



Simultaneous convergence in two metrics
by Eric Drake

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of
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Abstract:

On a metric space certain concepts of convergence for sequences are defined. Using these concepts, several relations between metrics on a set X are introduced and their interdependence studied. Included are the relations "equivalent" (identical topologies), "comparable" (one topology includes another), And "uniconvergent" (the identity map from X bearing one metric, to X bearing a second metric has closed graph). For X a commutative group, convergence in a translation-invariant metric d is more conveniently studied by introducing the associated metron (the function $p: X \rightarrow \mathbb{R}$ such that $\forall x \in X [p(x) = d(x, o)]$). The same is true of relations.

Five classes of metrons on a linear space are considered; metrons, scalar-continuous metrons (the product of scalar and vector is a continuous function of the scalar component), quasinorms (the product of scalar and vector is a jointly continuous function of the scalar and vector components), norms, and inner product norms. A typical question studied is whether, on a given linear space, all metrons of a given class bear a given relation to each other. For norms, only four of the relations studied remain distinct, while for complete norms all coincide.

It is proved that non-uniconvergent metrons on a one-dimensional space, and non-uniconvergent inner product norms on a space of countably-infinite Hamel dimension exist.

The scalar-continuous metron is considered in detail. On finite dimensional spaces it has simple continuity properties, yet allows surprising convergence behavior. Counterexamples illustrating non-comparable and incomplete scalar-continuous metrons on a one-dimensional space are constructed. Some questions relating to completion remain unsolved.

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ABSTRACT

On a metric space certain concepts of convergence for sequences are defined. Using these concepts, several relations between metrics on a set X are introduced and their interdependence studied. Included are the relations "equivalent" (identical topologies), "comparable" (one topology includes another), and "uniconvergent" (the identity map from X bearing one metric, to X bearing a second metric has closed graph). For X a commutative group, convergence in a translation-invariant metric d is more conveniently studied by introducing the associated metron (the function $p: X \rightarrow \mathbb{R}$ such that $\forall x \in X [p(x) = d(x, o)]$). The same is true of relations.

Five classes of metrons on a linear space are considered: metrons, scalar-continuous metrons (the product of scalar and vector is a continuous function of the scalar component), quasi-norms (the product of scalar and vector is a jointly continuous function of the scalar and vector components), norms, and inner product norms. A typical question studied is whether, on a given linear space, all metrons of a given class bear a given relation to each other. For norms, only four of the relations studied remain distinct; while for complete norms all coincide.

It is proved that non-uniconvergent metrons on a one-dimensional space, and non-uniconvergent inner product norms on a space of countably-infinite Hamel dimension exist.

The scalar-continuous metron is considered in detail. On finite dimensional spaces it has simple continuity properties, yet allows surprising convergence behavior. Counterexamples illustrating non-comparable and incomplete scalar-continuous metrons on a one-dimensional space are constructed. Some questions relating to completion remain unsolved.

INTRODUCTION

In the course of reading Mikhlin "The problem of the minimum of a quadratic functional" I encountered the following situation:

"A linear space D is embedded in each of two normed linear spaces H and H' . A sequence of elements in D is supposed convergent to a point u in the norm of H and to a point u' in the norm of H' . It is then proved that for the particular spaces and norms under consideration there is a one-one correspondence between such limit points u and u' ." (Mikhlin p.14).

It struck me that I was not familiar with general conditions that would guarantee such a result in even the simple case in which only one linear space is involved:

"A sequence of elements in a linear space is supposed convergent to a point u of the space in one metric, and to a point u' of the space in another metric. When are u and u' necessarily the same point?"

This problem formed the motivation for the present work; however, it forms but a small part of a wider ranging investigation.

In general terms, this thesis is concerned with how the properties of a metric control the convergence behavior of a sequence in a linear space.

Different types of metric and a variety of relations between metrics are considered. The typical question studied is whether, on a given linear space, all metrics of a given type bear a given relation to each other. An example of a statement of this nature is the well known result that, on a given finite-dimensional linear space, all norms are equivalent.

In chapter one a large part of the background for the thesis is presented, and a structure built up which enables the key problems to be identified. Some of this material will be familiar to mathematicians and is presented with as little elaboration as possible. Where no detailed proof or reference for a result is given, the demonstration should be found to require only a few, straightforward steps.

By contrast, the main results of the thesis follow from proofs involving rather detailed, though quite elementary, real analysis, and requiring some ingenuity in their construction. They are given, largely, in chapters two and three.

In chapter four, the more noteworthy results obtained are listed, and a set of further research problems related to this work is offered.

Appendix A identifies some of the notation used throughout.

The pre-requisites for reading the thesis are modest. The usual background of intermediate algebra and analysis is assumed. Though the thesis is in the field of functional analysis, the presentation is almost self-contained and little use is made of high-powered results from that discipline. However, some familiarity with the elements of axiomatic set theory and of point set topology is desirable; appendix B gives references for set theory, and appendix C outlines relevant results from topology.

References to the bibliography are given in the form:

(Wilansky (1) p.52).

References to numbered items in the thesis are given in the forms:

- 2,3 ---- Item 3 of section 2 of the same chapter;
- 1,2,3 ---- Item 3 of section 2 of chapter 1.

CHAPTER ONE: METRIC AND LINEAR SPACES.

We present in this introductory chapter certain basic concepts in the theories of metric and linear spaces. In the last section of this chapter we summarize the principal results obtained in this thesis.

S1 METRIC SPACES.

Let X denote any set, and R the real number system.

1.1 Definition. A function $d: X \times X \rightarrow R$ is a metric d on X iff

$\forall x, y, z \in X$:

- (a) $d(x, y) \geq 0$ (non-negative)
 $d(x, y) = 0 \iff x = y$ (total)
- (b) $d(x, y) = d(y, x)$ (symmetric)
- (c) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

Note (1) the non-negativity follows from the remainder of the definition (put $y = x$ in (c));

(2) the triangle inequality and symmetry imply

$$|d(x, z) - d(z, y)| \leq d(x, y) \leq d(x, z) + d(z, y).$$

A set X is a metric space iff a metric is defined in X .

We shall, for brevity, refer to a sequence of elements of X (precisely $\langle x^n \rangle_{n=1}^{\infty} \in X^{\omega}$) by writing $\langle x^n \rangle \in X^{\omega}$. The unordered elements of $\langle x^n \rangle$ are denoted by $\{x^n\} \subset X$.

1.2 Definition. Given a metric d on X we say that a sequence $\langle x^n \rangle$ of elements of X converges in metric d to an element a of X iff $d(x^n, a) \rightarrow 0$

i.e. $\forall \varepsilon > 0 \exists N \forall n > N [d(x^n, a) < \varepsilon]$.

1.3 Theorem. A sequence in X can converge in a fixed metric to at most one element of X .

Proof. Suppose $d(x^n, a) \rightarrow 0$ and $d(x^n, b) \rightarrow 0$.

Then $d(a, b) \leq [d(a, x^n) + d(x^n, b)] \rightarrow 0$.

Thus $d(a, b) = 0$ and $a = b$. //

1.4 Definition. A sequence $\langle x^n \rangle$ of elements of X is Cauchy in metric d iff $d(x^m, x^n) \rightarrow 0$,

i.e. $\forall \varepsilon > 0 \exists N \forall m, n > N [d(x^m, x^n) < \varepsilon]$.

1.5 Definition. Given a set X we say that metric d is complete in X (and X is complete in metric d) iff every sequence in X , Cauchy in d , converges in d to an element of X , i.e. iff

$\forall \langle x^n \rangle \in X^{\omega} : d(x^m, x^n) \rightarrow 0 \Rightarrow \exists a \in X [d(x^n, a) \rightarrow 0]$.

1.6 Definition. Two sequences $\langle x^n \rangle$, $\langle y^n \rangle$ of elements of X are

- 1) relatively convergent in metric d iff $d(x^n, y^n) \rightarrow 0$;
- 2) relatively Cauchy in metric d iff $d(x^m, y^n) \rightarrow 0$.

1.7 Theorem. For sequences $\langle x^n \rangle$, $\langle y^n \rangle$ of elements of X , and in terms of any one metric on X :

- 1) $\langle x^n \rangle$ convergent $\Rightarrow \langle x^n \rangle$ Cauchy;
- 2) $\langle x^n \rangle$, $\langle y^n \rangle$ relatively Cauchy $\Leftrightarrow \langle x^n \rangle$, $\langle y^n \rangle$ Cauchy and relatively convergent.

Proof.

- 1) Suppose $d(x^n, a) \rightarrow 0$.

Then $d(x^m, x^n) \leq [d(x^m, a) + d(a, x^n)] \rightarrow 0$

- 2) Suppose $d(x^m, y^n) \rightarrow 0$.

Then clearly $d(x^n, y^n) \rightarrow 0$,

$d(x^m, x^n) \leq [d(x^m, y^n) + d(y^n, x^n)] \rightarrow 0$, and similarly

$d(y^m, y^n) \rightarrow 0$.

Suppose $d(x^m, x^n) \rightarrow 0$ and $d(x^n, y^n) \rightarrow 0$.

Then $d(x^m, y^n) \leq [d(x^m, x^n) + d(x^n, y^n)] \rightarrow 0$. //

S2 RELATIONS BETWEEN METRICS ON A GIVEN SET.

Recall that a relation is a class of ordered pairs of sets.

If A is a relation and $\langle u, v \rangle \in A$, we shall write $u A v$.

2.1 Definition. Given any relation A we define:

- 1) A^{-1} , the inverse of A , by $A^{-1} = \{ \langle u, v \rangle \mid \langle v, u \rangle \in A \}$
i.e. $u A^{-1} v$ iff $v A u$.
- 2) $\bar{A}$, the symmetric restriction of A , by $\bar{A} = A \cap A^{-1}$
i.e. $u \bar{A} v$ iff $[u A v \text{ and } v A u]$.
- 3) A' , the symmetric extension of A , by $A' = A \cup A^{-1}$
i.e. $u A' v$ iff $[u A v \text{ or } v A u]$.

We may read $u A' v$ as " u, v are A-comparable", and $u \bar{A} v$ as " u, v are A-symmetric". In case A is an equivalence relation we may read $u A v$ as " u, v are A-equivalent".

2.2 Theorem. For any relations A, B , and for any class S :

- 1) $\bar{A} \subset A \subset A'$.
- 2) $\bar{A}$ reflexive on $S \iff A$ reflexive on $S \iff A'$ reflexive on S
- 3) $\bar{A}, A'$ are symmetric.
- 4) A transitive $\implies \bar{A}$ transitive.
- 5) $A \subset B \iff A^{-1} \subset B^{-1}$.
- 6) $A \subset B \implies [\bar{A} \subset \bar{B} \text{ and } A' \subset B']$.

Proof. 4) Suppose A transitive.

$$\begin{aligned}
 \text{Then } [u \bar{A} v \text{ and } v \bar{A} w] &\Leftrightarrow \begin{bmatrix} u A v \text{ and } v A u \\ \text{and } v A w \text{ and } w A v \end{bmatrix} \\
 &\Rightarrow [u A w \text{ and } w A u] \\
 &\Leftrightarrow u \bar{A} w. //
 \end{aligned}$$

We now introduce certain relations defined on a collection M of metrics on a given set X.

2.3 Definition. For metrics $d_1, d_2 \in M$:

- 1) $\underline{d_1 B d_2}$ iff $\exists k > 0 \forall x, y \in X [k d_1(x, y) \geq d_2(x, y)]$.
- 2) $\underline{d_1 C d_2}$ iff there exists a function $f: \mathbb{R} \rightarrow \mathbb{R}$, continuous at 0 with $f(0) = 0$, such that $\forall x, y \in X [f(d_1(x, y)) \geq d_2(x, y)]$.
- 3) $\underline{d_1 D d_2}$ iff $\forall \langle x^n \rangle, \langle y^n \rangle \in X^\omega [d_1(x^n, y^n) \rightarrow 0 \Rightarrow d_2(x^n, y^n) \rightarrow 0]$.
- 4) $\underline{d_1 E d_2}$ iff $\forall \langle x^n \rangle, \langle y^n \rangle \in X^\omega [d_1(x^m, y^n) \rightarrow 0 \Rightarrow d_2(x^m, y^n) \rightarrow 0]$.
- 5) $\underline{d_1 F d_2}$ iff $\forall \langle x^n \rangle \in X^\omega \forall a \in X [d_1(x^n, a) \rightarrow 0 \Rightarrow d_2(x^n, a) \rightarrow 0]$.
- 6) $\underline{d_1 G d_2}$ iff $\forall \langle x^n \rangle \in X^\omega \forall a \in X \exists b \in X$
 $[d_1(x^n, a) \rightarrow 0 \Rightarrow d_2(x^n, b) \rightarrow 0]$.
- 7) $\underline{d_1 H d_2}$ iff $\forall \langle x^n \rangle \in X^\omega [d_1(x^m, x^n) \rightarrow 0 \Rightarrow d_2(x^m, x^n) \rightarrow 0]$.
- 8) $\underline{d_1 I d_2}$ iff $\forall \langle x^n \rangle \in X^\omega \forall a \in X [d_1(x^n, a) \rightarrow 0 \Rightarrow d_2(x^m, x^n) \rightarrow 0]$.
- 9) $\underline{d_1 U d_2}$ iff $\forall \langle x^n \rangle \in X^\omega \forall a, b \in X$
 $[(d_1(x^n, a) \rightarrow 0 \text{ and } d_2(x^n, b) \rightarrow 0) \Rightarrow a = b]$.

