Existence, comparison and oscillation results for some functional differential equations
by Ernest DeCarteteret True

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics
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Abstract:
This paper is devoted to the qualitative study of solutions to functional differential equations of the form (A) x(n)(t) + f(t,x(g(t))) = Q(t) and (B) x(n)(t) + P(t,x(t),x(g(t)),x(t))x(n-1) (t) + Q(t,x(t),x(g(t)),x(t)) = 0 where x(t) denotes x(t),x(t),. . . ,x(n-1)(t), and f,Q,P, and g are known functions.

In Chapter II the method of successive approximations is employed to provide an existence and uniqueness theorem for solutions to (A) when f(t,x(g(t))) = a(t)x(g(t)) and Q(t) = O, subject to normal initial conditions. Under the suitable restrictions for g(t), the solution can be extended to the infinite interval (-∞,+∞).

Two comparison theorems are given for solutions to (A) in Chapter III, when f(t,x(g(t))) = a(t)P(x(g(t))). Here it is shown that if a(t) >_ 0 and continuous on [t0,+ ∞) and if there is a real number 0< γ < 1 such that for any continuous s(t) > γa(t), t >_ t0 the equation v(n)(t) + s(t)P(v(t)) = 0 has all its bounded solutions oscillatory, then all bounded solutions to x(n)(t) + a(t)P(x(g(t))) = Q(t) are also oscillatory.

In Chapter IV the maintenance of oscillation of solutions is examined for (A) under the effect of a small forcing term Q(t), while Chapter V is primarily devoted to maintaining oscillatory solutions to (B) under the effect of a small_non linear damping term P(t,x(t),x(g(t)),x(t)).

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ABSTRACT

This paper is devoted to the qualitative study of solutions to functional differential equations of the form

(A) $x^{(n)}(t) + f(t,x(g(t))) = Q(t)$ and

(B) $x^{(n)}(t) + P(t,x(t),x(g(t)),\overline{x}(t))x^{(n-1)}(t)$

$+ Q(t,x(t),x(g(t)),\overline{x}(t)) = 0$

where $\overline{x}(t)$ denotes $\dot{x}(t),\ddot{x}(t),\ldots,x^{(n-1)}(t)$, and f,Q,P, and g are known functions.

In Chapter II the method of successive approximations is employed to provide an existence and uniqueness theorem for solutions to (A) when $f(t,x(g(t))) = a(t)x(g(t))$ and $Q(t) \equiv 0$, subject to normal initial conditions. Under the suitable restrictions for $g(t)$, the solution can be extended to the infinite interval $(-\infty,+\infty)$.

Two comparison theorems are given for solutions to (A) in Chapter III, when $f(t,x(g(t))) = a(t)P(x(g(t)))$. Here it is shown that if $a(t) \geq 0$ and continuous on $[t_0,+\infty)$ and if there is a real number $0 < \gamma < 1$ such that for any continuous $s(t) \geq \gamma a(t)$, $t \geq t_0$, the equation $v^{(n)}(t) + s(t)P(v(t)) = 0$ has all its bounded solutions oscillatory, then all bounded solutions to $x^{(n)}(t) + a(t)P(x(g(t))) = Q(t)$ are also oscillatory.

In Chapter IV the maintenance of oscillation of solutions is examined for (A) under the effect of a small forcing term $Q(t)$, while Chapter V is primarily devoted to maintaining oscillatory solutions to (B) under the effect of a small non linear damping term $P(t,x(t),x(g(t)),\overline{x}(t))$.

CHAPTER I

INTRODUCTION

A natural way to generalize an ordinary differential equation is to replace the independent variable by a function of the independent variable as it appears in one or more places in the equation. Such a differential equation is called a functional differential equation and is the major topic of this paper. In particular, equations of the form $x^{(n)}(t) + f(t,x(g(t))) = Q(t)$ or
$x^{(n)}(t) + P(t,x(t),x(g(t)),\overline{x}(t))x^{(n-1)}(t)$
$+ Q(t,x(t),x(g(t)),\overline{x}(t)) = 0$ will be considered, where $\overline{x}(t)$ denotes $\dot{x}(t),\ddot{x}(t),\ldots,x^{(n-1)}(t)$, and $g(t)$ is some function of the independent variable t.

In recent years, many papers have been written on functional differential equations for the special case when $g(t) \leq t$, and these equations are called "differential equations with retarded arguments," or "delay equations." A good introduction to this subject is found in El'sgol'ts [2]. Throughout most of this paper, $g(t)$ will be continuous and tend to ∞ as $t \to \infty$, and arbitrary otherwise, except in the existence and uniqueness theory.

Only a small amount of work has been done on functional differential equations in which $g(t)$ is not necessarily of the delay type. An existence and uniqueness theorem has been presented by G.H. Ryder [10] for a solution to the equation $\dot{x}(t) = Ax(g(t))$ subject to the condition $x(t_0) = x_0$, where A is an $n \times n$ constant matrix, and $x(t)$ denotes an n vector. Later, Grefsrud [3] generalized the scalar case of the above equation to give an existence and uniqueness theorem for a solution to the second order equation $\ddot{x}(t) + a(t)x(g(t)) = 0$ subject to the conditions $x(t_0) = C_1$, $\dot{x}(t_0) = C_2$. The proofs and hypotheses for these results are given in Chapter II, in which an existence and uniqueness theorem is given for a solution to $y^{(n)}(t) + A(t)y(h(t)) = 0$ subject to the condition $y^{(k)}(t_0) = C_{k+1}$, $k = 0,1,\ldots,n-1$, in which the proof follows those of Ryder and Grefsrud for the cases when $n = 1,2$ respectively, and is stated here more as a matter of convenience and completeness, than of originality.

It should be mentioned that the above initial value problem can be translated to the origin by replacing t by $t + t_0$ and letting

$x(t) = y(t + t_0)$, $a(t) = A(t + t_0)$, $g(t) = h(t + t_0)$,

which yields the equation $x^{(n)}(t) + a(t)x(g(t)) = 0$ and initial conditions $x^{(k)}(0) = C_{k+1}$, $k = 0,1,\ldots,n-1$.

Another existence theorem has been presented by R.J. Oberg [9], which is a local existence theorem for the equation $\dot{x}(t) = f(t,x(t),x(g(t,x(t))))$ with $x(t_0) = x_0$, where a solution is guaranteed in some interval about a fixed point, t_0 of $g(t,x(t))$.

In addition to the existence and uniqueness theory, Grefsrud [3] has given conditions for which solutions to $x^{(n)}(t) + f(t,x(g(t))) = 0$ are either oscillatory or tend monotonically to zero. These results generalize some of the work done earlier by Bradley [1] and Waltman [12].

A differential equation with functional argument, $g(t)$, will be called linear (nonlinear) if the equation is linear (nonlinear) when $g(t)$ is replaced everywhere by t.

A solution on an interval I to a differential equation with functional argument, $g(t)$, is a function $x(t)$ which is defined on $I \cup g[I]$ and which satisfies the equation on I.

A solution $x(t)$ of a differential equation will be called oscillatory if $x(t)$ is a solution valid for all large t and has arbitrarily large zeros.

CHAPTER II

EXISTENCE AND UNIQUENESS

In this chapter an existence and uniqueness theorem of solutions to the n^{th} order functional differential equation

(1) $x^{(n)}(t) + a(t)x(g(t)) = 0$

subject to the initial conditions

(2) $x^{(k)}(0) = C_{k+1}$, $k = 0,1,\ldots,n - 1$.

will be presented. To do this the method of successive approximations will be applied to the equivalent integral equation

(3) $$x(t) = \sum_{k=1}^{n} \frac{C_k t^{k-1}}{(k - 1)!} - \int_0^t \frac{(t - s)^{n-1}}{(n - 1)!} a(s)x(g(s))ds,$$

which can be obtained easily from (1) and (2) by integrating (1) n times successively from 0 to t and interchanging the order of integration. Define

(4) $$x_0(t) = \sum_{k=1}^{n} \frac{C_k t^{k-1}}{(k - 1)!}$$
$$x_m(t) = x_0(t) - \int_0^t \frac{(t - s)^{n-1}}{(n - 1)!} a(s)x_{m-1}(g(s))ds, \quad m \geq 1.$$

Lemma II.1. Let $g(t)$ and $a(t)$ be continuous on $[-\sigma,T]$, $\sigma \geq 0$ and such that $g([-\sigma,T]) \subset [-\sigma,T]$. Then $x_m(t)$, defined by (4) is continuous on $[-\sigma,T]$ and

(5) $x_m(t) = x_0(t) + \sum_{k=1}^{m} (-1)^k g_k(t)$, $m \geq 1$, where $g_k(t)$

is defined by

$$g_1(t) = \int_0^t \frac{(t - s)^{n-1}}{(n - 1)!} a(s)x_0(g(s))ds$$

$$(6) \qquad = \int_0^t \frac{(t - s)^{n-1}}{(n - 1)} a(s) \sum_{k=1}^{n} \frac{C_k[g(s)^{k-1}}{(k - 1)!} ds$$

$$g_m(t) = \int_0^t \frac{(t - s)^{n-1}}{(n - 1)!} a(s)g_{m-1}(g(s))ds, \ m \geq 2.$$

PROOF. Since $a(t)$ and $g(t)$ are continuous on $[-\sigma,T]$, then $g_1(t)$ is continuous on $[-\sigma,T]$. Thus, assume $g_{k-1}(t)$ is continuous on $[-\sigma,T]$. Then, since $g([-\sigma,T]) \subset [-\sigma,T]$, $g_{k-1}(g(t))$ is also continuous on $[-\sigma,T]$. By (6) it follows that $g_k(t)$ is continuous on $[-\sigma,T]$, and hence, $g_m(t)$ is continuous on $[-\sigma,T]$ for all m by induction.

To establish (5) we have using (4) and (6)

$$x_1(t) = x_0(t) - \int_0^t \frac{(t - s)^{n-1}}{(n - 1)!} a(s)x_0(g(s))ds$$

$$= x_0(t) - \int_0^t \frac{(t - s)^{n-1}}{(n - 1)!} a(s) \sum_{k=1}^{n} \frac{C_k[g(s)]^{k-1}}{(k - 1)!} ds, \text{ or}$$

$x_1(t) = x_0(t) - g_1(t)$, $t \in [-\sigma,T]$.

$$x_2(t) = x_0(t) - \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} a(s)x_1(g(s))ds$$

$$= x_0(t) - \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} a(s)[x_0(g(s)) - g_1(g(s))]ds$$

$$= x_0(t) - \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} a(s)x_0(g(s))ds$$

$$+ \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} a(s)g_1(g(s))ds$$

$$= x_0(t) - g_1(t) + g_2(t),\ t \in [-\sigma,T].$$

Now assume

$$x_m(t) = x_0(t) + \sum_{k=1}^{m} (-1)^k g_k(t),\ m \geq 1,\ t \in [-\sigma,T].$$

Then

$$x_{m+1}(t) = x_0(t) - \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} a(s)x_m(g(s))ds$$

$$= x_0(t) - \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} a(s)[x_0(g(s))$$

$$+ \sum_{k=1}^{m} (-1)^k g_k(g(s))]ds$$

$$= x_0(t) - \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} a(s)x_0(g(s))ds$$

$$+ \sum_{k=1}^{m} (-1)^{k+1} \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} a(s)g_k(g(s))ds$$

$$= x_0(t) - g_1(t) + \sum_{k=1}^{m} (-1)^{k+1} g_{k+1}(t)$$

$$= x_0(t) + \sum_{k=1}^{m+1} (-1)^k g_k(t).$$

Thus, (5) is established by induction. Q.E.D.

The proof of the following theorem establishes the convergence of the sequence of approximations defined in (4) to a unique solution of (1) satisfying (2) on $[-\sigma, T]$, subject to suitable restrictions on $a(t)$ and $g(t)$.

