

Huygens principle applied to the cylindrical antenna boundary value problem
by Orville Kenneth Nyhus

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY in Electrical Engineering
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Abstract:
The subject of this thesis is the application of Huygens' principle to the cylindrical antenna boundary value problem, The research described herein yields an explicit expression for the impedance of a cylindrical antenna.

The content of the thesis may be summarized as follows: First, the principles involved in applying the mathematical form of Huygens' principle to the cylindrical antenna boundary value problem are presented,, Second, a Green's function is derived which satisfies the conditions required by the mathematical form of Huygens5 principle, Two forms of Green'S function are obtained. Third, expressions for the current distri-bution and impedance of the cylindrical antenna are obtained, Fourth, numerical results are presented and compared with published measured data and with other theoretical results. The numerical results demonstrate that the impedance characteristics of a cylindrical antenna are correctly described through the application of Huygens' principle,

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ABSTRACT

The subject of this thesis is the application of Huygens' principle to the cylindrical antenna boundary value problem. The research described herein yields an explicit expression for the impedance of a cylindrical antenna.

The content of the thesis may be summarized as follows: First, the principles involved in applying the mathematical form of Huygens' principle to the cylindrical antenna boundary value problem are presented. Second, a Green's function is derived which satisfies the conditions required by the mathematical form of Huygens' principle. Two forms of Green's function are obtained. Third, expressions for the current distribution and impedance of the cylindrical antenna are obtained. Fourth, numerical results are presented and compared with published measured data and with other theoretical results. The numerical results demonstrate that the impedance characteristics of a cylindrical antenna are correctly described through the application of Huygens' principle.

CHAPTER 1

INTRODUCTION

1.1 DESCRIPTION OF THE PROBLEM

An extremely important characteristic of an antenna is its driving point impedance. The value of this impedance must be known before the design of input or terminating networks can be accomplished. It is desirable to be able to calculate this impedance based on the construction and physical dimensions of the antenna. In most cases, this is a very complex problem.

The particular antenna to be dealt with in this thesis is the cylindrical antenna, shown in Figure 1.1. The terminals may be considered to be at any point along the conductor. Probably the most common case considers the terminals to be at the center. This is often called a dipole antenna and it may appear in the form of a suspended wire, or rigid rod.

A vertical monopole above a ground plane has nearly the same characteristics as a dipole, if one assumes an ideal ground plane. The radiation pattern is the same as a dipole in the half-space above the ground plane. That is, the monopole and its image in the ground plane essentially form a dipole insofar as the radiation pattern is concerned. The impedance of a monopole is half the dipole impedance.

The simplicity of the dipole and the monopole, together with their effective performance in practice, give them great utility. Consequently, the cylindrical antenna has been the subject of much research. One of the most difficult and most often attacked problems has been the analytic

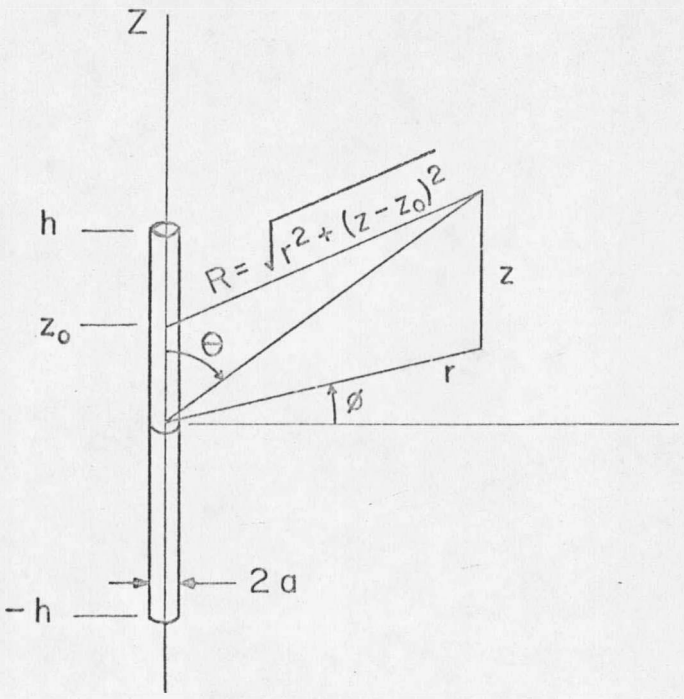


Fig. I.I Cylindrical antenna.

determination of the impedance based on the dimensions of the cylinder. This thesis describes further research into this problem.

1.2 HISTORICAL BACKGROUND

Work with antennas dates from the time of Hertz in the late 1880's. A review of progress in linear antenna analysis up to 1967 is given by Ronald W. P. King [35]. As King states, early antenna analysis was concerned with radiation properties at distances relatively far from the antenna. For this purpose the infinitesimal Hertzian dipole [26], [33], [35] was used as a convenient model. No attempt was made to determine the actual current distribution for finite linear antenna elements.

Solutions appeared as early as 1898 to boundary value problems for geometic shapes closely related to finite cylinders. These include prolate spheroids, or ellipsoids, [1], [45], [54], [58], [67], and the biconical antenna [33], [45], [58], [61]. The prolate spheroid approaches a line of finite length as its eccentricity approaches unity. Similarly, as the cone angle of the biconical antenna approaches zero, it becomes a finite line. The reason for choosing these geometric figures is that their surfaces may be represented in terms of a single variable in appropriate coordinate systems (Spheroidal coordinates for the spheroid and spherical coordinates for the biconical antenna are appropriate). This simplifies the analysis somewhat. However, these geometric figures are inadequate for representing actual cylinders, especially cylinders of relatively large radius compared to their length.

The late 1920's saw the beginning of analysis of finite linear

antenna elements with assumed current distributions. No attempt was made to analytically determine the actual current distribution, but a sinusoidal distribution of the form

$$I(z) = I(0) \sin k(h - |z|)$$

was assumed. As King points out [35], this was based partly on measured values, partly on its adequacy in special cases, and partly on the mistaken idea that a section of two-wire transmission line bent outward at the end to form a dipole has the same spatial waveform on the dipole as on the lossless tranmission line. More recent measurements show that the sinusoidal distribution is a good first approximation for antennas near on odd number of half wavelengths long and whose length to diameter ratio 2h/2a is large, but is considerably in error for antennas whose diameter is near the same order of magnitude as its length.

The assumed sinusoidal current distribution yielded much more accurate radiation patterns, especially for elements more than a half wavelength long, than were obtained with the infinitesimal dipole. It also opened the way for impedance calculations, particularly the EMF method [9], [12], [17], [33], [45]. This method yields accurate impedance values if the actual current distribution is known. With the assumed sinusoidal distribution, reasonable results are obtained for antenna lengths near an odd number of half wavelengths. However, for center-fed dipoles near an integral number of wavelengths long the sinusoidal function vanishes at the driving point. This would indicate that the impedance is infinite, which is not realistic. Therefore more accurate representation is needed for the actual current distribution

in order to obtain accurate impedance values for dipoles of arbitrary length.

In 1938 Hallén formulated the boundary value problem for the cylindrical antenna itself [23], [33], [35], [36], [45]. Many papers relating to this problem have appeared since, prominent among them being papers by King and Harrison [39] and by King and Middleton [41].

Hallén's formulation of the problem gives an integral equation in which the unknown current distribution function appears inside an integral. Techinques used to solve the integral equation include iteration [36], [39], [41], Fourier series [18], [65], variational methods [36], and numerical integration [49]. A review of some early methods of solving the cylindrical antenna problem is given by King and Harrison [40], and a comparison is given by Middleton and King [50].

Results obtained by these various methods for certain configurations are in good agreement. However, there are certain limitations on the dimensions of antennas to which these methods can be applied.

For example: In the iterative solution the numerical accuracy improves as higher order iterations are performed. But the complexity of the expressions involved limits the number of iterations which are conveniently performed. The highest order of iteration published is second order. A parameter used in the iterative solution is $\Omega = 2 \ln (2h/a)$ [39], [41], [45]. As Ω decreases (the antenna becomes thicker) more iterations are needed to maintain a certain level of numerical accuracy. Since the number of iterations is limited, the

antenna dimensions for which meaningful calculations can be obtained are also limited. The thickest antennas for which results are published have $\Omega = 10$ (i. e., $2h/a = 150$).

Attempts have been made to apply these techniques to multi-element arrays with cylindrical elements [36], [38]. Progress is limited, being confined to rather special two element arrays. As with the single element, the boundary value problem yields integral equations for the current in each element.

The EMF method, using assumed sinusoidal current distributions, has been applied to the two element array [9], [17], [33]. Reasonable results are obtained when the elements are near a half wavelength long. For center-fed elements near a wavelength long the method does not yield reasonable results because the sinusoidal distribution is not a good approximation for the actual current distribution in this case.

1.3 SCOPE OF THE THESIS

This thesis research is concerned with a method for the solution of the cylindrical antenna boundary value problem which apparently has not been investigated before. The previous formulations [36], [39], [41] of the problem have used known boundary values for the magnetic vector potential to form an integral equation for the current. It is shown in this thesis that it is possible, using the mathematical formulation of Huygens' principle, to express the current in terms of

an integral over the known boundary values of the magnetic vector potential. This is an explicit, closed form expression for the current distribution function. It should be expected that this will give numerical results which are not restricted by antenna dimensions.

Evaluation of the integral derived from Huygens' principle is a substantial problem in itself. Two approaches to evaluation of the integral are presented.

The scope of the thesis is fourfold: First, the application of Huygens' principle to the cylindrical antenna is presented. Second, techniques for evaluation of the integral expression obtained are described. Third, numerical values for the current distribution and impedance for the cylindrical antenna are presented which have been obtained using Huygens' principle. Fourth, aspects meriting additional study are discussed briefly.

The purpose of the thesis is to establish the fact that Huygens' principle can be applied to the cylindrical antenna boundary value problem. With this fact established, further research into this problem and into the possible application of Huygens' principle to other similar problems will be justified.

A comprehensive survey of the literature will not be presented because such a large number of publications deal with the cylindrical antenna. The reader is referred to reference [36] for an extensive bibliography.

CHAPTER 2

THE CYLINDRICAL ANTENNA BOUNDARY VALUE PROBLEM

2.1 INTRODUCTION

This chapter describes the model used for the cylindrical antenna. Consideration of the boundary conditions lead to an expression for the vector potential on the surface of the antenna.

The model, of necessity, contains some idealizations to simplify the problem. However, these idealizations are the ones most commonly used in the literature [41] and are not expected to appreciably affect the end results.

The presentation of boundary conditions is essentially the same as that presented elsewhere [39], but is included for review and completeness of the thesis.

2.2 THE MODEL

An illustration of the cylindrical antenna is shown in Figure 1.1. Such an antenna is usually made of a highly conducting material such as copper or aluminum. Although a term can be included in the analysis to account for finite conductivity [39], infinite conductivity will be assumed. For highly conductive materials, the antenna impedance depends almost entirely upon the radiation and induction fields surrounding the antenna rather than upon the conductivity of the antenna elements.

The excitation of the antenna is taken to be a voltage applied uniformly across an infinitesimal gap at the center of the antenna. No attempt will be made to describe how this might be accomplished in practice since the attachment of any real transmission line can be

expected to perturb the current distribution on the cylinder. The theoretical analysis to follow assumes the antenna to be completely isolated in free space.

The infinitesimal gap at the driving point can be expected to exhibit a large capacitance. This capacitive reactance will not be considered as part of the antenna impedance. The antenna impedance of interest is considered to be due entirely to the radiation and induction fields. The gap capacitance could be approximated as a parallel plate capacitor in parallel with the rest of the antenna. This matter has been discussed by Schelkunoff [62].

An exact formulation for the current distribution on the cylindrical antenna would have to include the fact that current flows radially on the ends of a solid cylinder or over the ends and inside a hollow cylinder. A hollow cylinder of typical size can be expected to be operating in a cut off waveguide mode internally so that the current flowing on the inside surface would be small. Solid cylinders of relatively small radii also have little current on the ends since the current must vanish at the centers of the end faces. A comparison of tubular and solid cylinders has been presented by Einarsson [19]. In the analysis to follow, it will be assumed that the current vanishes at the edges of the end faces rather than at the centers, that is, only axial current flow is assumed to exist on the antenna. This eliminates the need for solving a boundary value problem for the end faces of the cylinder. To neglect the current on the antenna ends is common and has been discussed in the literature [41].

The voltage at the gap is taken to be of the periodic time-dependent form

$V = V_m(\omega)\, e^{j\omega t}$.

The current will necessarily be of the same form,

$I(z) = I_m(z,\omega)\, e^{j\omega t}$.

$V_m(\omega)$ and $I_m(z,\omega)$ are complex variables in general. $I(z)$ is the current distribution function, that is, the amplitude of current as a function of position along the cylinder surface. Clearly azimuthal symmetry exists, hence there is no ϕ-dependence in the current distribution.

In subsequent analysis the $e^{j\omega t}$ term will be suppressed, as will the ω dependence of V_m and I_m. Therefore, it will be written $V_m = V_m(\omega)$ and $I_m(z) = I_m(z,\omega)$. The driving point impedance evaluated at the center point of the antenna is $Z = V_m/I_m(0)$.

2.3 BOUNDARY CONDITIONS

The boundary conditions at the cylinder surface are [58] $\bar{E}_t = 0$, $\bar{D}_n = \rho_s$, $\bar{B}_n = 0$, and $\bar{J}_s = \bar{n} \times \bar{H}_t$ where $\bar{E}_t$ is the tangential electric field intensity, $\bar{D}_n$ is the normal electric displacement vector, $\bar{B}_n$ is the normal magnetic induction field, $\bar{H}_t$ is the tangential magnetic field intensity, $\bar{J}_s$ is surface current density, and $\bar{n}$ is a unit normal vector directed outward from the conductor. These quantities are further related by $\bar{D} = \epsilon_o \bar{E}$ and $\bar{B} = \mu_o \bar{H}$ in free space. They also satisfy Maxwell's equations, which for sinusoidal fields with suppressed time

dependence $e^{j\omega t}$ become:

$$\nabla \times \bar{E} = -j\omega\bar{B}$$

$$\nabla \times \bar{H} = \sigma\bar{E} + j\omega\bar{D}$$

$$\nabla \cdot \bar{D} = \rho_s ,$$

and

$$\nabla \cdot \bar{B} = 0.$$

The electric field can be expressed in terms of the magnetic vector potential $\bar{A}$ defined by

$$\bar{B} = \nabla \times \bar{A} \qquad (2.1a)$$

$$\nabla \cdot \bar{A} = -j\frac{\omega}{c^2}\varphi \qquad (2.1b)$$

where c is the speed of light,

$$c = 1/\sqrt{\mu_o\epsilon_o} ,$$

and φ is the electric scalar potential. The electric field is

$$\bar{E} = -\nabla\varphi - j\omega\bar{A}$$

or

$$\bar{E} = -j\frac{\omega}{c^2}\left[\nabla(\nabla \cdot \bar{A}) + \frac{\omega^2}{c^2}\bar{A}\right]. \qquad (2.2)$$

Since only axial current is assumed to exist, the vector

potential [58]

$$\bar{A} = \frac{\mu_o}{4\pi} \int \frac{\bar{J}_e e^{-jkR}}{R} dV , \qquad (2.3)$$

can have only a z component. That is, the vector potential has components in only those directions in which conduction currents flow. In any event $A_r \ll A_z$, except possibly at the end faces, because the radial currents are small. The tangential electric field is the z component of the electric field along the cylinder, that is, $\bar{E}_t = E_z \bar{a}_z$. Thus, for the z component of Equation (2.2) at the surface of the cylinder ($\rho = a$),

$$E_z = \frac{-j\omega}{k^2} \left(\frac{\partial^2 A_z}{\partial z^2} + k^2 A_z \right)$$

where $k = \omega/c$. Since $\bar{E}_t = 0$, hence $E_z = 0$ at $\rho = a$, the differential equation

$$\frac{\partial^2 A_z}{dz^2} + k^2 A_z = 0$$

results. Solutions of this differential equation are of the form

$$A_z = \frac{-j}{c} [C_1 \cos kz + C_2 \sin kz] \qquad (2.4)$$

where C_1 and C_2 are constants and the multiplicative term $-j/c$ is included to simplify results later.

Symmetry exists about the center gap of the cylinder so that

$$I(z) = I(-z) \qquad (2.5a)$$

and

$$\bar{A}(z) = \bar{A}(-z) \qquad (2.5b)$$

Therefore, Equation (2.4) must be written

$$A_z = -\frac{j}{c}\left[C_1 \cos kz + C_2 \sin k|z|\right] \qquad (2.6)$$

in order to satisfy the symmetry conditions, Equations (2.5a, b).

The scalar potential boundary condition at the gap is [1]

$$V = \lim_{z \to 0} \{\varphi(+z) - \varphi(-z)\}.$$

From Equation (2.1b) one obtains, with $\bar{A} = A_z \bar{a}_z$,

$$\frac{\partial A_z}{\partial z} = -j \frac{\omega}{c^2} \varphi.$$

Also,

$$\frac{\partial A_z(+z)}{\partial z} = -j \frac{\omega}{c^2} \varphi(+z),$$

$$\frac{\partial A_z(-z)}{\partial(-z)} = -j \frac{\omega}{c^2} \varphi(-z),$$

and

$$\frac{\partial A_z(-z)}{\partial(-z)} = -\frac{\partial A_z(-z)}{\partial z}.$$

---

[1] At this point one can take account of a finite gap width 2ϵ by taking the limit as $z \to \epsilon$ instead of $z \to 0$ [42]. Subsequent integrations will then be on the intervals $-h \leq z \leq \epsilon$ and $\epsilon \leq z \leq h$.

Therefore,

$$\varphi(-z) = -\varphi(z) = -j\frac{c^2}{\omega}\frac{\partial A_z(+z)}{\partial z}$$

and

$$V = 2\lim_{z\to 0}\varphi(+z) = \frac{2jc^2}{\omega}\lim_{z\to 0}\frac{\partial A_z(+z)}{\partial z}. \qquad (2.7)$$

Now, from Equations (2.6) and (2.7), one obtains

$$\lim_{z\to 0}\frac{\partial A_z(+z)}{\partial z} = \frac{-jkC_2}{c} = \frac{-j\omega V}{2c^2}$$

so that

$$C_2 = \tfrac{1}{2}V$$

and

$$A_z = -\frac{j}{c}\left[C_1 \cos kz + \tfrac{1}{2}V \sin k|z|\right]. \qquad (2.8)$$

The constant C_1 cannot be determined at this time. It must be carried along throughout the analysis until an expression for the current distribution is obtained. Then C_1 can be determined by the condition that the current vanishes at the ends of the antenna.

2.4 THE VECTOR POTENTIAL AT THE ANTENNA SURFACE

The expression for vector potential in Equation (2.8) gives the exact form of the z-dependence at the antenna surface. Of course an r-dependence must also exist since fields will vary with distance from the antenna. No ϕ-dependence exists because of symmetry. Also,

a different z-dependence must exist beyond the ends of the antenna, but that is not of interest at this point. The important point is that the vector potential is precisely known, except for a constant, at every point on the antenna surface, the end faces being ignored.

The vector potential can also be expressed in terms of the integral, Equation (2.3),

$$A_z = \frac{\mu_o}{4\pi} \int_{-h}^{h} \frac{I(z')e^{-jkR_a}\,dz'}{R_a} \tag{2.9}$$

where

$$R_a = \sqrt{a^2 + (z-z')^2}$$

and $I(z')$ is an equivalent current on the axis of the cylinder.

Equating Equations (2.8) and (2.9) yields the integral equation which has been the subject of extensive research,

$$\frac{jc\mu_o}{4\pi} \int_{-h}^{h} \frac{I(z')e^{-jkR_a}}{R_a}\,dz' = C_1 \cos kz + \tfrac{1}{2}V \sin k\,|z|.$$

$I(z')$ is the unknown function in this integral equation. It is an equivalent current along the cylinder axis instead of the actual surface current that must exist on a good conductor. This is not a critical point for thin cylinders but is somewhat questionable for relatively thick cylinders.

Chapter 3 will present an application of Huygens' principle whereby the current distribution is obtained as an explicit closed form expression. The boundary values of the vector potential at the surface

of the cylinder, Equation (2.8), are the boundary conditions to be used in applying Huygens' principle.

CHAPTER 3

HUYGENS' PRINCIPLE APPLIED TO THE CYLINDRICAL ANTENNA

3.1 INTRODUCTION

Chapter 3 describes the principles upon which the thesis is based. The mathematical formulation of Huygens' principle is reviewed starting with the singular wave equation and Green's theorem. Huygens' principle is applied to the cylindrical antenna using the boundary values for the vector potential obtained in Chapter 2. This yields a general expression for the vector potential outside the antenna, that is, for $r \geq a$ and $-h < z < h$. Then an expression for the magnetic field is obtained from the vector potential and, finally, boundary conditions for the magnetic field yield an expression for the surface current on the antenna.

The expression for the surface current is essentially the goal of the thesis. The ultimate goal is the antenna driving point impedance, but once the current distribution is known the impedance can be obtained.

Although the principles presented in this chapter are straightforward, the implementation of these principles is a substantial task. The rest of the thesis following Chapter 3 is devoted to this task.

3.2 THE WAVE EQUATION AND GREEN'S THEOREM

A method of analysis used with boundary value problems uses Green's theorem to obtain expressions for potential based upon boundary values [6], [15], [55]. The wave equation plays an important role in the application of Green's theorem.

The vector potential defined by

$$\overline{B} = \nabla \times \overline{A},$$

$$\nabla \cdot \overline{A} = -j \frac{\omega}{c^2} \varphi$$

satisfies the wave equation

$$\nabla^2 \overline{A} + k^2 \overline{A} = -\mu_o \overline{J}. \tag{3.1}$$

In order to work with scalar quantities the x, y, z components of the vector potential will each be handled separately. Only expressions for the z component will be shown. Identical results obtain for the x and y components. For the z component the wave equation is

$$\nabla^2 A_z + k^2 A_z = -\mu_o J_z.$$

A description of coordinate notation is required at this point. Source coordinates are designated as x_o, y_o, and z_o and field coordinates as x, y, and z. The source point (x_o, y_o, z_o) and field point (x,y,z) are alternatively designated by the position vectors $\overline{R}_o$ and $\overline{R}$ respectively where $\overline{R}_o = x_o\overline{i} + y_o\overline{j} + z_o\overline{k}$ and $\overline{R} = x\overline{i} + y\overline{j} + z\overline{k}$.

Let a Green's function, G, be defined such that

$$\nabla^2 G + k^2 G = -\delta(\overline{R} - \overline{R}_o). \tag{3.2}$$

$\delta(\overline{R} - \overline{R}_o)$ is the Dirac delta function defined such that

$$\delta(\bar{R} - \bar{R}_o) = 0 \text{ if } \bar{R} \neq \bar{R}_o$$

and

$$\int_{V(vol)} \delta(\bar{R} - \bar{R}_o)\, dV_o = \begin{cases} 0 \text{ if } (x,y,z) \text{ is not in V(vol)} \\ 1 \text{ if } (x,y,z) \text{ is in V(vol).} \end{cases}$$

Furthermore,

$$\int_{V(vol)} F(\bar{R}_o)\, \delta(\bar{R} - \bar{R}_o)\, dV_o = \begin{cases} 0 \text{ if } (x,y,z) \text{ is not in V(vol)} \\ F(\bar{R}) \text{ if } (x,y,z) \text{ is in V(vol)} \end{cases} \qquad (3.3)$$

In cylindrical coordinates $\delta(\bar{R} - \bar{R}_o) = \frac{1}{r_o}\, \delta(r - r_o)\, \delta(\phi - \phi_o)\, \delta(z - z_o)$.

The Green's function is the point source response function. It may be used in the well-known convolution integral

$$\bar{A} = \mu_o \int \bar{J} G\, dV$$

where

$$G = \frac{e^{-jkR}}{4\pi R} .$$

A derivation of this Green's function is presented in Appendix A. This Green's function will be important later in the thesis.

Green's theorem may be written as [15]

$$\int_V (A_z \nabla_o G - G \nabla_o A_z)\, dV_o = \oint_S (G \nabla_o A_z - A_z \nabla_o G) \cdot d\bar{S}_o . \qquad (3.4)$$

∇_o designates differentiation with respect to x_o, y_o, z_o. It is important that the same variables, source coordinates in this case, be used for both integration and differentiation in Green's theorem.

With the wave equation, Green's function, and Green's theorem as background, an expression for the vector potential, A_z, may be developed in in the following manner. On the left hand side of Equation (3.4), add and subtract the quantity k^2GA_z,

$$\int_{V(vol)} A_z(\nabla_o^2 G + k^2 G)\,dV_o - \int_{V(vol)} G(\nabla_o^2 A_z + k^2 A_z)\,dV_o = \oint_S (G\nabla_o A_z - A_z \nabla_o G)\cdot d\bar{S}_o \,. \quad (3.5)$$

Using Equations (3.1), (3.2), and (3.3), Equation (3.5) becomes, on integrating the first term and rearranging,

$$A_z(\bar{R}) = \mu_o \int_{V(vol)} J_z G\,dV_o + \oint_S (A_z \frac{\partial G}{\partial n_o} - G \frac{\partial A_z}{\partial n_o})\,dS_o \quad (3.6)$$

where it is recognized that $\nabla_o G \cdot d\bar{S} = \frac{\partial G}{\partial n_o} dS$, $\frac{\partial G}{\partial n_o}$ being a normal derivative directed into the volume V. The volume V is enclosed by the closed surface S.

In general, a potential may be determined from one of three things, first, from distributed sources (using the convolution integral), second, from boundary values for the potential (the Direchlet problem), or, third, from boundary values for the normal derivative of the potential (the Neumann problem) or a combination of the three. Equation (3.6) is a very general expression for the vector potential in that it incorporates all three possibilities.

The convolution integral is obtained from Equation (3.6) by taking

the surface S to be a sphere with a radius tending to infinity. Because of the radiation condition [6], [64] the boundary values of A_z and $\frac{\partial A_z}{\partial n_o}$ must vanish on the surface S, leaving only the term

$$A_z(\bar{R}) = \mu_o \int_{V(vol)} J_z G dV_o$$

as being non-zero.

3.3 HUYGENS' PRINCIPLE

Consider a surface S chosen such that all sources are excluded from volume V in Equation (3.6). Then

$$A_z(\bar{R}) = \oint_S (A_z \frac{\partial G}{\partial n_o} - G \frac{\partial A_z}{\partial n_o}) dS_o. \qquad (3.7)$$

The vector potential everywhere in volume V is determined by the values of A_z and $\frac{\partial A_z}{\partial n_o}$ on the surface S. Surface S may be any convenient surface, not necessarily a physical boundary, on which the value of $\bar{A}$ and $\frac{\partial \bar{A}}{\partial n_o}$ are known. This is the mathematical formulation of Huygens' principle [4], [52], [64]. Huygens' principle can be stated as follows, quoting from Halliday and Resnick [24]: "All points on a wavefront can be considered as point sources for the production of spherical secondary wavelets. After a time t the new position of the wavefront will be the surface of tangency to these secondary wavelets." This principle is clearly shown in Equation (3.7) which is essentially a convolution integral with boundary values of $\bar{A}$ and $\frac{\partial \bar{A}}{\partial n_o}$ as the distributed source, other sources being excluded.

Usually both $\overline{A}$ and $\frac{\partial \overline{A}}{\partial n_o}$ are not known on a boundary. More frequently only $\overline{A}$ or $\frac{\partial \overline{A}}{\partial n_o}$ is known, or one is known for part of the boundary and the other for the remainder of the boundary. This does not preclude the use of Equation (3.7) however. Consider a Green's function which satisfies the singular wave equation,

$$\nabla^2 G + k^2 G = -\delta(\overline{R} - \overline{R}_o),$$

as before, and also is subject to either the condition that $G = 0$ on S or that $\frac{\partial G}{\partial n_o} = 0$ on S. With $G = 0$ on S Equation (3.7) becomes

$$A_z(\overline{R}) = \oint_S A_z \frac{\partial G}{\partial n_o} dS_o. \tag{3.8}$$

With

$$\frac{\partial G}{\partial n_o} = 0 \text{ on } S,$$

$$A_z(\overline{R}) = -\oint_S G \frac{\partial A_z}{\partial n_o} dS_o. \tag{3.9}$$

Therefore, knowledge of boundary values of either A_z or $\frac{\partial A_z}{\partial n_o}$ is sufficient to completely describe the vector potential inside volume V from which conduction current sources are excluded.

3.4 APPLICATIONS TO THE CYLINDRICAL ANTENNA

On the cylindrical antenna surface the vector potential is given by Equation (2.8). In view of Huygens' principle, in particular Equation (3.8), this is sufficient to completely describe the vector potential due to the cylindrical antenna, provided an appropriate closed surface

and a suitable Green's function can be specified.

