RATIONAL NUMBERS AND THE COMMON CORE STATE STANDARDS: A DESCRIPTIVE CASE STUDY

by

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ABSTRACT

Research suggests that some students and teachers have a problematic sense of number. While previous standards documents have emphasized facility with representations of rational numbers, the Common Core State Standards for Mathematics (CCSS) places comparatively more emphasis on rational numbers in general and the repeating decimal representation of rational numbers in particular. The limited body of literature on repeating decimal concepts suggests that teachers are ill-equipped to teach rational numbers in the way described in CCSS.

The purpose of this study was to describe the ways that inservice middle grades teachers, selected from a sample, understand rational numbers, how they interpreted a statement that rational numbers can be written as a repeating decimal, and how these understandings manifested during instruction. The setting is a mid-size town in the Rocky Mountain region.

The study was conducted in two phases. The interview phase consisted of 40–60 minute interviews with ten different inservice middle school teachers regarding their rational number sense. The observation phase involved observing four of the ten interview participants delivering a lesson on repeating decimals. Data were analyzed both between and within cases using a standardized open-ended interview protocol designed by the researcher as a framework for comparisons.

Results indicated that participants primarily understood rational numbers as a collection of mutually exclusive sets. Rational numbers were defined primarily in terms of their decimal representation, with a majority of participants describing rational numbers as "not irrational." The statement "A number is rational if it can be written as a repeating decimal" was interpreted as incomplete by many participants due to a sharp distinction drawn between terminating and repeating decimals. Each observed teacher displayed at least one understanding of rational number in classroom instruction that manifested during the interview, with many participants repeating their understandings verbatim. Recommendations for inservice teacher professional development are offered, as well as suggestions for future research.

CHAPTER 1

INTRODUCTION

1. Background

A great deal of research over the past thirty years has focused on improving the quality of mathematics teaching in the United States. The standards movement, which has its origins in the Curriculum and Evaluation Standards for School Mathematics (1989) (National Research Council, 2001, p. 34), has driven much of the scholarly discussion in this area. Another important voice was Deborah Ball, whose work over the past thirty years has advanced the importance of teacher knowledge in the effort to reform mathematics education nationally (Ball, 1988b, 1990a, 2000).

Over the course of many years and many publications, Deborah Ball made the case that teaching mathematics requires many forms of specialized knowledge about teaching, about mathematics, and about students (e.g., Ball, 1988a, 1988b, 1990a, 1990b, 1991, 2000; Ball & Bass, 2003). This *mathematical knowledge for teaching* (MKT) is described by Hill, Schilling, and Ball (2004) as "the mathematical knowledge needed to help students learn mathematics" (p. 15). Ball, Thames, and Phelps (2008) present MKT as a construct that encompasses two broad types of knowledge required for teaching mathematics effectively; i.e., pedagogical content knowledge and subject matter knowledge. MKT is strongly and positively associated with effective mathematics teaching (Hill et al., 2004; Hill, Rowan, & Ball, 2005; Hill et al., 2008) and as such is seen as an important construct for study in general, and the subject matter knowledge domain in particular.

The Curriculum and Evaluation Standards (1989) was "one facet of the mathematics education community's response to the call for reform in the teaching and learning of mathematics," a first step towards "a coherent vision of what it means to be mathematically literate," and "a set of standards to guide the revision of the school mathematics curriculum" (p. 1). A decade later, Principles and Standards for School Mathematics (2000) was offered, in part, as a way to "offer teachers, curriculum developers, and those responsible for establishing curriculum frameworks a way to focus curricula" (p. 7). The curriculum principle, one of the six principles for school mathematics laid out in Principles and Standards (2000), describes effective mathematics curricula as "coherent, focused on important mathematics, and well articulated across the grades" (p. 14). Each of these three features is intended to empower teachers as they guide students in unifying important ideas in the service of learning new and progressively more sophisticated mathematical ideas. The Curriculum Focal Points (2006) built on Principles and Standards (2000) and offered "a starting point in a dialogue on what is important at particular levels of instruction and as an initial step toward a more coherent, focused curriculum" (p. vii).

The Third International Mathematics and Science Study (TIMSS) analyzed data from 38 countries regarding classroom teaching and commonly used curricular materials (Kilpatrick, Martin, & Schifter, 2003). Results of this data analysis showed that "the traditional US curriculum is relatively repetitive, unfocused, and undemanding" (Kilpatrick et al., 2003, p. 11). This is perhaps nowhere more apparent than in the textbooks used, which have been widely criticized for their superficial treatment of numerous disconnected topics and nearly exclusive attention to procedures at the expense of conceptual understanding (Stodolsky, 1988; Schmidt, McKnight, & Raizen, 1997; Silver, 1998; Stigler & Hiebert, 1999). While fourth and

eighth graders in the US were above the international average on the 2011 TIMSS, they still lag behind many Asian and European countries (Provasnik et al., 2012).

References to number sense are interspersed throughout the NCTM standards documents, with particular attention paid to rational numbers and their representation. Curriculum and Evaluation Standards (1989) called for increased attention to place value, the meaning of fraction and decimal notation, and "relationships among representations of, and operations on, whole numbers, fractions, decimals, integers, and rational numbers" (p. 70). Principles and Standards (2000) developed the number and operation standard, which "describes deep and fundamental understanding of, and proficiency with, counting, numbers, and arithmetic, as well as an understanding of number systems and their structures" (p. 32). Curriculum Focal Points (2006, p. 10) identifies and describes three curriculum focal points in an effort "to guide integration of the focal points…to form a comprehensive mathematics curriculum." The number and operation focal point calls for understanding of key number concepts across all grade levels, beginning with basic ideas of correspondence and cardinality in kindergarten and culminating in grade seven with the use of "division to express any fraction as a decimal, including fractions that they must represent with infinite decimals" (p. 38).

Like its predecessors, the Common Core State Standards (CCSS) (2010) document proposes a set of standards that stress not only "conceptual understanding of key ideas" but also a continual "returning to organizing principles such as place value or the properties of operations to structure those ideas" (p. 4). The ambitiousness of these standards is made manifest in multiple ways, not the least of which is the attention paid to rational numbers. Students are to, by grade seven, "develop a unified understanding of number, recognizing fractions, decimals (that

have a finite or a repeating decimal representation), and percents as different representations of rational numbers" (p. 46).

There is nothing new about linking a definition to a particular representation. In fact, according to Zazkis and Sirotic (2010), "the definition of rational number relies on the existence of a certain representation" (p. 3). Indeed, a common definition of the rational numbers,

$$\mathbb{Q} = \left\{ \frac{p}{q} \middle| \, p, q \in \mathbb{Z}, \; q \neq 0 \right\} \tag{1.1}$$

is based explicitly on a particular representation (Bennett et al., 2004; Ruopp, 2009; Charles et al., 2008). The CCSS (2010) uses a similar definition, first defining fraction as "a number expressible in the form a/b where a is a whole number and b is a positive whole number" (p. 85) and then rational numbers as "a number expressible in the form a/b or $-a/b$ for some fraction a/b (p. 86). A representation that has received comparatively little attention in these standards documents is repeating decimals, defined by CCSS (2010) as "the decimal form of a rational number" (p. 87). The definition of terminating decimals as a decimal with repeating digit 0 brings repeating decimals to the forefront of the discussion of rational numbers.

Kieren (1976, p. 102) argues that rational numbers are presented as "objects of computation" and that as a result "children and adolescents miss many of the important interpretations of rational numbers." While natural number concepts "arise out of the natural activity of children," rational numbers and their properties are abstractly defined and are no longer intuitive. For this reason, he argues "that a variety of experiences with diverse interpretations of rational numbers are necessary." These varied experiences are important in problem solving (Behr, Lesh, Post, & Silver, 1983); in "conceptualizing a fraction as a point on a line" (Behr, Harel,

Post, & Lesh, 1992, p.113); in "unitizing," an important component of proportional reasoning (Lamon, 1996); and in transferring rational number knowledge to novel settings (Lamon, 2007).

Research on student understanding of rational numbers suggests this flexible understanding of rational numbers is largely absent. According to Mack (1990, 1993), student thinking about fractional representations of rational numbers is both severely constrained by whole number reasoning and disconnected from knowledge of formal symbols and procedures. Behr et al. (1983) describe a sample of seventy-seven fourth graders who "were generally incapable of conceptualizing a fraction as a point on a line" (p. 111). Research findings are even more alarming in the domain of decimal representations: many students are unable to conceive of decimal numbers as magnitudes or points on the number line (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981; Brown & Quinn, 2006; Vamvakoussi & Vosniadou, 2010). This research suggests some students are ill-equipped to develop the unified number sense called for by CCSS (2010).

Research on teacher's content knowledge of rational numbers suggests that it is similar to how Ball (1990a) described the mathematical understandings of preservice teachers: "rule-bound and thin" (p. 449). Post, Harel, Behr, and Lesh (1991) collected extensive data on middle school math teachers' understandings of rational number concepts in order to generate "profiles of mathematical understandings for teachers" (p. 183). An overall mean of less than seventy percent on a seventy-eight item test of "items which we feel were at the most rudimentary level," combined with more than half of the tested teachers failing to provide pedagogically sound explanations of simple word problems, led Post et al. (1991) to conclude that not only do teachers in this study not know enough mathematics, those who can solve school mathematics problems can not adequately explain their

reasoning. In her ground-breaking comparison of US and Chinese math teachers' understandings of the mathematics they teach, Ma (1999) found that less than half of the US teachers she studied could correctly perform the division $1\frac{3}{4} \div \frac{1}{2}$ and that none of them demonstrated a conceptual understanding of the algorithm they used to solve the problem.

Research regarding the topic of repeating decimals is quite limited; studies examining teacher understanding are even more scarce. The few studies that do exist suggest that inservice teachers' have idiosyncratic notions about repeating decimals (Sinclair, Liljedahl, & Zazkis, 2006; Burroughs & Yopp, 2010; Yopp, Burroughs, & Lindaman, 2011; Weller, Arnon, & Dubinsky, 2011). Some research suggests a belief among teachers that numbers change value with changes in representation (Khoury & Zazkis, 1994) or that different representations of the same number can be classified as different types of numbers (Burroughs & Yopp, 2010).

These studies, with the exception of Weller et al. (2011), which was a quasi-experimental curriculum study, were primarily interested in teachers' reactions to the statement $0.\overline{9} = 1$ and similar items. Missing from the literature is a detailed description of the nature of inservice teacher difficulty with general repeating decimal concepts.

2. The Problem

As mentioned above, the CCSS (2010) has placed increased emphasis on the decimal representation of rational numbers. Decimal representations of fractions are used as a conceptual bridge linking whole and rational numbers. Repeating decimals are defined as "the decimal form of a rational number" (p. 87) and, consequently, irrational numbers are those with non-terminating, non-repeating

decimal expansions. While there is a body of research regarding student understanding of definitions of rational number, it has focused primarily on undergraduate students, with special emphasis on preservice teachers (Pinto & Tall, 1996; Zazkis & Gadowsky, 2001; Edwards & Ward, 2004; Guven, Cekmez, & Karatas, 2011). Though there is much research regarding student understanding of finite decimal and fraction representation of rational numbers (Hiebert & Wearne, 1986; Ball, 1993; Moss & Case, 1999; Sinclair, 2001; Tzur, 2007; Johanning, 2008), there are comparatively few studies regarding how students understand repeating decimals. There is a similar dearth in the literature regarding inservice teacher knowledge of repeating decimal representations of rational numbers (Weller, Arnon, & Dubinsky, 2009; Burroughs & Yopp, 2010). It is not clear from the existing research the extent to which inservice teachers are prepared to engender the "unified sense of number" called for by the CCSS (2010, p. 46).

3. Purpose Statement

The purpose of this study was to explore how inservice middle school teachers, chosen from a sample, understood rational numbers and how those understandings manifested themselves during instruction. The study used interviews informed by a standardized open-ended interview protocol to describe the rational number sense of a sample of inservice middle school teachers. From this sample of teachers, a subsample was selected for observation and videorecording of instruction to identify ways in which the teachers' understanding manifested itself during instruction.

4. Research Questions

The following three questions guided this study:

1. What are the ways that inservice middle school teachers, selected from a sample, understand rational numbers?
2. How do these teachers interpret the statement that a rational number is a number that can be written as a repeating decimal?
3. How do these understandings manifest during instruction?

5. Significance of the Study

The Mathematical Education of Teachers II (2012), a report written by a committee designated by the Conference Board of the Mathematical Sciences, states that "the opportunity to understand the mathematics in the middle grades from a teacher's perspective" is a critical component of preparing to teach middle grades mathematics (p. 39). Specifically, MET II (2012) calls for middle grades teachers to be able to explain why rational numbers must have eventually repeating decimal representations, why the period of a repeating decimal representation of a fraction must be at most one less than the denominator, and to "explain why $0.999\ldots = 1$ in multiple ways" (p. 42). Current research provides evidence that the current and upcoming teacher corps is not well-prepared to engage ambitious goals such as these (Burroughs & Yopp, 2010; Weller et al., 2011; Yopp et al., 2011; Conner, 2013). The present study provides insight regarding the role of teacher knowledge of rational number in implementing one aspect of the CCSS (2010). Moreover,

this study provides insight into what content knowledge may be useful in content courses for preservice teachers, as well as what content knowledge would be useful in professional development with inservice teachers.

6. Context of the Study

This study describes the understanding of rational numbers and their representation held by a small sample of inservice middle grades teachers. Random sampling, the definitive standard for making generalizations in quantitative studies, was not employed in this study. As a result, the researcher has little basis for extending the study's findings to other populations. The school district from which teachers will be sampled was not representative; as a result there is no reason to assume that the findings of this study will hold in districts nationwide. While findings in this study do not generalize in the quantitative sense, the results will have wider implications due to the nature of the school district from which the purposefully chosen sample is taken.

The school district in question is located in the Rocky Mountain region of the United States. This school district has among the highest paid teaching positions in the state, and there is a great deal of competition for teaching positions in this district. The school district has been working to implement CCSS (2010) since the document was released. The district's curriculum has been largely rewritten to align with these standards. Teachers in this district have significant support: there are mathematics coaches who work with teachers at the elementary, middle, and secondary level and professional development to implement CCSS (2010) has been ongoing since 2011. The rational number sense of a sample of middle grades

teachers from this well supported school district can provide insight into what one might expect in a similarly or less well equipped school districts.

7. Defintion of Terms

Inservice Teacher: An individual who is licensed to teach at either the elementary or secondary level in the United States.

Middle Grades Teacher: A teacher who teaches in one of the two middle schools in the participating school district.

Rational Number: The rational numbers are defined as any number that can be written as one positive or negative whole number divided by another. For example, $\frac{1}{2}$ is a rational number, as is $3 = \frac{3}{1}$, as well as $-5 = \frac{-10}{2}$.

Trivial Repeating Decimal: A decimal with an infinite string of zeros appended to the end. For example, $0.75 = .75\overline{0}$ is a trivial repeating decimal.

Nontrivial Repeating Decimal: A decimal with an infinite string of nonzero decimal digits. For example, $0.\overline{3}$ is a nontrivial repeating decimal.

8. Assumptions

This study assumes that all participants have received appropriate training and preparation to teach at the level for which they are certified. In particular, this study assumes that each participant has received instruction in mathematics content appropriate to their certification. It is assumed that participants responded to interview questions truthfully and accurately, and that teachers did not significantly alter their behavior while observed by the researcher.

CHAPTER 2

REVIEW OF THE LITERATURE

1. Introduction

This chapter will begin with a description of the origins and evolutions of the construct known as mathematical knowledge for teaching. Following this discussion is a description of two important components of curricula: textbooks and standards, which is succeeded by an overview of theoretical perspectives on rational number learning and a more fine-grained analysis of student and teacher difficulties with fraction and decimal representations of rational numbers. The chapter ends with a discussion of connections between these representations.

2. Mathematical Knowledge for Teaching

Mathematical knowledge for teaching as a construct was first considered by Lee Shulman (1986). His tripartite framework of the knowledge required to teach mathematics consisted of three types of content knowledge: subject matter content knowledge, pedagogical content knowledge, and curricular knowledge. Deborah Ball (1990a) extended Shulman's framework by demonstrating that, in addition to knowledge of mathematics that any mathematically literate adult would have, teachers need a specialized knowledge of mathematics that was peculiar to teaching mathematics.

Lortie (1975) writes that "those who teach have normally had sixteen continuous years of contact with teachers and professors... the interaction, moreover, is not a passive observation–it is usually a relationship which has consequences for

the student" (p. 61). This "apprenticeship of observation" is far more influential on teacher practice than formal teacher education, according to Ball (1988b, p. 40). Moreover, she argues that "subject matter knowledge interacts with [teachers'] assumptions and explicit beliefs about teaching and learning, about students, and about context to shape the ways in which they teach mathematics to students" (p. 1).

Ball (1990a) also argued that, contrary to popular belief, school mathematics content is highly sophisticated and that much more than precollege and even college mathematics is required to teach effectively. As an example, she considers division: first in general, and then in the specific context of division with fractions, a topic described by Ma (1999, p. 55) as "at the summit of arithmetic." The research described in this paper is based on data collected from a questionnaire designed to explore preservice teachers' understanding of this sophisticated topic. Preservice teachers were asked to choose an appropriate story problem for the computation $\frac{1}{4} \div \frac{1}{2}$. Participants were then asked in an interview to perform the computation $1\frac{3}{4} \div \frac{1}{2}$ and write an appropriate story problem to represent the problem. Only 30% of survey respondents chose the correct story problem and very few prospective secondary and no prospective elementary teachers were able to devise a story problem. This "explicit conceptual understanding" (p. 459) is an integral part of subject matter knowledge for teaching, but these prospective teachers tended to see mathematics as "a largely arbitrary collection of facts and rules" (p. 460).

A decade later, Ball (2000) indicated that the content knowledge needed for effective teaching had not yet been identified. Decrying the common practice of defining necessary teacher knowledge in terms of desired student knowledge, she asserts that the knowledge must be sought in the context of teacher practice. She argues that unexpected "content demands emerge from analyzing the sorts of chal-

lenges with which teachers must contend in the course of practice, as they mediate students' ideas, make choices about representations of content, modify curriculum materials, and the like" (Ball, 2000, p. 245).

Citing divergent content strands appearing on nascent teacher certification exams, Hill et al. (2004) used extant teacher learning theory to build a survey designed to represent and measure the components of the knowledge required for effective mathematics teaching. Results suggested the existence of a separate type of knowledge beyond the common knowledge of content any mathematically literate adult would possess. Specialized content knowledge appeared in items requiring analysis of alternative algorithms or nonstandard procedures, representing numbers and operations with manipulatives, and explaining common mathematical properties such as tests for divisibility.

Ball et al. (2008) argue that pedagogical content knowledge is a useful construct because of its ability to make connections between content knowledge and teaching practice. However, "this bridge between knowledge and practice was still inadequately understood and the coherent framework Shulman (1986, p. 9) called for remained underdeveloped" (Ball et al., 2008, p. 389). Their approach to developing this framework was focused on teaching in general and on the knowledge and mathematical practices required for mathematics teaching in particular (Ball et al., 2008). Their framework is represented pictorially in Figure 2.1. *Mathematical knowledge for teaching* is defined by Ball et al. (2008) as "the mathematical knowledge needed to carry out the work of teaching mathematics" (p. 395). However, it is not a unidimensional construct, as Figure 2.1 indicates. Mathematical knowledge for teaching subsumes the original distinctions made by Shulman (1986), i.e., subject matter knowledge and pedagogical content knowledge, and further subdivides each of these into three subcategories.

Figure 5
Domains of Mathematical Knowledge for Teaching

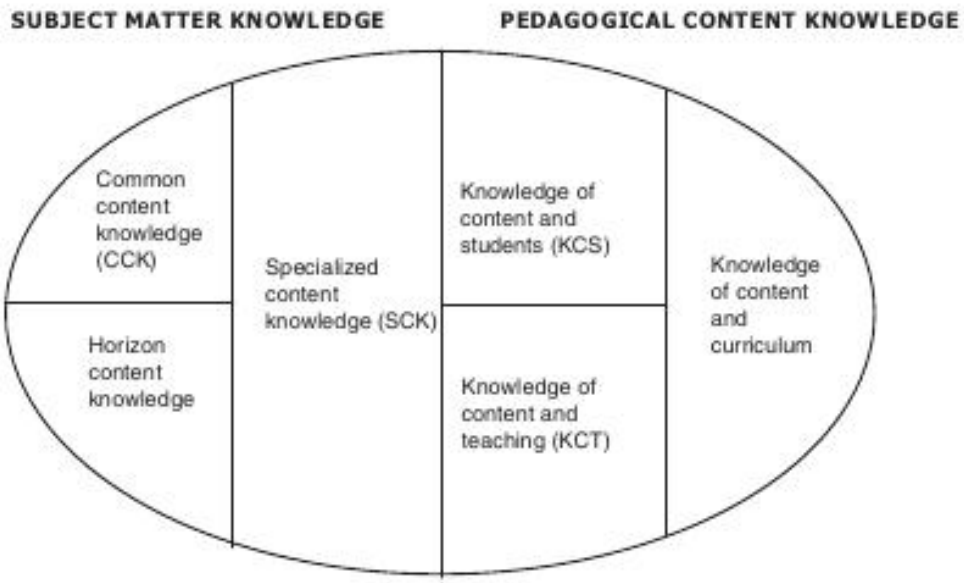


Figure 2.1. Content Knowledge Domain Map.

Under the banner of subject matter knowledge is *common content knowledge*, defined as "the mathematical knowledge and skill used in settings other than teaching" (Ball et al., 2008, p. 399); *specialized content knowledge*, defined as "the mathematical knowledge and skill unique to teaching" (Ball et al., 2008, p. 400); and *horizon knowledge*; defined as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403).

Pedagogical content knowledge exhibits three subconstructs: *knowledge of content and students*, defined as "knowledge that combines knowing about students and knowing about mathematics" (Ball et al., 2008, p. 401); *knowledge of content and*

teaching, defined as "knowledge that combines knowing about students and knowing about mathematics" (Ball et al., 2008, p. 401). *Curricular knowledge* (Shulman, 1986) was tentatively categorized as pedagogical content knowledge based on existing literature, with an acknowledgement that this type of knowledge was not yet fully understood.

This study focuses on subject matter knowledge necessary for effective rational number instruction at the middle grades level. Some examples of common content knowledge include definitions of the terms fraction, decimal, and rational number and familiarity with the nature of the decimal representations of rational numbers. Also considered is horizon content knowledge, such as an ability to classify numbers as rational or irrational and understanding the difference between colloquial and mathematical usage of vocabulary.

3. Curriculum

Kliebard (2004, pp. 23–25) describes four groups that shaped the American curriculum. The humanists, "guardians of an ancient tradition tied to the power of reason and the finest elements of the Western cultural heritage" fought a losing battle against the other three groups, each with "a different conception of what knowledge should be embodied in the curriculum and to what ends the curriculum should be developed." Developmentalists believed that the curriculum should be "reformed along the lines of a natural order of development in the child." Social efficiency educators sought to apply "standardized techniques of industry to the business of schooling" to eliminate waste and increase "specialization of skills and, thereofre, a far greater differentiation in the curriculum than had heretofore prevailed." Social meliorists, led by Lester Frank Ward, believed that the various

social injustices of the day "could all be addressed by a curriculum that focused directly on those very issues, thereby raising a new generation equipped to deal effectively with those abuses."

The passing of the National Defense Education Act in 1958 "represented an end of an era in several respects" according to Kliebard (2004, pp. 268–269). He declares that "professional educators were no longer to be given free rein in curriculum matters" and that curriculum now "would be developed first by experts at a center set up for that purpose, with the local school systems perceived as consumers of external initiatives," and hence "a version of the humanist position became dominant almost overnight." This turn of events established the "academic subjects as the basic building block of the curriculum"

Even more important than the academic subject in determining curriculum is the textbook. As early as the nineteenth century, "the widespread use of popular textbooks" exerted a "standardizing influence on the curriculum" (Kliebard, 2004, p. 2). Fuller (1928, p. iii) calls the textbook "the most important of the teacher's tools" and claims that "it is more decisive in day-to-day affairs than is the course of study outlined by the school system." Nearly seventy years later, Ornstein (1994) states in passing that "textbooks have come to drive the curriculum" (p. 70). This driving force is especially apparent in mathematics curricula, where the heavy reliance on textbooks "is perhaps more characteristic of the teaching of mathematics than of any other subject in the curriculum" (Robitaille & Travers, 1992, p.706).

Schmidt, Houang, and Cogan (2002) characterized the U.S. curriculum as lacking in focus, excessively repetitive, insufficiently challenging, and lacking coherence. In their own words, the curriculum U.S. teachers work with is "a mile wide and an inch deep" (p. 3). Mathematics textbooks, an important part of the curricu-

lum in the U.S., reflect similar deficiencies as the larger curriculum (Reys, Reys, & Chavez, 2004).

Citing a need to periodically revisit standards to maintain their viability, Principles and Standards (2000) "reflects input and influence from many different sources," including educational research, historical practice in mathematics education, values, beliefs, and experiences of educational stakeholders. This was followed by Curriculum Focal Points (2006), a publication intended "as a starting point in a dialogue on what is important at particular levels of instruction and as an initial step toward a more coherent, focused curriculum in this country" (p. vii). Building on the NCTM standards described above, the CCSS (2010) consists of standards for mathematical content and mathematical practice. The content standards "are a balanced combination of procedure and understanding;" the practice standards "describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years" (p. 8).