<u>Theorem II.1</u>. Let $|a(t)|\ |g(t)|^n \leq K$ for $t \in [-\sigma, T]$. Then under the assumptions of Lemma II.1, the sequence $\{x_n(t)\}$ defined by (4) converges uniformly on $[-\sigma, T]$ to a unique solution of (1) satisfying (2) provided $K < n!$

PROOF. Since $a(t)$ and $g(t)$ are continuous on $[-\sigma, T]$, there exists $K_1 > 0$ for which

$$|a(t)| \left| \sum_{k=1}^{n} \frac{C_k [g(t)]^{k-1}}{(n-1)!} \right| \leq K_1 \text{ on } [-\sigma, T].$$

Thus, from (6)

$$|g_1(t)| \le \left| \int_0^t \frac{|t-s|^{n-1}}{(n-1)!} \, |a(s)| \left| \sum_{k=1}^{n} \frac{C_k [g(s)]^{k-1}}{(k-1)!} \right| ds \right|$$

$$\le K_1 \left| \int_0^t \frac{|t-s|^{n-1}}{(k-1)!} \, ds \right| \le K_1 \frac{|t|^n}{n!}$$

$$\le K_1 \frac{I^n}{n!}, \text{ where } I = \max\{\sigma, T\}.$$

$$|g_2(t)| \le \left| \int_0^t \frac{|t-s|^{n-1}}{(n-1)!} \, |a(s)| \; |g_1(g(s))| \, ds \right|$$

$$\le \frac{K_1}{n!} \left| \int_0^t \frac{|t-s|^{n-1}}{(n-1)!} \, |a(s)| \; |g(s)|^n \, ds \right|$$

$$\le \frac{K_1}{n!} K \left| \int_0^t \frac{|t-s|^{n-1}}{(n-1)!} \, ds \right| \le \frac{K_1}{n!} K \frac{|t|^n}{n!}$$

$$\le \frac{K_1}{K} |t|^n \left(\frac{K}{n!}\right)^2 \le \frac{K_1}{K} I^n \left(\frac{K}{n!}\right)^2.$$

Now assume

$$|g_m(t)| \le \frac{K_1}{K} |t|^n \left(\frac{K}{n!}\right)^m, \quad m \ge 1. \quad \text{Then}$$

$$|g_{m+1}(t)| \le \left| \int_0^t \frac{|t-s|^{n-1}}{(n-1)!} \, |a(s)| \; |g_m(g(s))| \, ds \right|$$

$$\le \frac{K_1}{K} \left(\frac{K}{n!}\right)^m \left| \int_0^t \frac{|t-s|^{n-1}}{(n-1)!} \, |a(s)| \; |g(s)|^n \, ds \right|$$

$$\le \frac{K_1}{K} K \left(\frac{K}{n!}\right)^m \left| \int_0^t \frac{|t-s|^{n-1}}{(n-1)!} \, ds \right|$$

$$\leq \frac{K_1}{K} K\left(\frac{K}{n!}\right)^m \frac{|t|^n}{n!} \leq \frac{K_1}{K} |t|^n \left(\frac{K}{n!}\right)^{m+1}$$

$$\leq \frac{K_1}{K} I^n \left(\frac{K}{n!}\right)^{m+1}$$

Thus, by induction

$$|g_m(t)| \leq \frac{K_1}{K} I^n \left(\frac{K}{n!}\right)^m.$$

From (5)

$$|x_m(t)| \leq |x_0(t)| + \sum_{j=1}^{m} |(-1)^j g_j(t)|$$

$$\leq |x_0(t)| + \sum_{j=1}^{m} |g_j(t)|$$

$$\leq |x_0(t)| + \sum_{j=1}^{m} \frac{K_1}{K} I^n \left(\frac{K}{n!}\right)^j$$

$$\leq |x_0(t)| + \frac{K_1}{K} I^n \sum_{j=1}^{m} \left(\frac{K}{n!}\right)^j$$

Thus, $\{x_m(t)\}$ converges uniformly to some continuous function $x(t)$ on $[-\sigma,T]$ provided $K < n!$.

To show $x(t)$ is the required solution to (1) satisfying (2) on $[-\sigma,T]$, write

(7) $$x_{m+1}(t) = x_0(t) - \int_0^t \frac{(t - s)^{n-1}}{(n - 1)!} a(s)x_m(g(s))ds,$$

$-\sigma \leq t \leq T$.

Since $\lim_{m \to \infty} x_m(g(t)) = x(g(t))$ uniformly on $[-\sigma,T]$, then taking the limit of (7) as $m \to \infty$ yields

$$x(t) = x_0(t) - \int_0^t \frac{(t - s)^{n-1}}{(n - 1)!} a(s)x(g(s))ds,$$ and

$x(t)$ satisfies the integral equation (3). Thus, $x(t)$ is the required solution to (1) satisfying (2) on $[-\sigma,T]$.

To establish the uniqueness of a solution to (1) satisfying (2) on $[-\sigma,T]$, suppose $v(t)$ is any other solution to (1) on $[-\sigma,T]$ which satisfies (2). Then

$$v(t) = x_0(t) - \int_0^t \frac{(t - s)^{n-1}}{(n - 1)!} a(s)v(g(s))ds.$$

Since $g([-\sigma,T]) \subset [-\sigma,T]$, then

$$| v(t) - x_0(t) | \leq \left| \int_0^t \frac{| t - s |^{n-1}}{(n-1)!} | a(s) | \, | v(g(s)) | \, ds \right|$$

$$\leq \alpha \frac{| t |^n}{n!}, \ t \in [-\sigma,T],$$

$$\alpha = \max | a(s) | \, | v(g(s))$$

Also,

$$| v(t) - x_1(t) | \leq \left| \int_0^t \frac{| t - s |^{n-1}}{(n-1)!} | a(s) | \, | x_0(g(s)) - v(g(s)) | \, ds \right|$$

$$\leq \frac{\alpha}{n!} \left| \int_0^t \frac{| t - s |^{n-1}}{(n-1)!} | a(s) | \, | g(s) |^n \, ds \right|$$

$$\leq \frac{\alpha}{n!} K \frac{| t |^n}{n!} \leq \frac{\alpha}{n!} | t |^n \frac{K}{n!}$$

Now suppose

$$| v(t) - x_m(t) | \leq \frac{\alpha}{n!} | t |^n \left(\frac{K}{n!}\right)^m$$

Then

$$| v(t) - x_{m+1}(t) | \leq \left| \int_0^t \frac{| t - s |^{n-1}}{(n-1)!} | a(s) | \, | x_m(g(s)) - v(g(s)) | \, ds \right|$$

$$\leq \frac{\alpha}{n!} \left(\frac{K}{n!}\right)^m K \frac{| t |^n}{n!}$$

or

$$| v(t) - x_{m+1}(t) | \leq \frac{\alpha}{n!} | t |^n (\frac{K}{n!})^{m+1}$$

Thus, by induction,

$$| v(t) - x_m(t) | \leq \frac{\alpha}{n!} | t |^n (\frac{K}{n!})^m$$

Taking the limit as $m \to +\infty$,

$| v(t) - x(t) | \leq 0$ if $k < n!$ and hence,

$v(t) \equiv x(t)$ on $[-\sigma,T]$. Q.E.D.

The following corollary yields an upper bound for the error in approximating the solution to (1) by the m^{th} successive approximation used in the proof of Theorem II.1.

<u>Corollary</u>. Under the assumptions of Theorem II.1, if $x(t) = \lim_{m \to \infty} x_m(t)$ on $[-\sigma,T]$, then

$$| x(t) - x_m(t) | \leq \frac{K_1}{n! - K} |t|^n (\frac{K}{n!})^m, \quad K < n!$$

PROOF. From (5),

$$x_m(t) = x_0(t) + \sum_{k=1}^{m} (-1)^k g_k(t) \text{ and thus}$$

$$x(t) = x_0(t) + \sum_{k=1}^{\infty} (-1)^k g_k(t), \text{ so that}$$

$$x(t) - x_m(t) = \sum_{k=m+1}^{\infty} (-1)^k g_k(t), \text{ and}$$

$$| x(t) - x_m(t) | \le \sum_{k=m+1}^{\infty} | g_k(t) | \le \sum_{k=m+1}^{\infty} \frac{K_1}{K} | t |^n (\frac{K}{n!})^k$$

$$\le \frac{K_1}{K} | t |^n \frac{(\frac{K}{n!})^{m+1}}{1 - \frac{K}{n!}}, \text{ or}$$

$$| x(t) - x_m(t) | \le \frac{K_1}{n! - K} | t |^n (\frac{K}{n!})^m. \quad \text{Q.E.D.}$$

The following example illustrates that Theorem II.1 fails in general if $| a(t) | \; | g(t) |^n = n!$

Example II.1. Consider the initial value problem

(8) $x^{(n)}(t) - ax(\sqrt[n]{\frac{n!}{a}}) = 0$

(9) $x^{(k)}(0) = c_{k+1}$, $k = 0,1,2,\ldots,n - 1$.

Here, $a(t) = -a$, $a > 0$, constant, and $g(t) = \sqrt[n]{\frac{n!}{a}}$. Then $| a(t) | \; | g(t) |^n = n!$. Using (3),

$$x(t) = \sum_{k=1}^{n} \frac{c_k t^{k-1}}{(k - 1)!} + \int_0^t \frac{(t - s)^{n-1}}{(n - 1)!} ax(\sqrt[n]{\frac{n}{a}}) \, ds, \text{ or}$$

(10) $$x(t) = \sum_{k=1}^{n} \frac{c_k t^{k-1}}{(k - 1)!} + ax(\sqrt[n]{\frac{n!}{a}}) \frac{t^n}{n!}$$

If a solution is required on an interval about the origin containing the point $t_0 = \sqrt[n]{\frac{n!}{a}}$, then (10) yields

$$x(t_0) = \sum_{k=1}^{n} \frac{c_k t_0^{k-1}}{(k - 1)!} + \frac{ax(t_0)}{a}, \text{ or}$$

(11) $\sum_{k=1}^{n} \frac{C_k t_0^{k-1}}{(k-1)!} = 0$. Thus, a solution is possible only if $C_1, C_2, \ldots C_n$ are chosen so as to satisfy (11).

The next example shows that uniqueness cannot, in general, be guaranteed for a solution to (1). Satisfying (2) if $|a(t)|\ |g(t)|^n = n!$

<u>Example II.2</u>. The functions $x_1(t) = 0$ and $x_2(t) = t^n$ are both solutions to $x^{(n)}(t) - n!e^{nt}x(e^{-t}) = 0$ satisfying $x^{(k)}(0) = 0$, $k = 0,1,\ldots,n = 1$. Here $a(t) = -n!e^{nt}$, $g(t) = e^{-t}$ and $|a(t)|\ |g(t)|^n = n!$, while the remaining hypotheses of Theorem II.1 are satisfied on the interval $D_g = [-1,e]$.

<u>Example II.3</u>. Consider the problem

$$x^{(n)}(t) - \frac{(-1)^n n!}{2} \frac{(t+3)}{(t+1)^{n+1}} x\left(\frac{t+1}{2}\right) = 0$$

(12)

$$x(0) = 1,\ x^{(k)}(0) = (-1)^k k!,\ k = 1,2,\ldots,n-1.$$

Here, $a(t) = -\frac{(-1)^n n!}{2} \frac{(t+3)}{(t+1)^{n+1}}$, $g(t) = \frac{t+1}{2}$, and on the interval $[-\frac{1}{4},1]$,

$$|a(t)|\,|g(t)|^n = \frac{n!}{2^{n+1}} \frac{|t+3|}{|t+1|} = \frac{n!}{2^{n+1}} \left[\frac{t+3}{t+1}\right]$$

$$= \frac{n!}{2^{n-1}} \left[\frac{t+3}{4(t+1)}\right]$$

$$\leq n! \, [\frac{t + 3}{4(t + 1)}]$$

$$< n!, \text{ since}$$

$\frac{t + 3}{4(t + 1)} < 1$ if $t > -\frac{1}{3}$, which is satisfied on the interval $[-\frac{1}{4},1]$. Moreover, the range of $g(t)$ on $[-\frac{1}{4},1]$ is $[\frac{3}{8},1] \subset [-\frac{1}{4},1]$.

Thus, Theorem II.1 guarantees the existence of a unique solution to (12), and this solution is $x(t) = (t + 1)^{-1}$.

In fact, $x(t) = (t + 1)^{-1}$ is a solution to (12) on the interval $(-1,+\infty)$.

The previous example indicates that in some cases, solutions to (1) satisfying (2) can be continued beyond the given interval $[-\sigma,T]$ for which Theorem II.1 guarantees a unique solution. The following theorem shows that when $g(t)$ behaves properly, solutions to (1) satisfying (2) can be extended to $(-\infty,+\infty)$.

<u>Theorem II.2</u>. Let $g(t)$ and $a(t)$ be continuous for all t, with $|a(t)| \, |g(t)|^n < n!$ for all t. Suppose there exist positive monotone non-decreasing sequences $\{\sigma_i\}$ and $\{T_j\}$ for which $\lim_{i \to \infty} \sigma_i = \lim_{j \to \infty} T_j = +\infty$. Then if

$g([-\sigma_i, T_j] \subset [-\sigma_i, T_j]$ for all i and j, (1) possesses a unique solution satisfying (2) which is valid on $(-\infty, +\infty)$.