Recall that the volume enclosed by the closed surface of integration associated with Huygens' principle must exclude the conduction current sources. With the cylindrical antenna this means that the metalic cylinder itself must be excluded from the volume. A suitable closed surface may be formed in two parts. First, a large sphere centered on the antenna with a radius tending to infinity. The radiation condition insures that the boundary values of the vector potential vanish on this surface so as to cause the closed surface integral in Equation (3.8) to be zero for the sphere at infinity. The second part of the closed surface is the surface of the cylinder itself. The enclosed volume is all space outside the cylinder and inside the sphere at infinity. The vector potential everywhere in this volume can be determined by integrating Equation (3.8) over the antenna surface with the known boundary value of the vector potential, Equation (2.8).

The definition of a suitable Green's function to be used in Equation (3.8) with the cylindrical antenna is treated in Chapter 4.

Neglecting for the present the details of evaluating Equation (3.8) to obtain an expression for the vector potential, consider how the current distribution may be obtained. With a general expression for the vector potential available, the magnetic field can be determined as

$$\overline{H} = \frac{1}{\mu_o} \nabla \times \overline{A}.$$

Since, in this case, $\overline{A}$ has only a z-component

$$H_{\phi} = -\frac{1}{\mu_o}\frac{\partial A_z}{\partial r}$$

in cylindrical coordinates. The boundary condition relating the magnetic field $\bar{H}$ and surface current density $\bar{J}_s$ at a conducting boundary is $\bar{J}_s = \bar{n} \times \bar{H}$ where $\bar{n}$ is a unit normal vector directed outward from the conducting boundary. In this case

$$\bar{n} = \bar{a}_r$$

and

$$J_{sz} = -\frac{1}{\mu_o}\frac{\partial A_z}{\partial r}\bigg|_{r=a}.$$

Since azimuthal symmetry exists the total current is

$$I(z) = 2\pi a J_{sz},$$

or

$$I(z) = -\frac{2\pi a}{\mu_o}\frac{\partial A_z}{\partial r}\bigg|_{r=a}. \tag{3.10}$$

In terms of Equation (3.8) and Equation (2.8)

$$I(z) = -\frac{2\pi a}{\mu_o}\frac{\partial}{\partial r}\oint_S A_{zs}\frac{\partial G}{\partial n_o}\,dS_o \tag{3.11}$$

with

$$A_{zs} = -\frac{j}{c}\left[C_1 \cos kz + \tfrac{i}{2}V \sin k|z|\right]. \tag{3.12}$$

The expression in Equation (3.11) states the essence of the thesis. The current distribution is given explicitly in terms of the known

vector potential boundary values, Equation (3.12). The Green's function, G, satisfies the singular wave equation,

$$\nabla^2 G + k^2 G = -\delta(\bar{R} - \bar{R}_o),$$

and also the condition G = 0 on the surface of the antenna and the sphere at infinity. Expressions for the Green's function will be developed in Chapter 4.

Let the necessity of obtaining a general expression for the vector potential be clear at this point. Since boundary conditions on the magnetic field give the current distribution, and since the vector potential which determines the magnetic field has known values on the boundary, it might appear that these factors alone are sufficient to determine the current distribution. In fact, this would be sufficient if Equation (3.12) contained the radial dependence of A_z near the boundary. However, Equation (3.12) is a function of z only since it is an expression which is valid only at the boundary where r is a constant, r = a. And without the r-dependence of A_z the partial derivative in Equation (3.10) cannot be performed. So a more general expression for A_z is required. However, such an expression need be valid only in a region contiguous to the conductor boundary in order to determine the current distribution I(z).

CHAPTER 4

GREEN'S FUNCTION

4.1 INTRODUCTION

Chapter 4 contains the derivation of Green's function as required for applying Huygens' principle to the cylindrical antenna. Green's function, satisfying certain boundary conditions, may be derived in more than one way and the forms of the end results may appear to be different [48]. Two forms of Green's function are derived in this chapter. Application of these two forms of Green's function is taken up in Chapters 5 and 6 respectively.

The conditions that Green's function, G, must satisfy, as discussed in Chapter 3, are that G satisfy the singular wave equation

$$\nabla^2 G + k^2 G = -\delta(\overline{R} - \overline{R}_o),$$

and that G vanishes on the antenna surface and on a sphere at infinity. Since an expression for the vector potential need be valid only in the vicinity of the antenna in order to obtain the current distribution, valid expressions for G will be developed only for the region $-h \leq z \leq h$. Also, just as contributions to the vector potential from the end faces of the cylinder are ignored, so is the requirement that G vanishes on the end faces ignored. G will be defined on the interval $-h \leq z \leq h$ such that $G = 0$ at $r = a$ and $G \to 0$ as $r \to \infty$.

4.2 GREEN'S FUNCTION IN THE FIRST FORM

The first form of Green's function, G, is derived by letting G be the sum of two components, U and V, so that $G = U + V$. U satisfies the singular wave equation

$$\nabla^2 U + k^2 U = -\delta(\bar{R} - \bar{R}_o)$$

and is called a fundamental solution [48]. V satisfies the homgeneous wave equation

$$\nabla^2 V + k^2 V = 0$$

and is adjusted to take care of boundary conditions. The requirement that $G = 0$ on S (i. e., $G_s = 0$) is obtained by defining V such that

$$V_s = -U_s .$$

It is already known that the free space Green's function satisfies the singular wave equation (see Appendix A), so it is a suitable choice for the fundamental solution, i. e.,

$$U = \frac{e^{-jkR}}{4\pi R} .$$

The homogeneous wave equation is, in cylindrical coordinates,

$$\frac{1}{r_o} \frac{\partial}{\partial r_o} \left(r_o \frac{\partial V}{\partial r_o} \right) + \frac{1}{r_o^2} \frac{\partial^2 V}{\partial \phi_o^2} + \frac{\partial^2 V}{\partial z_o^2} + k^2 V = 0 .$$

By separating variables it is found that solutions of this equation are (see Appendix B),

$$V_1 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A^{(1)}_{m,n}\, g_{m,n}(r_o) \cos m\phi_o \cos \beta_n z_o \qquad (4.1a)$$

$$V_2 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A^{(2)}_{m,n}\, g_{m,n}(r_o) \cos m\phi_o \sin \beta_n z_o \qquad (4.1b)$$

$$V_3 = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} A^{(3)}_{m,n}\, g_{m,n}(r_o) \sin m\phi_o \cos \beta_n z_o \qquad (4.1c)$$

$$V_4 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A^{(4)}_{m,n}\, g_{m,n}(r_o) \sin m\phi_o \sin \beta_n z_o \qquad (4.1d)$$

where

$$g_{m,n}(r_o) = \begin{cases} H^{(2)}_m(\alpha_n r_o) & \beta_n < k \\ K_m(\alpha'_n r_o) & \beta_n > k \end{cases}$$

with

$$\beta_n = \frac{n\pi}{h},$$

$$\alpha_n = \sqrt{k^2 - \beta_n^2},$$

and

$$\alpha'_n = \sqrt{\beta_n^2 - k^2}.$$

Therefore, let

$$V = V_1 + V_2 + V_3 + V_4 .$$

The requirement that $G = 0$ for $r_o = a$ leads to

$$V\Big|_{r_o=a} = - U\Big|_{r=a}$$

and, by observing the orthogonality of the trigonometric functions, one obtains

$$A^{(1)}_{m,n} = \frac{-1}{\epsilon_m \epsilon_n \pi h g_{m,n}(a)} \int_{-h}^{h} \int_{-\pi}^{\pi} \frac{e^{-jkR_a}}{4\pi R_a} \cos m\phi_o \cos \beta_n z_o \, d\phi_o \, dz_o \qquad (4.2)$$

and similar expressions for $A^{(2)}_{m,n}$, $A^{(3)}_{m,n}$, and $A^{(4)}_{m,n}$ where

$$\epsilon_p = \begin{cases} 2 & p = 0 \\ 1 & p = 1,2,3,\ldots \end{cases}$$

From Equation (4.2) it is clear that the coefficients $A^{(1)}_{m,n}$, etc., are functions of r, ϕ, and z.

Green's function is symmetrical with respect to field and source points [6], [48], [52]. The physical interpretation of this is that the observed response of a point source is unchanged if the observer and the source exchange positions.

Because of the symmetry of Green's function one would expect to find that

$$A^{(1)}_{m,n}(r,\phi,z) = B^{(1)}_{m,n}\, g_{m,n}(r) \cos m\phi \cos \beta_n z,$$

$$A^{(2)}_{m,n}(r,\phi,z) = B^{(2)}_{m,n}\, g_{m,n}(r) \cos m\phi \sin \beta_n z,$$

$$A^{(3)}_{m,n}(r,\phi,z) = B^{(3)}_{m,n}\, g_{m,n}(r) \sin m\phi \cos \beta_n z,$$

$$A^{(4)}_{m,n}(r,\phi,z) = B^{(4)}_{m,n}\, g_{m,n}(r) \sin m\phi \sin \beta_n z,$$

the $B^{(i)}_{m,n}$ coefficients being constants. However, expressions for $A^{(i)}_{m,n}$ have not yet been obtained in a form wherein this explicit r,ϕ,z dependence is evident.

Evaluation of the coefficients for the Fourier series for the free space Green's function is presented in Chapter 7. In terms of the coefficients $P^{(i)}_{m,n}$ given in Chapter 7, the coefficients $A^{(i)}_{m,n}$ are

$$A^{(i)}_{m,n} = -\frac{P^{(i)}_{m,n}}{4\pi g_{m,n}(a)} . \qquad (4.3)$$

The first form of Green's function may be written

$$\begin{aligned} G = \frac{e^{-jkR}}{4\pi R} &+ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A^{(1)}_{m,n}\, g_{m,n}(r_o) \cos m\phi_o \cos \beta_n z_o \\ &+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A^{(2)}_{m,n}\, g_{m,n}(r_o) \cos m\phi_o \sin \beta_n z_o \\ &+ \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} A^{(3)}_{m,n}\, g_{m,n}(r_o) \sin m\phi_o \cos \beta_n z_o \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A^{(4)}_{m,n}\, g_{m,n}(r_o) \sin m\phi_o \sin \beta_n z_o . \end{aligned} \qquad (4.4)$$

4.3 GREEN'S FUNCTION IN THE SECOND FORM

The following derivation of Green's function is similar to that for the electrostatic case presented by Jackson [29]. Consider the singular wave equation

$$\nabla^2 G + k^2 G = -\frac{1}{r}\,\delta(r-r_o)\,\delta(\phi-\phi_o)\,\delta(z-z_o) \tag{4.5}$$

and expand the ϕ and z delta functions in Fourier series on $-\pi < \phi < \pi$ and $-h < z < h$.

$$\delta(\phi-\phi_o) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{jm(\phi-\phi_o)} = \sum_{m=0}^{\infty} \frac{\cos m(\phi-\phi_o)}{\epsilon_m \pi}$$

$$\delta(z-z_o) = \frac{1}{2h}\sum_{n=-\infty}^{\infty} e^{j\beta_n(z-z_o)} = \sum_{n=0}^{\infty} \frac{\cos \beta_n(z-z_o)}{\epsilon_n h}$$

with $\beta_n = \frac{n\pi}{h}$.

Solutions of the homogeneous wave equation can be written in a form similar to the delta function expansions

$$G = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{g_{m,n}(r,r_o)\cos m(\phi-\phi_o)\cos \beta_n(z-z_o)}{\epsilon_m \epsilon_n \pi h}. \tag{4.6}$$

Substitution of (4.6) into the singular wave equation (4.5) results in the equation

$$\sum_{m=0}^{} \sum_{n=0}^{} \left[\nabla_r^2 g_{m,n} - \frac{m^2 g_{m,n}}{r^2} + (k^2 - \beta_n^2)\, g_{m,n}\right] \frac{\cos m(\phi-\phi_o) \cos \beta_n(z-z_o)}{\epsilon_m \epsilon_n \pi h}$$

$$= -\frac{1}{r}\,\delta(r-r_o) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\cos m\,(\phi-\phi_o) \cos \beta_n(z-z_o)}{\epsilon_m \epsilon_n \pi h}\,.$$

Termwise

$$\nabla_r^2 g_{m,n} - \frac{m^2 g_{m,n}}{r^2} + (k^2 - \beta_n^2)\, g_{m,n} = -\frac{1}{r}\,\delta(r-r_o). \tag{4.7}$$

Now

$$\nabla_r^2 g_{m,n} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial g_{m,n}}{\partial r}\right)$$

and, with $g_{m,n}$ being a continuous function at r_o, integrating from r_o^- to r_o^+ on r in (4.7) gives

$$\left.\frac{\partial g_{m,n}}{\partial r}\right|_{r_o^+} - \left.\frac{\partial g_{m,n}}{\partial r}\right|_{r_o^-} = -\frac{1}{r_o}\,. \tag{4.8}$$

To satisfy the condition that G = 0 on the antenna surface, r = a, and to represent outwardly propagating waves, let

$$g_{m,n} = \begin{cases} g_{m,n}^{(1)} & \beta_n < k \\ g_{m,n}^{(2)} & \beta_n > k \end{cases}$$

where

$$g^{(1)}_{m,n} = \begin{cases} A_{m,n} H^{(2)}_m(\alpha_n r)\left[H^{(2)}_m(\alpha_n r_o)H^{(1)}_m(\alpha_n a) - H^{(1)}_m(\alpha_n r_o)H^{(2)}_n(\alpha_n a)\right] & (a \le r_o < r) \\ A_{m,n} H^{(2)}_m(\alpha_n r_o)\left[H^{(2)}_n(\alpha_n r)H^{(1)}_m(\alpha_n a) - H^{(1)}_m(\alpha_n r)H^{(2)}_m(\alpha_n a)\right] & (a \le r < r_o) \end{cases}$$

and

$$g^{(2)}_{m,n} = \begin{cases} A'_{m,n} K_m(\alpha'_n r)\left[I_m(\alpha'_n r_o)K_m(\alpha'_n a) - K_m(\alpha'_n r_o)I_m(\alpha'_n a)\right] & (a \le r_o < r) \\ A'_{m,n} K_m(\alpha'_n r_o)\left[I_m(\alpha'_n r)K_m(\alpha'_n a) - K_m(\alpha'_n r)I_m(\alpha'_n a)\right] & (a \le r < r_o) \end{cases}$$

and where

$$\alpha_n = \sqrt{k^2 - \beta_n^2}$$

and

$$\alpha'_n = \sqrt{\beta_n^2 - k^2} \quad .$$

$A_{m,n}$ and $A'_{m,n}$ are constants. The case of $\beta_n = k$ will not be treated at this time.

The constants $A_{m,n}$ and $A'_{m,n}$ must be adjusted so that $g_{m,n}$ satisfies Equation (4.8). We have

$$\left.\frac{\partial g^{(1)}_{m,n}}{\partial r}\right|_{r_o^+} = \alpha_n A_{m,n} H^{(2)'}_m(\alpha_n r_o)\left[H^{(2)}_m(\alpha_n r_o)H^{(1)}_m(\alpha_n a) - H^{(1)}_m(\alpha_n r_o)H^{(2)}_m(\alpha_n a)\right]$$

and

$$\left.\frac{\partial g^{(1)}_{m,n}}{\partial r}\right|_{r_o^-} = \alpha_n A_{m,n} H_m^{(2)}(\alpha_n r_o)\left[H_m^{(2)'}(\alpha_n r_o) H_m^{(1)}(\alpha_n a) - H_m^{(1)'}(\alpha_n r_o) H_m^{(2)}(\alpha_n a)\right]$$

or

$$\left.\frac{\partial g^{(1)}_{m,n}}{\partial r}\right|_{r_o^+} - \left.\frac{\partial g^{(1)}_{m,n}}{\partial r}\right|_{r_o^-} = \alpha_n A_{m,n} H_m^{(2)}(\alpha_n a)\left[H_m^{(1)'}(\alpha_n r_o) H_m^{(2)}(\alpha_n r_o)\right.$$

$$\left. - H_m^{(2)'}(\alpha_n r_o) H_m^{(1)}(\alpha_n r_o)\right]. \qquad (4.9)$$

The expression in square brackets in (4.9) can be simplified. Write

$$H_m^{(1)} H_m^{(2)'} = (J_m + jN_m)(J_m' - jN_n') = J_m J_m' + N_m N_m' + j(J_m' N_n - N_m' J_m)$$

$$H_m^{(2)} H_m^{(1)'} = (J_m - jN_m)(J_m' + jN_m') = J_n J_n' + N_m N_m' - j(J_n' N_m - N_m' J_m).$$

Then

$$H_m^{(2)} H_n^{(1)'} - H_m^{(2)'} H_m^{(1)} = -2j(J_m' N_m - N_m' J_m).$$

With the indentities [30]

$$J_m'(x) = -\frac{m}{x} J_m(x) + J_{m-1}(x),$$

$$N_m'(x) = -\frac{m}{x} N_m(x) + N_{m-1}(x),$$

and

$$N_{m-1}(x)J_m(x) - N_m(x)J_{m-1}(x) = \frac{2}{\pi x}$$

one can write

$$H_m^{(2)}(x)H_m^{(1)'}(x) - H_m^{(2)'}(x)H_m^{(1)}(x) = -2j[J_m'(x)N_m(x) - N_m'(x)J_m(x)]$$

$$= -2j[J_{m-1}(x)N_m(x) - N_{m-1}(x)J_m(x)]$$

$$= \frac{4j}{\pi x}.$$

Therefore

$$H_m^{(2)}(\alpha_n r_o)H_m^{(1)'}(\alpha_n r_o) - H_m^{(2)'}(\alpha_n r_o)H_m^{(1)}(\alpha_n r_o) = \frac{4j}{\pi \alpha_n r_o}. \qquad (4.10)$$

Equating (4.8) and (4.9) and utilizing (4.10) gives the constants

$$A_{m,n} = \frac{j\pi}{4H_m^{(2)}(\alpha_n a)}. \qquad (4.11)$$

Similarly

$$\left.\frac{\partial g^{(2)}_{m,n}}{\partial r}\right|_{r_o^+} - \left.\frac{\partial g^{(2)}_{m,n}}{\partial r}\right|_{r_o^-} = \alpha_n' A'_{m,n} K_m(\alpha_n' a)\left[K_m'(\alpha_n r_o) I_m(\alpha_n r_o)\right.$$

$$\left. - I_m'(\alpha_n r_o) K_m(\alpha_n r_o)\right]. \qquad (4.12)$$

The expression in square brackets can be simplified. With the identities [30]

$$I_m'(x) = \tfrac{1}{2}\left[I_{m-1}(x) + I_{m+1}(x)\right],$$

$$K_m'(x) = -\tfrac{1}{2}\left[K_{m-1}(x) + K_{m+1}(x)\right],$$

and

$$I_m(x)K_{m+1}(x) + I_{m+1}(x)K_m(x) = \frac{1}{x}$$

one obtains

$$I_m'(\alpha_n' r_o)K_m(\alpha_n' r_o) - K_m'(\alpha_n' r_o)I_m(\alpha_n' r_o) = \frac{1}{\alpha_n' r_o}\,. \qquad (4.13)$$

Equating (4.8) and (4.12) and utilizing (4.13) gives

$$A'_{m,n} = \frac{1}{K_m(\alpha_n' a)}\,. \qquad (4.14)$$

With the constants $A_{m,n}$ and $A'_{m,n}$ given by Equations (4.11) and (4.14) respectively one can write the second form of Green's function

$$G = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{g_{m,n}(r,r_o)\cos m(\phi-\phi_o)\cos \beta_n(z-z_o)}{\epsilon_m \epsilon_n \pi h} \qquad (4.15)$$

where, for $a \leq r_o < r$,

$$g_{m,n}(r,r_o) = \begin{cases} \dfrac{j\pi H_m^{(2)}(\alpha_n r)\left[H_m^{(2)}(\alpha_n r_o) H_m^{(1)}(\alpha_n a) - H_m^{(1)}(\alpha_n r_o) H_m^{(2)}(\alpha_n a)\right]}{4H_m^{(2)}(\alpha_n a)} & \beta_n < k \\ \dfrac{K_m(\alpha_n' a)\left[I_m(\alpha_n' r_o) K_m(\alpha_n' a) - K_m(\alpha_n' r_o) I_m(\alpha_n' a)\right]}{K_m(\alpha_n a)} & \beta_n > k. \end{cases}$$

(4.16)

Application of the second form of Green's function is presented in Chapter 6.

CHAPTER 5

APPLICATION OF THE FIRST FORM OF GREEN'S FUNCTION

5.1 INTRODUCTION

In this Chapter the mathematical form of Huygens' principle is applied to the cylindrical antenna using the first form of Green's function derived in Chapter 4. From Huygens' principle, Equation (3.8), the vector potential is given by

$$A_z(\bar{R}) = \oint_S A_{zs} \frac{\partial G}{\partial n_o} \, dS_o \, . \tag{5.1}$$

The vector potential on the antenna surface, Equation (2.8), is

$$A_{zs} = \frac{-j}{c} \left[C_1 \cos kz_o + \tfrac{1}{2} \bar{V}_i \sin k|z_o| \right]. \tag{5.2}$$

Green's function satisfying the requirement $G\big|_S = 0$ may be written, see Section 4.2,

$$G = U + V \tag{5.3}$$

where U is the fundamental solution and V is adjusted so that G satisfies boundary conditions. Suitable forms of U and V are derived in Chapter 4.

In the analysis to follow it will be convenient to separate the components of the integrand in Equation (5.1) so that, in view of Equation (5.3),

$$A_z(\bar{R}) = \oint_S A_{zs} \left[\frac{\partial U}{\partial n_o} + \frac{\partial V}{\partial n_o} \right] dS_o \, . \tag{5.4}$$

Let

$$A_{zu}(\bar{R}) = \oint_S A_z \frac{\partial U}{\partial n_o} dS_o \tag{5.5}$$

and

$$A_{zv}(\bar{R}) = \oint_S A_z \frac{\partial V}{\partial n_o} dS_o \,. \tag{5.6}$$

Then

$$A_z(\bar{R}) = A_{zu}(\bar{R}) + A_{zv}(\bar{R}) \,. \tag{5.7}$$

Recalling that $\bar{n}_o$ is a unit normal vector directed into the volume V enclosed by surface S [15], and that V is outside the cylindrical antenna, we have $\bar{n}_o = \bar{a}_r$ and Equations (5.5) and (5.6) may be written,

$$A_{zu}(\bar{R}) = \int_{-\pi}^{\pi} \int_{-h}^{h} A_{zs} \frac{\partial U}{\partial r_o} \Bigg|_{r_o = a} a \, d\phi_o dz_o \tag{5.8}$$

and

$$A_{zv}(\bar{R}) = \int_{-\pi}^{\pi} \int_{-h}^{h} A_{zs} \frac{\partial V}{\partial r_o} \Bigg|_{r_o = a} a \, d\phi_o dz_o \,. \tag{5.9}$$

5.2 THE VECTOR POTENTIAL DUE TO THE FUNDAMENTAL SOLUTION

Interchanging the order of integration and partial differentiation in Equation (5.8) gives

$$A_{zu}(\bar{R}) = a\int_{-h}^{h} A_{zs}\,dz_o \left\{ \frac{\partial}{\partial r_o} \int_{-\pi}^{\pi} U\, d\phi_o \right\} \Bigg|_{r_o = a}$$

As discussed in Appendix C,

$$\frac{\partial}{\partial r_o} \int_{-\pi}^{\pi} U\, d\phi_o \Bigg|_{r_o = a} \approx \frac{a}{2} e^{-jk\sqrt{r^2 + r_o^2 + (z-z_o)^2}} \left[\frac{-jk}{r^2 + a^2 + (z-z_o)^2} - \frac{1}{\left[r^2 + a^2 + (z-z_o)^2\right]^{\frac{3}{2}}} \right]$$

so that

$$A_{zu}(R) \approx \frac{a^2}{2}\int_{-h}^{h} A_{zs}\, e^{-jk\sqrt{r^2 + a^2 + (z-z_o)^2}} \left[\frac{-jk}{r^2 + a^2 + (z-z_o)^2} - \frac{1}{\left[r^2 + a^2 + (z-z_o)^2\right]^{\frac{3}{2}}} \right] dz_o. \qquad (5.10)$$

By writing the trigonometric functions in

$$A_{zs} = -\frac{j}{c}\left[C_1 \cos kz_o + \tfrac{1}{2} V \sin k|z_o|\right]$$

in terms of the exponential functions

$$\sin kz_o = \frac{e^{jkz_o} - e^{-jkz_o}}{2j}$$

and

$$\cos kz_o = \frac{e^{jkz_o} + e^{-jkz_o}}{2}$$

one obtains, in Equation (5.10), integrals of the form

$$F^{\pm}(A,B) = \int_A^B e^{-jk\left[\sqrt{r^2 + a^2 + (z-z_o)^2} \pm z_o\right]} \left\{ \frac{-jk}{r^2 + a^2 + (z-z_o)^2} - \frac{1}{\left[r^2 + a^2 + (z-z_o)^2\right]^{\frac{3}{2}}} \right\} dz_o . \qquad (5.11)$$

That is,

$$A_{zu}(R) = \frac{-ja^2}{4c} \left\{ C_1 \left[F^-(-h,h) + F^+(-h,h)\right] - j\tfrac{1}{2} V \left[F^-(0,h) - F^+(0,h) + F^-(0,-h) - F^+(0,-h)\right] \right\} \qquad (5.12)$$

The integrand of Equation (5.11) is a perfect differential (see Equations (7.23), (7.27) and (7.29)). Therefore

$$F^{\pm}(A,B) = \left. \frac{e^{-jk\left[\sqrt{r^2 + a^2 + (z-z_o)^2} \pm z_o\right]}}{\left[(z_o - z) \pm \sqrt{r^2 + a^2 + (z-z_o)^2}\right] \sqrt{r^2 + a^2 + (z-z_o)^2}} \right|_A^B \qquad (5.13)$$

Applying Equation (5.13) in Equation (5.12) gives

$$A_{zu}(R) = \frac{-ja^2}{4c} C_1 \{ \frac{e^{-jk\sqrt{r^2 + a^2 + (h-z)^2}}}{\sqrt{r^2 + a^2 + (h-z)^2}} [\frac{e^{jkh}}{(h-z) - \sqrt{r^2 + a^2 + (h-z)^2}}$$

$$+ \frac{e^{-jkh}}{(h-z) + \sqrt{r^2 + a^2 + (h-z)^2}}]$$

$$+ \frac{e^{-jk\sqrt{r^2 + a^2 + (h+z)^2}}}{\sqrt{r^2 + a^2 + (h+z)^2}} [\frac{e^{jkh}}{(h+z) - \sqrt{r^2 + a^2 + (h+z)^2}}$$

$$+ \frac{e^{-jkh}}{(h+z) + \sqrt{r^2 + a^2 + (h+z)^2}}] \}$$

$$- \frac{a^2}{4c} \frac{1}{2} V \{ \frac{e^{-jk\sqrt{r^2 + a^2 + (h-z)^2}}}{\sqrt{r^2 + a^2 + (h-z)^2}} [\frac{e^{jkh}}{(h-z) - \sqrt{r^2 + a^2 + (h-z)^2}}$$

$$- \frac{e^{-jkh}}{(h-z) + \sqrt{r^2 + a^2 + (h-z)^2}}]$$

$$+ \frac{e^{-jk\sqrt{r^2 + a^2 + (h+z)^2}}}{\sqrt{r^2 + a^2 + (h+z)^2}} [\frac{e^{jkh}}{(h+z) - \sqrt{r^2 + a^2 + (h+z)^2}}$$

$$- \frac{e^{-jkh}}{(h+z) + \sqrt{r^2 + a^2 + (h+z)^2}}]$$

$$- 2 \frac{e^{-jk\sqrt{r^2 + a^2 + z^2}}}{\sqrt{r^2 + a^2 + z^2}} \left[\frac{1}{z - \sqrt{r^2 + a^2 + z^2}} - \frac{1}{z + \sqrt{r^2 + a^2 + z^2}} \right] \} \tag{5.14}$$

Equation (5.14) can be rewritten in the form

$$A_{zu} = \frac{ja^2 C_1}{2c\,(r^2 + a^2)} \{ e^{-jk\sqrt{r^2 + a^2 + (h-z)^2}} \left[j \sin kh + \frac{(h-z) \cos kh}{\sqrt{r^2 + a^2 + (h-z)^2}} \right]$$

$$+ e^{-jk\sqrt{r^2 + a^2 + (h+z)^2}} \left[j \sin kh + \frac{(h+z) \cos kh}{\sqrt{r^2 + a^2 + (h+z)^2}} \right] \}$$

$$+ \frac{a^2 \frac{1}{2} V}{2c(r^2 + a^2)} \{ e^{-jk\sqrt{r^2 + a^2 + (h-z)^2}} \left[\cos kh + \frac{j(h-z) \sin kh}{\sqrt{r^2 + a^2 + (h-z)^2}} \right]$$

$$+ e^{-jk\sqrt{r^2 + a^2 + (h+z)^2}} \left[\cos kh + \frac{j(h+z) \sin kh}{\sqrt{r^2 + a^2 + (h+z)^2}} \right.$$

$$\left. - 2\, e^{-jk\sqrt{r^2 + a^2 + z^2}} \right\} \tag{5.15}$$

At the surface of the antenna, $r = a$, Equation (5.15) becomes, assuming $(h \pm z) >> a$, i. e., for values of z not near the ends of the antenna,

$$A_{zu} \approx \frac{j}{2c} \left[C_1 \cos kz + \tfrac{1}{2} V \sin k|z| \right] = - \tfrac{1}{2} A_{zs} . \tag{5.16}$$

For $z = \pm h$ Equation (5.15) gives

$$A_{zu} \approx -\tfrac{1}{4} A_{zs}\Big|_{z=\pm h} \tag{5.17}$$

5.3 THE VECTOR POTENTIAL DUE TO THE SOLUTION SATISFYING BOUNDARY CONDITIONS

From Equation (5.9)

$$A_{zv}(\bar{R}) = \int_{-h}^{h}\int_{-\pi}^{\pi} A_{zs} \frac{\partial V}{\partial r_o}\Big|_{r_o=a} a \, d\phi_o dz. \tag{5.18}$$

where

$$V = V_1 + V_2 + V_3 + V_4$$

and

$$V_1 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{m,n}\, g_{m,n}(r_o) \cos m\phi_o \cos \beta_n z_o,$$

$$V_2 = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A^{(2)}_{m,n}\, g_{m,n}(r_o) \cos m\phi_o \sin \beta_n z_o,$$

$$V_3 = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} A^{(3)}_{m,n}\, g_{m,n}(r_o) \sin m\phi_o \cos \beta_n z_o,$$

$$V_4 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A^{(4)}_{m,n}\, g_{m,n}(r_o) \sin m\phi_o \sin \beta_n z_o.$$

In terms of the Fourier coefficients derived in Chapter 7 (see Chapter 4 for derivation of $A^{(i)}_{m,n}$)

$$A^{(i)}_{m,n} = -\frac{p^{(i)}_{m,n}}{4\pi g_{m,n}(a)} \quad .$$

Upon interchanging the order of integration and differentiation in (5.18) and integrating termwise, one finds that only the m = 0 terms yield non-zero results because A_{zs} is not a function of ϕ and

$$\int_{-\pi}^{\pi} \cos m\phi_o \, d\phi_o = \begin{cases} 2\pi & m = 0 \\ 0 & m > 0 \end{cases}$$

Furthermore, terms involved in V_2 contribute nothing to the vector potential because A_{zs} is an even function of z_o. Consequently

$$A_z(\bar{R}) = \frac{\partial}{\partial r_o}\left[\sum_{n=0}^{\infty} 2\pi a A_n g_n(r_o) \int_{-h}^{h} A_{zs} \cos \beta_n z_o \, dz_o\right]\Bigg|_{r_o=a} \qquad (5.19)$$

where $g_n(r_o) = g_{0,n}(r_o)$ and $A_n = A^{(i)}_{0,n}$.

