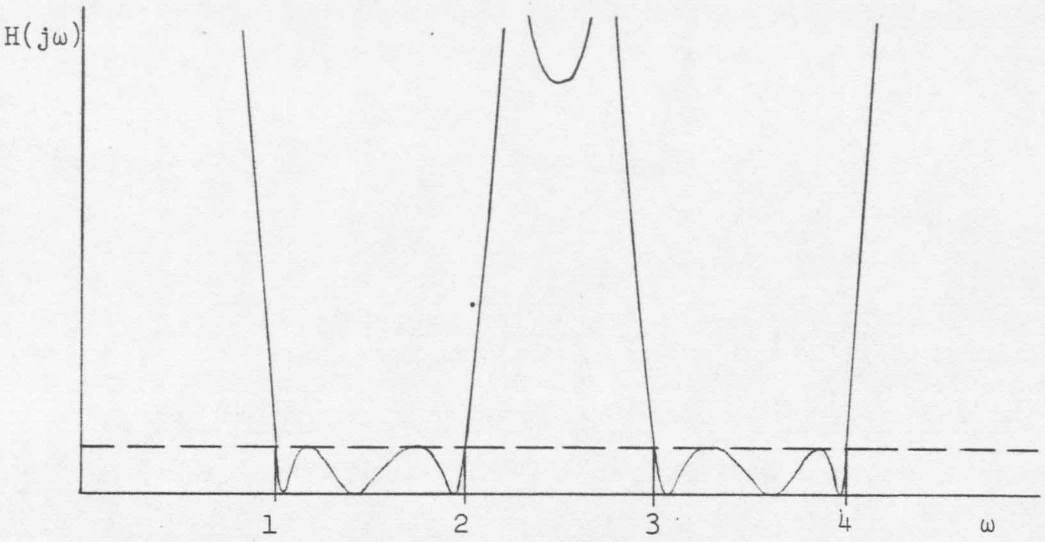


Figure 4-22. Two passbands, three ripples per band. Poles placed at $\omega = 0, 2.25, 4.25$.

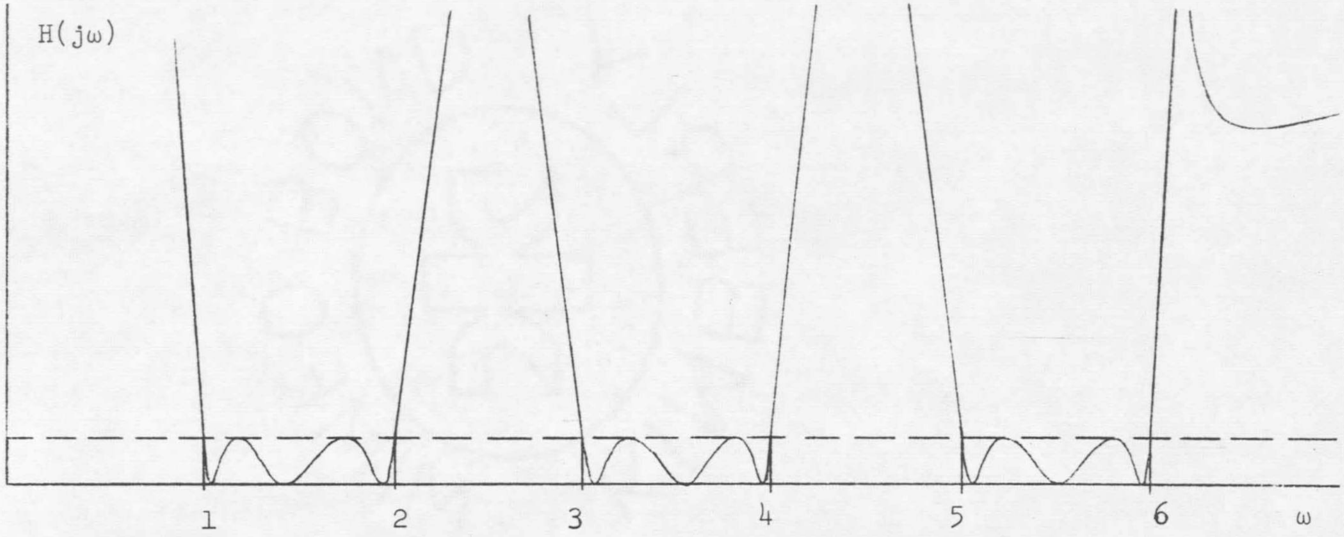


Figure 4-23. Three passbands, three ripples per band. Poles placed at $\omega = 0, 2.5, 4.5, \infty$.

The magnitude squared in the passband is given by

$$H(j\omega) = \frac{K^2 + 2K\cos\phi + 1}{K^2 + 2K + 1} \tag{4.50}$$

with maxima of unity and minima of $\frac{(K - 1)^2}{(K + 1)^2}$. For any K, it is seen that an identical function results if $\frac{1}{K}$ is substituted. Hence it is sufficient to consider K > 1 only.

Representing a bandpass F(p) as

$$F(p) = \frac{A(p) + B(p)\sqrt{p^2 + \omega_L^2}\sqrt{p^2 + \omega_H^2}}{A(p) - B(p)\sqrt{p^2 + \omega_L^2}\sqrt{p^2 + \omega_H^2}} \tag{4.51}$$