We introduce also the symmetric restriction and the symmetric extension (2.1) of each of the relations in 1) - 8); relation 9) is already symmetric, so $\bar{U} = U = U'$.

2.4 Theorem. The relations introduced in 2.3, defined on a collection M of metrics on a given set, have the following properties:

- 1) All are reflexive on M.
- 2) U and all relations of types $\bar{A}$ and A' are symmetric.
- 3) For $A = B, C, D, E, F, G, H$, relations of types A and $\bar{A}$ are transitive.
- 4) Thus $\bar{B}, \bar{C}, \bar{D}, \bar{E}, \bar{F}, \bar{G}, \bar{H}$ are equivalence relations on M.

Proof. Immediate from 2.1-2.3. //

2.5 Theorem. The implications portrayed in Figure 1 hold between the relations introduced in 2.3, all defined on the same collection of metrics on a given set (where, for example, for brevity the symbolism $\bar{B} \Rightarrow B$ is used to denote $d_1 \bar{B} d_2 \Rightarrow d_1 B d_2$ i.e. $\bar{B} \subset B$):

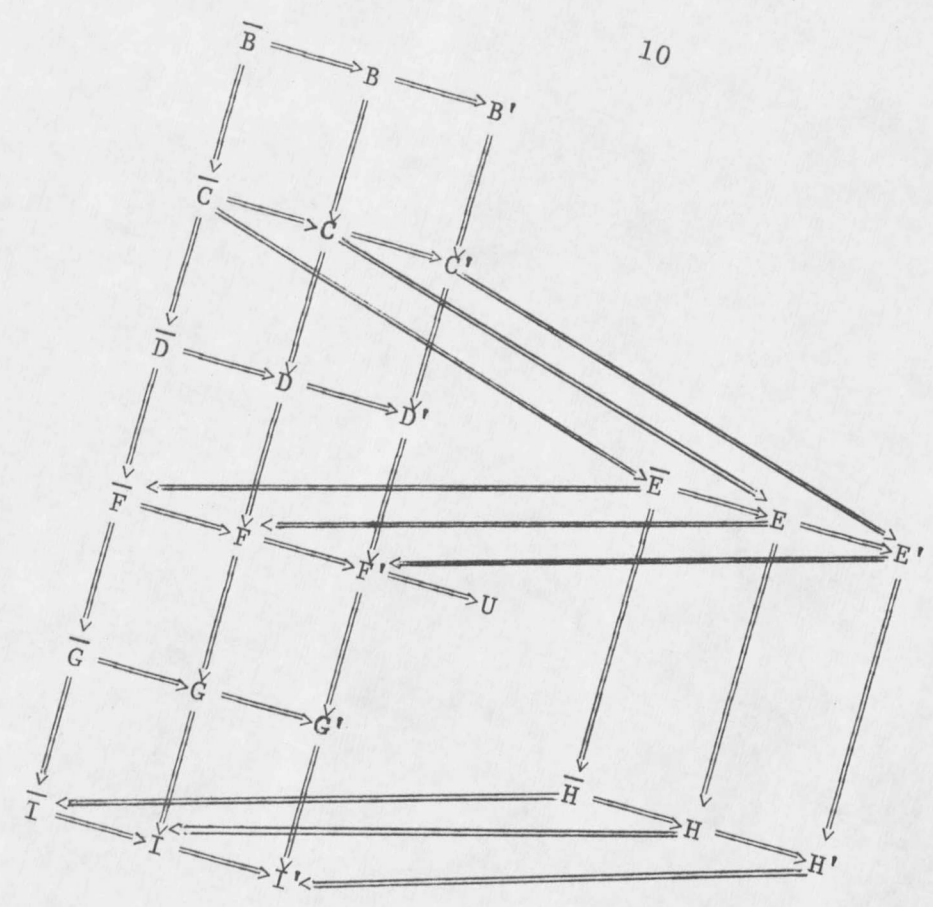


Figure 1.

Proof. The implications follow easily with some use of 1.3, 1.7 and 2.2. //

2.6 Theorem. Continuing 2.5, the further implications hold:

- 1) $F \Leftrightarrow [G \text{ and } U]$.
- 2) $[D \text{ and } H] \Rightarrow E$.
- 3) If d_2 is complete: $d_1 I d_2 \Leftrightarrow d_1 G d_2$;
if d_1 is complete: $d_1 I d_2 \Leftrightarrow d_1 H d_2$.
- 4) $[d_1 F d_3 \text{ and } d_2 F d_3] \Rightarrow d_1 U d_2$.

Proof. The statements follow easily with some use of 1.3 and 1.7. //

Certain of the relations introduced in 2.3 are important enough to be given names. In the literature, for the particular case of relation F (only):

$d_1 F d_2$ is read " d_1 is stronger than d_2 " or " d_2 is weaker than d_1 ";
 $d_1 F' d_2$ is read " d_1, d_2 are comparable" (rather than F -comparable);
 $d_1 \bar{F} d_2$ is read " d_1, d_2 are equivalent" (rather than $\bar{F}$ -equivalent).
 We shall read $d_1 U d_2$ as " d_1, d_2 are uniconvergent".

These relations are involved in the following characterizations.

2.7 Theorem. On a given set, metric d_1 is stronger than metric d_2 ($d_1 \bar{F} d_2$) iff the d_1 topology is stronger than (includes) the d_2 topology. Thus d_1 and d_2 are equivalent ($d_1 \bar{F} d_2$) iff the d_1 and d_2 topologies are identical. (Wilansky (2) p.27.)

2.8 Theorem. On a given set X , d_1 and d_2 are uniconvergent ($d_1 U d_2$) iff the identity map from X bearing the d_1 topology to X bearing the d_2 topology has closed graph. (Wilansky (1) p.195.)

S3 LINEAR SPACES.

3.1 Definition. A set X is a linear space over K , (X, K) , iff there exist operations of "addition" of the elements ("vectors") of X , denoted for vectors x and y by $x \dot{+} y$, and of "multiplication" of the elements of X by the elements ("scalars") of a commutative field K , denoted for a vector x and a scalar k by $k.x$, such that:

A) X is a commutative group with respect to addition;

B) $\forall h, k \in K \quad \forall x, y \in X$:

(a) $k.x \in X$

(b) $(h + k).x = h.x \dot{+} k.x$

(c) $k.(x \dot{+} y) = k.x \dot{+} k.y$

(d) $h.(k.x) = (hk).x$

(e) $1.x = x$ (where 1 denotes the scalar

multiplicative identity).

For simplicity we shall omit the dots and hereafter denote the addition of vectors x and y by $x + y$, and the multiplication of a vector x by a scalar k by kx .

We let: $-x$ denote the additive inverse of x ,

$x - y$ denote $x + (-y)$,

$0, o$ denote the zero elements of K, X respectively,

Q, R, C denote the fields of rational, real,

and complex numbers respectively.

3.2 Definition. A set L is a linear manifold of (X, K) iff $L \subset X$ and L is a linear space over K with the same operations of addition and multiplication as (X, K) .

For any (X, K) , $\{o\}$ is a linear manifold of (X, K) , the zero linear space.

3.3 Definition. Let (X, K) be given and let S be a subset of X .

A linear combination from S is an element $\sum_{i=1}^n k_i s^i$ of X where the coefficients $k_1, \dots, k_n$ are scalars, $s^1, \dots, s^n$ are distinct elements of S , and n is finite.

A linear combination is non-trivial iff at least one scalar coefficient is non-zero.

3.4 Definition. Let (X, K) be given. A subset S of X is linearly dependent iff 0 is a non-trivial linear combination from S ; otherwise linearly independent.

3.5 Definition. A Hamel basis for (X, K) is a subset S of X such that every element of X can be represented as a unique linear combination from S .

(By "unique" we mean that the scalar coefficient of any element s of S is the same in every linear combination from S containing s that represents a given element of X .)

3.6 Theorem. A subset S of X is a Hamel basis for (X, K) iff every element of X can be represented as a linear combination from S , and S is linearly independent.

Proof. Immediate (3.5). //

3.7 Theorem. 1) Every non-zero linear space has a Hamel basis.
(Wilansky (1) p.16).

2) Any two Hamel bases for a linear space are in one-one correspondence. (Wilansky (1) p.17).

3.8 Definition. The Hamel dimension of a linear space is the cardinality of any Hamel basis for the space.

3.9 Definition. Two linear spaces (X, K) and (X', K) are linearly isomorphic iff there exists a function $f: X \rightarrow X'$ which is

(a) bijective (one-one and onto),

(b) linear i.e. $\forall x, y \in X \quad \forall k \in K$

$$[f(x + ky) = f(x) + k f(y)].$$

3.10 Theorem. Two linear spaces over the same scalar field are linearly isomorphic iff they have the same Hamel dimension.

(Wilansky (1) p.20).

3.11 Example. 1) Let (R^n, R) denote the space of real n -tuples.

A Hamel basis may be taken as $\{\delta^k\}_{k=1}^n$

where $\delta^k = \langle 0, 0, \dots, 0, 1, 0, \dots, 0 \rangle$, the 1 occurring as the k^{th} coordinate.

Then every n -dimensional linear space is linearly isomorphic to (R^n, R) .

2) Let (R^∞, R) denote the space of real sequences with only a finite number of non-zero terms. A Hamel basis may be taken as $\{\delta^k\}_{k=1}^\infty$ where now $\delta^k = \langle 0, 0, \dots, 0, 1, 0, \dots \rangle$, the 1 occurring as the k^{th} coordinate.

Then every linear space of countably-infinite Hamel dimension is linearly isomorphic to (R^∞, R) .

S4 TRANSLATION-INVARIANT METRICS AND METRONS ON A COMMUTATIVE GROUP.

On a set which is a commutative group with respect to an operation of "addition" we may introduce the following two concepts.

4.1 Definition. On a commutative group X a metric d is a translation-invariant metric (t.i.m.) iff

$$\forall x, y, z \in X [d(x + z, y + z) = d(x, y)].$$

4.2 Definition. On a commutative group X a function $p: X \rightarrow \mathbb{R}$ is a metron (m.) iff $\forall x, y \in X$:

- (a) $p(x) \geq 0$; $p(x) = 0 \Leftrightarrow x = 0$ (positive definite)
- (b) $p(x) = p(-x)$ (symmetric)
- (c) $p(x + y) \leq p(x) + p(y)$ (sub-additive).

Note (1) $p(x) \geq 0$ follows from the remainder of the definition

(put $y = -x$ in (c));

(2) sub-additivity and symmetry imply

$$|p(x) - p(y)| \leq p(x - y) \leq p(x) + p(y).$$

We show the two concepts are closely related.

4.3 Theorem. Given a commutative group X , there is a one-one correspondence between the members of the set D of all translation-invariant metrics on X and the members of the set P of all metrons on X given by defining:

the metron p associated with a translation-invariant metric d
by

$$(1) \quad \forall x \in X [p(x) = d(x, o)];$$

the translation-invariant metric d associated with a metron p
by

$$(2) \quad \forall x, y \in X [d(x, y) = p(x - y)].$$

Proof. (a) For any $d \in D$ it is readily verified that the function p defined by (1) satisfies the requirements for a metron (4.2) i.e. $p \in P$.

(b) For any $p \in P$ it is readily verified that the function d defined by (2) satisfies the requirements for a metric (1.1). Furthermore $d(x + z, y + z) = p(x + z - y - z) = p(x - y) = d(x, y)$. Thus d is a translation-invariant metric i.e. $d \in D$.

(c) In addition to defining $p(x)$, (1) also defines a function $f: D \rightarrow P$; likewise a function $g: P \rightarrow D$ is defined by (2). We shall show that $f = g^{-1}$.

For any $d_1 \in D$, $f(d_1) = p \in P$ and $g(p) = d_2 \in D$. But
 $\forall x, y \in X [d_2(x, y) = p(x - y) = d_1(x - y, o) = d_1(x, y)]$. Hence
 $g \circ f$ is the identity map on D .

For any $p_2 \in P$, $g(p_2) = d \in D$ and $f(d) = p_1 \in P$. But
 $\forall x \in X [p_1(x) = d(x, o) = p_2(x - o) = p_2(x)]$. Hence $f \circ g$ is the
 identity map on P . Thus $f = g^{-1}$. //

When metrics are translation-invariant we shall find it
 simpler to work with the associated metrons than with the metrics
 themselves.

