MODELING PIEZOELECTRIC PVDF SHEETS WITH CONDUCTIVE POLYMER ELECTRODES

by

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ABSTRACT

The main concern of my research has been to find a good way to solve for the behavior of piezoelectric devices that are electroded not with metal electrodes (as has traditionally been the case) but with a conductive polymer material which has a much lower conductivity compared to metal. In this situation, if a time-varying voltage is applied at one end of the electrode, the voltage cannot be assumed to be uniform throughout the electrode because of the effects of resistivity. Determining the voltage in the electrodes as a function of time and position concurrently with the mechanical and electrical response of the piezoelectric material presents an added complexity. In this thesis the problem of the piezoelectric monomorph is considered. The piezoelectric sheet is PVDF, and the electrodes are PEDOT-PSS. As a first approximation the two problems of finding the voltage in the electrodes and the mechanical deformation in the piezoelectric material are decoupled. In order to determine the voltage distribution in the electrodes, the piezoelectric effects were neglected, which reduced the piezoelectric problem to a capacitor problem. Once the voltage function was determined the mechanical deformation of the PVDF sheet was calculated given the known voltage distribution as a function of position and time.

CHAPTER 1

INTRODUCTION

Piezoelectricity and PVDF

Piezoelectricity is a linear coupling between electrical and mechanical processes [1]. In the direct piezoelectric effect, when a piezoelecctric material is compressed, an electric polarization is formed across the material. In fact, the prefix piezo is derived from the Greek word for press [1]. The converse piezoelectric effect is when an applied electric field causes the piezoelectric to mechanically deform.

Piezoelectricity is made possible due to certain kinds of crystal structures which lack a center of symmetry. Materials which are piezoelectric come in several forms. There are single crystals, ceramics, and polymers (semi-crystalline) [2].

Poly(vinylidene-fluoride) (PVDF) is a piezoelectric polymer, which for actuator applications often comes in the form of a thin sheet (30 microns thick). It is stretched along the x direction to align the long chain molecules, and is poled in the z direction to align the electro-negative and electro-positive parts of the molecular units (the hydrogen and fluorine atoms) to create a strong piezoelectric constant, as shown in figure 1.1.

Because of the aligned ions, there is a charge polarization. When an electric field is applied across the PVDF sheet, the molecules will either stretch or contract, depending on the direction of the field, as shown in figures 1.2 and 1.3. The other dimensions of the PVDF sheet will also change. The thickness of the sheet is very small, but the length is substantial and even an elongation of only a small percent will be noticable. When an electric field is applied across two sheets that are glued

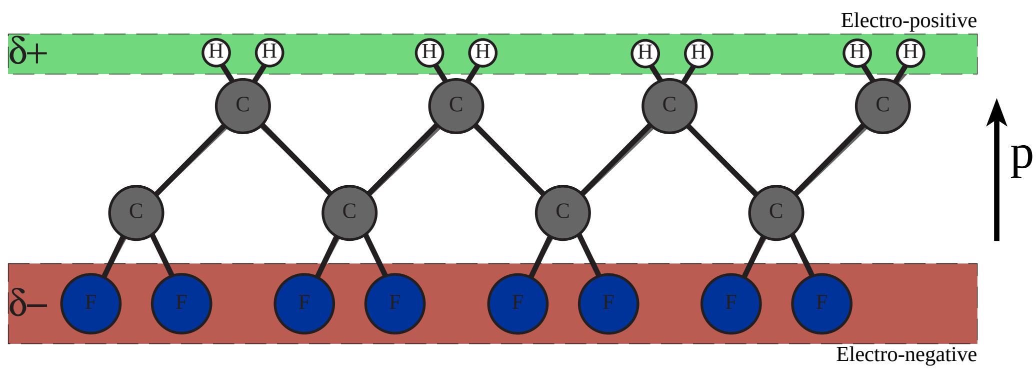

Figure 1.1: PVDF chain.

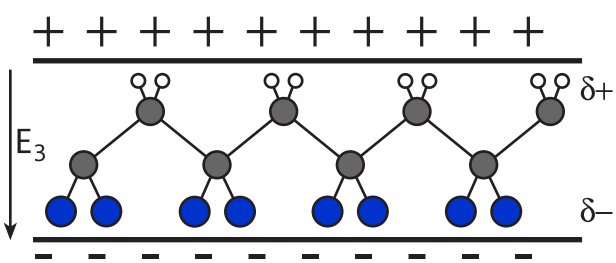

Figure 1.2: PVDF stretched.

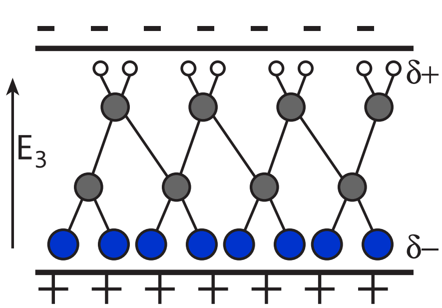

Figure 1.3: PVDF contracted.

together with opposite polarization, the sheets will bend out of the plane. This is a bimorph, and the deflection is much larger than the elongation of an individual sheet. This conversion from electric field to mechanical deformation and vice versa is very useful.

Conducting Polymers and PEDOT-PSS

In order to apply an electric field, the PVDF sheet must have applied electrodes. In small deformation applications the electrodes can be metal, which is very good because it has very high conductivity and the voltage throughout the electrode can be assumed to be uniform. The drawback with metal is that it is stiffer than the PVDF polymer and so hinders its deformation. Polymer electrodes are more flexible, but the conductivity is much lower, and so the voltage is not uniform. The higher the frequency, the more non-uniform the voltage will be. A big problem is certainly the amplitude attenuation along the electrode. Voltage amplitude is decreased by the resistivity (Ohm's law: $\Delta V = IR$).

One conducting polymer is PEDOT-PSS. Poly(ethylene dioxythiophene) is a conjugated polymer, and poly(styrene sulfonate) is a dopant which dramatically increases the conductivity of PEDOT [2]. See figure 1.5 for the molecular representation of PEDOT-PSS and figure 1.6 for a picture of PEDOT-PSS in it's liquid form.

There are several ways to apply the polymer electrodes onto the PVDF sheets. One method that has been attempted is spraying [2]. In this method a spray gun is used to spray liquid PEDOT-PSS onto the surface of the PVDF, which then is allowed to dry. See figure 1.4 to see an example of this technique. PVDF is hydrophobic, and since PEDOT-PSS comes in a water-diluted form, if the PEDOT-PSS is applied too

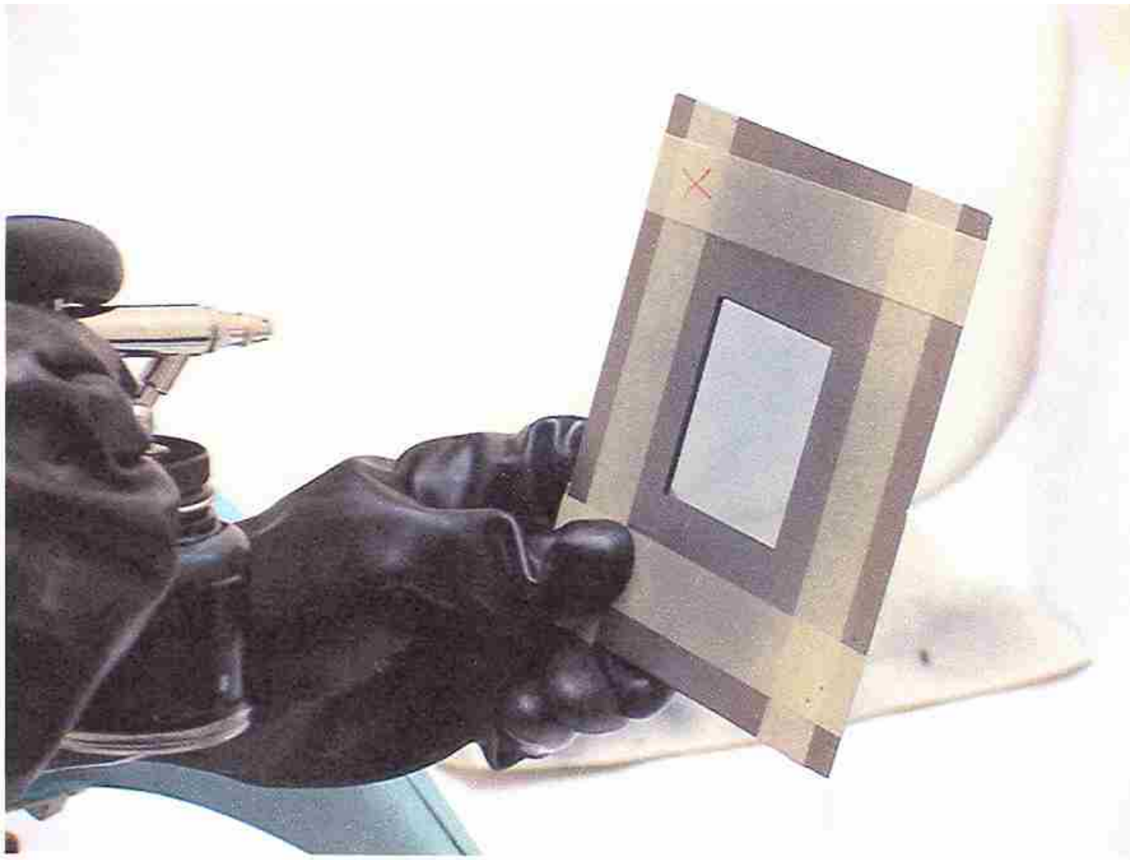

Figure 1.4: Air-brushing technique.

thickly it will bead up. A method that seems to work quite well is inkjet printing. In this method PEDOT-PSS is printed onto the PVDF sheets using an ordinary inkjet printer. The thickness of the applied layers can be easily controlled to produce uniform layers [3].

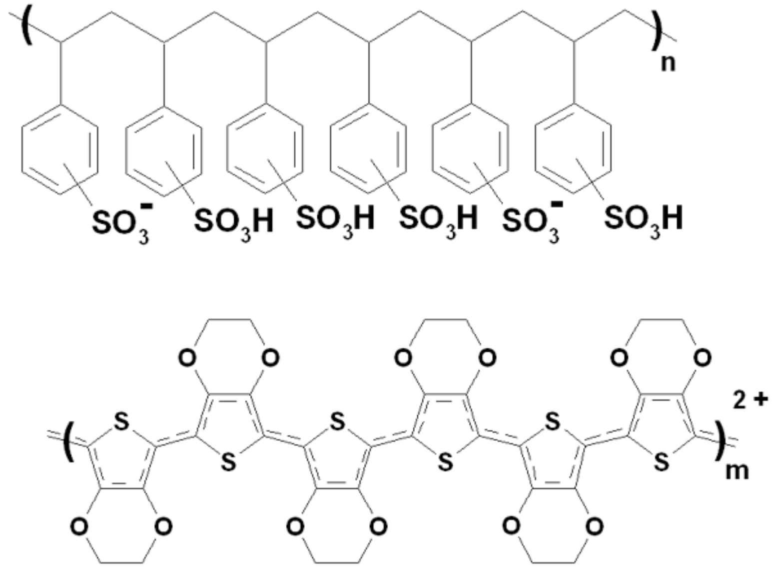

Figure 1.5: PEDOT-PSS molecular formula.

Figure 1.6: Liquid PEDOT-PSS.

CHAPTER 2

CONSTITUTIVE FORMULAS

There are four state variables which describe the electro-mechanical state of a piezoelectric actuator. The two mechanical state variables are stress (T), and strain (S). The two electric state variables are electric field (E) and electric displacement (D). In a piezoelectric material the two sets of state variables are coupled, which means both sets must be determined simultaneously, as one affects the other. The fundamental (S-E)-type relation [1, Table 2.1(b)] is given in equation (2.1). There are other, equally valid, sets of state variables to use. For example, instead of D, the electric polarization (P) can be used, in which case we would have an (S-P)type relation.

$$\boldsymbol{T} = \boldsymbol{c}^E \cdot \boldsymbol{S} - \boldsymbol{e} \cdot \vec{E} \tag{2.1a}$$

$$\vec{D} = \boldsymbol{e} \cdot \boldsymbol{S} + \boldsymbol{\varepsilon}^S \cdot \vec{E} \tag{2.1b}$$

The (S-E)type relation using full tensor index notation is given in equation (2.2).

$$T_{ij} = c^E_{ijkl} S_{kl} - e_{mij} E_m \tag{2.2a}$$

$$D_n = e_{nkl} S_{kl} + \varepsilon^S_{nm} E_m \tag{2.2b}$$

The stress is related to the applied and generated forces. The strain can be defined in terms of the mechanical displacements from equilibrium (u_i). Throughout this thesis the "1" direction refers to the x direction, the "2" direction to the y direction, and the "3" direction to the z direction. In every case the piezoelectric sheet is oriented so that the poled direction is the z direction, and the stretch direction is along x.

$$S_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.3}$$

For tensors that have symmetric pairs of indices, those pairs can be condensed into a single index. This allows expressions to be condensed by combining equal terms. For indices which can take on values of 1, 2, or 3, a condensed pair can take on the values 1 through 6. This condensed form is call matrix index notation.

$$11 \rightarrow 1, \quad 22 \rightarrow 2, \quad 33 \rightarrow 3, \quad 23 \rightarrow 4, \quad 13 \rightarrow 5, \quad 12 \rightarrow 6 \tag{2.4}$$

There is a convention for how to combine terms to form a single condensed term. Sometimes there is a factor of 2 or 4. This is done so that the final matrix index expression will look the same as the tensor index expression, without any extra factors in front of terms. The conversion for some quantities is given in equations (2.5) through (2.8).

$$e_{mij} \rightarrow e_{mn} \tag{2.5}$$

$$c_{ijkl} \rightarrow c_{\lambda\mu} \tag{2.6}$$

$$T_{ij} \rightarrow T_{\lambda} \tag{2.7}$$

$$S_{ij} \rightarrow S_{\lambda} \quad (i = j), \qquad 2S_{ij} \rightarrow S_{\lambda} \quad (i \neq j) \tag{2.8}$$

The matrix index notation version of the constitutive relation is given in equation (2.9).

$$T_{\lambda} = c^{E}_{\lambda\mu} S_{\mu} - e_{m\lambda} E_m \tag{2.9a}$$

$$D_n = e_{n\mu} S_{\mu} + \varepsilon^{S}_{nm} E_m \tag{2.9b}$$

The form of the matrices for the material constants are given below. The dielectric permittivity is ε, the piezoelectric constant is e, and the elastic stiffness is c. PVDF has mm2 symmetry, and so many entries are zero, which simplifies the resulting equations.

$$\varepsilon_{mn} = \begin{bmatrix} \varepsilon_1 & \cdot & \cdot \\ \cdot & \varepsilon_2 & \cdot \\ \cdot & \cdot & \varepsilon_3 \end{bmatrix} \tag{2.10}$$

$$e_{i\lambda} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & e_{15} & \cdot \\ \cdot & \cdot & \cdot & e_{24} & \cdot & \cdot \\ e_{31} & e_{32} & e_{33} & \cdot & \cdot & \cdot \end{bmatrix} \tag{2.11}$$

$$c_{\lambda\mu} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdot & \cdot & \cdot \\ c_{12} & c_{22} & c_{23} & \cdot & \cdot & \cdot \\ c_{13} & c_{23} & c_{33} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & c_{44} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & c_{55} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & c_{66} \end{bmatrix} \tag{2.12}$$

After explicitly writing out all the terms and substituting in the non-zero constants, we get the full general constitutive equations for mm2 symmetry.

$$T_1 = c_{11}S_1 + c_{12}S_2 + c_{13}S_3 - e_{31}E_3 \tag{2.13a}$$

$$T_2 = c_{12}S_1 + c_{22}S_2 + c_{23}S_3 - e_{32}E_3 \tag{2.13b}$$

$$T_3 = c_{13}S_1 + c_{23}S_2 + c_{33}S_3 - e_{33}E_3 \tag{2.13c}$$

$$T_4 = c_{44}S_4 - e_{24}E_2 \tag{2.13d}$$

$$T_5 = c_{55}S_5 - e_{15}E_1 \tag{2.13e}$$

$$T_6 = c_{66}S_6 \tag{2.13f}$$

$$D_1 = e_{15}S_5 + \varepsilon_1 E_1 \tag{2.14a}$$

$$D_2 = e_{24}S_4 + \varepsilon_2 E_2 \tag{2.14b}$$

$$D_3 = e_{31}S_1 + e_{32}S_2 + e_{33}S_3 + \varepsilon_3 E_3 \tag{2.14c}$$

$$S_1 = \frac{\partial u_1}{\partial x_1} \tag{2.15a}$$
$$S_2 = \frac{\partial u_2}{\partial x_2} \tag{2.15b}$$
$$S_3 = \frac{\partial u_3}{\partial x_3} \tag{2.15c}$$
$$S_4 = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \tag{2.15d}$$
$$S_5 = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \tag{2.15e}$$
$$S_6 = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \tag{2.15f}$$

Equations of motion:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial T_{ij}}{\partial x_j} \tag{2.16}$$

$$\rho \frac{\partial^2 u_1}{\partial t^2} = \frac{\partial T_1}{\partial x_1} + \frac{\partial T_6}{\partial x_2} + \frac{\partial T_5}{\partial x_3} \tag{2.17a}$$
$$\rho \frac{\partial^2 u_2}{\partial t^2} = \frac{\partial T_6}{\partial x_1} + \frac{\partial T_2}{\partial x_2} + \frac{\partial T_4}{\partial x_3} \tag{2.17b}$$
$$\rho \frac{\partial^2 u_3}{\partial t^2} = \frac{\partial T_5}{\partial x_1} + \frac{\partial T_4}{\partial x_2} + \frac{\partial T_3}{\partial x_3} \tag{2.17c}$$