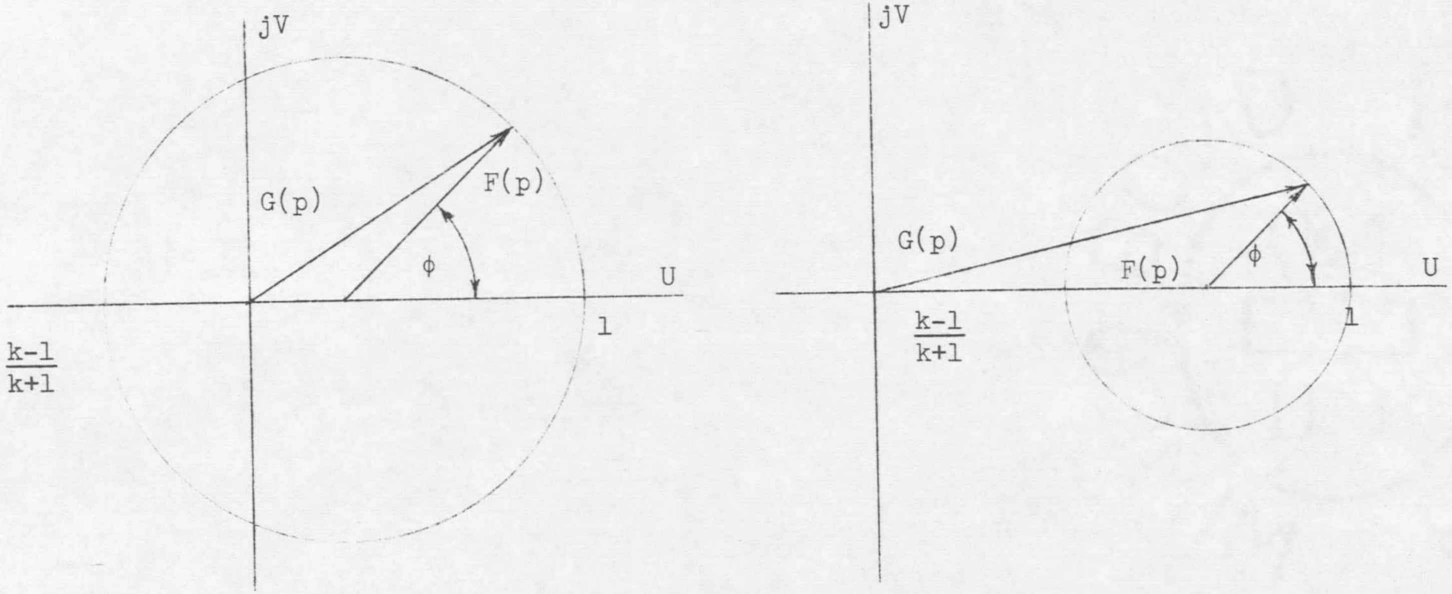
where A(p) and B(p) are even polynomials, we have

$$G(p) = \frac{(K + 1)A(p) - (K - 1)B(p)\sqrt{p^2 + \omega_L^2}\sqrt{p^2 + \omega_H^2}}{(K + 1)A(p) + (K + 1)B(p)\sqrt{p^2 + \omega_L^2}\sqrt{p^2 + \omega_H^2}}. \tag{4.52}$$

No longer is the numerator strictly rational but there is a dichotomy of real and imaginary parts in the passband, allowing formation of H(p) as a magnitude squared function.

$$H(p) = \frac{(K + 1)^2A^2(p) - (K - 1)^2B^2(p)(p^2 + \omega_L^2)(p^2 + \omega_H^2)}{(K + 1)^2A^2(p) - (K + 1)^2B^2(p)(p^2 + \omega_L^2)(p^2 + \omega_H^2)} \tag{4.53}$$

Figure 4-25 shows a general H(p). It is recognized as the reciprocal of a bandpass magnitude squared transfer function with zeros at $\omega = 0$, $\pm\omega_1$, $\pm\omega_2$, $\pm\omega_3$, ∞; passband minima of unity, maxima of $\frac{(K + 1)^2}{(K - 1)^2}$. This is an alternate way of forming transfer functions.



(a) $k < 1$. (b) $k > 1$.