4.4 Example. On any set X the discrete metric d is defined by:

$$\forall x, y \in X: d(x, y) = \begin{cases} 0 & (x = y) \\ 1 & (x \neq y). \end{cases}$$

When X is a commutative group the discrete metric is transla-
 tion-invariant, and the associated discrete metron p is defined by:

$$\forall x \in X: p(x) = \begin{cases} 0 & (x = o) \\ 1 & (x \neq o). \end{cases}$$

On a commutative group X bearing a metron p , definitions 1.2,
 1.4, 1.5, 1.6, become: A sequence $\langle x^n \rangle \in X^\omega$ converges in p to
 $a \in X$ iff $p(x^n - a) \rightarrow 0$ i.e. $\forall \varepsilon > 0 \exists N \forall n > N [p(x^n - a) < \varepsilon]$;
 A sequence $\langle x^n \rangle \in X^\omega$ is Cauchy in p iff $p(x^m - x^n) \rightarrow 0$ i.e.
 $\forall \varepsilon > 0 \exists N \forall m, n > N [p(x^m - x^n) < \varepsilon]$;

A metron p is complete in X (and X is complete in p) iff every sequence in X , Cauchy in p , converges in p to an element of X i.e. iff

$$\forall \langle x^n \rangle \in X^\omega: p(x^m - x^n) \rightarrow 0 \Rightarrow \exists a \in X. [p(x^n - a) \rightarrow 0];$$

Sequences $\langle x^n \rangle, \langle y^n \rangle \in X^\omega$ are

- 1) relatively convergent in p iff $p(x^n - y^n) \rightarrow 0$;
- 2) relatively Cauchy in p iff $p(x^m - y^n) \rightarrow 0$.

Theorem 1.3, 1.7 also have direct analogs.

Whenever a set X bears a metric d (metron p) it will be understood to bear the d topology (p topology i.e. the associated t.i.m. topology).

4.5 Definition. We denote:

a set X bearing a metric d by (X, d) ;

a commutative group X bearing a metron p by (X, p) ;

a linear space (X, K) bearing a metric d , a metric and linear space, by (X, K, d) ;

a linear space (X, K) bearing a metron p , a metron and linear space, by (X, K, p) ;

the set of real numbers bearing the Euclidean topology by R .

We now give a simple result that will be needed later.

4.6 Theorem. Let a commutative group X bear a metron q . Then a metron $p: (X, q) \rightarrow R$ is continuous everywhere on X iff p is continuous at zero.

Proof. Clearly p continuous everywhere $\Rightarrow p$ continuous at o .

Suppose p is continuous at o . Then $\forall a \in X \quad \forall \langle x^n \rangle \in X^\omega$:
 $q(a + x^n - a) = q(x^n - o) \rightarrow 0 \Rightarrow p(a + x^n - a) = p(x^n)$
 $\rightarrow p(o) = 0$
 $\Rightarrow |p(a + x^n) - p(a)| \rightarrow 0. //$

Consider the relations introduced in 2.3, but defined now on a collection M of translation-invariant metrics on a given commutative group X . Let $\underline{M}$ be the collection of metrons associated with the members of M by the bijection $f: M \rightarrow \underline{M}$ given by 4.3(1).

Corresponding to any relation A on M we introduce a relation $\underline{A}$ on $\underline{M}$ defined by:

$$\forall d_1, d_2 \in M [f(d_1) \underline{A} f(d_2) \Leftrightarrow d_1 A d_2].$$

The sets M and $\underline{M}$ are isomorphic with respect to the relations A and $\underline{A}$, and f is an isomorphism from M to $\underline{M}$.

The relation $\underline{A}$ between metrons corresponding to any given relation A between metrics introduced in 2.3 may be written directly in terms of metrons using $d(x, y) = p(x - y)$. Henceforth we shall employ this form only (4.7) of the relations, but for convenience relations between metrons will be identified by the same symbols

that originally identified the corresponding relations between metrics.

4.7 Definition. Let $\underline{M}$ be a collection of metrons on a given commutative group X . For metrons $p_1, p_2 \in \underline{M}$:

- 1) $\underline{p_1 B p_2}$ iff $\exists k > 0 \forall x \in X [k p_1(x) \geq p_2(x)]$.
- 2) $\underline{p_1 C p_2}$ iff there exists a function $f: R \rightarrow R$, continuous at 0 with $f(0) = 0$, such that $\forall x \in X [f(p_1(x)) \geq p_2(x)]$.
- 3) $\underline{p_1 D p_2}$ iff $\forall \langle x^n \rangle \in X^\omega [p_1(x^n) \rightarrow 0 \Rightarrow p_2(x^n) \rightarrow 0]$.
- 4) $\underline{p_1 E p_2}$ iff $\forall \langle x^n \rangle, \langle y^n \rangle \in X^\omega [p_1(x^m - y^n) \rightarrow 0 \Rightarrow p_2(x^m - y^n) \rightarrow 0]$.
- 5) $\underline{p_1 F p_2}$ iff $\forall \langle x^n \rangle \in X^\omega [p_1(x^n) \rightarrow 0 \Rightarrow p_2(x^n) \rightarrow 0]$.
- 6) $\underline{p_1 G p_2}$ iff $\forall \langle x^n \rangle \in X^\omega \exists c \in X$
 $[p_1(x^n) \rightarrow 0 \Rightarrow p_2(x^n - c) \rightarrow 0]$.
- 7) $\underline{p_1 H p_2}$ iff $\forall \langle x^n \rangle \in X^\omega [p_1(x^m - x^n) \rightarrow 0 \Rightarrow p_2(x^m - x^n) \rightarrow 0]$.
- 8) $\underline{p_1 I p_2}$ iff $\forall \langle x^n \rangle \in X^\omega [p_1(x^n) \rightarrow 0 \Rightarrow p_2(x^m - x^n) \rightarrow 0]$.
- 9) $\underline{p_1 U p_2}$ iff $\forall \langle x^n \rangle \in X^\omega \forall c \in X$
 $[(p_1(x^n) \rightarrow 0 \text{ and } p_2(x^n - c) \rightarrow 0) \Rightarrow c = 0]$.

We introduce also the symmetric restriction and the symmetric extension (2.1) of each of the relations in 1) - 8); relation 9) is already symmetric, so $\bar{U} = U = U'$.

We are thus considering, apparently, twenty-five different relations on $\underline{M}$. However, the following theorem shows that, for

metrons, we are considering no more than nineteen different relations.

4.8 Theorem. For a collection of metrons on a given commutative group X:

$$D \Leftrightarrow E \Leftrightarrow F,$$

and thus 2.5 reduces to the pattern of implications portrayed in Figure 2:

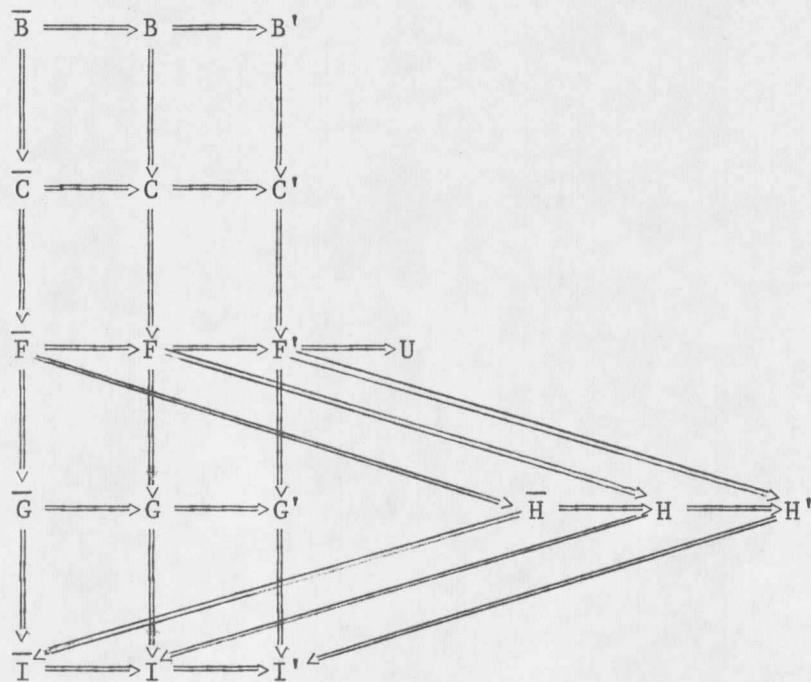


Figure 2.

Proof. $D \Leftrightarrow F$ is clear on comparing 4.7 3) and 5). We already have $E \Rightarrow F$ from 2.5.

To show $F \Rightarrow E$:

$$p_1^F p_2 \Leftrightarrow p_1 \text{ topology} \supset p_2 \text{ topology} \quad (2.7)$$

$$\Leftrightarrow \forall \varepsilon > 0 \exists \delta > 0 \quad \forall x \in X [p_1(x) < \delta \Rightarrow p_2(x) < \varepsilon]$$

$$\Leftrightarrow \forall \varepsilon > 0 \exists \delta > 0 \quad \forall \langle x^n \rangle, \langle y^n \rangle \in X^\omega$$

$$[p_1(x^m - y^n) < \delta \Rightarrow p_2(x^m - y^n) < \varepsilon]$$

$$\Rightarrow p_1^E p_2. //$$

S5 METRONS ON A LINEAR SPACE.

A metron defined on a linear space, in addition to possessing all of the properties 4.2(a),(b),(c) required of any metron on a commutative group, may also have some property deriving from the scalar multiplication.

For the scalar field we henceforth employ only R or C .

5.1 Definition. On a linear space (X,K) a metron p is a scalar-continuous metron (s.c.m.) iff

$$(d) \forall \langle t_n \rangle \in K^\omega \quad \forall x \in X [|t_n| \rightarrow 0 \Rightarrow p(t_n x) \rightarrow 0].$$

5.2 Definition. On a linear space (X,K) a metron p is a quasi-norm (q.n.) iff

$$(e) \forall \langle t_n \rangle \in K^\omega \quad \forall t \in K \quad \forall \langle x^n \rangle \in X^\omega \quad \forall x \in X$$

$$[(|t_n - t| \rightarrow 0 \text{ and } p(x^n - x) \rightarrow 0) \Rightarrow p(t_n x^n - tx) \rightarrow 0].$$

5.3 Definition. On a linear space (X, K) a metron p is a norm (n.) iff

$$(f) \quad \forall t \in K \quad \forall x \in X [p(tx) = |t| p(x)].$$

For a norm we will sometimes write $p(x)$ as $||x||$.

5.4 Definition. On a linear space (X, K) a function from $X \times X$ to K , whose value for $x, y \in X$ is denoted (x, y) , is an inner product iff

$$\begin{aligned} \forall x, y, z \in X \quad \forall t \in K: & \quad (i) \quad (x, x) \geq 0; \quad (x, x) = 0 \Leftrightarrow x = 0 \\ & \quad (ii) \quad (y, x) = \overline{(x, y)} \quad (\text{complex conjugate}) \\ & \quad (iii) \quad (x + ty, z) = (x, z) + t (y, z). \end{aligned}$$

5.5 Definition. On a linear space (X, K) a function $p: X \rightarrow R$ is an inner product norm (i.p.n.) iff there exists an inner product, whose value for $x, y \in X$ is denoted (x, y) , such that

$$(g) \quad \forall x \in X [p(x) = (x, x)^{\frac{1}{2}}].$$

5.6 Theorem. On a given linear space:

- 1) any inner product norm is a norm;
 - 2) any norm is a quasi-norm;
 - 3) any quasi-norm is a scalar-continuous metron;
 - 4) any scalar-continuous metron is a metron;
- i.e. $\text{i.p.n.} \Rightarrow \text{n.} \Rightarrow \text{q.n.} \Rightarrow \text{s.c.m.} \Rightarrow \text{m.}$

Proof. 1) (Wilansky (1) p.121);

2) (Wilansky (1) p.56). //

5.7 Definition. We shall refer to all metrons defined on a given linear space and of a given type (m., s.c.m., q.n., n., i.p.n.) as a class of metrons.

5.8 Theorem. On any linear space (X, K) we can construct an inner product norm.

Proof. For $X = \{0\}$ the theorem is trivial (5.4, 5.5). For any other (X, K) let $\{e^\alpha\}_{\alpha \in A}$ be a Hamel basis (3.5) where α is drawn from a possibly uncountable index set A . Let the representation of elements $x, y \in X$ as linear combinations from the Hamel basis be

$$x = \sum_{\alpha \in A} x_\alpha e^\alpha, \quad y = \sum_{\beta \in A} y_\beta e^\beta. \quad \text{Then if an inner product can be con-}$$

structed on the space, we shall have the representation:

$$(x, y) = \left(\sum_{\alpha \in A} x_\alpha e^\alpha, \sum_{\beta \in A} y_\beta e^\beta \right) = \sum_{\alpha \in A} \sum_{\beta \in A} x_\alpha \bar{y}_\beta (e^\alpha, e^\beta).$$

Conversely, the form $\sum_{\alpha \in A} \sum_{\beta \in A} k_{\alpha\beta} x_\alpha \bar{y}_\beta$, where the $k_{\alpha\beta}$ are any scalars

such that $k_{\beta\alpha} = \bar{k}_{\alpha\beta}$ and such that $x \neq 0 \Rightarrow \sum_{\alpha \in A} \sum_{\beta \in A} k_{\alpha\beta} x_\alpha \bar{x}_\beta > 0$,

yields an inner product, as is readily checked.