The Fourier series for A_{zs} is

$$A_{zs} = \sum_{n=0}^{\infty} D_n \cos \beta_n z_o$$

where

$$D_n = \frac{1}{\epsilon_n h} \int_{-h}^{h} A_{zs} \cos \beta_n z_o \, . \qquad (5.20)$$

Let

$$g_n'(a) = \left.\frac{\partial g_n(r_o)}{\partial r_o}\right|_{r_o=a}$$

Differentiating termwise and utilizing (5.20) gives

$$A_{zv}(\bar{R}) = \sum_{n=0}^{\infty} 2\pi a \epsilon_n h D_n g_n'(a) A_n(r,\phi,z,) \qquad (5.21)$$

where

$$g_n'(a) = \begin{cases} -\alpha_n H_1^{(2)}(\alpha_n a) & \beta_n < k \\ -\alpha_n' K_1(\alpha_n' a) & \beta_n < k. \end{cases}$$

The termwise differentiation leading to Equation (5.21) is permitted because the series in (5.21) converges uniformly. It is shown in Chapter 7 that D_n behaves as $1/n^2$ for large n. $g_n'(a)/g_n(a)$ behaves as n and $g_n'(a)A_{m,n}^{(1)}(r,\phi,z)$ behaves as $1/n$ since

$$g_n'(a)A_n^{(1)} = -\frac{g_n'(a)\, P_n^{(1)}}{4\pi g_n(a)}$$

and $P_n^{(1)}$ behaves as $1/n^2$ as shown in Chapter 7. Therefore the terms of the series in (5.21) behave as $1/n^3$ for large n, thus assuring uniform convergence by the Weierstrass M-test.

Approximately,

$$P_n \approx \frac{2}{\epsilon_n h} \begin{cases} \frac{-j\pi}{2} H_o^{(2)}(\alpha_n r) & \beta_n < k \\ K_o(\alpha_n' r) & \beta_n > k \end{cases}$$

Then, for small n,

$$g_n'(a) A_n^{(1)} \approx \frac{2}{\epsilon_n h} \cos \beta_n z$$

and

$$A_{zv}(\overline{R})\Big|_{r=a} \approx \sum_{n=0}^{\infty} D_n \cos \beta_n z = A_{zs}.$$

Combined with (5.16) this gives

$$A(\overline{R})\Big|_{r=a} = \tfrac{1}{2} A_{zs}. \qquad (5.22)$$

The general expression for the vector potential gives half the value of the boundary value at the antenna. The result (5.22) is derived from approximations for which no estimate of accuracy has been made.

5.4 EXPRESSIONS FOR THE CURRENT DISTRIBUTION AND THE DRIVING POINT IMPEDANCE

In Chapter 3 it is shown that

$$I(z) = -\frac{2\pi\epsilon}{\mu_o}\frac{\partial A_z}{\partial r}\bigg|_{r=a_o} \tag{5.23}$$

One can write

$$\frac{\partial A_z}{\partial r} = \frac{\partial A_{zu}}{\partial r} + \frac{\partial A_{zv}}{\partial r} .$$

Let

$$D_n = \frac{-j}{c}\left[C\,D_{cn} + \tfrac{1}{2}V\,D_{vn}\right],$$

$$A_{zu} = \frac{-j}{ac}\left[C\,P_u(r,z) + \tfrac{1}{2}V\,Q_u(r,z)\right],$$

and

$$A_{zv} = \frac{-j}{ac}\left[C\,P_v(r,z) + \tfrac{1}{2}V\,Q_v(r,z)\right]$$

where

$$D_{cn} = \frac{i}{\epsilon_n h}\int_{-h}^{h} \cos kz_o \cos \beta_n z_o \, dz_o,$$

$$D_{vn} = \frac{1}{\epsilon_n h} \int_{-h}^{h} \sin k|z_o| \cos \beta_n z_o \, dz_o,$$

$$P_u(r,z) = \frac{a^3}{2(r^2 + a^2)} \{ e^{-jk\sqrt{r^2 + a^2 + (h-z)^2}} \left[j \sin kh + \frac{(h-z) \cos kh}{\sqrt{r^2 + a^2 + (h-z)^2}} \right]$$

$$+ e^{-jk\sqrt{r^2 + a^2 + (h-z)^2}} \left[j \sin kh + \frac{(h+z) \cos kh}{\sqrt{r^2 + a^2 + (h+z)^2}} \right] \},$$

$$Q_u(r,z) = \frac{-ja^3}{2(r^2 + a^2)} \{ e^{-jk\sqrt{r^2 + a^2 + (h-z)^2}} \left[\cos kh + \frac{j(h-z) \sin kh}{\sqrt{r^2 + a^2 + (h-z)^2}} \right]$$

$$+ e^{-jk\sqrt{r^2 + a^2 + (h+z)^2}} \left[\cos kh + \frac{i(h+z) \sin kh}{\sqrt{r^2 + a^2 + (h+z)^2}} \right]$$

$$- 2e^{-jk\sqrt{r^2 + a^2 + z^2}} \},$$

$$P_v(r,z) = \sum_{n=0}^{\infty} 2\pi a^2 \epsilon_n h \, D_{cn} \, g_n'(a) \, A_n(r,z),$$

and

$$Q_v(r,z) = \sum_{n=0}^{\infty} 2\pi a^2 \epsilon_n h \, D_{vn} \, g_n'(a) \, A_n(r,z)$$

where

$$g_n'(a) = \begin{cases} -\alpha_n H_1^{(2)}(\alpha_n a) & \beta_n < k \\ -\alpha_n' K_1(\alpha_n' a) & \beta_n > k \end{cases}$$

Then

$$\frac{\partial A_{zu}}{\partial r} = \frac{-j}{ac}\left[C_1 P_u'(r,z) + \tfrac{1}{2} V Q_u'(r,z)\right]$$

and

$$\frac{\partial A_{zv}}{\partial r} = \frac{-j}{ac}\left[C_1 P_v'(r,z) + \tfrac{1}{2} V Q_v'(r,z)\right].$$

Combining terms

$$\frac{\partial A_z}{\partial r} = \frac{-j}{ac}\left[C_1 P'(r,z) + \tfrac{1}{2} V Q'(r,z)\right]$$

where

$$P'(r,z) = P_u'(r,z) + P_v'(r,z)$$

and

$$Q'(r,z) = Q'_u(r,z) + Q'_v(r,z) .$$

Therefore

$$I(z) = \frac{j2\pi}{R_s} \left[C_1 \, P'(a,z) + \tfrac{1}{2} V Q'(a,z) \right] ,$$

where $R_s = c\mu_o \approx 377$ ohms.

The requirement that $I(h) = 0$ yields

$$C_1 = - \tfrac{1}{2} V \frac{Q'(a,h)}{P'(a,h)}$$

and

$$I(z) = \frac{j\pi V}{R_s} \left[\frac{Q'(a,z)\, P'(a,h) - P'(a,z)\, Q'(a,h)}{P'(a,h)} \right] . \qquad (5.24)$$

The driving point impedance is

$$Z = \frac{V}{I(0)} = \frac{-jR_s}{\pi} \left[\frac{P'(a,h)}{Q'(a,0)\, P'(a,h) - P'(a,0)\, Q'(a,h)} \right] \qquad (5.25)$$

Differentiating the P and Q terms gives

$$P'_u(a,z) = - \tfrac{1}{4} \{ e^{-jk\sqrt{2a^2 + (h-z)^2}} \left(\left[\sin kh + \frac{(h-z) \cos kh}{\sqrt{2a^2 + (h-z)^2}} \right] \right.$$

$$\cdot \left[1 + \frac{jka^2}{\sqrt{2a^2 + (h-z)^2}}\right] + \frac{a^2(h-z)\cos kh}{\left[2a^2 + (h-z)^2\right]^{\frac{3}{2}}}\Bigg)$$

$$+ e^{-jk\sqrt{2a^2 + (h+z)^2}} \Bigg(\left[j \sin kh + \frac{(h+z)\cos kh}{\sqrt{2a^2 + (h+z)^2}}\right]$$

$$\cdot \left[1 + \frac{jka^2}{\sqrt{2a^2 + (h+z)^2}}\right] + \frac{a^2(h+z)\cos kh}{\left[2a^2 + (h+z)^2\right]^{\frac{3}{2}}}\Bigg) \Bigg\}$$

$$Q_u'(a,z) = \frac{j}{4}\Bigg\{e^{-jk\sqrt{2a^2 + (h-z)^2}} \Bigg(\left[\cos kh + \frac{j(h-z)\sin kh}{\sqrt{2a^2 + (h-z)^2}}\right]$$

$$\cdot \left[1 + \frac{jka^2}{\sqrt{2a^2 + (h-z)^2}} + \frac{ja^2(h-z)\sin kh}{\left[2a^2 + (h-z)^2\right]^{\frac{3}{2}}}\right)$$

$$+ e^{-jk\sqrt{2a^2 + (h+z)^2}} \Bigg(\left[\cos kh + \frac{j(h+z)\sin kh}{\sqrt{2a^2 + (h+z)^2}}\right]$$

$$\cdot \left[1 + \frac{jka^2}{\sqrt{2a^2 + (h-z)^2}}\right] + \frac{ja^2(h+z)\sin kh}{\left[2a^2 + (h+z)^2\right]^{\frac{3}{2}}}\Bigg)$$

$$- 2e^{-jk\sqrt{2a^2 + z^2}}\left[1 + \frac{jka^2}{\sqrt{2a^2 + z^2}}\right]\Bigg\},$$

$$P_v'(a,z) = \sum_{n=0}^{\infty} 2\pi a^2 \epsilon_n h\, D_{cn}\, g_n'(a)\, A_n'(a,z),$$

and

$$Q_v'(a,z) = \sum_{n=0}^{\infty} 2\pi a^2 \epsilon_n h\, D_{vn}\, g_n'(a)\, A_n'(a,z).$$

$A_n'(a,z)$ is given in Chapter 7. Analysis shows that the series representing $P_n'(a,z)$ and $Q_v'(a,z)$ converge uniformly, therefore term-wise differentiation is permissible.

The terms of the series for P_v' and Q_v' for which $\beta_n = k$ require special attention. This occurs when $h/\lambda = n/2$, that is, for integer wavelength antennas. Values of D_{cn} and D_{vn} are given in Chapter 7 as

$$D_{cn} = 1 \qquad \beta_n = k$$

$$D_{vn} = 0 \qquad \beta_n = k.$$

We have

$$g_n'(a)\, A_n'(a,z) = -\frac{g_n'(a)\, P_n'(a,z)}{4\pi\, g_n(a)}.$$

$P_n'(a,z)$ is well-behaved for $\beta_n = k$, but $g_n'(a)$ and $g_n(a)$ become singular as $\beta_n \to k$ (i. e., $\alpha_n \to 0$). However, for $\beta_n \to k^-$

$$\lim_{\beta_n \to k^-} \frac{g_n'(a)}{g_n(a)} = \lim_{\alpha_n \to 0} \frac{-\alpha_n H_1^{(2)}(\alpha_n a)}{H_0^{(2)}(\alpha_n a)} \approx \lim_{\alpha_n \to 0} \frac{-\alpha_n \left(j\frac{2}{\pi\alpha_n a}\right)}{1 + j\frac{2}{\pi}\ln\frac{2}{\gamma\alpha_n a}} = 0.$$

Similarly, for $\beta_n \to k^+$

$$\lim_{\beta_n \to k^+} \frac{g_n'(a)}{g_n(a)} = \lim_{\alpha_n' \to 0} \frac{-\alpha_n' K_1(\alpha_n' a)}{K_o(\alpha_n' a)} \approx \lim_{\alpha_n' \to 0} \frac{-\alpha_n' \left(\frac{1}{\alpha_n' a}\right)}{\ln \frac{2}{\gamma \alpha_n' a}} = 0 .$$

Therefore the terms of the series for which $\beta_n = k$ are zero.

For relatively thin antennas $ka \ll 1$ and the dominant terms in P and Q are

$$P(a,z) \approx -\tfrac{1}{4} \{ e^{-jk(h-z)} [\cos kh + j \sin kh] + e^{-jk(h+z)} [\cos kh + j \sin kh] \} = -\tfrac{1}{2} \cos kz$$

and

$$Q(a,z) \approx \frac{j}{4} \{ e^{-jk(h-z)} [\cos kh + j \sin kh] + e^{-jk(h+z)} [\cos kh + j \sin kh] - 2 e^{-jk|z|} \} = -\tfrac{1}{2} \sin k|z| .$$

Inserting these expressions for P and Q into (5.24) gives

$$I_z = \frac{j\pi V}{2R_s} \left[\frac{\sin kh \cos kz - \cos kh \sin k|z|}{\cos kh} \right]$$

$$= \frac{j\pi V}{2R_s} \frac{\sin k(h-|z|)}{\cos kh} .$$

A similar expression was obtained by King and Harrison [39] as a first approximation for the current. Following their analysis, one obtains

$$I(0) = \frac{j\pi V}{2R_s} \tan kh$$

and, with

$$I_{max} = I(0)/\sin kh,$$

$$I(z) = I_{max} \sin k(h-|z|).$$

I_{max} is a ficticious current for antennas with $h < \lambda/4$. This is the well-known sinusoidal current distribution. It represents the dominant part of the current distribution as verified by experimental measurements [5], [51].

CHAPTER 6

APPLICATION OF THE SECOND FORM OF GREEN'S FUNCTION

6.1 INTRODUCTION

In this Chapter Huygens' principle is applied to the cylindrical antenna with the second form of Green's function being used. The basic principles involved are the same as those in Chapter 5. The only basic difference is that a different form of Green's function is used here. Because of this, the functional forms obtained have a much different appearance.

6.2 A GENERAL EXPRESSION FOR THE VECTOR POTENTIAL

A general expression for the magnetic vector potential is obtained by employing the mathematical form of Huygens' principle, Equation (3.8),

$$A_z(\bar{R}) = \oint_S A_{zs} \frac{\partial G}{\partial n_o} \, dS_o \tag{6.1}$$

with, Equation (2.8)

$$A_{zs} = \frac{-j}{c} \left[C_1 \cos kz_o + \tfrac{1}{2} V \sin k|z| \right]. \tag{6.2}$$

Since $a \leq r_o < r$ one obtains from Equations (4.15) and (4.16)

$$G = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{g_{m,n}(r,r_o) \cos m(\phi-\phi_o) \cos \beta_n(z-z_o)}{\epsilon_m \epsilon_n \pi h} \tag{6.3}$$

where

$$g_{m,n}(r,r_o) = \begin{cases} \dfrac{j\pi H_m^{(2)}(\alpha_n r)\left[H_m^{(2)}(\alpha_n r_o)\, H_m^{(1)}(\alpha_n a) - H_m^{(1)}(\alpha_n r_o)\, H_m^{(2)}(\alpha_n a)\right]}{4H_m^{(2)}(\alpha_n a)} & \beta_n < k \\ \dfrac{K_m(\alpha_n r)\left[I_m(\alpha_n' r_o)\, K_m(\alpha_n' a) - K_m(\alpha_n' r_o)\, I_m(\alpha_n' a)\right]}{K_m(\alpha_n' a)} & \beta_n > k \end{cases} \tag{6.4}$$

In cylindrical coordinates

$$A_z(\bar{R}) = \int_{-h}^{h}\int_{-\pi}^{\pi} A_{zs}\,\frac{\partial G}{\partial r_o}\, a\, d\phi_o\, dz_o, \tag{6.5}$$

Because of the azimuthal symmetry in A_{zs} only the $m = 0$ terms in G will produce non-zero results upon termwise integration of (6.5) and

termwise differentiation of (6.3).[1] Therefore

$$A_z(\bar{R}) = \sum_{n=0}^{\infty} \frac{a}{\epsilon_n h} g_n'(r) \int_{-h}^{h} A_{zs} \cos \beta_n(z-z_o)\, dz_o. \qquad (6.6)$$

---

[1] Termwise integration appears to be justified because the series representing G is in the form of a Fourier series and Carslaw [11] has shown that termwise integration of a Fourier series is justified regardless of the convergence properties of the series.

Termwise differentiation of G is permissible since the derived series $\frac{\partial G}{\partial r_o}$ converges uniformly with respect to r_o. The convergence properties of $\frac{\partial G}{\partial r_o}$ are determined by the terms

$$T_{m,n} = \frac{\alpha_n' K_n(\alpha_n' r) \left[I_m'(\alpha_n' r_o) K_m(\alpha_n' a) + K_m'(\alpha_n' r_o) I_m(\alpha_n' a) \right]}{K_m(\alpha_n' a)}$$

Asymptotically

$$K_m(x) \approx \sqrt{\frac{\pi}{2x}}\, e^{-x}$$

$$I_m(x) \approx \frac{e^x}{\sqrt{2\pi x}} ,$$

and

$$\alpha_n' = \sqrt{(\tfrac{n\pi}{h})^2 - k^2} \approx \frac{n\pi}{h} .$$

$$T_{m,n} \approx \frac{1}{2\sqrt{r r_o}} \left[e^{-\alpha_n'(r-r_o)} + e^{-\alpha_n'(r+r_o-2a)} \right]$$

and

$$T_{m,n} \leq \frac{1}{a} e^{\frac{-n\pi}{h}(r-a)} .$$

This insures uniform convergence on $(r_o > a)$.

The cosine term can be expanded as

$$\cos \beta_n(z-z_o) = \cos \beta_n z \cos \beta_n z_o + \sin \beta_n z \sin \beta_n z_o .$$

A_{zs}, Equation (6.2), is an even function of z_o on $-h \leq z_o \leq h$.

Therefore the integrals of the sine terms are zero and

$$A_z(\bar{R}) = \sum_{n=0}^{\infty} \frac{a}{\epsilon_n h} g_n'(r) \cos \beta_n z \int_{-h}^{h} A_{zs} \cos \beta_n z_o dz_o . \qquad (6.7)$$

The Fourier series for A_{zs} on $-h < z_o < h$ is

$$A_{zs} = \sum_{n=0}^{\infty} D_n \cos \beta_n z_o \qquad (6.8)$$

where

$$D_n = \frac{1}{\epsilon_n h} \int_{-h}^{h} A_{zs} \cos \beta_n z_o dz_o . \qquad (6.9)$$

Inserting (6.9) in (6.7) gives

$$A_z(\bar{R}) = \sum_{n=0}^{\infty} a\, g_n'(r)\, D_n \cos \beta_n z. \qquad (6.10)$$

On differentiating Equation (6.4) one obtains[2]

$$g_n'(r) = \begin{cases} \dfrac{-j\pi\alpha_n H_o^{(2)}(\alpha_n r)\left[H_1^{(2)}(\alpha_n a) H_o^{(1)}(\alpha_n a) - H_1^{(1)}(\alpha_n a) H_o^{(2)}(\alpha_n a)\right]}{4H_o^{(2)}(\alpha_n a)} & \beta_n < k \\ \dfrac{\alpha_n' K_o(\alpha_n' r)\left[I_1(\alpha_n' a) K_o(\alpha_n' a) + K_1(\alpha_n' a) I_o(\alpha_n' a)\right]}{K_o(\alpha_n' a)} & \beta_n > k. \end{cases} \tag{6.11}$$

By Equations (4.10) and (4.13)

$$H_1^{(2)}(\alpha_n a) H_o^{(1)}(\alpha_n a) - H_1^{(1)}(\alpha_n a) H_o^{(2)}(\alpha_n a) = \frac{4j}{\pi\alpha_n a} \tag{6.12}$$

---

[2] The following relationships [30] are used in obtaining Equation (6.11):

$H_o^{(1)'}(x) = - H_1^{(1)}(x)$,

$H_o^{(2)'}(x) = - H_1^{(2)}(x)$,

$I_o'(x) = I_1(x)$, and

$K_o'(x) = - K_1(x)$.

and

$$I_1(\alpha_n' a)\, K_o(\alpha_n' a) + K_1(\alpha_n' a)\, I_o(\alpha_n' a) = \frac{1}{\alpha_n' a} . \tag{6.13}$$

Therefore

$$g_n'(r) = \begin{cases} \dfrac{H_o^{(2)}(\alpha_n r)}{a H_o^{(2)}(\alpha_n a)} & \beta_n < k \\ \dfrac{K_o(\alpha_n' r)}{a K_o(\alpha_n' a)} & \beta_n > k \end{cases} \tag{6.14}$$

At this point it is possible to check results because $A_z(\bar{R})$ should reduce to A_{zs} at $r = a$. From Equation (6.14) we have

$$g_n'(a) = \frac{1}{a} \qquad \beta_n \neq k \tag{6.15}$$

Consequently

$$A_z(\bar{R})\Big|_{r=a} = \sum_{n=0}^{\infty} D_n \cos \beta_n z \tag{6.16}$$

which is the Fourier series for A_{zs}.

6.3 EXPRESSIONS FOR THE CURRENT DISTRIBUTION AND THE DRIVING POINT IMPEDANCE

In Chapter 3 it is shown that for the cylindrical antenna the current distribution is given by Equation (3.10),

$$I(z) = \frac{-2\pi a}{\mu_o} \left.\frac{\partial A_z}{\partial r}\right|_{r=a} . \tag{6.17}$$

Equations (6.9) and (6.10) give

$$A_z(\bar{R}) = \sum_{n=0}^{\infty} a\, g_n'(r)\, D_n \cos \beta_n z_o \tag{6.18}$$

where

$$D_n = \frac{1}{\epsilon_n h} \int_{-h}^{h} A_{zs} \cos \beta_n z_o \, dz_o \tag{6.19}$$

and, Equation (6.2),

$$A_{zs} = \frac{-j}{c} \left[C_1 \cos kz_o + \tfrac{1}{2} V \sin k|z_o| \right]. \tag{6.20}$$

For convenience in the evaluation of the constant C_1 Equations (6.17), (6.18), and (6.19) may be written as follows: Let

$$D_n = \frac{-j}{c} \left[C_1 D_{cn} + \tfrac{1}{2} V D_{vn} \right] \tag{6.21}$$

where

$$D_{cn} = \frac{1}{\epsilon_n h} \int_{-h}^{h} \cos kz_o \cos \beta_n z_o \, dz_o \tag{6.22}$$

and

$$D_{vn} = \frac{1}{\epsilon_n h} \int_{-h}^{h} \sin k|z_o| \cos \beta_n z_o \, ds_o . \tag{6.23}$$

Let

$$A_z(\bar{R}) = \frac{-j}{c} \left[C_1 \, P(r,z) + \tfrac{1}{2} V \, Q(r,z) \right] \tag{6.24}$$

where

$$P(r,z) = \sum_{n=0}^{\infty} a \, g_n'(r) \, D_{cn} \cos \beta_n z_o \tag{6.25a}$$

and

$$Q(r,z) = \sum_{n=0}^{\infty} a \, g_n'(r) \, D_{vn} \cos \beta_n z_o . \tag{6.25b}$$

then

$$I(z) = \frac{j 2\pi}{c \mu_o} \left[C_1 \, P'(a,z) + \tfrac{1}{2} V \, Q'(a,z) \right] \tag{6.26}$$

where

$$P'(a,z) = a \left. \frac{\partial P(r,z)}{\partial r} \right|_{r=a} \tag{6.27a}$$

and

$$Q'(a,z) = a \left. \frac{\partial Q(r,z)}{\partial r} \right|_{r=a} . \tag{6.27b}$$

Since the current must vanish at the ends of the antenna

$$C_1 = - \tfrac{1}{2} V \frac{Q'(a,h)}{P'(a,h)} . \tag{6.28}$$

Therefore

$$I(z) = \frac{j\pi V}{R_s}\left[\frac{P'(a,h)\,Q'(a,z) - Q'(a,h)\,P'(a,z)}{P'(a,h)}\right] \tag{6.29}$$

where $R_s = c\mu_o \approx 377$ ohms.

Equation (6.29) is an explicit expression for the current distribution. From this expression one obtains the driving point impedance as

$$Z = \frac{V}{I(0)} = \frac{-jR_s}{\pi}\left[\frac{P'(a,h)}{P'(a,h)Q'(a,0) - Q'(a,h)\,P'(a,0)}\right] \tag{6.30}$$

6.4 EVALUATION OF THE DERIVATIVE OF AN INFINITE SERIES

The expressions for current and impedance each contain the terms $P'(a,z)$ and $Q'(a,z)$, both of which represent a derivative of an infinite series. Combining Equations (6.25a,b) and (6.27a,b) gives

$$P'(a,z) = \frac{\partial}{\partial r}\left[\sum_{n=0}^{\infty} a^2\, g_n'(r)\, D_{cn} \cos \beta_n z\right]\Bigg|_{r=a} \tag{6.31a}$$

and

$$Q'(a,z) = \frac{\partial}{\partial r}\left[\sum_{n=0}^{\infty} a^2\, g_n'(r)\, D_{vn} \cos \beta_n z\right]\Bigg|_{r=a} \tag{6.31b}$$

where $g_n'(r)$ is given by Equation (6.14) and D_{cn} and D_{vn} are given in Equation (6.22) and (6.23).