Maxwell equations:

$$\vec{\nabla} \cdot \vec{D} = \rho_f \tag{2.18}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.19}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{2.20}$$

$$\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}_f \tag{2.21}$$

Electromagnetic constitutive relations:

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \tag{2.22}$$

$$\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M} \tag{2.23}$$

$$\vec{J}_f = \sigma_{\mathrm{cond}} \vec{E} \tag{2.24}$$

Since the PVDF actuators are composed of non-magnetic materials, $M = 0$, and

$$\vec{H} = \vec{B}/\mu_0. \tag{2.25}$$

In the electrodes:

$$\vec{\nabla} \cdot \vec{E} = \rho_f/\varepsilon\varepsilon_0 \tag{2.26}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.27}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{2.28}$$

$$\frac{1}{\mu_0}\vec{\nabla} \times \vec{B} - \varepsilon\varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \sigma_{\text{cond}}\vec{E} \tag{2.29}$$

In the piezoelectric material:

$$\vec{\nabla} \cdot \vec{D} = 0 \tag{2.30}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.31}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{2.32}$$

$$\frac{1}{\mu_0}\vec{\nabla} \times \vec{B} - \frac{\partial \vec{D}}{\partial t} = 0 \tag{2.33}$$

CHAPTER 3

CAPACITOR MODEL

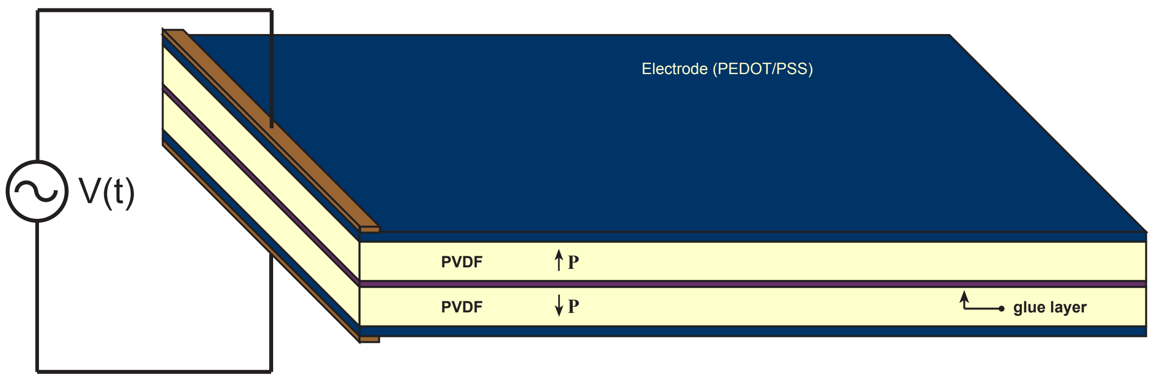

Figure 3.1: Schematic of a bimorph.

The voltage distribution in the electrodes of a bimorph can be quite complicated if the conductivity of the electrode material is not very high. As a first approximation the bimorph can be treated as a simple parallel-plate capacitor, as shown in figure 3.2. In this case the piezoelectric behavior is not included in the calculations.

Although we have currents and a time-changing electric field, and therefore a magnetic field, we can treat the electric field quasi-electrostatically, and neglect magnetic contributions, so that the electric field is equal to the negative gradient of the electric potential. The electric field across the thickness of the capacitor is E_3 (in the z

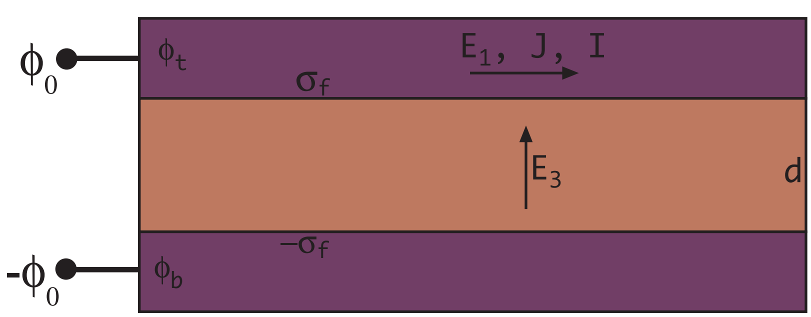

Figure 3.2: Capacitor schematic.

direction), while the electric field in the electrodes is approximated as only being in the x direction, and is called E_1. For this problem there is an applied voltage applied to the top and bottom electrodes at $x = 0$. In order to have a symmetric solution the applied voltage is such that the voltage on the bottom electrode is the negative of the voltage on the top electrode.

$$\vec{E} = -\vec{\nabla}\phi \tag{3.1}$$

$$E_3 = -\frac{\partial \phi}{\partial z} = -\frac{\phi_t - \phi_b}{d} = -\frac{2\phi_t}{d} \tag{3.2}$$

The electric field inside a parallel plate capacitor is proportional to the free surface charge density on the plates, where σ_t is the charge density on the top electrode, and σ_b is the charge density on the bottom electrode. Because of symmetry $\sigma_b = -\sigma_t$.

$$E_3 = -\frac{\sigma_t - \sigma_b}{2\varepsilon\varepsilon_0} = -\frac{\sigma_t}{\varepsilon\varepsilon_0} \tag{3.3}$$

By combining equations 3.2 and 3.3 we get an expression for the surface charge density in terms of the electric potential.

$$\sigma_t = \frac{2\varepsilon\varepsilon_0}{d}\phi_t \tag{3.4}$$

We can use the charge continuity equation and the equation for the current density in a simple conductor to derive an expression for the voltage distribution in the electrodes.

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho_f}{\partial t} \tag{3.5}$$

$$\vec{J} = \boldsymbol{\sigma}_{cond} \cdot \vec{E} \tag{3.6}$$

$$\vec{\nabla} \cdot \vec{J} = \vec{\nabla} \cdot \left(\boldsymbol{\sigma}_{cond} \cdot \vec{E}\right) = -\vec{\nabla} \cdot \left(\boldsymbol{\sigma}_{cond} \cdot \vec{\nabla}\phi\right) \tag{3.7}$$

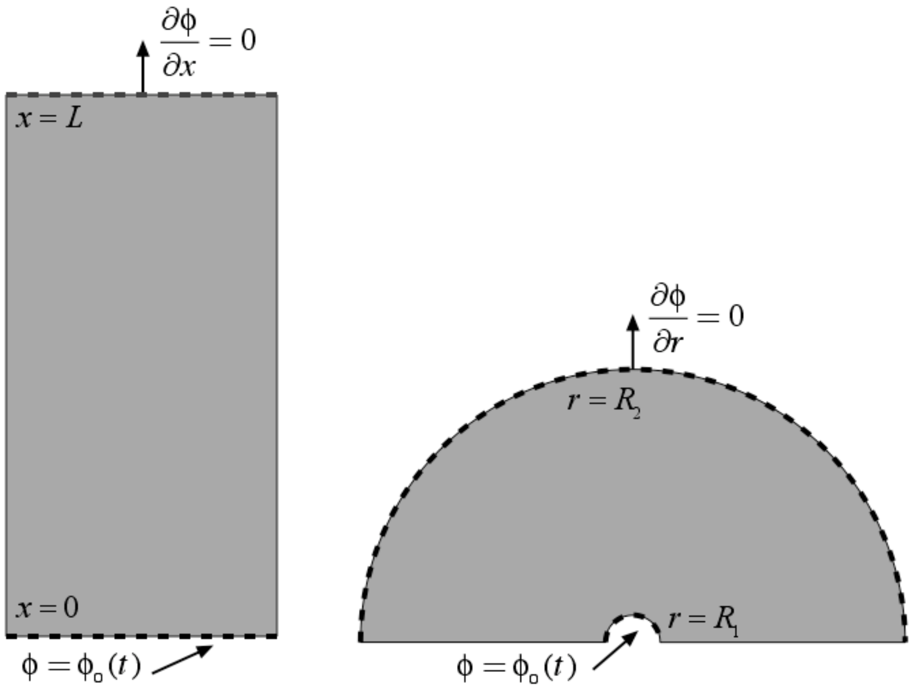

Figure 3.3: Boundary conditions for rectangular and cylindrical geometries.

The general differential equation for the electric potential, before any further approximations are made, is given below.

$$\vec{\nabla} \cdot \left(\boldsymbol{\sigma}_{cond} \cdot \vec{\nabla}\phi \right) = \frac{\partial \rho_f}{\partial t} \tag{3.8}$$

In general the conductivity is a tensor, but in this case the electrode material is isotropic. In both the rectangular and cylindrical cases, shown in figure 3.3, only the ∇_1 term survives because the current is assumed to flow only in either the x or radial direction due to symmetry. The volume charge density can be approximately expressed in terms of the surface charge density because the electrodes are so thin. The thickness of an electrode is w, its resistivity is ρ, and the thickness of the PVDF sheet is d. The permittivity of free space is ε_0, and the relative permittivity of PVDF is ε.

$$\rho_f \approx \frac{\sigma}{w} = \frac{2\varepsilon\varepsilon_0}{wd}\phi_t \tag{3.9}$$

Rectangular Case

For the rectangular case, where the current flows only in the x direction,

$$\frac{\partial \phi_t}{\partial t} = k \frac{\partial^2 \phi_t}{\partial x^2}, \quad k = \frac{wd}{2\rho\varepsilon\varepsilon_0} \tag{3.10}$$

with boundary conditions

$$\phi_t(0,t) = \frac{V_0}{2} \sin \omega t, \quad \left. \frac{\partial \phi_t}{\partial x} \right|_{x=L} = 0, \tag{3.11}$$

where V_0 is the amplitude of the applied voltage. For simplicity the input voltage function is sinusoidal. In general, of course, the input voltage can by any desired function.

The resulting differential equation, equation 3.10, has the same form as the one-dimensional diffusion equation. In the rectangular case the solution is comprised of sines and cosines. We can get the full solution by assuming a set of incident and reflected waves, and solving for the unknown coefficients.

$$\phi_t = C_{\text{inc}} f_1(x,t) + D_{\text{inc}} f_2(x,t) + C_{\text{ref}} f_3(x,t) + D_{\text{ref}} f_4(x,t) \tag{3.12}$$

$$f_1(x,t) = e^{-\alpha x} \cos(\alpha x - \omega t) \tag{3.13a}$$

$$f_2(x,t) = e^{-\alpha x} \sin(\alpha x - \omega t) \tag{3.13b}$$

$$f_3(x,t) = e^{\alpha x} \cos(\alpha x + \omega t) \tag{3.13c}$$

$$f_4(x,t) = e^{\alpha x} \sin(\alpha x + \omega t) \tag{3.13d}$$

Some derivatives of the f_i which will be useful later are listed below.

$$\dot{f} \equiv \frac{\partial f}{\partial t}, \qquad f' \equiv \frac{\partial f}{\partial x} \tag{3.14}$$

$$\dot{f}_1 = \omega f_2 \tag{3.15a}$$

$$\dot{f}_2 = -\omega f_1 \tag{3.15b}$$

$$\dot{f}_3 = -\omega f_4 \tag{3.15c}$$

$$\dot{f}_4 = \omega f_3 \tag{3.15d}$$

$$\ddot{f}_1 = -\omega^2 f_1 \tag{3.16a}$$

$$\ddot{f}_2 = -\omega^2 f_2 \tag{3.16b}$$

$$\ddot{f}_3 = -\omega^2 f_3 \tag{3.16c}$$

$$\ddot{f}_4 = -\omega^2 f_4 \tag{3.16d}$$

$$f_1' = -\alpha\left(f_1 + f_2\right) \tag{3.17a}$$

$$f_2' = \alpha\left(f_1 - f_2\right) \tag{3.17b}$$

$$f_3' = \alpha\left(f_3 - f_4\right) \tag{3.17c}$$

$$f_4' = \alpha\left(f_3 + f_4\right) \tag{3.17d}$$

$$f_1'' = 2\alpha^2 f_2 \tag{3.18a}$$

$$f_2'' = -2\alpha^2 f_1 \tag{3.18b}$$

$$f_3'' = -2\alpha^2 f_4 \tag{3.18c}$$

$$f_4'' = 2\alpha^2 f_3 \tag{3.18d}$$

$$\dot{f}_1' = \alpha\omega\left(f_1 - f_2\right) \tag{3.19a}$$

$$\dot{f}_2' = \alpha\omega\left(f_1 + f_2\right) \tag{3.19b}$$

$$\dot{f}_3' = -\alpha\omega\left(f_3 + f_4\right) \tag{3.19c}$$

$$\dot{f}_4' = \alpha\omega\left(f_3 - f_4\right) \tag{3.19d}$$

$$\dot{f}_1'' = -2\alpha^2\omega f_1 \tag{3.20a}$$

$$\dot{f}_2'' = -2\alpha^2\omega f_2 \tag{3.20b}$$

$$\dot{f}_3'' = -2\alpha^2\omega f_3 \tag{3.20c}$$

$$\dot{f}_4'' = -2\alpha^2\omega f_4 \tag{3.20d}$$

When we put our form of the solution through the differential equation we get the following requirement for α.

$$\alpha = \sqrt{\frac{\omega}{2k}} \tag{3.21}$$

By applying the first boundary condition we get the following conditions on the coefficients.

$$\begin{aligned} \phi_t(0,t) = (C_{\text{inc}} + C_{\text{ref}})\cos(\omega t) + (D_{\text{ref}} - D_{\text{inc}})\sin(\omega t) = \frac{V_0}{2}\sin(\omega t) \\ \rightarrow \qquad C_{\text{inc}} + C_{\text{ref}} = 0, \qquad D_{\text{ref}} - D_{\text{inc}} = \frac{V_0}{2} \end{aligned} \tag{3.22}$$

We can make the substitutions

$$C_{\text{ref}} = -C_{\text{inc}}, \qquad D_{\text{ref}} = \frac{V_0}{2} + D_{\text{inc}}. \tag{3.23}$$

After applying the second boundary condition we can fully specify the coefficients.

$$\begin{aligned} \phi'(L,t) &= \alpha\Big[-C_{\text{inc}}(f_1 + f_2) + D_{\text{inc}}(f_1 - f_2) \\ &\qquad + C_{\text{inc}}(f_4 - f_3) + \left(\frac{V_0}{2} + D_{\text{inc}}\right)(f_3 + f_4)\Big]_{x=L} \\ &= \alpha\Big[(D_{\text{inc}} - C_{\text{inc}})f_1 - (C_{\text{inc}} + D_{\text{inc}})f_2 \\ &\qquad + \left(\frac{V_0}{2} + D_{\text{inc}} - C_{\text{inc}}\right)f_3 + \left(\frac{V_0}{2} + C_{\text{inc}} + D_{\text{inc}}\right)f_4\Big]_{x=L} \\ &= 0 \end{aligned} \tag{3.24}$$

The f_i can be separated into terms of $\cos\omega t$ and $\sin\omega t$, and then equation (3.24) can be but into the form $C_c\cos\omega t + C_s\sin\omega t = 0$. Because $\cos\omega t$ and $\sin\omega t$ are orthogonal, both expressions C_c and C_s must individually equal zero.

$$f_1(L,t) = e^{-\alpha L}\cos(\alpha L)\cos(\omega t) + e^{-\alpha L}\sin(\alpha L)\sin(\omega t) \tag{3.25a}$$

$$f_2(L,t) = e^{-\alpha L}\sin(\alpha L)\cos(\omega t) - e^{-\alpha L}\cos(\alpha L)\sin(\omega t) \tag{3.25b}$$

$$f_3(L,t) = e^{\alpha L}\cos(\alpha L)\cos(\omega t) - e^{\alpha L}\sin(\alpha L)\sin(\omega t) \tag{3.25c}$$

$$f_4(L,t) = e^{\alpha L}\sin(\alpha L)\cos(\omega t) + e^{\alpha L}\cos(\alpha L)\sin(\omega t) \tag{3.25d}$$

$$\begin{aligned}
C_c &= (D_{\text{inc}} - C_{\text{inc}})\, e^{-\alpha L}\cos\alpha L - (C_{\text{inc}} + D_{\text{inc}})\, e^{-\alpha L}\sin\alpha L \\
&\quad + \left(\frac{V_0}{2} + D_{\text{inc}} - C_{\text{inc}}\right) e^{\alpha L}\cos\alpha L + \left(\frac{V_0}{2} + C_{\text{inc}} + D_{\text{inc}}\right) e^{\alpha L}\sin\alpha L \\
&= C_{\text{inc}}\Big[-e^{-\alpha L}\Big(\cos\alpha L + \sin\alpha L\Big) + e^{\alpha L}\Big(\sin\alpha L - \cos\alpha L\Big)\Big] \\
&\quad + D_{\text{inc}}\Big[e^{-\alpha L}\Big(\cos\alpha L - \sin\alpha L\Big) + e^{\alpha L}\Big(\cos\alpha L + \sin\alpha L\Big)\Big] \\
&\quad + \frac{V_0}{2}e^{\alpha L}\Big(\cos\alpha L + \sin\alpha L\Big) \\
&= 0
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
C_s &= (D_{\text{inc}} - C_{\text{inc}})\, e^{-\alpha L}\sin\alpha L + (C_{\text{inc}} + D_{\text{inc}})\, e^{-\alpha L}\cos\alpha L \\
&\quad - \left(\frac{V_0}{2} + D_{\text{inc}} - C_{\text{inc}}\right) e^{\alpha L}\sin\alpha L + \left(\frac{V_0}{2} + C_{\text{inc}} + D_{\text{inc}}\right) e^{\alpha L}\cos\alpha L \\
&= C_{\text{inc}}\Big[e^{-\alpha L}\Big(\cos\alpha L - \sin\alpha L\Big) + e^{\alpha L}\Big(\cos\alpha L + \sin\alpha L\Big)\Big] \\
&\quad + D_{\text{inc}}\Big[e^{-\alpha L}\Big(\cos\alpha L + \sin\alpha L\Big) + e^{\alpha L}\Big(\cos\alpha L - \sin\alpha L\Big)\Big] \\
&\quad + \frac{V_0}{2}e^{\alpha L}\Big(\cos\alpha L - \sin\alpha L\Big) \\
&= 0
\end{aligned} \tag{3.27}$$