For example, the form defined by $k_{\alpha\beta} = 1$ ($\alpha = \beta$), 0 ($\alpha \neq \beta$)
 i.e. $\forall x, y \in X [(x, y) = \sum_{\alpha \in A} x_{\alpha} \bar{y}_{\alpha}]$ is an inner product, and yields
 the inner product norm $p(x) = (\sum_{\alpha \in A} |x_{\alpha}|^2)^{\frac{1}{2}}$. //

5.9 Theorem. For a collection of norms on a given linear space

(X, K) : $B \Leftrightarrow I$. Thus $B \Leftrightarrow C \Leftrightarrow F \Leftrightarrow G \Leftrightarrow H \Leftrightarrow I$, and 4.8

reduces to the pattern of implications: $\bar{F} \Rightarrow F \Rightarrow F' \Rightarrow U$.

Proof. $B \Rightarrow I$ follows from 4.8.

To prove the converse let p_1, p_2 be norms on (X, K) and suppose $p_1 \not\equiv p_2$. Then $\forall k > 0 \exists x \in X [k p_1(x) < p_2(x)]$ holds (4.7). It follows that for each positive integer n we can find successively $k_n > 0$, $x^n \in X$, and $M_n > 0$ such that $k_n p_1(x^n) < p_2(x^n) = M_n p_1(x^n)$ where $k_1 = 1$, $M_1 > k_1$,

$$k_{n+1} = (n+1) \left(1 + \frac{M_n}{n}\right), \quad M_{n+1} > k_{n+1},$$

$$\text{i.e. } M_1 > 1, \quad \frac{M_{n+1}}{n+1} - \frac{M_n}{n} > 1.$$

For $y^n = \frac{x^n}{n p_1(x^n)}$ we have $p_1(y^n) = \frac{1}{n} \rightarrow 0$.

$$\begin{aligned} \text{But } \forall m, n: p_2(y^m - y^n) &\geq |p_2(y^m) - p_2(y^n)| \\ &= \left| \frac{p_2(x^m)}{m p_1(x^m)} - \frac{p_2(x^n)}{n p_1(x^n)} \right| \\ &= \left| \frac{M_m}{m} - \frac{M_n}{n} \right| \geq |m - n|. \end{aligned}$$

Thus $p_2(y^m - y^n) \neq 0$, and $p \nmid p_2$. //

When considering relations defined on a collection of complete metrons on a given commutative group, we already know:

$$G \Leftrightarrow H \Leftrightarrow I \quad (2.6).$$

For complete quasi-norms and complete norms on a given linear space we obtain stronger statements in virtue of the following result.

5.10 Theorem. Let X, Y be Fréchet spaces (complete quasi-normed spaces) and $f: X \rightarrow Y$ a linear map. Then f has closed graph iff f is continuous. (Wilansky (1) p.195, 200).

5.11 Theorem. For a collection of complete quasi-norms on a given linear space (X, K) : $\bar{F} \Leftrightarrow U$. Thus $\bar{F} \Leftrightarrow F \Leftrightarrow F' \Leftrightarrow U$, and 4.8 reduces to the pattern of implications portrayed in Figure 3:

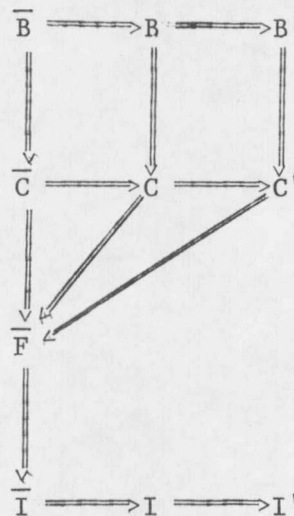


Figure 3.

Proof. Let p_1, p_2 be complete quasi-norms on (X, K) , and let

$f: (X, K, p_1) \rightarrow (X, K, p_2)$ be the identity map. Then

$$p_1 \cup p_2 \Leftrightarrow f \text{ has closed graph} \quad (2.8)$$

$$\Leftrightarrow f \text{ continuous} \quad (5.10)$$

$$\Leftrightarrow p_1 \text{ topology} \supset p_2 \text{ topology} \quad (\text{Wilansky (2) p.52})$$

$$\Leftrightarrow p_1^F p_2 \quad (2.7). \text{ But}$$

$$p_1 \cup p_2 \Leftrightarrow p_2 \cup p_1 \quad (4.7)$$

$$\Leftrightarrow p_2^F p_1 \quad (\text{as above}). \text{ Thus}$$

$$p_1 \cup p_2 \Leftrightarrow p_1^F p_2 \quad (2.1). //$$

5.12 Theorem. For a collection of complete norms on a given linear space, the twenty-five relations (4.7) become identical.

Proof. Combine 5.9 and 5.11. //

It is thus clear that the study of our twenty-five chosen relations reduces, in the case of Banach spaces (complete normed spaces), to the study of $\overline{F}$.

S6 POSITIVE ADDITIVE SPACES.

6.1 Definition. A set X is a positive additive space, $[X, R_+]$, iff there exist operations of "addition" of the elements of X , denoted for elements x and y by $x + y$, and of "multiplication" of the elements of X by the elements ("scalars") of the set R_+ of the non-negative real numbers, denoted for an element x and a scalar k by kx , such that:

- A) $\forall x, y, z \in X$: (a) $x + y \in X$
 (b) $(x + y) + z = x + (y + z)$
 (c) $x + y = y + x$
 (d) $x + y = x + z \Rightarrow y = z$
 $\exists w \in X \forall x \in X$: (e) $x + w = x$.
- B) $\forall h, k \in R_+ \forall x, y \in X$: (a) $kx \in X$
 (b) $(h + k)x = hx + kx$
 (c) $k(x + y) = kx + ky$
 (d) $h(kx) = (hk)x$
 (e) $1x = x$.

We let o denote the zero element of X referred to as w in A)(e).

(It is unique since:

$$x + w_1 = x = x + w_2 \Rightarrow w_1 = w_2.)$$

Then $\forall k \in R_+ \forall x \in X$: $kx = o \Leftrightarrow k = 0$ or $x = o$.

(For: $0x + o = 0x = (0 + 0)x = 0x + 0x \therefore o = 0x$,

$ko + o = ko = k(o + o) = ko + ko \therefore o = ko$,

$kx = o$ and $k \neq 0 \Rightarrow x = \frac{1}{k} (kx) = \frac{1}{k} (o) = o.)$

6.2 Definition. 1) On a commutative group X the zero semi metron is the function $p: X \rightarrow R$ defined by:

$$\forall x \in X [p(x) = 0].$$

2) On a linear space (X, K) the zero semi inner product is the function $f: X \times X \rightarrow K$ defined by:

$$\forall x, y \in X [f(x, y) = 0].$$

The utility of the concept of a positive additive space now appears.

6.3 Theorem. On a given linear space:

1) A class of metrons (m., s.c.m., q.n., n.), to which is adjoined the zero semi metron, forms a positive additive space over R_+ .

2) The class of inner products, to which is adjoined the zero semi inner product, forms a positive additive space over R_+ .

Proof. The statements follow easily from the definitions. //

6.4 Definition. 1) On a set X on which an operation of "addition" satisfying 6.1 A) is defined (an "additive space"), a function

$f: X \rightarrow R$ is sub-additive iff $\forall x, y \in X [f(x + y) \leq f(x) + f(y)]$;

2) On a positive additive space $[X, R_+]$ a function $f: X \rightarrow R$ is

concave iff $\forall r, t \in R_+ \forall x, y \in X$

$$[r + t = 1 \Rightarrow f(rx + ty) \geq r f(x) + t f(y)].$$

6.5 Theorem. On the positive additive space $[R_+, R_+]$ let a function $f: R_+ \rightarrow R$ be such that $f(0) = 0$. Then f concave $\Rightarrow f$ sub-additive.

Proof. Let $u, v \in R_+$. Clearly for $u = 0$ we have

$f(u + v) \leq f(u) + f(v)$. Suppose $u > 0$, and that f is concave.

Then

$$f(u) = f\left[\frac{v}{u+v} \cdot 0 + \frac{u}{u+v} \cdot (u+v)\right] \geq \frac{u}{u+v} \cdot f(u+v),$$

$$f(v) = f\left[\frac{u}{u+v} \cdot 0 + \frac{v}{u+v} \cdot (u+v)\right] \geq \frac{v}{u+v} \cdot f(u+v).$$

Thus $f(u) + f(v) \geq f(u+v)$. //

6.6 Theorem. On the positive additive space $[R_+, R_+]$ let a function

$f: R_+ \rightarrow R$ be concave and such that $\forall t \in R_+: f(t) \geq 0$;

$f(t) = 0 \Leftrightarrow t = 0$. Then for any metron p on a given linear space

(X, K) ; the function $q = f \circ p$ is a metron.

Proof. Requirements (a), (b) for a metron (4.2) are immediate.

To establish requirement (c) note that a concave, non-negative function on $[R_+, R_+]$ can never strictly decrease, for otherwise it would eventually become negative, a contradiction.

Thus $\forall x, y \in X: q(x + y) = f[p(x + y)]$

$$\begin{aligned} &\leq f[p(x) + p(y)] \\ &\leq f[p(x)] + f[p(y)] \quad (6.5) \\ &= q(x) + q(y). \quad // \end{aligned}$$

The construction of new metrons from old is facilitated by the next theorem.

6.7 Theorem. On a given linear space, starting with metrons $p, p_1, p_2, \dots$ of a given class, each of the constructions contained in column 1 of Table 1 yields a metron q of the same class and bearing the relations stated in column 2 to the original metrons, for the types of metron indicated in column 3.

	1	2	3
	Constructed metron	Relations to the original metrons	Types of metron
a	$q = t p \quad (t > 0)$	$p \bar{B} q$	m, scm, qn, n, ipn
b	$q = \frac{p}{1+p}$	$p B q, p \bar{C} q$	m, scm, qn
c	$q = p^t \quad (0 < t \leq 1)$	$p \bar{C} q$	m, scm, qn
d	$q = p_1 + p_2$ $q = (p_1^t + p_2^t)^{1/t}$ $(t \geq 1)$	$q B p_1, q B p_2$	m, scm, qn, n
e	$q = (p_1^2 + p_2^2)^{1/2}$	$q B p_1, q B p_2$	m, scm, qn, n, ipn
f	$q = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k}{1+p_k}$	$q F p_k,$ $(1 \leq k < \infty)$	m, scm, qn

Table 1.

Proof. The statements may be derived from the definitions with some use of the following results:

(a) - (c): 6.6;

(d) - (e): Minkowski inequality (Wilansky (1) p.5), 6.3 (2);

(f): Fréchet combination (Wilansky (1) p.54). //

In the preceding theorem we have not been able to construct a metron weaker than each of two arbitrary metrons. Each of the constructed metrons is stronger than any metron used in its formation. We employ one instance of this in proving the next theorem.

6.8 Theorem. On a given linear space (X, K) suppose that all metrons of a particular class (m., s.c.m., q.n., n., i.p.n.) are complete. Then all metrons of that class are uniconvergent.

Proof. Let p_1, p_2 be metrons of the given class. If $p_1 \not\sim p_2$ there is a sequence $\langle x^n \rangle \in X^\omega$ and a $a \in X$ such that $p_1(x^n) \rightarrow 0$, $p_2(x^n - a) \rightarrow 0$ but $a \neq 0$. By 6.7 (e), $q = (p_1^2 + p_2^2)^{1/2}$ is a metron of the same class as p_1 and p_2 , and stronger than p_1 and p_2 . Clearly $q(x^m - x^n) \rightarrow 0$ (1.7 (1)). But since q is complete, $q(x^n - b) \rightarrow 0$ for some $b \in X$. Then $p_1(x^n - b) \rightarrow 0$ and $p_2(x^n - b) \rightarrow 0$. Thus by 1.3, $0 = b = a$, a contradiction. //

S7 EMBEDDINGS.

7.1 Definition. Two metric spaces (X,d) and (X',d') are homeomorphic iff there exists a function $f: X \rightarrow X'$ which is

- (a) bijective,
- (b) bicontinuous (f and f^{-1} continuous).

7.2 Definition. Two metric spaces (X,d) and (X',d') are isometric iff there exists a function $f: X \rightarrow X'$ which is

- (a) bijective,
- (b) distance preserving i.e. $\forall x,y \in X: d(x,y) = d'(f(x),f(y))$.

7.3 Definition. Two metric and linear spaces (X,K,d) and (X',K,d') are linearly homeomorphic iff there exists a function $f: X \rightarrow X'$ which is

- (a) a linear isomorphism (3.9)
- (b) a homeomorphism (7.1).

7.4 Definition. Two metric and linear spaces (X,K,d) and (X',K,d') are linearly isometric, or isomorphic, iff there exists a function $f: X \rightarrow X'$ which is

- (a) a linear isomorphism (3.9),
- (b) an isometry (7.2).

7.5 Definition. We say (X, K, d) is isomorphically embedded in (Y, K, d'') iff (X, K, d) is isomorphic to (X', K, d') where $X' \subset Y$, and d' is d'' restricted to $X' \times X'$.

7.6 Theorem. Any metron and linear space (X, K, p) can always be isomorphically embedded in some metron and linear space (Y, K, q) of any higher Hamel dimension, where q is a metron of the same type (m., s.c.m., q.n., n., i.p.n.) as p .

Proof. Let (X, K, p) be given and let $\{e^\alpha\}_{\alpha \in A}$ be a Hamel basis for (X, K) . Let (Y, K) be a space of any higher Hamel dimension and let $\{f^\beta\}_{\beta \in B}$ be a Hamel basis for (Y, K) .