The question immediately arises of whether termwise differentiation is permissible in Equations(6.31a,b). It is permissible if the derived series converge uniformly. Differentiating termwise, we obtain

$$P'(r,z) = \sum_{n=0}^{\infty} T_{cn} \cos \beta_n z \qquad (6.32a)$$

and

$$Q'(r,z) = \sum_{n=0}^{\infty} T_{vn} \cos \beta_n z \qquad (6.32b)$$

where

$$T_{cn} = a^2 D_{cn} \frac{\partial g_n'(r)}{\partial r} \qquad (6.33a)$$

and

$$T_{vn} = a^2 D_{vn} \frac{\partial g_n'(r)}{\partial r} . \qquad (6.33b)$$

Differentiating Equation (6.14) gives

$$\frac{\partial g_n'(r)}{\partial r} = \begin{cases} \dfrac{-\alpha_n H_1^{(2)}(\alpha_n r)}{a H_o^{(2)}(\alpha_n a)} \\ \\ \dfrac{-\alpha_n' K_1(\alpha_n' r)}{a K_o(\alpha_n' a)} \end{cases}$$

The convergence of Equation (6.32a,b) is determined by the asymptoyic behavior of T_{cn} and T_{vn} for large n. D_{cn} and D_{vn} are given explicitly in Chapter 7, Equations (7.37a,b)

$$D_{cn} = \frac{2k(-1)^n \sin kh}{\epsilon_n h(k^2 - \beta_n^2)} \qquad (6.34a)$$

and

$$D_{vn} = \frac{2k\left[1-(-1)^n \cos kh\right]}{\epsilon_n h(k^2 - \beta_n^2)} \,. \qquad (6.34b)$$

Therefore, with $\epsilon_n = 1$ for $n > 0$,

$$\left|D_{cn}\right| \leq \frac{2k}{\beta_n^2 h}$$

and

$$\left|D_{vn}\right| \leq \frac{2k}{\alpha_n^2 h} \,.$$

The asymptotic behavior of the modified Hankel functions is [3], [30]

$$K_1(\alpha_n' r) \approx e^{-\alpha_n' r}\sqrt{\frac{\pi}{2\alpha_n' r}}$$

and

$$K_o(\alpha_n' a) \approx e^{-\alpha_n' a}\sqrt{\frac{\pi}{2\alpha_n' a}}$$

for large n. Therefore, with $a \leq r$,

$$\frac{K_1(\alpha_n' r)}{K_o(\alpha_n' a)} \approx \sqrt{\frac{a}{r}}\; e^{-\alpha_n'(r-a)} \leq \sqrt{\frac{a}{r}} \leq 1 \,.$$

Consequently

$$\left|\frac{\partial g_n'(r)}{\partial r}\right| \leq \frac{\alpha_n'}{a}$$

and

$$\left|T_{cn}\right| \leq \frac{2ka}{\alpha_n' h} \tag{6.35a}$$

and

$$\left|T_{vn}\right| \leq \frac{2ka}{\alpha_n' j} \quad . \tag{6.35b}$$

For large n

$$\alpha_n' = \sqrt{\beta_n^2 - k^2} \approx \frac{n\pi}{h} \quad .$$

Therefore $\left|T_{cn}\right|$ and $\left|T_{vn}\right|$ decrease as 1/n. A series of the form

$$\sum_{n=0}^{\infty} T_n \cos \beta_n z$$

converges on $0 < |z| < h$ if $\left|T_n\right|$ decreases as 1/n [11]. Since the inequalities (6.35a,b) are independent of r for $r \geq a$, the series in Equation (6.32a,b) converge uniformly with respect to r for $r \geq a$ and the termwise differentiation is justified.

To recapitulate, we have

$$P'(a,z) = \sum_{n=0}^{\infty} a^2 \frac{\partial g_n'(a)}{\partial r} D_{cn} \cos \beta_n z \qquad (6.36a)$$

and

$$Q'(a,z) = \sum_{n=0}^{\infty} a^2 \frac{\partial g_n'(a)}{\partial r} D_{vn} \cos \beta_n z \qquad (6.36b)$$

which converge on $0 < |z| < h$. However, values of $P'(a,0)$, $Q'(a,0)$, $P'(a,h)$, and $Q'(a,h)$ are needed to compute the current and impedance. Let us examine the convergence of (6.36a,b) at the ends of the interval $0 < |z| < h$, i. e., $z = 0$ and $|z| = h$. For $z = 0$, $\cos \beta_n z = 1$. The terms of the series for $P'(a,0)$ alternate in sign because of the $(-1)^n$ term in D_{cn}, Equation (6.34a). Therefore the series for $P'(a,0)$ is an alternating series converging as $1/n$. However, the terms for $Q'(a,0)$ do not alternate, see Equation (6.34b). Since the terms decrease approximately as $1/n$, this series may diverge as the harmonic series.

For $|z| = h$, $\cos \beta_n z = (-1)^n$. In this case, following an argument similar to that above, one finds that $Q'(a,h)$ is an alternating series converging approximately as $1/n$ and, therefore, $P'(a,h)$ may diverge as the harmonic series.

Because of these apparent convergence problems, care must be exercised in using Equations (6.36a,b) for computation of the current and impedance. However, Equation (6.31a,b) can be used provided

numerical differentiation is performed on the series in square brackets, without differentiating termwise. The terms in square brackets are

$$a\,P(r,z) = \sum_{n=0}^{\infty} a^2\, g_n'(r)\, D_{cn} \cos \beta_n z_o \qquad (6.37a)$$

and

$$a\,Q(r,z) = \sum_{n=0}^{\infty} a^2\, g_n'(r)\, D_{vn} \cos \beta_n z_o. \qquad (6.37b)$$

As shown above,

$$\left|D_{cn}\right| \leq \frac{2k}{\alpha_n{}^2 h}$$

and

$$\left|D_{vn}\right| \leq \frac{2k}{\alpha_n{}^2 h} .$$

Asymptotically

$$g_n'(r) = \frac{K_o(\alpha_n' r)}{K_o(\alpha_n' a)} \approx \sqrt{\frac{a}{r}}\; e^{-\alpha_n'(r-a)} \leq 1$$

for $r \geq a$. Consequently, the terms of the series in (6.37a,b) decrease as $1/n^2$ and the series converge uniformly with respect to z on $0 \leq |z| \leq h$. Since these series converge at the end-points of the

interval, they may be used for calculations of current and impedance.

Numerical differentiation techniques which may be used to evaluate Equations (6.31a,b) are presented in Chapter 7.

In spite of the apparent convergence problem with the series obtained by termwise differentiation, numerical results obtained from this series agree completely with the results obtained by numerical differentiation. Thus the estimates of convergence given above appear to be too stringent and, therefore, Equations (6.36a,b) are useful for calculations of current and impedance.

The terms of the series in Equation (6.36a,b) for which $\beta_n = k$ require special attention. In Chapter 7 it is shown that D_{cn} and D_{vn} remain finite for $\beta_n = k$. The Hankel functions in $\frac{\partial g_n'(a)}{\partial r}$ become singular for $\beta_n = k$, but it is shown in Chapter 5 that the expressions for $\frac{\partial g_n'(a)}{\partial r}$ approach a limit of zero as $\beta_n \to k$. Therefore the terms for which $\beta_n = k$ are zero.

CHAPTER 7

MATHEMATICAL AND NUMERICAL ANALYSIS

7.1 INTRODUCTION

This Chapter contains mathematical details and numerical techniques needed in carrying out the analysis described in the thesis. Topics discussed in this Chapter are: (1) A Fourier series for the free space Green's function and techniques for evaluating the Fourier coefficients, P_n. (2) Techniques for evaluating the derivative of the Fourier coefficients, $\frac{\partial P_n}{\partial r}$. (3) A Fourier series for the magnetic vector potential boundary value at the antenna surface. (4) Numerical derivative techniques.

7.2 A FOURIER SERIES FOR THE FREE SPACE GREEN'S FUNCTION

Consider the expansion

$$\frac{e^{-jkR}}{R} = \sum_{n=0}^{\infty} P_n^{(1)} \cos \beta_n z_o + \sum_{n=1}^{\infty} P_n^{(2)} \sin \beta_n z_o$$

where

$$\beta_n = \frac{n\pi}{h},$$

and

$$R = \sqrt{r^2 + (z-z_o)^2} \ .$$

By invoking the orthogonality of the trigonometric functions one obtains

$$P_n^{(1)} = \frac{1}{\epsilon_n h} \int_{-h}^{h} \frac{e^{-jkR}}{R} \cos \beta_n z_o \, dz_o \qquad (7.1)$$

where

$$\epsilon_n = \begin{cases} 2 \text{ if } b = 0 \\ 1 \text{ otherwise.} \end{cases}$$

The $P_n^{(2)}$ coefficients will not be dealt with in detail since numerical values are not needed for them in the thesis.

Equation (7.1) may be written in terms of integrals of the form[1]

---

[1] That is

$$P_n^{(1)} = \frac{1}{\epsilon_n h} \left\{ \frac{e^{j\beta_n z}}{2} \int_0^{k(h+z)} \Psi(x)dx - \frac{e^{-j\beta_n z}}{2} \int_0^{-k(h+z)} \Psi(x)dx \right.$$

$$\left. - \frac{e^{j\beta_n z}}{2} \int_0^{-k(h-z)} \Psi(x)dx + \frac{e^{-j\beta_n z}}{2} \int_0^{k(h-z)} \Psi(x)dx \right\}$$

where

$$\Psi(x) = \frac{\exp\left\{-j\left[\sqrt{x^2 + b^2} + \gamma_n x\right]\right\}}{\sqrt{x^2 + b^2}} .$$

$$I_n = \int_0^C \frac{\exp\{-j[\sqrt{x^2 + b^2} + \gamma_n x]\}}{\sqrt{x^2 + b^2}}\, dx \qquad (7.2)$$

where the dimensionless variables are

$x = k(z-z_o)$,

$b = kr$,

and

$\gamma_n = \beta_n/k$.

By changing variables twice in Equation (7.2), first

$x = b \sinh \alpha$, $(b > 0)$

$dx = b \cosh \alpha \, d\alpha$,

$\sqrt{x^2 + b^2} = b \cosh \alpha$,

and then $t = e^{\alpha}$,

$d\alpha = \frac{dt}{t}$,

$p = 1 + \gamma_n$,

$q = 1 - \gamma_n$, (see footnote 2)

---

[2] The analysis presented is valid for $0 < \gamma_n < 1$. For $\gamma_n > 1$, define $q = \gamma_{n-1}$ so that $q > 0$. The intermediate results differ in that the Bessel functions are replaced by modified Bessel functions. However, the end result, Equation (7.8) is the same for all γ_n.

one obtains

$$I_n = \int_1^{t_c} \exp\left\{ -j \frac{b}{2} \left[pt + q/t \right] \right\} \frac{dt}{t} \qquad (7.3)$$

where

$$t_c = \exp\left\{ \sinh^{-1} \left(\frac{c}{b} \right) \right\} = \exp\left\{ \ln \left[\frac{c}{b} + \sqrt{\left(\frac{c}{b}\right)^2 + 1} \right] \right\}$$

$$= \frac{c}{b} + \sqrt{\left(\frac{c}{b}\right)^2 + 1} \, .$$

Rewriting the integrand in Equation (7.3) gives

$$I_n = \int_1^{t_c} \exp\left\{ -\frac{b}{2} \sqrt{pq} \left[j\sqrt{\frac{p}{q}}\, t - \frac{1}{j\sqrt{\frac{p}{q}}\, t} \right] \right\} \frac{dt}{t} \, .$$

Then, with a further change of variables

$$g = b\sqrt{pq},$$

$$s = \sqrt{\frac{p}{q}}\, t, \quad (0 < s < \infty)$$

one obtains

$$I_n = \int_{\sqrt{\frac{p}{q}}}^{\sqrt{\frac{p}{q}}\, t_c} \frac{e^{\frac{g}{2}\left[js - \frac{1}{js} \right]}}{s} \, ds \, . \qquad (7.4)$$

The integrand of Equation (7.4) contains the generating function for Bessel functions, $f = e^{\frac{\xi}{2}\left[\tau - \frac{1}{\tau}\right]}$, which has the Laurent expansion [30], [52], [69]

$$e^{\frac{\xi}{2}\left[\tau - \frac{1}{\tau}\right]} = J_o(\xi) + \sum_{\nu=1}^{\infty} \left[\tau^{\nu} + (-\tau)^{-\nu}\right] J_{\nu}(\xi)$$

where $J_{\nu}(\xi)$ are Bessel functions of the first kind. Let

$$\xi = -g,$$

$$\tau = js.$$

Then

$$f = J_o(g) + \sum_{\nu=1}^{\infty} \left[(js)^{\nu} + (-js)^{-\nu}\right] (-1)^{\nu} J_{\nu}(g),$$

or

$$f = J_o(g) + \sum_{\nu=1}^{\infty} (-j)^{\nu}\left[s^{\nu} + s^{-\nu}\right] J_{\nu}(g).$$

f has an essential singularity at $s = 0$ ($\tau = 0$). f is analytic on the open interval $0 < s < \infty$. Since the limits of integration lie within this open interval the Laurent series may be integrated term by term. This will still be true if f is written in terms of the variable $t = \sqrt{\frac{q}{p}}\, s$,

$$f = J_o\left(b\sqrt{pq}\right) + \sum_{\nu=1}^{\infty} (-j)^{\nu}\left[\left(\sqrt{\frac{p}{q}}\, t\right)^{\nu} + \left(\sqrt{\frac{p}{q}}\, t\right)^{-\nu}\right] J_{\nu}\left(b\sqrt{pq}\right). \qquad (7.5)$$

Bessel functions may be written in the power series [48], [69]

$$J_\nu(b\sqrt{pq}) = (\frac{b\sqrt{pq}}{2})^\nu \sum_{r=0}^{\infty} \frac{(-\frac{pqb^2}{4})^r}{r!\,(r+\nu)!} .$$

Now $\xi^2 = pqb^2$

and let

$$F_\nu(\xi) = \sum_{r=0}^{\infty} \frac{(-\frac{\xi^2}{4})^r}{r!\,(r+\nu)!} . \qquad (7.6)$$

Then,

$$J_\nu\,(b\sqrt{pq}) = (b\frac{\sqrt{pq}}{2})^\nu\, F_\nu(\xi) . \qquad (7.7)$$

Inserting this expression (Equation 7.7) into the Equation (7.5) gives

$$f = F_o(\xi) + \sum_{\nu=1}^{\infty} (\frac{jb}{2})^\nu \left[p^\nu t^\nu + q^\nu t^{-\nu}\right] F_\nu(\xi) .$$

Then,

$$I_n = \int_1^{t_c} \left\{ F_o(\xi) + \sum_{\nu=1}^{\infty} (\frac{jb}{2})^\nu \left[p^\nu t^\nu + q^\nu t^{-\nu}\right] F_\nu(\xi) \right\} \frac{dt}{t}$$

or, integrating termwise,

$$I_n = \left[F_o(\xi)\,\ln(t)\Big|_1^{t_c} + \sum_{\nu=1}^{\infty} \frac{\left(\frac{-jb}{2}\right)^{\nu}\left[p^{\nu}t^{\nu} + q^{\nu}t^{-\nu}\right]}{\nu}\Bigg|_1^{t_c} F_{\nu}(\xi)\right] \qquad (7.8)$$

where

$$t_c = \sqrt{\left(\frac{c}{b}\right)^2 + 1} + \frac{c}{b}\,,$$

$$\xi^2 = pq\,b^2 = (1 - \gamma_n^{\,2})\,b^2\,,$$

and $F_{\nu}(\xi)$ is given by Equation (7.6).[3]

---

[3] In terms of the variables in Equation (7.2)

$$I_n = \left\{ F_o\left[\sqrt{(1-\gamma_n^{\,2})}b\right]\,\ln\left(\sqrt{\left(\frac{x}{b}\right)^2 + 1} + \frac{x}{b}\right)\right.$$

$$+ \sum_{\nu=1}^{\infty} \frac{\left(\frac{-jb}{2}\right)^{\nu}\left[(1+\gamma_n)^{\nu}\left(\sqrt{\left(\frac{x}{b}\right)^2 + 1} + \frac{x}{b}\right)^{\nu} - (1-\gamma_n)^{\nu}\left(\sqrt{\left(\frac{x}{b}\right)^2 + 1} + \frac{x}{b}\right)^{-\nu}\right]}{\nu}$$

$$\left. F_{\nu}\left[\sqrt{(1-\gamma_n^{\,2})}\,b\right]\right\}_{x=0}^{c}$$

This expression converges for all t_c since $t_c > 0$ for all finite values of C, thereby confining the integration to regions where the integrand is analytic.[4] This, then, is an exact expression for the integral defined in Equation (7.2).

A power series, $F(x) = \sum_{n=0}^{\infty} a_n x^n$, although it may converge for all x, may not be useful for numerical computations when x is large. When this is the case an asymptotic series, $F(x) = \sum_{n=0}^{N} b_n/x^n$, may be used for large x. N is chosen such that an acceptable error is obtained. A suitable value of x must be selected for making the transition from use of the power series to use of the asymptotic series.

In the power series for I_n, Equation (7.8), $(\frac{bpt_c}{2})^2$ are the dominant terms for positive C. For negative C the dominant terms are $(\frac{bq}{2t_c})^2$. Now

$$\frac{bpt_c}{2} = \frac{1}{2}(1 + \gamma_n)(\sqrt{C^2 + b^2} + C) \approx (\gamma_n + 1)\,C \text{ for } C >> b.$$

Similarly for negative $C = -C'$ $(C' > 0)$,

$$\frac{bq}{2t_c} = \frac{(1 - \gamma_n)b^2}{2[\sqrt{C'^2 + b^2} - C']} \approx (1 - \gamma_n)\,C' = (\gamma_n - 1)\,C \text{ for } |C| >> b.$$

---

[4] It would appear that the change of variable $s = \sqrt{p/q}\,t$ might make this analysis incorrect when $\gamma_n = 1$, i.e., $q = 1 - \gamma_n = 0$. When $\gamma_n = 1$, I_n may be expressed in terms of generalized sine- and cosine-integrals [36]. It can be shown that, for $\gamma_n = 1$, Equation (7.8) is the power series for the generalized sine- and cosine-integrals that represent I_n. So Equation (7.8) is valid for all γ_n.

For convenience, let $G_n = \gamma_n + 1$ if $C > 0$ and $G_n = \gamma_n - 1$ if $C < 0$. For rapid convergence (to eliminate the need for using a large number of significant figures in computations with digital computers) one requires $|G_n C|$ to be not too large, say $|G_n C| < K$, where K is determined by computer capabilities. Therefore, it will be desirable to have an asymptotic series for I_n to insure having a convenient means of computation regardless of the magnitudes of γ_n and C.

An asymptotic series for Equation (7.2) may be obtained using the principle of stationary phase [16]. Write Equation (7.2) in the form

$$I_n = \int_0^C e^{-jg(x)} f(x)\, dx$$

where

$$g(x) = \sqrt{x^2 + b^2} + \gamma_n x \qquad \text{and} \qquad f(x) = \frac{1}{\sqrt{x^2 + b^2}} .$$

$$g'(x) = \frac{x}{\sqrt{x^2 + b^2}} + \gamma_n$$

does not vanish for $0 \leq x \leq C$ and $\gamma_n > 1$. x and γ_n are both real.

Since $g'(x)$ does not vanish on $0 \leq x \leq C$ with $\gamma_n > 1$ there is no stationary point on this interval and integration by parts is permitted [16].

$$I_n = \frac{e^{-jg(x)} f(x)}{(-j)\, g'(x)} \Bigg|_0^C - \frac{e^{-jg(x)}}{(-j)^2\, g'(x)} \left(\frac{f(x)}{g'(x)}\right)' \Bigg|_0^C + \cdots$$

Let

$$\Phi_0(x) = \frac{f(x)}{g'(x)}$$

$$\Phi_1(x) = \frac{1}{g'(x)} \left(\frac{f(x)}{g'(x)}\right)' = \frac{1}{g'(x)} \Phi_0'(x)$$

$$\Phi_p(x) = \frac{1}{g'(x)} \Phi_{p-1}'(x)$$

Then

$$I_n = e^{-jg(x)} \sum_{p=0}^{P} \frac{(-1)^p \Phi_p(x)}{(-j)^{p+1}} \Bigg|_0^C + (-j)^{P+1} \int_0^C e^{-jg(x)} \Phi_P'(x)\, dx \tag{7.9}$$

The remainder after the P^{th} term is

$$R = (-j)^{P+1} \int_0^C e^{-jg(x)} \Phi_P'(x)\, dx$$

Since $g(x)$ and $\Phi_P(x)$ are real functions of x,

$$|R| \leq \left| \int_0^C \Phi_P'(x)\, dx \right| = |\Phi_P(x)|$$

Therefore the remainder after the P^{th} term is less than the magnitude of P^{th} term.

The first four terms of the asymptotic series give

$$I_n = je^{-j\left[\sqrt{x^2+b^2}+\gamma_n x\right]}\left\{\left[\frac{1}{x+\gamma_n\sqrt{x^2+b^2}}\right]+\left[\frac{j(\sqrt{x^2+b^2}+\gamma_n x)}{(x+\gamma_n\sqrt{x^2+b^2})}\right]\right.$$

$$+\left[\frac{1}{(x+\gamma_n\sqrt{x^2+b^2})^3}-\frac{3(\sqrt{x^2+b^2}+\gamma_n x)^2}{(x+\gamma_n\sqrt{x^2+b^2})^5}\right]$$

$$\left.+\left[\frac{j9(\sqrt{x^2+b^2}+\gamma_n x)}{(x+\gamma_n\sqrt{x^2+b^2})^5}-\frac{j\,5(\sqrt{x^2+b^2}+\gamma_n x)^3}{(x+\gamma_n\sqrt{x^2+b^2})^7}\right]\right\}\Bigg|_0^C$$

For $|C| >> b$ the lower limit, $x = 0$, will be dominant in estimating the remainder. In order to have a small remainder it is required that $|\gamma_n b| >> 1$.[5]

Consider the criteria for using the power series or the asymptotic series to compute I_n. As before, let $G_n = \gamma_n + 1$ if $C > 0$ and $G_n = \gamma_n - 1$ if $C < 0$. It is undesirable to use the power series if

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[5] Suppose $|\gamma_n b| = 10$ and $|C| >> b$, then the remainder after the fourth term is of the order

$$|R| \leq \left|\frac{9b}{(\gamma_n b)^5}-\frac{15b^3}{(\gamma_n b)^7}\right| < \frac{9}{10^5} < 10^{-4}.$$

It is assumed that $b < 1$, as it is for most cylindrical antennas. When the lower limit of integration in the asymptotic series is not zero, the estimate of the remainder may depend upon both limits. But, if the lower limit, say B, is large enough that $|G_n B| \geq 10$, the remainder must certainly be of the order shown above.

$|G_n C| > K$. The asymptotic series requires $|\gamma_n b| >> 1$. The criteria for using the power series or the asymptotic series depends upon three parameters, γ_n, C, and b. The two requirements $|G_n C| \leq K$ and $|\gamma_n b| >> 1$ are not compatible because parameters γ_n, C, and b may occur such that neither requirement is satisfied.

To remedy this problem, let us split the interval of integration as follows:

$$I_n = \int_0^B \frac{e^{-j\left[\sqrt{x^2+b^2}+\gamma_n x\right]}}{\sqrt{x^2+b^2}}\,dx + \int_B^C \frac{e^{-j\left[\sqrt{x^2+b^2}+\gamma_n x\right]}}{\sqrt{x^2+b^2}}\,dx \tag{7.10}$$

where $0<|B|<|C|$. The first integral on the right in Equation (7.10) will be represented by the power series, Equation (7.8), and the second integral by the asymptotic series, Equation (7.9). The requirements on B are, for the power series $|G_n B| < K$, and for the asymptotic series $|G_n B| >> 1$. (See footnote 5.) These two requirements are compatible if a reasonably large K, say K=10, can be tolerated. In this case $|\gamma_n b| > K$ is a suitable criterion for using the asymptotic series above. The transition from power series to asymptotic series may be summarized in three steps: (1) If $|G_n C| < K$ the power series alone may be used. (2) If $|G_n C| > K$ and $|\gamma_n b|$ is not sufficiently large to use the asymptotic series alone, choose B such that $|G_n B| = K$, i. e., $|B| = \frac{K}{|G_n|}$, and B has the same sign as C, and then use both the power series and asymptotic series as described above. (3) If $|\gamma_n b| >> 1$, say $|\gamma_n b| > K$, the asymptotic series may be used alone. (See footnote 5.)

Figure 7.1 illustrates the regions where the power series, the asymptotic series, or both may be used.

An expression for $P_n^{(1)}$, Equation (7.2), in terms of Hankel functions can be obtained. Write (see footnote 1, page 79)

$$P_n^{(1)} = \frac{1}{\epsilon_n h} \left\{ \frac{e^{j\beta_n z}}{2} \int_{-k(h-z)}^{k(h+z)} \Psi(x)dx + \frac{e^{-j\beta_n z}}{2} \int_{-k(h+z)}^{k(h-z)} \Psi(x)dx \right\} \tag{7.11}$$

where

$$\Psi(x) = \frac{\exp\{-j(\sqrt{x^2 + b^2} + \gamma_n x)\}}{\sqrt{x^2 + b^2}}.$$

Let

$$G_n(A,B) = \int_{-B}^{A} \Psi(x)dx. \tag{7.12}$$

Then

$$P_n^{(1)} = \frac{1}{\epsilon_n h} \left\{ \frac{e^{j\beta_n z}}{2} G_n[k(h+z), k(h-z)] + \frac{e^{-j\beta_n z}}{2} G_n[k(h-z), k(h+z)] \right\}. \tag{7.13}$$

With the two changes of variables used previously,

$x = b \sinh \alpha \qquad (b > 0)$

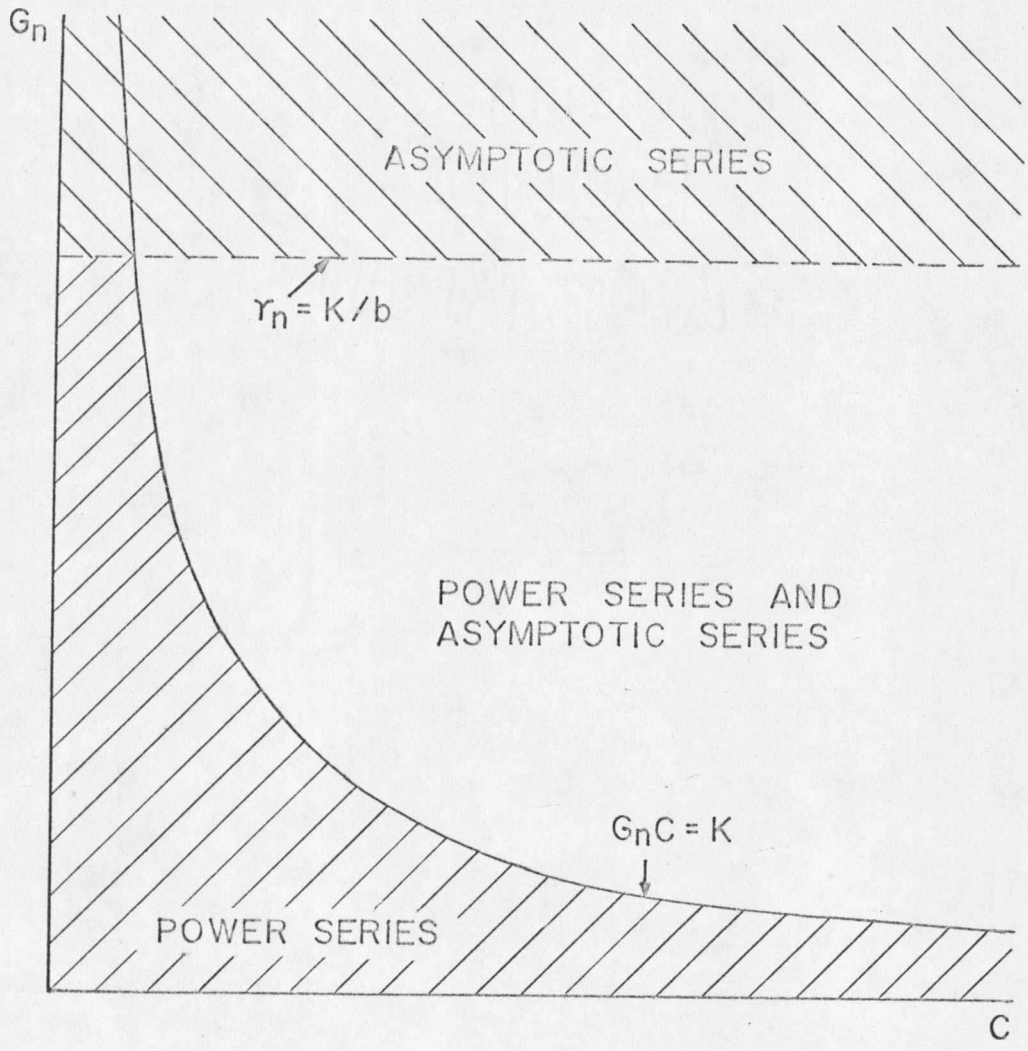


Fig. 7.1 Regions for use of power series, asymptotic series, or both.

and

$$t = e^{\alpha},$$

$$p = 1 + \gamma_n,$$

$$q = 1 - \gamma_n,$$

one obtains

$$G_n(A,B) = \int_{t_B}^{t_A} e^{-j\frac{b}{2}[pt + q/t]} \cdot \frac{dt}{t} \qquad (7.14)$$

where

$$t_A = \sqrt{(\frac{A}{b})^2 + 1} + \frac{A}{b}$$

and

$$t_B = \sqrt{(\frac{B}{b})^2 + 1} - \frac{B}{b} .$$

A and B are both positive. Therefore, $0 < t_B \leq 1$ and $1 \leq t_A < \infty$.