Equations (3.26) and (3.27) can be somewhat simplified.

$$C_{\text{inc}}\left(e^{\alpha L} - e^{-\alpha L}\right)\sin\alpha L + D_{\text{inc}}\left(e^{-\alpha L} + e^{\alpha L}\right)\cos\alpha L = -\frac{V_0}{2}e^{\alpha L}\cos\alpha L \tag{3.28}$$

$$C_{\text{inc}}\left(e^{-\alpha L} + e^{\alpha L}\right)\cos\alpha L + D_{\text{inc}}\left(e^{-\alpha L} - e^{\alpha L}\right)\sin\alpha L = \frac{V_0}{2}e^{\alpha L}\sin\alpha L \tag{3.29}$$

With these two equation and equation 3.23 we now have four equations for the four coefficients. The expressions for the coefficients are shown below.

$$\begin{aligned} C_{\text{inc}} &= V_0 \frac{\sin(\alpha L)\cos(\alpha L)}{e^{-2\alpha L} + 2\left(\cos^2\alpha L - \sin^2\alpha L\right) + e^{2\alpha L}} \\ &= \frac{V_0}{2} \frac{\sin 2\alpha L}{e^{-2\alpha L} + 2\cos 2\alpha L + e^{2\alpha L}} \end{aligned} \tag{3.30}$$

$$\begin{aligned} D_{\text{inc}} &= -\frac{V_0}{2} \frac{e^{\alpha L}\left[e^{\alpha L} + e^{-\alpha L}\left(\cos^2\alpha L - \sin^2\alpha L\right)\right]}{e^{-2\alpha L} + 2\left(\cos^2\alpha L - \sin^2\alpha L\right) + e^{2\alpha L}} \\ &= -\frac{V_0}{2} \frac{e^{2\alpha L} + \cos 2\alpha L}{e^{-2\alpha L} + 2\cos 2\alpha L + e^{2\alpha L}} \end{aligned} \tag{3.31}$$

$$C_{\text{ref}} = -\frac{V_0}{2} \frac{\sin 2\alpha L}{e^{-2\alpha L} + 2\cos 2\alpha L + e^{2\alpha L}} \tag{3.32}$$

$$D_{\text{ref}} = \frac{V_0}{2} \frac{e^{-2\alpha L} + \cos 2\alpha L}{e^{-2\alpha L} + 2\cos 2\alpha L + e^{2\alpha L}} \tag{3.33}$$

The full expression for the voltage on the top electrode is given below. The full potential difference across the sheet is twice ϕ_t.

$$\begin{aligned} \phi_t(x,t) = \frac{V_0}{2} &\frac{1}{e^{-2\alpha L} + 2\cos 2\alpha L + e^{2\alpha L}} * \\ &\left\{ e^{-\alpha x}\left[\sin 2\alpha L \cos(\alpha x - \omega t) - \left(e^{2\alpha L} + \cos 2\alpha L\right)\sin(\alpha x - \omega t)\right]\right. \\ &\left. + e^{\alpha x}\left[-\sin 2\alpha L \cos(\alpha x + \omega t) + \left(e^{-2\alpha L} + \cos 2\alpha L\right)\sin(\alpha x + \omega t)\right]\right\} \end{aligned} \tag{3.34}$$

The potential function is sinusoidal in time, and so it can be expressed as a sine function in t with an x dependent phase (relative to the input voltage).

$$\phi_t = |\phi_t| \sin(\omega t + \delta) = |\phi_t| \sin\delta \cos\omega t + |\phi_t| \cos\delta \sin\omega t \tag{3.35}$$

$$
\begin{aligned}
|\phi_t|\sin\delta = \frac{V_0}{2}\frac{1}{e^{-2\alpha L}+2\cos 2\alpha L+e^{2\alpha L}} * \Big[&\left(e^{-\alpha x}-e^{\alpha x}\right)\sin 2\alpha L\cos\alpha x \\
&+\Big(\left(e^{-2\alpha L}+\cos 2\alpha L\right)e^{\alpha x}-\left(e^{2\alpha L}+\cos 2\alpha L\right)e^{-\alpha x}\Big)\sin\alpha x\Big]
\end{aligned} \tag{3.36}
$$

$$
\begin{aligned}
|\phi_t|\cos\delta = \frac{V_0}{2}\frac{1}{e^{-2\alpha L}+2\cos 2\alpha L+e^{2\alpha L}} * \Big[&\left(e^{-\alpha x}+e^{\alpha x}\right)\sin 2\alpha L\sin\alpha x \\
&+\Big(\left(e^{-2\alpha L}+\cos 2\alpha L\right)e^{\alpha x}+\left(e^{2\alpha L}+\cos 2\alpha L\right)e^{-\alpha x}\Big)\cos\alpha x\Big]
\end{aligned} \tag{3.37}
$$

$$
\begin{aligned}
|\phi_t| &= \frac{V_0}{2}\sqrt{\frac{e^{-2\alpha(L-x)}+2\cos\left(2\alpha\left(L-x\right)\right)+e^{2\alpha(L-x)}}{e^{-2\alpha L}+2\cos 2\alpha L+e^{2\alpha L}}} \\
&= \frac{V_0}{2}\sqrt{\frac{\cosh\left(2\alpha\left(L-x\right)\right)+\cos\left(2\alpha\left(L-x\right)\right)}{\cosh 2\alpha L+\cos 2\alpha L}}
\end{aligned} \tag{3.38}
$$

$$
\begin{aligned}
\sin\delta &= \frac{\left(e^{-\alpha x}-e^{\alpha x}\right)\sin\left(\alpha\left(2L-x\right)\right)+\left(e^{-\alpha(2L-x)}-e^{\alpha(2L-x)}\right)\sin\left(\alpha x\right)}{\sqrt{\left(e^{-2\alpha L}+2\cos 2\alpha L+e^{2\alpha L}\right)\left(e^{-2\alpha(L-x)}+2\cos\left(2\alpha\left(L-x\right)\right)+e^{2\alpha(L-x)}\right)}} \\
&= -\frac{\sinh\left(\alpha x\right)\sin\left(\alpha\left(2L-x\right)\right)+\sinh\left(\alpha\left(2L-x\right)\right)\sin\left(\alpha x\right)}{\sqrt{\left(\cosh 2\alpha L+\cos 2\alpha L\right)\left(\cosh\left(2\alpha\left(L-x\right)\right)+\cos\left(2\alpha\left(L-x\right)\right)\right)}}
\end{aligned} \tag{3.39}
$$

$$
\begin{aligned}
\cos\delta &= \frac{\left(e^{-\alpha x}+e^{\alpha x}\right)\cos\left(\alpha\left(2L-x\right)\right)+\left(e^{-\alpha(2L-x)}+e^{\alpha(2L-x)}\right)\cos\left(\alpha x\right)}{\sqrt{\left(e^{-2\alpha L}+2\cos 2\alpha L+e^{2\alpha L}\right)\left(e^{-2\alpha(L-x)}+2\cos\left(2\alpha\left(L-x\right)\right)+e^{2\alpha(L-x)}\right)}} \\
&= \frac{\cosh\left(\alpha x\right)\cos\left(\alpha\left(2L-x\right)\right)+\cosh\left(\alpha\left(2L-x\right)\right)\cos\left(\alpha x\right)}{\sqrt{\left(\cosh 2\alpha L+\cos 2\alpha L\right)\left(\cosh\left(2\alpha\left(L-x\right)\right)+\cos\left(2\alpha\left(L-x\right)\right)\right)}}
\end{aligned} \tag{3.40}
$$

Graphs of the amplitude and phase of the symmetric solution for the top and bottom electrodes are shown below at various positions along the sheet. For the polar plots, the low frequency regions are on the outside, and the high frequency region is near the center of the spiral (at low amplitudes). It's important to note that although the plots for different x look similar, the graphs for smaller x go up to a much higher frequency. The closer to $x = 0$, the higher the required frequency to substantially reduce the amplitude.

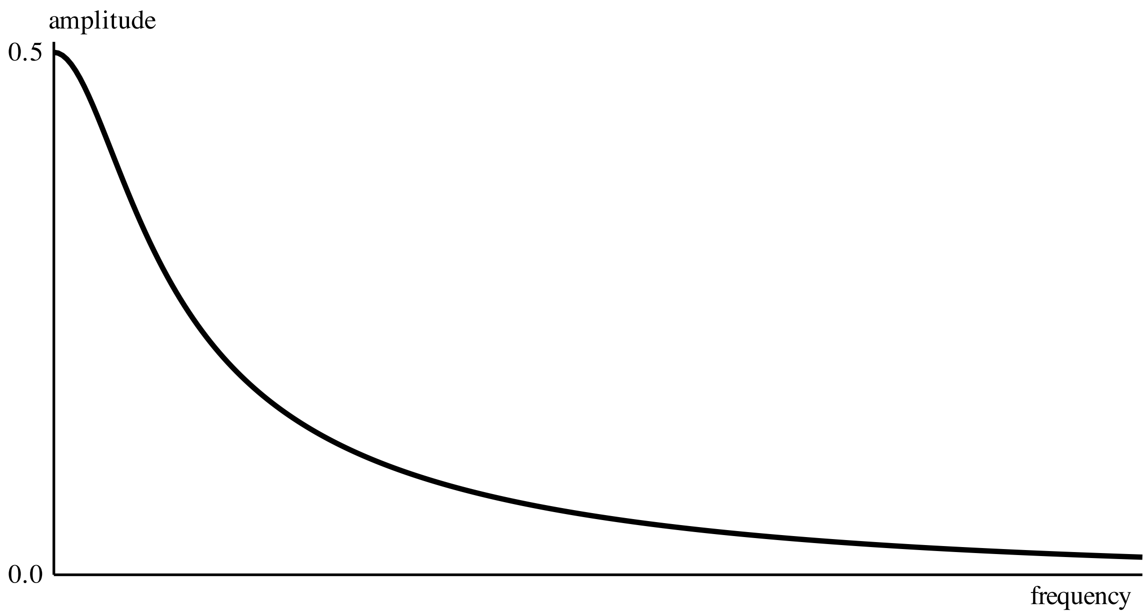

Figure 3.4: Amplitude vs. frequency for both electrodes at $x = L$.

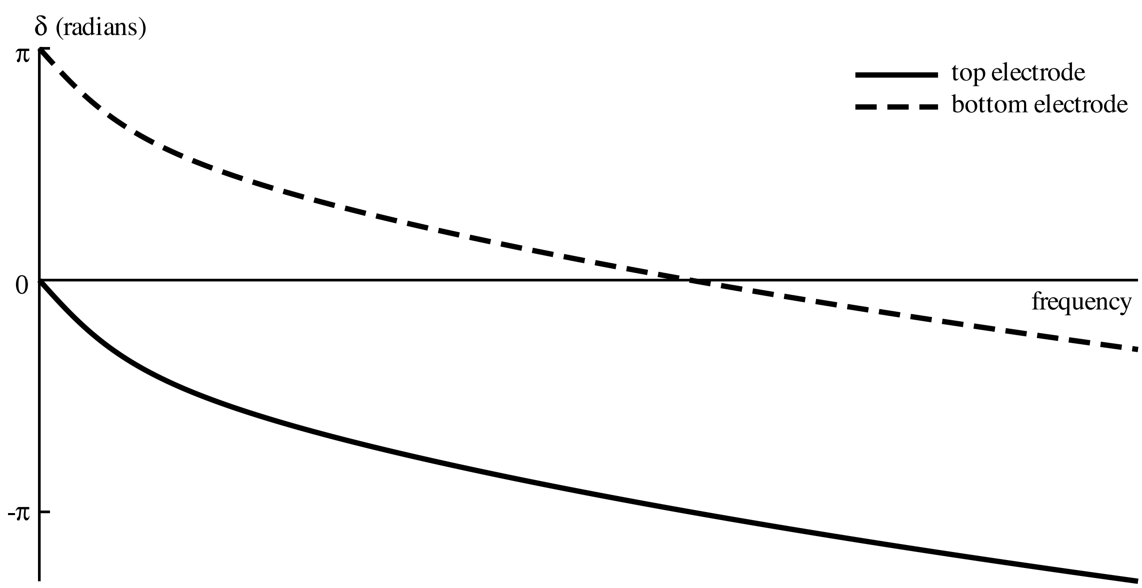

Figure 3.5: Phase vs. frequency at $x = L$.

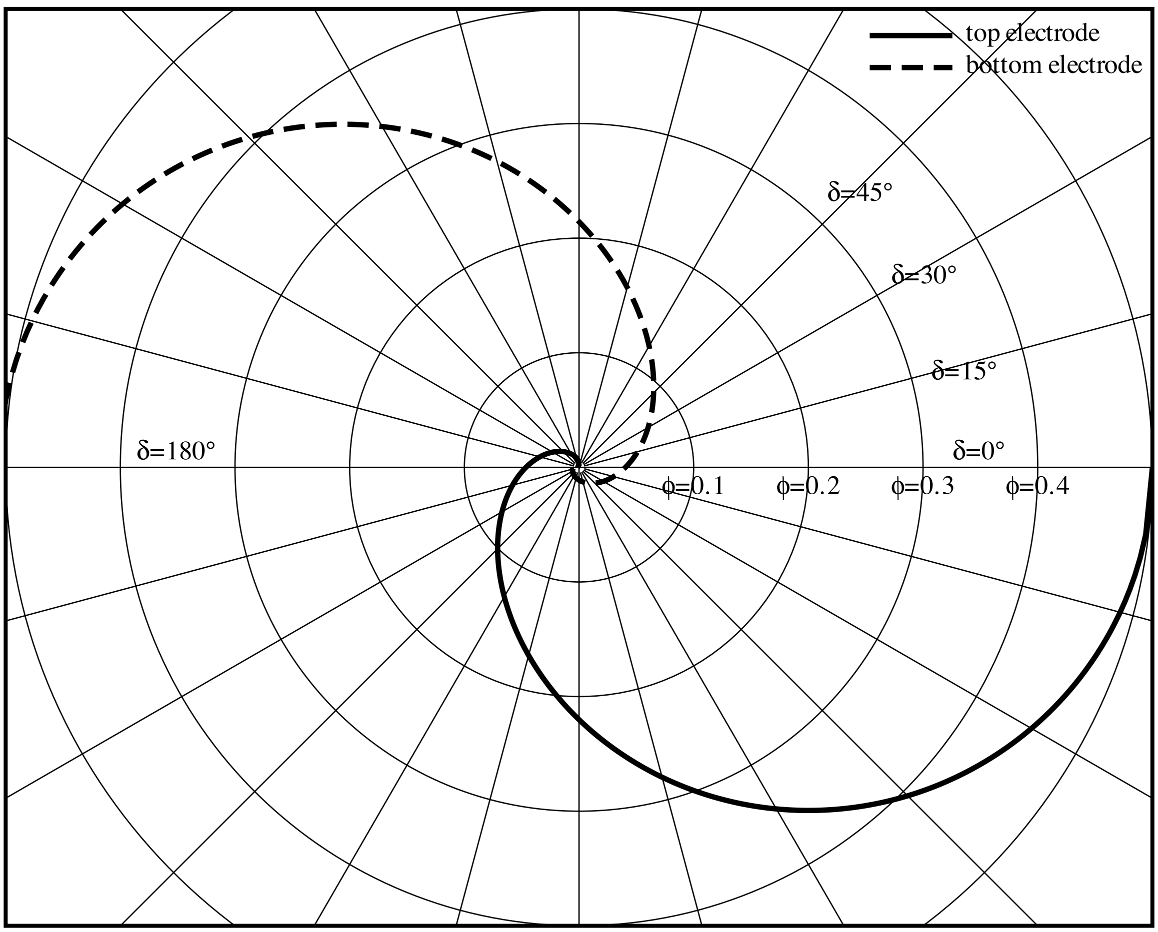

Figure 3.6: Amplitude/phase polar plot at $x = L$.