There exists a one-one function $g: A \rightarrow B$. Let C denote $g[A]$ (the range of g), and D denote $B \setminus C$ (the relative complement of C in B).

Let (U, K) , (V, K) be the linear manifolds of (Y, K) having Hamel bases $\{f^\beta\}_{\beta \in C}$, $\{f^\beta\}_{\beta \in D}$ respectively.

Obtain a linear isomorphism $h: (X, K) \rightarrow (U, K)$ by employing the unique representation of an element as a linear combination from

the Hamel basis as follows: for $x = \sum_{\alpha \in A} x_\alpha e^\alpha$ define

$$u = h(x) = \sum_{\alpha \in A} x_\alpha f^{g(\alpha)}.$$

Now define a metron p' on (U, K) by $p'(u) = p(x)$, and we have (X, K, p) , (U, K, p') isomorphic under h .

Finally extend p' to a metron q defined on (Y, K) in the following manner: any $y \in Y$ may be written as the unique combination

$y = u + v$ where $u \in U$, $v = \sum_{\beta \in D} v_{\beta} f^{\beta} \in V$; define

$$q(y) = \left(p'(u)^2 + \sum_{\beta \in D} |v_{\beta}|^2 \right)^{\frac{1}{2}}.$$

It is readily checked (4.2, 5.1-5.8) that q is a metron of the same type as p . //

S8 SUMMARY OF RESULTS.

8.1 Note. We consider five classes of metrons defined on a given linear space over R or C . In the following table, an entry states whether all (A), some (S) or none (N) of the members of the class specified at the left posses the property listed at the top. (S means that some do and some do not possess the property.)

The relations appearing in the heading have been selected either because of their topological importance $(\bar{F}, F', U; \text{ see } 2.7, 2.8)$, or because they represent extremes among the relations considered $(\bar{B}, I'; \text{ see } 4.8)$. The horizontal implications are as in 4.8, or consequences of 5.9. The vertical implications are as in 5.6.

The table is divided into the consideration of spaces of finite or of infinite Hamel dimension. Spaces of countably infinite and uncountable Hamel dimension need separate listing with regard to completeness but not with regard to relational behavior.

Since a metron and linear space can always be isomorphically embedded in some space of the same metron type and of any higher Hamel dimension (7.6), counter-examples to relational behavior valid in one space are valid for some space of the same metron type and of any higher Hamel dimension. Thus a statement made for relational behavior can change only from A to S as the dimension increases. (N does not appear for relations since all the relations are reflexive.)

Similarly, on a given linear space a statement made for relational behavior can change only from S to A as the relation is enlarged; and a statement made for relational behavior or completeness can change only from A to S or from N to S as the class of metrons is enlarged.

It is thus clear that the statements underlined in the table control the remaining entries. The rest of this thesis is devoted to a study of these underlined statements.

8.2 Table 2. PROPERTIES OF CLASSES OF METRONS DEFINED ON A GIVEN
LINEAR SPACE OVER $\mathbb{R}$ OR $\mathbb{C}$.

	CLASS OF METRONS	COMPLETE		U	I' $\longleftrightarrow$ F' $\longleftrightarrow$ $\bar{F}$ $\longleftrightarrow$ $\bar{B}$			
HAMEL DIMENSION FINITE	1) m. $\uparrow\uparrow$	S		<u>S</u>	S	S	S	S
	2) s.c.m. $\uparrow\uparrow$	<u>S</u>			<u>S</u>	S	S	S
	3) q.n. $\uparrow\uparrow$	<u>A</u>		A	A	A	<u>A</u>	<u>S</u>
	4) n. $\uparrow\uparrow$	A		A	A $\longleftrightarrow$ A		A $\longleftrightarrow$ A	
	5) i.p.n. $\uparrow\uparrow$	A		A	A $\longleftrightarrow$ A		A $\longleftrightarrow$ A	
HAMEL DIMENSION INFINITE	1) m. $\uparrow\uparrow$	<u>S</u>	S_0	S	S	S	S	S
	2) s.c.m. $\uparrow\uparrow$		$>S_0$	S	S	S	S	S
	3) q.n. $\uparrow\uparrow$	<u>N</u>		S	S	S	S	S
	4) n. $\uparrow\uparrow$	N		S	S $\longleftrightarrow$ S		S $\longleftrightarrow$ S	
	5) i.p.n. $\uparrow\uparrow$	N		<u>S</u>	S $\longleftrightarrow$ S		S $\longleftrightarrow$ S	

CHAPTER TWO: FINITE-DIMENSIONAL LINEAR SPACES.

In this chapter we establish the key statements (underlined) in the upper half of 1.8.2 Table. Wherever a counterexample is required we construct one for the simplest possible case, that of the one-dimensional space $(\mathbb{R}, \mathbb{R})$.

We recall first some results from the theory of normed spaces.

S1 NORMS.

1.1 Theorem. In any linear space bearing a norm $|| \cdot ||$ let $\{e^k\}_{k=1}^n$ be any finite, linearly independent set of vectors. Then there exists $\delta > 0$ such that for every choice of scalars $\{t_k\}_{k=1}^n$

$$|| \sum_{k=1}^n t_k e^k || \geq \delta \sum_{k=1}^n |t_k|. \quad (\text{Epstein p.82}).$$

1.2 Definition. On any linear space (X, K) , the α th coordinate functional on (X, K) relative to Hamel basis $\{e^\beta\}_{\beta \in B}$ is the function

$f_\alpha: X \rightarrow K$ defined for $\alpha \in B$ by: $\forall x \in X: f_\alpha(x) = t_\alpha$ where

$$x = \sum_{\beta \in B} t_\beta e^\beta.$$

Note that each coordinate functional is linear.

1.3 Theorem. On any n -dimensional normed space (X, K) each coordinate functional is continuous.

Proof. Let $\{e^k\}_{k=1}^n$ be any Hamel basis for the space. For

$\langle x^m \rangle \in X^\omega$, let $x^m = \sum_{k=1}^n t_k^m e^k$. Then

$$\|x^m\| \rightarrow 0 \Rightarrow \sum_{k=1}^n |t_k^m| \rightarrow 0 \quad (1.1)$$

$$\Rightarrow \forall k [|t_k^m| \rightarrow 0]. //$$

1.4 Theorem. Let $(X, K, || \cdot ||)$ be an n -dimensional normed space and $(X', K, || \cdot ||')$ be any normed space over the same field K . Then any linear map $f: X \rightarrow X'$ maps bounded sets onto bounded sets; indeed

$$\exists h > 0 \quad \forall x \in X [h \|x\| \geq \|f(x)\|'].$$

Proof. Let $\{e^k\}_{k=1}^n$ be a Hamel basis for (X, K) . For $x = \sum_{k=1}^n x_k e^k$,

$$\|f(x)\|' = \left\| \sum_{k=1}^n x_k f(e^k) \right\|'$$

$$\leq \sum_{k=1}^n |x_k| \|f(e^k)\|' \quad (\text{triangle inequality})$$

$$\leq M \sum_{k=1}^n |x_k| \quad (\text{where } M = \max_{1 \leq k \leq n} \|f(e^k)\|')$$

$$\leq (M/\delta) \|x\| \quad (\delta \text{ as in 1.1}).$$

Take $h = (M/\delta)$. //

1.5 Theorem. All n -dimensional normed spaces over K are linearly homeomorphic. (1.7.3).

Proof. In 1.4 let (X, K) and (X', K) each have Hamel dimension n , and let $f: X \rightarrow X'$ be any linear isomorphism (1.3.10). Then

$$\|x\| \rightarrow 0 \Rightarrow \|f(x)\|' \rightarrow 0, \text{ and conversely. } //$$

1.6 Corollary. On a given n -dimensional linear space all norms are equivalent.

1.7 Corollary. On a given n -dimensional linear space all norms are complete.

Proof. Let (X, K, p) be any n -dimensional normed space. Let $\{e^k\}_{k=1}^n$ be a Hamel basis for (X, K) , and define a norm $\|\cdot\|$ on (X, K) by:

$$\forall x \in X: \|x\| = \left(\sum_{k=1}^n |x_k|^2 \right)^{1/2} \text{ where } x = \sum_{k=1}^n x_k e^k \quad (1.5.8).$$

Then $(X, K, \|\cdot\|)$ is isomorphic (1.7.4) to K^n , and thus complete.

But on any metron space, $F \Rightarrow H$ (1.4.8), and thus (X, K, p) is complete. //

1.8 Corollary. On a given n -dimensional normed space, a subset is compact iff closed and bounded.

Proof. Use 1.4, 1.5, and the Heine-Borel theorem on K^n

(Bartle p.85). //

S2 METRONS.

2.1 Counterexample. On (R, R) there exist two non-uniconvergent metrons (Existence proof).

Proof. A Hamel basis for (R, Q) is uncountable. (If it were countable the set L_n of all rational linear combinations of n elements from the basis would be countable; then

$$R = \bigcup_{n=1}^{\infty} L_n \text{ would be countable, a contradiction.)}$$

Take any countable subset of any Hamel basis for (R, Q) .

Index and adjust the elements by multiplying by suitable rationals so that we have a sequence $S = \langle e^k \rangle_{k=1}^{\infty}$ with the property $|e^k - 1| \rightarrow 0$. The union of S with the remainder of the basis is a Hamel basis $\{e^{\alpha}\}$.

For $x \in R$ we have the representation $x = \sum x_{\alpha} e^{\alpha}$. Define $p: R \rightarrow R$ by requiring that $\forall x \in R$:

$$p(x) = \sum |x_{\alpha}| p(e^{\alpha}) \text{ where } p(e^{\alpha}) = \begin{cases} \frac{1}{k} & (e^{\alpha} = e^k \in S) \\ 1 & (e^{\alpha} \notin S). \end{cases}$$

Then $\forall x, y \in R$: (a) $0 \leq p(x) < \infty$; $p(x) = 0 \Leftrightarrow x = 0$

(by the uniqueness of basis representation).

(b) $p(-x) = p(x)$.

(c) $p(x + y) \leq p(x) + p(y)$.

(For $x = \sum x_\alpha e^\alpha$, $y = \sum y_\alpha e^\alpha$ we have:

$$\begin{aligned} p(x + y) &= p\left(\sum (x_\alpha + y_\alpha) e^\alpha\right) = \sum |x_\alpha + y_\alpha| p(e^\alpha) \\ &\leq \sum |x_\alpha| p(e^\alpha) + \sum |y_\alpha| p(e^\alpha) \\ &= p(x) + p(y). \end{aligned}$$

Thus p is a metron. Then defining q such that $\forall x \in R$: $q(x) = |x|$, we have $p(e^k - 0) \rightarrow 0$ and $q(e^k - 1) \rightarrow 0$. Thus $p \not\equiv q$. //

Note p is not a s.c.m., for $|e^k - 1| \rightarrow 0$ but $p(e^k - 1) \rightarrow p(1) \neq 0$.

S3 SCALAR-CONTINUOUS METRONS.

We present two theorems on the behavior of scalar-continuous metrons.

3.1 Theorem. Let any linear space (X, K) bear a metron q . For a metron $p: (X, K, q) \rightarrow R$ consider the statements:

- (1) p is continuous;
- (2) p is a scalar-continuous metron.

Then (a) if q is a s.c.m., (1) $\Rightarrow$ (2);

(b) if q is a norm and (X, K) is finite-dimensional,

(1) $\Leftrightarrow$ (2).

Proof. (a) Let q be s.c.m. and suppose (1). Then

$$\forall a \in X \quad \forall \langle t_n \rangle \in K^\omega: |t_n| \rightarrow 0 \Rightarrow q(t_n a) \rightarrow 0 \Rightarrow p(t_n a) \rightarrow 0.$$

Thus (1) $\Rightarrow$ (2).

(b) Let q be a norm, (X, K) have Hamel basis $\{e^k\}_{k=1}^s$, and

suppose (2). Then for $x^n = \sum_{k=1}^s t_k^n e^k$ we have

$$\forall k (1 \leq k \leq s): q(x^n) \rightarrow 0 \Rightarrow |t_k^n| \rightarrow 0 \quad (1.1)$$

$$\Rightarrow p(t_k^n e^k) \rightarrow 0 \text{ (by def. of s.c.m.)}.$$

$$\text{But } p(x^n) \leq \sum_{k=1}^s p(t_k^n e^k) \quad \text{(by triangle inequality).}$$

Thus $q(x^n) \rightarrow 0 \Rightarrow p(x^n) \rightarrow 0$, and we have p continuous at o and so, by 1.4.6, continuous everywhere. Thus (2) $\Rightarrow$ (1). //

3.2 Theorem. Let a finite-dimensional linear space (X, K) bear a scalar-continuous metron p and a norm $|| \cdot ||$. Then $\forall \langle x^n \rangle \in X^\omega$: $p(x^n) \rightarrow 0 \Rightarrow \forall m, M (0 < m \leq M) \exists N \forall n > N [||x^n|| < m \text{ or } M < ||x^n||]$.

Proof. Suppose that for some $\langle x^n \rangle \in X^\omega$ we have $p(x^n) \rightarrow 0$ while for some m, M we have $0 < m \leq ||x^n|| \leq M$ for an infinite set of values of n .