3.1. Textbooks

According to Schmidt et al. (1997), the number of topics presented in US math textbooks far exceeds the international average irrespective of grade level. This reflects a historical trend in US mathematics education, where the prevailing pedagogical wisdom dictated that mathematics concepts and skills are more easily learned when decomposed into "smaller facts and subskills" that students can more easily acquire (p. 17). Schmidt et al. (1997) argue that this practice is "less appropriate for the more complex, integrated mathematical expectations recommended in current mathematics reform proposals" (p. 17).

Based on examinations of elementary school mathematics textbooks and classroom observation, Stodolsky (1988) argues that "textbooks do not systematically attend to developing conceptual understanding" nor are they "meant to be used in conjunction with teacher instruction and explanation" (p. 123). Silver (1998) describes the US mathematics curriculum as paying "excessive attention to low-level knowledge and skills without sufficient attention to conceptual understanding or complex problem solving" (p. 5).

Behr et al. (1992) argue that many problematic understandings held by children and teachers "result from deficiencies in the curricular experiences provided in school" (p. 300). These deficiencies, broadly speaking, are a lack of experience with the role of the conceptual unit, a failure to examine arithmetic operations in terms of quantity, a narrow range of problems that support "less-constrained models of multiplication and division" (p. 300), insufficient attention to qualitative reasoning about magnitude, ordering, and results of operations, and inadequate exposure to computations that demonstrate "the invariance or variance of arithmetic operations" (p. 300).

Howson (1995) claims that mathematics textbooks present the material in a largely similar way. While the established pattern offers structure, he argues that this portrays mathematics as rigid and uniform. As a result, "the books actively foster only a very limited range of learning strategies: mainly learning by listening and by practising a restricted range of techniques in particular 'closed' situations" (p. 41).

Citing several studies that span multiple decades, Hiebert (2003, pp. 10–12) describes traditional pedagogy as "emphasis on teaching procedures" with "little attention during the lesson to helping students develop conceptual ideas or even connect the procedures they are learning with the concepts that show why

those procedures work." This method of instruction, combined with a curriculum that primarily "deals with calculating and defining…in a rather simplistic way," is described as a "traditional program." He argues that this type of program produces students with limited understanding of basic elementary mathematics and "uniformly low" performance on tasks requiring "reasoning, communicating, conjecturing, and justifying." Based in part on these and other findings from mathematics education, as well as cognitive science and educational policy and organization research, the developers of the Connected Mathematics Project (CMP) (2009b) created textbooks that "focus on big ideas of middle grades mathematics, teaching through student-centered exploration of mathematically rich problems, and continual assessment to inform instruction."

This study focused on the Connected Mathematics Project (CMP) (2009a) curriculum to the exclusion of other curricula. The research setting was a district whose middle schools identified CMP as their official textbook and hence the researcher used this curriculum to inform the design of the interview protocol used in data collection. This curriculum informed sampling choices as well, because the repeating decimal representation of rational numbers (the content of interest in this study) is addressed in Bits and Pieces III: Computing With Decimals and Percents (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2009). As a result, the researcher attempted to identify teachers who used this textbook so as to maximize opportunities to observe instruction about repeating decimals.

CMP was developed as a complete set of curricular materials informed by research from multiple disciplines ((CMP), 2009a). Many of the guiding principles that informed the development of this curriculum are echoed in Curriculum and Evaluation Standards (1989), Principles and Standards (2000), and Curriculum Focal Points (2006), such as a focus on developing problem solving skills, facility

with multiple representations of numbers, and coherence reflected in the accumulation of knowledge and skills as students progress. According to CCSS Transition Kit (2010, p. xii), CMP materials are designed and sequenced so that "by the end of grade 8, CMP students will have studied all of the content and skills in the Common Core State Standards (CCSS) for middle grades (Grades 6-8)." However, there are some differences with respect to sequencing of topics and skills. As a result, the publishers and authors of the CMP textbook series have created "a set of investigations for each grade level to further support and fully develop students understanding of the CCSS." Despite these efforts, teachers still do not have access to a full set of curricular materials that are fully aligned with CCSS (2010) and as a result many struggle to assemble a curriculum by piecing together selections from other materials.

3.2. NCTM

In 1989, NCTM introduced the Curriculum and Evaluation Standards for School Mathematics, describing it as "one facet of the mathematics education community's response to the call for reform in the teaching and learning of mathematics" (p. 1). The standards described in this document are regarded as the beginning of what has come to be known as the standards movement (National Research Council, 2001, p. 34). In an attempt to "create a coherent vision of what it means to be mathematically literate" and "create a set of standards to guide the revision of the school mathematics curriculum" (p. 1), a collection of fifty-four standards organized into two broad categories: curriculum standards by grade level (K–4; 5–8; 9–12) and standards for evaluation. These standards served to shift instructional focus away from "practice in manipulating expressions and practicing algorithms"

(p. 9) and towards developing students' ability "to explore, to conjecture, and to reason logically" (p. 6).

Curriculum and Evaluation Standards (1989) also introduced the notion of "mathematical power," which is defined as "an individual's ability to explore, conjecture, and reason logically, as well as the ability to use a variety of mathematical methods effectively to solve nonroutine problems" (p. 5). A student with mathematical power sees mathematics "as more than a collection of concepts and skills to be mastered; it includes methods of investigating and reasoning, means of communication, and notions of context. . . [and] involves the development of personal self-confidence" (p. 5).

In grades 5–8 of the Curriculum and Evaluation Standards (1989), students are called to "understand, represent, and use numbers in a variety of equivalent forms. . . develop number sense for whole numbers, fractions, decimals, integers, and rational numbers" and "investigate relationships among fractions, decimals, and percents" (p. 87). Repeating decimals are mentioned explicitly as "particularly appropriate" objects of study due to the ease with which one can pose and investigate the patterns that emerge when examining decimal representations of rational numbers in fraction form.

Responding to criticisms that Curriculum and Evaluation Standards (1989) lacked sufficient grounding in research and that the recommendations made were too broad, Principles and Standards (2000) was released. It "calls for a common foundation of mathematics to be learned by all students" (p. 5). Through the use of "common language, examples, and recommendations," multiple stakeholders can engage in a focused and productive dialogue about the direction of mathematics education improvement. The underlying framework is the six principles for school mathematics: equity, curriculum, teaching, learning, assessment, and technology.

These "particular features of high-quality mathematics education. . . describe the mathematical content and processes that students should learn" (p. 11) and point towards the five content and five process standards of number and operations, algebra, geometry, measurement, data analysis and probability, problem solving, reasoning and proof, communication, connections, and representation. The Principles and Standards (2000) promoted mathematics instruction that focused on making sense of mathematics (Tsamir & Tirosh, 2002; Strawhecker, 2005; Tzur, 2007).

The Number and Operations Standard of the Principles and Standards (2000) "describes deep and fundamental understanding of, and proficiency with, counting, numbers, and arithmetic, as well as an understanding of number systems and their structures" (p. 32). A recurring theme in the Number and Operations standards is that of representation. By the middle grades, students are expected to understand that numbers can be represented in different ways and that representation does not change a number's value. The rational number system is characterized in terms of representation; specifically, in terms of fractions, decimals, and percents. Unfortunately, explicit mention of repeating decimals is largely absent from these standards, with the only explicit mention coming in the Number and Operations standards for grades 9–12:

> Whereas middle-grades students should have been introduced to irrational numbers, high school students should develop an understanding of the system of real numbers. . . They should understand that irrational numbers can only be approximated by fractions or by terminating or repeating decimals. They should understand the difference between rational and irrational numbers.

This passage indicates what students are to be able to do, but lacks the "organizing structures for curriculum design and instruction at and across grade levels"

provided by the Curriculum Focal Points (2006, p. 5). These focal points are not "a list of objectives for students to master" but "major instructional goals and desirable learning expectations" (p. 10). This document briefly mentions repeating decimals in relation to grade 7:

> In grade 4, students used equivalent fractions to determine the decimal representations of fractions that they could represent with terminating decimals. Students now use division to express any fraction as a decimal, including fractions that they must represent with infinite decimals.

3.3. Common Core State Standards

A common theme in both the Principles and Standards (2000) and Curriculum Focal Points (2006) is a call for increased focus and coherence in curricular materials. The CCSS (2010) attempts to further realize this need by recasting the discussion about standards in terms of coherence and focus. Coherence is achieved by relying on "the key ideas that determine how knowledge is organized and generated within that discipline" and writing standards that "evolve from particulars. . . to deeper structures inherent in the discipline" (pp. 3–4). These key ideas serve as focal points for each grade level, providing the focus and coherence called for in previous standards documents.

The CCSS (2010) drew upon the process standards of the Principles and Standards (2000) and the strands of mathematical proficiency from the National Research Council (2001) to develop the eight standards for mathematical practice. These practices "describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter" and serve to connect the content and practice standards (p. 8).

The grade level standards "are not intended to be new names for old ways of doing business" (p. 5). This is especially evident in the treatment of repeating decimals in the CCSS (2010). One of the four critical areas for focus in Grade 7 calls for students to "develop a unified understanding of number, recognizing fractions, decimals (that have a finite or a repeating decimal representation) and percents as different representations of rational numbers" (p. 46). Grade 8 students are to further their understanding of rational number by showing that every rational number has an eventually repeating decimal expansion and to "convert a decimal expansion which repeats eventually into a rational number" (p. 54).

CCSS (2010) introduces rational numbers with fractions in Grade 3. Students are to interpret unit fractions (i.e., fractions of the form $\frac{1}{n}$ for some nonzero whole number n) as the building blocks of arbitrary fractions $\frac{m}{n}$. Concurrently students develop an understanding of the part-whole relationship between the numerator and denominator with careful attention to the inverse relationship between the denominator and the size of the fraction. They also begin to develop fraction number sense by representing numbers less than, greater than, and equal to one, as well as solve problems involving fractions.

In Grade 4, CCSS (2010) introduces equivalent fractions and operations with fractions. The key concept of multiple representations of the same number is reinforced by developing methods for producing and identifying equivalent fractions. Building on the conceptual understanding of a general fraction as an iterated unit fraction, students build, compose, and decompose fractions. The multiplicative interpretation of fraction is used to make sense of multiplication of a fraction by a whole number.

The Grade 5 standards build on students' understanding of fractions and equivalent fractions to rewrite fraction addition and subtraction as an equiva-

lent computation with like denominators. Number sense continues to develop as students learn to estimate sums and differences of fractions in addition to direct calculation. Using student understanding of fractions and multiplication and division as a foundation, students develop explanations of why fraction multiplication and division procedures make sense. At this level, students formally begin their study of finite decimals by using relationships between decimals and fractions (e.g., $\frac{1}{2} = \frac{5}{10} = 0.5$) to explain the rationale underlying multiplication and division algorithms for decimals, as well as perform computations with decimals to the hundredths place.

In Grade 6, CCSS (2010) students continue to use their conceptual understanding of fractions and multiplication and division to explain fraction operation procedures, as well as solve problems using these operations. Moreover, students are called to interpret the meaning of a quotient of two fractions using models and symbols. Students expand their understanding of the number system to include all rational numbers, with particular emphasis on negative fractions and negative integers. With the rational number system complete, students turn their attention to understanding these numbers as points on the number line, ordering these numbers, and identifying points in the coordinate plane with rational coordinates.

The penultimate stage of understanding of rational number in CCSS (2010) comes in Grade 7, where students are called to "develop a unified understanding of number, recognizing fractions, decimals (that have a finite or a repeating decimal representation), and percents as different representations of rational numbers" (p. 46). Not only do students extend the basic four operations to all rational numbers, they learn that they can express any rational number in decimal notation through long division and that the decimal form of a rational number must either terminate in zeros or eventually repeat.

The real number system is completed in Grade 8, where CCSS (2010) names those numbers that are not rational; namely, the irrational numbers. Students learn that every number can be expressed in decimal notation and that rational numbers must have eventually repeating decimal representations. As a consequence, any number written as a repeating decimal must be expressible as a rational number and students learn to move between representations. The rational numbers are framed as the numbers of approximation when students learn to order irrational numbers using rational approximations and estimate the value of irrational quantities.

A key difference between CCSS (2010) and Montana Common Core State Standards (MCSS) (2011) appears in Grade 8; specifically, 8.NS.1, where 8 refers to the grade level, NS refers to number system, and 1 refers to a particular standard. Both standards are quoted below, with the different passages in boldface:

> [Montana] 8.NS.1: Understand informally that every number has a decimal expansion; **the rational numbers are those with decimal expansions that terminate in 0s or eventually repeat. Know that other numbers are called irrational.**
>
> [National] 8.NS.1: Understand informally that every number has a decimal expansion; **for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.**

CCSS (2010) in general, and the standards discussed above in particular, played an important role in shaping the present study. When compared to Curriculum and Evaluation Standards (1989), Principles and Standards (2000), and Curriculum Focal Points (2006), significant differences in the ways rational numbers were presented became clear. The previous standards documents focused primarily on representations of rational numbers and facility with manipulating them. Very little

attention was paid to the repeating decimal representation of rational numbers, with the relationships between rational numbers and their decimal representations receiving even less attention. Comparatively more attention is paid to the repeating decimal representation by CCSS (2010), which explicitly states that "for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion that repeats eventually into a rational number."

4. Rational Numbers

The difficulty of teaching and learning rational numbers has been attributed in part to the complexity of the rational number construct. In his groundbreaking study of rational numbers, Kieren (1976) argues that a comprehensive understanding of the rational number construct requires a consideration of rational numbers from several different (but related) perspectives: as fractions, as equivalence classes, as ratios, as mappings or operators, as members of a quotient field, as measures or points on a number line, and as decimal fractions.

Kieren (1980, pp. 134–136) condensed his original seven interpretations into the five rational number subconstructs: part/whole, quotient, measure, operator, and ratio. The part/whole subconstruct constitutes "the traditional and modern bases for developing fraction meaning," the ratio subconstruct is concerned with "the quantitative comparisons of two qualities," the quotient interpretation allows for "quantification of the result of dividing a quantity into a given number of parts," the measure subconstruct focuses on "the arbitrary unit and its subdivision," and the operator interpretation views rational numbers that "map a set…multiplicatively onto another set." These five interpretations have largely stood the test of time (Lamon, 2007).

One of the difficulties of the rational number construct is semantic in nature: Lamon (2007) describes the colloquial and mathematical meanings of the word fraction and concludes by saying "a different rational number does not exist for each of the three fractions $\frac{2}{3}$, $\frac{6}{9}$, and $\frac{10}{15}$...these fractions are different numerals designating the same rational number. A single rational number underlies all of the equivalent forms of a fraction" (p. 635). She claims that students with well-developed rational number sense "should be comfortable in reasoning, computing, and problem solving in the domain of rational numbers" (p. 636).

One goal of the CCSS (2010) is to structure key ideas by "continually returning to organizing principles" (p. 4). One such key idea is number line diagrams, defined as "a diagram of the number line used to represent numbers and support reasoning about them" (p. 85). This is especially apparent in standards addressing rational numbers: standards regarding unit fractions and their iterates, equivalent fractions, and decimal representations of rational numbers are all to be understood primarily as locations on the number line.

4.1. Fraction Representation

Research suggests that the key idea of numbers as locations on the number line is one of the most difficult for students and teachers to grasp. When Chazan and Ball (1999) discussed representing eighths on a number line, they describe an entire class agreeing that the number of demarcations between 0 and 1 (seven) was the same as the number of pieces between 0 and 1. Behr et al. (1983) asked 77 fourth-graders to identify three-fourths of a region, number line, and set of discrete objects. The number-line model was the most problematic, prompting the researchers to conclude "children in this sample were generally incapable of conceptualizing a fraction as a point on a line" (pp. 112-113).

Khoury and Zazkis (1994) asked preservice teachers (both elementary and secondary) "Is the number 'one-half' in base three equal to the number 'one-half' in base five?" (p. 192). Seventy six percent of the participants argued, using multiple strategies, that these numbers were not equal. This suggests that these participants' "knowledge of place value and rational numbers is more syntactical than conceptual" (p. 203).

Zazkis and Sirotic (2010) used the theoretical perspective of relative transparency and opacity to study preservice secondary teachers' understanding of irrationality of numbers. For example, the number 784 can be represented as 28^2, a representation which is transparent with respect to its standing as a perfect square but is opaque with respect to its divisibility by 98. These researchers asked participants to determine if $M = 53/83$ is rational or irrational. Despite the relative transparency of this representation with respect to rationality, several peculiar responses emerged. Perhaps the most troubling was the following:

> Interviewer: And lets call this quotient M, if you perform this division on the calculator the display shows this number, 0.63855421687.
> Steve: And I assume it keeps going, thats just what fits on your calculator...
> Interviewer: Yeah, thats what the calculator shows, thats right. So is M rational or irrational?
> Steve: So this is the *quotient M*, yeah I would say its *irrational*.
> Interviewer: Because?
> Steve: Because *we can't see a repeating decimal*.

4.2. Decimal Representation

Ubuz and Yayan (2010) studied 63 inservice teachers' (grades one through five) subject matter knowledge regarding the decimal representation of rational numbers. To gauge participants' ability to label a given position on a number line in

decimal notation, five different number lines with different scales were drawn. Four different scales (tenths, hundredths, fifths, and twentieths) and five different ranges (5 to 6, 2.7 to 2.8, 3 to 4, 1 to 2, and 2 to 2.1) were given. Poorest performance (47 percent correct overall) was evidenced on the number line where segments were 1/20 in length. The position on the number line was 12 units to the right of 1; i.e., 12/20. Thirteen of the sixty three participants interpreted these 12 marks as 1/10 in length, hence labeling the position as 1.12. A similar interpretation error occurred (fourteen of sixty three) when the segments were 1/5 in length.

Martine and Bay-Williams (2003) used four different models to assess forty three sixth grade students' understanding of decimal values. Students were asked to represent the numbers 0.6 and 0.06 using a number line, a 10×10 grid, money, and place value. Only six of the forty three students could represent the numbers correctly using all four representations. The number line representation proved the most difficult; of the fourteen who used only one representation incorrectly, eleven were unable to correctly represent the numbers on a number line. The most common mistakes involved a misunderstanding of the relative sizes of the two numbers. One student indicated that 0.06 was "halfway between 0 and 0.6, confusing one-tenth the size of 0.6 with one-half the size of 0.6" (p. 246). Since it was not clear if the students' difficulty was with the model or of the relative magnitudes of the numbers, the researchers asked students to represent several numbers with a number line: the numbers 1 through 5, the number 2.5, the number 0.4, and the numbers 0.4 and 0.04. Only 20 percent were able to answer the last question correctly, which "indicates that students may appear to understand decimals using some models, but they lack a profound overall understanding of decimal concepts" (p. 246).

4.3. Connections Between Representations

According to Hiebert and Wearne (1986, pp. 208–209), "many students... do not recognize, for example, that .3 is equivalent to $\frac{3}{10}$ because both represent three-tenths of a unit. Rather, most students relate common fractions and decimal fractions directly through syntactic rules." The authors describe "syntactic rules" as rules for moving symbols on a page and argue that students can use these rules without any understanding of the meaning of those symbols. Markovits and Sowder (1991) found similar results when they asked fourteen ninth-grade students, "Are $1/4$ and 1.4 the same or are they different? Twelve of the fourteen said these were the same number. Twenty students were asked the value of the expression $5 + 1/2 + 0.5$. Three participants indicated that the sum was undefined because of the three different representations involved. "Barry" said "I don't see where [1/2] would fit in with any of the other ones, but I'm not sure about it" (p. 6).

O'Connor (2001) studied a group of fifth graders who "had only a dim understanding that some numerals in decimal notation denote the same number as some numerals in fraction notation" (p. 146). The instructor of the class, "by getting students to focus on the processes by which numbers in one format are transformed into the other format, the instructors hoped to get students to begin to see that 'decimals' and 'fractions' are alternative representational formats, not different types of numbers."

Transcripts from this study reveal a strategy for expressing fractions as decimals and vice versa. Given a decimal number, such as 0.235, a student may read the place values (two hundred thirty five thousandths) and recognize the equivalent fraction form $\frac{235}{1000}$ and vice versa. This strategy applies to any terminating decimal and to any fraction in simplest form and denominator that divides 10^n for

non-negative integers n. This subtlety, however, proves exceedingly difficult for the students to grasp:

> T: Alright, are you ready? Okay, can any fraction be turned into a decimal? Yes or no? And tell us why you think so. Mirjana?
>
> M: No, because three or eight or six are not the factors of ten, hundred, thousand, ten thousand, hundred thousand, so it can't be, uh…so this fraction one third or one eighth or one sixth can't turn, be turned into a fraction.
>
> T: So one third cannot be turned into a decimal?…Why not?
>
> M: Because it's not a factor of ten, hundred, thousand, ten thousand, hundred thousand, and so on…
>
> J: I think no because um if three, because threes are in the denominator so the denominator has to be a factor of ten, hundreds, thousands, and so on. And it's important because of…you can't change into a decimal because if you want to change into to a decimal, you have to have a factor. You have to have the denominator of the fraction to be a factor of ten, hundred, or thousand.

Like the preceding students, these students have equated "decimal" with "terminating decimal":

> T: So it works. [i.e. 1/3 can be converted to a decimal]
>
> S: No, because it keeps going.
>
> T: So does that make it not a decimal?
>
> S2: Yeah, but still because it's…but still it's a decimal.

This confounding of "decimal" with "terminating decimal" appears elsewhere in the literature (Tall & Schwarzenberger, 1978; Gardiner, 1985; Pinto & Tall, 1996; Sirotic & Zazkis, 2007) and is one of many implicit understandings students have that interferes with later learning. Another implicit understanding evidenced by

students was a tendency to interpret decimal numbers exclusively as parts of a whole:

> T: You bring up a very good point. You say that a repeating decimal is still a part of a whole. Even if it goes on forever. Do you agree with that? Gina?
>
> G: I don't. I don't agree with what Tyisha was saying that even if it goes on forever it's still a decimal, because I think that a decimal is actually a decimal because you don't know when it's going to stop. You don't know how many - cause I think a decimal is kind of like pieces of something of a whole. You don't know how many pieces it is that it's talking about [in the repeating decimal]. It just continues and continues, you don't know how many is in the whole, you don't know how many pieces in the whole, you just know that it just continues and it never ends. And I don't think that's really a decimal.

In the mind of this student, "decimal" is interpreted as a part/whole relationship, one of the several rational number subconstructs discussed in Section 4. This view is typically the focus in US instruction (Behr et al., 1983; Kieren, 1993; Ni & Zhou, 2005) and this situation illustrates why a robust understanding of rational numbers requires more than a part/whole understanding of rational numbers.

There is evidence from the literature that students and teachers alike see decimal representations and fraction representations as different types of numbers. Vamvakoussi and Vosniadou (2004) administered a questionnaire to assess sixteen ninth graders' understanding of rational number structure. Students were asked how many numbers were between the numbers 0.001 and 0.01, 3/8 and 5/8, and 0.005 and 0.006. Students who correctly answered those three questions were asked about 5/8 and 8.5, and 2/5 and 4/7 to assess the depth of that student's understanding of the density of rational numbers. Four of the sixteen students "answered that between two rational numbers, there are infinitely many numbers

in some, but not in all cases" (p. 463). The following quote exemplifies the meaning these participants attach to representation:

> Between these two (0.005 & 0.006), you can find 9 numbers, when they are in decimal form. But, if you turn them into fractions, you can find more: You can find infinitely many numbers in between.

Two participants said that there were an infinite number of numbers between $3/8$ and $5/8$, but considered "different representations of $4/8$ such as $4.0/8$, $\sqrt{16}/8$, and $32/64$" as different numbers (p. 462). This suggests that some students see equivalent fractions as different in some sense.

Hypothesizing that "aesthetically-rich learning environments enable children to wonder, to notice, to imagine alternatives, to appreciate contingencies and to experience pleasure and pride" (p. 26) Sinclair (2001) designed the "Colour Calculator," an online calculator that displays numerical results of computations in decimal notation as well as a rectangular grid of colored swatches, with one color corresponding to each of the Hindu-Arabic numerals 0–9. Fifteen eighth grade students explored the decimal representations of different rational numbers using the Colour Calculator. Student reactions to the activity suggested that there was a general sense that the decimal and fraction representation of rational numbers are different numbers. Several students "realised for the first time that a fraction and its corresponding decimal are the *same* [emphasis in original]" (p. 29). A "majority of the students realized. . . that [fractions] can have a denominator greater than 10 and they can even be *any* whole number over *any* [emphasis in original] whole number" (p. 30).

Sinclair et al. (2006) used the Colour Calculator to provide a different context in which to "work with fractions and their decimal representations interchangeably" (p. 179). They argue that the Colour Calculator emphasizes relationships between

the decimal and fractional representations of rational numbers. As evidence of this, Sinclair et al. (2006) describe how participants in their study who could use a well-known algorithm for expressing a repeating decimal in fraction form (see Beswick, 2004 or Weller et al., 2009 for details) explicitly stated that working with the Colour Calculator established a connection between the fraction obtained from the algorithm and the repeating decimal representation they started with. Similar to results obtained by Sinclair (2001), many participants expressed surprise that "the fraction they had inputted [to the calculator] and the decimal string that was being represented" were the same (p. 189).