Figure 4-24. $G(p) = \dfrac{F(p) + k(-1)^n}{k + 1}$, n even

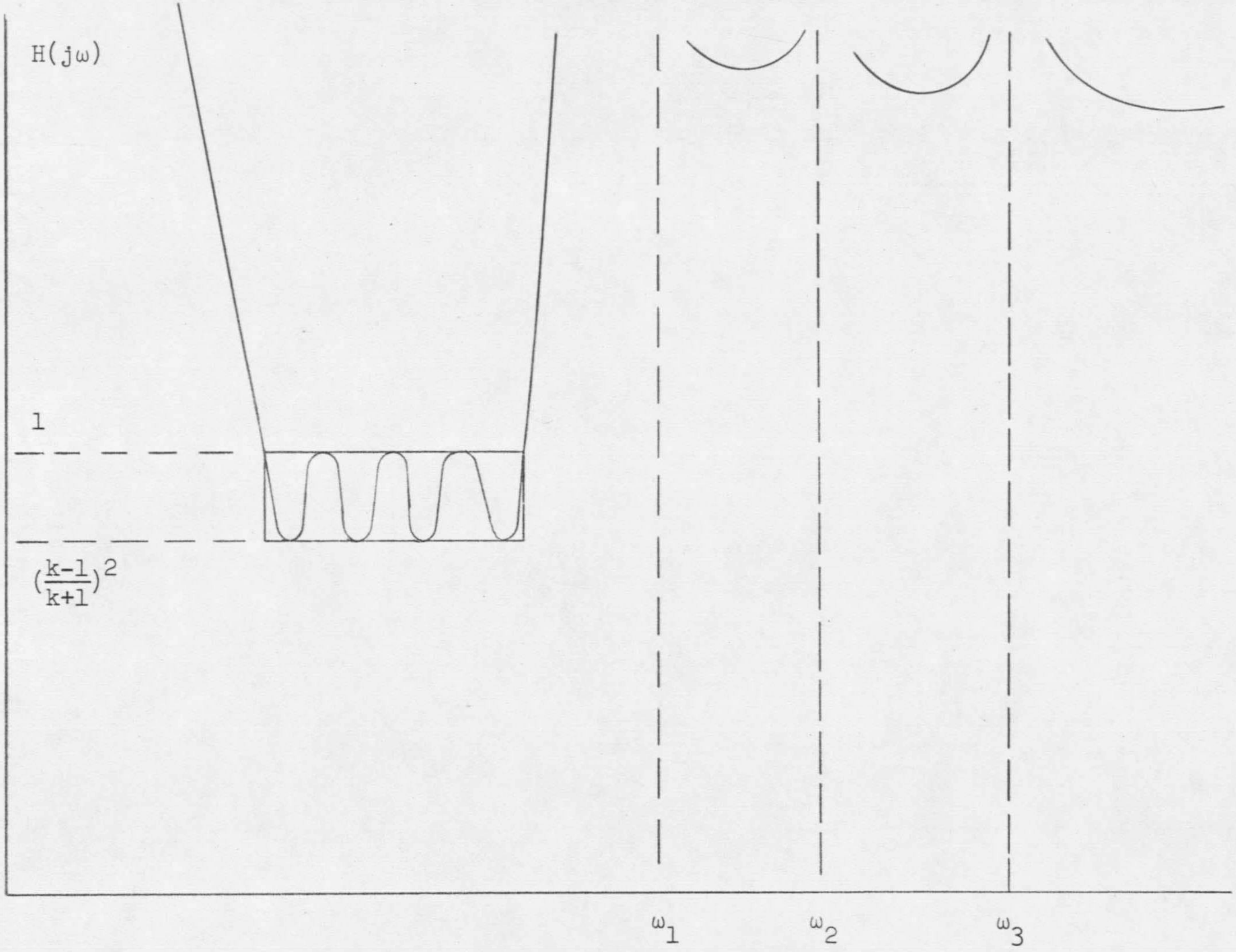


Figure 4-25. An inverted transfer function.

CHAPTER 5

SUMMARY AND RECOMMENDATIONS

5.1 SUMMARY OF RESULTS

It has been shown that casting Bennett's original transformation (3) in a different form leads to a straightforward method for generating equiripple magnitude squared functions in lowpass, bandpass and high-pass. The procedure is non-iterative and closed form. Equations for placing the arbitrary poles are first degree algebraic, involving only one unknown. There is no simpler form. As a result, hand computation of equiripple functions with arbitrary poles is very feasible. For this reason the method is recommended as either a more easily understood replacement for elliptic function generation or as a complement to it. The author has presented it to students of modern network synthesis for three years. He has found it to be readily accepted and understood by them. He also assumes it would be as appealing to the practicing engineer. Computer programming is uncomplicated due to an easily expressed recursion relation for coefficients.

Once the rotating vector principle is familiar, modified generating functions based on it can be readily constructed. General rational functions, both with and without symmetry about the origin, are generable. The simplicity of equations involved in pole placement is retained. By employing iterative procedures, equiripple passband and stopband functions can be constructed. Passband and stopband of any length, finite or infinite, can be accomodated. Extension of the rotating vector principle to more than one band gives a limited solution to the multiple equiripple passband problem in closed non-iterative form. Such functions present an identical number of zeros per band and interdependent pole locations.

Disadvantages of the method are also pointed out. There is no general means of application to problems requiring zeros outside the passband. All functions necessarily have the maximum possible number of ripples in the passband for a given degree. There also exists the possibility of non-convergence in the iterative procedure for equiripple passband and stopband functions. Such a problem was not encountered in any example.

5.2 RECOMMENDATIONS

Foremost in any recommendation for future effort is that of finding a closed form, non-iterative procedure for construction of functions with prescribed extrapassband zeros. Such a procedure should incorporate a simple means for locating desired poles. Continued application of such a function in an iterative manner would provide completely general equiripple passband and stopband functions with multiple bands and varying number of ripples per band. A procedure similar to that of McGee (13) would be followed. The significant difference is that McGee's procedure is iterative both in making a single passband equiripple and in making many passbands equiripple. The proposed procedure would be iterative only in the latter sense. Another endeavor is that of developing a rapid procedure for the construction of functions exhibiting equiripple delay. Lowpass delay functions were constructed by Humphereys (12), but to the author's knowledge, the bandpass delay problem has not received a general solution.

APPENDICES

APPENDIX A

A CONSTRUCTION PROBLEM ENCOUNTERED IN EXTRAPASSBAND ZERO REALIZATION

It was shown in Section 4.6.1 that a general form for magnitude squared functions that are equiripple over an interval $\omega_L \leq \omega \leq \omega_H$ is

$$H(p) = \frac{[K^2 \prod_{i=1}^{n} (p^2 + z_i^2)^2][\prod_{k=1}^{q} (p^2 + z_{ext_k}^2)^2]}{\{[K^2 \prod_{i=1}^{n} (p^2 + z_i^2)^2][\prod_{k=1}^{q} (p^2 + z_{ext_k}^2)^2] - [L^2 \prod_{i=1}^{n-1} (p^2 + \omega_i^2)^2](p^2 + \omega_L^2)(p^2 + \omega_H^2)D(p)\}} \tag{A.1}$$

where $\pm z_i$ are the undetermined passband zero locations; $\pm z_{ext_k}$ are the prescribed extrapassband zero locations, possibly complex conjugates in quadrantal symmetry; $\pm\omega_i$ are the undetermined passband locations of unity value for $H(p)$; ω_L and ω_H are the prescribed band edges, $D(p)$ is an unknown even polynomial of degree $4q$ whose zeros mark the extrapassband points at which $H(p) = 1$. $D(p)$ may have both simple and double roots on the $j\omega$ axis. It may also have roots in quadrantal symmetry displaced off the $j\omega$ axis. K and L are constants. For notational ease we define

$$A(p) \triangleq K \prod_{i=1}^{n} (p^2 + z_i^2) \tag{A.2}$$

$$B(p) \triangleq L \prod_{i=1}^{n-1} (p^2 + \omega_i^2) \tag{A.3}$$

$$C(p) \triangleq \prod_{k=1}^{q} (p^2 + z_{ext_k}^2). \tag{A.4}$$

Then

$$H(p) = \frac{A^2(p)C^2(p)}{A^2(p)C^2(p) - B^2(p)D(p)(p^2 + \omega_L^2)(p^2 + \omega_H^2)} \quad \text{(A.5)}$$

Formation of H(p) in the usual rotating vector process indicates that G(p) must be

$$G(p) = \frac{A(p)C(p)}{A(p)C(p) - B(p)\sqrt{D(p)}\sqrt{p^2 + \omega_L^2}\sqrt{p^2 + \omega_H^2}} \quad \text{(A.6)}$$

and F(p) must be

$$F(p) = \frac{A(p)C(p) + B(p)\sqrt{D(p)}\sqrt{p^2 + \omega_L^2}\sqrt{p^2 + \omega_H^2}}{A(p)C(p) - B(p)\sqrt{D(p)}\sqrt{p^2 + \omega_L^2}\sqrt{p^2 + \omega_H^2}} \quad \text{(A.7)}$$

It can be seen from Figure A-1 that between any two successive j axis zeros of D(p), F(p) is a unit vector in the complex plane. A second order zero is counted as two single zeros.