Then, since $\{x | 0 < m \leq ||x|| \leq M\}$ is compact (1.8), $\langle x^n \rangle$ will possess a subsequence $\langle x^{n_i} \rangle$ such that $||x^{n_i} - a|| \rightarrow 0$, where $0 < m \leq ||a|| \leq M$.

Then $p(x^{n_i} - a) \rightarrow 0$ (3.1 (b)).

But $p(x^{n_i}) \rightarrow 0$. Thus $a = o$ (1.1.3), a contradiction. //

The statement 3.2 Theorem suggests that it might be possible to exhibit a sequence on a finite-dimensional space that converges to zero in s.c.m. but diverges to infinity in norm. We show that this surprising behavior does in fact occur, and thus that not all s.c.m. on a finite-dimensional space are equivalent (3.5 Counterexample). Indeed, not all s.c.m. on a finite-dimensional space are comparable (3.6 Counterexample).

First we need two results from number theory.

3.3 Lemma. If x is a real number and n is a positive integer, there exist integers k and h such that:

$$|kx - h| \leq 1/(n+1) \quad \text{where } 1 \leq k \leq n \quad (\text{LeVeque p.125}).$$

3.4 Corollary. If x is an irrational number, y a real number, and ϵ any positive real number, there exist integers k and h such that:

$$|(kx - h) - y| \leq \epsilon \quad \text{where } 1 \leq |k|.$$

Proof: If $y = 0$ the statement follows from 3.3 Lemma. If $y \neq 0$ use 3.3 Lemma to find integers p and q such that:

$$|px - q| \leq 1/(n+1) \quad \text{where } 1/(n+1) \leq |y| \quad \text{and } 1 \leq p \leq n. \quad \text{Let}$$

$$|px - q| = \delta \quad \text{where } 0 < \delta \leq 1/(n+1) \leq |y|. \quad \text{Then if } M \text{ denote}$$

the greatest integer $\leq |y|/\delta$ we have $1 \leq M$. Then

$$0 \leq |y| - M\delta < \delta \leq 1/(n+1) \quad \text{i.e. } \left| |y| - M|px - q| \right| < 1/(n+1).$$

Thus if $|k| = Mp$ and $|h| = M|q|$ we can choose the signs of k and

h so that: $|(kx - h) - y| < 1/(n+1) \quad \text{where } 1 \leq M \leq |k| \leq Mn.$

And if we choose n so that $1/(n+1) \leq \min\{|y|, \epsilon\}$ we have

$$|(kx - h) - y| < \epsilon \quad \text{where } 1 \leq |k|. \quad //$$

3.5 Counterexample. We show two s.c.m. on a one-dimensional space that are comparable but non-equivalent.

Proof. Consider on $(\mathbb{R}, \mathbb{R})$: $p(x) = |\sin x| + |\sin \pi x|$
 $||x|| = |x|.$

Now $p(x) = 0 \Leftrightarrow x = k\pi$ and $x = h$ (k, h integers)

$\Leftrightarrow x = h = k = 0$ (since π irrational).

And it is readily checked that the remaining requirements for s.c.m. (1.4.2, 1.5.1) are also satisfied by p .

Since $\forall x: |\sin x| \leq |x|$, $|x - k\pi| < \varepsilon \Rightarrow |\sin x| = |\sin(x - k\pi)| < \varepsilon$.

For each positive integer n use 3.3 Lemma to find integers k_n

and h_n such that $|k_n\pi - h_n| < 1/(n+1)$ where $1 \leq k_n$. Then taking

$x^n = h_n$ we have: $p(x^n) = |\sin h_n| + |\sin \pi h_n| < 1/(n+1)$. Then

$p(x^n) \rightarrow 0$ while clearly $||x^n|| = |h_n| \rightarrow \infty$. //

Note further in this counterexample, that every neighborhood of the origin in the p topology is unbounded in norm $|| \cdot ||$.

3.6 Counterexample. We modify 3.5 Counterexample to show two s.c.m. on $(\mathbb{R}, \mathbb{R})$ that are non-comparable. Indeed, we show more; we exhibit two s.c.m., p_1 and p_2 , such that a sequence convergent in either is not necessarily Cauchy in the other, $p_1 \not\sim p_2$.

Proof. Let $p_1(x) = |\sin(x/2)| + |\sin(\pi x/2)|$

$$p_2(x) = |\sin(x/3)| + |\sin(\pi x/3)|.$$

For each positive integer n use 3.4 Corollary to find integers k_n , h_n such that:

$$|3k_n\pi - (3h_n + 1)| < 1/n \text{ i.e. } |(k_n\pi - h_n) - 1/3| < 1/3n \text{ where } 1 \leq |k_n|.$$

$$\text{Then taking } x^n = \begin{cases} 0 & (n \text{ odd}) \\ 2(3h_n + 1) & (n \text{ even}), \end{cases}$$

$$\text{we have } p_1(x^n) = \begin{cases} 0 & (n \text{ odd}) \\ |\sin(3h_n + 1)| < 1/n & (n \text{ even}) \end{cases}$$

and $p_1(x^n) \rightarrow 0$. But for m odd and n even,

$$\begin{aligned} p_2(x^m - x^n) &= \left| \sin\left[\frac{2(3h_n + 1)}{3}\right] \right| + \left| \sin\left[\frac{2\pi(3h_n + 1)}{3}\right] \right| \\ &\geq \left| \sin\left[\frac{2\pi}{3}\right] \right| = \frac{\sqrt{3}}{2}. \end{aligned}$$

Thus $\langle x^n \rangle$ is not Cauchy in p_2 .

Similarly, for each positive integer n find integers r_n, s_n such that:

$$|2r_n\pi - (2s_n + 1)| < 1/n \text{ where } 1 \leq |r_n|.$$

$$\text{Then taking } y^n = \begin{cases} 0 & (n \text{ odd}) \\ 3(2s_n + 1) & (n \text{ even}), \end{cases}$$

we have $p_2(y^n) \rightarrow 0$. But for m odd and n even, $p_1(x^m - x^n) \geq 1$.

Thus $\langle y^n \rangle$ is not Cauchy in p_1 . //

3.7 Counterexample. We show that p in 3.5 is not complete.

Proof. Let $p(x) = |\sin x| + |\sin \pi x|$. For each positive integer n use 3.4 Corollary to find integers k_n, h_n such that:

$$|(k_n\pi - h_n) - 1/2| < 1/n \text{ where } 1 \leq |k_n|. \text{ Then taking } x^n = h_n \text{ we have: } p(x^m - x^n) = |\sin(h_m - h_n)| < \frac{1}{m} + \frac{1}{n} \rightarrow 0.$$

Thus $\langle x^n \rangle$ is Cauchy in p . However, for any fixed $c \in \mathbb{R}$,

$$p(x^n - c) = |\sin(h_n - c)| + |\sin(\pi c)|$$

$$\rightarrow |\sin(c + 1/2)| + |\sin(\pi c)|. \text{ If the limiting value were}$$

zero we should have for some integers r, s : $c = r\pi - 1/2 = s$, a

contradiction since π is irrational. Thus $p(x^n - c) \neq 0$ for any c ,

and p is not complete. //

S4 QUASI-NORMS.

4.1 Counterexample. We show two quasi-norms, p_1 and p_2 , on $(\mathbb{R}, \mathbb{R})$ such that $p_1 \not\sim p_2$, indeed $p_1 \not\sim p_2$.

Proof. Let $p_1(x) = |x|$,

$$p_2(x) = |x|^{\frac{1}{2}},$$

Then $|x| \rightarrow 0 \Rightarrow \frac{p_2(x)}{p_1(x)} \rightarrow +\infty$, thus $p_1 \not\sim p_2$;

$$|x| \rightarrow \infty \Rightarrow \frac{p_1(x)}{p_2(x)} \rightarrow +\infty, \text{ thus } p_2 \not\sim p_1. //$$

4.2 Theorem. On a given finite-dimensional linear space all quasi-norms are equivalent.

Proof. On a finite-dimensional linear space (X, K) let p be a quasi-norm and $|| \cdot ||$ a norm. Then 3.1 (b) shows that

$||x|| \rightarrow 0 \Rightarrow p(x) \rightarrow 0$ i.e. $|| \cdot || \ll p$. Now suppose $\langle x^n \rangle \in X^\omega$ is such that $p(x^n) \rightarrow 0$ but $||x^n|| \not\rightarrow 0$. Then, by 3.2, $\langle x^n \rangle$ contains a subsequence $\langle y^n \rangle$ such that $||y^n|| \rightarrow \infty$. Take $t_n = 1/||y^n||$.

Then $|t_n| \rightarrow 0$, $p(y^n) \rightarrow 0$, and so $p(t_n y^n) = p(y^n/||y^n||) \rightarrow 0$ (1.5.2).

But $||y^n/||y^n|||| = 1$, and $\{x \mid ||x|| = 1\}$ is compact in $|| \cdot ||$

(1.8). Thus $\langle y^n/||y^n|| \rangle$ contains a subsequence $\langle z^n \rangle$ such that

$||z^n - c|| \rightarrow 0$ where $||c|| = 1$. But then $p(z^n - c) \rightarrow 0$ (since

$|| \cdot || \ll p$), and, by 1.1.3, $c = 0$, a contradiction. Thus $p \ll || \cdot ||$. //

4.3 Theorem. On a given finite-dimensional linear space all quasi-norms are complete.

Proof. On a given finite-dimensional linear space (X, K) let p be a quasi-norm and $|| \cdot ||$ a norm.

$$\forall \langle x^n \rangle \in X^\omega \exists c \in X: p(x^m - x^n) \rightarrow 0 \Leftrightarrow ||x^m - x^n|| \rightarrow 0 \quad (4.2, 1.4.7)$$

$$\Leftrightarrow ||x^n - c|| \rightarrow 0 \quad (1.7)$$

$$\Leftrightarrow p(x^n - c) \rightarrow 0 \quad (4.2). //$$

We now show that, on a finite-dimensional space, the quasi-norms are precisely those scalar-continuous metrons such that the coordinate functionals (1.2) are continuous.

4.4 Theorem. Let (X, K, p) be a finite-dimensional space bearing a scalar-continuous metron p . Consider the statements:

- (1) p is a quasi-norm;
- (2) for every Hamel basis for (X, K) each coordinate-functional is continuous;
- (3) there exists a Hamel basis for (X, K) such that each coordinate functional is continuous.

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

Proof. (1) $\Rightarrow$ (2): Let $|| \cdot ||$ be a norm on (X, K) . Then, relative to any Hamel basis for (X, K) , each coordinate functional on $(X, K, || \cdot ||)$ is continuous (1.3), and thus each coordinate functional on (X, K, p) is continuous (4.2, 1.2.7).

(2) $\Rightarrow$ (3): Immediate (1.3.6 (1)).

(3) $\Rightarrow$ (1): Suppose each coordinate functional on (X, K, p) relative to Hamel basis $\{e^k\}_{k=1}^h$ is continuous.

For $\langle x^n \rangle \in X^\omega$, $x \in X$, $\langle s^n \rangle \in K^\omega$, $s \in K$ we have the representations (with the summation on k): $x = \sum_k t_k e^k$, $x^n = \sum_k t_k^n e^k$,

$$s_n x^n - s x = \sum (s_n - s)(t_k^n - t_k) e^k + \sum (s_n - s) t_k e^k + \sum s(t_k^n - t_k) e^k.$$

$$\begin{aligned} \text{Then } p(s_n x^n - s x) &\leq \sum p[(s_n - s)(t_k^n - t_k) e^k] + \sum p[(s_n - s) t_k e^k] \\ &\quad + \sum p[s(t_k^n - t_k) e^k]. \end{aligned}$$

Now suppose $p(x^n - x) \rightarrow 0$, $|s_n - s| \rightarrow 0$. Then

$\forall k [|t_k^n - t_k| \rightarrow 0]$ (by continuity of coordinate functionals), and

$p(s_n x^n - s x) \rightarrow 0$ (1.5.1). Thus p is a quasi-norm (1.5.2). //

CHAPTER THREE: INFINITE-DIMENSIONAL LINEAR SPACES.

In this chapter we establish the key statements (underlined) in the lower half of 1.8.2 Table. Wherever a counterexample is required we construct one for the simplest possible case, that of the space $(\mathbb{R}^\infty, R)$ of countably-infinite Hamel dimension (1.3.11).

S1 METRONS.

1.1 Example. The discrete metron (1.4.4) on a linear space of any Hamel dimension is complete.

Proof. Immediate (1.1.5). //

S2 QUASI-NORMS.

2.1 Theorem. No quasi-normed space of countably-infinite Hamel dimension is complete (Wilansky (1) p. 205 Cor.6).

S3 INNER PRODUCT NORMS.

3.1 Counterexample. We show two inner product norms, p_1 and p_2 , on $(\mathbb{R}^\infty, R)$ that are not uniconvergent, $p_1 \not\sim p_2$.