The equality $1 = 0.\overline{9}$ has been a frequent source of controversy and confusion for students, teachers, and researchers alike. Tall (1977) investigated the equality through the lens of convergence of infinite sequences and series. Of thirty-six first-year mathematics students responding to a questionnaire, twenty-nine knew that $\lim_{n\to\infty}\left(1+\frac{9}{10}+\frac{9}{100}+\ldots+\frac{9}{10^n}\right)=2$ but twenty of the thirty-six said $0.\overline{9}$ was "just less than one" (p. 11). Moreover, of seven students who gave a valid definition of the limit of a sequence, only one responded that $0.\overline{9} = 1$. However, when the same group of students were told that $0.\overline{3} = 0.333\ldots$ and $0.\overline{9} = 0.999\ldots$, twenty-four students wrote $0.\overline{9} = 1$ or $0.\overline{9} = \frac{1}{1}$. The author postulates that these conflicting responses are a manifestation of the students' incomplete understanding of the limit process.

Keynes, Lindaman, and Schmitz (2009) investigated student understanding of different notations for repeating decimal numbers. A group of exceptional high school students participated in an advanced course of mathematical study, including college level calculus. These students, along with several different cohorts of calculus students at a large midwestern university, completed a survey instrument designed to assess understanding of three different notations for repeating deci-

mal numbers; e.g., $\frac{5}{9} = .\overline{5} = .555\ldots = \lim_{n\to\infty} \underbrace{.555\ldots5}_{n}$. A majority of students recognized that $0.\overline{7} = \frac{7}{9}$ but insisted that $0.\overline{9} \neq \frac{9}{9} = 1$. Further, the notation ".555..." was not associated with the underbrace limit notation used in the study. One student indicated that the "..." notation was "confusing and ambiguous" (p. 9) but the overline and "braced n" notation were not.

The majority of studies investigating $1 = 0.\overline{9}$ have concerned calculus or real analysis students (Burroughs & Yopp, 2010). Weller et al. (2009) point out that "many reports and studies suggest that college students, even those who have completed calculus courses, have difficulties with the limit concept at least as serious as their misunderstandings of repeating decimals" (p. 6). To this end, Weller et al. (2009) designed an instructional intervention for prospective elementary and middle grades teachers "that facilitates a deeper understanding of the connections between repeating decimals and the fractions or integers to which they correspond but does not depend on an understanding of limits" (p. 6). The experimental curriculum was delivered parallel to a traditional curriculum. Participants completed a survey after instruction to measure change in beliefs on two fronts: belief that $0.\overline{9} = 1$ and belief that every repeating decimal corresponds to a fraction or integer. According to the metrics used in data analysis, participants in the experimental section "made considerable progress [in] developing a deep understanding, both procedural and conceptual, of the relation between a fraction and its decimal expansion" (p. 23).

Weller et al. (2011) conducted a follow-up study to investigate three issues that were underexplored in their 2009 study: repeating decimals as numbers, belief that every repeating decimal corresponds to a fraction or integer, and belief about $0.\overline{9} = 1$. Several participants indicated that they did not believe repeating deci-

mals were numbers precisely because their decimal expansion is unending. One participant denied that repeating decimals could even be plotted on the number line, ostensibly because their exact value could not be determined. Using similar reasoning, one participant indicated that a repeating decimal cannot correspond to a fraction. Participants who rejected $0.\overline{9} = 1$ evidenced belief in an infinitesimal quantity between $0.\overline{9}$ and 1.

Using multiple studies that investigated the equality among calculus and real analysis students (Tall & Schwarzenberger, 1978, Edwards, 1997, and Dubinsky, Weller, McDonald, & Brown, 2005) as a framework, Burroughs and Yopp (2010) conducted an exploratory study investigating "variation in the existing repeated decimal conceptions among prospective elementary teachers" (p. 28). Many of the misconceptions the prospective elementary teachers held regarding repeating decimals were similar to those held by calculus students but had their origins in elementary school instruction.

Burroughs and Yopp (2010) interviewed five prospective elementary teachers regarding the equality $1 = 0.\overline{9}$. When asked if they believed $1 = 0.\overline{9}$, all five rejected the equality. Two participants, despite acknowledging that both whole numbers and fractions can have different representations (e.g., $2 = 2.0$, $1 = 2/2$), explicitly asserted that one has a unique representation and hence was not equal to $0.\overline{9}$. When asked to write $1/3$ as a repeating decimal and to rewrite the equation as a multiplication problem (i.e., $1/3 = 0.\overline{3} \Rightarrow 1 = 3 \times 0.\overline{3}$), one participant rejected the "benchmark equality" $0.\overline{3} = 1/3$ (p. 24). Another student rejected the inverse relationship between multiplication and division. Moreover, Burroughs and Yopp (2010) found evidence that "calculus and algebra concepts taught in later grades did not dissolve these misconceptions and quite possibly worsened them" (p. 39). These results reveal a problematic sense of number that, left untreated, may

lead teachers to "undermine aspects of important elementary school mathematics" (p. 24).

One question unexplored by Burroughs and Yopp (2010) was how knowledge of elementary school curricula held by inservice teachers influences their understanding of repeating decimals. Yopp et al. (2011) interviewed three inservice elementary school teachers to "gather data about participants' knowledge of whole numbers, fractions and repeating decimals and the relationships among these concepts" (p. 305). This part of the interview was used to assist the participant in developing three arguments for the equality $0.\overline{9} = 1$. The intent was to place the participants in a state of cognitive disequilibrium, defined as "the state of confusion that results when individuals are faced with information that conflicts with their prior held belief or understanding" (p. 305).

The participant with "over 20 years teaching experience, Maude, provided the opportunity for rich analysis" (p. 306). Like participants in Burroughs and Yopp (2010), Maude denied that $0.\overline{3} = \frac{1}{3}$. Also, her denial led her to make contradictory statements, such as stating that $0.\overline{3}$ is bigger than $\frac{1}{3}$, $0.\overline{6}$ is bigger than $\frac{2}{3}$, but $0.\overline{3} + 0.\overline{6}$ is less than 1. Maude's interview responses revealed an important difference between her and preservice elementary teachers; namely, Maude's understanding of real numbers is "intertwined with her sense of measurement" (p. 309):

> Interviewer: Some people think that one is equal to point nine repeating and some people don't. I'd like to hear what you think about it and why.
> Maude: I dont think its equal. I dont think its equal because, uh, I think that would be confusing to kids, to say that 99 cents can be rounded up to a dollar. You know, just right to money is immediately what I would think of. Umm. But definitely when you get to more scientific things I can see where that might be seen that way, you know, in different high

> level sciences, possibly, but, even then you'd think they'd want to be more particular about the size or the number.

The researchers assert the existence of a pattern. "When Maude is confronted with a question about repeating decimals, she situates the number in a sensory scenario, and when some of the information about the number doesn't situate well, Maude adjusts the number to accommodate her sensory perspective" (p. 309).

MET II (2012) "uses the CCSS as a framework for describing the mathematics that middle grades teachers, both prospective and practicing, should study and know" (p. 40). For grades 6–8, essential understandings regarding the number system make specific reference to repeating decimal concepts, including the equality $0.\overline{9} = 1$. Middle grades teachers are called to "explain why decimal expansions of fractions eventually repeat" and show "how decimals that eventually repeat can be expressed as fractions" (p. 41). To illustrate these ideas, the report recommends that teachers use "the standard US algorithm to explain why the length of the string of repeating digits in the decimal expansion of a fraction is at most 1 less than the denominator" and that teachers learn multiple ways to explain why $0.\overline{9} = 1$.

5. Summary

Content knowledge is an important component of the mathematical knowledge necessary for effective teaching. Researchers have identified at least three forms of content knowledge, all of which are seen as important for developing the unified sense of number described in CCSS (2010). In the United States, curriculum consists of textbooks and standards, with the former historically having a greater influence on what is taught than the latter. Mathematics textbooks used in

the United States have a long history of treating numerous topics superficially at the expense of developing conceptual understanding. Numerous standards documents have been published to address this deficiency (and other issues); among the most influential are Curriculum and Evaluation Standards (1989), Principles and Standards (2000), and Curriculum Focal Points (2006). The latest iteration of standards documents, CCSS (2010), frames the discussion of rational numbers substantially differently than previous standards documents, with more emphasis on rational numbers in general and the repeating decimal representation of rational numbers in particular. Existing research of student and teacher understanding of rational numbers, their representation, and connections between these representations suggests that the current teacher corps may be ill-equipped to implement the learning goals described in CCSS (2010) regarding rational number. To provide insight into implementation of these standards, a detailed description of how a variety of inservice middle grades teachers understand rational numbers is necessary.

CHAPTER 3

METHODS

1. Introduction

The purpose of this study was to describe how individual inservice middle school teachers, sampled from a particular school district, understood rational numbers and how those understandings appeared during instruction. The study employed standardized open-ended interviews to illuminate the rational number sense of a sample of inservice middle school teachers. From these teachers, a subsample was chosen for videorecored observation of instruction on repeating decimal concepts to identify ways in which the teachers' understanding revealed itself during instruction. The following three questions guided this study:

1. What are the ways that inservice middle school teachers, selected from a sample, understand rational numbers?

2. How do these teachers interpret the statement that a rational number is a number that can be written as a repeating decimal?

3. How do these understandings manifest during instruction?

This research describes how teachers understand rational numbers and how their understandings manifest during instruction. While many studies document teacher knowledge of rational numbers e.g., Hiebert & Wearne, 1986; Ball, 1990a, 1990b; Ubuz & Yayan, 2010), far fewer document teachers' understanding of equivalence among representations. In addition, few studies currently exist which address how teachers understand the repeating decimal representation of rational

numbers and even fewer make connections between what teachers know and what they teach to their students. This study explored how some inservice middle school teachers understood rational numbers, their representation, and the ways in which those understandings manifest during instruction.

2. Research Design

According to Cresswell (2007, p. 40), qualitative research is appropriate when the researcher seeks "to understand the contexts or settings in which participants in a study address a problem or issue." Qualitative methods allow the researcher to "explore a problem rather than to use predetermined information from the literature or rely on results from other research studies" and are often employed "because quantitative measures and the statistical analyses simply do not *fit* the problem" (p. 40, emphasis in original). Additionally, quantitative measures do not fully capture interpersonal interactions and are not very sensitive to differences between individuals.

Gay, Mills, and Airasian (2009) define case study research in terms of four conditions: it is "a qualitative approach to studying a phenomenon," it is "focused on a unit of study or a bounded system," it is "not a methodological choice, but a choice of what to study," and it is "an all-encompassing research method." They suggest that case study research is appropriate "for researchers interested in providing causal explanations, such as describing the process by which a particular innovation had a particular effect on the participants of the setting" (pp. 426–427). Stake (2005, p. 443) claims that a case study is not defined "by the methods of inquiry used" but "by interest in an individual case." My interest in how individual teachers understood rational numbers, their representation, and how those

understandings might appear during instruction led me to choose the case study approach. With this methodological decision made, I had to consider what type of case study would be appropriate given these interests. Since my ultimate goal was to provide insight into teacher understanding of rational numbers and how those understandings manifested during instruction, I chose to study multiple different teachers. In the words of Stake (2005, p. 446), I "believed that understanding them [would] lead to better understanding, and perhaps better theorizing, about a still larger collection of cases." Stake (2005) calls this kind of case study a *multiple case study*; for the reasons specified above, I chose this approach.

3. Sampling

Cresswell (2007, p. 126) describes convenience sampling as as "sites or individuals from which the researcher can access and easily collect data." Because of limited financial resources, I chose to sample from a school district that was nearby and readily accessible. My familiarity with teachers in this district facilitated gaining access to the research site, as I had existing connections with some teachers in this district.

Since the population of interest was middle grades teachers, I initially invited all middle grades mathematics teachers in the selected school district to participate in a 30–45 minute interview regarding their views on teaching and on rational numbers. Of the 22 middle grades mathematics teachers in the district, 10 agreed to participate. Those who declined offered one of two reasons: either they did not have enough time or they did not feel like the content of interest was part of their curriculum. To maximize the number of comparisons I could make between interviews and observations, I invited all ten interview participants to participate in

the observation phase. Four of these ten agreed to participate; those who declined said that they were too busy to be observed.

4. Data Collection

Qualitative researchers commonly employ four methods of data collection: observation, interviews, documents, and audiovisual materials (Cresswell, 2007). I used three types of data collection strategies to answer the research questions posed in this study; namely, interviews, documents, and observation.

4.1. Interviews

According to Patton (2002), "we interview people to find out from them those things that we cannot directly observe" (p. 340). To obtain a fuller characterization of teacher knowledge of rational numbers as it relates to CCSS (2010), I conducted what Patton (2002) describes as a "standardized open-ended interview" (p. 344). I designed an interview protocol using as a framework recommendations from Patton (2002). In what follows, I will explain the design process of the interview protocol and the rationale for design choices I made.

A standardized open-ended interview protocol uses carefully and fully articulated questions. Including this much detail in the interview protocol allows the dissertation committee, the institutional review board, and the school administration to review before granting approval. The interview can be conducted in an efficient manner because the questions to be asked are specified ahead of time and data "analysis is facilitated by making responses easy to find and compare." The protocol was still considered open-ended, however, because I allowed myself to state questions differently depending on the context of the conversation. I also

allowed participants to explore tangential ideas and questions as I deemed appropriate.

According to Patton (2002), categorizing questions asked in an interview "forces the interviewer to be clear about what is being asked and helps the interviewee respond appropriately (p. 348). The six categories of questions he describes are experience and behavior, opinions and values, feeling, knowledge, sensory, and background/demographic. I began the interview with a combination of experience and behavior and background/demographic questions so as to "encourage the respondent to talk *descriptively*" because "opinions and feelings are likely to be more grounded and meaningful once the respondent has verbally 'relived' the experience" (p. 352, emphasis in original). After these questions came questions about the participants' knowledge of rational numbers and their representation.

Patton (2002) also invites careful consideration of the wording of interview questions. He argues that "a truly open-ended question does not presuppose which dimension of feeling or thought will be salient for the interviewee" (p. 354). A typical mistake in question construction is the use of dichotomous questions, defined as those that "provide the interviewee with a grammatical structure suggesting a 'yes' or 'no' answer" (p. 354). I carefully constructed open-ended questions to allow "the person being interviewed to select from among that person's full repertoire of possible responses those that are most salient" (p. 354). I also reviewed the transcripts from my pilot study (see Section 4.3) to determine if any of my questions were potentially dichotomous.

One goal of this research was to assess teacher knowledge with respect to statements made about rational numbers in CCSS (2010). Of particular interest were standards addressing the repeating decimal representation of rational numbers

that were largely absent in previous standards documents. These standards and how they informed construction of the interview protocol will be discussed below.

Based on a careful reading of the number system standards for sixth, seventh, and eighth grade, I concluded that one of the overarching goals of the CCSS (2010) was for students to subsume terminating decimals as a special case of repeating decimals. This goal stood out upon comparison of the two following standards; the former from grade seven and the latter from grade eight:

> CCSS 7.NS 2d: Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.
> CCSS 8.NS 1a: Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually and convert a decimal expansion which repeats eventually into a rational number.

Based on the lack of emphasis on the repeating decimal representation of rational numbers in previous standards documents, I believed that middle grades teachers would possibly have difficulty viewing terminating decimals as a special case of repeating decimals. For this reason, I deliberately chose to omit the word terminating when discussing repeating decimals in the interview protocol.

I also found one key difference between CCSS (2010) and MCSS (2011) that informed the construction of the interview protocol in an important way. That difference can be seen by comparing the following standards; the boldfaced text has been added for emphasis:

> [Montana] 8.NS.1: Understand informally that every number has a decimal expansion; **the rational numbers are those with decimal expansions that terminate in 0s or eventually repeat. Know that other numbers are called irrational.**

> [National] 8.NS.1: Understand informally that every number has a decimal expansion; **for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.**

My initial interpretation of the statement in MCSS (2011) was that this document was defining rational numbers as eventually repeating decimal numbers. For this reason, and because of the widespread practice in elementary and middle school where definitions are taken as descriptions of concepts rather than formal definitions, I presented the following statement to teachers as a definition of rational number for consideration: "A number is rational if it can be written as a repeating decimal." It is important to note that, in the standard presentation of the rational numbers, this statement is not a definition of rational number but rather a logical consequence of defining rational numbers as ratios of integers.

4.2. Interview Protocol

The interview protocol (see Appendix A) contained two broad sections. The first section corresponds to questions one through four, which served to establish a rapport with the participant as well as identify key demographic and other contextual information. Questions five through nine are intended to provide multiple perspectives on how the participant understood rational number. Questions 10 through 13 served to provide multiple perspectives on how the participant interpreted the statement "A number is rational if it can be written as a repeating decimal." A matrix describing the design of the interview protocol is pictured in Table 3.1. The interview protocol was designed to provide a full picture of how participants understand rational number and the statement about rational numbers and their decimal representation. Each participant described their under-

Table 3.1. Interview Protocol Alignment Matrix

Questions	1–4	5–7	8–9	10	11
Task	Background	Definitions	Concept Map	Interpretation	Classification

standing of rational number through their responses to four tasks. The first task will hereafter be referred to as the "definitions task", and consisted of participants describing, in their own words, their definition of the terms "fraction," "decimal," and "rational number."

To provide further insight into participants' understanding of rational number, they were asked to complete questions eight and nine of the interview protocol, which will be hereafter referred to as the "concept map" task. Participants were given three four inch by six inch laminated index cards and were directed to write the word "fraction" on one, the word "decimal" on the second, and the words "rational number" on the third. The participants were then invited to attach the index cards to the whiteboard (each card had a magnet attached to it) and, using words, pictures, or whatever means they felt appropriate, to describe the relationships they saw between these three concepts.

Question ten, hereafter referred to as the "interpretation task", consisted of participants receiving a piece of paper with the following statement written on it: DEFINTION: A number is *rational* if it can be written as a repeating decimal. Participants were asked to offer an opinion of that definition, with follow up questions asked as deemed necessary by the interviewer. The final task, hereafter referred to as the "classification task", consisted of participants using the definition from the interpretation task to identify from a list of ten different numbers those that were rational.

4.3. Pilot Study

I conducted a pilot study to assess the suitability of the interview protocol I devised. A student teacher, "Kendra," volunteered to participate in the pilot. The pilot study provided insight into Kendra's sense of rational number:

> Interviewer: Rational number, how would you define rational number?
> Kendra: A rational number–I was just going over this with my kids... a rational number, it's like the Venn diagram, includes um, integers and whole numbers... a rational number is a number that cannot go on indefinitely.
> Interviewer: Okay, so a rational number in sum is, would you say it again? I missed it?
> Kendra: It includes integers and whole numbers... and counting numbers, and it cannot go on indefinitely.

Kendra does not have a mathematically precise definition of rational number in her active repertoire of knowledge, instead relying on a more colloquial description of rational number that relies on different features of numbers. When interpreting the repeating decimal characterization of rational number, Kendra draws a sharp distinction between terminating and repeating decimals:

> Interviewer: All right... some people define rational number as numbers that can be written as a repeating decimal... as someone who studies math, what do you think about that?
> Kendra: No.
> Interviewer: No?
> Kendra: Um, rational numbers have to be definite, they have to have a definite ending... so like one sixth, it is a rational number but if you put it into rational form or put it into fraction form it becomes irrational.

Like the prospective secondary teachers studied by Zazkis and Sirotic (2010), Kendra evidenced belief that the nature of a number changes when it is repre-

sented differently. This belief appeared again when I asked Kendra to classify the number $\frac{1.2}{5.74}$ as rational or irrational:

> Kendra: If it has a finite ending after oh, four or five, it has a finite ending, then it is rational, if it keeps going on, then it is irrational.
> Interviewer: Okay, so a finite ending, like you said four or five, did you mean four or five digits or the numeral four or five?
> Kendra: Four or five digits, but even after, um, even after ten, I mean, it, it has a finite ending.
> Interviewer: So you think that one's rational, or you're not sure?
> Kendra: If I keep going [with the division procedure].

In the parlance of Zazkis and Sirotic (2010), the representation $\frac{1.2}{5.74}$ is opaque with respect to its status as a rational number. The equality $\frac{1.2}{5.74} = \frac{120}{574}$ embodies a representation that makes its status as a rational number transparent, provided that one knows that rational numbers are ratios of integers. Kendra's incomplete understanding of rational numbers ostensibly obscures this more transparent representation.

4.4. Observation

Observation "entails the systematic noting and recording of events, behaviors, and artifacts (objects) in the social setting" (Marshall & Rossman, 2011, p. 139). The design of the study necessitated classroom observations, as one of the goals of this research was to compare and contrast teacher knowledge of rational numbers to how they taught rational numbers. Cresswell (2007) describes observing as consisting of a series of steps: selecting a site for observation, identifying who or what to observe, determining a role as observer, designing an observation protocol, recording information about the site, being introduced, and withdrawing from

the site. My observations were structured in a similar way to these steps, but with some important differences, as will be described below.

The site for observation was determined by participants, as I could only observe participants who agreed to such an observation. Because I was primarily interested in instruction, I focused my attention on the teacher and observed for as long as the teacher was teaching repeating decimal concepts. As a result, my observations ranged from one to three 45 minute class periods. I wanted to view the instruction and content presented from the perspective of a mathematics educator and to make comparisons between what was said during the interviews and what was said during instruction. However, I also wanted to respect the work being done in the classroom and be of assistance to the participants and their students in any way I could that did not compromise the data collection. For this reason, I chose to strike a balance between observing from the emic (participant view) and etic (observer view) perspectives.

I chose to use videorecordings as a way of recording field notes during observations. These recordings provided a record of the physical setting, the nature of the interactions between students and the teacher, and a verbatim account of the words exchanged during instruction. I used the results of data analysis from the interview phase to determine what to watch for during observations. Specifically, I looked for similarities and differences with respect to the different perspectives on rational number that I found during data analysis, which are discussed in Chapter 4.

5. Procedures

I approached the superintendent of the participating school district as per the school district's policy regarding the conducting of research. I provided an abstract of the project, copies of informed consent documents for all participants, copies of parental consent forms, and a detailed time line indicating when research events will occur so that the purpose of the research was clear. After obtaining permission from the district administration, I approached the principals of the two middle schools with the same documents so as to obtain permission to talk with the teachers in their respective buildings. Each principal made initial contact with the teachers in their buildings as per the district administration's request; with their help and that of the middle school mathematics coach, I compiled a list of middle grades mathematics teachers. I sent an informal email message to each potential participant that briefly described the project and what their participation would involve.

All ten interviews were scheduled within the first four weeks of the district's school year, which was necessary for several reasons. First, my conversations with teachers suggested that there was considerable variability in the sequencing of topics and it was important that I conducted the interviews before participants completed formal instruction regarding repeating decimals. This was important because I deliberately chose to conduct interviews before observations. While discussing rational number concepts with participants before they taught those concepts likely influenced the way they conducted their instruction, I determined that the benefit of building a relationship with the participant was of greater value and offset these costs.

Interviews proceeded largely in the way suggested by the interview protocol. I asked each participant each question on the interview protocol, but only used the prompts when I felt it was necessary. Based on the organic nature of conversation, I sometimes asked questions in a slightly different order or using slightly different language. The most common instance of this was when I asked about definitions. Sometimes I asked participants how they defined the terms fraction, decimal, and rational number, while other times I asked them how they would define those terms to a student, basing my choice on whatever was most natural given the nature of the conversation.

Teachers who agreed to be observed were asked to share an approximate time frame during which I could observe them teaching a lesson related to the repeating decimal representation of rational numbers. The first two observations occurred approximately one month after the interviews with those participants; the third after approximately two months; and the fourth after approximately five months. The four participants who agreed to be observed exhibited a great deal of variability along many dimensions. The fewest years of teaching experience was five, with the most being 29. I observed instruction at sixth, seventh, and eighth grade levels ranging from primarily teacher-centered instruction to a flipped classroom model where students watched instructional videos and completed exercises independently with teacher assistance as needed. Despite having only obtaining four participants, I was satisfied with the variability obtained from these four observations.

6. Data Analysis

Unlike quantitative research designs, qualitative data analysis begins "from the initial interaction with participants and continues that interaction and analysis throughout the entire study" (Gay et al., 2009, p. 448). The data collection continually interacts with analysis: the researcher collects data, studies the data, compares the data with newly collected data, summarizes the data, and uses those data to refine the next phase of data collection. Once data collection is complete, however, the researcher turns his focus exclusively to data analysis. I will first describe how I analyzed the interviews, followed by the observations.

6.1. Analysis of Interviews

I began analysis of the interviews by watching the videorecordings and produce verbatim transcripts of each interview. This process allowed me to become very familiar with contextual details of each case, such as the participant's disposition, perspectives on teaching and on mathematics, and the way they talked about and thought about rational numbers. With the transcripts in hand, I read each one individually, underlining passages that seemed important and making margin notes about my reactions to the data. After completing this for each participant, I set aside these transcripts so as to have a record of my initial impressions of the data. This record of my initial impressions was useful in later data analysis, as I could compare my initial impressions of the data to perspectives developed later in data analysis.

After this initial pass through the data, I began a within case analysis. I read the transcripts from each interview again, underlining passages that seemed important and writing notes in the margin that recorded my reactions to what I was

reading. Next I compared the definitions of fraction, decimal, and rational number each participant gave to how they described these words being related in the concept mapping task, making note of how these were consistent and inconsistent with each other. I made similar comparisons between responses to the interpretation task and the classification task.