The problem is now to construct F(p) as a product of individual $F_i(p)$, using the same procedure as that used when extrapassband zeros were not present. In that instance

$$F_i(p) = \frac{m_i\sqrt{p^2 + \omega_L^2} + \sqrt{p^2 + \omega_H^2}}{-m_i\sqrt{p^2 + \omega_L^2} + \sqrt{p^2 + \omega_H^2}} \quad \text{(A.8)}$$

and

$$F(p) = \prod_{i=1}^{n} F_i(p) = \frac{A(p) + B(p)\sqrt{p^2 + \omega_L^2}\sqrt{p^2 + \omega_H^2}}{A(p) - B(p)\sqrt{p^2 + \omega_L^2}\sqrt{p^2 + \omega_H^2}} \quad \text{(A.9)}$$

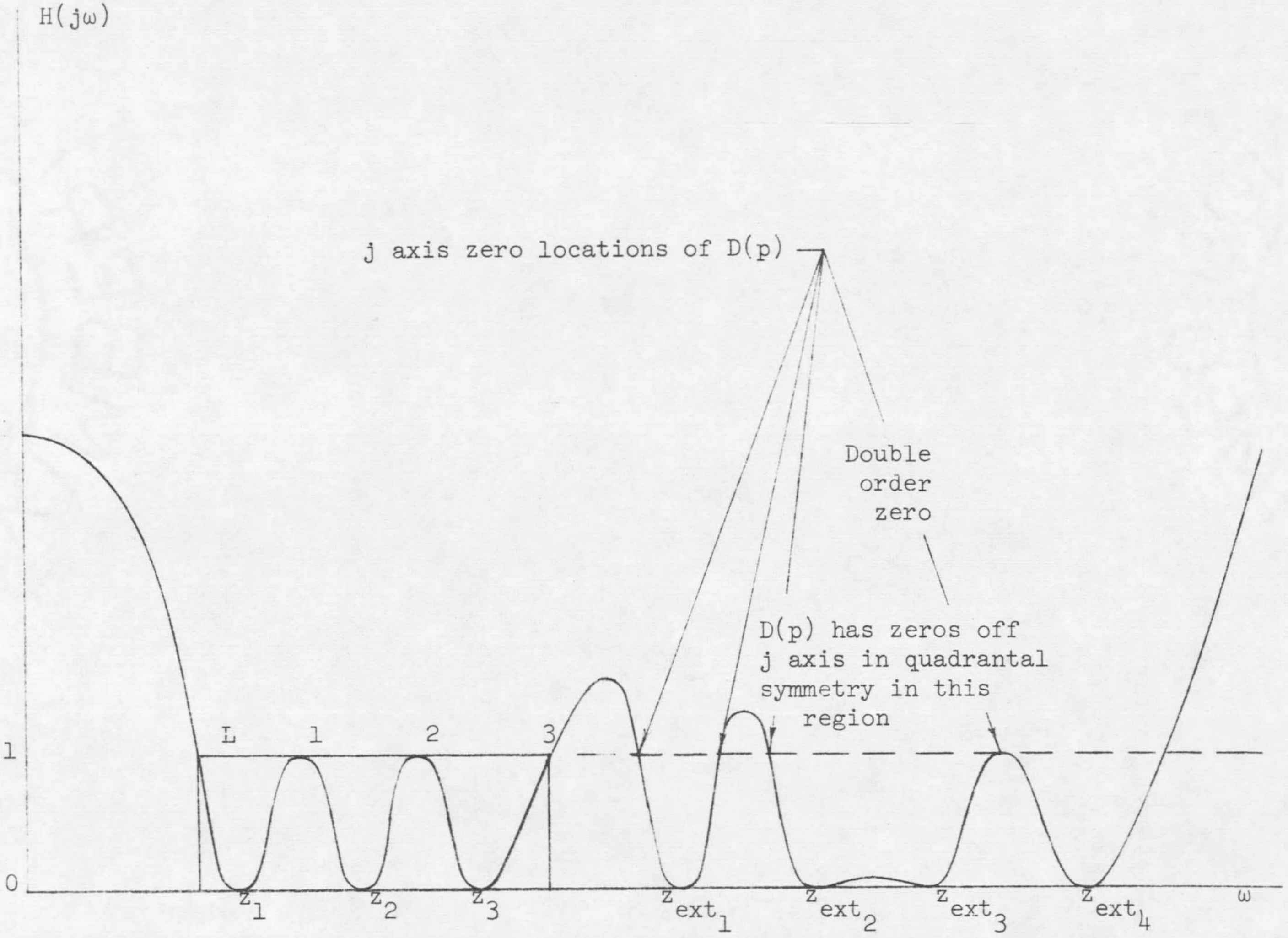


Figure A-1. A General $H(p)$ with Extrapassband Zeros

This is a restricted case of the F(p) now sought in that $C(p) \equiv \sqrt{D(p)} \equiv 1$. The simplest H(p) is one with a single zero in the passband, a set of prescribed extrapassband zeros and a prescribed denominator. F(p) consistent with the formation of such a function is

$$F(p) = \frac{(p^2 + z_1^2)C(p) + \sqrt{D(p)}\sqrt{p^2 + \omega_L^2}\sqrt{p^2 + \omega_H^2}}{(p^2 + z_1^2)C(p) - \sqrt{D(p)}\sqrt{p^2 + \omega_L^2}\sqrt{p^2 + \omega_H^2}} \qquad (A.10)$$

F(p) is a product of

$$F_1(p) = \frac{C_1(p)\sqrt{p^2 + \omega_L^2} + D_1(p)\sqrt{p^2 + \omega_H^2}}{-C_1(p)\sqrt{p^2 + \omega_L^2} + D_1(p)\sqrt{p^2 + \omega_H^2}} \qquad (A.11)$$

and

$$F_2(p) = \frac{C_2(p)\sqrt{p^2 + \omega_L^2} + D_2(p)\sqrt{p^2 + \omega_L^2}}{-C_2(p)\sqrt{p^2 + \omega_L^2} + D_2(p)\sqrt{p^2 + \omega_L^2}} \qquad (A.12)$$

$C_1(p)$, $C_2(p)$, $D_1(p)$ and $D_2(p)$ are expressions whose forms are to be determined.