PROOF. Fix i and j. By Theorem II.1, let x(t) be the unique solution to (1) satisfying (2) on $[-\sigma_i, T_j]$. Then $x(t) = \lim_{m \to \infty} x_m(t)$, where the $x_m(t)$ are defined as in (4) on $[-\sigma_i, T_j]$.

Now consider the sequence defined by (4) again on $[-\sigma_{i+1}, T_{j+1}]$ as

$$x_0(t) = \sum_{k=1}^{n} \frac{C_k t^{k-1}}{(k - 1)!}$$

$$x_m(t) = x_0(t) - \int_0^t \frac{(t - s)^{n-1}}{(n - 1)!} a(s) x_{m-1}(g(s)) ds, \quad m \geq 1.$$

Then $\lim_{m \to +\infty} x_m(t) = X(t)$ on $[-\sigma_{i+1}, T_{j+1}]$.

But since $[-\sigma_i, T_j] \subset [-\sigma_{i+1}, T_{j+1}]$, $X(t) = x(t)$ on $[-\sigma_i, T_j]$ by uniqueness, and, therefore, x(t) can be extended to $[-\sigma_{i+1}, T_{j+1}]$ and thus to $(-\infty, +\infty)$.

CHAPTER III

COMPARISON THEOREMS

In this chapter two comparison theorems are presented for oscillatory solutions to the differential equation

(13) $x^{(n)}(t) + a(t)f(x(g(t))) = Q(t)$.

Both theorems generalize a result due to A.G. Kartsatos [4], who exhibited a comparison theorem for (13) without the functional argument $g(t)$.

The function $Q(t)$ in (13) acts as a periodic forcing term subject to the following condition.

(i) There exists a function $R(t)$ for which $R^{(n)}(t) = Q(t)$ for all $t \geq t_0$, and $R(t_n) = \lambda_1$, $R(t_n^*) = -\lambda_2$, $-\lambda_2 \leq R(t) \leq \lambda_1$, for all $t \geq t_0$, where $\{t_n\}$ and $\{t_n^*\}$ are any two sequences for which $\lim\limits_{t \to \infty} t_n = \lim\limits_{t \to \infty} t_n^* = +\infty$.

It is worth noting that $Q(t) \equiv 0$ satisfies (i), with $R(t) \equiv 0$ and $\lambda_1 = \lambda_2 = 0$.

<u>Theorem III.1</u>. Suppose $Q(t)$ satisfies (i) and

(ii) $a(t) \geq 0$ and continuous on $[t_0, +\infty)$, and for any $s(t) \geq a(t)$, $t \geq t_0$, the equation

(14) $v^{(n)}(t) + s(t)f(v(g(t))) = 0$ has all its bounded solutions (respectively, solutions) oscillatory.

(iii) $f(x)$: $(-\infty, +\infty) \to (-\infty, +\infty)$ is continuous, increasing in x, and $xf(x) > 0$ for $x \neq 0$.

(iv) $\lim_{t \to \infty} g(t) = +\infty$.

Then all bounded solutions (respectively, solutions) to (13) which are valid for all large t are oscillatory.

PROOF. Two cases will be considered.

Case 1: Suppose first, all bounded solutions of (14) are oscillatory. Assume by contradiction that $x(t)$ is a bounded non-oscillatory solution to (13); i.e., suppose $x(t) > 0$ for $t \geq \alpha \geq t_0$. Since $g(t) \to +\infty$, there is some $K > 0$ such that $0 < x(t) < K < +\infty$ and $0 < x(g(t)) < K$ for all $t \geq \beta \geq \alpha$ for some β.

Consider $w(t) = x(t) - R(t)$, which is a solution to

(15) $w^{(n)}(t) + a(t)f[w(g(t)) + R(g(t))] = 0, \ t \geq \beta.$

Since $x(g(t)) = w(g(t)) + R(g(t)) > 0$ for $t \geq \beta$, then by (iii), $f[w(g(t)) + R(g(t))] > 0$ and hence $w^{(n)}(t) = -a(t)f[w(g(t)) + R(g(t))] < 0$. Moreover, $w(t)$ is bounded, $-\lambda_1 \leq w(t) \leq K + \lambda_2$, $t \geq \beta$.

If n is even, then $\dot{w}(t) > 0$ for $t \geq \gamma_1 \geq \beta$, and $R(t) \geq -\lambda_2$ implies $x(t) = w(t) + R(t) \geq w(t) - \lambda_2$. Since $\dot{w}(t) > 0$, $w(t)$ is increasing and for all $t \geq \tilde{t}_n^* \geq \gamma_1$, $w(t) \geq w(\tilde{t}_n^*)$ so that

$x(t) = w(t) + R(t) \geq w(t) - \lambda_2 \geq w(\tilde{t}_n^*) - \lambda_2 = w(\tilde{t}_n^*) + R(\tilde{t}_n^*)$ $= x(\tilde{t}_n^*) > 0$. Thus, $w(t) - \lambda_2 < 0$ for all $t \geq \tilde{t}_n^*$.

If n is odd, then $\dot{w}(t) > 0$ for all $t \geq \gamma_2 \geq \beta$ and $w(t)$ is decreasing. Also, $R(t) \geq -\lambda_2$ and hence, $0 < x(t) = w(t) + R(t) \geq w(t) - \lambda_2$, $t \geq \gamma_2$. If $w(\tau) - \lambda_2 \leq 0$ for some $\tau \geq \gamma_2$, then since $w(t)$ is decreasing, $w(t) - \lambda_2 \leq 0$ for all $t \geq \tau$. In particular, there is some $\tilde{t}_n^* \geq \tau$ for which $w(\tilde{t}_n^*) - \lambda_2 \leq 0$. But $w(\tilde{t}_n^*) - \lambda_2$ $= w(\tilde{t}_n^*) + R(\tilde{t}_n^*) = x(\tilde{t}_n^*) > 0$, which is a contradiction.

In either case we now have $w(t) + R(t) \geq w(t) - \lambda_2 > 0$ for large t.

Let $v(t) = w(t) - \lambda_2$. Then $v^{(n)}(t) = w^{(n)}(t)$ and (15) becomes

$$v^{(n)}(t) + a(t)f[v(g(t)) + \lambda_2 + R(g(t))]$$
$$= v^{(n)}(t) + a(t)\,\frac{f[v(g(t)) + \lambda_2 + R(g(t))]}{f[v(g(t))]}\,f[v(g(t))], \text{ or}$$

(16) $v^{(n)}(t) + s(t)f(v(g(t))) = 0$, where

$$s(t) = a(t)\,\frac{f[v(g(t)) + \lambda_2 + R(g(t))]}{f[v(g(t))]} \geq a(t) \text{ by (iii)}$$

Since $v(t)$ is a solution to (16), then by hypothesis $v(t)$ is oscillatory, which contradicts $v(t) > 0$. Thus, $x(t)$ must be oscillatory.

The proof is similar if one assumes $x(t) < 0$ for large t.

Case 2: Suppose now all solutions to (14) are oscillatory. Let $x(t)$ be a solution to (13), and suppose $x(t)$ is unbounded and non-oscillatory; e.g., $x(t) > 0$, $x(g(t)) > 0$ and unbounded for $t \geq \alpha \geq t_0$. Again, let $w(t) = x(t) - R(t)$, which satisfies

(15) $w^{(n)}(t) + a(t)f[w(g(t)) + R(g(t))] = 0$, $t \geq \alpha$.

Since $R(t)$ is bounded and $x(t)$ is unbounded, then for t sufficiently large, $x(t) = w(t) + R(t) \geq w(t) - \lambda_2 > 0$, and the result follows as in Case 1. Again, the proof is similar if one assumes $x(t) < 0$. Q.E.D.

The next theorem compares the oscillatory solutions of (13) with those of a differential equation which has no functional argument.

Theorem III.2. Suppose $Q(t)$ satisfies (i) and

(ii) $a(t) \geq 0$ and continuous on $[t_0, +\infty)$ and there exists a real number γ, $0 < \gamma < 1$ such that for any $s(t) \geq \gamma a(t)$, $t \geq t_0$, the equation

(17) $v^{(n)}(t) + s(t)f(v(t)) = 0$ has all its bounded solutions oscillatory.

(iii) $f(x)$: $(-\infty, +\infty) \to (-\infty, +\infty)$ is continuous, increasing in x and $xf(x) > 0$ for $x \neq 0$.

(iv) $\lim\limits_{t \to \infty} g(t) = +\infty$.

Then if n is even, all bounded solutions $x(t)$ to (13) valid for large t are oscillatory.

If n is odd, all bounded solutions $x(t)$ to (13) valid for large t are oscillatory or $\underline{\text{Lim}}\, |x(t)| = 0$.

PROOF. Suppose all bounded solutions to (17) are oscillatory. Assume, by contradiction, that $x(t)$ is a bounded non oscillatory solution to (13); i.e., suppose $x(t) > 0$ for all $t \geq \alpha \geq t_0$. Since $g(t) \to +\infty$, there exists a number K such that $0 < x(t) < K < +\infty$, and $0 < x(g(t)) < K$, $t \geq \beta \geq \alpha$ for some β.

Consider $w(t) = x(t) - R(t)$, which is a solution to

(15) $w^{(n)}(t) + a(t)f[w(g(t)) + R(g(t))] = 0, \; t \geq \beta.$

Since $x(g(t)) = w(g(t)) + R(g(t)) > 0$, then $f[w(g(t)) + R(g(t))] > 0$ by (iii) and hence $w^{(n)}(t) = -a(t)f[w(g(t)) + R(g(t))] < 0$, $t \geq \beta$. Moreover, $w(t)$ is bounded, $-\lambda_1 \leq w(t) \leq K + \lambda_2$, $t \geq \beta$. Proceeding as in Case 1 for the proof of Theorem III.1, we obtain $w(t) + R(t) \geq w(t) - \lambda_2 > 0$ for large t.

Let $v(t) = w(t) - \lambda_2$. Then $v^{(n)}(t) = w^{(n)}(t)$ and (15) becomes $v^{(n)}(t) + a(t)f[v(g(t)) + \lambda_2 + R(g(t))]$

$$= v^{(n)}(t) + a(t)\,\frac{f[v(g(t)) + \lambda_2 + R(g(t))]}{f[v(t)]}\,f[v(t)] \text{ or}$$

(17) $v^{(n)}(t) + s(t)f(v(t)) = 0$, where

$$s(t) = a(t)\,\frac{f[v(g(t)) + \lambda_2 + R(g(t))]}{f[v(t)]}$$

$$= a(t)\,\frac{f[w(g(t)) + R(g(t))]}{f[w(t) - \lambda_2]}$$

We now show $s(t) \geq \gamma a(t)$, for every γ, $0 < \gamma < 1$. Since $w^{(n)}(t) < 0$ and $w(t)$ is bounded, then $\dot{w}(t) > 0$ if n is even, and $\dot{w}(t) < 0$ if n is odd. Thus, $w(t)$ is monotone and bounded and, therefore, has a limit as $t \to +\infty$.

Let $\lim_{t \to \infty} w(t) = L$. Then $\lim_{t \to \infty} w(g(t)) = L$.

If $L \neq \lambda_2$, then $w(g(t)) + R(g(t)) \geq w(g(t)) - \lambda_2$ and f increasing implies

$$\frac{f[w(g(t)) + R(g(t))]}{f[w(t) - \lambda_2]} \geq \frac{f[w(g(t)) - \lambda_2]}{f[w(t) - \lambda_2]} \quad \text{and}$$

$$\lim_{t \to \infty} \frac{f[w(g(t)) - \lambda_2]}{f[w(t) - \lambda_2]} = \frac{f(L - \lambda_2)}{f(L - \lambda_2)} = 1$$

Thus, there exists a number T, such that given γ, $0 < \gamma < 1$, for all $t \geq T$,

$$\frac{f[w(g(t)) - \lambda_2]}{f[w(t) - \lambda_2]} > \gamma .$$

Then $s(t) = a(t) \dfrac{f[w(g(t)) + R(g(t))]}{f[w(t) - \lambda_2]} \geq a(t) \dfrac{f[w(g(t)) - \lambda_2]}{f[w(t) - \lambda_2]}$

$$\geq \gamma a(t).$$

By hypothesis v(t) is a solution to (17) and is, there fore, oscillatory, which contradicts $v(t) > 0$ for large t. Thus, x(t) must be oscillatory.