Watson shows that [69]

$$\int_0^{\infty} e^{-j[au + b/u]} \frac{du}{u} = -j\pi H_o^{(2)}(2\sqrt{ab}) .$$

Therefore, assuming $\gamma_n < 1$, i. e., $pq > 0$, write

$$G_n(A,B) = \begin{cases} G_n^{(1)}(A,B) & \gamma_n < 1 \\ G_n^{(2)}(A,B) & \gamma_n = 1 \\ G_n^{(3)}(A,B) & \gamma_n > 1 \end{cases} \tag{7.15}$$

and

$$G_n^{(1)}(A,B) = -j\pi H_o^{(2)}(b\sqrt{pq}) - \int_0^{t_B} e^{-j\frac{b}{2}[pt + q/t]} \frac{dt}{t}$$

$$- \int_{t_A}^{\infty} e^{-j\frac{b}{2}[pt + q/t]} \frac{dt}{t} .$$

With the change of variable

$$v = \frac{1}{t}$$

we obtain

$$\int_0^{t_B} e^{-j\frac{b}{2}[pt + q/t]} \frac{dt}{t} = \int_{1/t_B}^{\infty} e^{-j\frac{b}{2}[qv + p/v]} \frac{dv}{v} .$$

If one imposes the restriction that $|z| < h-10a$, that is, only field points more than ten radii from the ends of the antenna will be considered, then the approximation can be made that

$$\int_{t_A}^{\infty} e^{-j\frac{b}{2}[pt + q/t]} \frac{dt}{t} \approx \int_{t_A}^{\infty} \frac{e^{-j\frac{b}{2}pt}}{t} dt = -Ci\left(\frac{bpt_A}{2}\right) - j\left[\frac{\pi}{2} - Si\left(\frac{bpt_A}{2}\right)\right].$$

Similarly

$$\int_{1/t_B}^{\infty} e^{-j\frac{b}{2}[qv + p/v]} \frac{dt}{t} \approx -Ci\left(\frac{bq}{2t_B}\right) - j\left[\frac{\pi}{2} - Si\left(\frac{bq}{2t_B}\right)\right].$$

Then

$$G_n^{(1)}(A,B) \approx -j\pi H_o^{(2)}(b\sqrt{pq}) + Ci\left(\frac{bpt_A}{2}\right) + Ci\left(\frac{bq}{2t_B}\right)$$

$$-j\left[\pi - Si\left(\frac{bpt_A}{2}\right) - Si\left(\frac{bq}{2t_B}\right)\right]. \qquad (7.16)$$

For $\gamma_n > 1$ we obtain

$$G_n^{(3)}(A,B) \approx 2K_o(b\sqrt{pg'}) + Ci\left(\frac{bpt_A}{2}\right) + Ci\left(\frac{bq'}{2t_B}\right)$$

$$-j\left[Si\left(\frac{bpt_A}{2}\right) - Si\left(\frac{bq'}{2t_B}\right)\right] \qquad (7.17)$$

where $p = \gamma_n + 1$ and $q' = \gamma_n - 1$.

For $\gamma_n = 1$, $p = 2$ and $q = q' = 0$. Therefore

$$G_n^{(2)}(A,B) = \int_{t_B}^{t_A} e^{-jbt}\ \frac{dt}{t} = Ci(bt_A) - Ci(bt_B) - j[Si(bt_A) - Si(bt_B)]. \tag{7.18}$$

Equation (7.15) together with Equations (7.16), (7.17), and (7.18) give $G_n(A,B)$ for all γ_n (i. e., all β_n since $\gamma_n = \beta_n/k$) and for $|z|$ not close to the ends of the antenna.

A useful approximation for $\gamma_n \neq k$ is

$$G_n(A,B) \approx \begin{cases} -j\pi H_o^{(2)}(\alpha_n r) & \beta_n < k \\ 2K_o(\alpha_n' r) & \beta_n > k \end{cases} \tag{7.19}$$

where

$$\alpha_n = \sqrt{k^2 - \beta_n^2}$$

and

$$\alpha_n' = \sqrt{\beta_n^2 - k^2}\ .$$

An estimate of the behavior of $P_n^{(1)}$ for large n can be obtained by integrating (7.2) by parts. We have

$$\epsilon_n h P_n^{(1)} = g(z_o) \frac{\sin \beta_n z_o}{\beta_n} \Bigg|_{-h}^{h} - \frac{1}{\beta_n} \int_{-h}^{h} g'(z_o) \sin \beta_n z_o dz_o \qquad (7.20)$$

where

$$g(z_o) = \frac{e^{-jkR}}{R} .$$

But $\sin (\pm \beta_n h) = \sin (\pm n\pi) = 0$, therefore the first term on the right hand side of (7.20) is zero. Integrating by parts again

$$\epsilon_n h P_n^{(1)} = g'(z_o) \frac{\cos \beta_n z_o}{\beta_n^2} \Bigg|_{-h}^{h} - \frac{1}{\beta_n^2} \int_{-h}^{h} g'' (z_o) \cos \beta_n z_o dz_o . \qquad (7.21)$$

The integrand of (7.21) has an upper bound independent of n determined by the inequality

$$\left| \int_{-h}^{h} g'' (z_o) \cos \beta_n z_o dz_o \right| \leq \left| \int_{-h}^{h} \left| g'' (z_o) \right| dz_o \right| .$$

Therefore

$$\left| \epsilon_n h P_n^{(1)} \right| \leq \frac{1}{\beta_n^2} \left\{ \left| g'(z_o) \Big|_{-h}^{h} \right| - \left| \int_{-h}^{h} \left| g'' (z_o) \right| dz_o \right| \right\}$$

and $P_n^{(1)}$ decreases as $1/n^2$ for large n.

7.3 THE DERIVATIVE OF THE FOURIER COEFFICIENTS

Evaluation of the current distribution and driving point impedance in Chapter 5 requires the evaluation of

$$A_n^{(1)'}(a,z) = \frac{1}{4\pi g_n(a)} \; P_n^{(1)'}(r,z) \Big|_{r=a}$$

where

$$P_n^{(1)'}(r,z) = \frac{1}{\epsilon_n h} \int_{-h}^{h} \frac{\partial g(r,z)}{\partial r} \cos \beta_n z_o \, dz_o \qquad (7.22)$$

and

$$g(r,z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-jkR_a(\phi)}}{R_a(\phi)} \, d\phi_o \approx \frac{e^{-jkR_a}}{R_a}$$

with

$$R_a(\phi) = \sqrt{r^2 + a^2 - 2ar \cos(\phi - \phi_o) + (z - z_o)^2}$$

and

$$R_a = \sqrt{r^2 + (z - z_o)^2} \, .$$

From the approximation, Equation (7.19),

$$P_n^{(1)} \approx \frac{1}{\epsilon_n h} \begin{cases} -j\pi H_o^{(2)}(\alpha_n r) & \beta_n < k \\ 2K_o(\alpha_n' r) & \beta_n > k \end{cases}$$

one obtains

$$P_n^{(1)'}(a,z) \approx \frac{1}{\epsilon_n h} \begin{cases} j\pi\alpha_n H_1^{(2)}(\alpha_n a) & \beta_n < k \\ -2\alpha_n' K_1(\alpha_n' a) & \beta_n > k . \end{cases}$$

On the other hand, we have the approximate expression for $g(r,z)$

$$P_n^{(1)}(r,z) \approx \frac{1}{\epsilon\pi\epsilon_n h} \int_{-h}^{h} \frac{e^{-jkR_a}}{R_a} \cos\beta_n z_o \, dz_o$$

Let $\xi = (z-z_o)$ and $dz_o = -d\xi$. Then

$$P_n^{(1)}(r,z) = \frac{1}{4\pi\epsilon_n h} \int_{z-h}^{z+h} \frac{e^{-jk\sqrt{r^2+\xi^2}}}{\sqrt{r^2+\xi^2}} \cos\beta_n(\xi-z)\, d\xi$$

$$= \frac{1}{4\pi\epsilon_n h} \int_{0}^{h+z} \frac{e^{-jk\sqrt{r^2+\xi^2}}}{\sqrt{r^2+\xi^2}} \cos\beta_n(\xi-z)\, d\xi$$

$$+ \frac{1}{4\pi\epsilon_n h} \int_{0}^{h-z} \frac{e^{-jk\sqrt{r^2+\xi^2}}}{\sqrt{r^2+\xi^2}} \cos\beta_n(\xi+z)\, d\xi .$$

Just as $P_n^{(1)}$ may be written in terms of the integral given in Equation (7.3), so $P_n^{(1)'}(r,z)$ may be written in terms of [6]

$$I_n'(r,c) = \int_0^{C'} e^{-j\left[k\sqrt{r^2+\xi^2}+\beta_n\xi\right]} \left\{ \frac{-jkr}{r^2+\xi^2} - \frac{r}{\left[r^2+\xi^2\right]^{\frac{3}{2}}} \right\} d\xi. \tag{7.23}$$

Let

$x = k\xi$,

$b = kr$,

$C = kC'$,

and

$\gamma_n = \beta_n/k$.

---

[6] That is,

$$P_n^{(1)}(r,z) = \frac{1}{\epsilon_n h} \left\{ \frac{e^{j\beta_n z}}{2} \int_0^{k(h+z)} \Psi(x)dx - \frac{e^{-j\beta_n z}}{2} \int_0^{-k(h+z)} \Psi(x)dx \right.$$

$$\left. - \frac{e^{j\beta_n z}}{2} \int_0^{-k(h-z)} \Psi(x)dx + \frac{e^{-j\beta_n z}}{2} \int_0^{k(h-z)} \Psi(x)dx \right\}$$

where

$$\Psi(x) = \exp\left\{-j\left[\sqrt{x^2+b^2}+\gamma_n x\right]\right\} \left(\frac{j}{x^2+b^2} + \frac{1}{\left[x^2+b^2\right]^{\frac{3}{2}}} \right).$$

Then

$$I_n'(b,C) = -k\int_0^{\alpha_c} e^{-jb[\cosh\alpha + \gamma_n \sinh\alpha]} \left\{ \frac{j}{\cosh\alpha} + \frac{1}{b\cosh^2\alpha} \right\} d\alpha .$$

With a further change of variables

$t = e^{\alpha}$,

$d\alpha = dt/t$,

$p = 1 + \gamma_n$,

$q = 1 - \gamma_n$,

$t_c = \sqrt{(\frac{C}{b})^2 + 1} + \frac{C}{b}$

one obtains, since $p + q = 2$,

$$I_n'(b,C) = -k\int_1^{t_C} e^{-j\frac{b}{2}[pt + \frac{q}{t}]} \left\{ \frac{j(p+q)}{t^2+1} + \frac{4t}{b(t^2+1)^2} \right\} dt \qquad (7.24)$$

Integration by parts gives

$$\int_0^{t_C} e^{-j\frac{b}{2}[pt + \frac{q}{t}]} \frac{jp}{t^2+1}\, dt = -\frac{2e^{-j\frac{b}{2}[pt + \frac{q}{t}]}}{b(t^2+1)} \Bigg|_1^{t_C}$$

$$+ \int_1^{t_C} e^{-j\frac{b}{2}[pt + \frac{q}{t}]} \left\{ \frac{jq}{t^2(t^2+1)} - \frac{4t}{b(t^2+1)^2} \right\} dt . \qquad (7.25)$$

Insertion of (7.25) into (7.24) gives

$$I_n'(b,C) = \frac{2ke^{-j\frac{b}{2}\left[pt + \frac{q}{t}\right]}}{b(t^2+1)} \Bigg|_1^{t_C} - jqk \int_1^{t_C} e^{-j\frac{b}{2}\left[pt + \frac{q}{t}\right]} \frac{dt}{t^2} . \qquad (7.26)$$

If $\gamma_n = 1$ (i. e., $\beta_n = k$), then $q = 0$ and

$$I_n'(b,C) = \frac{2ke^{-jbt}}{b(t^2+1)} \Bigg|_1^{t_C} = \frac{ae^{-jk\left[\sqrt{r^2+\xi^2}+\xi\right]}}{\sqrt{r^2+\xi^2}\left[\xi+\sqrt{\xi^2+r^2}\right]} \Bigg|_1^{C'} . \qquad (7.27)$$

Integrating (7.26) by parts gives

$$I_n'(b,C) = \frac{2ke^{-j\frac{b}{2}\left[pt + \frac{q}{t}\right]}}{b(t^2+1)} \Bigg|_1^{t_C} - \frac{2k}{b} e^{-j\frac{b}{2}\left[pt + \frac{q}{t}\right]} \Bigg|_0^{t_C}$$

$$-jpk \int_1^{t_C} e^{-j\frac{b}{2}\left[pt + \frac{q}{t}\right]} dt$$

$$= \frac{-2kt^2 e^{-j\frac{b}{2}\left[pt + \frac{q}{t}\right]}}{b(t^2+1)} \Bigg|_1^{t_C} - jpk \int_1^{t_C} e^{-j\frac{b}{2}\left[pt + \frac{q}{t}\right]} dt. \qquad (7.28)$$

If $\gamma_n = -1$ (i. e., $\beta_n = -k$), then $p = 0$ and

$$I_n'(b,C) = \frac{-2kt^2\ e^{-j\frac{b}{t}}}{b(t^2+1)}\Bigg|_0^{t_C} = \frac{ae^{-jk\left[\sqrt{r^2+\xi^2}-\xi\right]}}{\sqrt{r^2+\xi^2}\left[\xi-\sqrt{r^2+\xi^2}\right]}\Bigg|_0^{C} \qquad (7.29)$$

Following the same techniques used in Section 7.2, let

$$D = \int_1^{t_C} e^{-j\frac{b}{2}\left[pt+\frac{q}{t}\right]}\,dt$$

$$= \int_1^{t_C} \left\{F_o(\xi) + \sum_{\nu=1}^{\infty}\left(\frac{-jb}{2}\right)^{\nu}\left[p^{\nu}t^{\nu} + q^{\nu}t^{-\nu}\right]F_{\nu}(\xi)\right\}dt\,.$$

Integrating termwise

$$D = \left[F_o(\xi)\ t\Big|_1^{t_C} - \frac{jb}{2}F_1(\xi)\left\{\frac{pt^2}{2} + g\ \ln\ t\right\}\Bigg|_1^{t_C}\right.$$

$$\left. + \sum_{\nu=2}^{\infty}\left(\frac{-jb}{2}\right)^{\nu}\left(\frac{p^{\nu}t^{\nu+1}}{\nu+1} + \frac{q^{\nu}t^{1-\nu}}{1-\nu}\right)\Bigg|_1^{t_C} F_{\nu}(\xi)\right].$$

To form an asymptotic series write

$$D = \int_0^C e^{-j\left[\sqrt{b^2 + x^2} + \gamma_n x\right]} \left[\frac{x}{\sqrt{x^2 + b^2}} + 1\right] dx .$$

Let

$$g(x) = \sqrt{b^2 + x^2} + \gamma_n x$$

and

$$f(x) = \frac{x}{\sqrt{x^2 + b^2}} + 1 .$$

Then

$$D = \int_0^C e^{-jg(x)} f(x)\, dx$$

As in Section 7.2

$$D = e^{-jg(x)} \sum_{p=0}^{P} \frac{(-1)^p \Phi_p(x)}{(-j)^{p+1}} \Bigg|_0^C + (-j)^{p+1} \int_0^C e^{-jg(x)} \Phi_p'(x)\, dx .$$

For the first five terms of the asymptotic series

$$\Phi_0 = \frac{\sqrt{x^2 + b^2} + x}{x + \gamma_n \sqrt{x^2 + b^2}}$$

$$\Phi_1 = (\sqrt{x^2 + b^2} + x) \left\{ \frac{1}{[x + \gamma_n \sqrt{x^2 + b^2}]^2} - \frac{\sqrt{x^2 + b^2} + \gamma_n x}{[x + \gamma_n \sqrt{x^2 + b^2}]^3} \right\}$$

$$\Phi_2 = (\sqrt{x^2 + b^2} + x) \left\{ \frac{2}{[x + \gamma_n \sqrt{x^2 + b^2}]^3} - \frac{3[\sqrt{x^2 + b^2} + \gamma_n x]}{[x + \gamma_n \sqrt{x^2 + b^2}]^4} \right.$$

$$\left. + \frac{3[\sqrt{x^2 + b^2} + \gamma_n x]^2}{[x + \gamma_n \sqrt{x^2 + b^2}]^5} \right\}$$

$$\Phi_3 = -(\sqrt{x^2 + b^2} + x) \left\{ \frac{1}{[x + \gamma_n \sqrt{x^2 + b^2}]^4} + \frac{3[\sqrt{x^2 + b^2} + \gamma_n x]}{[x + \gamma_n \sqrt{x^2 + b^2}]^5} \right.$$

$$\left. - \frac{15[\sqrt{x^2 + b^2} + \gamma_n x]^2}{[x + \gamma_n \sqrt{x^2 + b^2}]^6} + \frac{15[\sqrt{x^2 + b^2} + \gamma_n x]^3}{[x + \gamma_n \sqrt{x^2 + b^2}]^7} \right\}$$

$$\Phi_4 = -(\sqrt{x^2 + b^2} + x) \left\{ \frac{4}{[x + \gamma_n \sqrt{x^2 + b^2}]^5} - \frac{31[\sqrt{x^2 + b^2} + \gamma_n x]}{[x + \gamma_n \sqrt{x^2 + b^2}]^6} \right.$$

$$\left. + \frac{15[\sqrt{x^2 + b^2} + \gamma_n x]^2}{[x + \gamma_n \sqrt{x^2 + b^2}]^7} + \frac{105[\sqrt{x^2 + b^2} + \gamma_n x]^3}{[x + \gamma_n \sqrt{x^2 + b^2}]^8} - \frac{105[\sqrt{x^2 + b^2} + \gamma_n x]^4}{[x + \gamma_n \sqrt{x^2 + b^2}]^9} \right\}.$$

The criteria for using the power series, the asymptotic series, or both is the same as described in Section 7.2.

The behavior of $P_n^{(1)'}(r,z)$ for large n can be determined by integrating (7.22) by parts as was done in the previous section to determine the behavior of $P_n^{(1)'}(r,z)$. It is found that $P_n^{(1)}(r,z)$ also decreases as $1/n^2$ for large n.

7.4 A FOURIER SERIES FOR THE VECTOR POTENTIAL BOUNDARY VALUE

The vector potential at the surface of the cylindrical antenna is

$$A_{zs} = \frac{-j}{c} \left[C_1 \cos kz_o + \tfrac{1}{2} V \sin k|z_o| \right]. \qquad (7.30)$$

The Fourier series representation of A_{zs} is

$$A_{zs} = \sum_{n=0}^{\infty} D_n \cos \beta_n z_o \qquad (7.31)$$

where

$$D_n = \frac{1}{\epsilon_n h} \int_{-h}^{h} A_{zs} \cos \beta_n z \, dz \tag{7.32}$$

with

$$\epsilon_n = \begin{cases} 2 & n=0 \\ 1 & n=1,2,3,\ldots \end{cases} .$$

Let

$$D_n = \frac{-j}{c} \left[C_1 D_{cn} + \tfrac{1}{2} V D_{vn} \right] \tag{7.33}$$

where

$$D_{cn} = \frac{1}{\epsilon_n h} \int_{-h}^{h} \cos kz \cos \beta_n z \, dz \tag{7.34a}$$

and

$$D_{cn} = \frac{1}{\epsilon_n h} \int_{-h}^{h} \sin k|z| \cos \beta_n z \, dz . \tag{7.34b}$$

Since the integrands of Equations (7.34a,b) are even functions of z we have

$$D_{cn} = \frac{2}{\epsilon_n h} \int_0^h \cos kz \cos \beta_n z \, dz \qquad (7.35a)$$

and

$$D_{vn} = \frac{2}{\epsilon_n h} \int_0^h \sin kz \cos \beta_n z \, dz . \qquad (7.35b)$$

By employing the identities

$$\cos kz \cos \beta_n z = \tfrac{1}{2} \left[\cos (k + \beta_n) z + \cos (k - \beta_n) z\right]$$

and

$$\sin kz \cos \beta_n z = \tfrac{1}{2} \left[\sin (k + \beta_n) z + \sin (k - \beta_n) z\right]$$

and integrating, one obtains

$$D_{cn} = \frac{1}{\epsilon_n h} \left[\frac{\sin (k + \beta_n) h}{k + \beta_n} + \frac{\sin (k - \beta_n) h}{k - \beta_n}\right], \qquad \beta_n \neq k \qquad (7.36a)$$

and

$$D_{vn} = \frac{1}{\epsilon_n h} \left[\frac{1 - \cos (k + \beta_n) h}{k + \beta_n} + \frac{1 - \cos (k - \beta_n) h}{k - \beta_n}\right]. \qquad \beta_n \neq k \qquad (7.36b)$$

If $\beta_n = k$ we obtain, with $\epsilon_n = 1$ for $n \neq 0$,

$$D_{cn} = \frac{2}{h} \int_0^h \cos^2 kz \, dz = 1$$

$$D_{vn} = \frac{2}{h} \int_0^h \sin kz \cos kz \, dz = 0.$$

With $\beta_n = n\pi/h$, Equations (7.36a,b) take a simpler form. We have

$$\sin (k \pm \beta_n)h = \sin kh \cos \beta_n h \pm \cos kh \sin \beta_n h$$

and

$$\cos (h \pm \beta_n)k = \cos kh \cos \beta_n h \pm \sin kh \sin \beta_n h.$$

With $\beta_n = n\pi/h$,

$$\sin \beta_n h = \sin n\pi = 0$$

and

$$\cos \beta_n h = \cos n\pi = (-1)^n.$$

Therefore,

$$D_{cn} = \frac{2k\ (-1)^n \sin kh}{\epsilon_n h\ (k^2 - \beta_n^2)} \tag{7.37a}$$

and

$$D_{vn} = \frac{2k\ [1-(-1)^n \cos kh]}{\epsilon_n h\ (k^2 - \beta_n^2)} . \tag{7.37b}$$

For $\beta_n = k$, D_{cn} and D_{vn} are

$$D_{cn} = \frac{2}{\epsilon_n h} \int_0^h \cos^2 \beta_n z\, dz = 1 \tag{7.38a}$$

since $\epsilon_n = 1$ for $n \neq 0$ and n cannot be zero if $\beta_n = k$, and

$$D_{vn} = \frac{2}{\epsilon_n h} \int_0^h \sin \beta_n z \cos \beta_n z\, dz = 0. \tag{7.38b}$$

The case of $\beta_n = k$ occurs if

$$\frac{h}{\lambda} = \frac{n}{2}$$

which will occur for integer wavelength antennas.

7.5 NUMERICAL DIFFERENTIATION

This section reviews the numerical differentiation techniques used in the thesis. The lozenge diagram in Figure 7.2, taken from Kunz [46], gives expressions for the first derivative to seventh differences. As Kunz points out, the best results are expected when a nearly equal number of points are taken before and after the point at which the derivative is desired. Therefore, following the dashed line in Figure 7.2 we obtain, to sixth differences,

$$\frac{dy}{dx} \approx \Delta y_0 - \frac{1}{2}\Delta^2 y_{-1} - \frac{1}{6}\Delta^3 y_{-1} + \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{30}\Delta^5 y_{-2} - \frac{1}{60}\Delta^6 y_{-3}. \quad (7.39)$$

Formulas obtained from the lozenge diagram in Figure 7.2 represent the derivative of a polynomial which approximates the tabulated function. With the N^{th} difference included, the approximating polynomial is of N^{th} order. Consequently Equation (7.39) is exact for functions up to and including sixth order.

Since Hankel functions are involved in the expressions in the thesis to be numerically differentiated, computer data was obtained to determine the order of accuracy obtained by using Equation (7.39) with Hankel functions and modified Hankel functions. The accuracy of numerical differentiation of $H_o^{(2)}(x)$ and $K_o(x)$ can easily be evaluated since $H_o^{(2)\prime}(x) = -H_1^{(2)}(x)$ and $K_o'(x) = -K_1(x)$. The numerical derivatives of Hankel functions and modified Hankel functions obtained using Equation (7.39) were within 1% of the exact expressions for arguments of $1 \leq x \leq 10$ and within 0.001% for $10^{-5} \leq x \leq 1$ with $h = 0.1x$.

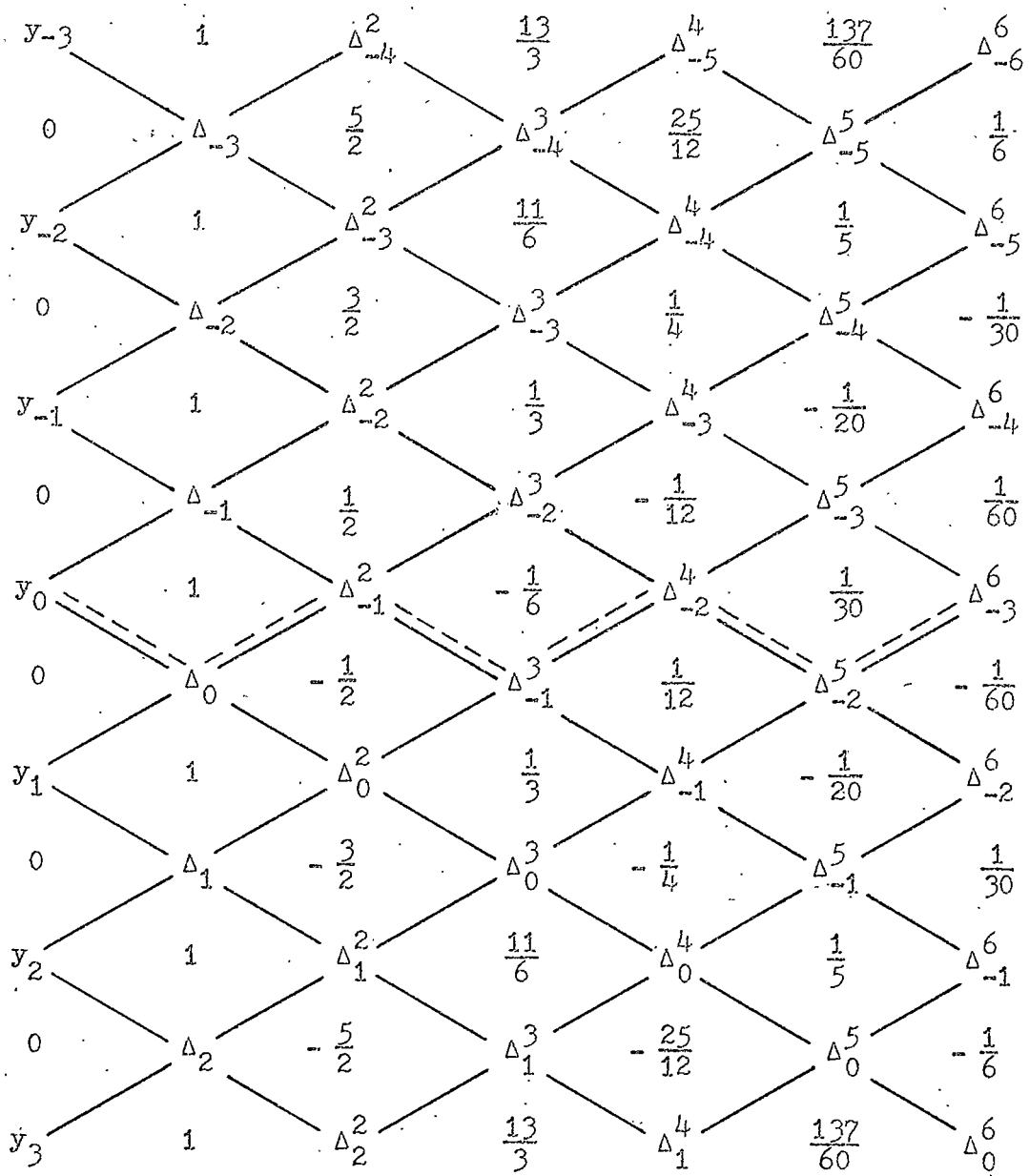


Fig. 7.2 Lozenge for the first derivative at y_0. All formulas obtained from this diagram are to be multiplied by 1/h, and Δ^m_s stands for $\Delta^m y_s$.

CHAPTER 8

NUMERICAL RESULTS AND COMPARISONS

8.1 INTRODUCTION

This chapter presents numerical values for the current distribution and impedance of a cylindrical antenna calculated by the methods described in the thesis. These values are compared with experimental results obtained by Morita [51] and Barzilai [5] and with the King-Middleton Theory [36], [41].

The results presented in this chapter were obtained with the use of a high-speed digital computer. The infinite series in the expressions for current and impedance were truncated at twenty-five terms.

These results were obtained with the second form of Green's function. A factor of two has been incorporated to account for Gibb's phenomenon as described in Section 8.3. Identical results are obtained with the first form of Green's function when a heuristic change is made in the expression for the current. This is discussed in Section 8.3 also.

8.2 THE MEASURED VALUES TO BE USED FOR COMPARISON OF THEORY AND EXPERIMENT

The measured values to be used in comparing the theorical values with experimental data have been obtained by Morita [51] and Barzilai [5]. In order to make meaningful comparisons and to understand the implications of the comparisons, it is necessary to know what measurements were made and what the measured quantities actually represent. For this reason a review of the measuring techniques is presented.