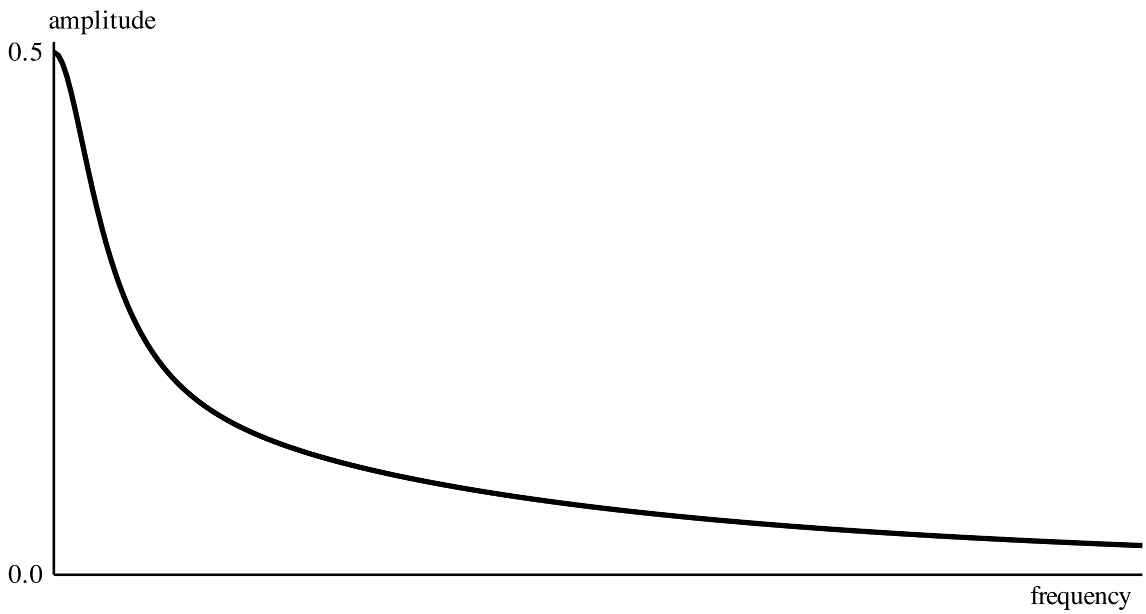

Figure 3.7: Amplitude vs. frequency for both electrodes at $x = 0.5L$.

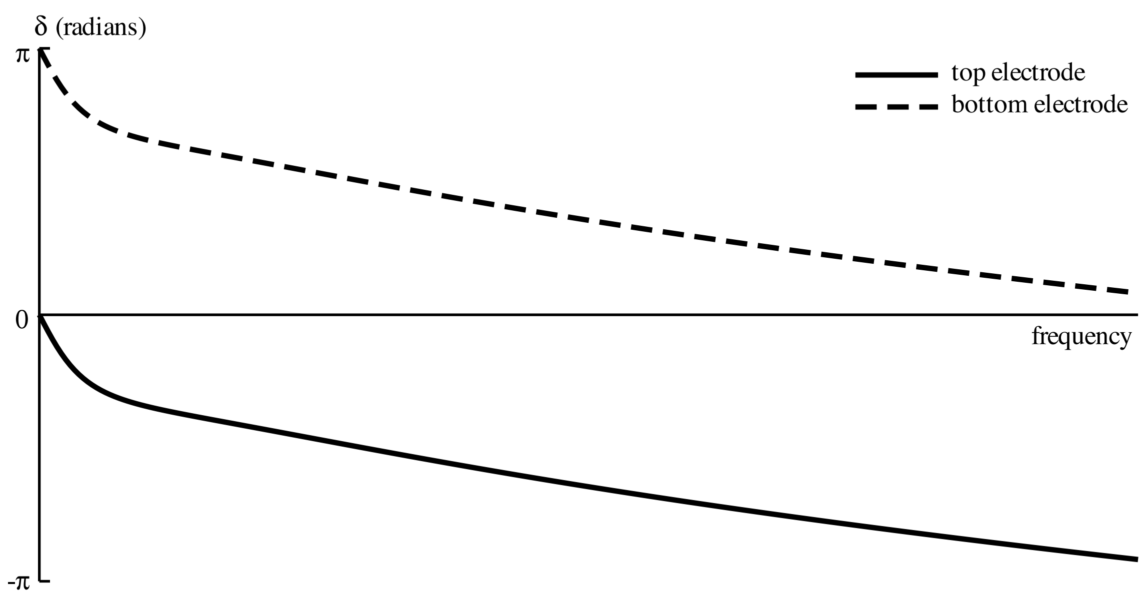

Figure 3.8: Phase vs. frequency at $x = 0.5L$.

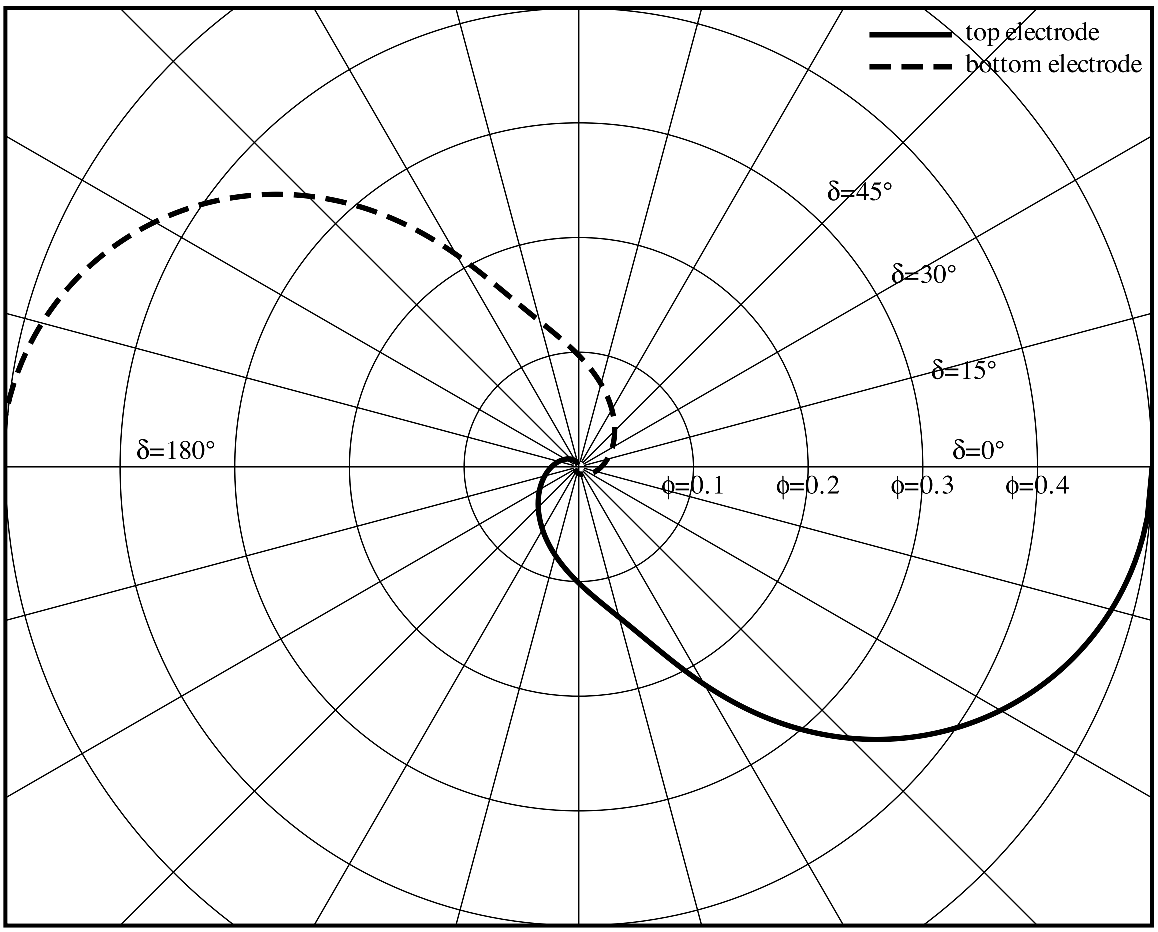

Figure 3.9: Amplitude/phase polar plot at $x = 0.5L$.

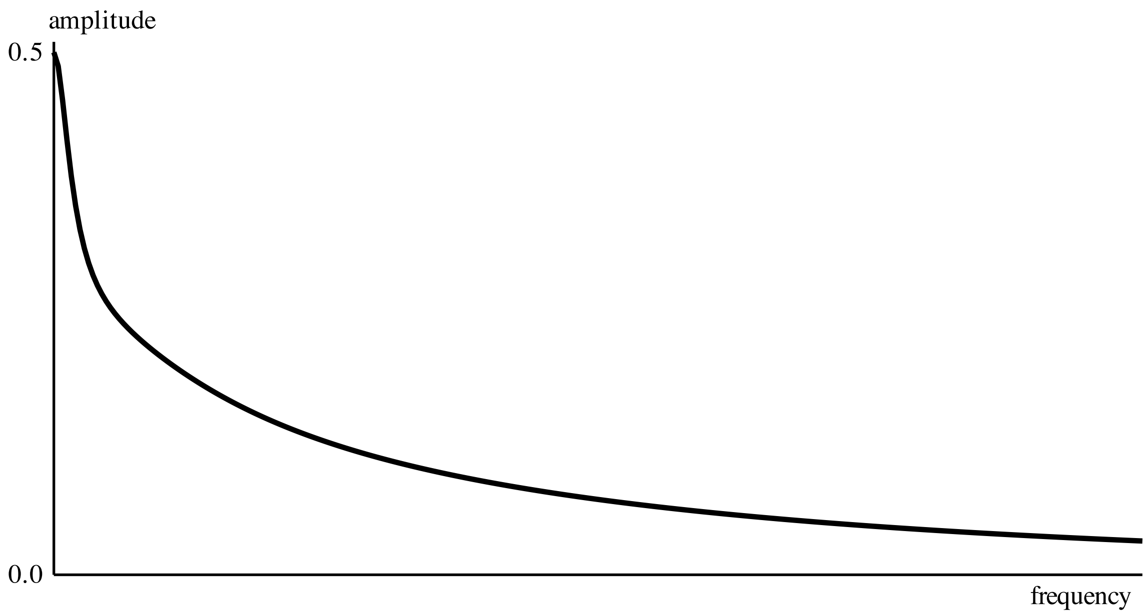

Figure 3.10: Amplitude vs. frequency for both electrodes at $x = 0.3L$.

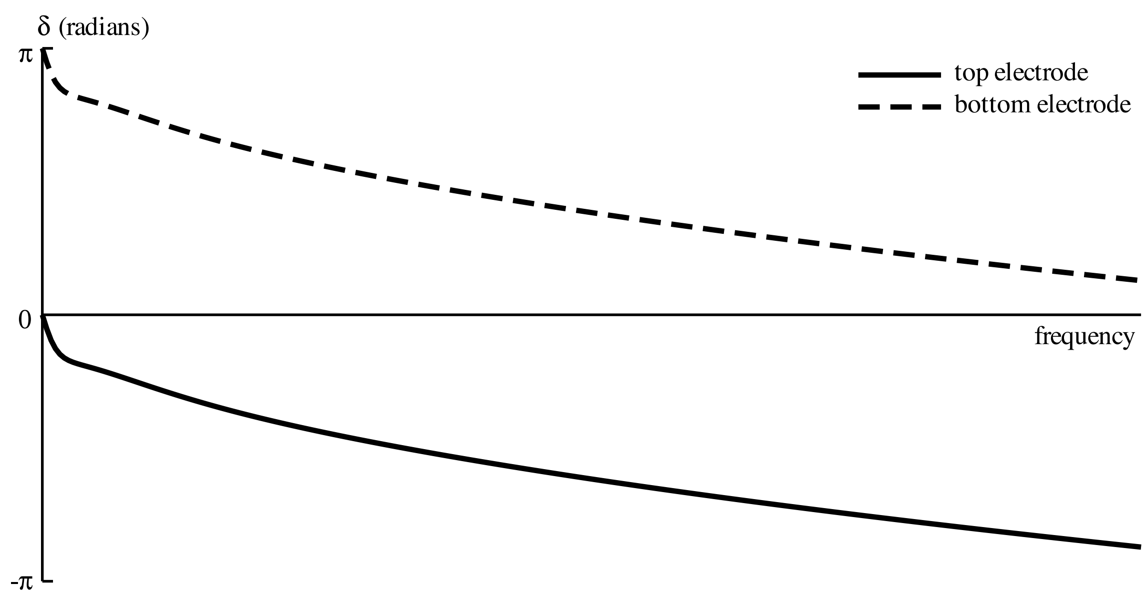

Figure 3.11: Phase vs. frequency at $x = 0.3L$.

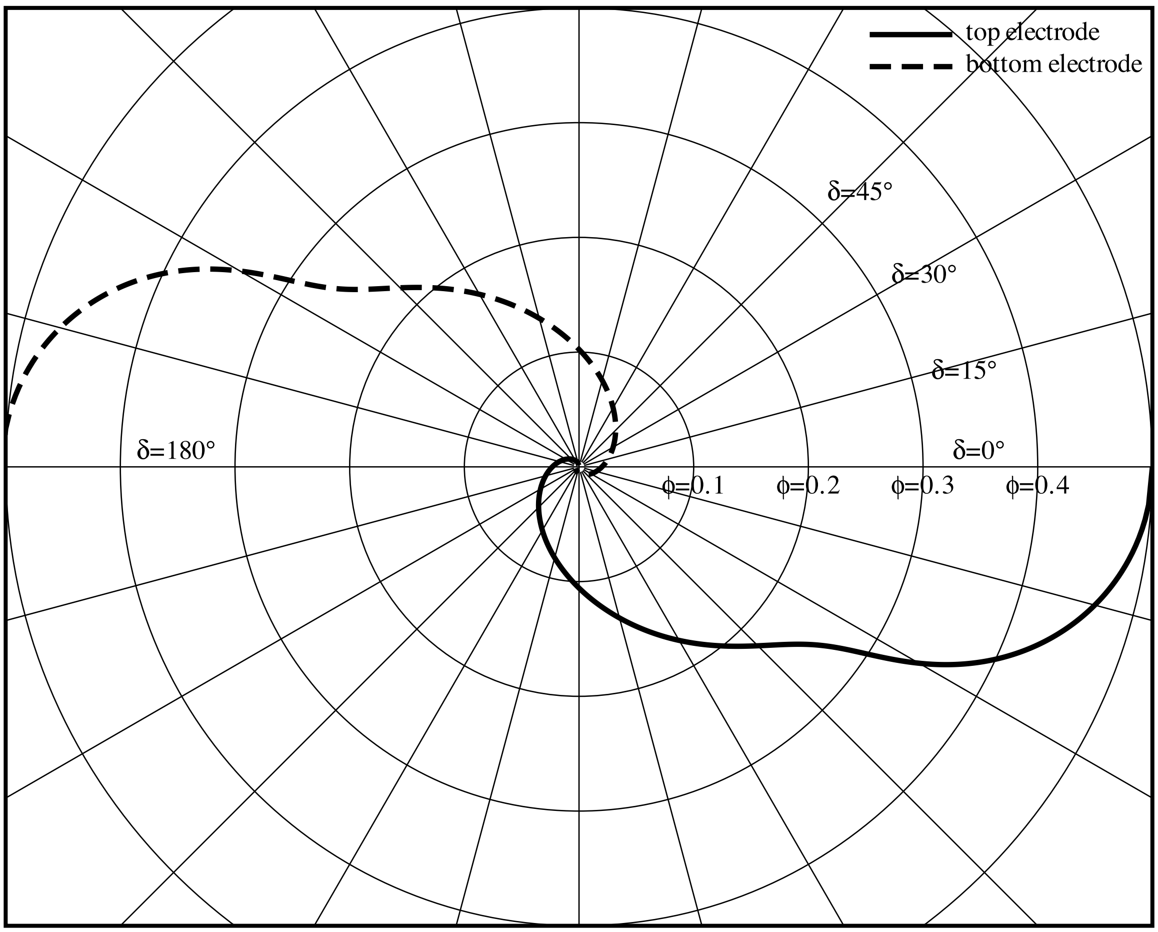

Figure 3.12: Amplitude/phase polar plot at $x = 0.3L$.

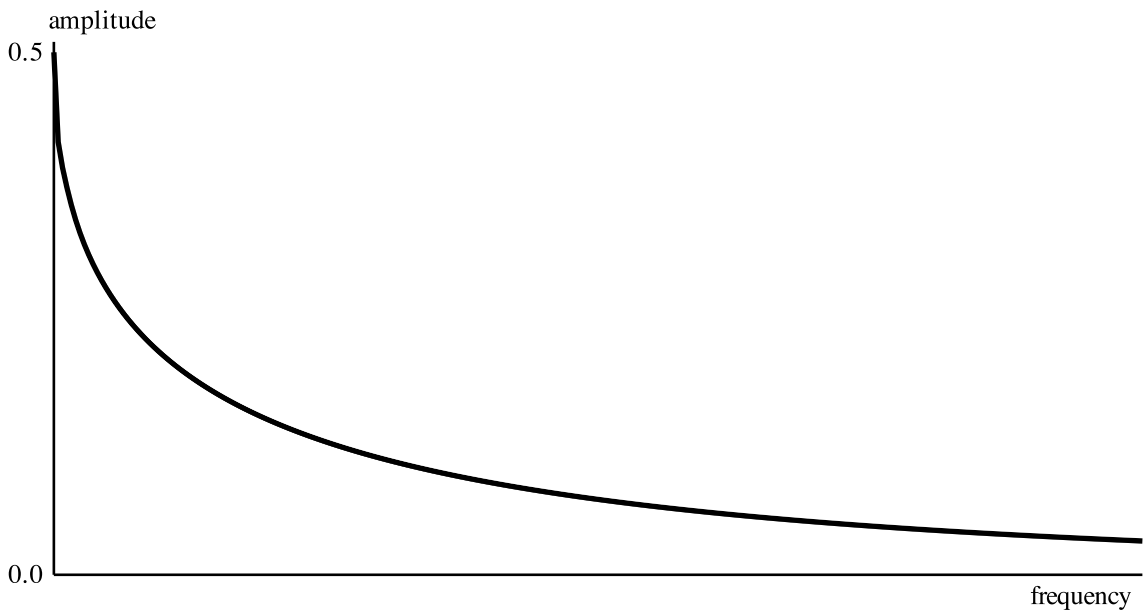

Figure 3.13: Amplitude vs. frequency for both electrodes at $x = 0.1L$.