The next step was a cross-case analysis that spanned all participants. I compared each participants' responses to each of the tasks from the interview protocol and began looking for similarities and differences. I attached labels to groupings of text and assembled a matrix that associated each code with a participant and one of the tasks from the interview protocol. Codes that represented similar ideas were grouped together into categories. These categories were refined through negative case analysis; i.e., I looked for participant responses that were not accounted for by my categories and revised my categories accordingly. As I established categories, I solicited feedback on my choices from my advisor and my colleagues to ensure that my interpretations were not merely artifacts of my own perspectives. The final step of data analysis was to identify understandings of and perspectives on rational number that emerged from the categories.

7. Trustworthiness

Citing "a dearth of knowledge about how to apply rigor in the naturalistic paradigm" (p. 76), Lincoln and Guba (1986) proposed a structure for evaluating the quality of qualitative research that is parallel to that used in the quantitative paradigm. Rather than address a study's validity, they suggest several criteria for evaluating a study's trustworthiness. In what follows I describe the measures taken in this research to ensure its trustworthiness.

7.1. Rich Data

Becker (1970, pp.51–52) defines rich data "as containing great detail and specificity about the events studied. . . ." These sort of data. . .

> . . . counter the twin dangers of respondent duplicity and observer bias by making it difficult for respondents to produce data that uniformly support a mistaken conclusion, just as they make it difficult for the observer to restrict his observations so that he sees only what supports his prejudices and expectations.

I took several measures to ensure the richness of the data collected for this study. I conducted interviews with each of the ten participants; these interviews ranged from 40 to 60 minutes in length. The videorecordings allowed me to produce verbatim transcripts of each interview and taken together provided a great deal of information about the context in which the interviews took place. All ten participants provided samples of written mathematics when they completed the classification task and also completed a concept mapping activity where they discussed their conceptual understanding of rational numbers. Additionally, I observed instruction of four of the ten participants for one to three class lessons.

7.2. Respondent Validation

Respondent validation is "systematically soliciting feedback about your data and conclusions from the people you are studying" (Maxwell, 2013, p. 126). Respondent validation protects against misinterpreting the words and perspectives of participants, and is "an important way of identifying your biases and misunderstandings of what you observed" (p. 127). Despite the power of respondent validation, it is important to view participant feedback as "*evidence* regarding the validity of your account" (p. 127, emphasis in original). I performed respondent

validation throughout the interviews by rephrasing participant's words and asking them for feedback regarding the correctness of my interpretations. When the meaning of a participant's words was unclear, I contacted them for clarification.

7.3. Negative Case Analysis

Another form of evidence are those data that "cannot be accounted for by a particular interpretation or explanation" (Maxwell, 2013, p. 127). Conducting negative case analysis was an integral part of data analysis in this study. At each stage of data analysis, I looked for data that conflicted with my hunches, my codes, and my categories. My colleagues and advisor also offered their insight and opinions about my data analysis choices and offered feedback regarding the correctness of my interpretations.

7.4. Triangulation

Maxwell (2013) defines triangulation as "using different methods as a check to one another, seeing if methods with different strengths and limitations all support a single conclusion" (p. 102). According to Patton (2002), the use of one method to the exclusion of others makes a study more susceptible to the errors inherent in that method. I interviewed all ten participants and collected samples of written mathematics and diagrams from all participants. I conducted observations with four of the ten participants to triangulate my results.

7.5. The Researcher

I received a Bachelor of Arts degree in Secondary Mathematics teaching in 2004, after which time I taught prealgebra to seventh grade students and first-year algebra to eighth grade students. After this, I enrolled in a Master of Arts in Teaching

Mathematics degree program. As part of this program, I taught remedial algebra to undergraduates and conducted an action research project regarding the effectiveness of implementing active learning techniques in a remedial algebra classroom. After I completed my MAT degree, I enrolled in a mathematics education PhD program. I studied real analysis, probability theory, linear algebra, applied mathematics, and linear modeling. I also assisted a classmate in his dissertation project by conducting standardized open ended interviews of some of his research participants.

This background information highlights the biases that I bring to the present study. These biases are rooted in my teaching experience with secondary and college students, as well as my study of mathematics at the PhD level. I have seen different students at different levels make the same mistakes repeatedly, such as adding numerators and denominators and obtaining a common denominator for multiplication of fractions. Moreover, I have observed in my teaching experience a near universal preference for decimal over fraction representations among my students in their written work. My advanced mathematical training has made me cognizant of the perils of imprecision when doing and teaching mathematics. My teaching experience has led me to expect students to rely on colloquial, rather than precise, definitions of rational number, such as "fractions are parts of a whole" and "decimals are always numbers between whole numbers."

CHAPTER 4

RESULTS

1. Introduction

This study described how inservice middle school teachers described their understanding of rational numbers, how they interpreted a statement about the decimal form of rational numbers, and how those understandings manifested during instruction. The research was conducted in two phases. The interview phase consisted of videorecorded interviews with ten inservice middle school teachers using a standardized open-ended interview protocol designed by the researcher. Four of those teachers agreed to participate in the observation phase, during which the researcher videorecorded a lesson on rational number concepts taught by one of the teachers.

This study was designed to answer the following questions:

1. What are the ways that inservice middle school teachers, selected from a sample, understand rational numbers?

2. How do these teachers interpret the statement that a rational number is a number that can be written as a repeating decimal?

3. How do these understandings manifest during instruction?

This chapter presents the results of qualitative data analysis in two sections. The first section begins with a tabular summary of demographic information about each participant in the interview phase. After the summary, results from analysis of individual responses to the four tasks described in Chapter 3 are presented. This

Table 4.1. Interview Participant Overview

Participant Pseudonym	Teaching Certificate	Number of Years Teaching Experience	
		General	Mathematics
Amy	K-8	7	5
Adam	K-8	9	2
Bill	K-8	10	6
Betsy	K-8	11	5
Charlie	K-8	7	7
Claire	K-8	12	12
Dawn	K-8	10	7
Dillon	K-8	29	29
Emily	K-8	2	2
Francine	K-8	17	4

section concludes with comparisons across all ten participants in the interview phase. The second section presents results from the observation phase, beginning with a detailed description of each of the four cases and concluding with results from each of the four observations. The chapter concludes with a discussion of the perspective on rational number that emerged from the categories, followed by a summary.

2. Results from the Interviews by Participant

2.1. Amy

Amy's definition of fraction emphasized the relationship between the part and the whole. In her words, students have a tendency "[to] get locked into this frame of thinking where a fraction is just like a strip or a circle divided into pieces and so when you start presenting them with problems that are outside of that context

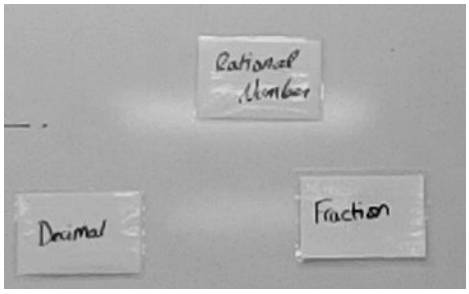


Figure 4.1. Amy's concept map.

they don't understand it still represents a fraction." Amy's definition of decimal was framed by the conceptual difficulties her students have with this representation. She said, "it's easier for them to say three-fourths…I can represent that in multiple ways, whereas if I gave them seventy-five hundredths…there's a disconnect in that they're between being able to represent fractions and being able to represent decimals." To combat this, she linked decimals "to fractions, and [tells the students] about how…there's a fluidity between fractions and decimals and how you can represent them differently."

Amy explained her understanding of rational number with an amusing anecdote:

> Amy: (Laughs) [A colleague] started talking rational numbers last year and [the students] like freaked out.
> Interviewer: Why's that?
> Amy: Because [the students] thought it was, like, some paradigm shift to their…understanding of numbers…[so I] talked to [the students] about, you know, when people were developing their understanding of numbers they classified them according to different properties…so rational number is just providing an over arching classification for all these different types of numbers that you have already been studying.

Amy's concept map depicting her understanding of the ways fraction, decimal, and rational number are related is pictured in Figure 4.1. Amy used the three cards as vertices of a triangle. Rational number is placed at the top because "rational

numbers are an overarching classification." However, she said "not all fractions are rational numbers and not all decimals are rational numbers." After encountering difficulty producing an example, Amy took a different approach:

> Amy: So not all decimals are rational numbers, because if it's a non-repeating decimal that never terminates, it is not considered rational, and now that I think about fractions, I think all fractions are classified as rational numbers, but when you convert them into decimals they would no longer be considered rational... but I don't know if that's really true or not.

Amy's provisional thinking that a number's representation changes its status as rational appeared again when she considered the numbers $\frac{1.2}{5.74}$, $\frac{\pi}{4}$, $\sqrt{2}$, $\frac{5\frac{1}{4}}{47\frac{1}{2}}$ during the classification task.

> Amy: So these four numbers would be considered irrational in decimal form... so in one sense it makes sense that they would be considered irrational in this form as well because how can a number go between rational and irrational... if my definition is based on decimals, then they could be rational, so I feel like I need to refine my definition.

Amy's response to the classification task provided a more clearly articulated statement of her definition of rational number.

> Amy: So, that would be one type of rational number, but you could have a rational number with a decimal that terminates, you could have a whole number that's a rational number, you could have a fraction that's a rational number, so I think if you're talking about that specific context in which what type of decimals become rational, that could be a good definition, but... my students, they would think like point three three three three three three three repeating, they wouldn't expand that context into like, point one two three, one two three, one two three, so I feel like you would need to clarify some of that.

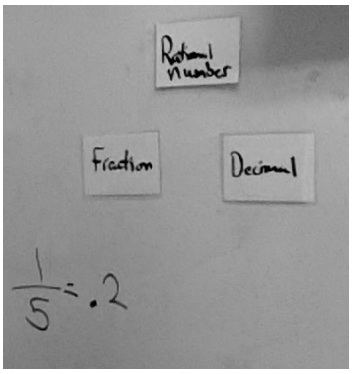


Figure 4.2. Adam's concept map.

2.2. Adam

Adam's definitions of decimal and fraction were succinct. A fraction was "an amount of a whole" and decimal was "a way of sorting place values, showing, the difference of the ones and the tenths place." Adam had comparatively more to say about rational numbers.

> Interviewer: What about if a student asked you to define rational number, what would you say?
> Adam: A rational number is, is something that makes sense, it can, it's got a definitive answer to it, whereas irrational, has, well, definitive is, that's a poor choice of words…irrational numbers have that non-repeating pattern, that continues…you can't predict what's coming next, whereas a rational number you could, as far as the decimal places go, you could predict the end.

Adam's concept map depicting his understanding of the ways fraction, decimal, and rational number are related is pictured in Figure 4.2. After placing the cards in a column with fraction, followed by decimal, and ending with rational number, Adam rearranged the cards as vertices of a triangle. He said "with rational numbers, you could produce a rational number with a fraction or fraction is a rational number." He added:

> Adam: Decimals are involved in rational numbers... as well as... finding out the, the number from a fraction, you would be able to, use decimals to help define that.
> Interviewer: So when you say computing the fraction, what would that be?
> Adam: Like dividing it out, using, calculation.

Adam's response to the interpretation task showed that he viewed terminating and repeating decimals as fundamentally different:

> Adam: [reading from his textbook] rational number: a number that can be written as a fraction in the form a over b, where a and b are integers and b cannot be equal to zero.
> Interviewer: Hmm. What do you think of that definition?
> Adam: It's not the same as [the repeating decimal characterization] because... that repeating thing seems to eliminate some of what they have here.
> Interviewer: You mean the a over b?
> Adam: Yeah, so a could be 1 and b could be 2, and... so yeah, if I was an eighth grader I would look at this and say, well what if I went 1 over 2 then that would be five tenths and, that in my eyes is not a repeating decimal.

Adam used both the statement about the decimal representation of rational number and the definition from his textbook to complete the classification task. He correctly classified π and $\frac{\pi}{4}$ after reminding himself that π is not an integer and hence not rational. Adam experienced conflict with regard to 2.33 "'cause [2.33 is] not an integer" but eventually resolved the conflict, saying "I think I could come up with a fraction that gets that and both [a and b are] integers." He similarly decided zero was rational "because I can go, zero over one... and b is not zero." The numbers that were expressed explicitly as repeating decimals were easily classified

by Adam as rational; however, the numbers that were not presented explicitly as repeating decimals were more problematic:

> Adam: (referring to $\frac{1.2}{5.74}$) One point two (writes down irrational) but I'm gonna double check [using a calculator]...in case I get something funky out of this...irrational...because...two zero nine zero five nine two three three, there doesn't seem to be any repeating pattern here
> Interviewer: But you thought that even before you put it in the calculator.
> Adam: Yeah, I'm thinking of the, integers...and so same thing, I'm gonna go with this one, same thing [about $\frac{5\frac{1}{4}}{47\frac{1}{2}}$], irrational...'cause they're not integers.

2.3. Bill

A recurring theme in my conversation with Bill was a desire to connect new knowledge to existing knowledge. Bill consistently viewed interview questions through this lens, including my question about how he defines decimal.

> Bill: It's a way we write something in order to clearly understand, and differentiate, different types of numbers...a whole number might have pieces and a decimal is...a rule that we use so that we're all talking the same language, to express pieces, portions, parts. And of course I would do that with showing them something...one way we talk about a part of things [draws a circle on the desk, divides it in two] is we can say there's two parts, and this is one of the two parts.

Bill defined rational number in terms of ratios, saying "at a seventh grade level, you know I would talk about being able to express a rational number in a fractional form as, one of the kind of definitions of a rational number. He also advocated the use of examples to make the concept of rational number more accessible to students.

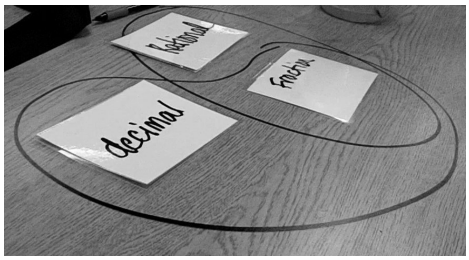


Figure 4.3. Bill's concept map.

> ...I wanna give an example...pi's one that they're familiar with, to some degree, isn't it a non-rational number?...I would make sure, I would give 'em some examples, whole numbers, you know, negative numbers.

Bill's perspective on fractions, decimals, and rational numbers came into sharper focus as he worked on the concept mapping task. His concept map depicting this relationship is shown in Figure 4.3.

> Bill: Yeah, maybe there's a hierarchy or not, I don't know...well, fractions are rational numbers, some decimals are rational numbers...so these live together (draws a circle around rational number and fraction)...fractions can be written as decimals...if I were explaining it to a seventh grader...I would say one half equals point five...see how fractions can be decimals.

Bill's understanding of rational number, accompanied with his ready acceptance of the statement about the decimal form of rational numbers, enabled him to quickly dispatch with all of the classification tasks except for [$\frac{\pi}{4}$]:

> Bill: That can't be [rational], right?
> Interviewer: Why do you think that?
> Bill: Well because I can't...I'd have to make [π] a rational number before I, divide it by four...I'm not positive, but I would say that...if I take a not rational number and divide it by a rational number, I can't do that, I don't know how to do that...I don't know how to get a final answer for [$\frac{\pi}{4}$].

Bill classified $\frac{\pi}{4}$ as irrational because he believed it was impossible to perform the indicated division to generate a decimal representation.

2.4. Betsy

Betsy's definition of decimal was "a way to look at a part or a whole with a part." Moreover, she interpreted it as a way of representing a number, "just like a fraction is a way to represent a number, a decimal can be a way that you represent a number that's either a part of a whole or a whole and a part." Betsy defined fraction in terms of decimals, saying they are "the same thing…if you're looking at the tail end of a decimal, the way you say it is the same as what a fraction looks like." As an example, Betsy wrote .10 and said "so [.10] would be [$\frac{10}{100}$] and, that's really what this is, in terms of a fraction, it's the same thing…[they] say the same thing but look really different, and they're both based on this idea of, you know, the place value, of, the last digit."

While fractions and decimals were a way of representing numbers to Betsy, she saw rational number as a concept that connects several other ideas from middle school mathematics:

> Betsy: So a number that, you would, that could be placed on the number line, it would include integers, so, whole numbers, and their opposites, and decimals, and fractions would be part of rational numbers, because those could be placed on that number line.

While Betsy's statement about the number line is true, it suggests that the number line is populated solely by rational numbers. This idea came up again when Betsy completed the concept mapping task (see Figure 4.4):

> Betsy: Well I would say that [rational number] is the overall arching umbrella of [fraction and decimal] because [fraction and decimal] fit

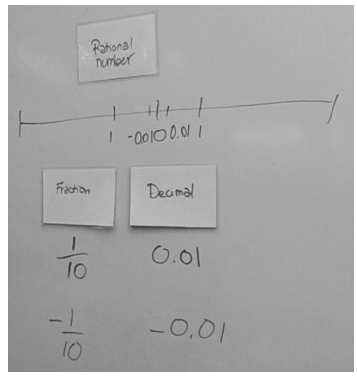


Figure 4.4. Betsy's concept map.

> within…because [fraction] is one way to say the same thing as [decimal]…so both [1/10 and 0.01] this is just two different ways to represent this number…and both of these could fall within [rational number]…all of these could be placed along a number line and they would, they would make sense.

Betsy's reaction to the statement about the decimal form of rational numbers showed that she viewed terminating and repeating decimals as different. She said, "a number's rational if it can be written as a repeating decimal, so, does that mean that if it can't be written as a repeating decimal that it's not rational?" As a counter example, she offers 1/10, but quickly acknowledges that "I guess you could say…that the decimal one tenth, if I put zeros on the back of that (writes six zeros after 0.01), that would be repeating," but only "if you interpret it loosely…to me a repeating decimal…is not a place holder."

During the classification task, Betsy added zeros to the end of a decimal string to determine whether or not a given number was rational. When a number was not presented in decimal notation, Betsy used a hand held calculator to generate a decimal approximation, which she uniformly interpreted as equal to the given number. Her work on classifying $\frac{5\frac{1}{4}}{47\frac{1}{2}}$ exemplifies the details of this strategy:

> Interviewer: So then, $5\frac{1}{4}$ is 5.25 and that's rational and 47.5, that's rational, I wonder–what would happen when you did the division now? I wonder if that's still rational?
> Betsy: (types on calculator) so…it ends up being (display shows 0.0110526315)…a really long string, but again, out here or if you wanted to, you could add a zero…so by that definition I'd say yes.

Based on the output from the calculator, Betsy concluded that $\frac{5\frac{1}{4}}{47\frac{1}{2}} = 0.0110526315 = 0.01105263150\overline{0}$ and hence classified this number as rational. This correct conclusion was based on incorrect reasoning, as $\frac{5\frac{1}{4}}{47\frac{1}{2}} = 0.1\overline{105263157894736842}$.

2.5. Charlie

Charlie described his understanding of fraction in two different ways. "The simplest…is a fraction usually represents a part, of something bigger, although it might represent that something bigger and, part of another." He also said that "especially in fifth and sixth grade, I would remind [students] that a fraction is a division problem…and it's also another way to express that idea of dividing things." Charlie's definition of decimal was closely related to his definition of fraction.

> …A decimal can be thought of as a number that's between two other numbers, two whole numbers, if you think of two whole numbers, we need a way to…account for what's between those, and so that's what a decimal number is, something between two whole numbers…

When asked how he defined rational number, Charlie responded in terms of his ability to distinguish between rational and irrational numbers:

> Charlie: To explain that to kids…is something that I don't understand, the distinction. I mean, I can give you examples…I know that π is an irrational number…it's a decimal that goes on forever without repeating, and that's an irrational number. I can't think of other examples at the

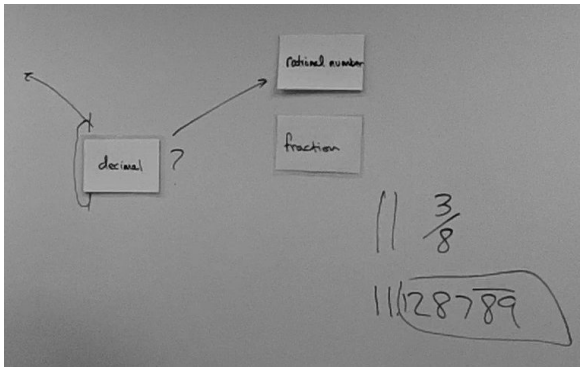


Figure 4.5. Charlie's concept map.

> moment, but…to distinguish between that and a rational number…I couldn't do it.

While he didn't offer an explicit definition of rational number in this exchange, the reference to the decimal representation of an irrational number suggested that Charlie would define rational number in terms of its decimal representation. As he worked on the concept mapping task, however, a more colloquial understanding of rational number emerged:

> Charlie: My initial thinking is that decimals and fractions in my mind, were rational, but, I know that there are some decimal numbers that are irrational…like π…so…I feel that fraction works [under rational number] better, because, when you express, some of the division that's going on as fractions, you can call that a precise, number…somehow that seems more rational to me than a decimal that goes on forever and keeps changing.

In Charlie's experience, students saw a number like $11\frac{3}{8}$ as "much more concrete" whereas a repeating decimal like $11.128\overline{789}$ "is more on the irrational side because I really can't wrap my mind around all these digits." Charlie's comment about rational numbers being "precise" came into sharper focus when asked to interpret the statement about the decimal form of a rational number:

> Charlie: Well, you know, you're talking to a word guy, so... the math definitions are gonna be different, of course–my mind immediately goes to reasonable... for rational... that it's a logical number... so when you say it's written as a repeating decimal, that's a... definition that I can apply, but it's not one that I can really appreciate and understand.

Charlie was already familiar with the irrationality of π and argued by extension that since π is "an irrational number... even if I divide that by four I don't think that's gonna convert it into something... that's gonna go on forever without repeating too." He classified $\frac{\sqrt{4}}{3}$ as rational because "that's just two thirds" and 0 as rational because "I'm at zero now, and I'm thinking wow, that's a really intriguing concept, but I can write that as zero point, zero with a zero going on forever." $\frac{1.2}{5.74}$ and $\sqrt{2}$ were classified as irrational based on the first nine decimal digits of their decimal expansion generated by a hand held calculator. In both cases, Charlie justified his response because of the absence of any discernible pattern.

2.6. Claire

To Claire, a "fraction, of course, is a part out of a whole... the whole can be one whole piece of paper, but the whole could be three coffee cups... so a fraction is a portion of that whole, whether it be something that you cut out of, one whole, or whether it be one out of the three." Claire described decimal as "in its essence... it is a portion of a whole–very, very similar... Just because it's represented differently than a fraction is, it's very much a fraction... It could be the whole plus... you could have your whole, and more, you know, but not, quite, another whole." When asked how she would define rational number, Claire indicated that this was content knowledge she did not use on a regular basis:

> Claire: Typically, my reference for rational numbers, you know, come[s] from the fifth grade... instead of necessarily like a sixth, seventh,

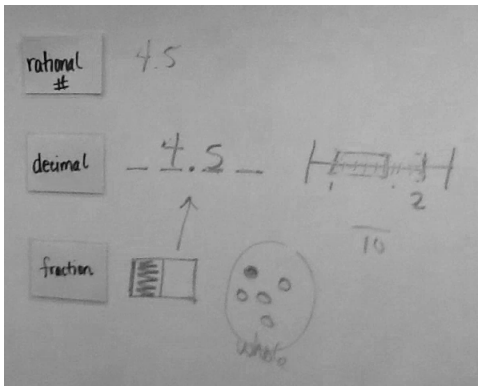


Figure 4.6. Claire's concept map.

> eighth–
> Interviewer: Is it different?
> Claire: It's not, it's not different, however, it doesn't come up much. . . not at all. . . you know, for [fifth grade students]. . . I don't know this, when do they get into irrational numbers?

Claire revealed more about how she understood rational number as she completed the concept mapping task. She began by placing all three cards in a column as pictured in Figure 4.6. Claire saw fraction as either "a part of the whole" or as "one of a set of things." Decimals were related to fractions "because, I could also do my decimal on the number line." At this point Claire drew the picture depicted in Figure 4.7 and said decimal "is, part of a fraction, because, I could also do my decimal. . . on the number line. . . yes it can be a fraction as well." Having laid this framework, Claire reveals that she sees fraction, decimal, and rational number as a hierarchy of abstraction.

> Claire: And then rational numbers. . . to me, those actually become numbers, you know, it kind of goes from. . . a picture [points to fraction], into something more, more abstract [points to decimal], where it's a representation of something. . . with numbers [writes 4.5 on the number line by decimal]. . . [4.5] then, to me, is just the actual number.

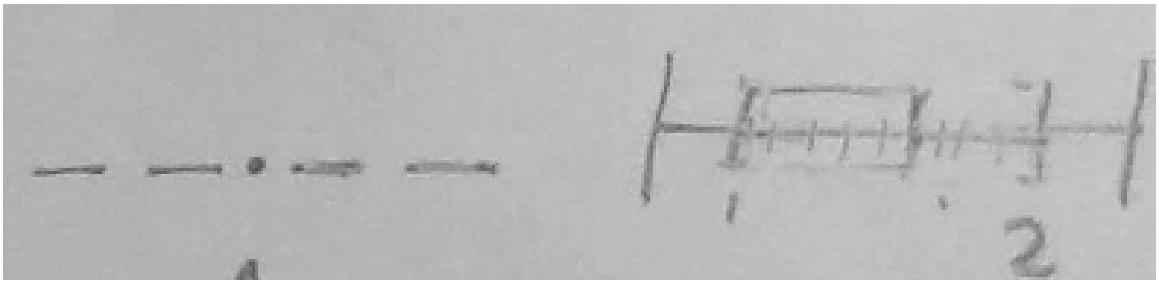


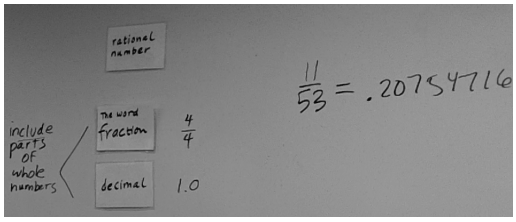
Figure 4.7. How Claire saw decimals related to fractions.