If $L = \lambda_2$, then n is odd, for if n were even, then $\dot{w}(t) > 0$ implies w(t) is increasing and $w(t) > \lambda_2$. Moreover, $w(t) = x(t) - R(t) > \lambda_2$, and for each term of the sequence $\{t_n^*\}$,

$w(t_n^*) = x(t_n^*) - R(t_n^*) = x(t_n^*) + \lambda_2 > \lambda_2$. Since

$\lim_{n \to \infty} w(t_n^*) = \lambda_2$ also, then $\lim_{n \to \infty} x(t_n^*) = 0$, and, therefore,

$\underline{\lim}\, x(t) = 0$. The proof is similar if one assumes $x(t) < 0$.

Q.E.D.

If all solutions to (17) are oscillatory, then all unbounded solutions to (13) are also oscillatory if the additional assumptions are made in Theorem III.2.

(v) $g(t) \geq t - C$ for large t, C a positive constant.

(vi) There exist constants β, $\delta > 0$ such that

$f(\lambda x) \geq \lambda^{\beta} f(x)$ if $x > 0$ and

$f(\lambda x) \leq \lambda^{\delta} f(x)$ if $x < 0$, λ constant.

Theorem III.1 and Theorem III.2 can be generalized by replacing Equation (13) with

(18) $x^{(n)}(t) + a(t)f[x(g(t)),x(t),\dot{x}(t),\ldots,x^{(n-1)}(t)] = Q(t)$

and the proofs follow in a similar manner. In Theorem III.1, for example, (14) would then become

(19) $v^{(n)}(t) + s(t)f[v(g(t)),v(t) + \lambda_2 + R(t), \dot{v}(t) + \dot{R}(t), \ldots,v^{(n-1)}(t) + R^{(n-1)}(t)] = 0.$

However, oscillation theorems for (19) are nowhere as abundant as those for (14) or (17).

In order to make use of Theorem III.1 and Theorem III.2, it is necessary to be familiar with oscillation theorems for equations of the form (14) and (17). The next theorem is an oscillation theorem due to G. Grefsrud [3] for the equation

(19) $x^{(n)}(t) + f(t,x(g(t))) = 0$ in which (14) is a special case.

Theorem III.3. Assume the following.

(i) $g(t) \geq t - c$, for large t, $c > 0$ and constant,

(ii) $f(t,y)$ continuous on $S = [0,+\infty) \times (-\infty,+\infty)$,

(iii) $a(t)\Phi(y) \leq f(t,y)$ if $y > 0$,

$b(t)\Psi(y) \geq f(t,y)$ if $y < 0$,

(iv) $a(t) \geq 0$, $b(t) \geq 0$ and locally integrable on $[0,+\infty)$, $a(t) \not\equiv 0 \not\equiv b(t)$ on any subinterval of $[0,+\infty)$,

(v) $\Phi(y)$, $\Psi(y)$ are non decreasing and $y\Phi(y) > 0$, $y\Psi(y) > 0$ on $(-\infty,+\infty)$ for $y \neq 0$,

(vi) There exist constants β, $\delta > 0$ for which $\Phi(\lambda y) = \lambda^{\beta}\Phi(y)$, $\Psi(\lambda y) = \lambda^{\delta}\Psi(y)$, λ constant

(vii) for some $\alpha > 0$,

$$\int_{\alpha}^{\infty} \frac{du}{\Phi(u)} < +\infty \quad \text{and} \quad \int_{-\alpha}^{-\infty} \frac{du}{\Psi(u)} < +\infty,$$

(viii) $\int_0^{\infty} t^{n-1}a(t)dt = \int_0^{\infty} t^{n-1}b(t)dt = +\infty$.

If n is even, every solution $x(t)$ of (19) valid for large t is oscillatory, while if n is odd, every solution valid for large t iseither oscillatory or tends monotonically to zero together with its first $n - 1$ derivatives.

There are a number of oscillation theorems for certain equations of the form (17), one of which is due to Ryder and Wend [11].

Example III.1. It is a simple matter to show that if $x(t)$ is any bounded solution valid for large t to

(20) $\ddot{x}(t) + x(t+\sin t) = \cos t$, then $x(t)$ is oscillatory.

Here, $a(t) = 1$, $f(x) = x$, $Q(t) = \cos t$, $g(t) = t + \sin t$. If $s(t) \geq \gamma$, $0 < \gamma < 1$, then it is well known that all solutions to the equation $\ddot{x}(t) + s(t)x(t) = 0$ are oscillatory. Thus, by Theorem III.2, all bounded solutions to (20) are also oscillatory.

Example III.2. Consider the equation

(21) $x^{(4)}(t) + \frac{2}{t^4}[x(e^t)]^5 = \sin(2t+1)$

Here, $a(t) = \frac{2}{t^4}$, $f(x) = x^5$, $Q(t) = \sin(2t+1)$, $R(t) = -\frac{1}{4}\sin(2t+1)$, and $g(t) = e^t$.

Let $s(t) \geq \gamma a(t)$, $0 < \gamma < 1$, and consider the equation

(22) $v^{(4)}(t) + s(t)[v(t)]^5 = 0.$

Since $\int_{t_0}^{\infty} t^3 s(t)dt \geq \int_{t_0}^{\infty} t^{-1}dt = +\infty$, it follows by a theorem due to A.G. Kartsatos [5], that all solutions to (22) valid for large t are oscillatory. Again, by Theorem III.2, all bounded solutions valid for large t to (21) are also oscillatory.

CHAPTER IV.

OSCILLATION UNDER THE EFFECT OF A SMALL FORCING TERM

The theorems of this chapter will be devoted to sufficient conditions for the maintenance of oscillation of solutions to functional differential equations of the form

(23) $x^{(n)}(t) + f(t,x(g(t))) = Q(t)$, n even, where $Q(t)$ represents a small forcing term.

The following two lemmas, which can be found in Ryder and Wend [11], summarize the possible behavior of nonoscillatory solutions and will simplify the proofs of the theorems in this chapter and Chapter V.

<u>Lemma IV.1</u>. Suppose $u(t) \in C^k[a,+\infty)$, $u(t) \geq 0$ and $u^{(k)}(t)$ is monotone on $[a,+\infty)$. Then exactly one of the following is true.

(i) $\lim\limits_{t \to \infty} u^{(k)}(t) = 0$,

(ii) $\lim\limits_{t \to \infty} u^{(k)}(t) > 0$ and $u(t), \dot{u}(t), \ldots, u^{(k-1)}(t)$ tend to ∞ as $t \to +\infty$.

<u>Lemma IV.2</u>. Suppose $u(t) \in C^n[a,+\infty)$, $u(t) \geq 0$ and $u^{(n)}(t) \leq 0$ on $[a,+\infty)$. Then exactly one of the following is true.

(I) $(-1)^k u^{(n-k)}(t) \leq 0$, $k = 0,1,\ldots,n-1$,

(II) There exists an odd integer $2i-1$, $1 \leq 2i-1 \leq n-1$ such that $(-1)^k u^{(n-k)}(t) \leq 0$, $k = 0,1,\ldots,2i-1$, $\lim_{t \to \infty} u^{(n-2i+1)}(t) \geq 0$, $\lim_{t \to \infty} u^{(n-2i)}(t) > 0$, and $u(t), \dot{u}(t), \ldots, u^{(n-2i-1)}(t)$ tend to ∞ as $t \to \infty$.

Now consider Equation (23) where the following assumptions are made on $Q(t)$ and $f(t,x)$.

(i) $Q(t)$ is real valued on $I = [t_0, +\infty)$ and for some function $R(t) : I \to R = (-\infty, +\infty)$, $R^{(n)}(t) = Q(t)$ and $\lim_{t \to \infty} R(t) = 0$.

(ii) $f(t,x) : I \times R \to R$ and there exist four continuous funtions

$P_i(t)$, $G_i(x)$, $i = 1,2$ for which $P_i(t) : I \to [0, +\infty)$ $i = 1,2$;

$G_1(x) > 0$ for $x > 0$;

$G_2(x) < 0$ for $x < 0$; and

$P_1(t)G_1(x) \leq f(t,x)$ if $x > 0$, while $f(t,x) \leq P_2(t)G_2(x)$ if $x < 0$.

The next two theorems generalize the results of A.G. Kartsatos [6] to differential equations with the functional argument $g(t)$.

Theorem IV.1. Assume (i) and (ii) hold and

(iii) $g(t)$ is continuous on $I = [t_0, \infty)$ and $\lim_{t \to \infty} g(t) = +\infty$,

(iv) $xf(t,x) > 0$ whenever $x \neq 0$, on $I \times R$

(v) $\int_{t_0}^{\infty} t^{n-1} P_i(t)\,dt = +\infty$, $i = 1,2$.

Then if n is even, every bounded solution $x(t)$ of (23) valid for large t is either oscillatory or $\lim_{t \to \infty} |x(t)| = 0$.

PROOF: Let $x(t)$ be a bounded, non-oscillatory solution to (23) valid for large t; i.e., suppose $0 < x(t) < M$ and hence by (iii) $0 < x(g(t)) < M$ for all $t \geq t_1 \geq t_0$ for some M.

Let $u(t) = x(t) - R(t)$. Then since $\lim_{t \to \infty} R(t) = 0$,

$u(t)$ is also bounded for large t. Moreover, $u^{(n)}(t) = x^{(n)}(t) - Q(t) = -f(t, x(g(t)))$ implies

(24) $u^{(n)}(t) + f(t, u(g(t)) + R(g(t))) = 0$.

We now show that (24) cannot have a bounded solution $u(t)$ such that $u(t) + R(t) > 0$ unless $u(t) < 0$, which yields the desired contradiction, unless $x(t) \to 0$.

It follows from (24) and (iv) that $u^{(n)}(t) = -f(t, u(g(t)) + R(g(t))) = -f(t, x(g(t))) < 0$, $t \geq t_1$.

Thus, $u^{(n)}(t) < 0$, $u(t)$ is bounded and $u(t) \geq 0$ for large t. By Lemma IV.2, it follows that

(25) $(-1)^k u^{(n-k)}(t) \leq 0$, $k = 0,1,\ldots,n-1$, and, therefore,

$$\lim_{t \to \infty} \dot{u}(t) = 0.$$

Thus, suppose $\lim_{t \to \infty} u(t) = \alpha > 0$. Then by (i),

(26) $$\lim_{t \to \infty} [u(t) + R(t)] = \alpha.$$

Now consider $G_1(u(t) + R(t))$. Since G_1 is continuous,

$\lim_{t \to \infty} G_1(u(t) + R(t)) = G_1(\alpha)$ and

$\lim_{t \to \infty} G_1(u(g(t)) + R(g(t))) = G_1(\alpha)$. Thus, choose $\epsilon > 0$,

but $\epsilon < G_1(\alpha)$. Then there exists $t_2 \geq t_1$ such that

(27) $$0 < A = G_1(\alpha) - \epsilon < G_1(u(g(t)) + R(g(t))) < G_1(\alpha) + \epsilon.$$

Now consider the equation

(28) $$[t^{n-1}u^{(n-1)}(t)]' = -t^{n-1}[f(t,u(g(t)) + R(g(t)))] + (n-1)t^{n-2}u^{(n-1)}(t)$$

An integration of (28) from t_2 to $t \geq t_2$ yields

$$(29)\quad t^{n-1}u^{(n-1)}(t) = t_2^{n-1}u^{(n-1)}(t_2)$$

$$- \int_{t_2}^{t} s^{n-1} f(s, u(g(s)) + R(g(s)))ds$$

$$+ (n-1) \int_{t_2}^{t} s^{n-2}u^{(n-1)}(s)ds$$

$$\leq t_2^{n-1}u^{(n-1)}(t_2) - \int_{t_2}^{t} s^{n-1}AP_1(s)ds$$

$$+ (n-1) \int_{t_2}^{t} s^{n-2}u^{(n-1)}(s)ds.$$

Since $t^{n-1}u^{(n-1)}(t) \geq 0$, and $\lim\limits_{t \to \infty} \int_{t_2}^{t} s^{n-1}P_1(s)ds = +\infty$,

(29) yields

$$(30)\quad \lim_{t \to \infty} \int_{t_2}^{t} s^{n-2}u^{(n-1)}(s)ds = +\infty.$$

Now integrate (30) by parts to obtain

$$(31)\quad \int_{t_2}^{t} s^{n-2}u^{(n-1)}(s)ds = s^{n-2}u^{(n-2)}(s) \Big|_{t_2}^{t}$$

$$- (n-2)\int_{t_2}^{t} s^{n-3}u^{(n-2)}(s)ds.$$

Again, by (25) $t^{n-2}u^{(n-2)}(t) \leq 0$ and (31) implies

$$(32)\quad \lim_{t \to \infty} \int_{t_2}^{t} s^{n-3}u^{(n-2)}(s)ds = -\infty.$$

From the results of (30) and (32), it is easy to conclude

$$(33)\quad \int_{t_2}^{\infty} s^{n-k}u^{(n-k+1)}(s)ds = \begin{cases} +\infty, & k = 2,4,\ldots,n-2 \\ -\infty, & k = 3,5,\ldots,n-1 \end{cases}$$

Thus, for $k = n - 1$,

$$(34)\quad \int_{t_2}^{\infty} s\ddot{u}(s)ds = -\infty.$$

Integrating (34) by parts yields

$$\int_{t_2}^{t} s\ddot{u}(s)ds = t\dot{u}(t) - t_2\dot{u}(t_2) - u(t) + u(t_2),$$

and from (34) we can conclude

$$\lim_{t \to \infty} [t\dot{u}(t) - u(t)] = -\infty.$$

However, $\dot{u}(t) \geq 0$ by (25) and, therefore, $\lim_{t \to \infty} [u(t)] = +\infty$ which contradicts the fact that $u(t)$ is bounded.