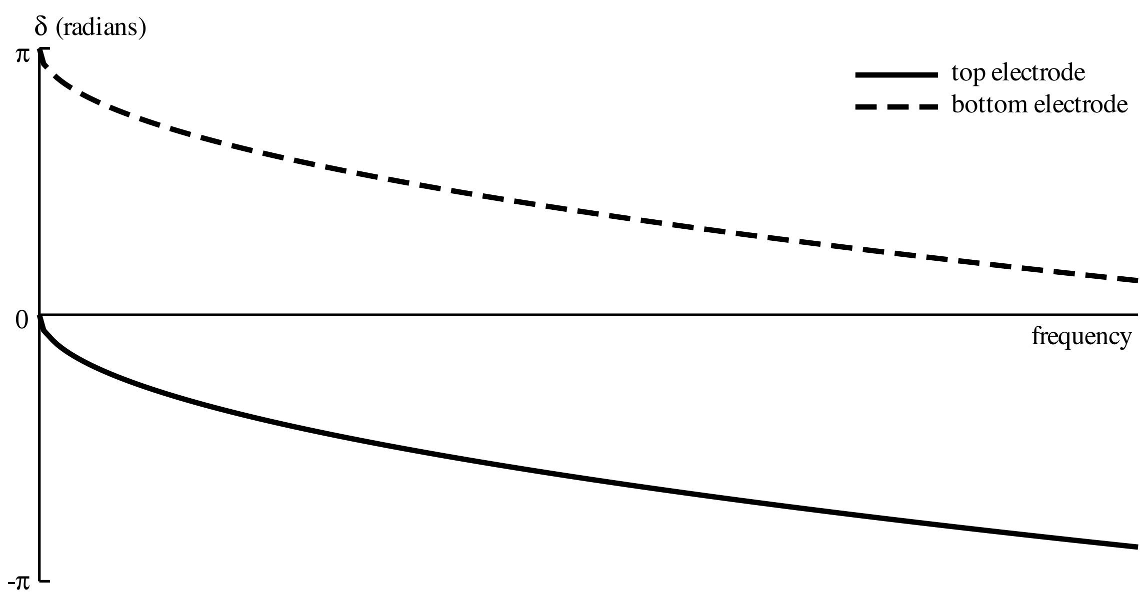

Figure 3.14: Phase vs. frequency at $x = 0.1L$.

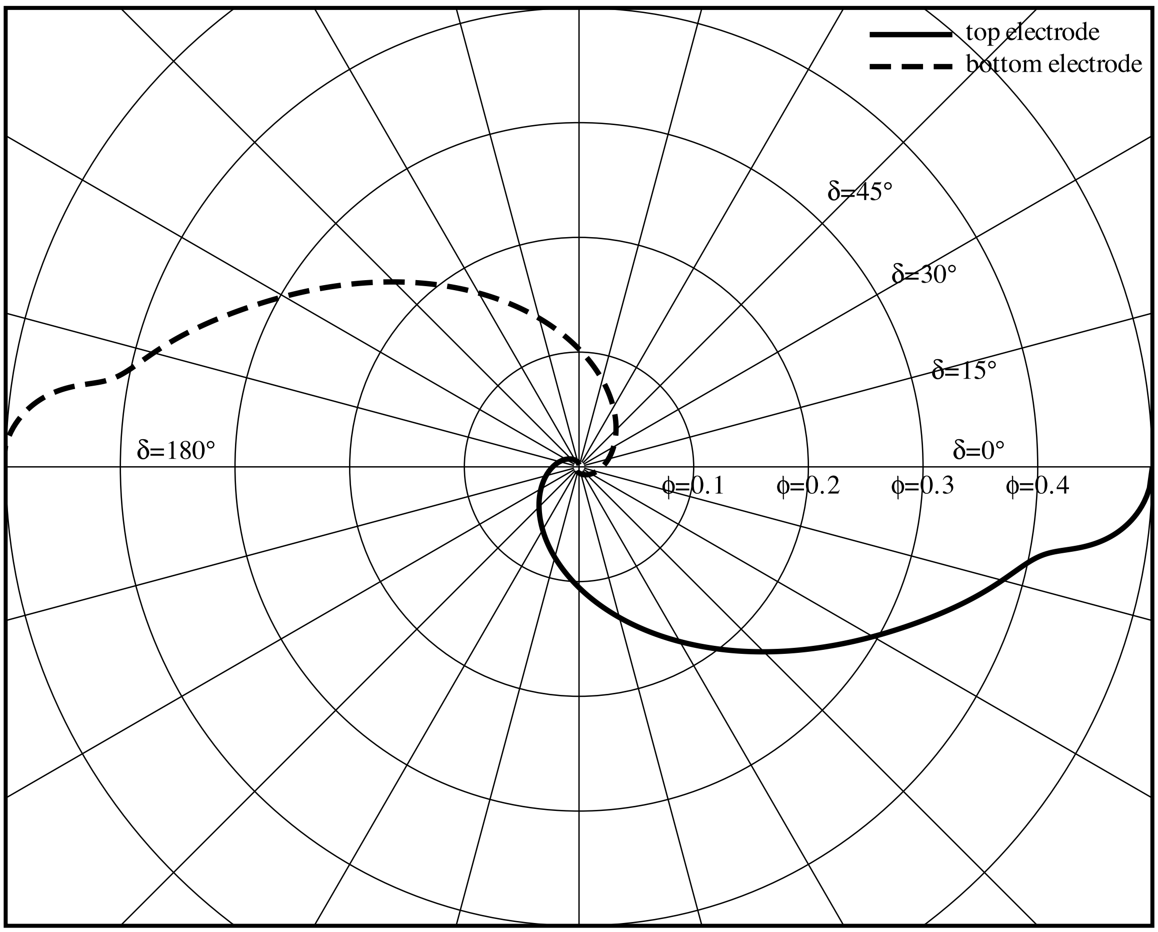

Figure 3.15: Amplitude/phase polar plot at $x = 0.1L$.

In the actual case the bottom electrode is grounded at $x = 0$. This is done because experimentally it's easier to ground one electrode instead of splitting the applied voltage. In order to get the asymmetric solution we just add $(V_0/2)\sin\omega t$ to the symmetric solution. The new boundary condition at $x = 0$ becomes $\phi_t(0,t) = V_0 \sin\omega t$ and $\phi_b(0,t) = 0$. It's interesting to note that at high enough frequencies the amplitude of the "grounded" electrode can be greater than the top electrode for $x > 0$. In the expressions below $\tilde{\phi}$ represents the asymmetric solution, while ϕ represents the symmetric solution. It's good to know the asymmetric solution to verify experimental measurements, but what really matters is the potential difference across the sheet, which is the same as for the symmetric solution.

$$\tilde{\phi}_t = \frac{V_0}{2}\sin\omega t + \phi_t \tag{3.41}$$

$$\tilde{\phi}_b = \frac{V_0}{2}\sin\omega t - \phi_t \tag{3.42}$$

Just as with the symmetric solution, we can break up the potential into terms of $\cos\omega t$ and $\sin\omega t$.

$$\tilde{\phi}_t = |\tilde{\phi}_t|\sin(\omega t + \tilde{\delta}_t) = |\tilde{\phi}_t|\sin\tilde{\delta}_t\cos\omega t + |\phi_t|\cos\tilde{\delta}_t\sin\omega t \tag{3.43}$$

$$\tilde{\phi}_b = |\tilde{\phi}_b|\sin(\omega t + \tilde{\delta}_b) = |\tilde{\phi}_b|\sin\tilde{\delta}_b\cos\omega t + |\phi_b|\cos\tilde{\delta}_b\sin\omega t \tag{3.44}$$

$$|\tilde{\phi}_t|\sin\tilde{\delta}_t = |\phi_t|\sin\delta \tag{3.45}$$

$$|\tilde{\phi}_t|\cos\tilde{\delta}_t = \frac{V_0}{2} + |\phi_t|\cos\delta \tag{3.46}$$

$$|\tilde{\phi}_t| = \left[|\phi_t|^2 + \frac{V_0^2}{4} + V_0|\phi_t|\cos\delta\right]^{1/2} \tag{3.47}$$

$$|\tilde{\phi}_b|\sin\tilde{\delta}_b = -|\phi_t|\sin\delta \tag{3.48}$$

$$|\tilde{\phi}_b|\cos\tilde{\delta}_b = \frac{V_0}{2} - |\phi_t|\cos\delta \tag{3.49}$$

$$|\tilde{\phi}_b| = \left[|\phi_t|^2 + \frac{V_0^2}{4} - V_0|\phi_t|\cos\delta\right]^{1/2} \tag{3.50}$$

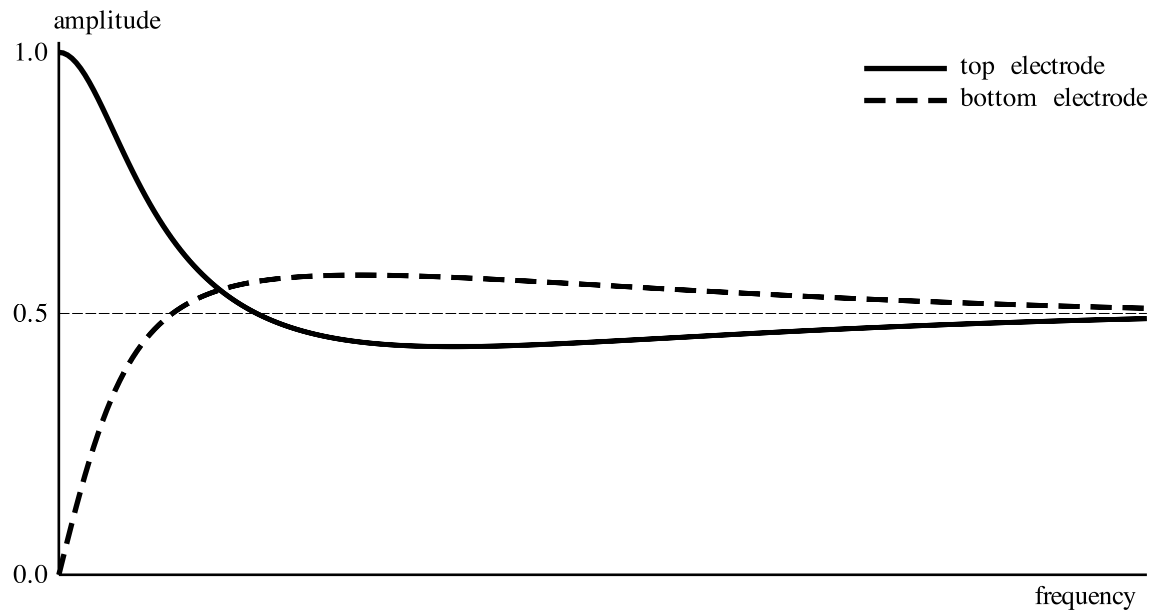

Figure 3.16: Amplitude vs. frequency at $x = L$.

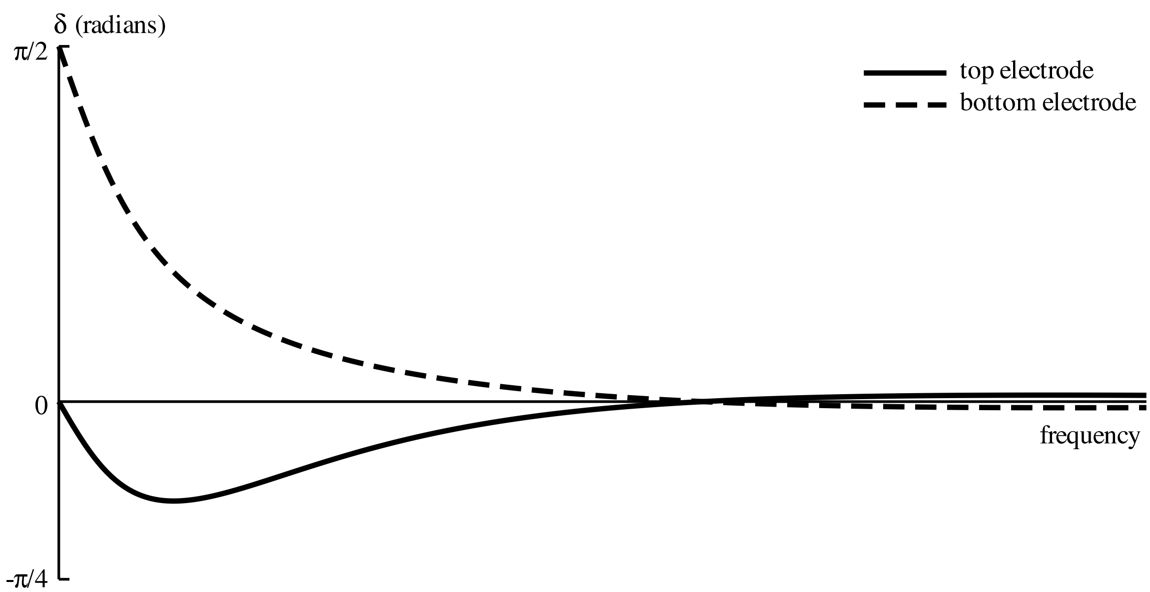

Figure 3.17: Phase vs. frequency at $x = L$.

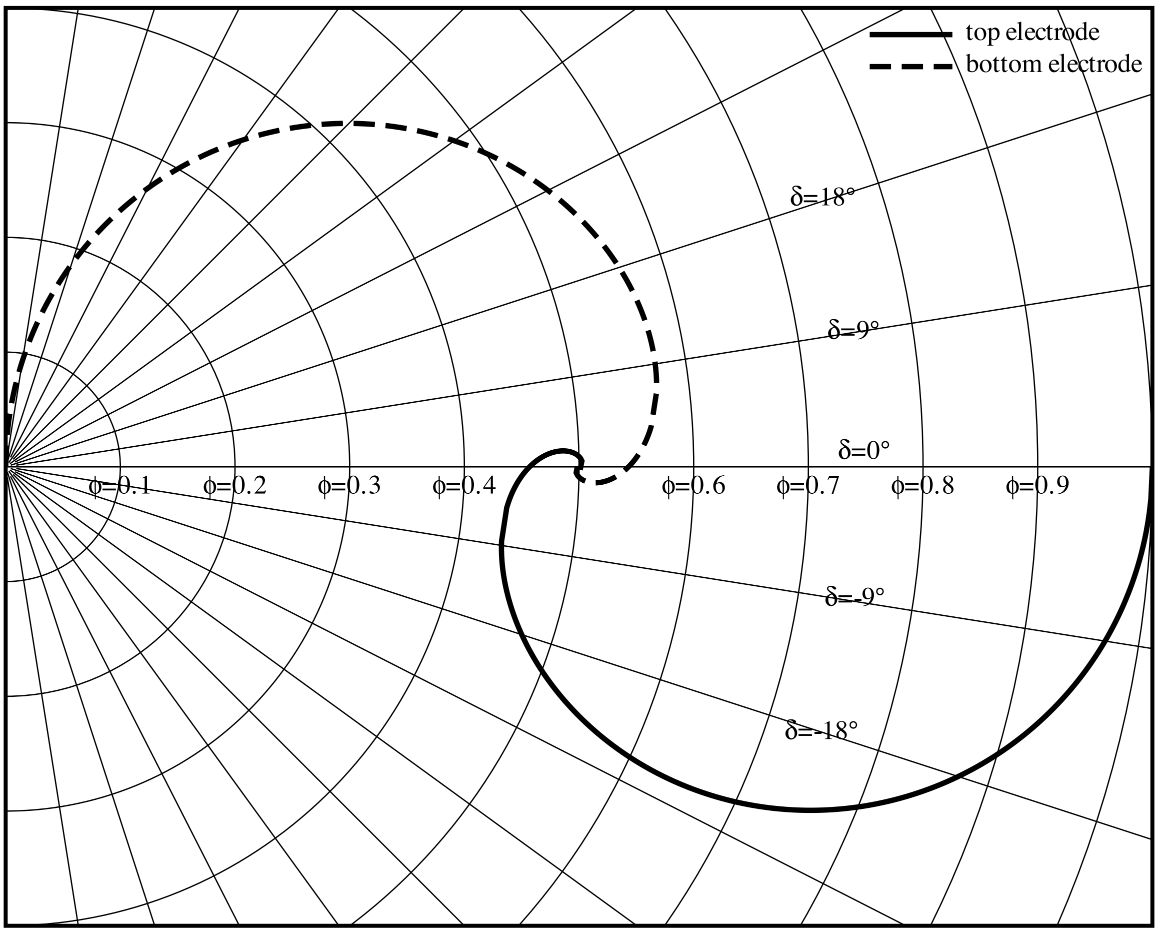

Figure 3.18: Amplitude/phase polar plot at $x = L$.