Proof. Let (R^∞, R) have the Hamel basis $\{\delta^k\}_{k=1}^\infty$. (1.3.11). For

$x, y \in R^\infty$ we have the representations $x = \sum_k x_k \delta^k$, $y = \sum_k y_k \delta^k$.

$$\text{Define } (x, y)_1 = \sum_{k=1}^{\infty} \frac{x_k y_k}{k^2}; \quad p_1(x) = \left(\sum_{k=1}^{\infty} \frac{x_k^2}{k^2} \right)^{1/2};$$

$$(x, y)_2 = \sum_{k=2}^{\infty} \frac{x_k y_k}{k^2} + \left(\sum_{k=1}^{\infty} x_k \right) \left(\sum_{k=1}^{\infty} y_k \right);$$

$$p_2(x) = \left(\sum_{k=2}^{\infty} \frac{x_k^2}{k^2} + \left(\sum_{k=1}^{\infty} x_k \right)^2 \right)^{1/2}.$$

$$\text{Then } p_1(\delta^n) = 1/n, \quad p_2(\delta^n - \delta^1) = \begin{cases} 0 & (n = 1) \\ 1/n & (n \geq 2) \end{cases}$$

Thus $p_1(\delta^n - o) \rightarrow 0$, $p_2(\delta^n - \delta^1) \rightarrow 0$, but $o \neq \delta^1$. //

3.2 Example. A space of uncountable Hamel dimension bearing a complete inner product norm: ℓ^2 (Wilansky (1) p.77).

3.3 Counterexample. A space of uncountable Hamel dimension bearing an incomplete inner product norm.

Proof. Let X be the set of elements of ℓ^1 (Wilansky (1) p. 289), and let p be the norm of ℓ^2 . Then (X, R, p) is isomorphically embedded in ℓ^2 .

Let $\{\delta^k\}_{k=1}^\infty$ be as in 1.3.11.

Consider $\langle x^n \rangle \in X^\omega$ defined by: $\forall n [x^n = \sum_{k=1}^n (\delta^k/k)]$. Then

$\langle x^n \rangle$ is Cauchy in p , but $p(x^n - x) \rightarrow 0$ where

$x = \langle x_1, x_2, \dots, x_k, \dots \rangle$ with $\forall k [x_k = 1/k]$. But $x \in \ell^2$, $\notin \ell^1$,

and thus (X, R, p) is not complete. //

CHAPTER FOUR: REVIEW AND FURTHER PROBLEMS.

S1 REVIEW.

On looking back, certain results that we have obtained appear particularly interesting and required some ingenuity for their derivation. We would draw attention to:

Chapter One : The coincidence of relations B, C, D, E, F, G, H, I, when dealing with norms on a given linear space (1.5.9).

Chapter Two : The existence of two non-uniconvergent metrons on (R, R) (2.2.1). The results for scalar-continuous metrons on finite-dimensional spaces (2.3.1 - 2.3.7)

Chapter Three: The counterexample showing two non-uniconvergent inner product norms on (R^{∞}, R) (3.3.1).

S2 FURTHER PROBLEMS

- 1) Are there characterizations of a topological nature for all the relations introduced in 1.2.3 (c.f. 2.7, 2.8)?
- 2) What other relations between metrics might be significant?
- 3) What other metrons might be significant?
- 4) What relations coincide in a class of metrons (c.f. 1.4.7, 1.5.9)?
- 5) Develop the structure of positive additive spaces.
- 6) Can a metron weaker than each of two arbitrary metrons be constructed?
- 7) Is there a class of metrons such that all are comparable but not all are equivalent?
- 8) Can a concrete example of two non-uniconvergent metrons on $(\mathbb{R}, \mathbb{R})$ be exhibited?
- 9) Do two non-uniconvergent s.c.m. on a finite dimensional space exist?
- 10) What is the completion of a finite dimensional space bearing an incomplete s.c.m.?
- 11) Does a complete s.c.m. on a space of countably-infinite Hamel dimension exist?
- 12) Consider convergence in an embedded space (Mikhlin p.14).
- 13) What applications exist for s.c.m. in applied mathematics?

APPENDIX A: NOTATION

$p \Rightarrow q$	p only if q
$p \Leftrightarrow q$, p iff q	p if and only if q
$\forall x$	for each x
$\exists x$	there exists an x such that
$=$	logical identity, equality
$x \in X$	x is an element of X
$x \notin X$	x is not an element of X
$\{x, y, \dots\}$	the set containing $x, y, \dots$
$\{x \mid \dots\}$	the set of all x such that...
$\emptyset$	the empty set
$X \subset Y$, $Y \supset X$	X is a subset of Y
$\cup$	union of sets
$\cap$	intersection of sets
$Y \setminus X$	complement of X in Y
$\{x^\alpha\}_{\alpha \in A}$, $\{x^\alpha\}$	the set of all x^α for $\alpha \in$ index set A
$\{x^n\}_{n=1}^\infty$, $\{x^n\}$	set of all x^n for n a positive integer
Q	the rational numbers
R	the real numbers
C	the complex numbers
ω	the natural numbers, non-negative integers
$\aleph_0$	ω , the first infinite cardinal

$x < y$	real number x is less than real number y
$x \leq y$	real number x is less than or equal to real number y
$ z $	absolute value of complex number z
$\overline{z}$	conjugate of complex number z
$\langle x, y \rangle$	ordered pair of elements x and y
$X \times Y$	Cartesian product of X and Y
$\langle x^k \rangle_{k=1}^n$	
$\langle x_1, x_2, \dots, x_n \rangle$	ordered n -tuple
R^n	set of real ordered n -tuples
C^n	set of complex ordered n -tuples
K^n	set of field K ordered n -tuples
R^∞	set of all real finite sequences
δ^k	(1.3.11)
$u A v$	$\langle u, v \rangle \in \text{relation } A$
$u \not A v$	$\langle u, v \rangle \notin \text{relation } A$
A^{-1}	inverse of relation A
$\overline{A}$	(1.2.1)
A'	(1.2.1)
$f: X \rightarrow Y$	f is a function on X into Y
$f[S]$	$\{f(x) \mid x \in S\}$
Y^X	set of all functions on X into Y

f^{-1}	inverse function of function f
$f \circ g$	composite of functions g and f
$\langle x^n \rangle_{n=1}, \langle x^n \rangle$	sequence
$x^n \rightarrow x$	sequence $\langle x^n \rangle$ converges to x
$x^n \not\rightarrow x$	sequence $\langle x^n \rangle$ does not converge to x
(X, K)	linear space X over field K
$(X, d), (X, p)$	set X bearing metric d , metron p
$(X, K, d), (X, K, p)$	linear space (X, K) bearing metric d , metron p .
$[X, R_+]$	positive additive space X (1.6.1)
0	zero of real and complex numbers.
o	zero of linear space
$\ x\ $	norm of x
(x, y)	inner product of x and y
f_α	α th coordinate functional (2.1.2)
ℓ^1	set of absolutely summable complex sequences
ℓ^2	set of square summable complex sequences.
//	end of proof

APPENDIX B: SET THEORY

Only a modest background in set theory is needed to comprehend this thesis. In fact, acquaintance with Boolean algebra, relations, functions, products, orderings, the axiom of choice, and cardinal numbers, as in commonly obtained from elementary or intermediate courses will suffice.

For a deeper understanding a grasp of axiomatic set theory is desirable. The Mostowski-Kelley-Morse system is presented in (Kelley) and (Monk). The several axiomatic systems are compared in (Hatcher) and (Fraenkel, Bar-Hillel, Levy).

APPENDIX C: TOPOLOGY

In this appendix we outline those elements of topology that are needed for comprehension of the thesis. Detailed expositions may be found in (Dugundji) and (Wilansky (1) and (2)).

S1 TOPOLOGICAL SPACE.

1.1 Definition. Let X be a set. A collection T of subsets of X is a topology for X iff

- (a) $\emptyset \in T, X \in T$;
- (b) $\forall G_1, G_2 \in T [G_1 \cap G_2 \in T]$;
- (c) $\forall S \subset T [\cup\{G \mid G \in S\} \in T]$.

1.2 Definition. A set X is a topological space, (X, T) , iff a topology T is given for X . The members of T are called open sets.

1.3 Definition. Given (X, T) , a set $F \subset X$ is closed iff $X \setminus F$ is open.

1.4 Definition. Given (X, T) , a family $B \subset T$ is a base for T iff each member of T is the (arbitrary) union of members of B .

1.5 Definition. Given (X, T) , a family $S \subset T$ is a sub-base for T iff there exists some base B for T such that every member of B is the finite intersection of members of S .

1.6 Theorem. Given any collection S of subsets of X , there exists a unique topology T for X such that S is a sub-base for T . T is then called the topology generated by S . (Dugundji p. 65).

1.7 Definition. Given (X, T) , $x \in X$, a set $N \subset X$ is:

a neighborhood (nbd.) of x iff $\exists G \in T [x \in G \subset N]$;

a deleted neighborhood of x iff $x \notin N$ and $N \cup \{x\}$ is a neighborhood of x .

1.8 Theorem. A set is open iff it is a neighborhood of each of its points (Wilansky (2) p.18).

1.9 Definition. Given (X, T) , $x \in X$, a collection L of subsets of X is a neighborhood base at x iff each member of L is a neighborhood of x and for every neighborhood N of x there exists a set $M \in L$ with $M \subset N$.

1.10 Theorem. (1) The intersection of any family of closed sets is a closed set.

(2) The union of finitely many closed sets is a closed set.

(Dugundji p.69).

1.11 Definition. Given (X, T) , $A \subset X$, a point $x \in X$ is a closure point of A iff each neighborhood of x meets A . The set of all closure points of A is called the closure, $\bar{A}$, of A .

1.12 Theorem. $\bar{A}$ is the intersection of all closed sets F such that $A \subset F$. (Dugundji p.70).

1.13 Definition. Given (X, T) , $A \subset X$, a point $x \in X$ is a cluster point of A iff each deleted neighborhood of x meets A . Thus every cluster point of A is a closure point of A .

1.14 Theorem. A set is closed iff it contains all its cluster points. (Dugundji p.71).

1.15 Definition. Given (X, T) , $A \subset X$, a set $D \subset X$ is dense in A iff $A \subset \bar{D}$.

1.16 Definition. Given topologies T, T' for a set X we say T is stronger than T' , T' is weaker than T , iff $T \supset T'$.

1.17 Theorem. Let T, T' be topologies for a set X . Then $T \supset T'$ iff for every subset A of X it is true that the closure of A in $T \subset$ the closure of A in T' . (Wilansky (2) p.19).

1.18 Definition. Given (X, T) , $Y \subset X$. The relative topology T_Y for Y is $\{Y \cap G \mid G \in T\}$. It is readily checked that T_Y is a topology for Y .

S2 CONVERGENCE.

2.1 Definition. A set D is a directed set with respect to a relation $\leq$ on D , $[D, \leq]$, iff $\leq$ is

- (a) reflexive on D ,
- (b) transitive,
- (c) directive on D i.e.

$\forall \delta_1, \delta_2 \in D \exists \delta_3 \in D [\delta_1 \leq \delta_3 \text{ and } \delta_2 \leq \delta_3]$. We write $\delta_1 \geq \delta_2$ iff $\delta_2 \leq \delta_1$.

2.2 Definition. Given sets D and X , a function $f: D \rightarrow X$ is a net in X , (more fully, a net from D to X), iff D is a directed set.

For $\delta \in D$ we denote $f(\delta)$ by x^δ ; and we denote a net from D to X by $\langle x^\delta \rangle \in X^D$.

Thus a sequence in X is a net from ω to X .

For any $\delta_0 \in D$ we may call $\{\delta \mid \delta \geq \delta_0\}$ a tail of D .

2.3 Definition. Given $A \subset X$ and a directed set $[D, \leq]$, we say a net $\langle x^\delta \rangle \in X^D$ is:

- (1) eventually in A iff $\exists \delta_0 \in D \forall \delta \geq \delta_0 [x^\delta \in A]$;
- (2) frequently in A iff $\forall \delta_0 \in D \exists \delta \geq \delta_0 [x^\delta \in A]$.

2.4 Theorem. Given a net $\langle x^\delta \rangle \in X^D$, let $A_1, A_2, \dots, A_n$ be finitely many subsets of X and suppose that for each $k = 1, 2, \dots, n$, $\langle x^\delta \rangle$ is in A_k eventually. Then $\langle x^\delta \rangle$ is in $\bigcap_{k=1}^n A_k$ eventually. (Wilansky (2) p.39).

2.5 Definition. Given (X, T) , a net $\langle x^\delta \rangle \in X^D$ and $x \in X$, we say that $\langle x^\delta \rangle$ converges to x in T , in symbols $x^\delta \rightarrow x$ in T , iff for every neighborhood N of x , $\langle x^\delta \rangle$ is eventually in N .

We then say that x is a limit point of $\langle x^\delta \rangle$.

2.6 Definition. Given (X, T) , a net $\langle x^\delta \rangle \in X^D$ and $x \in X$, we say that x is an accumulation point of $\langle x^\delta \rangle$ iff for every neighborhood N of x , $\langle x^\delta \rangle$ is frequently in N .

Thus every limit point of $\langle x^\delta \rangle$ is an accumulation point of $\langle x^\delta \rangle$.

2.7 Theorem. Given (X, T) , $A \subset X$ and $x \in X$, then $x \in \bar{A}$ iff there exists a net in A converging to x . (Wilansky (2) p.40).