Thus, it must be the case that $u(t) \leq 0$ eventually, and, therefore, $u(t) = x(t) - R(t) \leq 0$ which implies $\lim_{t \to \infty} x(t) \leq \lim_{t \to \infty} R(t) = 0$. Since $x(t)$ was assumed positive for large t, it follows that $\lim_{t \to \infty} x(t) = 0$.

The proof is similar if one assumes $x(t) < 0$ for large t. Q.E.D.

Theorem IV.2. Assume (i) and (ii) hold, P. 29, and

(iii) $g(t)$ is continuous on $I = [t_0, +\infty)$, $g(t) \geq t - C$ for large t, where C is any positive constant,

(iv) $xf(t,x) > 0$ whenever $x \neq 0$ on $I \times R$,

(v) $G_i(x)$, $i = 1,2$ as given in (ii) are increasing and

$$\int_{\epsilon}^{\infty} \frac{ds}{G_1(s)} < +\infty, \quad \int_{-\epsilon}^{-\infty} \frac{ds}{G_2(s)} < +\infty \text{ for every } \epsilon > 0,$$

(vi) $\int_{t_0}^{\infty} t^{n-1} P_i(t)\,dt = +\infty$, $i = 1,2$.

Then if n is even, every solution $x(t)$ of (23) valid for large t is either oscillatory or $\lim_{t \to \infty} |x(t)| = 0$.

PROOF: Let $x(t)$ be any solution of (23) valid for large t. If $x(t)$ is bounded, the conclusion of the theorem follows from Theorem IV.1. Thus, suppose $x(t)$ is non-oscillatory and unbounded; e.g., suppose there exists t_1 such that $x(t) > 0$, $x(g(t)) > 0$ for all $t \geq t_1$.

Let $u(t) = x(t) - R(t)$. Since $R(t) \to 0$, there exists $t_2 \geq t_1$ and $\epsilon > 0$ such that

$$(35)\quad \begin{aligned} &u(t) > 0,\ 0 < u(t) - \epsilon < u(t) + R(t),\ t \geq t_2 \text{ and} \\ &u(g(t)) > 0,\ 0 < u(g(t)) - \epsilon < u(g(t)) + R(g(t)), \\ &t \geq t_2. \end{aligned}$$

Also, $u^{(n)}(t) = x^{(n)}(t) - Q(t) = -f(t,x(g(t)))$, or

$$\begin{aligned} 0 &= u^{(n)}(t) + f[t,u(g(t)) + R(g(t))] \\ &\geq u^{(n)}(t) + P_1(t)G_1[u(g(t)) + R(g(t))] \\ &\geq u^{(n)}(t) + P_1(t)G_1(u(g(t)) - \epsilon), \text{ since } G_1 \text{ is increasing.} \end{aligned}$$

Thus,

$$u^{(n)}(t) \leq - P_1(t)G_1(u(g(t)) - \epsilon) < 0, \text{ for } t \geq t_2.$$

By Lemma IV.2, there are two cases to consider.

Case 1: $(-1)^k u^{(n-k)}(t) \leq 0$, $k = 1,2,\ldots,n - 1$.

Here $u(t)$ is non-decreasing, so that by (iii), $u(g(t)) \geq u(t - C)$, and (35) yields $u(g(t)) + R(g(t)) \geq u(g(t)) - \epsilon \geq u(t - C) - \epsilon > 0$ for $t \geq t_2 + C$. Since G_1 is increasing,

$$G_1(u(g(t)) + R(g(t))) \geq G_1(u(t - C) - \epsilon)$$

Now let $t \geq t_3 \geq t_2 + C$, and let

$$F(t) = \frac{t^{n-1}u^{(n-1)}(t)}{G_1(u(t - C) - \epsilon)} .$$ Then

$$\dot{F}(t) = \frac{[t^{n-1}u^{(n)}(t) + (n - 1)t^{n-2}u^{(n-1)}(t)]}{G_1(u(t - C) - \epsilon)}$$

$$+ t^{n-1}u^{(n-1)}(t) \frac{d}{dt} [\frac{1}{G_1(u(t - C) - \epsilon)}]$$

$$= \frac{-t^{n-1}f(t,u(g(t)) + R(g(t)))}{G_1(u(t - C) - \epsilon)} + \frac{(n - 1)t^{n-2}u^{(n-1)}(t)}{G_1(u(t - C) - \epsilon)}$$

$$+ t^{n-1}u^{(n-1)}(t) \frac{d}{dt} [\frac{1}{G_1(u(t - C) - \epsilon)}]$$

$$\leq \frac{-t^{n-1}P_1(t)G_1(u(g(t)) + R(g(t)))}{G_1(u(t - C) - \epsilon)} + \frac{(n - 1)t^{n-2}u^{(n-1)}(t)}{G_1(u(t - C) - \epsilon)}$$

$$+ t^{n-1}u^{(n-1)}(t) \frac{d}{dt} [\frac{1}{G_1(u(t - C) - \epsilon)}]$$

Thus,

$$(36)\quad \dot{F}(t) \leq -t^{n-1}P_1(t) + \frac{(n - 1)t^{n-2}u^{(n-1)}(t)}{G_1(u(t - C) - \epsilon)}$$

$$+ t^{n-1}u^{(n-1)}(t) \frac{d}{dt} [\frac{1}{G_1(u(t - C) - \epsilon)}]$$

An integration of (36) from t_3 to $t \geq t_3$ yields

$$(37)\quad F(t) \leq F(t_3) - \int_{t_3}^{t} s^{n-1}P_1(s)ds$$

$$+ (n-1)\int_{t_3}^{t} \frac{s^{n-2}u^{(n-1)}(s)}{G_1(u(s-C)-\epsilon)}\,ds$$

$$+ \int_{t_3}^{t} s^{n-1}u^{(n-1)}(s)\,d\left[\frac{1}{G_1(u(s-C)-\epsilon)}\right],$$ where the

last integration is considered in the Riemann-Stieltjes sense, and $\frac{1}{G_1(u(t-C)-\epsilon)}$ is a decreasing function of t. Moreover, since $u^{(n-1)}(t) \geq 0$, the last integral in (37) is non positive and can be dropped from (37) to yield

$$(38)\quad F(t) \leq F(t_3) - \int_{t_3}^{t} s^{n-1}P_1(s)ds$$

$$+ (n-1)\int_{t_3}^{t} \frac{s^{n-2}u^{(n-1)}(s)}{G_1(u(s-C)-\epsilon)}\,ds.$$

Since the first integral on the right in (38) approaches $+\infty$, we obtain

$$(39)\quad \lim_{t\to\infty}\left[F(t) - (n-1)\int_{t_3}^{t} \frac{s^{n-2}u^{(n-1)}(s)}{G_1(u(s-C)-\epsilon)}\,ds\right] = -\infty.$$

For convenience, let

$$q(t) = \int_{t_3}^{t} \frac{s^{n-2}u^{(n-1)}(s)}{G_1(u(s - C) - \epsilon)} ds.$$ Then $t\dot{q}(t) = F(t)$ and

(40) $$\lim_{t \to \infty} [t\dot{q}(t) - (n - 1)q(t)] = - \infty.$$

Since $t\dot{q}(t) \geq 0$ for large t, then (40) implies $\lim_{t \to \infty} [q(t)] = + \infty$. That is,

(41) $$\lim_{t \to \infty} \int_{t_3}^{t} \frac{s^{n-2}u^{(n-1)}(s)}{G_1(u(s - C) - \epsilon)} ds = + \infty.$$

An integration of (41) by parts yields

(42) $$\int_{t_3}^{t} \frac{s^{n-2}}{G_1(u(s - C) - \epsilon)} u^{(n-1)}(s) ds$$

$$= \frac{s^{n-2}u^{(n-2)}(s)}{G_1(u(s - C) - \epsilon)} \Bigg|_{t_3}^{t} - (n - 2)\int_{t_3}^{t} \frac{s^{n-3}u^{(n-2)}(s)}{G_1(u(s - C) - \epsilon)} ds$$

$$- \int_{t_3}^{t} s^{n-2}u^{(n-2)}(s) d\left[\frac{1}{G_1(u(s - C) - \epsilon)}\right]$$

where again, the last integral on the right is considered in the Riemann-Stieltjes sense and is non negative, since $u^{(n-2)}(s) \leq 0$.

Thus, from (42), we have

$$(43)\quad \int_{t_3}^{t} \frac{s^{n-2}}{G_1(u(s - C) - \epsilon)} \, u^{(n-1)}(s)\,ds$$

$$\leq \frac{s^{n-1}u^{(n-2)}(s)}{g_1(u(s - C) - \epsilon)}\Bigg|_{t_3}^{t} - (n - 2)\int_{t_3}^{t} \frac{s^{n-3}u^{(n-2)}(s)}{G_1(u(s - C) - \epsilon)}\,ds$$

The first term on the right of (43) is eventually negative, and from (41), the left side of (43) tends to $+\infty$, which implies the last integral on the right of (43) tends to $-\infty$; i.e.,

$$(44)\quad \lim_{t \to \infty} \int_{t_3}^{t} \frac{s^{n-3}u^{(n-2)}(s)}{G_1(u(s - C) - \epsilon)}\,ds = -\infty .$$

By induction, observing (41) and (44), we have

$$(45)\quad \int_{t_3}^{\infty} \frac{t^{n-(m+1)}u^{(n-m)}(t)}{G_1(u(t - C) - \epsilon)}\,dt = \begin{Bmatrix} +\infty, & m = 1,3,5,\ldots,n - 1 \\ -\infty, & m = 2,4,6,\ldots,n - 2 \end{Bmatrix}$$

For $m = n - 1$,

$$(46)\quad \int_{t_3}^{\infty} \frac{\dot{u}(s)}{G_1(u(s - C) - \epsilon)}\,ds = +\infty ,$$ Since $\ddot{u}(s) \leq 0$,

then $\dot{u}(s)$ is non-increasing, so that $\dot{u}(s) \leq \dot{u}(s - C)$ and from (46),

$$+\infty = \int_{t_3}^{\infty} \frac{\dot{u}(s)}{G_1(u(s - C) - \epsilon)}\, ds \le \int_{t_3}^{\infty} \frac{\dot{u}(s - C)}{G_1(u(s - C) - \epsilon)}\, ds$$

$$= \int_{t_3}^{\infty} \frac{dv}{G_1(v)}\ , \text{ where } v = u(t - C) - \epsilon .$$

This last equation is a contradiction to hypothesis (v).