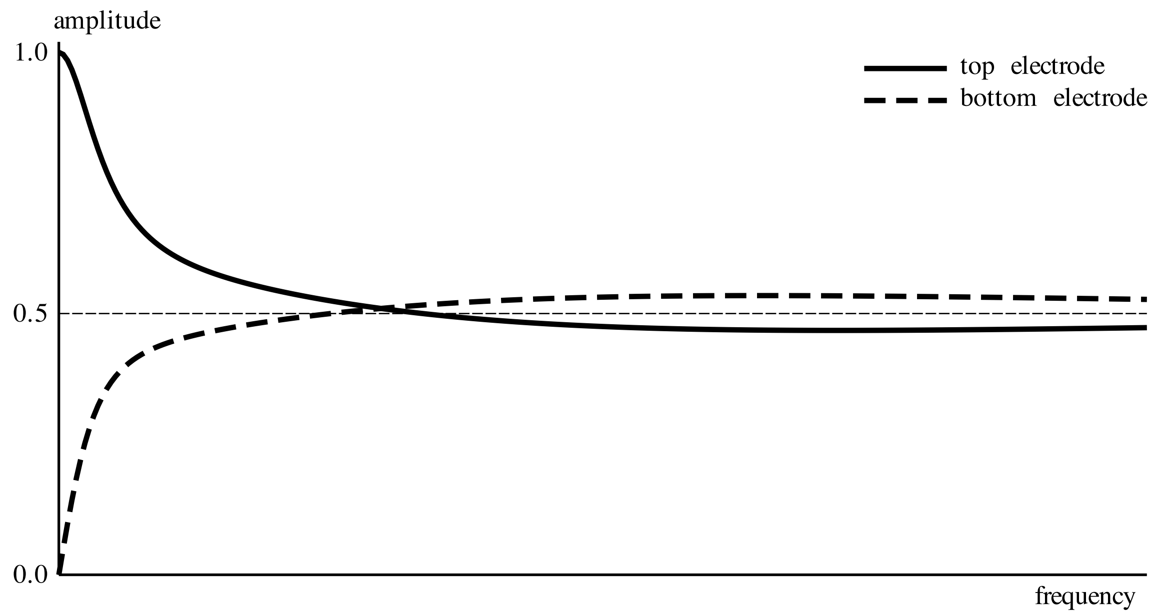

Figure 3.19: Amplitude vs. frequency at $x = 0.5L$.

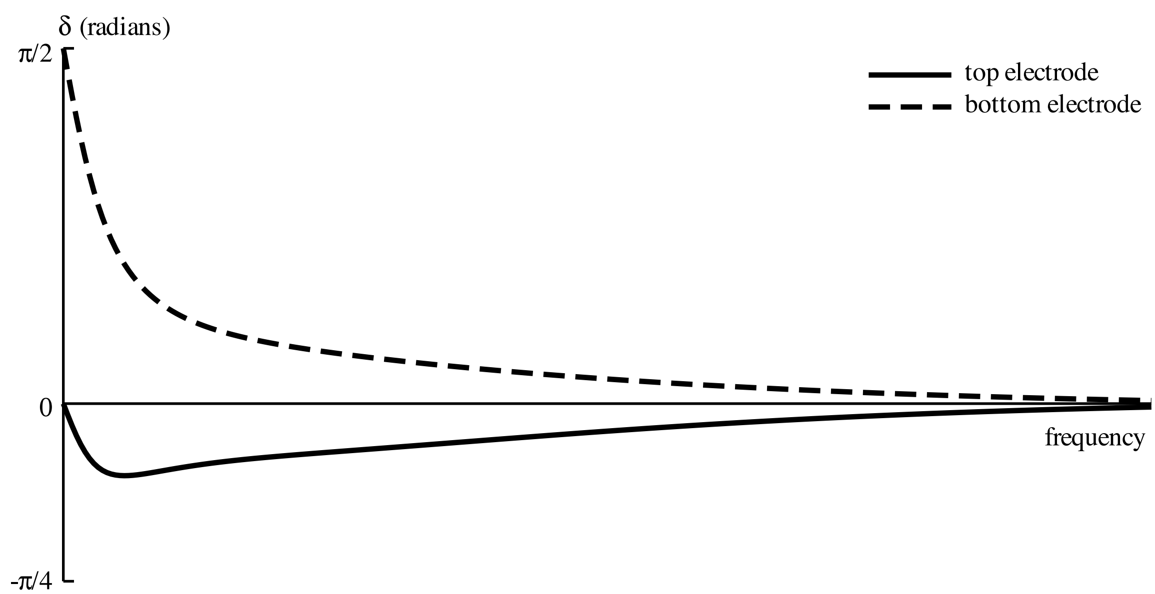

Figure 3.20: Phase vs. frequency at $x = 0.5L$.

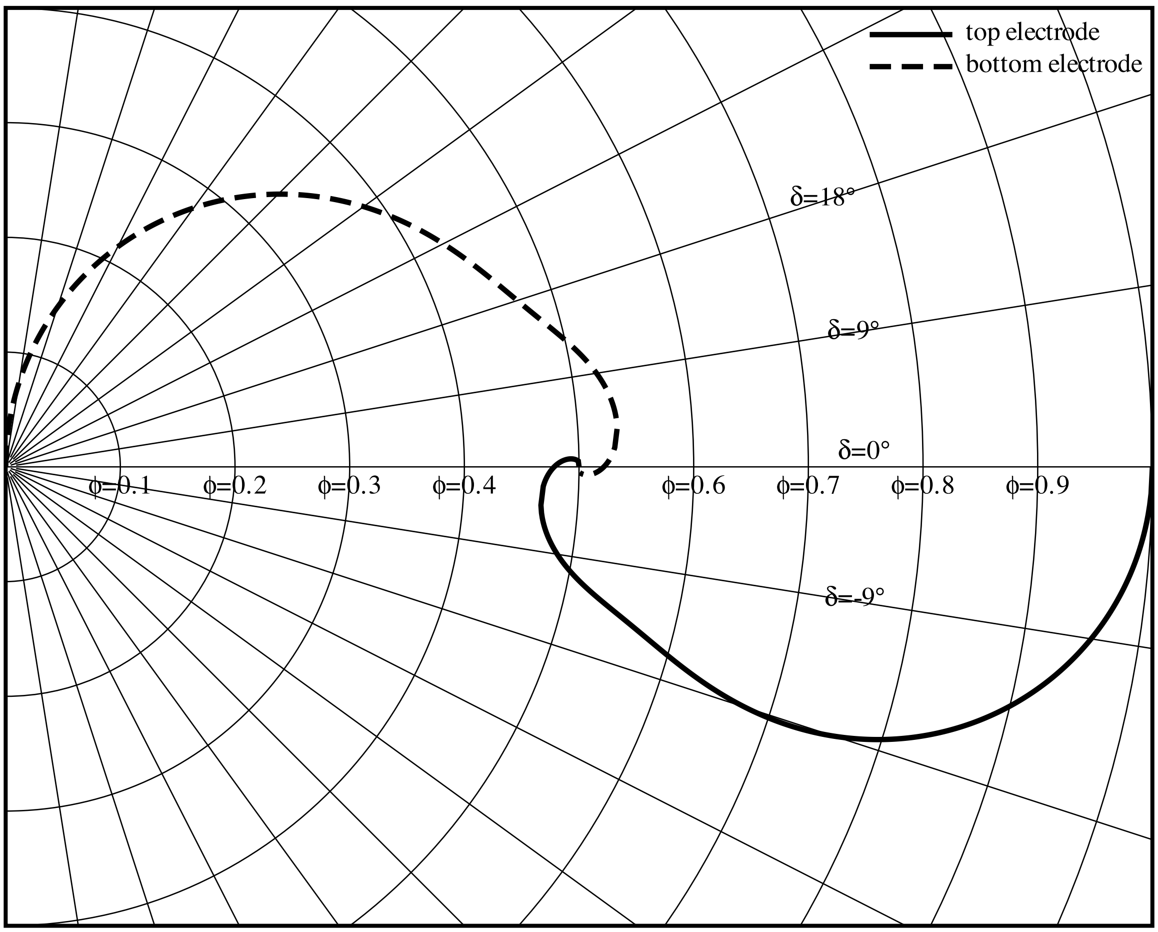

Figure 3.21: Amplitude/phase polar plot at $x = 0.5L$.

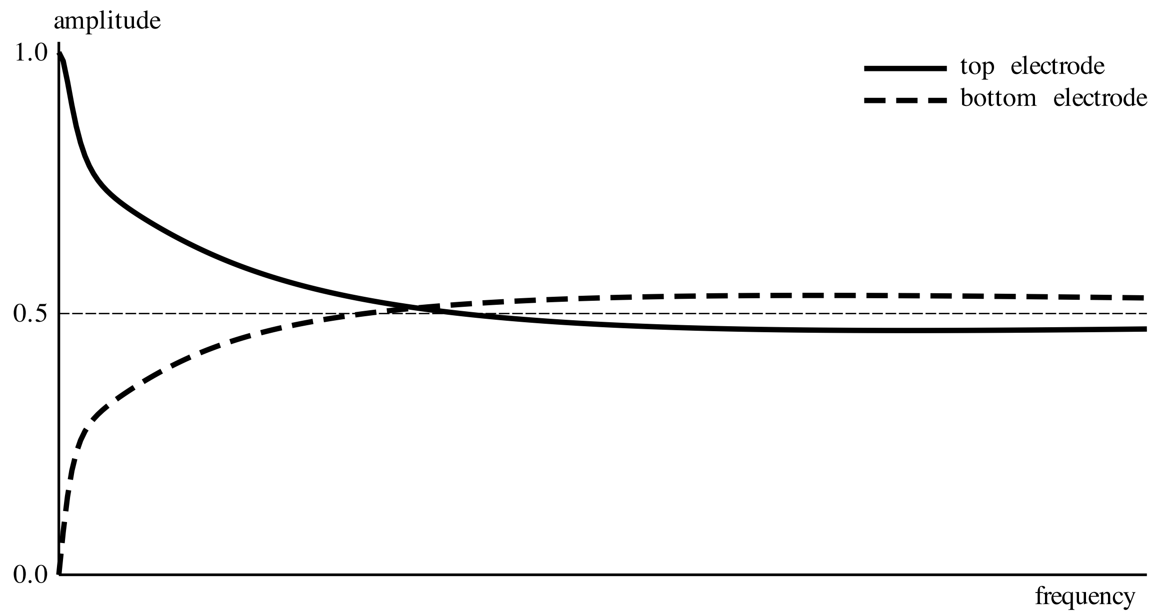

Figure 3.22: Amplitude vs. frequency at $x = 0.3L$.

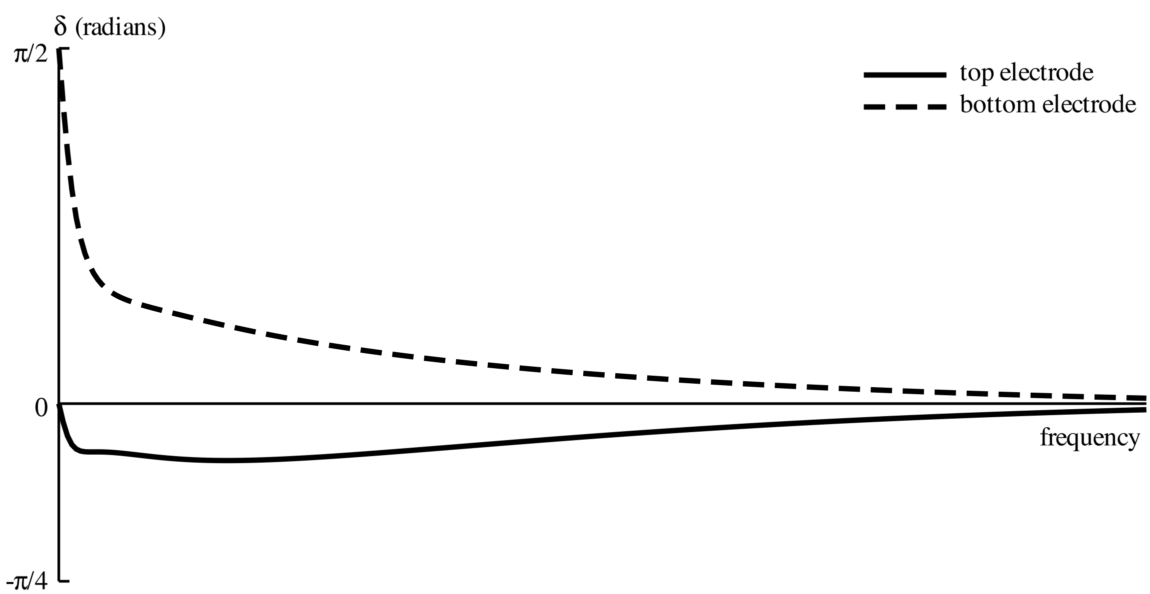

Figure 3.23: Phase vs. frequency at $x = 0.3L$.

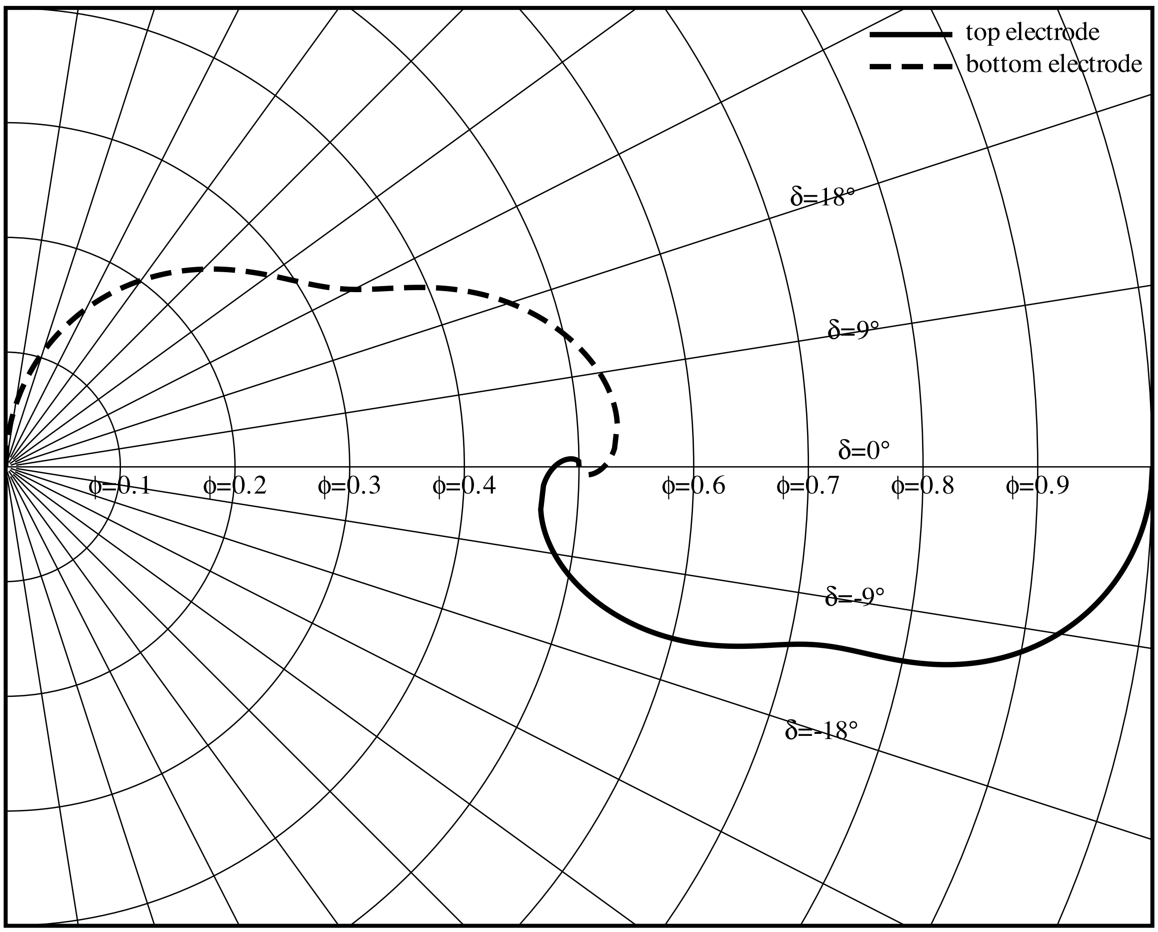

Figure 3.24: Amplitude/phase polar plot at $x = 0.3L$.

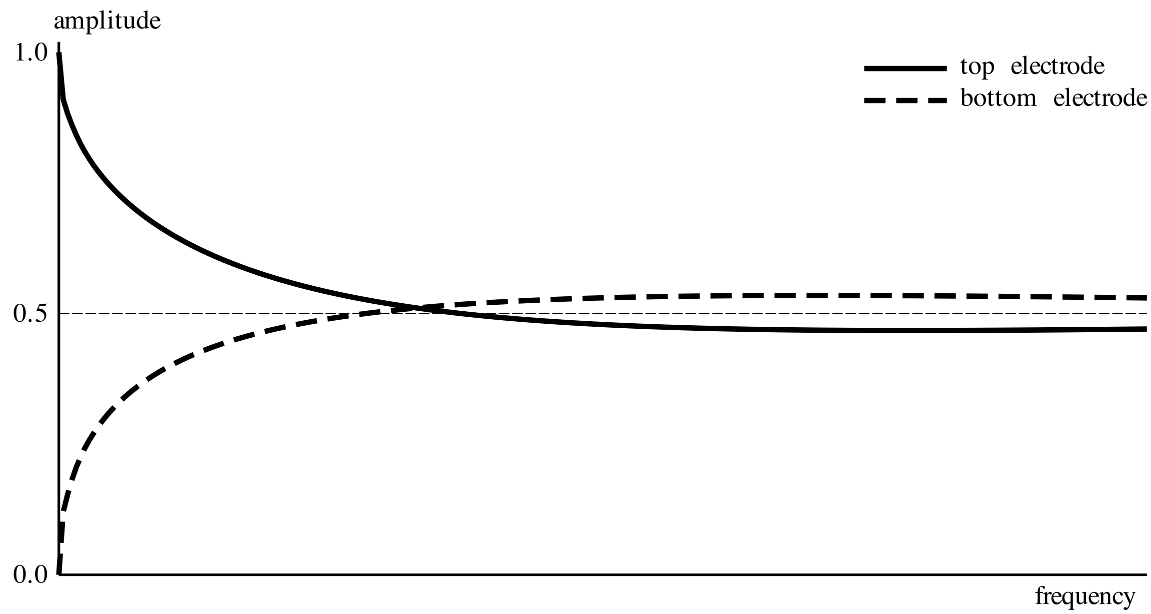

Figure 3.25: Amplitude vs. frequency at $x = 0.1L$.