> Interviewer: So it sounds like you're saying that, a rational number is like, context free–there are no, is that it?
> Claire: In my experience, yes.

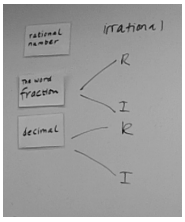
Claire was unfamiliar with the statement about the decimal form of rational numbers. As Claire completed the classification task, it became clear that she interpreted "repeating decimal" as referring to nontrivial repeating decimals. She classified $0.\overline{3}$ and $0.\overline{9}$ as rational because "that would be, for me the obvious ones, because of that decimal, coming up to a rational number, seeing that line, immediately means to me rational number." Using a calculator, Claire determined that $\frac{\sqrt{4}}{3}$ was rational "because [its decimal representation] repeats." That number, along with $0.\overline{3}$ and $0.\overline{9}$, were the only three Claire classified as rational.

2.7. Dawn

Aside from part to part comparisons that occur in the context of ratios, Dawn said that she would define fraction as "part of a whole," where the whole "could be a square, could be a rectangle, could be anything." Decimals were used to "represent things that are not part–that are not a whole." To Dawn, "you would never really use a decimal unless you were representing something with–that was not a whole. . . otherwise why bother. . . I guess it's just going from whole numbers to not whole numbers."



(a) Dawn's first concept map.



(b) Dawn's second concept map.

Figure 4.8. Dawn's two concept maps.

When asked how she defined rational number, Dawn first said what a rational number was not, saying "If I'm gonna teach irrational, I've gotta look it up, what does that mean?" Shortly after this, she offered the following definition:

> Dawn: For my understanding of rational number…all whole numbers are rational numbers, so all integers are rational numbers…a decimal that terminates, yeah. That's it. Decimals that terminate, or whole numbers…anything with a repeating, or like π, they go blahdeblah on forever, or numbers that repeat, forever, are irrational.

Dawn's first attempt at a concept map is pictured in Figure 4.8a. Beyond indicating that decimal and fraction "are…really kind of the same thing…in that they're parts of whole numbers," Dawn was unable to draw any certain conclusions about how these three ideas were related. After saying "I feel like certain fractions and certain decimals will fit under rational numbers and certain ones won't," she changed her approach. The resulting concept map is pictured in Figure 4.8b and is described below:

> Dawn: I was thinking of [the rational numbers] as a larger group…but maybe not, maybe we have like branches or something…we could say, any fractions will split to either, R or I…I don't know…but the same number…whether it's the decimal representation or the fraction representation it's going to be the same, it's gonna either be rational or irrational.
> Interviewer: So given a number…no matter how you represent it,

> it's. . . not a different number?
> Dawn: No no no, I don't know–I'm saying that right now, but then I'm like, I've never really thought about that.

Dawn's reaction to the repeating decimal characterization of rational number was one of surprise, as she initially thought that any number with an infinite decimal expansion was irrational. To her, terminating and repeating decimals were fundamentally different:

> Dawn: This is just part of a definition. . . a number is rational if it can be written as a repeating decimal? Well it could be, it doesn't have to be, right? So to me this is saying. . . it has to be written as a repeating decimal to be rational. . . to me that's saying if and only if it can be written as a repeating decimal, is rational.
> Interviewer: Oh, you're saying there are lots of other numbers that wouldn't be repeating decimals that are still rational. . . like 1/4 equals 0.25.
> Dawn: Yeah, or one, or two.

2.8. Dillon

Dillon said that "the easiest" way to define a fraction was "a whole number over a whole number. . . because then you can say eight thirds." Dillon elaborated, saying that "a fraction is a relationship of two numbers" and offered as an example three fourths, saying "We could say three is three fourths as big as four, four is four thirds as big as three." Dillon defined decimal as "powers of ten," but only after saying that decimals were "so related to percents. . . I almost can't separate the two because. . . of the way I've taught them." He taught his students that "percent means out of a hundred. . . and that's the connection between decimals and percents. . . and to the, to the hundredths place, boom that's all."

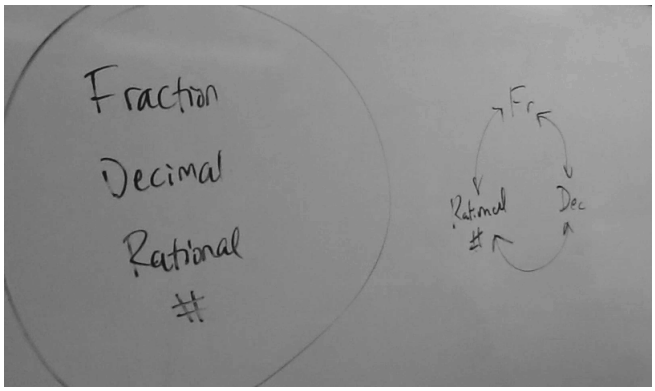


Figure 4.9. Dillon's concept map.

It is interesting to note that Dillon's definition of fraction was, up to sign, equivalent to the one offered in CCSS (2010); i.e., a rational number is "a number expressible in the form a/b or $-a/b$ for some fraction a/b. Despite these similarities, however, Dillon defined rational number in terms of decimal representations:

> Dillon: The only way I describe rational as opposed to irrational... with my sixth graders is, π, you know, it's irrational... about as close as we're gonna talk about it is really not π, but we say ... three and a seventh and then I have 'em divide one by seven, you see that repeating pattern... so I tell em rational numbers are basically fractions that end up repeating their pattern or they don't repeat, like three fourths stops at point seven five.

Without any prompting, Dillon offered his own clarifying statement: "You could find it on a number line." When I repeated his statement, Dillon added to it by saying "and irrationals don't." Dillon said more about this during the observation phase, which is discussed in Section 4.4.

Dillon revealed more about how he defines rational number as he completed the concept mapping task. His concept map is pictured in Figure 4.9 and is described below.

> Dillon: I really have to say, all rational numbers, can be, represented in decimal form and in fraction form... so I'd put 'em all in

> the same…set…we can even do it this way…we could put fraction, decimal, and rational number…okay?…They're multiple representations…of the same concept…of the same value.

Dillon's reaction to the statement about the decimal form of rational numbers was unqualified acceptance, which is not surprising, given his working definition of rational number. Dillon used previous knowledge to classify $\sqrt{2}$, π, and $\frac{\pi}{4}$ as irrational. He easily identified $\frac{5\frac{1}{4}}{47\frac{1}{2}}$, $\frac{1.2}{5.74}$, and $\frac{\sqrt{4}}{3}$ as rational by observing that each expression can be written as a ratio of whole numbers. Dillon also handled 2.33 easily, saying "I can put zeros after that." Despite recognizing $0.\overline{3}$ as a repeating decimal and hence rational, Dillon was unsure whether or not $0.\overline{3}$ could be placed on the number line. He drew the diagram pictured in Figure 4.10 and described it as follows:

> Dillon: All right. If this is zero, and this is, one…there's point five…one, two, three, four, so then we gotta get to point, three, but then we're gonna get to point three [between 0.3 and 0.4], okay, so…one two three four…
> Interviewer: So you're doing three tenths, and then three hundredths?
> Dillon: Yeah, and then we'll just keep going and, man, we're not gonna get away from….3333333 forever.
> Interviewer: So you're saying that we can't draw [$0.\overline{3}$] on the number line?
> Dillon: Well this is a really good question…I've been told if the decimal repeats, it's rational so, I'm taking everybody's word for it…But now that I'm trying to…it can't go [to the right of 0.4] and it can't go [to the left of 0.3], so it's gotta be [brings his hands together in a V over the number line].

Dillon also demonstrated some hesitation about classifying zero as rational, as his definition of rational number as "a whole number over a whole number" was in question because he didn't know if zero is a whole number. This uncertainty

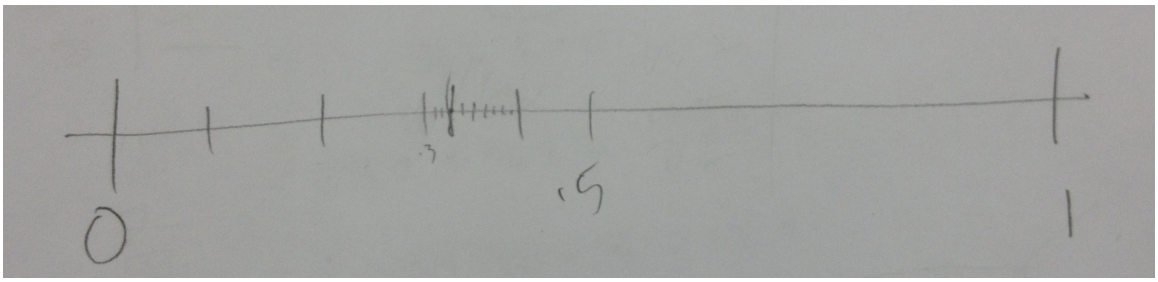


Figure 4.10. Dillon's diagram of $0.\overline{3}$.

prevented him from applying his definition. It is interesting to notice that Dillon was able to classify 2.33 as rational by appending zeros but he did not consider applying this strategy to classify 0 as rational.

2.9. Emily

Emily said that she defined fraction as "a part to a whole, so it's essentially two whole numbers that are, relating to each other…The top is a whole number, the bottom is a whole number" This was reiterated when Emily defined decimal:

> Emily: Decimal is just two numbers that have been divided…and then you've actually gone through di–it's no longer a fraction, 'cause essentially two numbers that are dividing each other are fractions, one number that's dividing another is a fraction, so a decimal is just the result of that fraction.

In the context of defining fractions and decimals, Emily drew a sharp distinction between the two, suggesting that one representation is transformed into the other. This perspective was context dependent, however, as Emily adopted a very different perspective when describing a typical lesson in her classroom:

> Emily: I've posed problems such as, like, negative three plus, a negative four is equal to negative seven. However, what if I put, what if I do, three, negative three tenths plus negative four tenths, what is that going to equal? Once it gets to a decimal, instantly, (imitating a student)

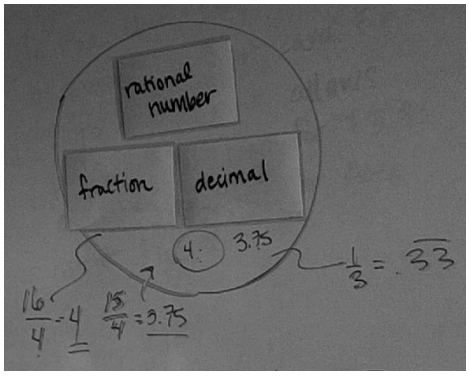


Figure 4.11. Emily's concept map.

> oh my gosh, my brain is gonna explode.
> Interviewer: So [the students] see fraction and decimal representations of number–those are different, somehow, than integers.
> Emily: Yeah, exactly. And so developing that concept, that–oh my goodness, these are actually almost the same thing, the only difference is place value.

Emily's definition of rational number was similar to her definition of fraction but emphasized the process of division. This idea first emerged when I asked her directly about rational number:

> Emily: So when we were talking about rational numbers, I feel like rational numbers are essentially a whole number divided, or it's in some sort–it's in some form of division…Or you're looking at something that could just be a repeating decimal that goes on forever like one over three. Rational number is essentially all of those things combined (laughs) It's essentially, a number in some state of division.

It was not clear what "all of those things" included until Emily completed the concept mapping task. Her concept map is pictured in Figure 4.11.

> Emily: It all depends on, what the fraction is…if you're looking at sixteen over four…[it] ends up being a whole number and sure, it's a rational number but it's, a more special rational number because, you do, fifteen over four, you're now looking at, three point seven five…It's just

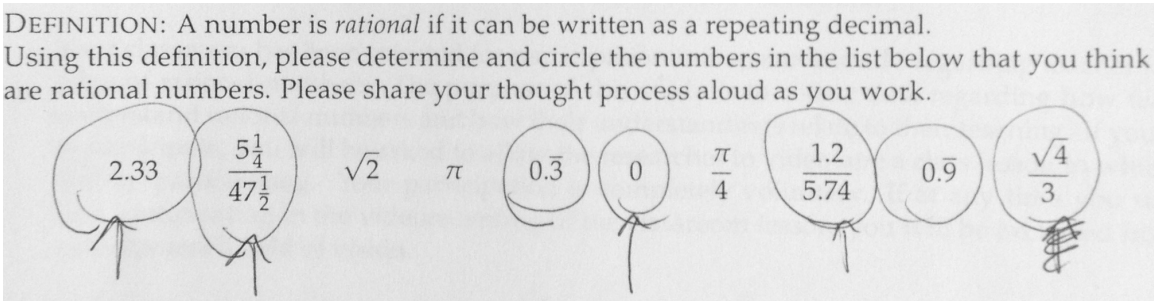
> division of numbers, some of 'em just end up, being almost a perfect division versus a less perfect division, or you're looking at one third, which is an even less perfect division 'cause you're looking at a repeating decimal.

Emily's[1] initial reaction to the interpretation task was that "I think it's incredibly vague. . . but I think it could work, I don't see why not." Emily's response to the classification task is pictured in Figure 4.12 and is pictured to offer further insight into exactly what Emily finds vague about the statement that a rational number is one that can be written as a repeating decimal. As is clear from the figure, Emily correctly classified each number listed. The numbers that are circled are the ones that Emily identified as rational and the numbers that have an arrow pointing at them are the numbers that Emily identified as *terminating decimals*:

> Emily: All of these [numbers circled in Figure 4.12] would be repeating if you wanted to. . . You could always add zeros for as long as you want, so technically it's repeating. . . I mean [2.33] is a terminating decimal. . . I should be specific, repeating decimal being beyond–like, numbers that repeat, that are different than zero?

Emily said that 2.33 was only "technically" a repeating decimal, unlike $0.\overline{3}$, which had "the same number repeated over and over." While Emily acknowledged multiple times during the interview that terminating decimals could be interpreted as repeating decimals, it was clear from the expressions on her face that she viewed this interpretation with skepticism.

[1]Emily had less than five minutes to complete this portion of the interview because of multiple interruptions during our conversation. Her responses may have been different had she not been faced with a group of students coming in.


DEFINITION: A number is *rational* if it can be written as a repeating decimal.
Using this definition, please determine and circle the numbers in the list below that you think are rational numbers. Please share your thought process aloud as you work.

2.33 $\frac{5\frac{1}{4}}{47\frac{1}{2}}$ $\sqrt{2}$ π $0.\overline{3}$ 0 $\frac{\pi}{4}$ $\frac{1.2}{5.74}$ $0.\overline{9}$ $\frac{\sqrt{4}}{3}$

Figure 4.12. Emily's results from the classification task.

2.10. Francine

For Francine, the underlying concept connecting her understanding of fractions and decimals was division. "When I start fractions…I have students stand up front, and I will say, four out of five had jeans on…so that [the students] understand you know, the numerator and the denominator." Francine describes decimals as "easier to teach" than fractions because "it's just an easier concept of, you know, let's divide the denominator into the numerator." This exchange suggests that Francine conceptualizes decimal as the result of division. Decimal is the conceptual lens through which Francine understood rational number, as indicated by the following exchange:

> Interviewer: So what about, I don't know if you use this word in your classroom, rational number?
> Francine: Yes. And I always have to be, reminded–rational numbers are nonrepeating…correct? Irrational repeat…the decimal goes on and on and on, it's infinite, it doesn't stop, and rational numbers, they do stop.

Francine characterized rational numbers as those whose decimal expansions "stop" and irrational numbers as those whose decimal expansions "go on and on [they are] infinite." Moreover, she appeared to see these categories as mutually exclusive. This hypothesis was confirmed by Francine's concept map, which is pictured in Figure 4.13:

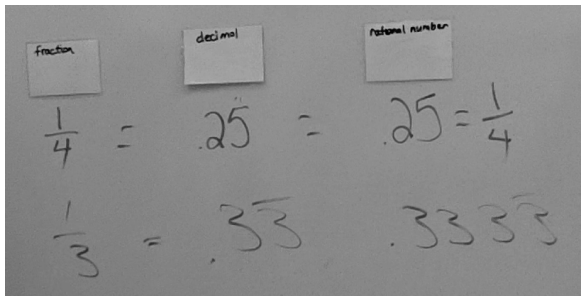


Figure 4.13. Francine's concept map.

> Francine: Well, if I have one fourth... of something, I also have (writes .25 beneath the decimal card) one fourth of something... which my rational number, would still be it's a rational number because [.25] is not repeating... I'm not quite sure how I would relate it along, except I would say that [.25] is a rational number... and they're all equal.
> Interviewer: So... if you have a fraction, you can write it as a decimal... and both those are rational numbers, is that what you would say?
> Francine: Well [1/4 and 0.25] are, one third wouldn't be a rational number... because one third, is (writes $\frac{1}{3} = .3\overline{3}$), it goes on and on and on and on... which is what the line means... but for every fraction there's a decimal, and there's either a rational or an irrational number that goes with it.

Francine's interpretation of rational numbers as those with terminating decimal expansions was true in the sense that every terminating decimal corresponds to a rational number. Taken in combination with her (correct) belief that rational and irrational numbers are mutually exclusive sets, Francine's understanding of rational numbers could easily be completed through a discussion of the fact that every rational number must have an eventually repeating decimal representation. As she completed the classification task, however, Francine interpreted repeating decimals as rational and any other type of decimal as irrational:

> Francine: Just looking at the fourth and the half [in $\frac{5\frac{1}{4}}{47\frac{1}{2}}$], I know that those are not repeating decimals... so those would be irrational... The

> line over [the three in $0.\overline{3}$] which means [$0.\overline{3}$] is a repeating decimal, that would be rational... I would say zero would have to be irrational, that it wouldn't repeat, 'cause there's nothing to repeat... and so, [$0.\overline{9}$], I would say is repeating because of the line over the top... And [$\frac{\sqrt{4}}{3}$] being, two thirds, would be repeating because it's, thirds... so that would be rational.

3. Results from the Interviews Across Participants

Results from the cross-case analysis begin with a description of categories that resulted from coding transcripts and data reduction. A summary of categories and labels can be found in Table F.1. Responses to the concept mapping task were analyzed separately, with the results of that data analysis presented in Section 3.7. The chapter concludes with a discussion of what the analysis reveals about participant understanding of rational numbers and their representation.

3.1. Definition Task: Fraction

Participant responses to the first definition task were categorized as fraction as part of a whole, fraction as the same as a decimal, fraction as a number between two whole numbers, fraction as a division problem, and fraction as a whole number over a whole number. Each of these categories represents a participant's understanding of what a fraction is and will be explained in greater detail below, with excerpts from interview transcripts given to situate the category in the context of the interview.

Eight of the ten participants described a fraction as part of a whole. This means that, when the participant was asked how they would define fraction for their students, they described a fraction in those terms. Dawn's responses to the fraction task perfectly demonstrate what it means to understand fraction in this way, as

she consistently referred to fractions in this way throughout the interview. When asked how she would define fraction to a student, she said "Well I guess initially we'd say it's part of a whole." After emphasizing the special case of ratios in which "you could be a part compared to a part," Dawn asserted that "I guess not initially... any time you're gonna see a fraction it's gonna be a part of a whole...." Claire also referred to fractions as part of a whole consistently throughout the interview. In addition to indicating that "fraction, of course, is a part out of a whole," Claire emphasized that "the whole can be different things, the whole can be one whole piece of paper, but the whole could be three coffee cups...." This flexible understanding of the whole was characteristic of those who understood fractions as part of a whole.

Two of the ten participants understood fraction to be the same as a decimal. This means that, at some point in their description of how they would define fraction to their students, they said that they view the two (fraction and decimal) as equivalent in some sense. To Betsy, fraction and decimal were "the same thing, both are based on place value... what's interesting is if you're looking at the tail end of a decimal, the way you say it is the same as what a fraction looks like." Bill connected fraction to decimal as well, but in a slightly different way, saying that a decimal "is a rule that we use so that we're all talking the same language, to express pieces, portions, parts... I might show them, one way we talk about a part of things is we can say there's two parts, and this is one of the parts."

Two of the ten participants understood fraction to mean a whole number over a whole number. People who understood fraction this way exhibited two overarching perspectives on fraction. One was the ability to represent a number as one whole number "over" another, where "over" indicates division. The other was a perspective on fraction as a relationship between two numbers. Dillon's de-

scription of fraction as "a relationship of two numbers" exemplifies this category. Emily also understood a fraction to refer to a whole number over a whole number. In her words, a fraction is "essentially two whole numbers that are relating to each other. . . they're whole numbers, the top, is a whole number, the bottom is a whole number."

In addition to describing a fraction as a part of a whole, Charlie also exhibited two other understandings of fraction. After saying that he would start by explaining to students that "a fraction usually represents a part of something bigger," Charlie described a fraction as a number that " might represent that something bigger and, part of another." This represents an understanding of fraction as a number between two whole numbers. Charlie also mentioned that "especially in fifth and sixth grade, I would remind [the students] that a fraction is also a division problem." This response showed that Charlie understood fraction as an instruction to divide the numerator by the denominator.

3.2. Definition Task: Decimal

Participant responses to the second definition task were categorized as decimal as numbers between whole numbers, decimal as the result of division, decimal as the same as fraction, decimal as related to place values, and decimal as connected to percents. Each of these categories represents a way of understanding the word decimal and will be explained in greater detail below, with excerpts from interview transcripts given to situate the understanding in the context in which it emerged.

Six of the ten participants understood decimal to refer to numbers between numbers. This means that a participant described decimal numbers as a way of describing numbers that were greater than one whole number but less than the next whole number. Betsy exemplified this category in her response to the definitions

task, saying that decimals were "a way to look at a part or a whole with a part." The use of the word whole was characteristic of this understanding of decimal as numbers between numbers, which suggested that these participants thought of decimals as a way of "filling the gaps" between whole numbers. Charlie was quite explicit about this, saying that a decimal "can be thought of as a number that's between two other numbers, two whole numbers, if you think of two whole numbers, we need a way to account for what's between those, and so that's what a decimal number is, something between two whole numbers."

Charlie's definition of decimal suggests that he saw both fractions and decimals in a similar way. In fact, when asked how they define decimal, three of ten participants understood that decimals are the same as fractions. Betsy was the most explicit about this similarity, saying that "just like a fraction is a way to represent a number, a decimal can be a way that you represent a number that's either a part of a whole or a whole and a part."

Three participants understood decimal as a result of a division. This means that when asked how he or she defined decimal, these participants described it as the decimal expansion resulting from dividing one number by another. Francine described this process exactly when asked how she defined decimal for her students, saying that "decimals are easier to teach. . . maybe it's just an easier concept of, you know, let's divide the denominator into the numerator." Emily also understood decimal to be the result of division, saying "Decimal is just two numbers that have been divided."

Dillon was the only participant who connected decimal to percent in his definition. Before even offering a definition of decimal, Dillon exclaimed "to me decimal is so related to percents. . . I almost can't separate the two." He explained that since percent meant out of one hundred, whenever he saw a percentage "like

three percent, I tell the kids that's three hundredths, because percent means out of a hundred. . . and that's the connection between decimals and percents, I always drill that. . . over and over." This shows that Dillon understood percents as a special type of decimal.

Four of ten participants understood decimal as referring to place value. This understanding was shown by responding to the definition of decimal task by making reference to place value structures. The relationship between decimals and place value varied among participants. Adam said that a decimal was "a way of sorting place values, showing, the difference between the ones and the tenths place." Dillon's perspective on decimals was the exemplar case of this understanding. Dillon said "decimal has to do with place value of powers of ten, and as we start with one as the unit, then if we multiply that by ten we get ten, and by ten we got a hundred. . . and then if I divide one by ten, and I divide it by a hundred, and a thousand, I move the other way around that decimal."

3.3. Definition Task: Rational Number

Participant responses to the rational number definition task were categorized as not irrational and rational number as extrinsic to middle school. These categories represent two different ways of understanding rational number, and excerpts from interview transcripts are given to demonstrate the context in which these understandings emerged.

Six of the ten participants understood rational number as an overset. This means that they defined rational number in a way that included two or more different types of numbers. Amy's definition of rational number perfectly characterized this understanding, saying that numbers can be classified "according to different properties, and so we have whole numbers, we have fractions, we have decimals,

we have positives, we have negatives, we have all these different things, so rational number is just providing an overarching classification." A participant did not have to include all rational numbers in their grouping to hold this understanding of rational number, as Dawn's definition demonstrated. She said "all whole numbers are rational numbers... all integers are rational numbers... decimals that terminate, or whole numbers... are rational numbers."

Betsy and Dillon understood rational number in a way that is related to the notion of overset but with an important distinction. For example, Betsy understood a rational number to be "a number that, you would, that can be placed on the number line." For Betsy, the vehicle for grouping seemingly disparate types of numbers is the number line. Dillon defined rational numbers as "fractions that end up repeating their pattern or they don't repeat, like three fourths stops at point seven five" and added, as if to clarify, "you could find [a rational number] on a number line."