The measurement techniques used by Morita and Barzilai are similar. Their experimental antennas differ in that Barzilai worked with a center-

fed dipole while Morita used a monopole above a ground plane. Both used a hollow pipe slotted lengthwise for the antenna. A small loop protruding from the slot is moved along the length of the antenna. The output of the loop, which is proportional to the tangential magnetic field and hence proportional to the surface current, is rectified by a crystal diode. The DC current in the crystal is the measured quantity. The reader is referred to Morita's article [51] for details, particularly in regard to the measurement of the phase of the current. With such a measuring technique only relative values of the antenna surface current can be obtained. But this is sufficient to make meaningful comparisons of experiment and theory if measured impedance, or admittance, values are available. Morita has given measured antenna impedance values.

Barzilai and Morita normalized the relative current distributions to coincide with the theoretical current distribution curves at a point of maximum current amplitude. This gives an indication of how well the shapes of the measured and theoretical curves match.

Further comparison between experiment and theory can be made by comparing measured impedance values with theoretical impedance values. If the theoretical current distribution curve has the same shape as the normalized experimental current distribution, and if the theory correctly predicts the antenna impedance, one can conclude that the theory is probably correct.

The reason for using this rather indirect method of comparing the current distributions is that it is generally difficult to measure the

actual values of current on the antenna. Likewise, the terminal voltage is difficult to measure. However, the terminal impedance of the transmission feed-line at the antenna can be measured and it is the ratio of antenna terminal voltage and current. Thus the impedance provides a means of scaling the measured relative current in terms of amperes per volt.

No attempt has been made to estimate the effect of the terminal zone, or center gap, on the measured impedance.

8.3 CALCULATION OF THEORETICAL VALUES

As stated in the introduction of this chapter, a high speed digital computer was used to compute the theoretical values presented in this chapter. The infinite series involved were truncated at twenty-five terms. No estimate of the remainder has been obtained. Calculations were made with up to 120 terms of the series. Terms at truncation were of the order of 0.1% to 1.0% of the leading terms. Using larger numbers of terms changed the results very little. The general tendencey was found to be that the current increased slightly and the impedance correspondingly decreased slightly as more terms were included. It is not clear whether this change represents an improvement in accuracy or merely the accumulation of more insignificant figures into the final results by the computer. Regardless of these computational problems, the results clearly have the characteristics shown by experiment to be correct for the current distribution and impedance of a cylindrical antenna.

It appears that Gibb's phenomenon occurs in the infinite series involved in the expressions for the current distribution. The current obtained by summing the series as given in Chapter 6 for the second form of Green's function is too small by a factor of $\frac{1}{2}$. This is actually more apparent in the impedance since impedance is the only measured quantity available with an absolute reference scale. Of course, the impedance is then too large by a factor of 2. Recall that the current is obtained from the magnetic field just outside the antenna. Just inside the antenna, which is assumed to be a perfect conductor, all fields must vanish. This represents a finite discontinuous jump in the magnetic field at the antenna boundary. From studies of Gibb's phenomenon [8], [11], it is known that at a point of discontinuity a Fourier series converges to the mean of the values of the function on either side of the discontinuity. In this case we expect the series to give

$$H_\phi(r=a) = \frac{H_\phi(r=a^+) + H_\phi(r=a^-)}{2} .$$

Since

$$H_\phi(r=a^-) = 0,$$

$$H_\phi(r=a) = \frac{H_\phi(r=a^+)}{2} .$$

Since the current is directly proportional to $H_\phi(r=a)$, it will also reflect this factor of $\frac{1}{2}$.

Calculations based on the first form of Green's function exhibit this apparent Gibb's phenomenon also. An additional anomaly also appears in the first form of Green's function. Recall that the first form of Green's function is expressed in two parts so that $G = U + V$. It is found that the numerical results for the portion of the solution due to V, when corrected for the factor of $\frac{1}{2}$ attributed to Gibb's phenomenon, correctly represents the current and impedance. In fact, these results are identical with those obtained by the second form of Green's function for antennas of dimensions such that $\Omega \geq 10$. However, the portion of the solution due to U introduces considerable error. The reason for this has not been established conclusively. However, the derivations of the expressions for the vector potential, magnetic field, and finally current involved the integration of

$$A_{zu} = \int_S A_{zs} \frac{\partial U}{\partial n_o} dS . \qquad (8.1)$$

Now U is singular when source and field points coincide. This is of no concern when the source and field points are separated. In fact, one does not apply such an integral to field points which coinside with source points.

In applying Huygens' principle to the cylindrical antenna, and using the first form of Green's function, Equation (8.1) was used in

obtaining an expression for the magnetic vector potential outside the antenna. In this case field and source points are separated and Equation (8.1) is valid. Certain approximations are made regarding this integral (see Chapter 5 and Appendix C). In the final step of obtaining an expression for the current, a limit is approached for the radial derivative of the vector potential just outside the antenna, thus obtaining the magnetic field at the boundary, and, finally, the current. In taking this limit, one essentially brings the source and field points indefinitely close together. It should not be surprising if the approximations made are in error under these severe conditions. The numerical results indicate that the U-portion of the solution contributes too greatly to the expression for current, giving current on the order of an ampere where a few milliamperes are expected. This matter deserves further study, especially in view of the satisfactory results obtained with the second form of Green's function. The results obtained with the second form of Green's function clearly show that it is feasible to apply Huygens' principle to the cylindrical antenna boundary value problem.

8.4 ANTENNA CURRENT DISTRIBUTIONS

Curves are shown in Figures 8.1-8.7 which compare theoretical and expermental values of antenna current. The measured relative current values have been normalized to coincide with the theoretical curve at a point of maxium current amplitude. The phase angle of the current is also normalized at the same point. The theoretical values were

calculated for the particular values of Ω ($\Omega = 2 \ln \frac{2h}{a}$) used in the experiments. Since the radii of the experimental antennas remain constant, Ω varies with antenna length.

Clearly, the theoretical curves show the correct trend.

8.5 ANTENNA IMPEDANCE CHARACTERISTICS

Theoretical impedance values are obtained by assuming a unit excitation voltage. Then $Z = \frac{1}{I(0)}$ where $I(0)$ is the antenna terminal current. Curves are shown in Figures 8.8-8.13 comparing results with measured values and with the King-Middleton second order theory.

The calculated impedance certainly shows the correct trend. The agreement with experiment regarding antenna lengths at which resonance and anti-resonance occur is remarkable. It is in closer agreement with experiment than the King-Middleton theory (see Figure 8.9).

The asserted Gibb's phenomenon is based upon the impedance of the half-wave dipole which is well-known to be about $72 + j43$ ohms [33]. Values of impedance near anti-resonance appear to be too small by a factor of two. Had the factor of one-half attributed to Gibb's phenomenon not been introduced, the agreement of calculated and measured values of impedance would be better near anti-resonance, but then it would differ by a factor of two near resonance. The cause of this discrepancy is not yet apparent.

In spite of these difficulties, the correct trend is certainly present in the theory. The important point has been established that Huygens' principle can be used to obtain explicit expressions for the current distribution and impedance of the cylindrical antenna.

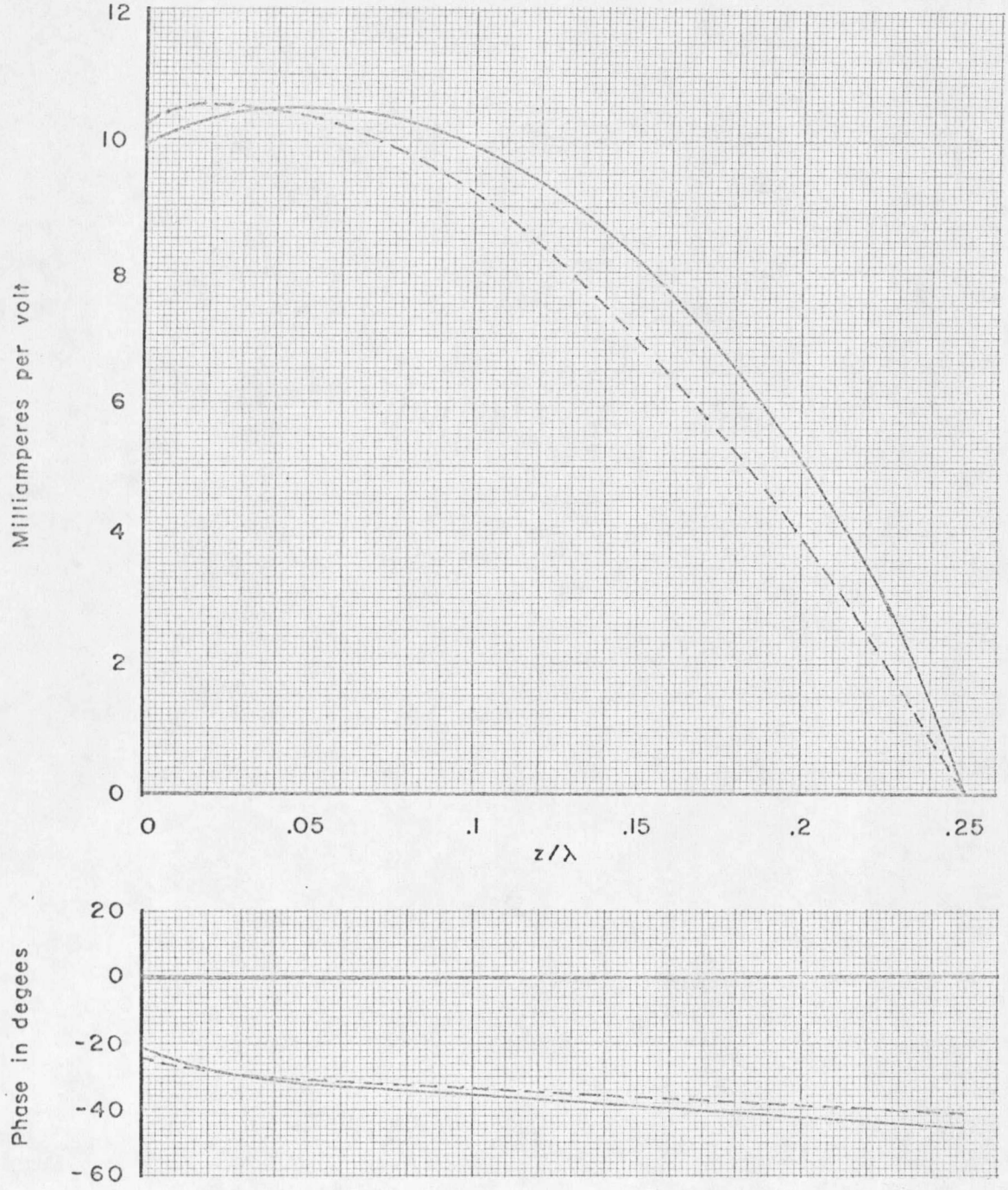


Fig. 8.1 Comparison of measured and theoretical current distribution, $2h = 0.5\lambda$, $\Omega = 10.12$. —— theoretical; ---- measured (Morita).

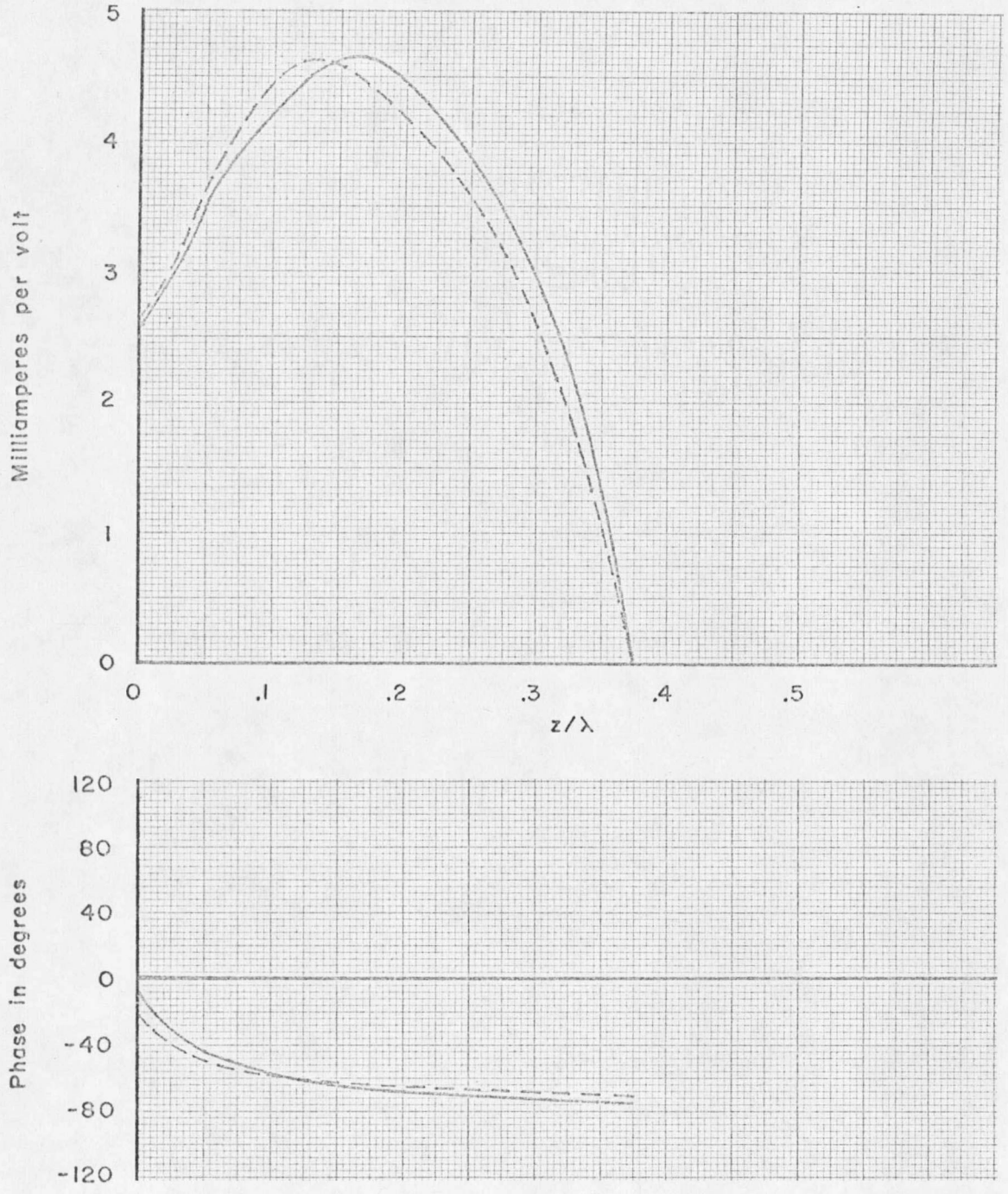


Fig. 8.2 Comparison of measured and theoretical current distribution, $2h=0.75\lambda$, $\Omega=10.76$. —— theoretical; ---- measured (Morita).

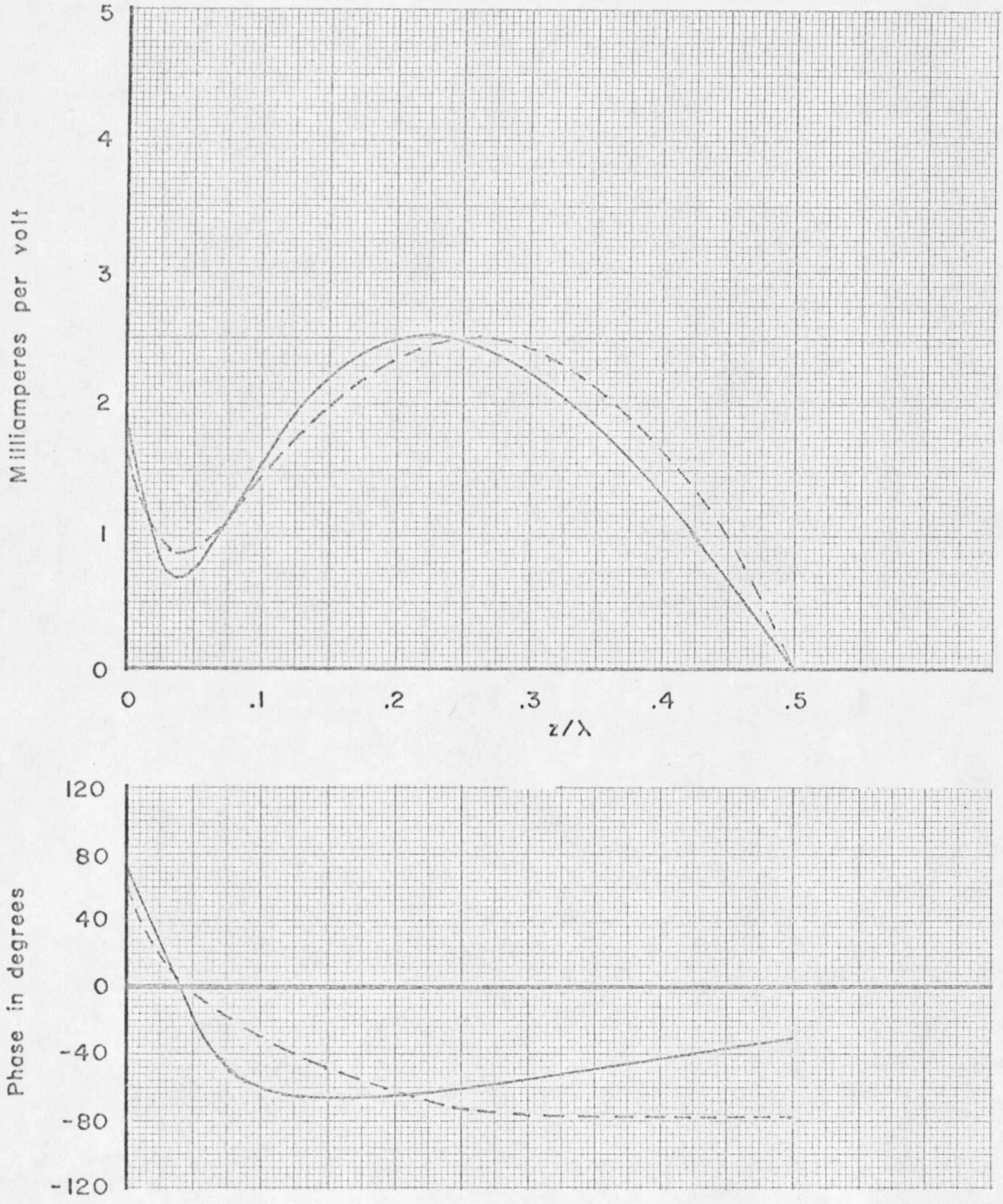


Fig. 8.3 Comparison of measured and theoretical current distribution, $2h=1.0\lambda$, $\Omega=11.5$. —— theoretical; ---- measured (Morita).

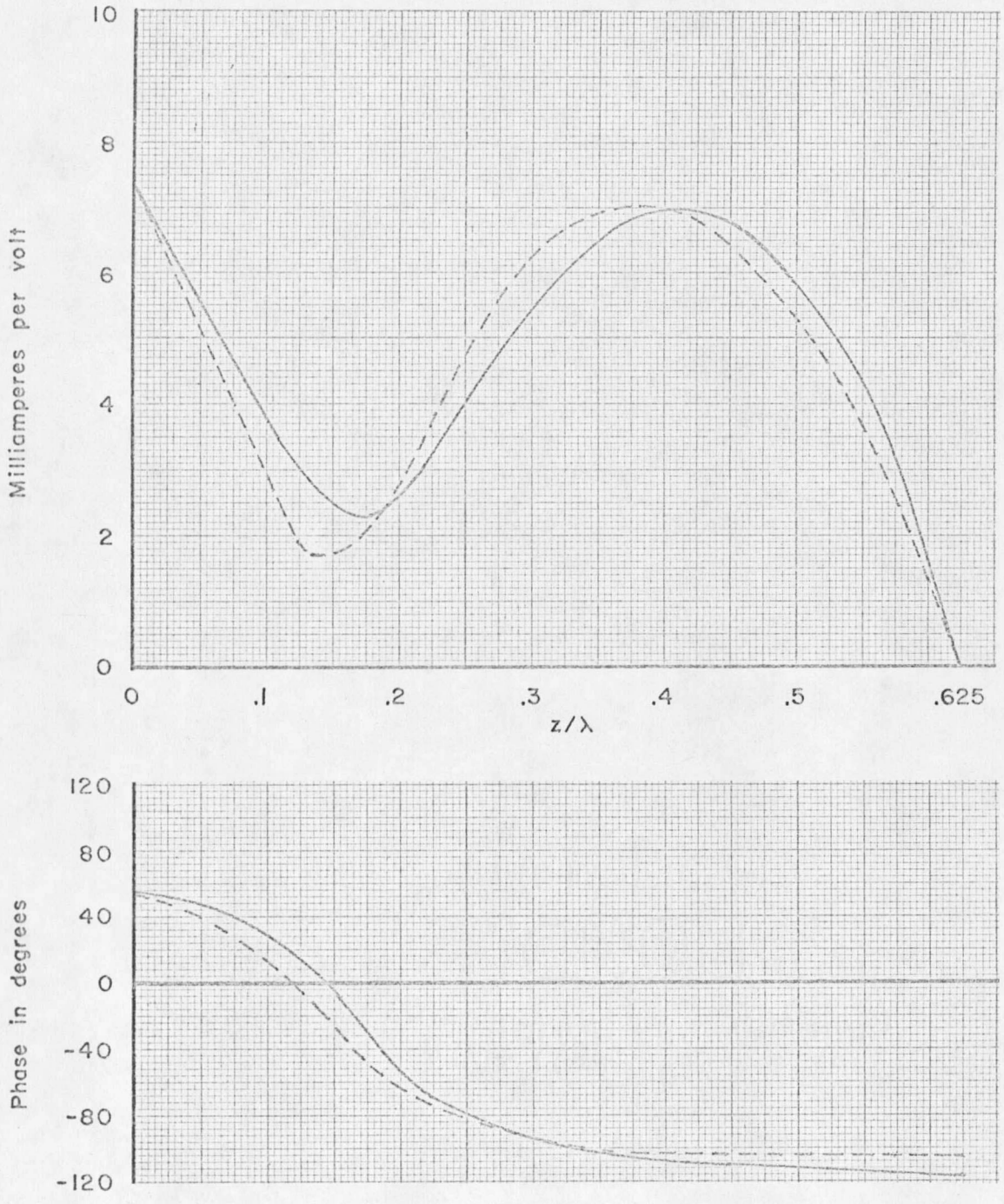


Fig. 8.4 Comparison of measured and theoretical current distribution, $2h=1.25\lambda$, $\Omega=11.94$. —— theoretical; ---- measured (Morita).

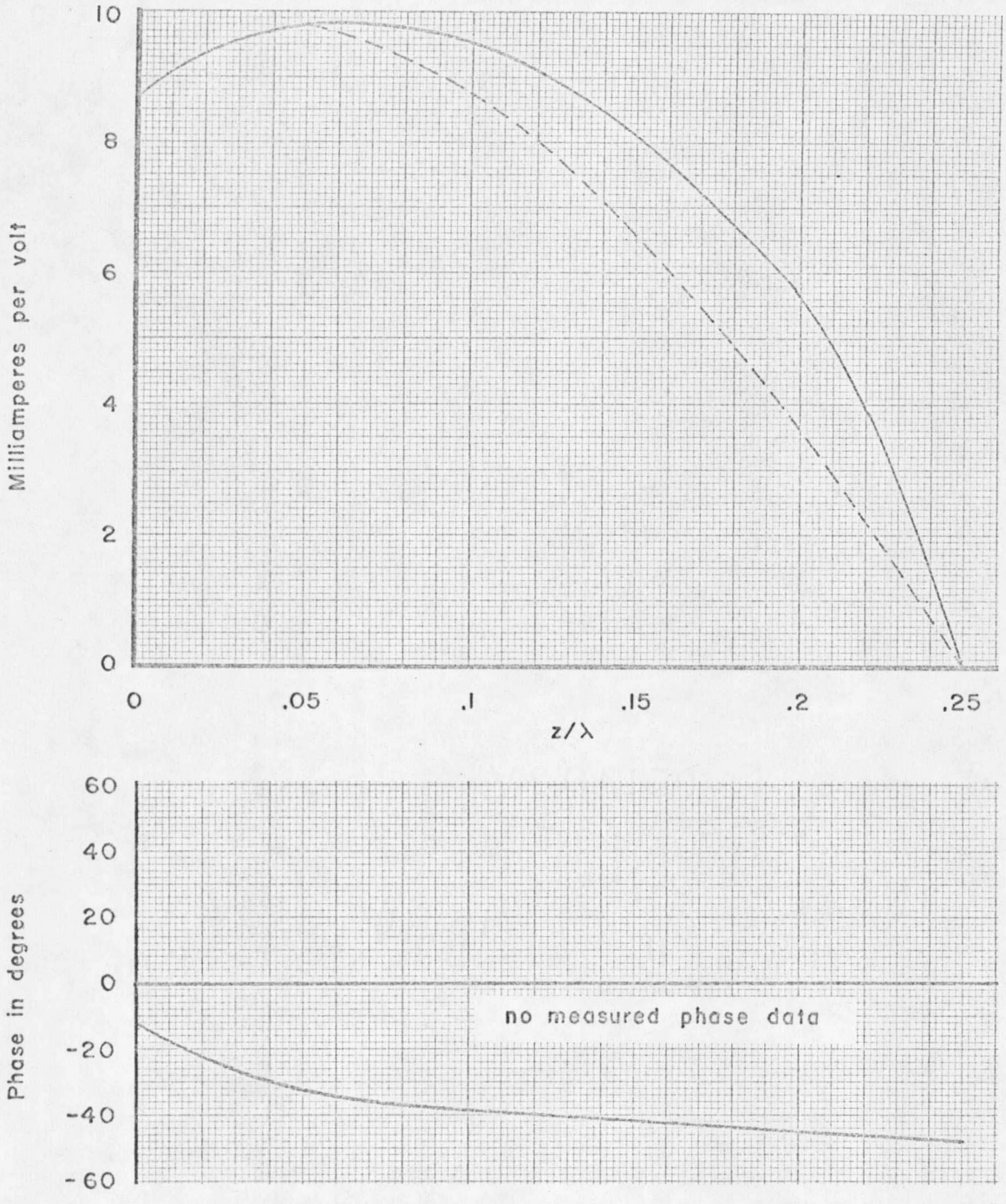


Fig. 8.5 Comparison of measured and theoretical current distribution, $2h = 0.5\lambda$, $\Omega = 8.3$. —— theoretical; ---- measured (Barzilai).

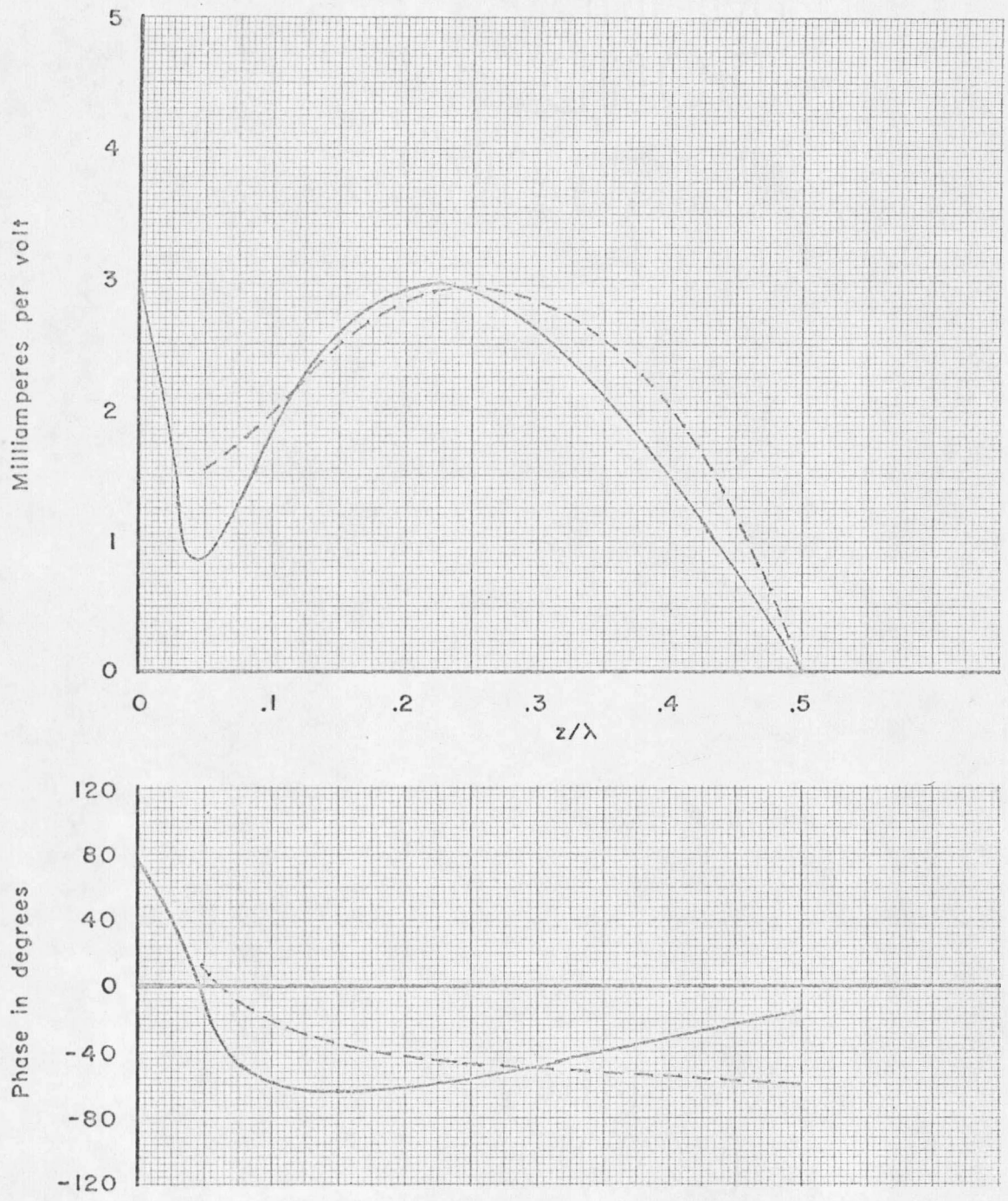


Fig. 8.6 Comparison of measured and theoretical current distribution, $2h=1.0\lambda$, $\Omega=9.7$. —— theoretical; ---- measured (Barzilai).