Multiplying out

$$F(p) = F_1(p)\, F_2(p)$$

$$= \frac{\{C_1(p)C_2(p)(p^2 + \omega_L^2) + D_1(p)D_2(p)(p^2 + \omega_H^2)\} + \{(C_1(p)\, D_2(p) + C_2(p)D_1(p)\sqrt{p^2 + \omega_L^2}\sqrt{p^2 + \omega_H^2}\}}{C_1(p)C_2(p)(p^2 + \omega_L^2) + D_1(p)D_2(p)(p^2 + \omega_H^2)\} - \{(C_1(p)D_2(p) + C_2(p)D_1(p)\sqrt{p^2 + \omega_L^2}\sqrt{p^2 + \omega_H^2}\}} \qquad (A.13)$$

Upon equating coefficients of $\sqrt{p^2 + \omega_L^2}\sqrt{p^2 + \omega_H^2}$ in equations A.10 and A.13, one gets

$$C_1(p)D_2(p) + C_2(p)D_1(p) = \sqrt{D(p)}. \tag{A.14}$$

Excluding the trivial case of $D(p) =$ constant, this can be satisfied only for

$$C_1(p)D_2(p) = k\sqrt{D(p)} \tag{A.15}$$

and

$$C_2(p)D_1(p) = (1-k)\sqrt{D(p)} \tag{A.16}$$

in which $C_1(p)$, $C_2(p)$, $D_1(p)$ and $D_2(p)$ are square roots of polynomials, such polynomials having the same roots as $D(p)$ when combined as in equations A.15 and A.16. In order to choose $C_1(p)$, $C_2(p)$, $D_1(p)$ and $D_2(p)$ so that the procedure for constructing $F(p)$ will work, one must already know the zeros of $D(p)$. Such information is not assumed known ab initio.

Extension of this reasoning to $H(p)$ with larger number of passband zeros leads to the identical problem: $F_i(p)$ cannot be constructed unless the zeros of $D(p)$ are known. This points out clearly the dependence of the rotating vector principle upon knowledge of band edges, i.e., unity value points.

APPENDIX B

ON THE NON-EXISTENCE OF $F(p)$ WITH PRESCRIBED PROPERTIES

It is shown here that no $F(p)$ with the following properties exists:

1. In the passband $F(p)$ is a unit vector that rotates the proper number of times.

2. $F(p)$ contains all desired pole locations as singularities.

3. $F(p)$ takes on values of either $+1$ (when $G(p) = \frac{F(p) - 1}{2}$) or -1 (when $G(p) = \frac{F(p) + 1}{2}$) at $\omega = z_{ext_i}$, $i = 1, 2, \cdots, q$. This will cause $G(p) = 0$ at $\omega = z_{ext_i}$.

4. $F(p)$ will yield $H(P)$ with zeros at $\omega\ z_{ext_i}$, $i = 1, 2, \cdots, q$, and poles at the same locations as those of $F(p)$.

Figure B-1(a) shows the appearance of such a suggested $F(p)$ for zeros at $\omega = \pm z_{ext_1}$, $\pm z_{ext_2}$ and poles at $\omega = \pm\omega_1$, $\pm\omega_2$. Figure B-1(b) shows the corresponding $G(p)$. To arrive at $H(p)$ such that passband zeros are located for equiripple properties in the passband, $G(p)$ must be multiplied by $\bar{G}(p)$. Since, in the passband,

$$G(p) = \frac{F(p) + (-1)^n}{2}$$

$$= \frac{e^{j\phi(\omega)} + (-1)^n}{2} \qquad \text{(B-1)}$$

we see that

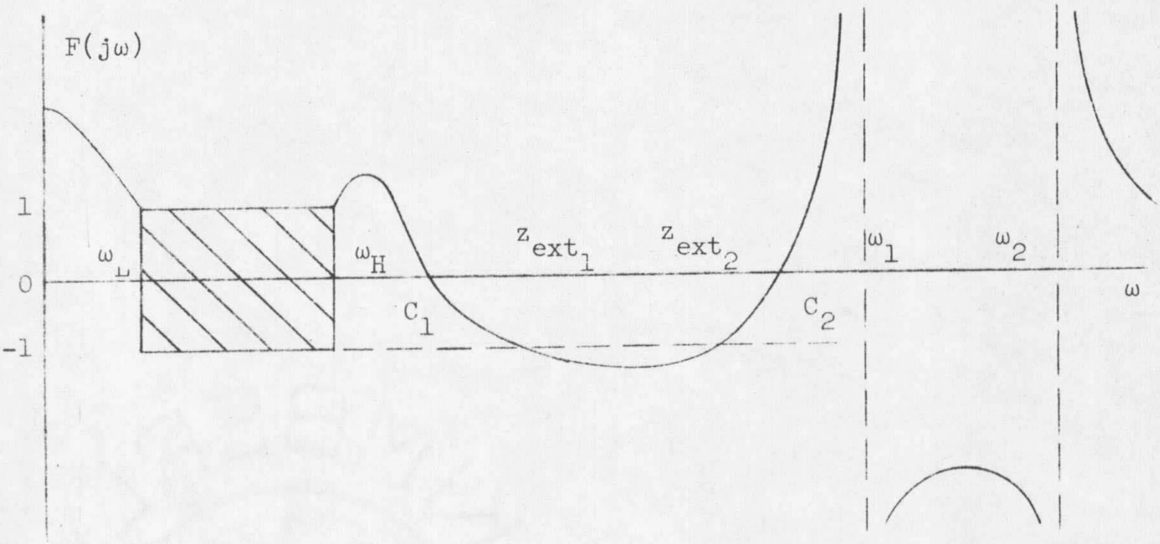
$$\bar{G}(p) = \frac{\bar{F}(p) + (-1)^n}{2}$$