Eight of the ten participants' definitions of rational numbers were categorized as not irrational. This category represents a way of understanding rational numbers as what they are not. Adam did exactly this when he said "irrational numbers have that non-repeating pattern that continues... you can't predict what's coming next, whereas a rational number you could, as far as the decimal places go, you could predict the end...." The veracity of a participant's definition was not considered in classifying it as not irrational, as can be seen by Francine's definition. Her definition was "rational numbers are nonrepeating... correct? Irrational repeat, I mean... the decimal goes on and on and on it's, infinite... it doesn't stop... with rational numbers, they do stop." A common theme among participants who understood rational numbers as not irrational numbers was to characterize rational or irrational numbers in terms of decimal representations. When asked how he

would define rational number, Charlie said "I know that π is an irrational number…it's a decimal that goes on forever without repeating, and that's an irrational number."

An unexpected perspective on rational number emerged from the not irrational category; namely, five of the eight participants in this category did not view the distinction between rational and irrational numbers as important. Adam exemplifies this perspective when he says "I did rational and irrational numbers for a brief time, and…I find it to be something that I'm not sure makes a big difference to kids at this level." When I asked him to elaborate, he said "I don't think that they encounter…and work with irrational numbers to the point that they would really need to know, the use of one over the other…I'm not even sure if I would know what that would be, but, I would think that knowing the difference of them isn't gonna affect their learning." While he did not offer any normative judgements about the value of rational number instruction, Dillon viewed the distinction as peripheral to the curriculum. He says "the only way [he] describes rational as opposed to irrational" is by juxtaposing a well-known irrational number (π) with a fraction ($\frac{22}{7}$) that is commonly used for the purposes of computation. Charlie implied that this knowledge was not necessary for his work as a teacher when he said "to distinguish between [an irrational] and a rational number, I'm not–I couldn't do it."

3.4. Interpretation Task

Participant responses to the interpretation task were categorized as incomplete or as willing to accept. These categories represent two different interpretations of the statement "A number is rational if it can be written as a repeating decimal." The first interpretation was to reject the statement as false because of a

belief that terminating and repeating decimals are fundamentally different. Amy's interpretation was prototypical of this interpretation: "I think that's a partial definition... that would be one type of rational number, but you could have a rational number with a decimal that terminates." Adam consults his textbook, which said a rational number is "a number that can be written as a fraction in the form a over b, where a and b are integers, and b cannot be equal to zero." He said "the repeating thing seems to eliminate some of what they have here... so a could be 1 and b could be 2... then that would be five tenths and, that in my eyes is not a repeating decimal."

The other five participants interpreted the statement about the decimal form of a rational number as correct, with some important variations. Bill and Dillon both accepted the statement as true because their definition of rational number agreed with with the statement. Charlie and Claire accepted the statement as true because neither had a definition of rational number in their active repertoire of knowledge. Francine's interpretation was one of surprise, as the statement contradicted her definition of rational numbers as terminating decimals and irrational numbers as infinite decimals. Francine decided to revise her definition of rational number so that it agreed with the statement given in the interpretation task.

3.5. Classification Task

Participant responses to the classification task are displayed in Table 4.2. Participant responses to the classification task were categorized as nontransparent, add zeros, flexible, and representation versus classification. Each of these categories represents a different interpretation of what a repeating decimal is and will be explained in greater detail below, with excerpts from interview transcripts given to situate the understanding in the context in which it emerged.

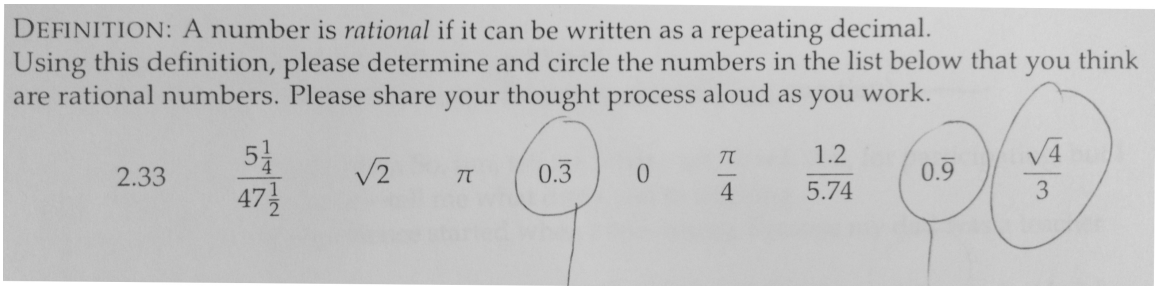
Table 4.2. Classification Task Results

	2.33	$\frac{5\frac{1}{4}}{47\frac{1}{2}}$	$\sqrt{2}$	π	$0.\overline{3}$	0	$\frac{\pi}{4}$	$\frac{1.2}{5.74}$	$0.\overline{9}$	$\frac{\sqrt{4}}{3}$	% Correct
Amy	R	DOR	DOR	I	R	R	DOR	DOR	R	R	60
Adam	R	I	I	I	R	R	I	I	R	R	80
Bill	R	R	I	I	R	R	I	R	R	R	100
Betsy	R	R	I	R	R	NR	R	R	R	I	60
Charlie	R	NR	I	I	R	R	I	I	R	R	70
Claire	I	I	I	I	R	I	I	I	R	R	60
Dawn	I	I	I	I	R	NR	NR	NR	NR	R	30
Dillon	R	R	I	I	R	NR	I	R	R	R	90
Emily	R	R	I	I	R	R	I	R	R	R	100
Francine	R	I	I	R	R	I	R	I	R	R	50

Note. R refers to rational, I refers to irrational, DOR refers to depends on representation, and NR refers to no response.

Five of ten participants responses to the classification task were categorized as nontransparent. These participants understood repeating decimal as referring to an easily discernible pattern that would always be identifiable by using a hand held calculator. Charlie, who defined rational numbers as not irrational, exemplified this understanding of repeating decimal in his classification of $\frac{1.2}{5.74}$ as irrational. The decimal representation of this number is repeating; however, it has period thirty, which is not made transparent by the nine digit display on the hand held calculator Charlie used (Zazkis & Sirotic, 2010). Charlie explained his choice to classify this number as irrational by saying "because this one looks to me like it's not repeating, it, it doesn't appear to be a pattern to me."

Dawn, who also defined rational numbers as not irrational, reached a similar conclusion regarding $\frac{5\frac{1}{4}}{47\frac{1}{2}}$. She (incorrectly) simplified this complex fraction, obtaining $\frac{1}{95}$ and, using a hand held calculator, obtained 0.010526315. She concluded


DEFINITION: A number is *rational* if it can be written as a repeating decimal.
Using this definition, please determine and circle the numbers in the list below that you think are rational numbers. Please share your thought process aloud as you work.

2.33 $\quad \frac{5\frac{1}{4}}{47\frac{1}{2}}$ $\quad \sqrt{2}$ $\quad \pi$ $\quad 0.\overline{3}$ $\quad 0$ $\quad \frac{\pi}{4}$ $\quad \frac{1.2}{5.74}$ $\quad 0.\overline{9}$ $\quad \frac{\sqrt{4}}{3}$

Figure 4.14. Claire's responses to the classification task.

that "because of the five, I'm–I mean, I don't know, I'm just gonna guess that it probably terminates" and since there is no visible repeating pattern, concluded that this number is irrational. Using the ratio of integers definition he described in Section 2.2, Adam offered another perspective on what it means to classify a number incorrectly based on a nontransparent representation. Since neither $5\frac{1}{4}$ nor $47\frac{1}{2}$ are integers, Adam concluded that $\frac{5\frac{1}{4}}{47\frac{1}{2}}$ is irrational. Adam immediately classified $\frac{1.2}{5.74}$ as irrational because it did not match the definition given in his textbook (i.e., rational numbers are ratios of integers) but he decided to "double check" by using a hand held calculator. He said "because I see a–decimal, two zero nine zero five nine two three three there doesn't seem to be any repeating pattern here." To the response, "But you thought that even before you put it in the calculator," Adam replied, "yeah, I'm thinking of the, integers."

Claire, who defined rational numbers as not irrational, revealed that, to her, only decimals of the form $0.\overline{a}$ were repeating decimals. This claim was based on her responses to the classification task, pictured in Figure 4.14. Claire classified $0.\overline{3}$, $0.\overline{9}$ as rational because they were explicitly presented as repeating decimals. Claire used a calculator to determine $\frac{\sqrt{4}}{3} = 0.\overline{6}$ and hence classified it as rational as well. The remaining seven numbers were represented in a way that was opaque with respect to their status as repeating decimals. Claire did not indicate that changing

their representation would allow her to apply the statement about the decimal form of a rational number. Francine, who defined rational numbers as terminating decimals and irrational numbers as infinite decimals, also interpreted repeating decimals exclusively as decimals of the form $0.\overline{a}$. This narrow understanding of what repeating encompasses led Francine to classify zero as irrational, saying "I would say zero would have to be irrational, that it wouldn't repeat, 'cause there's nothing to... repeat."

Two participants' responses to the classification task were categorized as add zeros. This category represents an understanding of repeating decimals as those that have an easily discernible pattern or as decimals that repeat in zero. These participants both appended a string of zeros after a postulated "last decimal place" to classify terminating and nonterminating decimals as rational. Betsy, whose definition of rational number was classified as overset, correctly identified 2.33 as rational by writing it as $2.33\overline{0}$. This approach failed when Betsy applied it to π, saying "and then the π, three point one, one point blah blah blah put a zero on the end, by that definition, I'd say yes [π is rational]." Betsy used the same approach to erroneously classify $\frac{\pi}{4}$ as rational. Emily, who defined rational numbers as a result of division, used language similar to Betsy in her approach to the classification task, saying "all of these would be, repeating if you wanted to... you could always add zeros for as long as you want, so technically it's repeating."

Two participants' responses to the classification task were categorized flexible. This category represents an internalized understanding of rational number that was connected to other mathematical understandings, including the fact that every rational number must have an eventually repeating decimal representation. Participants who saw rational number this way classified numbers from the classification task using whatever understanding of rational number was appropriate

for the context. Bill connected rational to the root word "ratio" and said "at a seventh grade level, you know I would talk about being able to express a rational number in a fractional form as, one of the kind of definitions of a rational number." He moved quickly through the classification task, stating with confidence whether each number was rational or irrational. Bill puzzled over $\frac{\pi}{4}$, saying "I would say that . . . if I take a, a not rational number and divide it by a rational number. . . I don't know how to get a final answer for this one." While $\frac{\pi}{4}$ is in a fractional form, Bill knew that π is not rational. The repeating decimal characterization failed him as well, because he could not envision dividing an unending decimal by four.

Dillon also had a internalized definition of rational number, saying that he defined rational numbers as "fractions that end up repeating their pattern or they don't repeat, like three fourths stops at point seven five." Recall that Dillon defined fraction as a whole number over a whole number explicitly because it included improper fractions. Like Bill, Dillon used this definition and the statement about the decimal form of a rational number interchangeably. For example, Dillon classified 2.33 as rational because "I can put zeros after that. . . so I, I would say that that's rational." He dispatched quickly with both $\frac{5\frac{1}{4}}{47\frac{1}{2}}$ and $\frac{1.2}{5.74}$ because both could be written as whole numbers over whole numbers. Despite his ability to use definitions flexibly, Dillon remained undecided about zero because he was unsure whether or not zero was a whole number and hence was unsure if he could form a ratio of whole numbers whose quotient was zero. While Dillon immediately identified $0.\overline{3}$ as a repeating decimal and hence rational, he seemed to have some conflict with his other definition of rational number as one that can be found on a number line. Dillon revealed through a sketch of the unit interval (see 4.10) that he did not believe it was possible to plot $0.\overline{3}$ on the number line. He encapsulated his conflict by saying "I've been told if the decimal repeats, it's rational, so I'm taking

everyone's word for it…but now that I tried to [place it on the number line] but it's gotta–it, it can't go beyond [0.33] and it can't go beyond [0.34].

Amy's response to the classification task was categorized representation versus classification. This category represents an understanding of rational number as being a feature of the number's representation. This understanding first manifested when Amy input $\frac{5\frac{1}{4}}{47\frac{1}{2}}$ into a hand held calculator. Her conclusion was "irrational if you turn it into a decimal." She reached the same conclusion about $\frac{1.2}{5.74}$, and then questioned the irrrationality of both $\sqrt{2}$ and $\frac{\pi}{4}$. The source of her confusion can be traced back to Amy's definitions of fraction, rational number, and irrational number. When completing the concept map task, Amy said "not all decimals are rational numbers, because if it's a nonrepeating decimal that never terminates, it is not considered rational, and now that I think about fractions, I think all fractions are classified as rational numbers, but when you convert them into decimals they would no longer be considered rational." Ergo, when Amy was confronted with a "fraction" that has a nonrepeating, nonterminating decimal representation, she was forced to conclude that the number was rational in fraction form but "becomes irrational" in decimal form. After clarifying her thinking on this matter, Amy qualified her statement, saying "it makes sense that [$\frac{5\frac{1}{4}}{47\frac{1}{2}}$, $\frac{1.2}{5.74}$, $\sqrt{2}$, $\frac{\pi}{4}$] would be considered irrational [in both fraction and decimal form] because how can a number go between rational and irrational…but if my definition is based on decimals, then they could be rational, so I feel like I need to refine my definition."

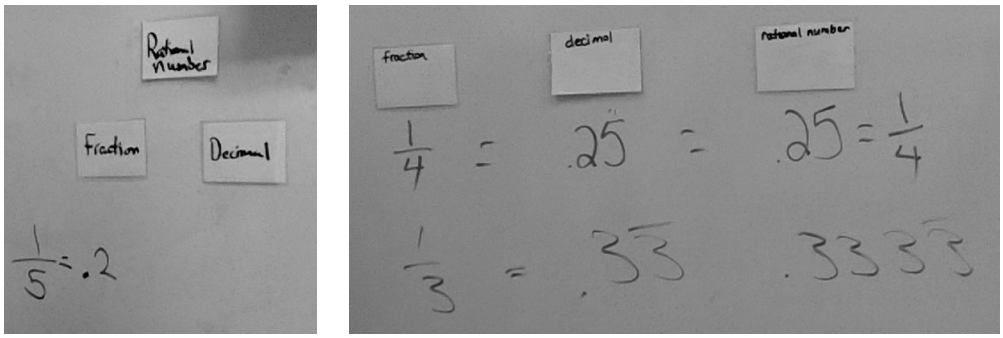
3.6. Terminating Versus Repeating Decimals

CCSS (2010, p. 87) defines repeating decimal as "the decimal form of a rational number." Under this definition, terminating decimals are considered a special case of repeating decimals where the repeating digit is zero. When asked to interpret

the statement "A number is rational if it can be written as a repeating decimal," many participants interpreted the statement in a less inclusive way.

Adam initially classified zero as a repeating decimal but after consulting his mathematics textbook (which defines rational numbers as ratios of integers), Adam concluded that the two definitions "are completely different because…if I was an eighth grader I would say…one over two, then that would be five tenths and that in my eyes is not a repeating decimal." Dawn rejected the statement about the decimal form of a rational number on similar grounds, describing it as "just part of a definition." She initially agreed that the rational number 1 can be represented as a repeating decimal by appending zeros, but, like Adam, after some consideration says of the repeating decimal characterization that "this is just part of a definition…to me this is saying…it has to be written as a repeating decimal to be rational…that's not true."

Emily classified decimals as either terminating or repeating. When asked if 2.33 is "repeating because you can add zeros to it," Emily reluctantly agreed, saying "yes, technically" but maintained that 2.33 was different from a nontrivial repeating decimal like $0.\overline{3}$ because, in that case, "that's gonna be the same number repeating over and over." Betsy initially rejected the repeating decimal characterization, saying "does that mean that if it can't be written as a repeating decimal that it's not rational?" After some thought, Betsy convinced herself that, "if you interpret it loosely," 1/10 could be considered repeating by appending zeros, but she maintained her skepticism, saying "to me a repeating decimal is…not a place holder…I'd do something like…the traditional…three three three."



(a) Adam's concept map. *(b)* Francine's concept map.

Figure 4.15. Adam and Francine's concept maps.

3.7. Concept Mapping Task

Participant responses for the concept mapping task varied widely. Participants who had similar responses across the other tasks will be identified and compared to better understand their responses.

Adam and Francine were very similar in their responses across the other tasks. Both defined fractions as part of a whole, decimals as a result of division, and rational numbers as not irrational numbers. Adam and Francine both struggled to classify numbers as rational when the given representation was not transparent with respect to rationality. Adam and Francine's concept maps are pictured side by side in Figure 4.15.

Despite having very similar responses to the other tasks, Adam and Francine had very different ideas about how fraction, decimal, and rational number were related as evidenced by the concept mapping task. Adam began the concept map by placing all three cards in a vertical column, starting with fraction and ending with rational number. He thought for several seconds, then rearranged the cards to appear as they do in Figure 4.15a. At this point he said "My first thought is that with rational number, you could produce a rational number with a fraction or fraction is a rational number…while decimals would be involved in rational

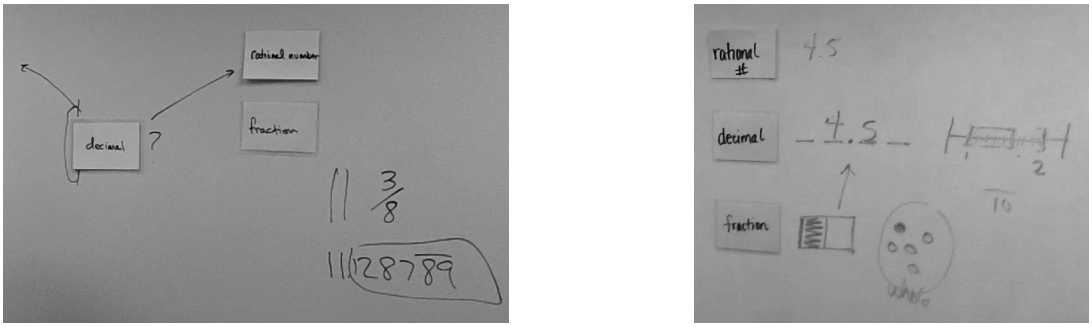
numbers, and, helping to define those rational numbers, as well as when computing fractions, finding out the number from a fraction, you would be able to use decimals to help define that." Adam confirmed that "finding the number from a fraction" refered to "dividing it out using [long division]." His concept map and his language implied that Adam believed all fractions are rational numbers and that decimals were indirectly related to rational numbers because they "help define" fractions.

Francine began the concept mapping task by placing the cards in one row, beginning with fraction and ending with rational number. Almost immediately she said,

> Okay. Well, if I have, one fourth [writes 1/4 under fraction] of a frac– one fourth of something, I also have [writes .25 under decimal] one fourth of something... which my rational number, would still be... I'm not quite sure how I would relate [rational number] along, except I would say that [.25] is a rational number. I guess my question is... it's still point two five, if I would, write it as point two five as a rational number, but, both [.25 and 1/4] are rational numbers... and they're all equal.

Taken together, Francine's uncertainty about how to "relate it along" and whether or not she would "write it as point two five as a rational number" imply that she saw fraction, decimal, and rational number as different ways of representing numbers.

Charlie and Claire also had very similar responses across tasks. Both saw fractions primarily as part of a whole, both described decimal explicitly as numbers between numbers, both interpreted rational numbers as not irrational numbers, and both indicated that they didn't discuss rational number much in their classrooms.



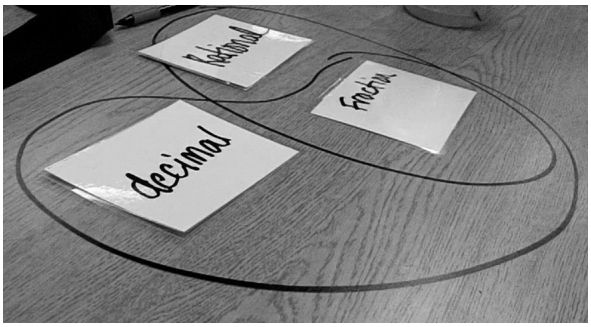
(a) Charlie's concept map. *(b)* Claire's concept map.

Figure 4.16. Charlie and Claire's concept maps.

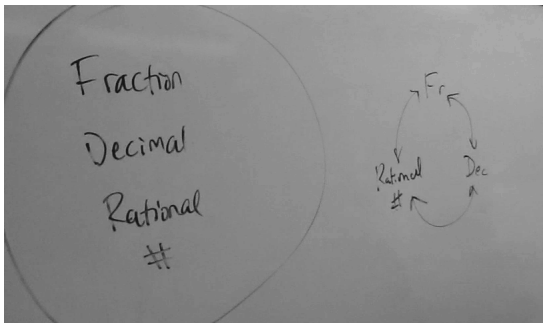
Their respective concept maps, however, reveal dramatically different perspectives on rational number and are pictured side by side in Figure 4.16.

Charlie arranged the cards to form the vertices of an equilateral triangle with rational number at the top. He said that his "initial thinking is that decimals and fractions in my mind, were rational. But, I know that there are some decimal numbers that are irrational. . . so what I was gonna do is say that [decimal and fraction] both belong [under rational number]." Charlie seemed to believe that there was a relationship between all three, saying "That's interesting, how to relate that, in some way that includes all of them." He then put rational number under fraction and decimal off to the left, saying that "when you express some of the division that's going on as fractions, you can call that a precise number. . . somehow that seems more rational to me than a decimal that goes on forever and keeps changing." By this, Charlie meant that, from his point of view, a terminating decimal was more concrete and hence, more "rational." Charlie concluded by saying "a lot of [decimals] are rational, in my opinion, the way I'm thinking about it, and then there are a few that are. . . irrational."

Claire arranged the cards in a column, with rational number at the top and fraction at the bottom. Claire described fraction as a part of a whole, where the



(a) Bill's concept map.



(b) Dillon's concept map.

Figure 4.17. Bill and Dillon's concept maps.

whole can either be continuous or discrete. She then described a strategy she used previously with her fifth grade students to understand decimals by connecting them to fractions. About rational numbers, Claire says "I think of more as, to me, those actually become numbers. It kind of goes from that [fraction diagram] to... something more abstract [the place value chart] where it's a representation of something, without a picture... and [rational number] then, is just the actual number." To Claire, the terms fraction, decimal, and rational number were related, but the relationship was based on moving from more to less concrete representations of quanitity. The first level (fraction) was "a picture, a visual"; the second level (decimal) is "something more abstract, where it's a representation, of something... with numbers" and the apex was rational number, which, to Claire, is "just the actual number."

While Bill and Dillon had somewhat different perspectives on fraction and decimal, both participants had flexible definitions of rational number. They also both accepted the repeating decimal characterization readily and completed the classification task with relative ease. A comparison of their concept maps reveals some interesting differences; their concept maps are pictured side by side in Figure 4.17. Bill began by placing the three cards to form the vertices of an isosceles triangle

with rational number at the top. After some thought, he abruptly moved fraction directly below rational number, asserting that "fractions are rational numbers." He then placed decimal immediately below fraction and then quickly moved it to the left, saying "some decimals are rational numbers." He then drew a circle around rational number and fraction, saying "these live together." After noting that "fractions can be written as decimals," Bill drew a kidney shaped region connecting decimal and fraction to emphasize the different character of that relationship. After puzzling over "how to graphically represent" that relationship, Bill decided to "put it in the context of…if I were explaining it to a seventh grader. He quickly turned to the root word of rational, ratio:

> Bill: We do speak specifically about how a ratio is a fraction, a fraction is a ratio…here at seventh grade, or rates could be ratios…you know, this number (3/2) doesn't mean anything…until I put words in it…is this three miles per hour, now it's a rate…is it, you know, is it three halves of pizza…So the same thing could be a rate, miles per hour…it's still a fraction, but they don't–but when I talk about it as a rate it might confuse them…but then sometimes it starts to click, 'oh it's just a fraction with different units.'

Dillon began his concept map by asking rhetorically "can all fractions be represented in decimal form…are they the same thing?" Noting that "1.0 is a decimal" and "4 over 4 is a fraction," Dillon concluded that fractions and decimals are the same, drawing an equals sign between the two words. Regarding rational numbers, Dillon said "I really have to say, all rational numbers, can be represented in decimal form and a fraction form." At this point, Dillon draws a circle around all three words, saying "I'd put them all in the same…set, yeah." I ask Dillon if he means that all pairwise equalities between the three hold, he agrees, and offers a secondary diagram to clarify. Both these diagrams can be seen in Figure 4.17.

Like Bill, Dillon believed that every fraction is a rational number; moreover, Dillon made the true statement that every rational number can be represented in fraction and decimal notation. Unlike Bill, however, he did not consider the converse; e.g, whether or not every number expressed as a fraction or as a decimal is rational.

4. Results from the Observations

This section presents the results of analysis of data collected from observations of four different participants (Amy, Dawn, Emily, and Dillon). I will provide an overview of the lesson structure, content, and perceived objectives. I will then enumerate the understandings manifested during instruction that also emerged during that participant's interview, followed by excerpts from the lesson to put those understandings in context.