2.8 Theorem. 1) Let T and T' be topologies for a set X . Then

$$T \supset T' \Leftrightarrow \forall \text{ directed set } D \quad \forall \langle x^\delta \rangle \in X^D \quad \forall x \in X$$

$$[x^\delta \rightarrow x \text{ in } T \Rightarrow x^\delta \rightarrow x \text{ in } T'].$$

2) Thus on a given set, two topologies with the same convergent nets and limits thereof are identical.

Proof. 1) Suppose $T \supset T'$ and that $x^\delta \rightarrow x$ in T . Let N be a T' neighborhood of x . Then N is a T neighborhood of x and so $\langle x^\delta \rangle$ is eventually in N .

Conversely, suppose that for all nets in X , $x^\delta \rightarrow x$ in $T \Rightarrow x^\delta \rightarrow x$ in T' . Then $\forall A \subset X$, the closure of A in $T \subset$ the closure of A in T' (2.7). But then $T \supset T'$ (1.17). //

2.9 Definition. Given (X, T) , (Y, T') and $x \in X$, a function $f: X \rightarrow Y$ is:

continuous at point x iff $\forall$ neighborhood N of $f(x)$

$[f^{-1}[N] \text{ is a neighborhood of } x];$

continuous on X iff f is continuous at each point of X .

2.10 Theorem. Given (X, T) and (Y, T') , a function $f: X \rightarrow Y$ is continuous on X iff $\forall G \in T' [f^{-1}[G] \in T]$. (Wilansky (2) p.52).

2.11 Theorem. Given (X, T) , (Y, T') and $x \in X$, a function $f: X \rightarrow Y$ is continuous at x iff $\forall$ directed set D

$$\forall \langle x^\delta \rangle \in X^D [x^\delta \rightarrow x \text{ in } T \Rightarrow f(x^\delta) \rightarrow f(x) \text{ in } T'].$$

(Wilansky (2) p.52).

S3 TOPOLOGICAL PROPERTIES.

3.1 Definition. Given (X, T) and (Y, T') , a function $f: X \rightarrow Y$ is:

a homeomorphism from X into Y iff

(1) f is one-one,

(2) f is continuous and $f^{-1}: f[X] \rightarrow X$ is continuous;

a homeomorphism from X onto Y iff, in addition, f is onto.

3.2 Definition. Two topological spaces are homeomorphic iff there exists a homeomorphism from one onto the other.

3.3 Definition. A topological property is a property which may be possessed by a topological space, and which is preserved by every homeomorphism from any topological space onto any other topological space.

3.4 Definition. A topological space (X, T) is Hausdorff iff

$\forall x_1, x_2 \in X [x_1 \neq x_2 \Rightarrow \exists \text{ nbd } N_1 \text{ of } x_1 \exists \text{ nbd } N_2 \text{ of } x_2 [N_1 \cap N_2 = \emptyset]].$

3.5 Theorem. Given (X, T) , (Y, T') and a one-one, continuous map $f: X \rightarrow Y$, then Y Hausdorff $\Rightarrow X$ Hausdorff. Thus "Hausdorff" is a topological property. (Dugundji p.140).

3.6 Theorem. Given (X, T) , X is Hausdorff iff every convergent net in X has exactly one limit point. (Wilansky (1) p.146).

3.7 Definition. Given sets X , $A \subset X$, a collection C of subsets of X is a cover of A iff $A \subset \bigcup \{B \mid B \in C\}$. Given a cover C of $A \subset X$, a subset C' of C is a subcover iff C' is a cover of A . If X is a topological space, a cover C is an open cover iff every member of C is open.

3.8 Definition. Given (X, T) , $A \subset X$, A is compact iff every open cover of A has a finite subcover.

3.9 Theorem. Given (X, T) , (Y, T') and a continuous map $f: X \rightarrow Y$, then X compact $\Rightarrow Y$ compact. Thus compactness is a topological property. (Wilansky (2) p.82).

3.10 Theorem. A compact subset in a Hausdorff space is closed. (Wilansky (2) p.82).

3.11 Definition. Given directed sets $[D, \leq]$ and $[B, \leq]$, a function $u: B \rightarrow D$ is finalizing iff $\forall \delta \in D \exists \beta_0 \in B \forall \beta \geq \beta_0 [u(\beta) \geq \delta]$.
i.e. iff the net u is eventually in every tail of D .

3.12 Definition. Let directed sets $[D, \leq]$ and $[B, \leq]$, sets X and Y , and a net $\langle x^\delta \rangle \in X^D$ be given. Then a net $\langle y^\beta \rangle \in Y^B$ is a subnet of $\langle x^\delta \rangle$ iff there exists a finalizing map $u: B \rightarrow D$ such that $\forall \beta \in B [y^\beta = x^{u(\beta)}]$.

3.13 Theorem. Every subnet of a convergent net converges to the same limit points as the net. (Wilansky (1) p.157).

3.14 Theorem. A point is an accumulation point of a net iff the net has a subnet converging to the point. (Wilansky (1) p.158).

3.15 Theorem. Given (X, T) , $A \subset X$, the following statements are equivalent:

- (1) A is compact;
 - (2) Every net in A has an accumulation point in A ;
 - (3) Every net in A has a subnet converging to a point in A .
- (Wilansky (1) p.161).

S4 SUP, WEAK, AND PRODUCT TOPOLOGIES.

4.1 Definition. Let a set X be given a non-empty collection C of topologies. The topology having sub-base $\{\cup T \mid T \in C\}$ is called the sup topology by $C, \bigvee C$.

4.2 Theorem. Let a set X be given a non-empty collection C of topologies. Let T' denote $\bigvee C$, and let a net $\langle x^\delta \rangle \in X^D$ and a point $x \in X$ be given. Then $x^\delta \rightarrow x$ in T' $\Leftrightarrow x^\delta \rightarrow x$ in every $T \in C$. (Wilansky (1) p.148).

4.3 Definition. Let a set Y , a topological space (X, T) , and a function $f: Y \rightarrow X$ be given. Then the topology on Y having sub-base $\{f^{-1}[G] \mid G \in T\}$ is called the weak topology by $f, w(Y, f)$.

4.4 Definition. Let a set Y , a collection $\{(X_\alpha, T_\alpha)\}_{\alpha \in A}$ of topological spaces, and for each X_α one or more functions $f: Y \rightarrow X_\alpha$ be given. Let the collection of all these functions be denoted F . Then the topology $\bigvee \{w(Y, f) \mid f \in F\}$ on Y is called the weak topology by $F, w(Y, F)$.

4.5 Theorem. Let a set Y be given the weak topology $w(Y, F)$ by a family F of maps from Y to a collection of topological spaces. Let a net $\langle y^\delta \rangle \in Y^D$, and $y \in Y$ be given. Then
 $y^\delta \rightarrow y \iff \forall f \in F [f(y^\delta) \rightarrow f(y)]$. (Wilansky (1) p.149).

4.6 Theorem. With the notation of 4.4, each $f \in F$ is continuous on Y bearing $w(Y, F)$, and $w(Y, F)$ is the weakest topology on Y such that this is true. (Wilansky (1) p.150).

4.7 Definition. Let an index set B be given, and for each $\beta \in B$ a set X_β . The (direct) product, $P X_\beta$, is the set of all functions $f: B \rightarrow \cup X_\beta$ such that $\forall \beta \in B [f(\beta) \in X_\beta]$.

For each $\alpha \in B$, the α th projection is the map $p_\alpha: P X_\beta \rightarrow X_\alpha$ given by: $\forall f \in P X_\beta [p_\alpha(f) = f(\alpha)]$.

4.8 Definition. The product topology for a product of topological spaces is the weak topology by the family of all projections.

4.9 Theorem. Let $X = P X_\beta$ be a product of topological spaces, $\langle x^\delta \rangle \in X^D$ be a net in X , and $x \in X$. The $x^\delta \rightarrow x$ in the product topology $\iff \forall \beta \in B [p_\beta(x^\delta) \rightarrow p_\beta(x)]$. (Wilansky (1) p.151).

4.10 Definition. Given (X, T_x) , (Y, T_y) , and (Z, T_z) , a function $f: X \times Y \rightarrow Z$ is jointly continuous iff $\forall$ directed set D

$$\forall \langle x^\delta \rangle \in X^D \quad \forall \langle y^\delta \rangle \in Y^D \quad \forall x \in X \quad \forall y \in Y: \\ (x^\delta \rightarrow x \text{ and } y^\delta \rightarrow y) \Rightarrow f(x^\delta, y^\delta) \rightarrow f(x, y).$$

4.11 Theorem. With the notation of 4.10, let T_{xy} be the product topology of $X \times Y$. Then f is jointly continuous iff $f: (X \times Y, T_{xy}) \rightarrow (Z, T_z)$ is continuous. (Wilansky (1) p.152).

S5 FIRST COUNTABLE SPACES.

5.1 Definition. A topological space (X, T) is:

first countable at point $x \in X$ iff it has a countable neighborhood base at x ;

first countable iff it is first countable at each of its points.

Clearly, first countable is a topological property.

5.2 Theorem. Let (X, T) be first countable at point $x \in X$. Then any net converging to x contains a sequence converging to x .

Proof. Let the net $\langle x^\delta \rangle \in X^D$ be such that $x^\delta \rightarrow x$. Let a countable neighborhood base at x be $L = \{N_n \mid n \in \omega\}$. Replace this by the shrinking countable neighborhood base at x

$$L' = \{N'_n \mid n \in \omega\}, \text{ where } \forall n [N'_n = \bigcap_{k=1}^n N_k] \text{ and thus}$$

$$\forall n [N'_{n+1} \subset N'_n].$$

Then for each $n \in \omega$, $\langle x^\delta \rangle$ is eventually in N'_n and we may select $y^n \in N'_n \subset \{x^\delta\}$. Then the net $\langle x^\delta \rangle$ contains the sequence $\langle y^n \rangle$, and $y^n \rightarrow x$. //

Note, however, that this sequence need not be a subsequence (i.e. a sequence which is a subnet) of the net. Indeed, a net need not contain any subsequences. For let $[D, \leq]$ consist of the elements of the first uncountable ordinal Ω with the usual order. Then if a map $u: \omega \rightarrow \Omega$ were finalizing we should have

$$\forall \alpha \in \Omega \exists n_0 \in \omega \forall n \geq n_0 [u(n) \geq \alpha]. \text{ But then}$$

$$\Omega = \bigcup_{n=1}^{\infty} \{\alpha \in \Omega \mid \alpha \leq u(n)\}, \text{ a countable union of countable sets, and}$$

thus countable, a contradiction.

5.3 Corollary. Let (X, T) be first countable at $x \in X$, and let A be any subset of X . Then there exists a sequence in A converging to x iff there exists a net in A converging to x .

Proof. 5.2. //

5.4 Remark. In general, when first countable spaces are involved, "net" may be replaced by "sequence" in theorems dealing with convergence. Specifically, making use of 5.1-5.3 and modifying the proofs suitably, this will be found to hold: when (X, T) is first countable for 2.7, 2.8 1), 2.11, 3.6, 3.15; when both topologies are first countable for 2.8 2).

S6 SEMIMETRIC AND METRIC SPACES.

Let X denote any set, and R the real number system.

6.1 Definition. A function $d: X \times X \rightarrow R$ is a semimetric d on X iff $\forall x, y, z \in X$:

- (a) $d(x, y) \geq 0$; $d(x, x) = 0$
- (b) $d(x, y) = d(y, x)$
- (c) $d(x, y) \leq d(x, z) + d(z, y)$.

A semimetric is a metric iff

$\forall x, y \in X [d(x, y) = 0 \Rightarrow x = y]$.

A semimetric space, (X, d) , is a set X on which is defined a semimetric d .

6.2 Example. On C^n , let $x = \langle x_1, x_2, \dots, x_n \rangle$,
 $y = \langle y_1, y_2, \dots, y_n \rangle$. Then the function $d: C^n \rightarrow R$ defined by:
 $\forall x, y \in C^n [d(x, y) = \left(\sum_{k=1}^n |x_k - y_k|^2 \right)^{1/2}]$ is a metric, the Euclidean
metric on C^n .

6.3 Definition. Given a semimetric space (X, d) , for $a \in X$ and $r > 0$ the ball with center a and radius r , $N(a, r)$, is
 $\{x \in X | d(x, a) < r\}$.

A set included in some ball is said to be semimetrically
bounded.

6.4 Theorem. | Given a semimetric space (X, d) , the collection of sets
 $\{G \subset X | \forall x \in G \exists r > 0 [N(x, r) \subset G]\}$ is a topology for the space,
the d topology. (Wilansky (2) p.15).

A semimetric space (X, d) will be understood to bear the d
topology.

6.5 Theorem. Balls in a semimetric space are open.

(Wilansky (2) p.15).

6.6 Theorem. Given a semimetric space (X, d) , a net

$\langle x^\delta \rangle \in X^D$, and $x \in X$, then $x^\delta \rightarrow x$ iff $d(x^\delta, x) \rightarrow 0$.

(Wilansky (2) p.40).

6.7 Theorem. Every metric space is Hausdorff.

Proof. Given a metric space (X, d) let a, b be any two distinct points of X . Let $r = \frac{1}{2} d(a, b)$.

Then the balls $N(a, r)$ and $N(b, r)$ are disjoint. //

6.8 Theorem. Every semimetric space is first countable.

(Wilansky (2) p.27).

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