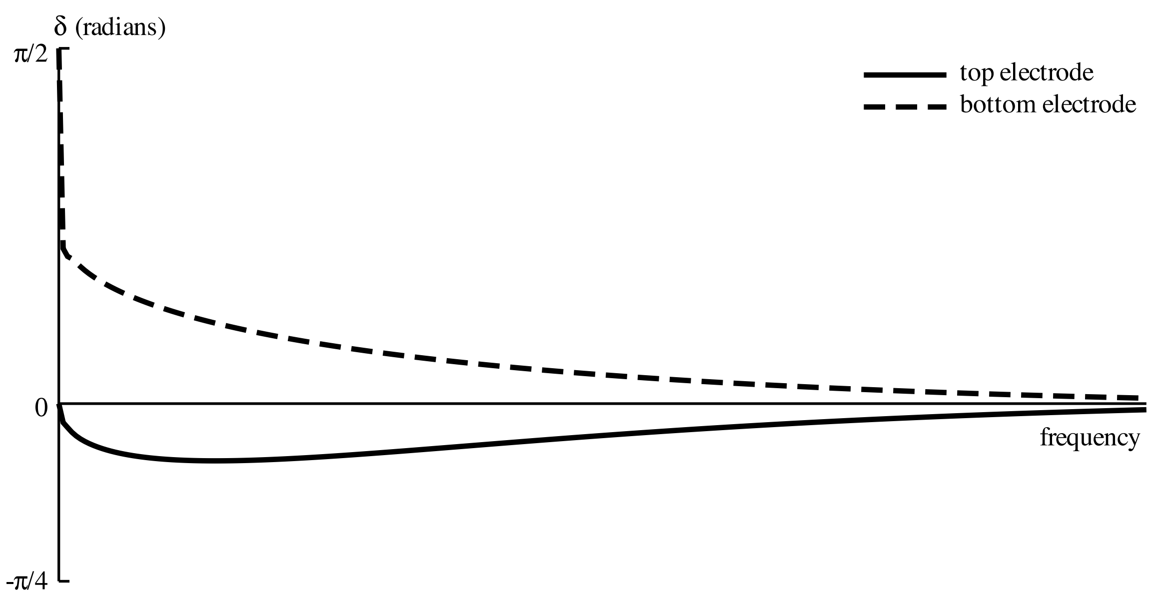

Figure 3.26: Phase vs. frequency at $x = 0.1L$.

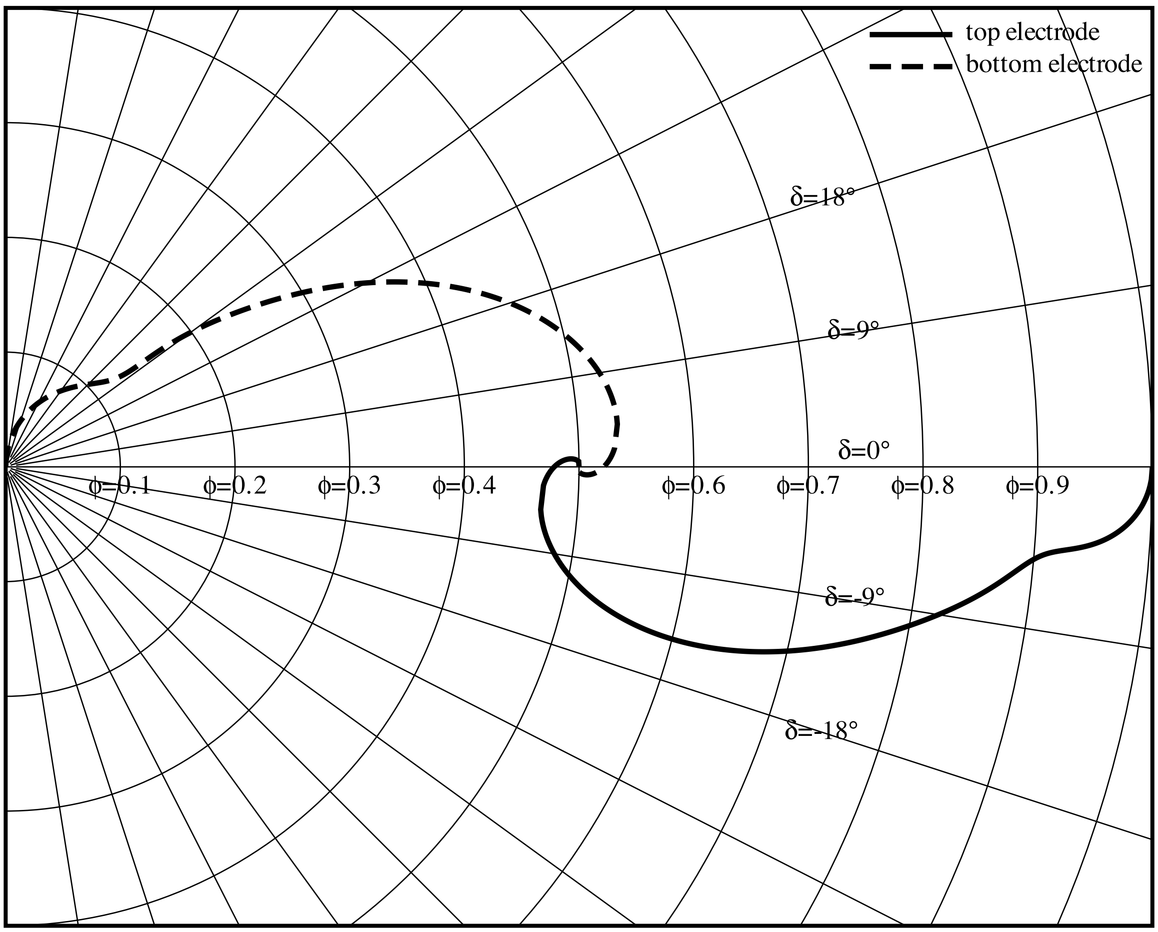

Figure 3.27: Amplitude/phase polar plot at $x = 0.1L$.

Cylindrical Case

In the cylindrical case the solution is modified Bessel functions. The differential equation can be solved by separation of variables. The time part is a simple exponential, $e^{i\omega t}$, but because we want the sine part, the final solution will be the imaginary part of the composite function.

The solution expressed in real form is quite complicated, so here it is left in complex form. The graphs for the solutions are very nearly identical to the graphs for the rectangular solutions.

$$\frac{\partial \phi_t}{\partial t} = k\left[\frac{\partial^2 \phi_t}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_t}{\partial r}\right], \quad \phi_t(R_1, t) = \frac{V_0}{2}\sin\omega t, \quad \left.\frac{\partial \phi_t}{\partial r}\right|_{r=R_2} = 0 \tag{3.51}$$

$$\begin{aligned} \phi_t\left(r,t\right) &= \mathrm{Im}\left[\frac{V_0}{2}e^{i\omega t}\frac{\mathrm{I}_0\big(\sqrt{i}\alpha r\big)\,\mathrm{K}_1\big(\sqrt{i}\alpha R_2\big) + \mathrm{I}_1\big(\sqrt{i}\alpha R_2\big)\,\mathrm{K}_0\big(\sqrt{i}\alpha r\big)}{\mathrm{I}_0\big(\sqrt{i}\alpha R_1\big)\,\mathrm{K}_1\big(\sqrt{i}\alpha R_2\big) + \mathrm{I}_1\big(\sqrt{i}\alpha R_2\big)\,\mathrm{K}_0\big(\sqrt{i}\alpha R_1\big)}\right] \\ &= |\phi_t|\sin\left(\omega t + \delta\right) = |\phi_t|\sin\delta\cos\omega t + |\phi_t|\cos\delta\sin\omega t \end{aligned} \tag{3.52}$$

$$|\phi_t|\sin\delta = \frac{V_0}{2}\,\mathrm{Im}\left[\frac{\mathrm{I}_0\big(\sqrt{i}\alpha r\big)\,\mathrm{K}_1\big(\sqrt{i}\alpha R_2\big) + \mathrm{I}_1\big(\sqrt{i}\alpha R_2\big)\,\mathrm{K}_0\big(\sqrt{i}\alpha r\big)}{\mathrm{I}_0\big(\sqrt{i}\alpha R_1\big)\,\mathrm{K}_1\big(\sqrt{i}\alpha R_2\big) + \mathrm{I}_1\big(\sqrt{i}\alpha R_2\big)\,\mathrm{K}_0\big(\sqrt{i}\alpha R_1\big)}\right] \tag{3.53}$$

$$|\phi_t|\cos\delta = \frac{V_0}{2}\,\mathrm{Re}\left[\frac{\mathrm{I}_0\big(\sqrt{i}\alpha r\big)\,\mathrm{K}_1\big(\sqrt{i}\alpha R_2\big) + \mathrm{I}_1\big(\sqrt{i}\alpha R_2\big)\,\mathrm{K}_0\big(\sqrt{i}\alpha r\big)}{\mathrm{I}_0\big(\sqrt{i}\alpha R_1\big)\,\mathrm{K}_1\big(\sqrt{i}\alpha R_2\big) + \mathrm{I}_1\big(\sqrt{i}\alpha R_2\big)\,\mathrm{K}_0\big(\sqrt{i}\alpha R_1\big)}\right] \tag{3.54}$$

$$|\phi_t| = \sqrt{\left(|\phi_t|\sin\delta\right)^2 + \left(|\phi_t|\cos\delta\right)^2} \tag{3.55}$$

CHAPTER 4

MECHANICAL DEFORMATION

Now that we have solved for the electric potential, we can now include the piezoelectric effect and solve for the mechanical deformation. For the case of the monomorph the deformation will only be in the plane (no bending). For a thin-beam approximation we can say that the monomorph is also thin in the y dimension, and so we can limit our interest to the x displacement. From the constitutive relation, equation 2.1a, the piezoelectric contribution to the deformation is proportion to the electric field. The field $E_3(x,t)$ applied across the thickness d is $-2\phi_t(x,t)/d$, using equation (3.34) for $\phi(x,t)$. We abbreviate the notation by writing $E_3(x,t)$ as

$$E_3(x,t) = a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4, \tag{4.1}$$

where the f_i are the same as previously defined, and the a_i are defined below.

$$\begin{aligned} a_1 &= -2C_{\text{inc}}/d, \qquad & a_2 &= -2D_{\text{inc}}/d \\ a_3 &= -2C_{\text{ref}}/d, \qquad & a_4 &= -2D_{\text{ref}}/d \end{aligned} \tag{4.2}$$

We assume that the y and z dimensions of the monomorph (the width and thickness) are small enough so that the inertially caused stresses T_2 and T_3 are zero.

$$T_2 = c_{12}S_1 + c_{22}S_2 + c_{23}S_3 - e_{32}E_3 = 0 \tag{4.3}$$

$$T_3 = c_{13}S_1 + c_{23}S_2 + c_{33}S_3 - e_{33}E_3 = 0 \tag{4.4}$$

From these two equations we can solve for S_1 and S_2, and then plug these expression back into the equation for T_1.

$$\begin{bmatrix} c_{22} & c_{23} \\ c_{23} & c_{33} \end{bmatrix} \begin{bmatrix} S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} e_{32}E_3 - c_{12}S_1 \\ e_{33}E_3 - c_{13}S_1 \end{bmatrix} \tag{4.5}$$

$$\begin{bmatrix} S_2 \\ S_3 \end{bmatrix} = \frac{1}{c_{23}^2 - c_{22}c_{33}} \begin{bmatrix} -c_{33} & c_{23} \\ c_{23} & -c_{22} \end{bmatrix} \begin{bmatrix} e_{32}E_3 - c_{12}S_1 \\ e_{33}E_3 - c_{13}S_1 \end{bmatrix} \tag{4.6}$$

$$S_2 = \frac{\left(c_{23}e_{33} - c_{33}e_{32}\right)E_3 + \left(c_{12}c_{33} - c_{13}c_{23}\right)S_1}{c_{23}^2 - c_{22}c_{33}} \tag{4.7}$$

$$S_3 = \frac{\left(c_{23}e_{32} - c_{22}e_{33}\right)E_3 + \left(c_{13}c_{22} - c_{12}c_{23}\right)S_1}{c_{23}^2 - c_{22}c_{33}} \tag{4.8}$$

$$\begin{aligned} T_1 &= c_{11}S_1 + c_{12}S_2 + c_{13}S_3 - e_{31}E_3 \\ &= S_1\left[c_{11} + \frac{c_{12}\left(c_{12}c_{33} - c_{13}c_{23}\right) + c_{13}\left(c_{13}c_{22} - c_{12}c_{23}\right)}{c_{23}^2 - c_{22}c_{33}}\right] \\ &\quad + E_3\left[-e_{31} + \frac{c_{12}\left(c_{23}e_{33} - c_{33}e_{32}\right) + c_{13}\left(c_{23}e_{32} - c_{22}e_{33}\right)}{c_{23}^2 - c_{22}c_{33}}\right] \end{aligned} \tag{4.9}$$

<u>Undamped Case</u>

The constant coefficients in equation (4.9) can be abbreviated as d_1 and d_3, respectively.

$$T_1 = d_1S_1 + d_3E_3 \tag{4.10}$$

$$d_1 = c_{11} + \frac{c_{12}^2c_{33} + c_{13}^2c_{22} - 2c_{12}c_{13}c_{23}}{c_{23}^2 - c_{22}c_{33}} \tag{4.11}$$

$$d_3 = -e_{31} + \frac{\left(c_{13}c_{23} - c_{12}c_{33}\right)e_{32} + \left(c_{12}c_{23} - c_{13}c_{22}\right)e_{33}}{c_{23}^2 - c_{22}c_{33}} \tag{4.12}$$

Substituting this expression for T_1 into our equation of motion for u_1, and setting T_4, T_5, and T_6 equal to zero because there are no shearing forces (only elongation in the x_1 direction), we get

$$\rho\frac{\partial^2 u_1}{\partial t^2} = \frac{\partial T_1}{\partial x_1} = d_1\frac{\partial^2 u_1}{\partial x_1^2} + d_3\frac{\partial E_3}{\partial x_1}. \tag{4.13}$$

From now on u_1 is just u and x_1 is just x. The function that satisfies equation (4.13) before applying boundary conditions is called u_{ih}. To this function we can add a function u_{h} which satisfies equation (4.16), and which together with u_{ih} will ensure that the boundary conditions are satisfied.

$$u = u_{\mathrm{ih}} + u_{\mathrm{h}} \tag{4.14}$$

$$\rho\frac{\partial^2 u_{\mathrm{ih}}}{\partial t^2} - d_1\frac{\partial^2 u_{\mathrm{ih}}}{\partial x^2} = d_3\frac{\partial E_3}{\partial x} \tag{4.15}$$

$$\rho\frac{\partial^2 u_{\mathrm{h}}}{\partial t^2} - d_1\frac{\partial^2 u_{\mathrm{h}}}{\partial x^2} = 0 \tag{4.16}$$

Since the derivatives of the f_i give back the f_i with factors, the solution u_{ih} will be a combination of the f_i.

$$u_{\mathrm{ih}} = c_1 f_1(x,t) + c_2 f_2(x,t) + c_3 f_3(x,t) + c_4 f_4(x,t) \tag{4.17}$$

After putting this form for u_{ih} through the differential equation, we get the following requirements for the c_i.

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = C\begin{bmatrix} a_2 - a_1 \\ -(a_1 + a_2) \end{bmatrix} \tag{4.18}$$

$$\begin{bmatrix} A & -B \\ B & A \end{bmatrix}\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = C\begin{bmatrix} a_3 + a_4 \\ a_4 - a_3 \end{bmatrix} \tag{4.19}$$

$$A = -\rho\omega^2, \quad B = 2\alpha^2 d_1, \quad C = \alpha d_3 \tag{4.20}$$

The form of the coefficient matrices are simple to invert. We get the following solutions for the c_i.

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{C}{A^2 + B^2}\begin{bmatrix} A & -B \\ B & A \end{bmatrix}\begin{bmatrix} a_2 - a_1 \\ -(a_1 + a_2) \end{bmatrix} \tag{4.21}$$

$$\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \frac{C}{A^2 + B^2}\begin{bmatrix} A & B \\ -B & A \end{bmatrix}\begin{bmatrix} a_3 + a_4 \\ a_4 - a_3 \end{bmatrix} \tag{4.22}$$

A less abbreviated expression for the c_i is shown below.