Case 2: Now suppose there is an odd integer $2i - 1$, $1 \le 2i - 1 \le n - 1$ for which $(-1)^k u^{(n-k)}(t) \le 0$, $k = 1,2,\dots,2i - 1$, and $u^{(n-2i)}(t) \ge 0$ for large t. Then (45) still holds for $m = 2i - 1$ and we have

$$(47) \quad \int_{t_3}^{\infty} \frac{t^{n-2i} u^{(n-2i+1)}(t)}{G_1(u(t - C) - \epsilon)}\, dt = +\infty .$$

Before proceeding any further, we show there exists a positive constant M such that

$t^{n-2i} u^{(n-2i+1)}(t) \le M\dot{u}(t - C)$ for t sufficiently large. To do this, consider the Taylor polynomial for $\dot{u}(t)$ on the interval $[t_3, t - C]$. Since $u(t) \in C^{n-1}[t_0, +\infty]$, there exists a τ, $t_3 \le \tau \le t - C$ for which $\dot{u}(t - C)$

$$= \dot{u}(t_3) + \ddot{u}(t_3)[t - C - t_3] + \dots + u^{(n-2i+1)}(\tau)\ \frac{[t - C - t_3]^{n-2i}}{(n - 2i)!}$$

By Lemma IV.2, $\dot{u}(t), \ddot{u}(t), \ldots, u^{(n-2i)}(t)$ are all eventually non negative, so that

$$(50) \quad \dot{u}(t - C) \geq u^{(n-2i+1)}(\tau) \frac{[t - C - t_3]^{n-2i}}{(n - 2i)!}$$

Now for $k = 2i - 2$, $u^{(n-2i+2)}(t) \leq 0$ implies $u^{(n-2i+1)}(t)$ is non increasing as well as non negative so that (50) becomes

$$\dot{u}(t - C) \geq u^{(n-2i+1)}(t) \frac{[t - C - t_3]^{n-2i}}{(n - 2i)!}, \text{ or}$$

$$(t - C - t_3]^{n-2i} u^{(n-2i+1)}(t) \leq K\dot{u}(t - C) \text{ for all } t \geq \tau.$$

Multiplying through by $\dfrac{t^{n-2i}}{[t - C - t_3]^{n-2i}}$,

$$t^{n-2i} u^{(n-2i+1)}(t) \leq \dot{u}(t - C) \frac{K t^{n-2i}}{[t - C - t_3]^{n-2i}}$$

$$= \dot{u}(t - C) K \left[\frac{1}{1 - \frac{C + t_3}{t}}\right]^{n-2i}$$

and for t sufficiently large, there exists a positive constant $M \geq K\left[\dfrac{1}{1 - \frac{C + t_3}{t}}\right]^{n-2i}$ for which

(51) $t^{n-2i}u^{(n-2i+1)}(t) \leq M\dot{u}(t - C)$.

With the use of (49), (47) can now be written as

$$+\infty = \int_{t_3}^{\infty} \frac{t^{n-2i}u^{(n-2i+1)}(t)}{G_1(u(t - C) - \epsilon)}\,dt \leq M \int_{t_3}^{\infty} \frac{\dot{u}(t - C)}{G_1(u(t - C) - \epsilon)}\,dt$$

$$= M \int_{t_3}^{\infty} \frac{dv}{G_1(v)} = +\infty.$$

where again $v(t) = u(t - C) - \epsilon$, and this last equation is a contradiction to hypothesis (v). Q.E.D.

Remarks: Theorem IV.1 and Theorem IV.2 can be generalized to the case when n is odd, in which one should conclude that all solutions considered either oscillate or tend monotonically to zero.

Also, in addition to the assumptions made in Theorem IV.2 (Theorem IV.3), if R(t) is assumed oscillatory, then every bounded solution (every solution) of (23) is oscillatory. The proof of this follows by assuming a solution x(t) to be eventually positive and arriving at a contradiction by showing $x(t) \leq R(t)$ for large t.

CHAPTER V

OSCILLATION UNDER THE EFFECT OF A SMALL NONLINEAR DAMPING

We now consider the functional differential equation

$$(52)\quad x^{(n)}(t) + P(t,x(t),x(g(t)),\overline{x}(t))x^{(n-1)}(t) + Q(t,x(t),x(g(t)),\overline{x}(t)) = 0$$

where $\overline{x}(t)$ denotes $\dot{x}(t),\ddot{x}(t),\ldots,x^{(n-1)}(t)$.

If $P \equiv 0$, sufficient conditions are given in [3] for solutions to (52) to be oscillatory subject to appropriate conditions on Q. The object of this chapter is to impose conditions on the nonlinear damping term P so as to maintain oscillatory solutions to (52).

If one considers the second order linear equation

$$(53)\quad \ddot{x}(t) + A\dot{x}(t) + Bx(t) = 0,$$ A, B positive constants,

the oscillation of solutions to (53) is determined by the size of A as compared to the size of B. A similar approach could be investigated for the functions P and Q in (52). However, the approach here will be to require P to become small in some sense for large t, while the conditions imposed on Q will be independent of those imposed on P. In addition, the special case $P \equiv 0$ will be included in the following results.

The following conditions on P and Q will be considered.

(i) P and Q are continuous functions on $[t_0, +\infty) \times R^{n+1} \to R$, where $R = (-\infty, +\infty)$, and $xQ(t,x,y,x_1,\ldots,x_{n-1}) > 0$ if $x \neq 0$.

(ii) There exist continuous functions $k(t) \geq 0$, $m(t) \geq 0$ for which $-k(t) \leq P(t,x,y,x_1,x_2,\ldots x_{n-1}) \leq m(t)$ on $[t_0, +\infty)$ and for some $\alpha \geq t_0$ and $\bar{t} \geq t_0$,

$$\int_\alpha^\infty t^{n-1}k(t)\,dt < +\infty, \quad \lim_{t\to\infty} \int_\alpha^t \exp[-\int_{\bar{t}}^s m(u)\,du]\,ds = +\infty.$$

(iii) $a(t)\Phi(x,y) \leq Q(t,x,y,x_1,\ldots x_{n-1})$ if $x > 0$ and $\Phi(x,y) > 0$ if $x > 0$, $y > 0$, $Q(t,x,y,x_1,\ldots,x_{n-1}) \leq b(t)\Psi(x,y)$ if $x < 0$ and $\Psi(x,y) < 0$ if $x < 0$, $y < 0$, $a(t)$, $b(t)$ are non-negative, continuous, and locally integrable on $[t_0, +\infty)$, and $\Phi(x,y)$, $\Psi(x,y)$ are continuous and non-decreasing in x and y on $R \times R$.

The following lemma can be found in Kartsatos and Onose [7] for the special case in which $g(t) = t$.

Lemma V.1. In addition to (i), assume that

(a) $g(t)$ is continuous on $[t_0, +\infty)$, $\lim_{t \to \infty} g(t) = +\infty$,

(b) there exist continuous functions $k(t) \geq 0$, $m(t) \geq 0$ for which $-k(t) \leq P \leq m(t)$ on $[t_0, \infty)$,

(c) $\lim_{t \to \infty} \int_{\bar{t}}^{t} \exp[-\int_{\bar{t}}^{s} m(u)du]ds = +\infty$ for any $\bar{t} > t_0$.

Then every nonoscillatory solution $x(t)$ of (52) valid for large t is such that $x(t)x^{(n-1)}(t) > 0$ and

$$|x^{(n-1)}(t)| \leq |x^{(n-1)}(t_1)| \exp[\int_{t_1}^{t} k(s)ds], \quad t \geq t$$

PROOF: Let $x(t)$ be a nonoscillatory solution to (52) valid for large t; e.g., suppose $x(t) > 0$, $x(g(t)) > 0$ for all $t \geq t_1$ for some $t_1 \geq t_0$. We first show that $x^{(n-1)}(t)$ is eventually of one sign for all $t \geq t_1$. Thus, suppose $x^{(n-1)}(\tau) = 0$ for some $\tau \geq t_1$. Then from (52), $x^{(n)}(\tau) = -Q(\tau, x(\tau), x(g(\tau)), \bar{x}(\tau)) < 0$, and, therefore, $x^{(n)}(t) < 0$ for every zero of $x^{(n-1)}(t)$. Thus, $x^{(n-1)}(t)$ can have at most one zero on $[t_1, +\infty)$.

Now suppose $x^{(n-1)}(\bar{t}) < 0$ for some $\bar{t} > t_1$. Then $x^{(n-1)}(t) < 0$ for all $t \geq \bar{t}$, and hence,

$$(53)\quad x^{(n)}(t) + P(t,x(t),x(g(t)\overline{x}(t))x^{(n-1)}(t)$$
$$= -Q(t,x(t),x(g(t)),\overline{x}(t)) < 0$$

Division of (53) by $x^{(n-1)}(t) < 0$ and integrating from $\overline{t}$ to $t > \overline{t}$ yields,

$$(54)\quad \ln\left|\frac{x^{(n-1)}(t)}{x^{(n-1)}(\overline{t})}\right| \geq -\int_{\overline{t}}^{t} P(s,x(s),x(g(s)),\overline{x}(s))ds$$

$$\geq -\int_{\overline{t}}^{t} m(s)ds, \text{ from which we get}$$

$$(55)\quad x^{(n-1)}(t) \leq x^{(n-1)}(\overline{t})\exp[-\int_{\overline{t}}^{t} m(s)ds].$$

Now integrate (55) from $\overline{t}$ to t, to obtain

$$(56)\quad x^{(n-2)}(t) \leq x^{(n-2)}(\overline{t}) + x^{(n-1)}(\overline{t})\int_{\overline{t}}^{t}\exp[-\int_{\overline{t}}^{s} m(u)du]ds.$$

By (c), since $x^{(n-1)}(\overline{t}) < 0$, then in taking the limit of $x^{(n-2)}(t)$ in (56), we get $\lim_{t\to\infty} x^{(n-2)}(t) = -\infty$.

But this implies $\lim_{t\to\infty} x(t) = -\infty$, which contradicts $x(t) > 0$. Thus, $x^{(n-1)}(t) \geq 0$ on $[t_1,+\infty)$.

To show $x^{(n-1)}(t) \neq 0$ on $[t_1,+\infty]$, suppose $x^{(n-1)}(\overline{t}) = 0$, $\overline{t} > t_1$. Then again by (52), $x^{(n)}(\overline{t}) < 0$, and, therefore, $x^{(n-1)}(t)$ is strictly decreasing on some interval containing $\overline{t}$, which contradicts $x^{(n-1)}(t) \geq 0$. Thus, $x^{(n-1)}(t) > 0$, and, therefore, $x(t)x^{(n-1)}(t) > 0$ on $[t_1,+\infty)$.

Now integrate (52) from t_1 to t, observing first that on $[t_1,+\infty)$

$$x^{(n)}(t) = -P(t,x(t),x(g(t)),\overline{x}(t))x^{(n-1)}(t)$$

$$- Q(t,x(t),x(g(t)),\overline{x}(t)) \leq k(t)x^{(n-1)}(t),$$ from which

$$x^{(n-1)}(t) \leq x^{(n-1)}(t_1) + \int_{t_1}^{t} k(s)x^{(n-1)}(s)ds$$

By Gronwall's Inequality,

$$|x^{(n-1)}(t)| \leq |x^{(n-1)}(t_1)| \exp[\int_{t_1}^{t} k(s)ds], \ t \geq t_1.$$

The proof is similar if one assumes $x(t) < 0$.

Q.E.D.

We are now ready to state a theorem on the properties of bounded solutions to (52). The following theorem and Theorem V.2 generalize the results of Kartsatos and Onose [7].

Theorem V.1. In addition to (i) - (iii), assume

(iv) $g(t)$ is continuous on $[t_0, +\infty)$ and $\lim_{t \to \infty} g(t) = +\infty$.

(v) $\int_{t_0}^{\infty} t^{n-1} a(t)\,dt = \int_{t_0}^{\infty} t^{n-1} b(t)\,dt = +\infty$.

Then if n is even, every bounded solution of (52) valid for large t is oscillatory, while if n is odd, every bounded solution of (52) valid for large t either oscillates or tends monotonically to zero along with its first $n - 2$ derivatives.

PROOF: Suppose $x(t)$ is a bounded nonoscillatory solution of (52); i.e., suppose $x(t) > 0$ and $x(g(t)) > 0$ for all $t \geq t_1 \geq t_0$. Then by Lemma V.1, $x^{(n-1)}(t) > 0$ and

$$(57)\quad x^{(n-1)}(t) \leq x^{(n-1)}(t_1)\exp\left[\int_{t_1}^{t} k(s)\,ds\right]$$

$$\leq x^{(n-1)}(t_1)\exp\left[\int_{t_1}^{\infty} k(s)\,ds\right] = N_x < +\infty \text{ by}$$

(ii).

Moreover, by (iv), there is a number $t_2 \geq t_1$ such that $g(t) \geq t_1$ for all $t \geq t_2$.

Since $x(t)$ is bounded, there are two cases to consider, according to Lemma IV.2.

Case 1: If n is even, then $(-1)^{i+1}x^{(i)}(t) > 0$, $i = 1,2,\ldots,n-1$ for all $t \geq t_1$. Then for $i = 1$, $\dot{x}(t) > 0$ and hence, for all $t \geq t_2 \geq t_1$, $x(t) \geq x(t_1) > 0$ and $x(g(t)) \geq x(t_1) > 0$. Also, both $x(t)$ and $x(g(t))$ are bounded on $[t_2,+\infty)$. Since $\Phi(x(t),x(g(t)))$ is continuous on $[t_2,+\infty)$, there exist constants L, $M > 0$ for which

(58) $L < \Phi(x(t),x(g(t))) < M$ for $t \geq t_2$.