$$= \frac{e^{-j\phi(\omega)} + (-1)^n}{2}$$

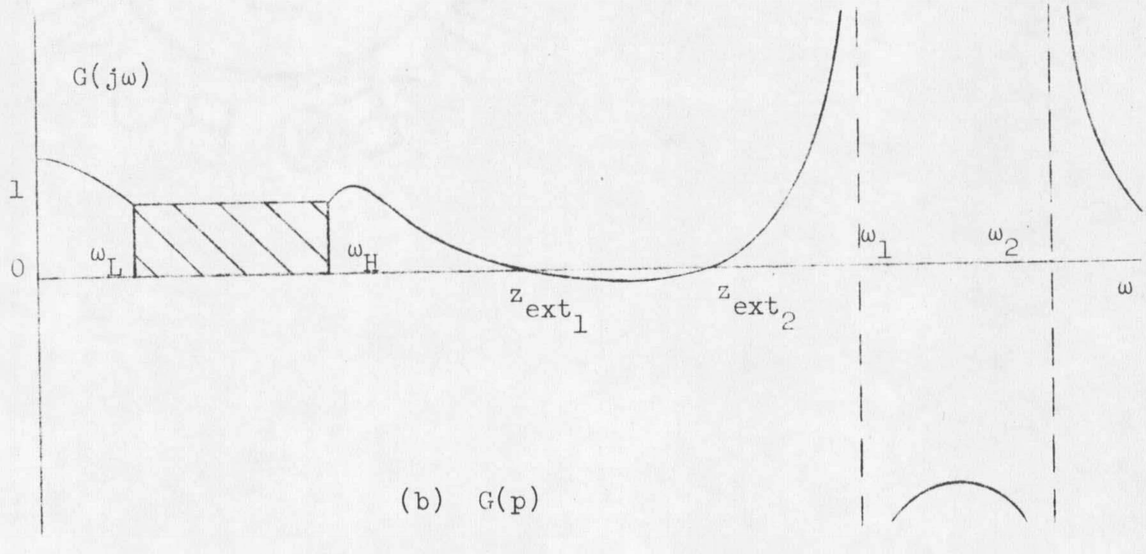
$$= \frac{\frac{1}{e^{j\phi(\omega)}} + (-1)^n}{2}$$

$$= \frac{\frac{1}{F(p)} + (-1)^n}{2} \qquad \text{(B-2)}$$

showing that $\bar{F}(p) = 1/F(p)$. It is necessary that $F(j\omega_H) = 1$ and $F(jz_{ext_1}) = -1$. Since $F(j\omega)$ is real and continuous in this range, it is evident that $H(p)$ will have simple poles at $\omega = \pm\omega_{C_1}$, $\pm\omega_{C_2}$ which were assumed to be not required. Therefore there is no $F(p)$ that is a rotating vector in the passband and real outside the passband that will produce extrapassband zeros with specified pole locations.



(a) F(p)



(b) G(p)

Figure B-1. Proposed F(p) and G(p)

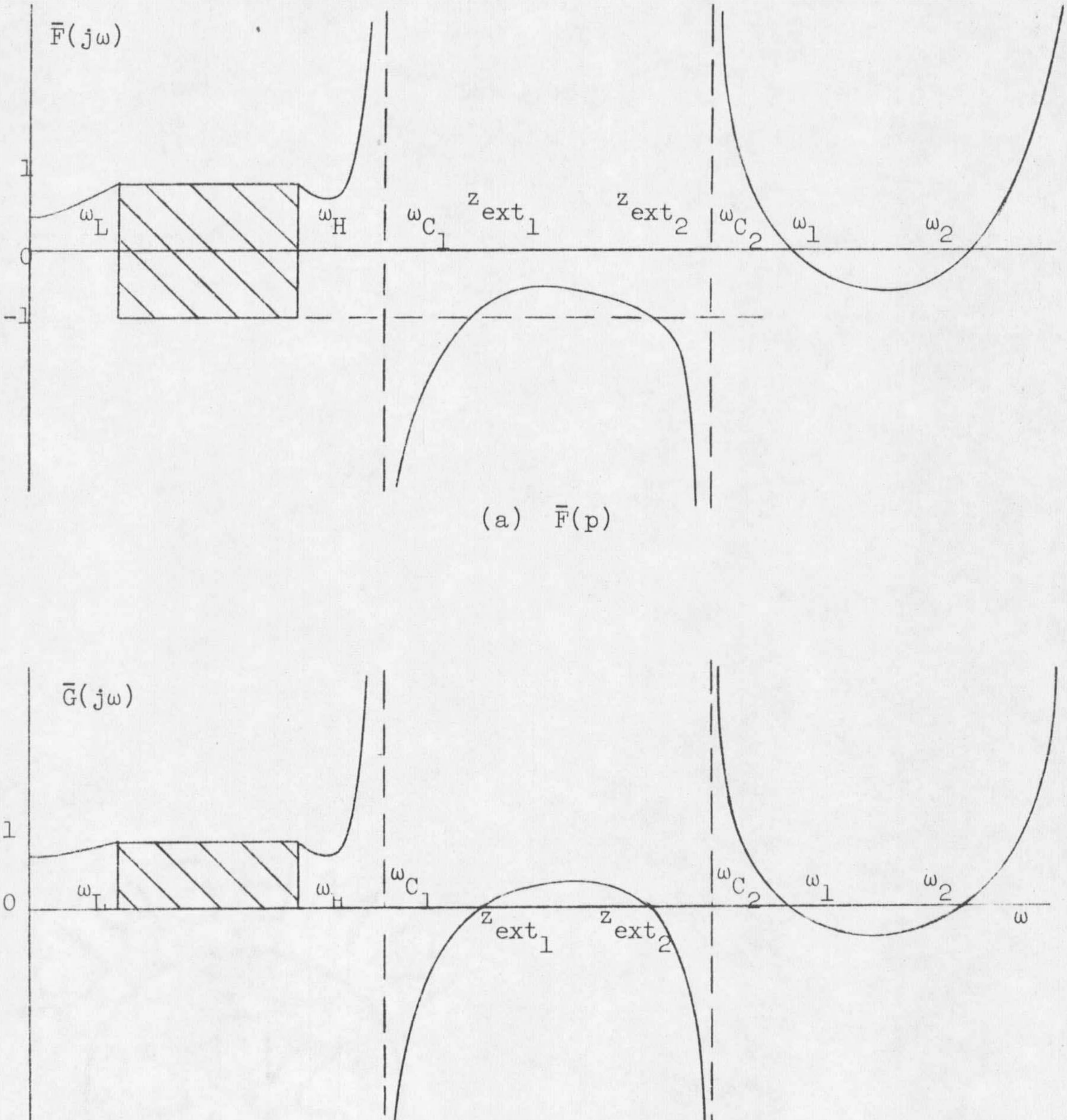


Figure B-2. Resultant $\bar{F}(p)$ and $\bar{G}(p)$

APPENDIX C

POLE LOCATIONS OF MULTIPLE BAND FILTERS

In Section 4-7, multiple band filters were described. F(p) for constructing such filters is