4.1. Amy

My observation of Amy was unique because of the structure of her classroom. Amy employed a "flipped" classroom model, the specifics of which she and I discussed during her interview. Students were assigned problems that they worked on during class at their own pace. Instead of Amy providing instruction to the entire class, students watched instructional videos recorded by Amy beforehand that were aligned to the specific learning objectives students were working to master. Amy's primary role during class time was, as she described it, "providing mini lessons to small groups of students…providing independent or individual instruction…checking in with each student to make sure they're on track." After attempting to videorecord one class session of this type I abandoned the approach for two reasons. One was that I felt that following Amy around the classroom with

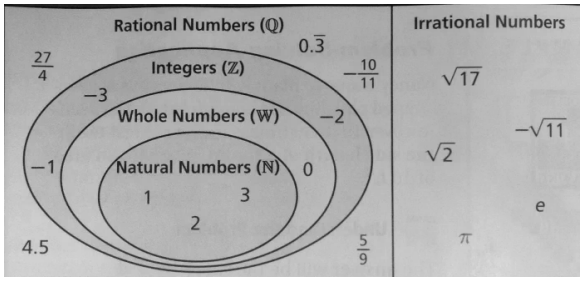


Figure 4.18. Diagram of the real number system used in Amy's instruction.

a video camera would be a distraction from her work and the second was that not all students had parent permission to be videorecorded. Instead, I analyzed two instructional videos recorded by Amy. These videos provide instruction regarding two learning objectives: defining rational numbers and expressing fractions in decimal notation. This analysis will focus on the first video, the goal of which was, in Amy's words, "talking about rational numbers, it's gonna cover goal number one, I can define a rational number."

Amy's instructional video ran for ten minutes and twenty one seconds, and it began with her describing "the different classifications that exist for numbers" as depicted in the diagram pictured in Figure 4.18. After discussing the structure of the diagram, Amy defined rational number and then used the rest of the time working through several examples of "deciding what classification a number belongs to."

After stating the purpose of the video, the diagram pictured in Figure 4.18 appeared on the screen. Amy introduced the diagram by saying "this chart helps us see the different classifications that exist for numbers, and how our number system has developed, through time." Natural numbers were introduced as "simply the numbers we count with," while the whole numbers were described as those that "start with zero, and continue with all the natural numbers." The integers "are

where we have our negative numbers added to the number system." At this point Amy introduced a strategy for classifying numbers:

> Amy: So when you are deciding where a number is placed, you start in the middle and determine is it a natural number, yes or no? Then you think, is it a whole number, yes or no? Is it an integer, yes or no? And then is it a rational number, yes or no?

Amy then said "now we need to define rational numbers. And when we look at rational numbers, they are comprised of natural numbers, whole numbers, integers, those are all rational numbers." This description of rational numbers was consistent with how Amy defined rational numbers during the interview; namely, rational numbers as a collection of one or more different types of numbers.

Amy's interpretation of the repeating decimal characterization of rational number as incomplete also manifested during instruction, as indicated by the following excerpt:

> Amy: But we have one more aspect we need to look at, to decide about rational numbers, and that is decimals... 4.5 is what is called a terminating decimal... this decimal ends, and so that makes it a rational number. Another example of a rational number with a decimal would be $0.\overline{3}$ with this bar over it. This bar tells us it is repeating... if it's a repeating decimal, that also means that it is rational. The final type could be a repeating decimal with a pattern, so you can see [$0.\overline{125}$] goes one two five, one two five, one two five. Since that pattern repeats, it is also a rational number.

Like during the interview, Amy described rational number as a collection of different representations. She also taught her class that different representations of rational numbers are different types of rational numbers. Also noteworthy is the fact that the repeating patterns described in the video are not representative of all possible repeating decimals (e.g., $\frac{1}{6} = 0.1\overline{6}$).

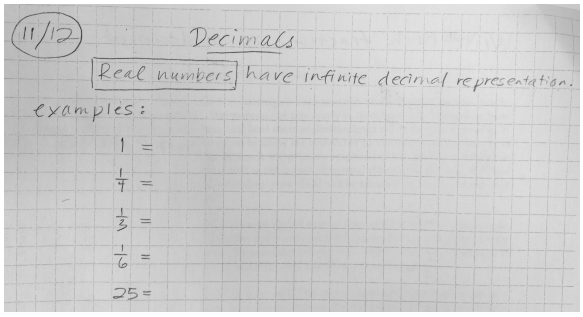


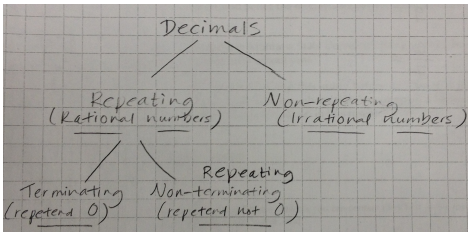
Figure 4.19. Dawn's definition of real numbers.

4.2. Dawn

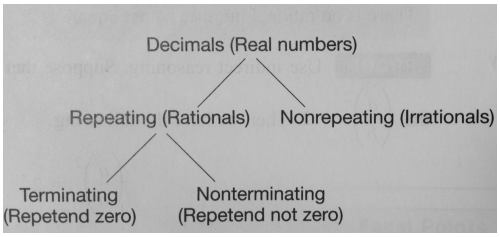
Dawn's lesson on rational number spanned three days. Most of the first day was spent completing instruction from a previous class meeting; only about ten minutes of instruction concerned rational numbers. The second day was not video-recorded as the instruction was led by a practicum student who had been assigned to Dawn's classroom. The third day devoted a full class period to instruction on rational number concepts. Both the first and third day were structured similarly: Dawn combined direct instruction with individual and group work, as well as student presentation of solutions to problems on the third day.

The first day of rational number instruction began with Dawn using a document camera to project the image pictured in Figure 4.19 on the whiteboard. Dawn used these examples to explain what is meant by the statement "Real numbers have infinite decimal representation." For example, Dawn says "One, can be represented with infinite decimal representation, how?", to which a student responded "One point oh repeating." Dawn continued the discussion, soliciting ideas from the students as to how each of the numbers pictured in Figure 4.19 could be represented with an infinite decimal.

After some discussion of the different ways in which the five numbers pictured in Figure 4.19 could be categorized, a student said about the decimal representa-



(a) Dawn's flowchart.



(b) Flowchart from Musser et al. (2002).

Figure 4.20. Two different flowcharts.

tions of one third and one sixth that "they repeat, but not in zero." At this point, Dawn introduces the flowchart pictured in Figure 4.20a. Dawn asked the students to copy this flowchart into their notebooks, saying "there's two different kinds, they can either repeat, repeating or non-repeating." After a student said that repeating decimals are rational, Dawn agreed, and continues, "if they don't they're called irrational…there's nothing else over on the branch of irrational numbers. And that's not that they're not important, it's just that we're not going there just yet. So what we are gonna look at, are the repeating numbers, the rational numbers." In this exchange, Dawn presented rational numbers in terms of their decimal representation, which is consistent with what she said during the interview.

As the students copied down the flowchart, a student began the following exchange. At this time, it is useful to compare Dawn's flowchart (refer to Figure 4.20b) with the original flowchart, which Dawn indicated she found in a textbook she used when earning her teacher license (Musser et al., 2002, p. 374):

> Student: So, for under repeating that splits into two, does it split into terminating–
> Dawn: Yeah, okay. So then under here, yeah, underneath rational it splits again. Now I added the word–I just said terminating and non-terminating, but then I realized this was confusing so I added the word repeating.
> Student: So should we write repeating on–both?

> Dawn: Yeah, you should. It'll help you. Nope, just on the one on the right.

When Dawn said she "realized this was confusing," she was referring to the previous class period when she taught the same lesson to a different group of students. Since I was not present for that lesson, I can not say with certainty what was confusing about the words "terminating" and "nonterminating."

Another understanding of rational number described by Dawn during the interview was decimals as numbers between whole numbers. This understanding appeared several times during the third day of rational number instruction, which began with two long division problems ($6249 \div 23$ and $48231 \div 41$) written on the board with the instructions, "Compute the quotients using long division." Both problems were written in a manner suggesting the use of the standard long division algorithm taught in US schools. Despite the instructions to use long division, many students were apparently using a different procedure. After observing this, Dawn made a remark that exemplified her understanding of decimals as numbers between numbers:

> Dawn: We need a strategy, to divide numbers, besides a calculator...My intent with this is to when we start dividing numbers that don't have a whole number answer, and it has decimals that we have to go into...How to then turn it into decimal places, and how we can tie that into these concepts of irrational and rational numbers, okay?

Dawn did tie long division into concepts of irrational and rational numbers, while again manifesting her understanding of decimals as numbers between numbers. This all occured in the context of Dawn explaining the long division algorithm to her class, beginning with generating a decimal representation of $\frac{1}{4}$. After the class confirmed that four does not divide one, Dawn asks the class what to do.

A student said to "put a decimal point after the one and bring the zero down to the one on the bottom now." Dawn said "Perfect. So decimal here because we have to go under whole numbers, obviously." At least one student learned to view decimals in this way from Dawn's work on computing the quotient $6249 \div 23$. After determining the hundreds, tens, and ones digit of the quotient, the same student as before observed that "now we're to the decimals." Dawn asks the student what should happen next and he repeated, "We go to decimals."

After generating four decimal places of the quotient $6249 \div 23$, Dawn said "we can go on for a very long time." After finding a mistake in one of the long division steps, she added "Now my tendency was to think well it's not repeating and it's not terminating but, by our definition we really should be a rational number, why?" She went on to remind students of the definition offered by the practicum teacher the previous day; i.e., a number is rational if it can be written as a ratio of integers:

> Dawn: I want us to know how to... divide numbers, we will do more coming up, how to take it beyond the decimal place, and then to determine rational/irrational. Technically, we should not be able to generate an irrational number by dividing two whole numbers, based on the definition from yesterday. What was the definition from yesterday?
> Student: All rational numbers equal p over q, but q cannot be zero.
> Dawn: Good. So it says all rational numbers should be derived by dividing two integers, right?

While Dawn correctly stated that all rational numbers are quotients of integers, she missed an opportunity to apply the contrapositive of this statement when students argue that $\frac{\sqrt{3}}{9}$ is rational as follows:

> Student: So, our fraction was, the square root of three divided by nine, and we got, for the square root, was one point seven three two, and so

> then we did, nine goes into one zero times, so we can't do that, and then nine goes into seventeen, once...
> Dawn: Nice job on the long division!
> Student: Yeah, okay yeah, we got eight, and then we carried down the three, eighty three, and then nine went into eighty three nine times–
> Dawn: Good, and I like the way you showed it multiple times, yeah, it hit four, and then every time it's forty minus thirty six is four, forty minus thirty six is four... it's never gonna change, good!
> Student: Okay, and then we figured out that it was rational and nonterminating.
> Dawn: Rational, nonterminating, do you guys agree? (students murmur yes) Rational, nonterminating, nicely done!

Here Dawn demonstrated an incomplete grasp of the distinction between an irrational number $\sqrt{3}$ and a rational approximation thereof (1.732). As a result, when the student performed the division indicated by $\frac{1.732}{9}$ and concluded that $\frac{\sqrt{3}}{9}$ is rational, Dawn did not see the mistake and the misconception remained unaddressed.

4.3. Emily

Emily's lesson spanned three days and employed a structure very similar to the typical lesson she described during the interview; i.e., a combination of exploration and direct instruction. Day one began with board work; specifically, the task was to "provide an example of a terminating decimal [and] an example of a repeating decimal." (The concept had been introduced near the end of the previous class period.)

After discussing the board work as a class, Emily drew the class' attention to a table on the white board, which is pictured in Figure 4.21. This table had two columns: the first column was headed "fractions that convert to repeating deci-

Figure 4.21. Emily's table.

mals" and the second column was headed "fractions that convert to terminating decimals." Students duplicated these tables on large sheets of paper before Emily introduced the purpose of the lesson; namely, "we're solely trying to figure out today, what causes, a fraction to turn into a terminating decimal versus a repeating decimal." After soliciting some hypotheses from students, Emily distributed sheets of paper with different fractions printed on them. Students were directed to cut these fractions out and place them in the appropriate column on their table. The rest of the class period (and the other two periods) had students creating and refining conjectures about what fractions have terminating versus repeating decimal representations. Emily stepped in to guide the conversation periodically; when she was not doing that, she worked with small groups of students.

During the interview, Emily described decimals as "two numbers that have been divided." Moreover, she indicated that "then you've actually gone through di–the–it's no longer a fraction, 'cause essentially…one number that's dividing another is a fraction, so a decimal is just the result of that fraction." This understanding of decimal as the result of division manifested no less than fourteen times during instruction. The following will interweave representative examples of these manifestation as well as describe the progress of the lesson to situate each manifestation in context.

The first instance occurred when Emily asked "what are some examples of repeating decimals?" After a pause, Emily probed the class, saying "maybe a fraction that turns into a repeating decimal? Anybody have a fraction that will turn into a repeating decimal?" At this point students offered such examples as $\frac{1}{3}$ and $\frac{1}{6}$. Emily used the words "turn into" and "convert to" interchangeably as she continued instruction, including when she described the purpose of the lesson:

> Emily: So we have fractions that convert to repeating decimals, fractions that convert to terminating decimals. We're solely trying to figure out today what causes a fraction, to turn into a terminating decimal versus a repeating decimal, that is what we are going to try to discover.

Emily asked the students if they "have any thoughts on the matter." A student offered a conjecture related to the parity of the denominator, which was quickly shown to be false because $1/6$ had a repeating decimal representation. At this point a student asked about $1/7$ and, after some disagreement about the output from hand held calculators, Emily suggested using long division to settle the question. After computing five decimal digits of the quotient $1/7$, Emily asked "why are we still doing this?" A student responds "because it's still not repeating decimal." Emily responded "because it's still not repeating. We wanna see if we can get to a point where it's going to repeat."

After two more iterations of the long division algorithm and obtaining another 1 in the quotient several students indicated that they believed the decimal expansion would repeat. Emily opted to continue the procedure, saying "we want to be positive it's going to repeat. Are we sure it's gonna follow this same pattern?" A student agrees, saying "we shouldn't be positive that it doesn't repeat. We should put another zero on the end and do it anyway." After obtaining four more deci-

mal digits, Emily and the class concluded that the decimal for 1/7 is a repeating decimal.

Day 2 was spent primarily on students developing and refining their conjectures as to "What numbers will turn something into a terminating decimals; what numbers will turn things into a repeating decimal when you're looking at fractions." Day 3 was primarily led by Emily in an effort to "completely figure out the exact reason as to why a fraction terminates or repeats, as in when converted to a decimal." The idea that decimals are a result of division manifested here again, when Emily said "I had a lot of people starting to say, well I don't think the numerator... makes any sort of difference, to if it's going to be terminating or repeating... when they turn into decimals."

Emily focused students on this idea by distributing a sheet with unit fractions whose denominators ranged from 1 to 20. After allowing students to explore the decimal representations of these, the class made several interesting observations, which were summarized by Emily:

> Emily: So I hear multiples of threes, equal repeating, right? I also hear multiples of two equal terminating but it sounds like there is a caveat to this...
> Student: Well, lots of those are multiples of prime numbers, aren't they?
> Emily: [Fourteen] is a multiple of a prime number? What prime number?
> Student: Seven.
> Emily: Seven, and two's technically a prime number, too, but which one is causing that to repeat... what about multiples of eleven... it's a prime number, so... is it going to repeat or terminate?
> Student: One divided by twenty two is repeating.
> Emily:... How about this? Eleven over twenty two, that should be repeating, right? It's a multiple of eleven on the bottom. What's wrong with this?

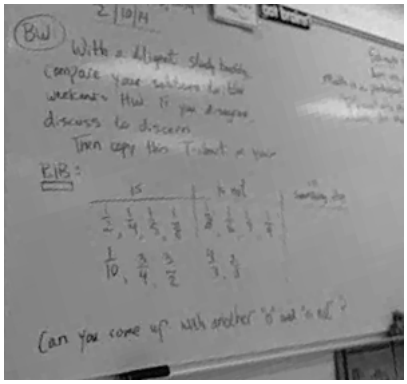


Figure 4.22. Dillon's boardwork.

> Student: It's terminating!
> Emily: Because…we're eliminating that eleven, and what does that eleven cause? Repetition!

Emily ended class by saying "we're going to refine this idea of what numbers cause repetition, what numbers cause termination of our, fractions, when they're converted to decimals." Emily's understanding of fractions "turning into" decimals suggests that she has reified fractions and decimals. As a result, the distinction between number and representations of number is blurred and students are exposed to an understanding of fractions and decimals as different "types" of numbers.

4.4. Dillon

Dillon's lesson proceeded in three phases that were described as launch, review, and learn. During the launch phase, students worked on the board work assignment while Dillon checks homework from the previous class day. The board work served to motivate the new idea or concept that was to be studied during the learn phase. The review phase consisted of Dillon working through solutions to the homework from the previous class day. The learn phase consisted of Dillon using the board work task to teach the new skill or concept he wished to intro-

duce. The instructional period is approximately fifty minutes long. Of those fifty minutes, approximately eleven minutes were spent on the launch phase, approximately twenty nine minutes were spent on the review phase, and the remainder (approximately nine minutes) were spent on the learn phase.

The board work task is pictured in Figure 4.22. BW (board work) is circled, and written next to it is "With a diligent study buddy compare your solutions to the weekend's HW. If you disagree, discuss to discern. Then copy this T-chart in your [Big Idea Book]:" The T-chart has three columns: one with the heading "is," a second with the heading "is not," and the third with the heading "something else." The "is" column has the fractions 1/2, 1/4, 1/5, 1/8, 1/10, 3/4, 3/2 ; the "is not" column has the fractions 1/3, 1/6, 1/7, 1/9, 2/3, 4/3. Below this is a question: Can you come up with another 'is' and 'is not?'

Dillon manifested some of the same understandings of rational number during instruction that he did during the interview. One of these understandings was a fraction as a relationship between two numbers. During the interview, Dillon described this relationship using the example three fourths, saying "three is three fourths as big as four [and] four is four thirds as big as three." During observation, while discussing how to compute the quotient $1.2 \div 4.8$, Dillon said "There's four twelves in forty–eight, and there's also one fourth of forty–eight in twelve." Another understanding of rational number Dillon described during the interview was about decimals and how he connected them to percents. In particular, Dillon indicated that he used percentages as a way to help students better understand numbers with many decimal digits. This perspective on decimals appeared several times during the lesson, especially when Dillon discussed repeating decimals:

> Dillon: Folks, the decimal for one seventh. . . one, four, two, eight, five, seven. . . to infinity. . . folks, if you ate one seventh of a pie, you ate about

> fourteen percent of the pie, right?...One ninth. What's the decimal for one ninth?...zero point one one one one one forever. It's about eleven percent.

During the interview, Dillon said that he told his students that "rational numbers are basically fractions that end up repeating their pattern or they don't repeat...." This perspective manifested itself in the following exchange during the observation:

> Dillon: What does that mean if a decimal is terminal?...It stops. What are these decimals if they don't stop? What's the opposite of terminating?...Repeating! Write that down, repeating. Now, write this word [irrational] down. Write this word down.
> Student: I know what irrational means.
> Dillon (referring to the decimal expansion of π): Okay, listen kids: there's no pattern, and it keeps going forever...that's irrational. Rational, irrational.

Of particular interest, however, was a statement Dillon made during a lesson that I did not videorecord. After the observation was complete, Dillon invited me to stay for the same instruction presented to a different class. Near the end of this lesson, Dillon was attempting to explain to his students the difference between rational and irrational numbers. To motivate this, he asked "What's the opposite of rational?" The desired response was irrational, which is true in the sense that $\mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}) = \emptyset$. As the class period was wrapping up, Dillon reiterates the point he just made by saying "You'll never find [irrational numbers] on the number line!" A student responded by asking "But why not? The number line is every number!" Dillon did not respond to this query, for reasons unknown. This exchange was atypical for Dillon, who consistently demonstrated an interest in student thinking during the interview and instruction. Why he chose not to engage this student in conversation is unclear.

5. Revealing Understanding

Several different understandings of rational number and representations of rational number emerged from the categories discussed in this chapter. This section describes what the research has revealed about how participants understand rational number.

In his seminal paper detailing the complexities of rational numbers, Kieren (1976, p. 102) argued that rational numbers were primarily viewed as "objects of computation." Participants in this study largely embraced this understanding of rational number, which led to many surprising responses to the tasks from the interview protocol. When asked how they defined rational number, participants interpreted the question procedurally, either as how they would teach their students the difference between rational and irrational numbers or as how to classify a given number as rational or irrational. As can be seen from the results of data analysis, attempting to distinguish between rational and irrational numbers or classifying a given number as rational or irrational without a solid conceptual understanding of these concepts is unlikely to be successful.

Participants viewed rational numbers as a collection of different types of numbers, a perspective that is both mathematically valid and pedagogically useful. However, participants in this study emphasized these differences at the expense of discussing similarities within the set of rational numbers. They consistently used differences within the set of rational numbers as a context for teaching concepts related to rational numbers. This was clear in the definitions offered of fraction, decimal, and rational number, as well as how they interpreted the statement that rational numbers can be written as repeating decimals. By focusing on different

ways in which they could partition the rational numbers, participants failed to present rational number as a unifying and organizing principle.

The definitions of rational number offered by participants in this study suggest that they do not have a deep conceptual understanding of what a rational number is. This is most clear in the near unanimity with which teachers defined rational numbers as not irrational numbers. This tautological statement is especially problematic because none of the participants in this study demonstrated a clear understanding of what an irrational number was. The other common definition of rational number as a particular type of decimal was also problematic. Participants interpreted repeating decimal in different ways; most common was that a repeating decimal is one that has an easily discernible pattern that can be seen in the display of a hand held calculator.

Participants responses to the interview tasks suggested a tenuous grasp on the difference between number and representations of number. This is most clear in Amy's belief that changing a number's representation could change the number's status as rational or irrational. Emily's repeated references to fractions "turning into decimals" leaves an impression that, at some level, Emily may view fractions and decimals as different kinds of numbers, rather than different representations of the same number.

One of the difficulties in learning mathematics stems from conflict between colloquial and mathematical meanings of words. For example, mathematicians use the conjunction 'or' in an inclusive sense, whereas 'or' is typically used in the exclusive sense in ordinary conversation. Participants in this study clearly understood the words 'terminating' and 'repeating' as referring to mutually exclusive categories, much as these words are typically used in a colloquial sense. While the distinction between terminating and repeating decimals is both mathemati-

cally and pedagogically valuable, there is also value in understanding repeating decimals as subsuming terminating decimals as a special case.

6. Summary

Participants described their understanding of rational numbers by completing several tasks in the context of a standardized open-ended interview. Their responses were categorized into several different categories and comparisons both between and within cases were made. From these categories several different ways of understanding rational numbers and their representation emerged. Teachers who agreed to be observed manifested many of the same understandings that they described during instruction. Several conclusions based on these findings can be drawn, as well as several recommendations for teacher educators and mathematics education researchers. These conclusions and recommendations are the subject of the next and final chapter of this report.

CHAPTER 5

DISCUSSION

1. Introduction

This chapter presents conclusions drawn from the results of the data analysis discussed in Chapter 4 and implications thereof. Following these conclusions is a list of recommendations for inservice teacher professional development as well as suggestions for areas of future research.

2. Conclusions

Examination of the ten teachers in this study has produced a number of conclusions that are described in this section. It is understood that specific results from this relatively narrow case study cannot be broadly generalized. However, many of the behaviors exhibited by teachers in this study are likely to be duplicated by other teachers. Conclusions reported here therefore have implications for developing a deeper conceptual understanding of rational number in other middle school teachers.

2.1. Definitions May Differ from Descriptions

In this study, participants' definitions of rational number contrasted sharply with their descriptions of rational number. Participants phrased their definitions of fraction, decimal, and rational number in terms of different representations; e.g., numbers that can be written as a fraction or as a terminating or repeating decimal. The majority of participants viewed rational numbers as a collection of different "types" of numbers that were mutually exclusive. Moreover, participant under-

standing of rational number varied considerably with context, as was seen by the analysis of the concept mapping task. Asking participants about the same concept from different perspectives yielded very different responses. This result suggests that these teachers have yet to transition to what David Tall (1992, p. 495) describes as advanced mathematical thinking. As one transitions "from a position where concepts have an intuitive basis founded on experience, to one where they are specified by formal definitions and their properties re-constructed through logical deductions" it is typical to experience "a wide variety of cognitive conflict" which was certainly the case in this study.

2.2. Inadequate Definitions of Rational Number

The majority of participants framed their definition of rational number using irrational numbers; i.e., a rational number is one that is "not irrational." While this is mathematically valid, it is also circular and assumes that the person employing it understands what an irrational number is. None of the participants demonstrated a firm of irrational numbers, and even if they did, defining rational numbers as not irrational numbers is inefficient. Participants who used this frame of reference also characterized rational and irrational numbers according to their decimal representations but did not demonstrate any awareness of *why* a rational number must have an eventually repeating decimal representation. Additionally, participants who could clearly articulate a definition of rational number interpreted the statement about the decimal form of a rational number correctly. These participants also demonstrated considerable flexibility when completing the classification task.

Those who drew sharp distinctions between terminating and repeating decimals demonstrated relatively less flexibility in their interpretations of the statement about rational numbers and their decimal form and were less successful

in their efforts to complete the concept mapping task. Without this conceptual connection, teachers struggled to classify numbers as rational or irrational and relied on colloquial understandings of repeating decimal such as decimals that are "predictable" or that have a "pattern." Additionally, two participants (Dillon and Emily) both demonstrated interesting understandings of repeating decimals. For Dillon, this manifested in his belief that $0.\overline{3}$ could not be placed on the number line because he could not complete the process of subdividing the number line. Emily saw both terminating and repeating decimals as a process that was the result of division; moreover, she defined rational numbers as a process, defining a rational number as "a number in some state of division." Like the participants described by Keynes et al. (2009), both these participants seemed to be locked into a process view of decimal numbers.