Now multiply (52) by t^{n-1} and integrate, using (57) and (58) to obtain

$$(59)\quad \int_{t_2}^{t} s^{n-1}x^{(n)}(s)ds = -\int_{t_2}^{t} s^{n-1}Px^{(n-1)}(s)ds - \int_{t_2}^{t} s^{n-1}Q\,ds$$

$$\leq \int_{t_2}^{t} s^{n-1}k(s)N_x\,ds$$

$$-\int_{t_2}^{t} s^{n-1}a(s)\Phi(x(s),x(g(s)))ds$$

$$\leq N_x \int_{t_2}^{t} s^{n-1}k(s)ds - L\int_{t_2}^{t} s^{n-1}a(s)ds.$$

Integration of the left side of (59) yields

$$(60)\quad t^{n-1}x^{(n-1)}(t) - (n-1)\int_{t_2}^{t} s^{n-2}x^{(n-1)}(s)ds$$

$$\leq t_2^{n-1}x^{(n-1)}(t_2) + N_x \int_{t_2}^{t} s^{n-1}k(s)ds - L\int_{t_2}^{t} s^{n-1}a(s)ds.$$

With the use of hypotheses (ii) and (v), we may conclude that

$$(61)\quad \lim_{t\to\infty} \int_{t_2}^{t} s^{n-2}x^{(n-1)}(s)ds = +\infty.$$

Another integration by parts from (61) gives

$$\int_{t_2}^{t} s^{n-2}x^{(n-1)}(s)ds = t^{n-2}x^{(n-2)}(t) - t_2^{n-1}x^{(n-2)}(t_2)$$

$$- (n-2)\int_{t_2}^{t} s^{n-3}x^{(n-2)}(s)ds.$$

Since $x^{(n-2)}(t) < 0$, it follows from (61) that

$$(62)\quad \lim_{t\to\infty} \int_{t_2}^{t} s^{n-3}x^{(n-2)}(s)ds = -\infty.$$

Thus, by induction, observing (61) and (62), we have

$$(63)\quad \int_{t_2}^{\infty} t^{n-(m+1)}x^{(n-m)}(t)dt = \begin{cases} +\infty, & m = 1,3,\ldots,n-1 \\ -\infty, & m = 2,4,\ldots,n-2 \end{cases}$$

In particular, for $m = n - 1$,

$$\int_{t_2}^{\infty} \dot{x}(t)dt = \lim_{t \to \infty} [x(t) - x(t_2)] = +\infty,$$ which contradicts

the boundedness of $x(t)$.

Thus, for n even, $x(t)$ must be oscillatory.

Case 2: If n is odd, then $(-1)^{i+1}x^{(i)}(t) < 0$, $i = 1,2,\ldots,n - 1$, for all $t \geq t_2$. Thus, for $i = 1$, $\dot{x}(t) < 0$ and hence, $x(t)$ is monotone decreasing and bounded below by 0. If $\lim_{t \to \infty} x(t) = 0$, the proof is complete, since Lemmas IV.1, IV.2 yield $\lim_{t \to \infty} x^{(i)}(t) = 0$, $i = 1,2,\ldots,n - 2$.

Suppose then, $\lim_{t \to \infty} x(t) = \alpha > 0$. Then $\lim_{t \to \infty} x(g(t)) = \alpha > 0$. As before, since Φ is continuous, there exist constants $L, M > 0$ for which $L < \Phi(x(t),x(g(t))) < M$ on $[t_2,+\infty)$, and proceeding as in Case 1, we obtain

$$(64) \quad \int_{t_2}^{\infty} t^{n-(m+1)}x^{(n-m)}(t)dt = \begin{Bmatrix} +\infty, & m = 1,3,\ldots,n - 2 \\ -\infty, & m = 2,4,\ldots,n - 1 \end{Bmatrix}$$

Then for $m = n - 1$,

$\int_{t_2}^{\infty} \dot{x}(t)dt = \lim_{t \to \infty} [x(t) - x(t_2)] = -\infty$, which contradicts the boundedness of $x(t)$ on $[t_2, +\infty)$.

Thus, $\lim_{t \to \infty} x(t) = 0$ and hence, $\lim_{t \to \infty} x^{(i)}(t) = 0$, $i = 0,1,\ldots,n-2$. Q.E.D.

Before investigating solutions of (52) which may not be bounded, a lemma will be given which is very similar to Theorem III.7 in Grefsrud [3], and for this reason, the proof will not be given here. In Grefsrud's theorem, Φ and Ψ are functions of one variable, whereas Φ and Ψ are functions of two variables here. Conditions will be imposed on Φ and Ψ by way of the second variable here, whereas similar conditions were imposed on Φ and Ψ in Grefsrud's theorem by way of the single variable.

<u>Lemma V.2</u>. In addition to (i) and (iii) assume

(a) $g(t) \geq t - C$ for large t, $C > 0$ constant, and $g(t)$ is continuous for large t.

(b) There exist constants $\beta, \delta > 0$ such that

$\Phi(x,\lambda y) \geq \lambda^{\beta}\Phi(x,y)$,

$\Psi(x,\lambda y) \leq \lambda^{\delta}\Psi(x,y)$, λ constant.

Let $x(t)$ be a non-oscillatory solution to (52) valid for large t. [If n is odd, assume $\lim_{t \to \infty} x(t) \neq 0$].

Then there is a constant $\mu > 0$ such that $\Phi(x(t),x(g(t))) \geq \mu\Phi(x(t),x(t))$ if $x(t)$ is eventually positive, $\Psi(x(t),x(g(t))) \leq \mu\Psi(x(t),x(t))$ if $x(t)$ is eventually negative, for t sufficiently large.

With the use of Lemma V.2, we can now prove

Theorem V.2. Suppose conditions (i) - (iii) hold and (a), (b) hold in Lemma V.2, and in addition

(c) For some $\alpha > 0$,

$$\int_{\alpha}^{\infty} t^{n-1}a(t)dt = \int_{\alpha}^{\infty} t^{n-1}b(t)dt = +\infty.$$

(d) $\displaystyle\int_{\alpha}^{\infty} \frac{du}{\Phi(u,u)} < +\infty, \quad \int_{-\alpha}^{-\infty} \frac{du}{\Psi(u,u)} < +\infty.$

Then if n is even, every solution $x(t)$ of (52) valid for large t is oscillatory, while if n is odd, $x(t)$ is either oscillaotry or tends monotonically to zero along with its first $n - 2$ derivatives as $t \to \infty$.

PROOF: Suppose $x(t)$ is non-oscillatory; i.e., suppose $x(t) > 0$, $x(g(t)) > 0$ for all $t \geq t_1 \geq t_0$ for some t_1. If $x(t)$ is bounded on $[t_1, \infty)$, the proof follows from Theorem V.1. Thus, assume $x(t)$ is unbounded. Now by Lemma V.1 and (ii),

(65) $0 < x^{(n-1)}(t) \leq x^{(n-1)}(t_1)\exp[\int_{t_1}^{t} k(s)ds]$

$$\leq x^{(n-1)}(t_1)\exp[\int_{t_1}^{\infty} k(s)ds] = N_x < +\infty.$$

We will consider two possible cases.

Case 1: $(-1)^m x^{(n-m)}(t) \leq 0$, $m = 1,2,\ldots,n-1$ for all $t \geq t_1$.

Case 2: There is an odd integer $2i - 1$, $i \leq 2i - 1 \leq n - 1$, for which $(-1)^m x^{(n-m)}(t) \leq 0$, $m = 1,2,\ldots,2i - 1$, while $x^{(n-2i)}(t) > 0$ and $x(t),\dot{x}(t),\ldots,x^{(n-2i-1)}(t)$ tend to ∞ as $t \to \infty$.

In Case 1, if n is odd, then for $m = n - 1$, $\dot{x}(t) \leq 0$ which contradicts $x(t) > 0$ and unbounded.

If n is even in Case 1 or 2, then $\dot{x}(t) \geq 0$ and hence, $x(t)$ tends to $+\infty$ monotonically. Then there exists $t_2 \geq t_1$ for which $x(t) \geq x(t_1)$ and $x(g(t)) \geq x(t_1)$, $t \geq t_2$ and hence,

(66) $\Phi(x(t),x(g(t))) \geq \Phi(x(t_1),x(t_1))$, $t \geq t_2 \geq t_1$.

By Lemma V.2, there is a constant $\mu > 0$ for which

(67) $\dfrac{Q(t,x(t),x(g(t)),\overline{x}(t))}{\Phi(x(t),x(t))} \geq \mu \dfrac{Q(t,x(t),x(g(t)),\overline{x}(t))}{\Phi(x(t),x(g(t)))}$

$$\geq \mu a(t).$$

Now multiply (52) by $t^{n-1}/\Phi(x(t),x(t))$ and integrate from t_2 to $t \geq t_2$, using (66) and (67) to obtain

$$(68)\quad \int_{t_2}^{t} \frac{s^{n-1}}{\Phi(x(s),x(s))} x^{(n)}(s)ds$$

$$= -\int_{t_2}^{t} \frac{s^{n-1}Px^{(n-1)}(s)}{\Phi(x(s),x(s))} ds - \int_{t_2}^{t} \frac{s^{n-1}Q(s,x(s),x(g(s)),\overline{x}(s))}{\Phi(x(s),x(s))} ds$$

$$\leq \int_{t_2}^{t} \frac{s^{n-1}k(s)N_x}{\Phi(x(s),x(s))} ds - \int_{t_2}^{t} s^{n-1}\mu a(s)ds$$

$$\leq \frac{N_x}{\Phi(x(t_1),x(t_1))} \int_{t_2}^{t} s^{n-1}k(s)ds - \mu \int_{t_2}^{t} s^{n-1}a(s)ds.$$

By hypotheses (ii) and (c), the right side of (68) tends to $-\infty$.

Next, integrate the left side of (68) by parts.

$$(69)\quad \int_{t_2}^{t} \frac{s^{n-1}}{\Phi(x(s),x(s))} x^{(n)}(s)ds$$

$$= \frac{t^{n-1}}{\Phi(x(t),x(t))} x^{(n-1)}(t) - C_1$$

$$- \int_{t_2}^{t} x^{n-1}(s)d\left[\frac{s^{n-1}}{\Phi(x(s),x(s))}\right]$$

$$= \frac{t^{n-1}x^{(n-1)}(t)}{\Phi(x(t),x(t))} - C_1 - (n - 1)\int_{t_2}^{t} \frac{s^{n-2}x^{(n-1)}(s)}{\Phi(x(s),x(s))}\,ds$$

$$- \int_{t_2}^{t} s^{n-1}x^{(n-1)}(s)\,d\left[\frac{1}{\Phi(x(s),x(s))}\right]$$

where the last integral is considered in the Riemann-Stieltjes sense, and $C_1 = \frac{t_2^{n-1}x^{(n-1)}(t_2)}{\Phi(x(t_2),x(t_2))}$. Furthermore, since $x^{(n-1)}(t) > 0$ and $1/\Phi(x(t),x(t))$ is a decreasing function of t, when $\dot{x}(t) \geq 0$, it follows that $\int_{t_2}^{t} s^{n-1}x^{(n-1)}(s)\,d\left[\frac{1}{\Phi(x(s),x(s))}\right] \leq 0$, and w may conclude from (68) and (69) that, since $\frac{t^{n-1}x^{(n-1)}(t)}{\Phi(x(t),x(t))} > 0$, then

(70) $$\lim_{t \to \infty} \int_{t_2}^{t} \frac{s^{n-2}x^{(n-1)}(s)}{\Phi(x(s),x(s))}\,ds = +\infty.$$

Now consider Case 1, in which $(-1)^m x^{(n-m)}(t) \leq 0$, $m = 1,2,\ldots,n - 1$.

Another integration by parts from (70) yields, as in (69),

$$(71)\quad \int_{t_2}^{t} \frac{s^{n-2}x^{(n-1)}(s)}{\Phi(x(s),x(s))}\,ds = \frac{t^{n-2}x^{(n-2)}(t)}{\Phi(x(t),x(t))} - C_2$$

$$- (n-2) \int_{t_2}^{t} \frac{s^{n-3}x^{(n-2)}(s)}{\Phi(x(s),x(s))}\,ds$$

$$- \int_{t_2}^{t} s^{n-2}x^{(n-2)}(s)\,d\left[\frac{1}{\Phi(x(s),x(s))}\right],$$

where again, the last integral is considered in the Riemann-Stieltjes sense and is non-negative, since $x^{(n-2)}(t) \leq 0$. From (70), we conclude in (71) that

$$(72)\quad \lim_{t \to \infty} \int_{t_2}^{t} \frac{s^{n-3}x^{(n-2)}(s)}{\Phi(x(s),x(s))}\,ds = -\infty.$$

By induction, observing (70) and (72), for n even

$$(73)\quad \int_{t_2}^{\infty} \frac{t^{n-(m+1)}x^{(n-m)}(t)}{\Phi(x(t),x(t))}\,dt = \begin{cases} +\infty, & m = 1,3,\ldots,n-1 \\ -\infty, & m = 2,4,\ldots,n-2 \end{cases}$$

In particular, for $m = n - 1$, (73) yields

$$\int_{t_2}^{\infty} \frac{\dot{x}(t)}{\Phi(x(t),x(t))}\,dt = \int_{t_2}^{\infty} \frac{du}{\Phi(u,u)} = +\infty,$$ which contradicts

hypothesis (d).