$$F(p) = \prod_{i=1}^{n} \frac{m_i\sqrt{p^2+\omega_1^2}\sqrt{p^2+\omega_4^2}\cdots + \sqrt{p^2+\omega_2^2}\sqrt{p^2+\omega_3^2}\cdots}{-m_i\sqrt{p^2+\omega_1^2}\sqrt{p^2+\omega_4^2}\cdots + \sqrt{p^2+\omega_2^2}\sqrt{p^2+\omega_3^2}\cdots} \qquad (C.1)$$

The denominator of F(p) determines finite frequency poles explicitly. Infinite frequency poles are produced when the denominator is of lower degree than the numerator. A single representative denominator term will be studied with regard to pole locations. Finite pole locations come from setting this term equal to zero.

$$-m_i\sqrt{p^2+\omega_1^2}\sqrt{p^2+\omega_4^2}\cdots + \sqrt{p^2+\omega_2^2}\sqrt{p^2+\omega_3^2}\cdots = 0 \qquad (C.2)$$

Suppose that a two band filter is desired. Equation C.2 becomes

$$-m_i\sqrt{p^2+\omega_1^2}\sqrt{p^2+\omega_4^2} + \sqrt{p^2+\omega_2^2}\sqrt{p^2+\omega_3^2} = 0. \qquad (C.3)$$

An equation exhibiting the same zeros as C.3 is

$$-m_i(P^2+\omega_1^2)(p^2+\omega_4^2) + (p^2+\omega_2^2)(p^2+\omega_3^2) =$$
$$-m_i^2[p^4 + (\omega_1^2+\omega_4^2)p^2 + \omega_1^2\omega_4^2] + [p^4 + (\omega_2^2+\omega_3^2)p^2 + \omega_2^2\omega_3^2] = 0 \qquad (C.4)$$

This is a fourth degree equation and has, by the fundamental theorem of algebra, four roots. Given a set of band edges, a single m_i controls four pole locations. If $m_i = 1$, equation C.4 is second degree, having

2 finite roots, the other two poles being produced at infinity. There are many particular cases in which all 4 poles appear at infinity. If $m = 1$ and $\omega_1^2 + \omega_4^2 = \omega_2^2 + \omega_3^2$ only a constant is left. An example is $\omega_1 = 1$, $\omega_2 = 2$, $\omega_3 = 3$, $\omega_4 = \sqrt{12}$.

Therefore pole locations depend upon two things: (1) the value of the ω_k and (2) the value of m_i. Band edges are usually prescribed, leaving m_i as the determining factor. For a particular filter with q bands, individual denominator terms have 2q zeros in general. Each m_i then controls locations of 2q poles but does not exercise independent control of the poles, i.e., the poles come in sets. If one pole of a set is chosen as an arbitrary location, all other poles of the set appear and must be accepted as part of the function.

As an example, let $\omega_1 = 1$, $\omega_2 = 2$, $\omega_3 = 3$ and $\omega_4 = 4$. Equation C.4 becomes for $p = j\omega$

$$-m_i^2(\omega^4 - 14\omega^2 + 16) + (\omega^4 - 13\omega^2 + 36) = 0. \tag{C.5}$$

Suppose that double poles at infinity are desired. For such poles $m_i = 1$ and we have

$$(17\omega^2 - 16) + (13\omega^2 + 36) = 0 \tag{C.6}$$

which gives as finite values of ω, $\omega = \pm j\sqrt{5}$ or $p = \pm\sqrt{5}$ in addition to the specified pole pair at infinity.

Some general characteristics of the pole set can be examined.

Rewriting equation C.5

$$m_i^2 = \frac{(4 - \omega^2)(p - \omega^2)}{(1 - \omega^2)(16 - \omega^2)} \tag{C.7}$$

we see that m_i^2 can be considered as a complex function of p and plotted as in Figure C-1. From familiarity with such types of plots one can sketch m_i^2 for positive ω (m_i^2 is an even function and this is all we need.). Figure C.2 shows that only for values of m_i^2 greater than 9/4 will all poles lie on the $j\omega$ axis.