2.3. Rational Numbers as Extrinsic to Middle School

Some participants did not see the concept of rational number as part of their curriculum, which may explain why they seemed not to have a working definition of rational number in their active repertoire of knowledge. To answer questions about rational number, these participants relied on their intuition and colloquial meanings of words such as "rational," "terminating," and "repeating." These participants were not unwilling to engage in critical thought about their understanding; on the contrary, participants were uniformly interested in developing a deeper understanding of this content. This reveals a key feature of teacher disposition that should be encouraged at all levels of teacher training and development; i.e., a commitment to and passion for lifelong learning.

2.4. Colloquial Versus Mathematical Meaning of Words

Participants displayed a wide variety of interpretations of the statement "A number is rational if it can be written as a repeating decimal." Many participants drew a sharp distinction between terminating and repeating decimals, which led them to interpret this statement as incomplete because they saw it as excluding a whole class of rational numbers; i.e., those with terminating decimal representations. This interpretation may be due to relying on colloquial understandings of the words "terminating" and "repeating" and an unfamiliarity with the mathematical parsimony typically employed by mathematicians when defining concepts.

Several participants interpreted the word "rational" colloquially and described rational numbers as numbers that made sense, or, in the case of Charlie, as "logical" numbers. Interestingly, Charlie interpreted the word "irrational" colloquially as well, saying that a fraction like $11\frac{3}{8}$ was "more concrete" than a repeating decimal like $11.128\overline{789}$, which he described as "more on the irrational side because I can't really wrap my mind around all those digits." It may be that Charlie views rationality and irrationality as existing on a continuum, much as a person's actions can be viewed as more or less rational or irrational.

2.5. Different Understandings of the Word Repeating

Several participants understood "repeating decimal" as referring to one of two canonical forms; i.e., decimals of the form $0.\overline{a}$ or $0.\overline{a_1a_2a_3}$. This narrow view of what constitutes a repeating decimal caused participants great difficulty in classifying numbers as rational or irrational and provided further evidence that participants were largely unaware of *why* rational numbers must have eventually repeating

decimal representations. Knowing that, for example, a rational number can be interpreted as a division of integers and that when dividing an integer a by a non-zero integer b, there are at most $b - 1$ possible remainders may have alerted a participant to other possible configurations of repeating decimals. With a solid grasp of this result about the decimal representation of rational numbers, teachers would likely be in a stronger position to develop the rational number sense of their students.

2.6. Reliance on Calculators

In this study, a common strategy for completing the classification task was reliance on the string of decimal digits produced by a hand held calculator. When participants were unable to discern a pattern in the decimal string produced by the calculator, they concluded that the number they entered into the calculator was irrational. Many participants reached this conclusion regardless of the given representation. This result highlights the importance of familiarity with the ratio of integers definition of rational number in addition to logical consequences of that definition (e.g., that rational numbers must have eventually repeating decimal representations). Additionally consider that any decimal string given by a hand held calculator is finite and hence must represent a rational number. Students using rational approximations of irrational quantities who are unaware of the difference could develop misconceptions about irrational numbers, much like the students in Dawn's classroom who concluded that $\frac{\sqrt{3}}{9}$ was rational by using a rational approximation of $\sqrt{3}$ generated by a hand held calculator.

2.7. Points of Divergence from the Norm

Each participant began the interview by expressing clear mathematical understandings that conformed with established mathematical norms. However, each participant reached a point in the interview where his or her understandings "diverged" from the accepted standard in the mathematical community. There were three "points of divergence from the norm" for this group of participants. For Charlie, Dawn, and Francine, the point of divergence from the norm was the definition of rational number task. Amy, Adam, Claire, and Emily all diverged from the norm when asked to complete the concept mapping task. The remaining three participants (Bill, Betsy, and Dillon) continued to respond to the interview protocol questions clearly and articulately until reaching the classification task, at which point it was difficult to follow their explanations. These three participants are similar in at least one other way; i.e., all three had a well-articulated definition of rational number readily available and referred to it throughout the interview. This result suggests that the ability to clearly articulate a definition of rational number may be a useful proxy for measuring a person's understanding of rational numbers.

2.8. How Understanding Manifests During Instruction

Interestingly, the four observed teachers in this study demonstrated nearly identical understandings of rational number in the classroom as they exhibited during the interviews. Participants demonstrated some correct mathematical understandings, such as Dillon's understanding of a fraction as a relationship between two numbers and Amy's understanding of rational number as "an overar-

ching classification" for the numbers studied in the elementary and middle school curriculum. Mathematically incorrect understandings were also presented, such as Amy's belief that rational numbers belonging to $\mathbb{Q} \setminus \mathbb{Z}$ have exactly three configurations and Dillon's assertion that irrational numbers do not have locations on the real number line. This implies that, at least for these teachers, there is a level of consistency between what they say about their understanding and how they teach mathematics to their students. In turn, this implies that (with appropriate revisions and pilot testing as described in Section 3), future researchers can use the interview protocol from this study as a diagnostic tool without conducting observations.

3. Recommendations for Practice

Based on the results of this study, several recommendations can be made relevant to those involved in preservice and inservice teacher education in the area of rational numbers.

Professional development should be designed for middle school teachers that specifically attends to each of the "points of divergence from the norm" discussed in Section 2.7. Some specific recommendations based on the results of this study are presented below, with one recommendation for each of the aforementioned "points of divergence from the norm":

Professional development could be organized around "position-driven discussions." These discussions involve a teacher leading students in an examination of a particular question (called the framing question) that is substantial yet has a limited number of answers (O'Connor, 2001). Participants in this sort of discussion are directed to answer the question (i.e., take a position of their own choosing) and provide evidence in support of that position. The following three framing ques-

tions could be used as the basis for three related position-driven discussions about rational numbers.

1. What is the definition of the word fraction?

2. What is the definition of the word decimal?

3. What is the definition of a rational number?

Many participants struggled to make connections between fraction, decimal, and rational number. This was especially clear for Amy and Francine during the concept mapping task, during which both participants displayed a lack of understanding of the distinction between number and representations of number. To address this problem, professional developers could reproduce the diagram Amy used in her lesson on rational numbers (see Figure 4.18) on the floor of a classroom. The leaders of the professional development would assign each attendee a numeral and ask them to position themselves in the appropriate region. This concrete model could reinforce the similarities and differences between different representations of numbers.

The classification task was a struggle for all the participants; even those who correctly classified all the numbers demonstrated some uncertainty about their decisions. The observed difficulties were due largely to a lack of understanding of the equivalence of the ratio of integers definition of rational number and the logical consequence of that definition that every rational number must have an eventually repeating decimal representation. This is a rich mathematical topic and inservice teachers would benefit from a structured investigation of this topic. To contextualize the definition of a rational number as a ratio of integers, professional developers could refer to the discussion of rational numbers as an extension of the

integers conducted by Courant and Robbins (1996, pp. 52–53). To "reduce the problem of measuring to the problem of counting," one selects a unit of measurement, counts the number of units that "together make up the quantity to be measured," and in the case that the quantity "lies between two successive multiples of this unit... take a further step by introducing new sub-units, obtained by sub-dividing the original unit into a number n of equal parts." This subdivision is represented symbolically by the expression $\frac{1}{n}$, and the rational number $\frac{m}{n}$ represents m groups of $\frac{1}{n}$ units. To develop a deeper understanding of why every rational number must have an eventually repeating decimal representation, teachers could work through an investigation similar to the following:

1. Use a calculator to find a decimal representation of the fractions 21/45, 62/125, 63/90, 326/400, 39/60, and 54/300.

2. Find the prime factorization of the numerator and denominator of these fractions.

3. Express each of these fractions in simplest form, leaving the numerator and denominator in factored form.

4. Consider the denominators of the fractions whose decimal representations repeat with 0s. What (if anything) do they have in common? Repeat for the denominators of the fractions whose decimal representations repeat with nonzero digits.

5. Discuss with a colleague: What is the key idea that determines whether a fraction's decimal representation repeats with 0s or nonzero digits?

The three "points of divergence from the norm" discussed above represent three broad difficulties with rational numbers held by participants in this study:

inadequate knowledge of the definition of rational number, blurred distinctions between number and representations of numbers, and difficulty identifying rational numbers. Results from this study suggest that the latter two difficulties are a symptom of the first; i.e., a person who does not have a clear understanding of what a rational number is will likely confound number and representations of number, as well as struggle to identify a given number as rational.

None of the participants in this study demonstrated a clear understanding of what an irrational number is, yet the majority of them defined rational numbers as not irrational numbers. A more powerful way of relating rational numbers to irrational numbers is presented in the Grades 6-8 Learning Progressions for the Common Core State Standards in Mathematics (2013) (`http://commoncoretools.me/wp-content/uploads/2013/07/ccssm_progression_NS+Number_2013-07-09.pdf`). This document describes how, after understanding that rational numbers have infinite repeating decimal representations, students might consider the possibility of infinite decimals that do not repeat. These numbers should still have a location on the number line despite the fact that their decimal representations cannot be described fully. A study of these learning progressions could form the basis for effective professional development for inservice middle grades teachers regarding the concept of irrational number.

The interview protocol used in this study was designed to explore how teachers understand rational numbers and their representation. The interviewer deliberately avoided questioning the validity of teachers' understandings due to the goals of the study. However, if this interview protocol were used merely as a way of engaging teachers in a two-way conversation, it could form the basis of effective professional development. The following changes should be made before using this interview protocol in this way:

1. The questions about definitions should be rephrased to include more scaffolding. Rather than asking teachers the very broad question of how they define fraction (or decimal or rational number), questioning should begin with more narrow questions progressively becoming more broad as the interviewer develops a fuller understanding of the teachers' perspective.

2. The concept mapping activity employed the use of laminated index cards upon which participants wrote the words fraction, decimal and rational number to avoid imposing on participants the interviewer's views of how those ideas were related. An unexpected consequence of this decision was the imposition of a discrete, mutually exclusive structure that may have limited participants' responses. Other configurations should be considered; as a starting point, consider writing the words on a page with directions indicating that participants should construct a diagram that represents how they see the ideas as being related using a classroom whiteboard. Additionally, since so many participants viewed rational number in relation to irrational number, future implementations of this task should include the phrase "irrational number."

3. Presenting the statement "A number is rational if it can be written as a repeating decimal" to participants as a definition of rational number introduced confusion that was not necessarily related to conceptual understanding of mathematics. Some participants may have been interpreting "terminating" and "repeating" in a colloquial sense. From this perspective, it is quite reasonable to draw a sharp distinction between terminating and repeating decimals. The statement should be presented as a statement for interpretation, not as a definition to accept or reject. Related to this issue was the confusion

between colloquial and mathematical uses of words; with the most notable examples being "terminating" and "repeating." This is a common area of difficulty in learning mathematics (Vinner, 1991) and is worth addressing via appropriate professional development. For many of the participants in the present study, a simple conversation about the common practice of using the word repeating to include terminating as a special case would likely have been sufficient. For other audiences, a reasonable starting point would be a conversation about similar conventions in the English language, which has many words that are used to convey different ideas in different contexts (e.g., lie, fair, bass, etc.).

4. The classification task should be revised to include more examples of irrational numbers, including numbers that exhibit a "pattern" that cannot be generated by division of integers. Some examples would include 2.718281828... and 0.0100100010001....

Stylianides and Stylianides (2014) introduce the notion of pedagogy-related mathematics tasks as a vehicle to teach mathematical knowledge for teaching. Participant perspectives on rational number found in this study could be used to build a class of pedagogy-related mathematics tasks centered on the rational number construct.

4. Recommendations for Future Research

Corbin and Strauss (2008) describe a process known as *theoretical sampling* in which the researcher collects data regarding a particular phenomenon, analyzes the data, returns to the field to collect data based on what was learned from the previous round of analysis, and continues until reaching *saturation*, a point at which

no new understandings emerge from data analysis. A study that attempts to saturate all possible views of rational numbers by theoretically sampling from the population of middle grades teachers should be conducted and used as a basis for improving understanding of rational number among both students and teachers. The interview protocol used in the present study should undergo further piloting and refinement, incorporating the recommendations made in Section 2.

The present study focused on how teachers understood rational numbers and to what extent those understandings manifested during instruction. A similar descriptive case study should be conducted that describes how a sample of students at the middle and high school levels understand rational number.

The Number System standards detailed in CCSS (2010) seek to develop in students a conceptual understanding of number, beginning by attending to different types of numbers within the rational number system and progressing towards the ultimate goal of unifying all the different types of numbers studied in the elementary and middle school curriculum. A longitudinal study tracking how students' understanding of rational number develops as they progress through their K-12 instruction, guided by CCSS (2010), would be a welcome addition to the research literature.

5. Summary

Participants in this study (i.e., inservice middle grades teachers) understood rational numbers primarily as a collection of mutually exclusive sets of numbers. This focus on differences came at the expense of attending to what was similar about rational numbers, which was not surprising given that the last three generations of standards documents focused on facility with different representations

of rational numbers and given the well-documented focus in textbooks on computation and procedural understanding. The ways that teachers in this study understood rational numbers were reflected clearly and at numerous times during instruction. Both mathematically valid understandings and misconceptions manifested during instruction, which suggests that inservice teachers need ongoing professional development that attends to the conceptually rich and complex construct known as rational numbers.

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APPENDICES

APPENDIX A

INTERVIEW PROTOCOL

Inservice Teacher Interview Protocol

First, I would like to thank you for your willingness to participate. I really appreciate it!

I would like to know more about you as a teacher.

1. What drew you to the profession of teaching?
   (a) *Prompt*: When did you first develop interest?
   (b) *Prompt*: What people influenced your interest?
2. What grade levels have you taught?
   (a) *Prompt*: What content?
   (b) *Prompt*: What did you like? Dislike?
3. In your opinion, what characteristics does an effective teacher have?
   (a) *Prompt*: What do they do?
   (b) *Prompt*: What do they know?
   (c) *Prompt*: What does the classroom look like?

Tell me more about your current teaching situation

4. How many distinct courses are you teaching now? (e.g., how many preps?)
   (a) Do you use *Bits and Pieces III* _____ ...
      i. How do you structure that class?
      ii. What other resources do you use besides *Bits and Pieces III*?

Okay, thanks for answering those questions. What else do you feel like sharing? Now I would like to change gears a bit and focus on some ideas from mathematics as they relate to the course we were just talking about.

5. Suppose one of your students in this course asked you to define *number*. How would you respond?
6. Suppose a student asked you to define (fraction/decimal/rational number). How would you respond?
7. So you said a (fraction/decimal/rational number) is
   _____
   (a) How long has that been your definition?

(b) Where did you encounter that definition?

i. A class you took? When?

ii. A textbook? Which one?

Okay, great. Thanks for answering those questions. If you're willing, I would like to get a visual representation of how you think about fractions, decimals, and rational numbers.

8. I'm going to give you three pieces of laminated paper and a wet-erase marker. Please write the word "fraction" on one, "decimal" on another, and "rational number" on the third.

9. Please stick the cards on the whiteboard. Here's a dry-erase marker. Please use the marker and the cards to draw me a picture of how you see these three ideas and how they relate to each other. You can use words, arrows, pictures, or anything you think is appropriate. I would really like you to describe aloud your thought process as well.

Thank you for sharing with me how you think about those three concepts; it was really helpful. Now I would like your opinion about another definition.

10. Some people define *rational number* as those numbers that can be written as a repeating decimal

    (a) What is your opinion of this definition as a math teacher?

    (b) Pretend you were planning a lesson about rational numbers and you saw this definition in your textbook. How do you think you would react?

Thanks for sharing your opinion about that definition with me. Now I would like to ask you some questions about specific numbers.

11. Please circle all of the numbers in the following list that you think are rational, according to your understanding of this definition. Please share your thought process aloud as you work.

$$2.33 \qquad \frac{5\frac{1}{4}}{47\frac{1}{2}} \qquad \sqrt{2} \qquad \pi \qquad 0.\overline{3} \qquad 0 \qquad \frac{\pi}{4} \qquad \frac{1.2}{5.74} \qquad 0.\overline{9} \qquad \frac{\sqrt{4}}{3}$$

12. Can you write $\frac{1}{3}$ as a decimal? Why or why not?

13. Does $0.\overline{9} = 1$? Why or why not?

DEFINITION: A number is *rational* if it can be written as a repeating decimal. Using this definition, please determine and circle the numbers in the list below that you think are rational numbers. Please share your thought process aloud as you work.

2.33	$\frac{5\frac{1}{4}}{47\frac{1}{2}}$	$\sqrt{2}$	π	$0.\overline{3}$	0	$\frac{\pi}{4}$	$\frac{1.2}{5.74}$	$0.\overline{9}$	$\frac{\sqrt{4}}{3}$

APPENDIX B

INFORMED CONSENT–TEACHER INTERVIEW

INFORMED CONSENT FORM – TEACHER INTERVIEW

You are being asked to participate in a research study regarding your understanding of rational numbers. The purpose of this study is to collect data regarding how teachers understand rational numbers and how their understandings relate to their teaching. If you agree to participate, you will be asked to participate in a videotaped interview about your classroom and your teaching practice. The interview will last 30 – 45 minutes. Your participation is completely voluntary and you may stop participating at any time.

The content of the study should cause no more discomfort to you than you would experience in everyday life. All information and media related to you will be kept strictly confidential. Your name will not be associated in any way with the research findings. If any publications should result from this research, only pseudonyms will be used.

While this study is of no direct benefit to you, your participation could result in a better understanding of how teacher's knowledge impacts their students' learning. Your help with this research will help me understand better the interactions between teacher knowledge, instruction, and learning. Those who participate fully in the interview will receive a gift card to Old Chicago restaurant in appreciation.

The Department of Mathematical Sciences at Montana State University has granted permission for this research to occur during the 2013-2014 school year.

If you would like additional information concerning this study before or after it is finished, please feel free to contact Roger Fischer at the address, phone number, or email address below. Thank you for your consideration, time, and participation in this research project.

Roger Fischer
2-214 Wilson Hall, Montana State University
406.994.5362
`rfischer@math.montana.edu`

If you have additional questions about the rights of human subjects you can contact the Chair of the Institutional Review Board, Mark Quinn, (406) 994-4707 [`mquinn@montana.edu`].

. . . . . .

AUTHORIZATION: I have read the above and understand the discomforts, inconvenience, and risk of the study. I, _____ (*name of subject*), agree to participate in this research. I understand that I may later refuse to participate and that I may withdraw from the study at any time. I have received

a copy of this consent form for my own records.

Signed: _____ Date: _____
Email address: _____
Principal Investigator: Roger Fischer

APPENDIX C

INFORMED CONSENT–TEACHER OBSERVATION

Informed Consent Form – Teacher Observation

You are being asked to participate in a research study regarding your understanding of rational numbers. The purpose of this study is to collect data regarding how teachers understand rational numbers and how their understandings relate to their teaching. You have been contacted because you previously participated in a videotaped interview regarding your ideas about teaching in general and rational numbers in particular. If you agree to participate, you will be asked to allow the researcher to observe and videotape a regular class lesson that you teach regarding rational number concepts. Your participation is completely voluntary. Should you decide to discontinue participation, the videocamera will be turned off immediately and the video data file will be destroyed.

The content of the study should cause no more discomfort to you than you would experience in everyday life. All information and media related to you will be kept strictly confidential. Your name will not be associated in any way with the research findings. If any publications should result from this research, only pseudonyms will be used.

While this study is of no direct benefit to you, your participation could result in a better understanding of how teacher's knowledge impacts their students' learning. Your help with this research will help me understand better the interactions between teacher knowledge, instruction, and learning. Those who participate fully in the observation will receive a gift card to a local restaurant in appreciation.

The Department of Mathematical Sciences at Montana State University has granted permission for this research to occur during the 2013-2014 school year.

If you would like additional information concerning this study before or after it is finished, please feel free to contact Roger Fischer at the address, phone number, or email address below. Thank you for your consideration, time, and participation in this research project.

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. . . . . .

AUTHORIZATION: I have read the above and understand the discomforts, inconvenience, and risk of the study. I, _____ (*name of subject*), agree to participate in this research. I understand that I may later refuse to participate and that I may withdraw from the study at any time. I have received a copy of this consent form for my own records.

Signed: _____ Date: _____
Email address: _____
Principal Investigator: Roger Fischer

APPENDIX D

INFORMED CONSENT FORM – STUDENT

INFORMED CONSENT FORM – STUDENT

Your classroom has been selected for participation in a research study regarding teacher knowledge of rational numbers. The purpose of this study is to collect data regarding how teachers understand rational numbers and how their understandings relate to their teaching. If you agree to participate, you will be asked to allow the researcher to videotape a class lesson in which you will be participating. Your participation is completely voluntary. If at any time you wish to stop participating in the videorecording of the classroom lesson, you will be excluded from the videocamera's field of vision.

The content of the study should cause no more discomfort to you than you would experience in everyday life. All information and media related to you will be kept strictly confidential. Your name will not be associated in any way with the research findings. If any publications should result from this research, only pseudonyms will be used.

While this study is of no direct benefit to you, your participation could result in a better understanding of how teacher's knowledge impacts their students' learning. Your help with this research will help me understand better the interactions between teacher knowledge, instruction, and learning. Those who participate fully in the study will receive a gift card to a local restaurant in appreciation.

The Department of Mathematical Sciences at Montana State University has granted permission for this research to occur during the 2013-2014 school year.

If you would like additional information concerning this study before or after it is finished, please feel free to contact Roger Fischer at the address, phone number, or email address below. Thank you for your consideration, time, and participation in this research project.

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If you have additional questions about the rights of human subjects you can contact the Chair of the Institutional Review Board, Mark Quinn, (406) 994-4707 [`mquinn@montana.edu`].

. . . . . .

AUTHORIZATION: I have read the above and understand the discomforts, inconvenience, and risk of the study. I, _____ (*name of subject*), agree to participate in this research. I understand that I may later refuse

to participate and that I may withdraw from the study at any time. I have received a copy of this consent form for my own records.

Signed: _____ Date: _____
Email address: _____
Principal Investigator: Roger Fischer

APPENDIX E

INFORMED CONSENT–PARENTS

Informed Consent Form – Parents

Your child's classroom is being asked to participate in a research study regarding how teachers understand rational numbers. The purpose of this study is to collect data regarding how teachers understand rational numbers and how their understandings relate to their teaching. If you agree to participate, you will be asked to allow the researcher to videotape a class lesson in which your child is participating. Instruction is the primary emphasis of the interview and your child will not be asked to do anything that is not a part of normal classroom activity. If either you or your child objects to the videorecording, your child will be situated so that he or she is at no time in the camera's field of vision.

The content of the study should cause no more discomfort to your child than he or she would experience in everyday life. All information and media related to your child will be kept strictly confidential. Your child's name will not be associated in any way with the research findings. If any publications should result from this research, only pseudonyms will be used.

While this study is of no direct benefit to your child, his or her participation could result in a better understanding of how teachers' knowledge impacts their students' learning. Participation in this research will help me understand better the interactions between teacher knowledge, instruction, and learning. Those who participate fully in the study will receive a gift card to the MSU Bookstore in appreciation.

The Department of Mathematical Sciences at Montana State University has granted permission for this research to occur during the 2013-2014 school year.
If you would like additional information concerning this study before or after it is finished, please feel free to contact Roger Fischer at the address, phone number, or email address below. Thank you for your consideration, time, and participation in this research project.

Roger Fischer
2-214 Wilson Hall, Montana State University
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`rfischer@math.montana.edu`

If you have additional questions about the rights of human subjects you can contact the Chair of the Institutional Review Board, Mark Quinn, (406) 994-4707 [`mquinn@montana.edu`].

. . . . . .

AUTHORIZATION: I have read the above and understand the discomforts, inconvenience, and risk of the study. I, _____ (*name*

of parent/guardian), give permission for my child to participate in this research. I understand that I may later refuse to allow my child to participate and that I may withdraw my child from the study at any time. I have received a copy of this consent form for my own records.

Signed: _____ Date: _____
Email address: _____

Principal Investigator: Roger Fischer

APPENDIX F

CROSS CASE ANALYSIS SUMMARY

Table F.1. Cross Case Analysis Summary

Participant	Definitions Task			Interpretation Task	Classification Task
	Fraction	Decimal	Rational		
Amy	PW	SAF	OVERSET	INC	RVC
		PV	NI		
Adam	PW	PV	NI	INC	NT
		ROD	EXT		
		NBN			
Bill	PW	NBN	OVERSET	ACCEPT	FLEXIBLE
	SAD		NI		
Betsy	SAD	NBN	OVERSET	INC	ADD ZEROS
	SAF	SAF			
Charlie	PW	PW	NI	ACCEPT	NT
	NBN	NBN	EXT		
	DIV	SAF			
Claire	PW	PW	NI	LIMITED	NT
		NBN	EXT	ACCEPT	
		SAF			
Dawn	PW	NBN	OVERSET	INC	NT
			NI		
			EXT		
Dillon	W/W	RPV	OVERSET	ACCEPT	FLEXIBLE
		CTP	NI		
			EXT		
Emily	PW	ROD	ROD	INC	ADD ZEROS
			OVERSET		
Francine	PW	ROD	NI	LIMITED	NT
				ACCEPT	