Now suppose Case 2 applies, so that

$(-1)^m x^{(n-m)}(t) \leq 0$, $m = 1,2,\ldots,2i-1$, while $x^{(n-2i)}(t) > 0$.

Then $\dot{x}(t) \geq 0$ eventually, and (73) still holds for $m = 2i - 1$, which yields, since $2i - 1$ is odd,

$$(74) \int_{t_2}^{\infty} \frac{t^{n-2i} x^{(n-2i+1)}(t)}{\Phi(x(t),x(t))} dt = +\infty.$$

Now, as in the proof of Theorem IV.2, by considering a Taylor polynomial for $\dot{x}(t)$ on $[t_2, t]$, there is a positive constant M for which

$$(75) \quad x^{(n-2i+1)}(t)t^{n-2i} \leq M\dot{x}(t), \quad t \geq t_2.$$

Using (75) in (74),

$$\int_{t_2}^{\infty} \frac{M\dot{x}(t)}{\Phi(x(t),x(t))} dt = M \int_{t_2}^{\infty} \frac{du}{\Phi(u,u)} = +\infty,$$ which again contradicts hypothesis (d).

The proof is similar if one assumes $x(t) < 0$ and unbounded. Q.E.D.

We now give another theorem concerning oscillatory solutions to (52) for the case $n = 2$, with slightly different conditions on the functions $a(t)$, $b(t)$, $\Phi(x,y)$, and $\Psi(x,y)$.

Theorem V.3. Suppose (i) and (iii) hold for $n = 2$, and suppose

(a) $\int_{\alpha}^{\infty} a(t)dt = \int_{\alpha}^{\infty} b(t)dt = +\infty$ for every $\alpha \geq t_0$.

(b) $g(t)$ is continuous and $\lim_{t \to \infty} g(t) = +\infty$.

(c) There exist continuous functions $k(t) \geq 0$, $m(t) \geq 0$ for which $-k(t) \leq P(t,x(t),x(g(t)),\dot{x}(t)) \leq m(t)$, $t \geq t_0$ and for some $\alpha \geq t_0$,

$$\int_{\alpha}^{\infty} k(t)dt < +\infty \text{ and } \lim_{t \to \infty} \int_{\alpha}^{t} \exp[-\int_{\bar{t}}^{s} m(u(du]ds = +\infty$$

for every $\bar{t} \geq t_0$.

If $n = 2$, then every solution $x(t)$ of (52) valid for large t is oscillatory.

PROOF: Suppose $x(t)$ is a non-oscillatory solution to (52); i.e., suppose $x(t) > 0$ and $x(g(t)) > 0$ for $t \geq t_1 \geq t_0$. Since $g(t) \to +\infty$, there is a $t_2 \geq t_1$ for which $g(t) \geq t_1$ for all $t \geq t_2$. Also, by Lemma V.1, $\dot{x}(t) > 0$ and

$$(76) \quad \dot{x}(t) \leq \dot{x}(t_1)\exp[\int_{t_1}^{t} k(s)ds] \leq \dot{x}(t_1)\exp[\int_{t_1}^{\infty} k(s)ds]$$

$$= N_x < \infty.$$

Thus, $x(t)$ is monotone increasing for $t \geq t_1$ and hence, for all $t \geq t_2$, we have $x(t) \geq x(t_1)$, and $x(g(t)) \geq x(t_1)$. Then by (iii),

(77) $\Phi(x(t),x(g(t))) \geq \Phi(x(t_1),x(t_1))$.

An integration of (52) from t_2 to $t \geq t_2$ yields

$$\dot{x}(t) = \dot{x}(t_2) - \int_{t_2}^{t} P(s,x(s),x(g(s)),\dot{x}(s))\dot{x}(s)ds$$

$$- \int_{t_2}^{t} Q(s,x(s),x(g(s)),\dot{x}(s))ds$$

$$\leq \dot{x}(t_2) + \int_{t_2}^{t} k(s)\dot{x}(s)ds$$

$$- \int_{t_2}^{t} a(s)\Phi(x(s),x(g(s)))ds$$

$$\leq \dot{x}(t_2) + N_x \int_{t_2}^{t} k(s)ds - \Phi(x(t_1),x(t_1)) \int_{t_2}^{t} a(s)ds.$$

By hypotheses (a) and (c), the first integral on the right is finite, while the last term on the right tends to $-\infty$, and hence $\lim_{t \to \infty} \dot{x}(t) = -\infty$, which contradicts $\dot{x}(t) > 0$, $t \geq t_1$.

With the use of Lemma V.1 and appropriate conditions on P, we can extend a theorem due to Ryder and Wend [11] for the equation (52).

Before stating the theorem, the following lemma is given, a proof of which may be found in an article written by I.T. Kiguradze [8].

Lemma V.3. If $x(t), \dot{x}(t), \ldots, x^{(n-1)}(t)$ are absolutely continuous and of constant sign on the interval $[t_0, +\infty)$ and $x(t)x^{(n)}(t) \leq 0$, then there exists an integer ℓ, $0 \leq \ell \leq n - 1$, which is even if n is odd and odd if n is even, so that

$$|x(t)| \geq \frac{(t - t_0)^{n-1}}{(n-1)\ldots(n-\ell)} |x^{(n-1)}(2^{n-\ell-1}t)|, \quad t \geq t_0.$$

Theorem V.4. Suppose (i) and (iii) hold and

(a) There is a continuous function $m(t) \geq 0$ on $[t_0, +\infty)$ for which $0 \leq P(t, x, y, x_1, \ldots, x_{n-1}) \leq m(t)$, and

$$\lim_{t \to \infty} \int_{\bar{t}}^{t} \exp[-\int_{\bar{t}}^{s} m(u)\,du]\,ds = +\infty, \text{ for some } \bar{t} \geq t_0.$$

(b) There exist positive constants λ_0, M, N and constants β, γ, $0 \leq \beta, \gamma \leq 1$ for which

$$\Phi(x,\lambda y) \geq M\lambda^{\beta}\Phi(x.y),\ y > 0,$$

$$\Psi(x,\lambda y) \leq N\lambda^{\gamma}\Psi(x,y),\ y < 0,\ \lambda \geq \lambda_0 > 0, \text{ and}$$

$$\int^{\infty} t^{(n-1)\beta} a(t)dt = \int^{\infty} t^{(n-1)\gamma} b(t)dt = +\infty,$$

(c) $g(t) \geq t - C$ for large t, C a positive constant, and $g(t)$ continuous on $[t_0, +\infty)$.

Let $x(t)$ be a solution to (52) valid for large t. Then if n is even, $x(t)$ is oscillatory, while if n is odd, $x(t)$ is oscillatory or tends to zero with its first $n - 1$ derivatives.

PROOF: Suppose $x(t)$ is a non-oscillatory solution of (52). Assume $x(t) > 0$ and $x(t - C) > 0$ for all $t \geq t_1$. By Lemma V.1, $x^{(n-1)}(t) > 0$, $t \geq t_1$, and by (a), Equation (52) yields

(78) $x^{(n)}(t) = -Px^{(n-1)}(t) - Q \leq -Q \leq -a(t)\Phi(x(t),x(g(t))) \leq 0.$

Thus, $x(t) > 0$ and $x^{(n)}(t) < 0$.

Suppose n is even. Then by Lemma IV.2, $\dot{x}(t) \geq 0$, so $x(t)$ is non-decreasing. Also, $x^{(n)}(t) \leq 0$, so $x^{(n-1)}(t)$ is positive and non-increasing on $[t_1, +\infty)$. Thus, by Lemma V.3,

(79) $x(t) \geq x(2^{1-n}t) \geq At^{n-1}x^{(n-1)}(t)$, for $t \geq 2^n t_1 = t_2$,

where $A = 2^{-n^2}/(n-1)!$

Since $x(t)$ is non-decreasing, $t - C \geq t_1 - C$ implies $x(t - C)/x(t_1 - C) \geq 1$ and for any $\lambda_0 > 0$,

$$\frac{\lambda_0}{x(t_1 - C)} x(t - C) = kx(t - C) \geq \lambda_0, \quad t \geq t_2.$$

Thus,

$$\Phi(x(t),x(g(t))) \geq \Phi(x(t),x(t - C)) = \Phi(x(t),kx(t - C)/k)$$

$$\geq Mk^\beta (x(t - C)]^\beta \Phi(x(t),1/k)$$

$$\geq Mk^\beta [x(t - C)]^\beta \Phi(x(t_1),1/k),$$ by

the monotonicity of Φ and $x(t)$. Now, let $B = Mk^\beta \Phi(x(t_1), 1/k)$. Then from (78),

(80) $x^{(n)}(t) + Ba(t)x^\beta(t - C) \leq 0$ for $t \geq t_2$.

Also, from (78), (80) becomes

(81) $x^{(n)}(t) + BA^\beta a(t)[t - C]^{(n-1)\beta}[x^{(n-1)}(t - C)]^\beta \leq 0$, $t \geq t_2 + C$.

Now divide by $[x^{(n-1)}(t)]^\beta$ and integrate from $t_3 = t_2 + C$ to t, to obtain

(82) $$\int_{t_3}^{t} \frac{x^{(n)}(t)}{[x^{(n-1)}(t)]^\beta} dt + BA^\beta \int_{t_3}^{t} a(s)(s - C)^{(n-1)\beta}\left[\frac{x^{(n-1)}(s-c)}{x^{(n-1)}(s)}\right]^\beta ds \leq 0.$$

Since $x^{(n-1)}(t)$ is non-increasing, then

$\frac{x^{(n-1)}(s - C)}{x^{(n-1)}(s)} \geq 1$, and hence,

$$(83)\quad \int_{x^{(n-1)}(t_3)}^{x^{(n-1)}(t)} \frac{du}{u^\beta} + BA^\beta \int_{t_3}^{t} (s - C)^{(n-1)\beta} a(s)ds \leq 0.$$

Now if $s \geq t_2 + C$, then $s - C \geq t_2 \frac{s}{t_2 + C}$ and (83) becomes

$$(84)\quad \int_{x^{(n-1)}(t_3)}^{x^{(n-1)}(t)} \frac{du}{u^\beta} + BA^\beta \frac{t_2^{(n-1)\beta}}{t_3^{(n-1)\beta}} \int_{t_3}^{t} s^{(n-1)\beta} a(s)ds \leq 0.$$

But $0 > \int_{x^{(n-1)}(t_3)}^{x^{(n-1)}(t)} \frac{du}{u^\beta} \geq \int_{b}^{0} \frac{du}{u^\beta}$, $0 < b < \infty$, and the latter integral is finite if $\beta < 1$. Thus, as $t \to +\infty$, (84) yields a contradiction to the hypothesis $\lim_{t \to \infty} \int^{t} s^{(n-1)\beta} a(s)ds = +\infty$. Thus, $x(t)$ is oscillatory if n is even.

The case in which $x(t)$, $x(t - C)$ are negative, $t \geq t_1$ is handled in a similar manner and yields a contradiction to $\int^{\infty} t^{(n-1)\gamma} b(t)dt = +\infty$.

Now suppose n is odd, and suppose $x(t)$ does not approach zero. Then $|x^{(n-1)}(t)|$ is still non-increasing and

$$|x(t)| = \left|\frac{x(t)}{x(2^{1-n}t)}\right| |x(2^{1-n}t)|$$

$$\geq \inf_{t \geq t_2} \left\{\left|\frac{x(t)}{x(2^{1-n}t)}\right|\right\} \cdot |x(2^{1-n}t)|$$

$$\geq \inf_{t \geq t_2} \left\{\left|\frac{x(t)}{x(2^{1-n}t)}\right|\right\} A\,|x^{(n-1)}(t)|\; t^{n-1}, \quad t \geq t_2.$$

Thus, $|x(t)| \geq B_1 t^{n-1} |x^{(n-1)}(t)|$ for constant B_1, and the preceding part of the proof for n even yields a contradiction to the existence of a non-oscillatory solution of (52).

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