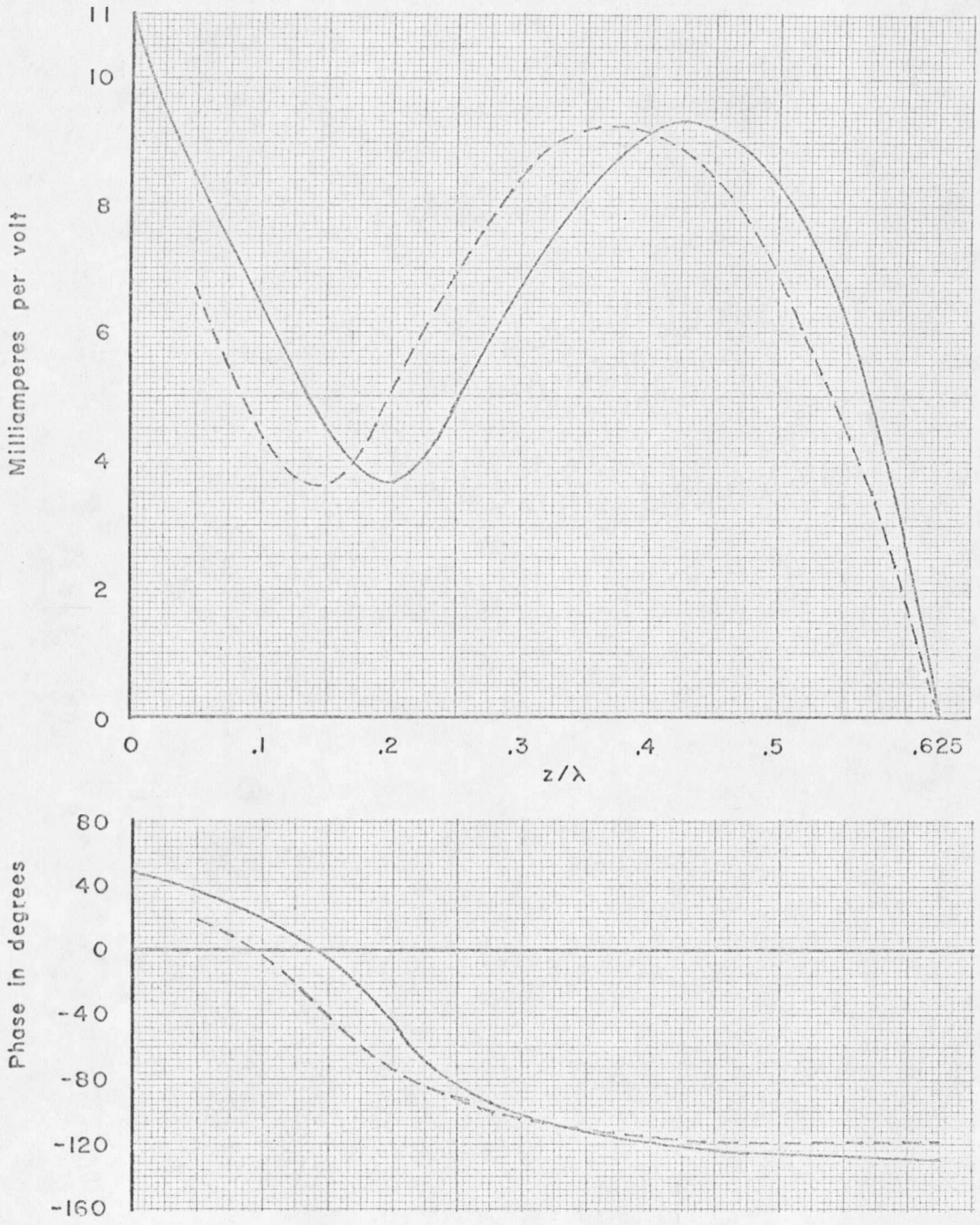


Fig. 8.7 Comparison of measured and theoretical current distribution, $2h=1.25\lambda$, $\Omega=10.2$. —— theoretical; ---- measured (Barzilai).

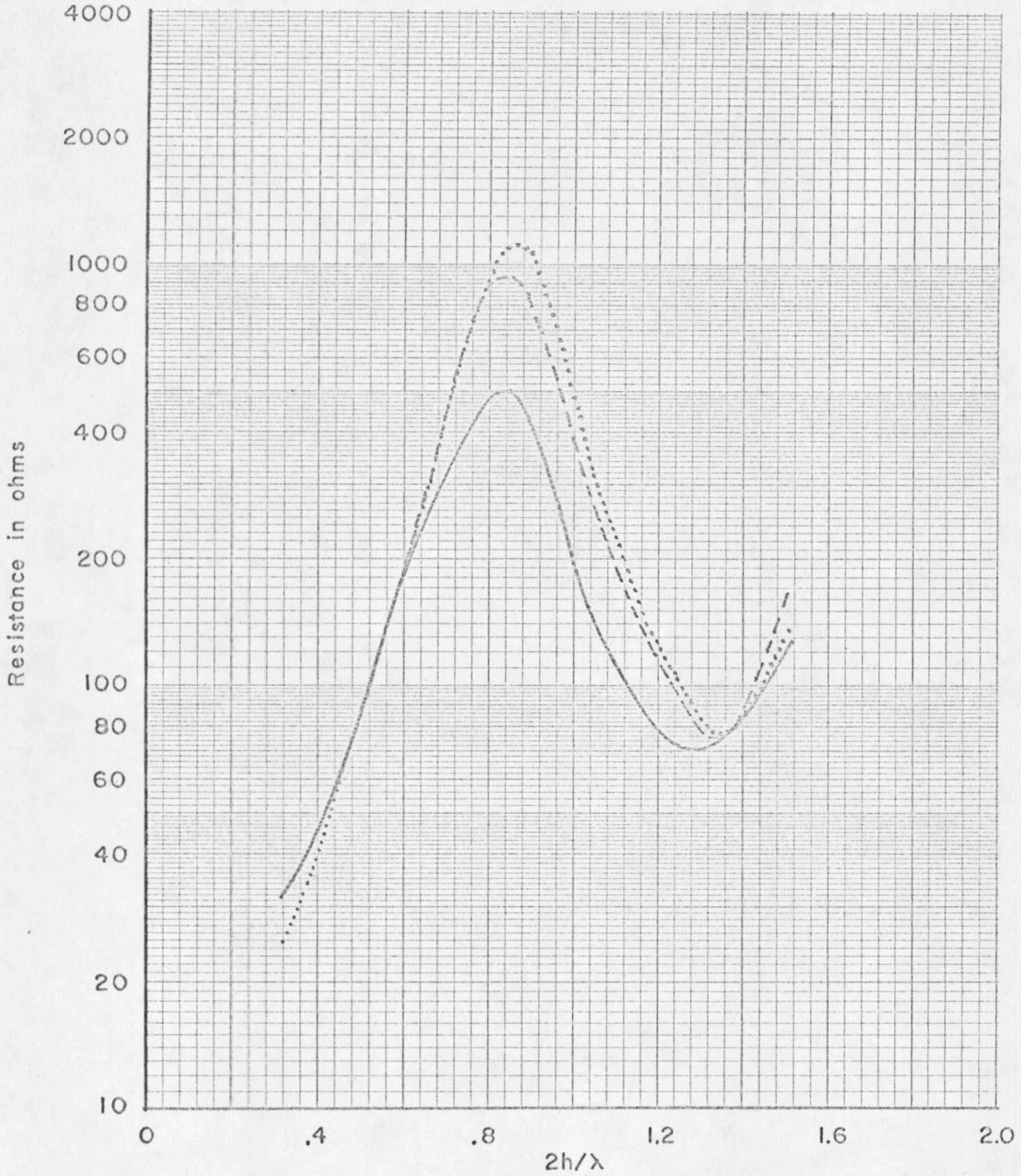


Fig. 8.8 Comparison of measured and theoretical antenna resistance, radii $a=0.0032\lambda$.
—— thesis results;
---- measured (Morita);
····· King-Middleton theory [36].

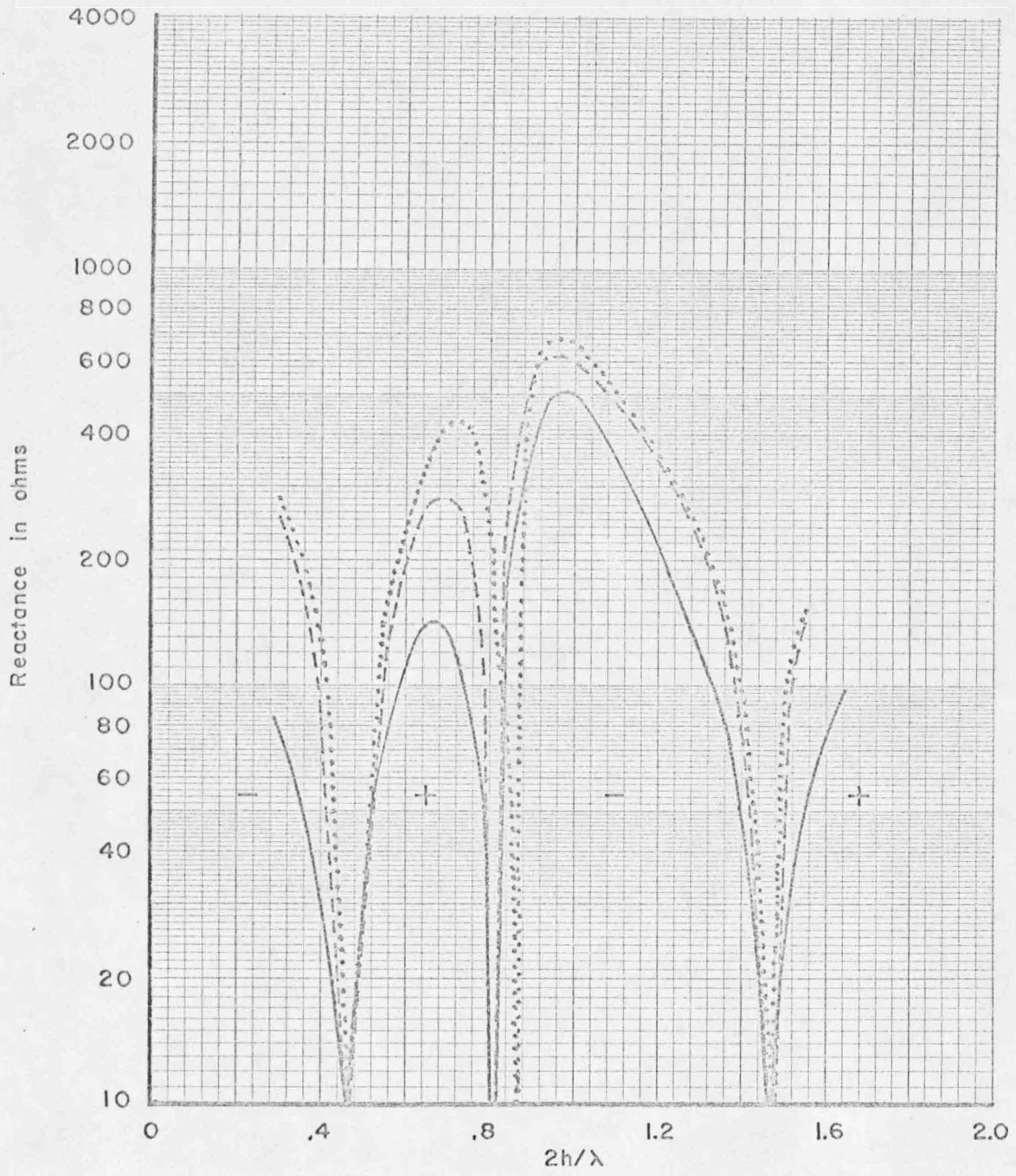


Fig. 8.9 Comparison of measured and theoretical antenna reactance, radii a=0.0032λ.
——thesis results;
----measured (Morita);
····· King-Middleton theory [36].

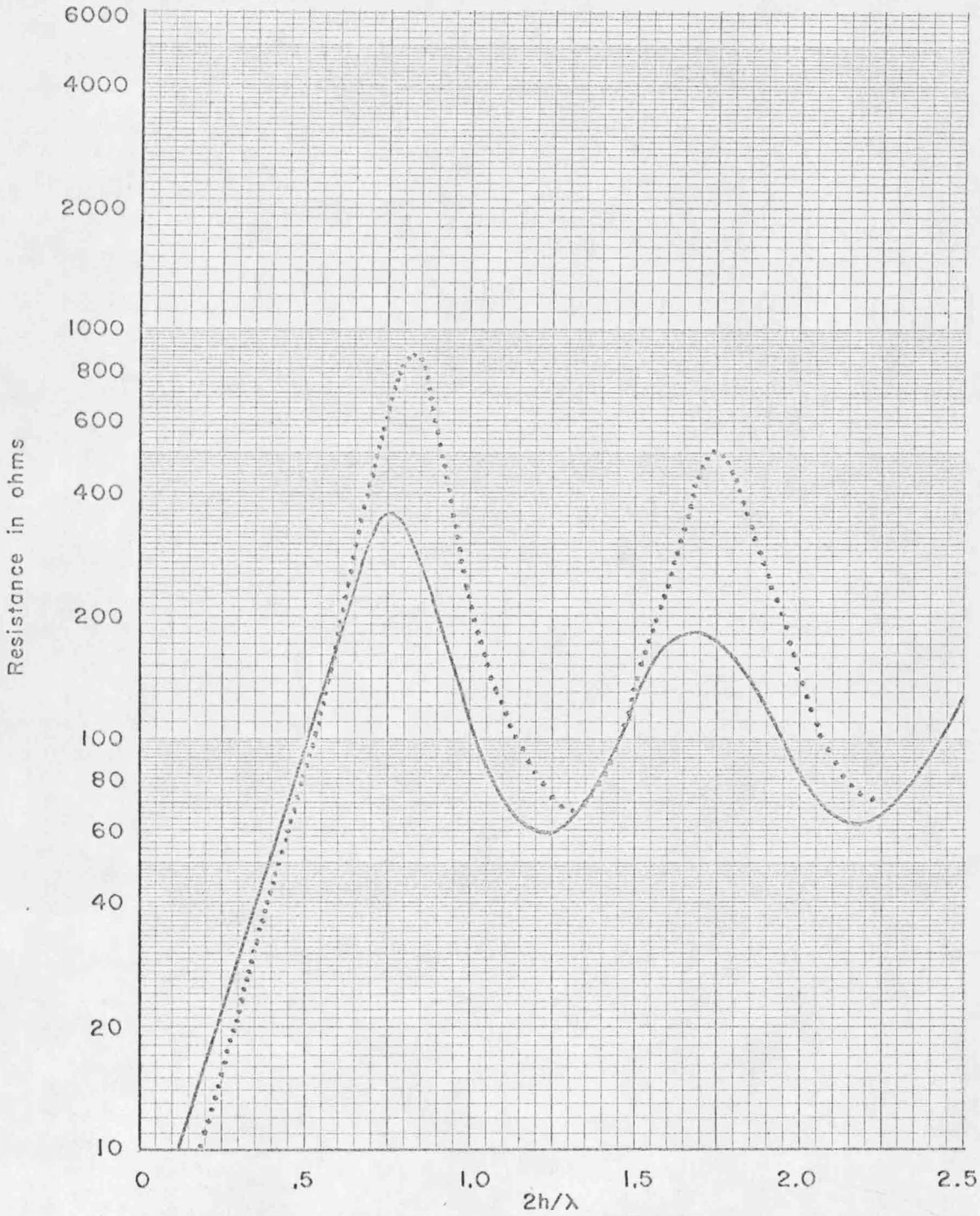


Fig. 8.10 Comparison of antenna resistance with King-Middleton theory, $\Omega = 10$.
—— thesis results; ····· King-Middleton [36].

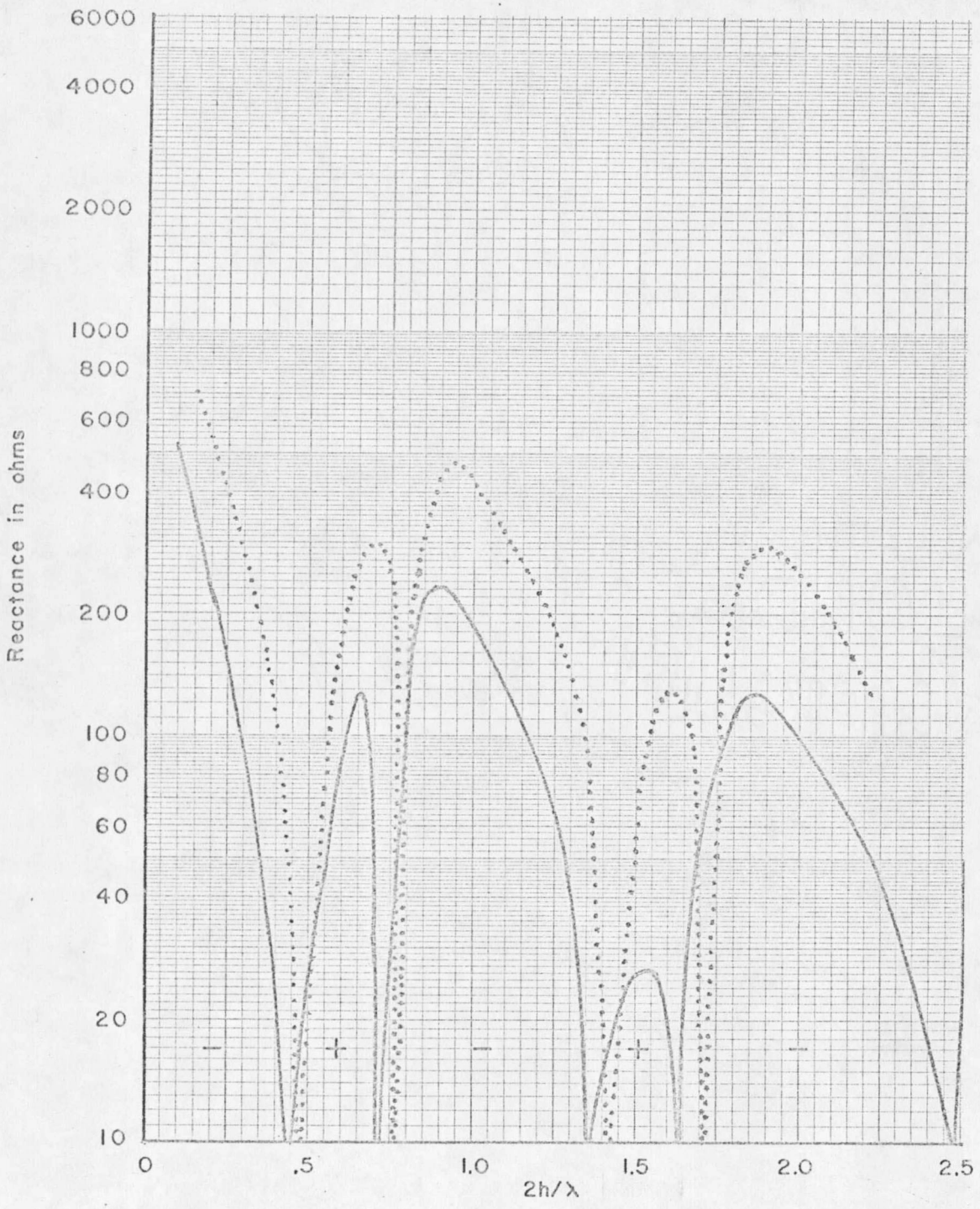


Fig. 8.11 Comparison of antenna reactance with King-Middleton theory, $\Omega = 10$.
—— thesis results; ····· King-Middleton [36].

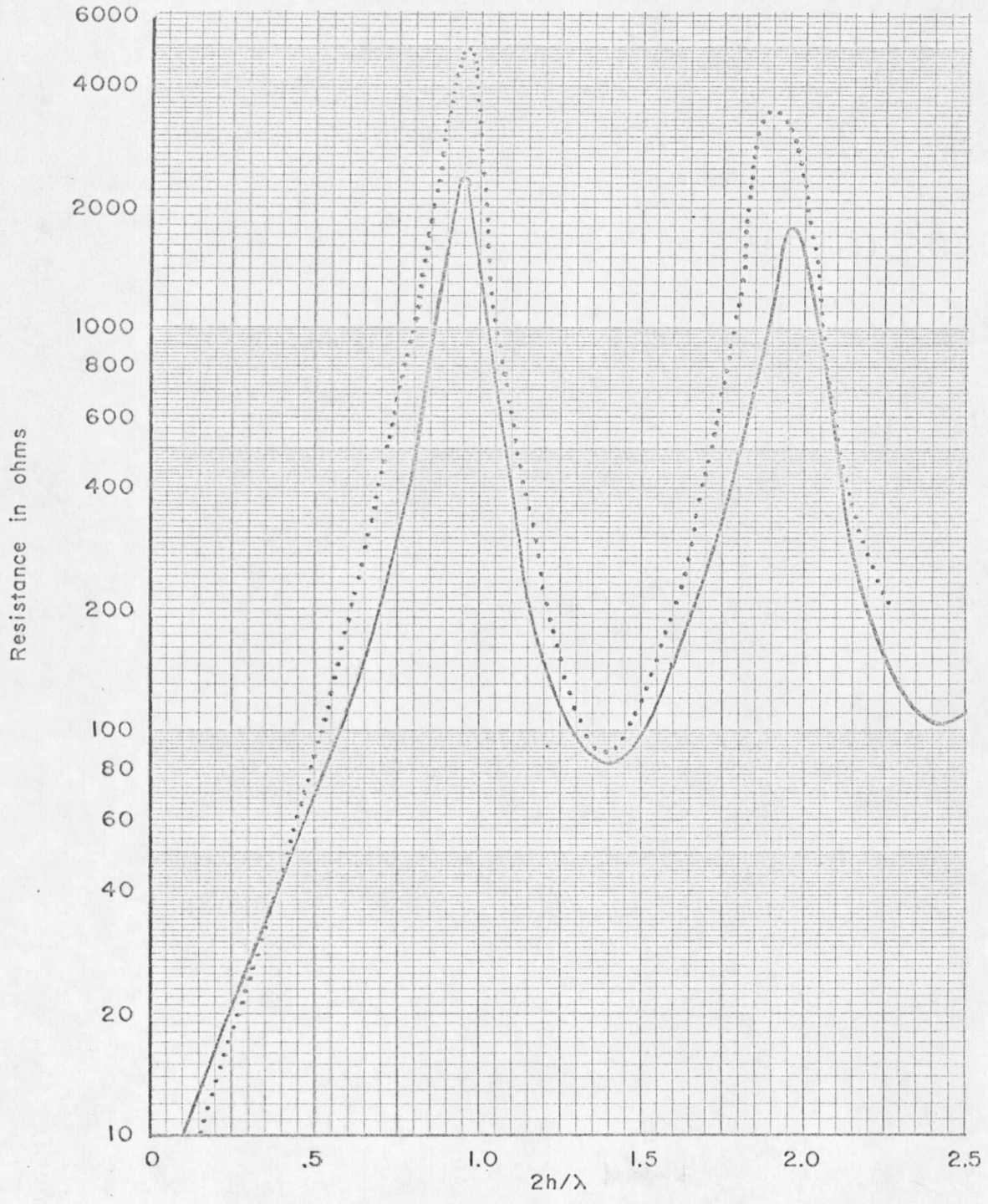


Fig. 8.12 Comparison of antenna resistance with King-Middleton theory, $\Omega = 20$.
—— thesis results; ····· King-Middleton [36].

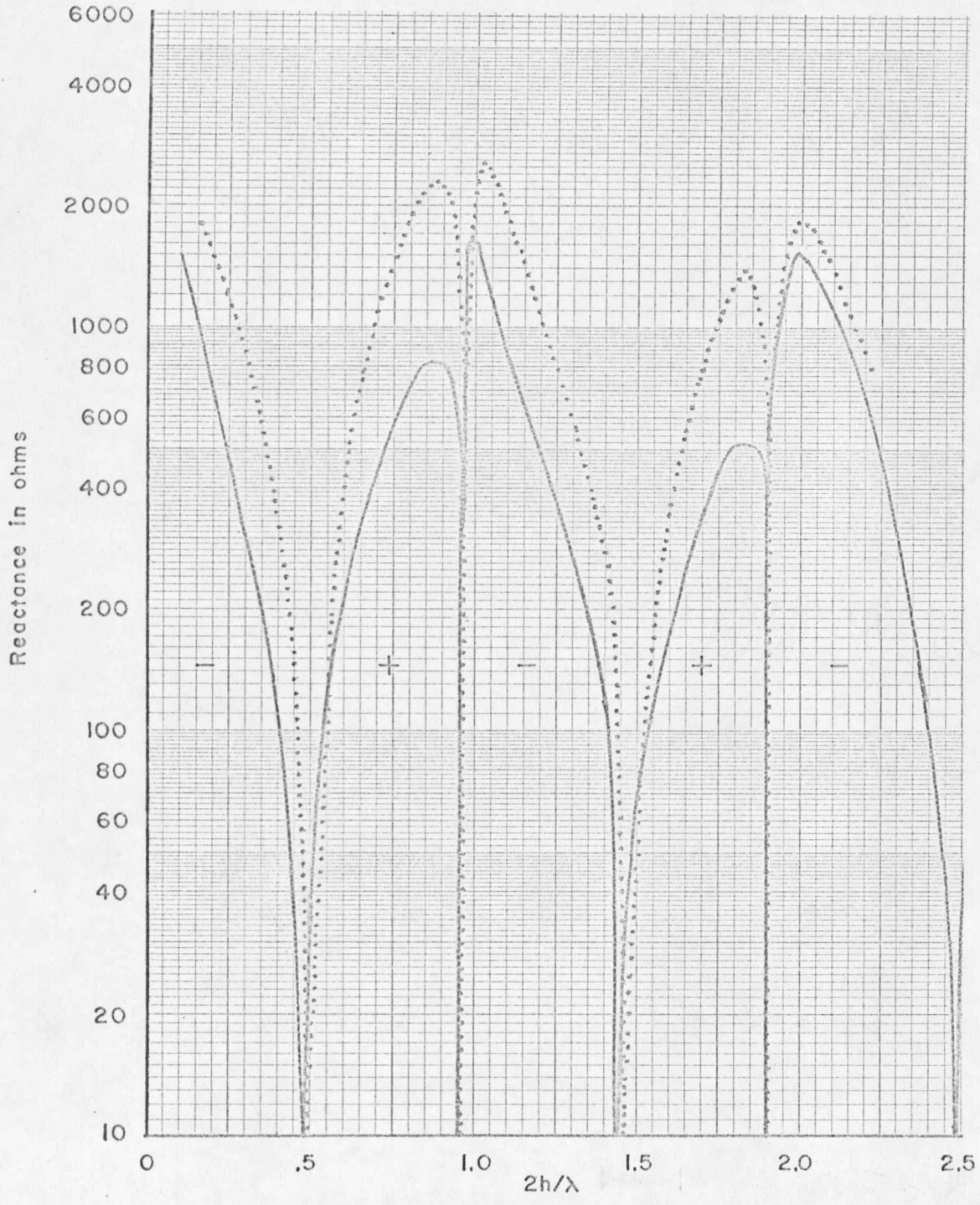


Fig. 8.13 Comparison of antenna reactance with King-Middleton theory, $\Omega = 20$.
—— thesis results; ····· King-Middleton [36].

CHAPTER 9

SUMMARY AND SUGGESTED ADDITIONAL RESEARCH

9.1 SUMMARY OF THE THESIS RESEARCH

The thesis presents a solution of the cylindrical antenna boundary value problem which apparently has not been investigated before. This solution is posed in terms of the mathematical form of Huygens' principle. Well-known boundary values of the magnetic vector potential at the antenna surface are used to obtain an expression for the vector potential in the vicinity of the antenna. Given such an expression for the vector potential, one can determine the magnetic field in the vicinity of the antenna and invoke boundary conditions to obtain an expression for the surface current on the antenna. Finally, with the current determined, the impedance can be obtained.