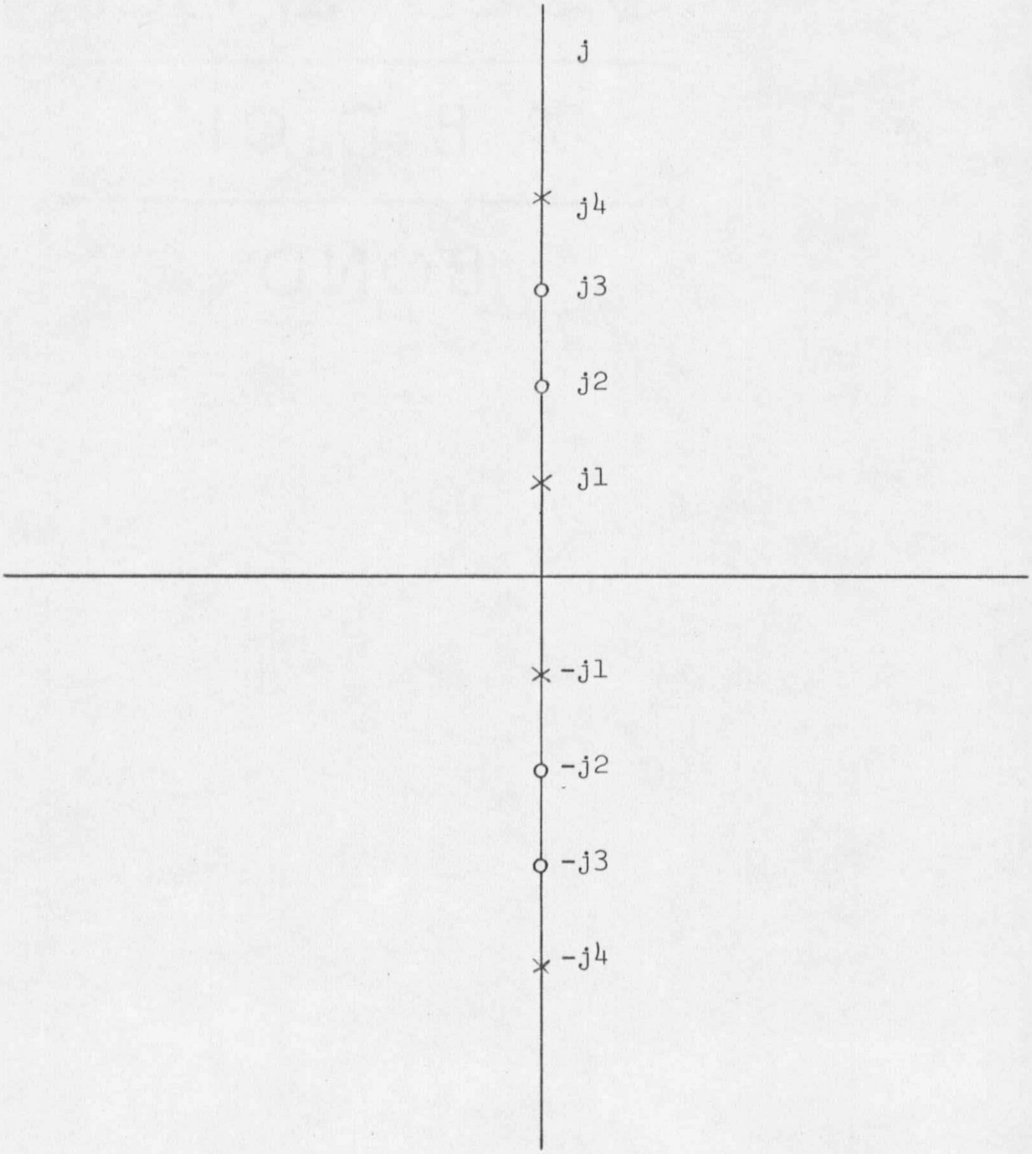


Figure C-1. Plot of $m^2 = \dfrac{(p^2 + 4)(p^2 + 9)}{(p^2 + 1)(p^2 + 16)}$

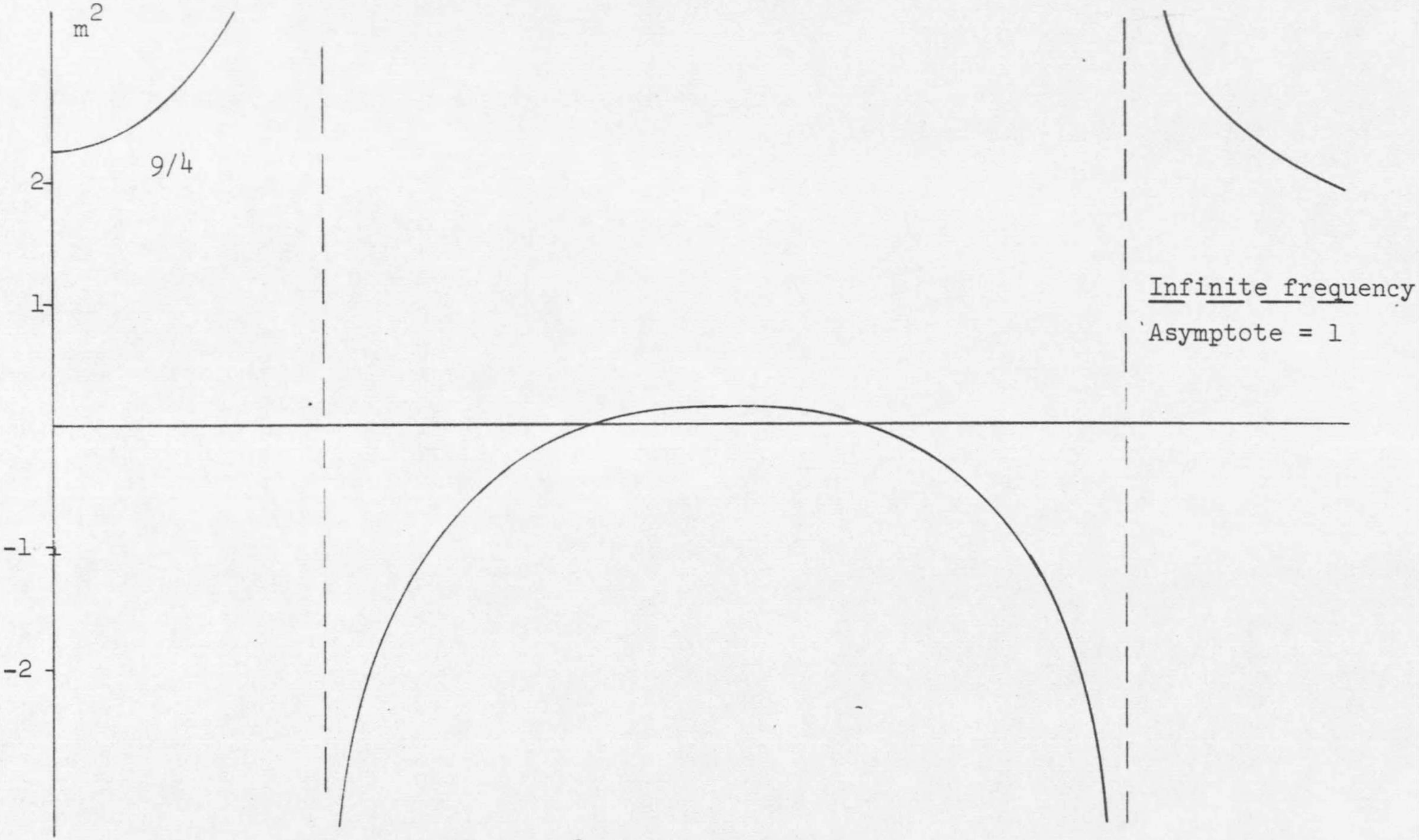


Figure C-2. Plot of m^2 as a function of ω.

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