The use of Huygens' principle requires a properly defined Green's function. Green's function must satisfy the wave equation

$$\nabla^2 G + k^2 G = -\delta(\overline{R} - \overline{R}_o).$$

For the application of Huygens' principle to the cylindrical antenna problem, Green's function must also vanish on a sphere at infinity and on the surface of the antenna. Two forms of Green's function are presented in the thesis. Each is derived by well-known techniques.

Numerical values of current and impedance obtained with the second form of Green's function agree well with measured values when Gibb's phenomenon is accounted for. The first form of Green's function which is written in two terms $G = U + V$, yields results identical with those of the second Green's function if the part of the solution due to the

U term is diminished or ignored. This indicates that some of the approximations used may be in error in the immediate vicinity of the antenna. The results obtained with the second form of Green's function clearly show that it is feasible to apply Huygens' principle to the cylindrical antenna problem.

9.2 ASPECTS MERITING ADDITIONAL STUDY

The aspects meriting additional study may be separated into two groups; aspects contained within the realm of the thesis research and aspects beyond the scope of the thesis.

First, aspects related to the thesis research include: A study of the remainders of the truncated infinite series in the expressions for the antenna current. A study of the anomaly encountered in the U term of the first form of Green's function. Verification of the apparent Gibb's phenomenon which occurs in the expressions for magnetic field at the antenna surface.

Second, aspects which are beyond the scope of the thesis include the application of Huygens' principle to more complex cylindrical antenna problems. These may be considered in three groups.

First, an extension may be made to very thick cylinders, with diameters much larger that the length, for example. The monopole equivalent of this may be used as a low profile antenna on fast moving vehicles. Solution of the boundary value problem in this case will require determination of the vector potential boundary values on the antenna end faces. With this, and a suitable Green's function, Huygens'

principle may be applied.

Second, a very important field of investigation is the determination of the self- and mutual-impedance characteristics for an array of cylindrical antennas. These impedance characteristics are needed not only for impedance matching purposes but, more importantly, for antenna beam pattern analysis and synthesis.

As yet no satisfactory, generally applicable method of computing self- and mutual-impedance for an array exists, even for only two elements. It seems worthwhile to investigate the use of Huygens' principle for a two element array as a next higher level application in antenna theory.

Third, an area of investigation that has attracted much interest with the advent of space exploration concerns the characteristics of an antenna in a plasma. Articles appearing in the literature include both theoretical and experimental investigations [25], [31]. The King-Middleton theory, to which reference is made in the thesis [41], has been adapted to an ionized medium [25]. It is possible that application of Huygens' principle may be advantageous in this case.

The three examples above are indicative of the types of problems that may be approached with Huygens' principle. It may be that many other applications exist as well.

APPENDIX A

THE FREE SPACE GREEN'S FUNCTION

The free space Green's function, U, is the point source response function. It is the spatial analog to the one-dimensional delta function response commonly used in time domain analysis. The free space Green's function could be called a three dimensional impulse response.

U satisfies the wave equation

$$\nabla^2 U + k^2 U = -\delta(\bar{R} - \bar{R}_o)$$

where $\bar{R} = x\bar{i} + y\bar{j} + z\bar{k}$ and $\bar{R}_o = x_o\bar{i} + y_o\bar{j} + z_o\bar{k}$ designate the field point and source point respectively. Let $x_o = y_o = z_o = 0$.

The Fourier transform pair for U is

$$U(\bar{R}) = \frac{1}{(2\pi)^3} \int u(\bar{\xi})\, e^{j\bar{\xi}\cdot\bar{R}}\, d^3\xi$$

$$u(\bar{\xi}) = \int U(\bar{R})\, e^{-j\bar{\xi}\cdot\bar{R}}\, d^3R$$

where $\bar{\xi} = \alpha\bar{i} + \beta\bar{j} + \gamma\bar{k}$.

The response of a point source is expected to be symmetrical about the source. This implies symmetry in the transform domain as well [7]. Let us choose the z-axis of spherical coordinates in $\bar{\xi}$-space along vector $\bar{R}$. Then

$$U(\bar{R}) = \frac{1}{(2\pi)^2 jR} \int_{-\infty}^{\infty} u(\bar{\xi})\, e^{j\xi R}\, \xi\, d\xi$$

and

$$u(\bar{\xi}) = -\frac{2\pi}{j\xi} \int_{-\infty}^{\infty} U(\bar{R})\, e^{-j\xi R}\, R\, dR\ .$$

Transforming the wave equation gives, together with the radiation condition [6] $\lim_{R \to \infty} R\left(\frac{dU}{dR} + jkU\right) = 0$,

$$u(\bar{\xi}) = \frac{1}{\xi^2 - k^2}\ .$$

Therefore

$$U(\bar{R}) = \frac{1}{(2\pi)^2 jR} \int_{-\infty}^{\infty} \frac{e^{j\xi R}}{\xi^2 - k^2}\, \xi\, d\xi\ .$$

This integral may be evaluated by means of the residue theorem [14]. The poles are at $\xi = \pm k$. In order to have outwardly propagating waves, choose a contour of integration as shown in Figure A.1. In terms of the residue

$$U(\bar{R}) = \frac{e^{-jkR}}{4\pi R}$$

where

$R = \sqrt{x^2 + y^2 + z^2}$ (see footnote 1).

$U(\overline{R})$ is the well-known Green's function which appears in the convolution integrals for magnetic vector potential

$$\overline{A} = \int_{V(vol)} \frac{\mu \overline{J} e^{-jkR}}{4\pi R} \, dV$$

and for scalar potential

$$\varphi = \int_{V(vol)} \frac{\rho e^{-jkR}}{4\pi R} \, dV .$$

---

[1] In a strict sense, the contour shown in Figure A.1 should not deflect around the poles. However, if the contour passes through the poles, the Cauchy principle value yields both incoming and outgoing spherical waves. This is not acceptable from a physical viewpoint. In a real propagation medium some attenuation is expected so that the propagation constant is complex, that is, $k = \beta - j\alpha$. Then the poles lie in the second and fourth quadrants of the ξ-plane, slightly above and below the real axis. In this case, with an arbitrarily small non-zero α, the residue is the same as that for the contour of Figure A.1 and only outwardly propagating waves result. The contour of Figure A.1 is a symbolic representation of poles near the contour of integration as described by Mathews and Walker [48].

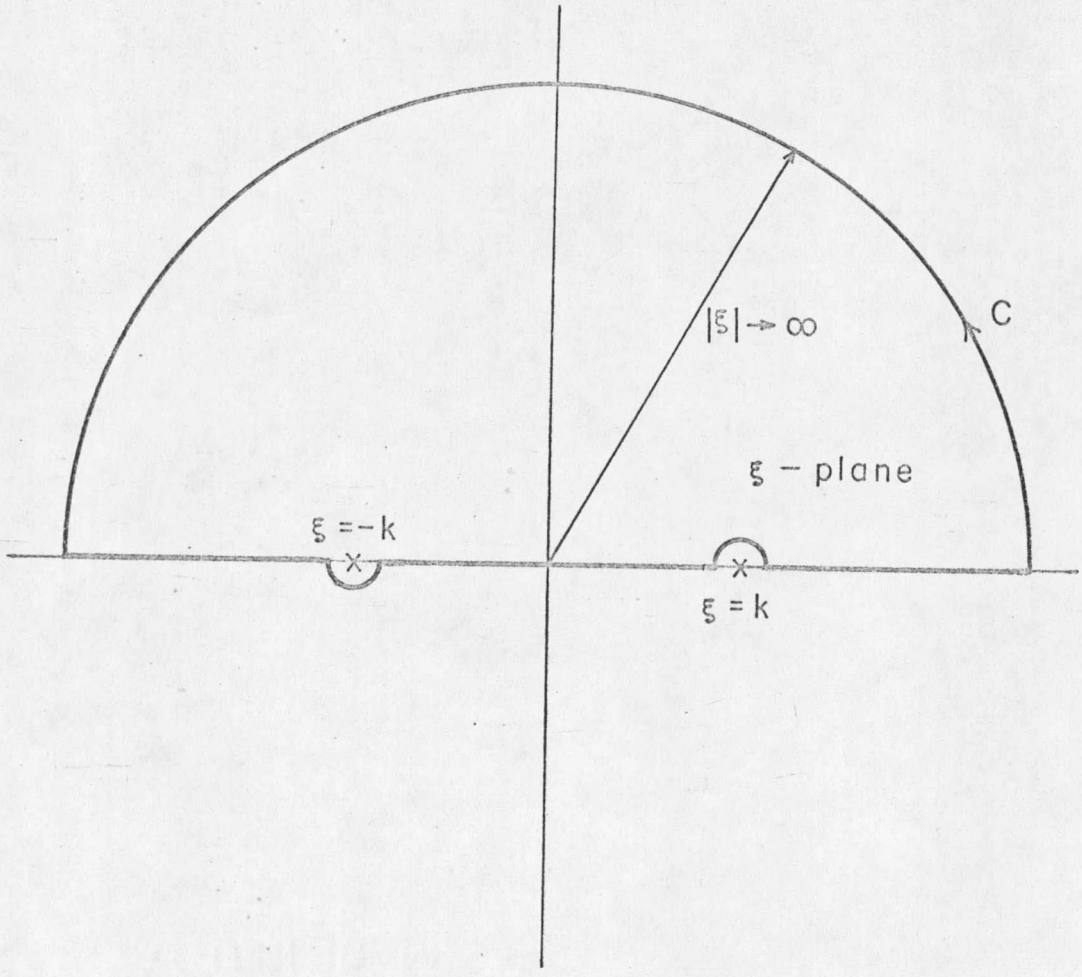


Fig. A.1 Contour of integration for the free space Green's function.

APPENDIX B

SOLUTIONS OF THE HOMOGENEOUS WAVE EQUATION IN CYLINDRICAL COORDINATES

Solutions of the homogeneous wave equation are adequately treated in the literature. A review of this subject is included here for the convenience of the reader.

The homogeneous wave equation may be written

$$\nabla^2 V + k^2 V = 0.$$

In cylindrical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} + k^2 V = 0.$$

Solutions of this equation may be obtained by separating variables.

Let

$$V(r,\phi,z) = R(r)\,\Phi(\phi)\,Z(z).$$

Then

$$\frac{\Phi Z}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{RZ}{r^2}\frac{d^2\Phi}{d\phi^2} + R\Phi\frac{d^2 Z}{dz^2} + k^2 R\Phi Z = 0. \qquad (B.1)$$

Dividing (B.1) by $R\Phi Z$ gives

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2\Phi}\frac{d^2\Phi}{d\phi^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} + k^2 = 0. \qquad (B.2)$$

Equation (B.2) must be true for all z, therefore

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = -\beta^2 \qquad (B.3)$$

where β is a constant. Solutions for (B.2) are

$$Z = A \cos \beta z + B \sin \beta z \qquad (B.4)$$

where A and B are constants. Inserting (B.3) into (B.2) and multiplying by r^2 gives

$$\frac{r}{R}\frac{d}{dr}(r\frac{dR}{dr}) + (k^2 - B^2)\, r^2 + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0. \qquad (B.5)$$

Equation (B.5) must be true for all ϕ, therefore

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = - m^2 \qquad (B.6)$$

where m is a constant. Solutions of (B.6) are

$$\Phi = C \cos m\phi + D \sin m\phi. \qquad (B.7)$$

where C and D are constants. Inserting (B.6) into (B.5) gives

$$\frac{r}{R}\frac{d}{dr}(r\frac{dR}{dr}) + (k^2 - \beta^2)\, r^2 - m^2 = 0. \qquad (B.8)$$

Equation (B.8) may be written

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\alpha^2 r^2 - m^2) = 0 \qquad (B.9)$$

where $\alpha = \sqrt{k^2 - \beta^2}$. Equation (B.9) is well-known as Bessel's equation [48], [57]. Solutions of (B.9) are

$$R = E\, J_m(\alpha r) + F\, N_m(\alpha r)$$

where E and F are constants and $J_m(\alpha r)$ is called a Bessel function and $N_m(\alpha r)$ is called a Neumann function (sometimes called a Bessel function of the second kind). A particular linear combination of Bessel and Neumann functions is

$$R = J_m(\alpha r) - jN(\alpha r) = H_m^{(2)}(\alpha r). \qquad \text{(B.10)}$$

$H_m^{(2)}(\alpha r)$ is a Hankel function of the second kind. For large values of αr

$$H_m^{(2)}(\alpha r) \approx \sqrt{\frac{2}{\pi \alpha r}}\; e^{-j\alpha r}\; e^{j\left(\frac{m\pi}{2}+\frac{\pi}{4}\right)}.$$

Thus $H_m^{(2)}(\alpha r)$ represents outwardly propagating waves for $\alpha^2 > 0$.

For $\alpha^2 = -\alpha'^2 < 0$ solutions of (B.9) are $I_m(\alpha' r)$ and $K_m(\alpha' r)$, which are a modified Bessel function and a modified Hankel function respectively. Corresponding to

$$R = H_m^{(2)}(\alpha r)$$

which represents outwardly propagating waves for $\alpha^2 > 0$, we have

$$R = K_m^{(2)}(\alpha' r) \qquad \text{(B.11)}$$

for $\alpha^2 < 0$. Asymtotically

$$K_m(\alpha' r) \approx \sqrt{\frac{2}{\pi \alpha' r}}\, e^{-\alpha' r}$$

which is an exponentially decreasing function.

A complete set of solutions for the homogeneous wave equation may be written by combining Equations (B.4), (B.7), (B.10), and (B.11). Thus

$$V = V_1 + V_2 + V_3 + V_4$$

where

$$V_1 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A^{(1)}_{m,n}\, g_{m,n}(r) \cos m\phi \cos \beta_n z,$$

$$V_2 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A^{(2)}_{m,n}\, g_{m,n}(r) \cos m\phi \sin \beta_n z$$

$$V_3 = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} A^{(3)}_{m,n}\, g_{m,n}(r) \sin m\phi \cos \beta_n z$$

$$V_3 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A^{(4)}_{m,n}\, g_{m,n}(r) \sin m\phi \sin \beta_n z$$

and

$$g_{m,n} = \begin{cases} H^{(2)}_m(\alpha_m r) & \beta_n < k \\ K_m(\alpha'_n r) & \beta_n > k \end{cases}$$

with

$$\alpha_n = \sqrt{k^2 - \beta_n^2}$$

and

$$\alpha'_n = \sqrt{\beta^2_n - k^2}\,.$$

APPENDIX C

APPROXIMATE EXPRESSIONS USED WITH THE FREE SPACE GREEN'S FUNCTION

The approximation is often made that [35]

$$K(r,x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-jkR(\phi)}}{R(\phi)} \, d\phi \qquad (C.1)$$

is

$$K(r,x) \approx \frac{e^{-jkR}}{R} \qquad (C.2)$$

where

$$R(\phi) = \sqrt{r^2 + a^2 - 2ra\cos\phi + x^2}$$

and

$$R = \sqrt{r^2 + x^2} \, .$$

In the thesis it is necessary to have some estimate of the accuracy of this approximation. Also, it is necessary to obtain an expression for $\frac{\partial K(r,x)}{\partial r}$. Both of these topics are considered in Appendix C.

The study of the approximation in Equation (C.2) includes comparison of values for (C.2) with values obtained by numerically integrating (C.1) by Simpson's rule and with values obtained with a Fourier series for (C.1) given by Duncan and Hinchey [18].

The Fourier series given by Duncan and Hinchey is [18]

$$K(a,x) = \frac{D_o}{2} + \sum_{m=1}^{\infty} D_m \cos \frac{m\pi}{2h} x, \text{ for } 0 \leq x \leq 2h .$$

Let

$$\beta_m = \frac{m\pi}{2h}, \ \alpha_m = \sqrt{k^2 - \beta_n^2} \ , \text{ and } \alpha_m' = \sqrt{\beta_m^2 - k^2} .$$

The Fourier coefficients are given by

$$D_m = \frac{1}{h} \sqrt{\frac{\pi}{2}} \left[G(\beta_m) - G_2(\beta_m) \right]$$

where

$$G(\beta_m) = \begin{cases} -j\sqrt{\frac{\pi}{2}} \ J_o(\alpha_m a) \ H_o^{(2)}(\alpha_m a) & \beta_n < k \\ \\ \sqrt{\frac{2}{\pi}} \ I_o(\alpha_m' a) \ K_o(\alpha_m' a) & \beta_n > k, \end{cases}$$

$$G_2(\beta_m) = \begin{cases} -\sqrt{\frac{1}{2\pi}} \left(Ci[(k+\beta_m)2h] + Ci[(k-\beta_m)2h] \right. \\ \qquad \left. + j\pi - jSi[(k+\beta_m)2h] - jSi[(k-\beta_m)2h] \right) & \beta_n < k \\ -\sqrt{\frac{1}{2\pi}} \left(Ci[(\beta_m+k)2h] + Ci[(\beta_m-k)2h] \right. \\ \qquad \left. - jSi[(\beta_m+k)2h] + jSi[(\beta_m-k)2h] \right) & \beta_n > k. \end{cases}$$

For the case of $\beta_m = k$

$$D_m \approx \frac{1}{2h} \left\{ \ln\left(\frac{4h}{\gamma ka^2}\right) + Ci(4kh) - jSi(4kh) \right\}$$

where

$\gamma = e^C = 0.577...$

and C is Euler's number,

$C = 1.781...$

The numerical results comparing Equation(C.2) with (C.1) with $r = a$ by Simpson's rule and by the Duncan-Hinchey Fourier series are

shown in Figure (C.1a) in terms of the differences in percent between results. The Duncan-Hinchey Fourier series was truncated at n = 100.

The derivative of K(r,x) is

$$\frac{\partial K(r,x)}{\partial r} = \frac{-1}{2\pi} \int_{-\pi}^{\pi} e^{-jkR(\phi)} \left\{ \frac{jk}{R^2(\phi)} + \frac{1}{R^3(\phi)} \right\} (r - a\cos\phi)\, d\phi \qquad (C.3)$$

Write (C.3) as two integrals

$$\frac{\partial K(r,x)}{\partial r} = -\frac{r}{2\pi} \int_{-\pi}^{\pi} e^{-jkR(\phi)} \left\{ \frac{jk}{R^2(\phi)} + \frac{1}{R^3(\phi)} \right\} d\phi$$

$$+ \frac{a}{2\pi} \int_{-\pi}^{\pi} e^{-jkR(\phi)} \left\{ \frac{jk}{R^2(\phi)} + \frac{1}{R^3(\phi)} \right\} \cos\phi\, d\phi. \qquad (C.4)$$

For $x >> r \geq a$, $R(\phi)$ remains nearly constant as ϕ varies from $-\pi$ to π. Therefore the integrand of the first term in (C.4) remains nearly constant also while the integrand of the second term oscillates approximately as $\cos\phi$. These facts, together with insight derived from the approximation in Equation (C.2), leads one to expect that

$$\frac{\partial K(r,x)}{\partial r} \approx -\, r\, e^{-jkR} \left\{ \frac{jk}{R^2} + \frac{1}{R^3} \right\}. \qquad (C.5)$$

Comparison of numerical results for Equation (C.3) and (C.5) with $r = a$ are shown in Figure C.2a. Equation (C.3) was computed by Simpson's rule.

Figures C.1b and C.2b show comparisons made with the approximations

$$K(r,x) \approx \frac{e^{-jkR'}}{R'}$$

and

$$\frac{\partial K(r,x)}{\partial r} \approx - r\, e^{-jkR'} \left\{ \frac{jk}{R'^2} + \frac{1}{R'^3} \right\}$$

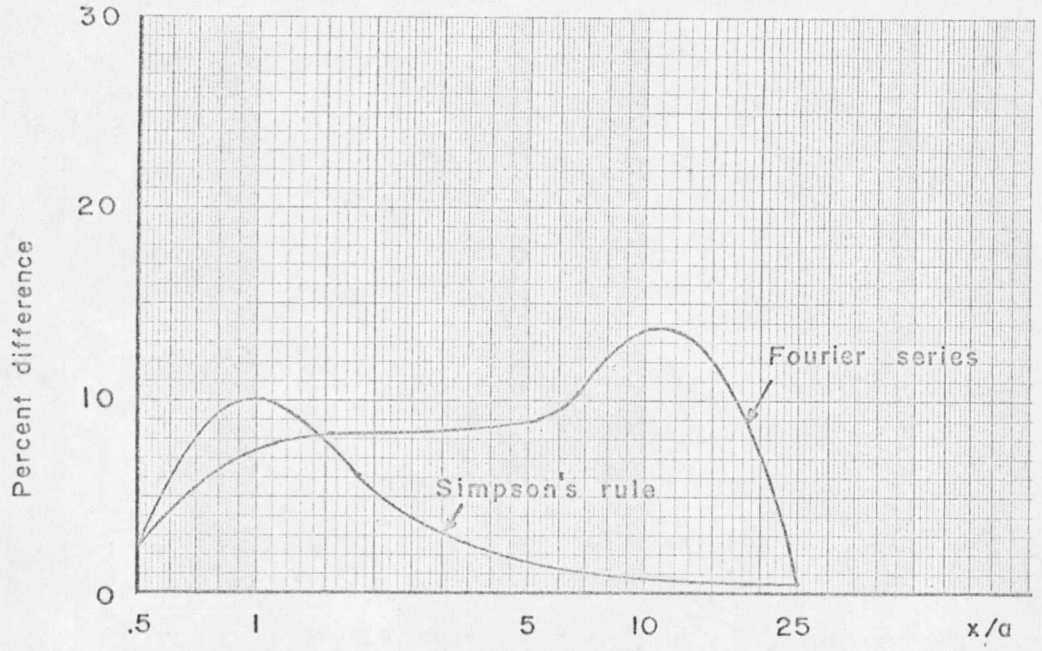
where

$$R' = \sqrt{r^2 + a^2 + x^2}$$

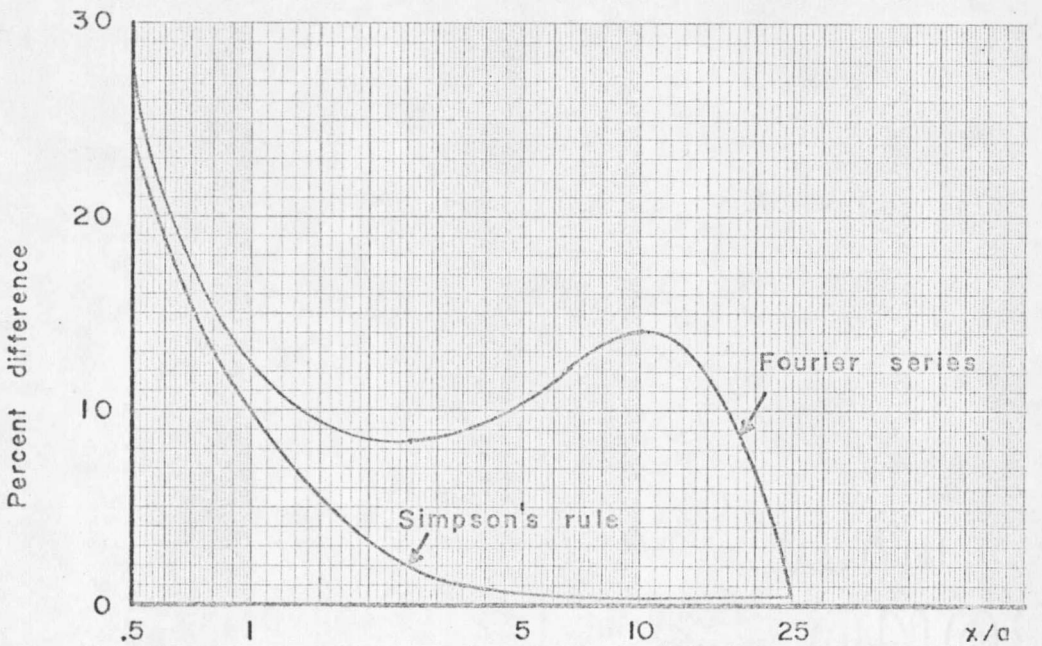
with

$r = a$.

Comparing these results with those for $R = \sqrt{r^2 + x^2}$ with $r = a$ indicate that the results for $K(r,x)$ are better with R, but the results for $\frac{\partial K(r,x)}{\partial r}$ are better with R'.

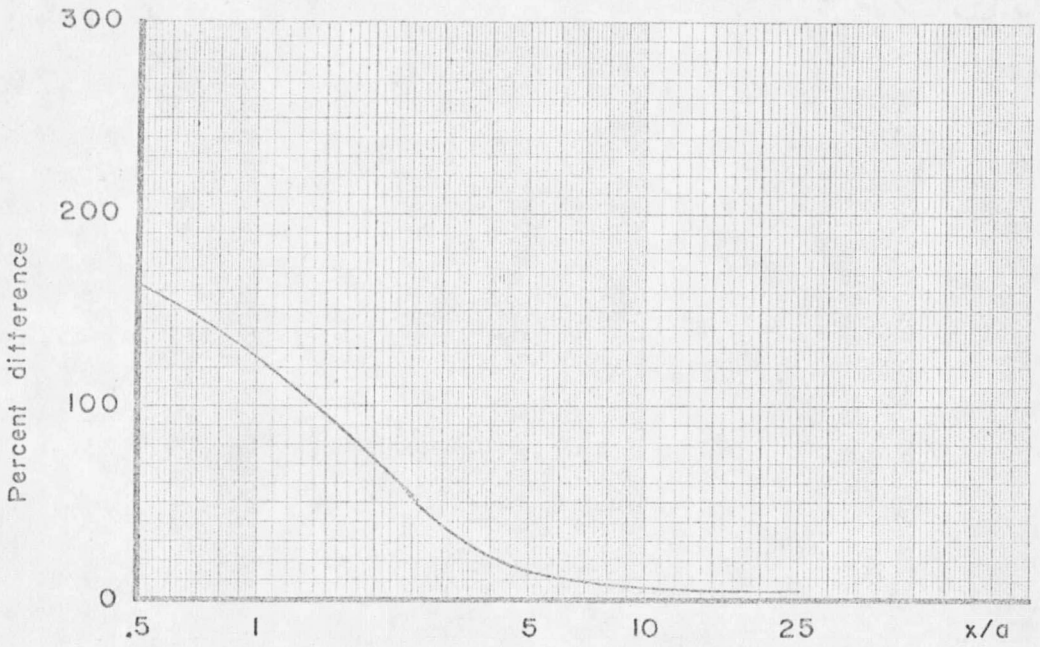


(a) $R=\sqrt{a^2+x^2}$

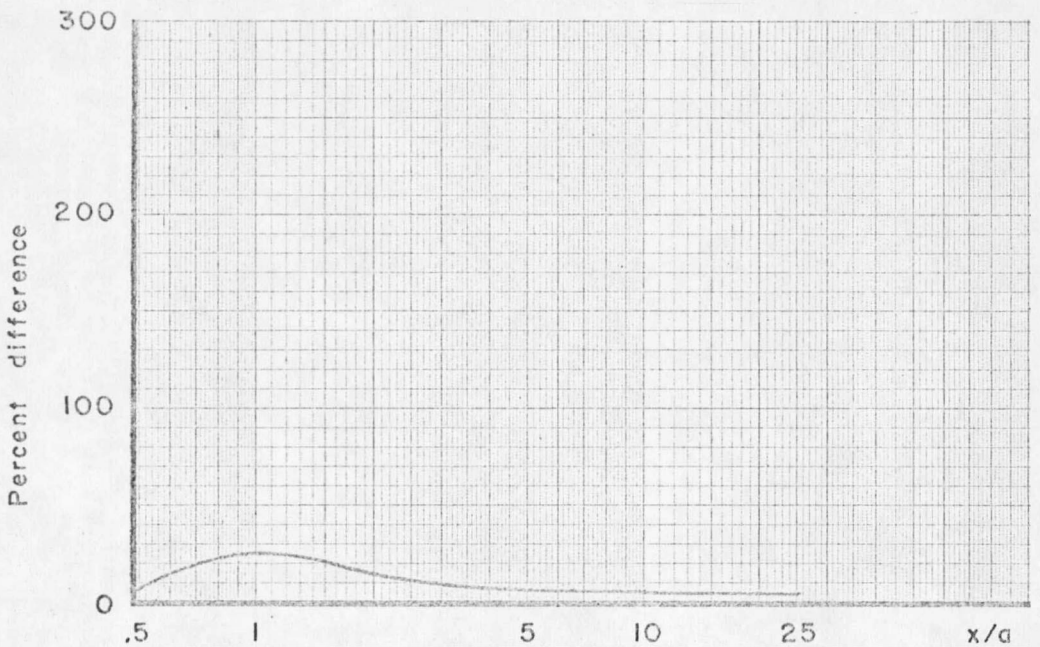


(b) $R'=\sqrt{2a^2+x^2}$

Fig. C.1 Comparison of free space Green's function approximations.



(a) $R=\sqrt{a^2+x^2}$



(b) $R'=\sqrt{2a^2+x^2}$

Fig. C.2 Comparison of free space Green's function derivative approximations.

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