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \frac{\alpha d_3}{\rho^2\omega^4 + 4\alpha^4 d_1^2}\begin{bmatrix} \rho\omega^2(a_1 - a_2) + 2\alpha^2 d_1(a_1 + a_2) \\ \rho\omega^2(a_1 + a_2) + 2\alpha^2 d_1(a_2 - a_1) \\ -\rho\omega^2(a_3 + a_4) + 2\alpha^2 d_1(a_4 - a_3) \\ \rho\omega^2(a_3 - a_4) - 2\alpha^2 d_1(a_3 + a_4) \end{bmatrix} \tag{4.23}$$

Now we need to find u_h. As a guess we choose the following form for the homogeneous solution.

$$u_\mathrm{h} = g_1 h_1(x,t) + g_2 h_2(x,t) + g_3 h_3(x,t) + g_4 h_4(x,t) \tag{4.24}$$

$$\begin{aligned} h_1 &= \cos\left(qx - \omega t\right), \qquad & h_2 &= \sin\left(qx - \omega t\right) \\ h_3 &= \cos\left(qx + \omega t\right), \qquad & h_4 &= \sin\left(qx + \omega t\right) \end{aligned} \tag{4.25}$$

After putting this form for u_h through the homogeneous differential equation, we get the following requirement for q:

$$q = \omega \sqrt{\frac{\rho}{d_1}} \tag{4.26}$$

In this problem we assume that the left end of the monomorph is clamped, so that $u(0,t) = 0$, and the right end is free, so that $T_1(L,t) = 0$. After applying the first boundary condition we get the following conditions on the g_i, the coefficients for the homogeneous solution.

$$\begin{gathered} u\left(0,t\right) = 0 = \left(g_1 + g_3 + c_1 + c_3\right)\cos\left(\omega t\right) + \left(g_4 - g_2 + c_4 - c_2\right)\sin\left(\omega t\right) \\ \rightarrow \quad g_3 = -\left(g_1 + c_1 + c_3\right), \quad g_4 = g_2 + c_2 - c_4 \end{gathered} \tag{4.27}$$

Now we apply the second boundary condition.

$$T_1(L,t) = 0 = d_1 \frac{\partial u}{\partial x}\bigg|_{x=L} + d_3 E_3(L,t) \tag{4.28}$$

$$\begin{aligned} & f_1(L,t)\Big[d_1\alpha\left(c_2 - c_1\right) + d_3 a_1\Big] + f_2(L,t)\Big[-d_1\alpha\left(c_1 + c_2\right) + d_3 a_2\Big] \\ & + f_3(L,t)\Big[d_1\alpha\left(c_3 + c_4\right) + d_3 a_3\Big] + f_4(L,t)\Big[d_1\alpha\left(c_4 - c_3\right) + d_3 a_4\Big] \\ & + d_1 q\Big[g_2 h_1(L,t) - g_1 h_2(L,t) + g_4 h_3(L,t) - g_3 h_4(L,t)\Big] \\ & = 0 \end{aligned} \tag{4.29}$$

The resulting expressions are going to start getting very messy, so I'm going to start abbreviating many of the constant coefficients.

$$\zeta_1 = d_1 q = \omega\sqrt{\rho d_1} \tag{4.30}$$

$$\gamma_1 = d_1\alpha\left(c_2 - c_1\right) + d_3 a_1, \quad \gamma_2 = -d_1\alpha\left(c_1 + c_2\right) + d_3 a_2$$
$$\gamma_3 = d_1\alpha\left(c_3 + c_4\right) + d_3 a_3, \quad \gamma_4 = d_1\alpha\left(c_4 - c_3\right) + d_3 a_4 \tag{4.31}$$

Since the c_i aren't too complicated, the γ_i can be expressed in a less abbreviated form.

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix} = \frac{\rho\omega^2 d_3}{\rho^2\omega^4 + 4\alpha^4 d_1^2} \begin{bmatrix} \rho\omega^2 a_1 + 2\alpha^2 d_1 a_2 \\ \rho\omega^2 a_2 - 2\alpha^2 d_1 a_1 \\ \rho\omega^2 a_3 - 2\alpha^2 d_1 a_4 \\ \rho\omega^2 a_4 + 2\alpha^2 d_1 a_3 \end{bmatrix} \tag{4.32}$$

Given these abbreviated coefficients, and equation (4.62), equation (4.29) can be written as follows.

$$\begin{aligned} &\gamma_1 f_1(L,t) + \gamma_2 f_2(L,t) + \gamma_3 f_3(L,t) + \gamma_4 f_4(L,t) + \zeta_1 g_2 h_1(L,t) \\ -\ &\zeta_1 g_1 h_2(L,t) + \zeta_1\left(g_2 + c_2 - c_4\right) h_3(L,t) + \zeta_1\left(g_1 + c_1 + c_3\right) h_4(L,t) \\ &= 0 \end{aligned} \tag{4.33}$$

We pick up a couple more constants.

$$\gamma_5 = \zeta_1\left(c_2 - c_4\right), \quad \gamma_6 = \zeta_1\left(c_1 + c_3\right) \tag{4.34}$$

$$\begin{bmatrix} \gamma_5 \\ \gamma_6 \end{bmatrix} = \frac{\alpha\omega d_3\sqrt{\rho d_1}}{\rho^2\omega^4 + 4\alpha^4 d_1^2} \begin{bmatrix} \rho\omega^2\left(2a_1 + a_2 + a_4\right) - 2\alpha^2 d_1\left(2a_1 - a_2 - a_4\right) \\ \rho\omega^2\left(2a_1 - a_2 - a_4\right) + 2\alpha^2 d_1\left(2a_2 + a_2 + a_4\right) \end{bmatrix} \tag{4.35}$$

Since the $f_i(L,t)$ and the $h_i(L,t)$ are sinusoids in time, we once again get an expression of the form $C_c \cos\omega t + C_s \sin\omega t = 0$, where in order to satisfy equation (4.33) for all time, both C_c and C_s must equal zero.

$$\begin{aligned} &\gamma_1 e^{-\alpha L}\cos\alpha L + \gamma_2 e^{-\alpha L}\sin\alpha L + \gamma_3 e^{\alpha L}\cos\alpha L \\ +\ &\gamma_4 e^{\alpha L}\sin\alpha L + \gamma_5\cos qL + \gamma_6\sin qL \\ &= -2g_2\zeta_1\cos qL = K_1 \end{aligned} \tag{4.36}$$

$$\begin{aligned} &\gamma_1 e^{-\alpha L}\sin\alpha L - \gamma_2 e^{-\alpha L}\cos\alpha L - \gamma_3 e^{\alpha L}\sin\alpha L \\ +\ &\gamma_4 e^{\alpha L}\cos\alpha L - \gamma_5\sin qL + \gamma_6\cos qL \\ &= -2g_1\zeta_1\cos qL = K_2 \end{aligned} \tag{4.37}$$

With equations (4.36), (4.64), and (4.62) we can now fully solve for the g_i.

$$g_1 = -\frac{K_2}{2\zeta_1 \cos qL} \tag{4.38a}$$

$$g_2 = -\frac{K_1}{2\zeta_1 \cos qL} \tag{4.38b}$$

$$g_3 = \frac{K_2 - 2\gamma_6 \cos qL}{2\zeta_1 \cos qL} \tag{4.38c}$$

$$g_4 = \frac{2\gamma_5 \cos qL - K_1}{2\zeta_1 \cos qL} \tag{4.38d}$$

We see that the solution blows up (divide by zero) when $\cos qL = 0$. This condition occurs at the resonance frequencies, when $qL = (2n+1)\pi/2$.

$$\omega_{\text{Res}} = \frac{(2n+1)\,\pi}{2L}\sqrt{\frac{d_1}{\rho}}, \quad n = 0, 1, 2, \ldots \tag{4.39a}$$

$$f_{\text{Res}} = \frac{(2n+1)}{4L}\sqrt{\frac{d_1}{\rho}}, \quad n = 0, 1, 2, \ldots \tag{4.39b}$$

Since the solution u is sinusoidal in time, we can find the amplitude and phase.

$$u = |u| \sin(\omega t + \delta) = |u| \sin\delta \cos\omega t + |u| \cos\delta \sin\omega t \tag{4.40}$$

$$\begin{aligned} |u| \sin\delta = & c_1 e^{-\alpha x} \cos\alpha x + c_2 e^{-\alpha x} \sin\alpha x + c_3 e^{\alpha x} \cos\alpha x + c_4 e^{\alpha x} \sin\alpha x \\ & + g_1 \cos qx + g_2 \sin qx + g_3 \cos qx + g_4 \sin qx \end{aligned} \tag{4.41}$$

$$\begin{aligned} |u| \cos\delta = & c_1 e^{-\alpha x} \sin\alpha x - c_2 e^{-\alpha x} \cos\alpha x - c_3 e^{\alpha x} \sin\alpha x + c_4 e^{\alpha x} \cos\alpha x \\ & + g_1 \sin qx - g_2 \cos qx - g_3 \sin qx + g_4 \cos qx \end{aligned} \tag{4.42}$$

The amplitude of the displacement versus angular frequency is shown if figure 4.1. The displacement at $\omega = 0$ is a finite value, as shown in the graph, though it may seem at first glance that the displacement goes to infinity. This displacement was calculated for a monomorph of $L = 0.05$m, and applied voltage amplitude of 1V.

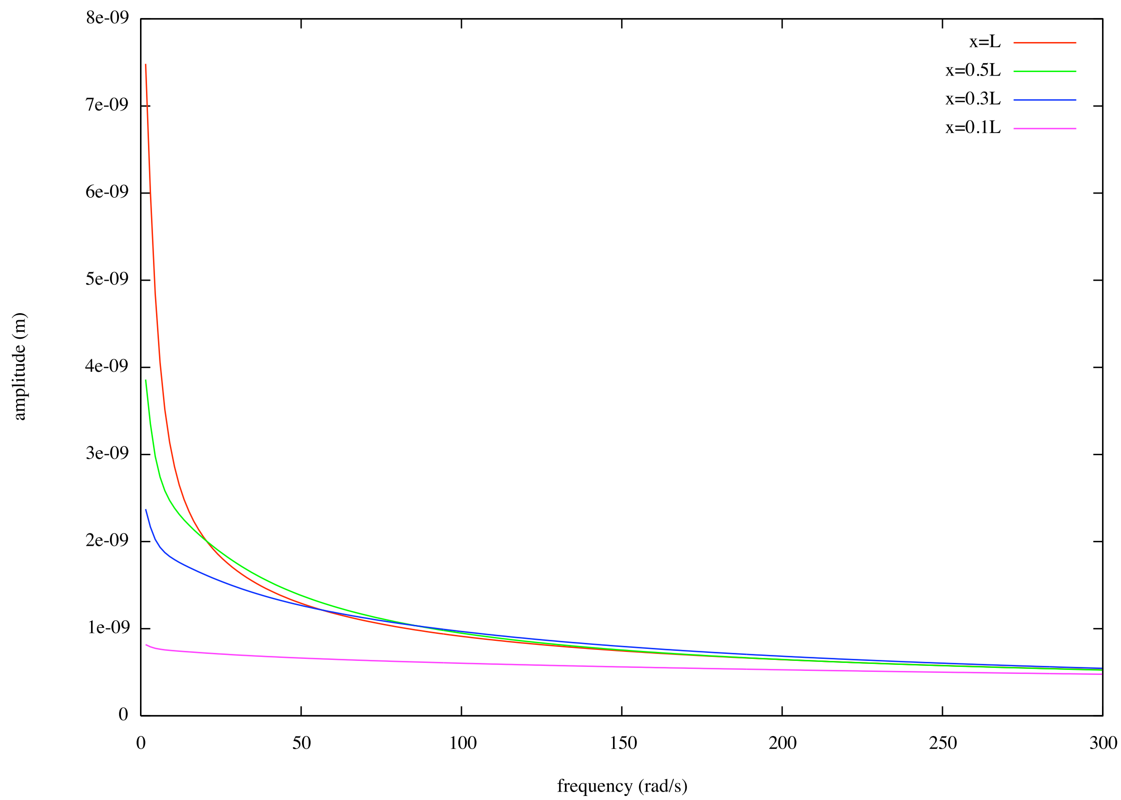

Figure 4.1: Displacement amplitude vs. frequency.

Damped Case

As a step closer to reality we can add a damping term to the differential equation. This should prevent the solution from blowing up at the resonance frequencies.

$$T_1 = d_1 S_1 + d_2 \frac{\partial S_1}{\partial t} + d_3 E_3 \tag{4.43}$$

There are several choices for the form of the damping term, so it depends on the physics model. Here d_2 is a positive damping constant. Now we go through the entire process as for the undamped case.

$$\rho \frac{\partial^2 u_{\mathrm{ih}}}{\partial t^2} - d_1 \frac{\partial^2 u_{\mathrm{ih}}}{\partial x^2} - d_2 \frac{\partial^3 u_{\mathrm{ih}}}{\partial x^2 \partial t} = d_3 \frac{\partial E_3}{\partial x} \tag{4.44}$$

$$\rho \frac{\partial^2 u_{\mathrm{h}}}{\partial t^2} - d_1 \frac{\partial^2 u_{\mathrm{h}}}{\partial x^2} - d_2 \frac{\partial^3 u_{\mathrm{h}}}{\partial x^2 \partial t} = 0 \tag{4.45}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \frac{C}{A^2 + B^2} \begin{bmatrix} A\left(a_2 - a_1\right) + B\left(a_1 + a_2\right) \\ -A\left(a_1 + a_2\right) + B\left(a_2 - a_1\right) \\ A\left(a_3 + a_4\right) + B\left(a_4 - a_3\right) \\ A\left(a_4 - a_3\right) - B\left(a_3 + a_4\right) \end{bmatrix} \tag{4.46}$$

$$A = 2\omega\alpha^2 d_2 - \rho\omega^2, \quad B = 2\alpha^2 d_1, \quad C = \alpha d_3$$

The only difference for the form of the homogeneous solution is that each term is multiplied by an exponential in x.

$$u_{\mathrm{h}} = g_1 h_1(x,t) + g_2 h_2(x,t) + g_3 h_3(x,t) + g_4 h_4(x,t) \tag{4.47}$$

$$\begin{aligned} h_1 &= e^{-bx}\cos\left(qx - \omega t\right), & h_2 &= e^{-bx}\sin\left(qx - \omega t\right) \\ h_3 &= e^{bx}\cos\left(qx + \omega t\right), & h_4 &= e^{bx}\sin\left(qx + \omega t\right) \end{aligned} \tag{4.48}$$

Some derivatives of the h_i which will be useful later are listed below.

$$\dot{h}_1 = \omega h_2 \tag{4.49a}$$

$$\dot{h}_2 = -\omega h_1 \tag{4.49b}$$

$$\dot{h}_3 = -\omega h_4 \tag{4.49c}$$

$$\dot{h}_4 = \omega h_3 \tag{4.49d}$$

$$\ddot{h}_1 = -\omega^2 h_1 \tag{4.50a}$$

$$\ddot{h}_2 = -\omega^2 h_2 \tag{4.50b}$$

$$\ddot{h}_3 = -\omega^2 h_3 \tag{4.50c}$$

$$\ddot{h}_4 = -\omega^2 h_4 \tag{4.50d}$$

$$h_1' = -bh_1 - qh_2 \tag{4.51a}$$

$$h_2' = qh_1 - bh_2 \tag{4.51b}$$

$$h_3' = bh_3 - qh_4 \tag{4.51c}$$

$$h_4' = qh_3 + bh_4 \tag{4.51d}$$

$$h_1'' = \left(b^2 - q^2\right) h_1 + 2bqh_2 \tag{4.52a}$$

$$h_2'' = -2bqh_1 + \left(b^2 - q^2\right) h_2 \tag{4.52b}$$

$$h_3'' = \left(b^2 - q^2\right) h_3 - 2bqh_4 \tag{4.52c}$$

$$h_4'' = 2bqh_3 + \left(b^2 - q^2\right) h_4 \tag{4.52d}$$

$$\dot{h}_1' = \omega\left(qh_1 - bh_2\right) \tag{4.53a}$$

$$\dot{h}_2' = \omega\left(bh_1 + qh_2\right) \tag{4.53b}$$

$$\dot{h}_3' = -\omega\left(qh_3 + bh_4\right) \tag{4.53c}$$

$$\dot{h}_4' = \omega\left(bh_3 - qh_4\right) \tag{4.53d}$$

$$\dot{h}_1'' = \omega\Big(-2bqh_1 + \left(b^2 - q^2\right) h_2\Big) \tag{4.54a}$$

$$\dot{h}_2'' = -\omega\Big(\left(b^2 - q^2\right) h_1 + 2bqh_2\Big) \tag{4.54b}$$

$$\dot{h}_3'' = -\omega\Big(2bqh_3 + \left(b^2 - q^2\right) h_4\Big) \tag{4.54c}$$

$$\dot{h}_4'' = \omega\Big(\left(b^2 - q^2\right) h_3 - 2bqh_4\Big) \tag{4.54d}$$

After putting our new form for u_h through the damped homogeneous differential equation, we get the following conditions on the g_i.

$$\begin{gathered}\begin{bmatrix} F & G \\ -G & F \end{bmatrix}\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} F & -G \\ G & F \end{bmatrix}\begin{bmatrix} g_3 \\ g_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ F = \left(q^2 - b^2\right) d_1 - \rho\omega^2, \quad G = 2bqd_1 - \left(q^2 - b^2\right)\omega d_2 \end{gathered} \tag{4.55}$$

For non-zero g_i, the determinants of the coefficient matrices must be zero.

$$F^2 + G^2 = 0 \tag{4.56}$$

The only way for this to be satisfied is for both F and G to both be zero.

$$F = 0, \quad G = 0 \tag{4.57}$$

This provides two equations to find b and q.

$$bq = \frac{\rho\omega^3 d_2}{2\left(d_1^2 + \omega^2 d_2^2\right)} \tag{4.58}$$

$$q^2 - b^2 = \frac{\rho\omega^2 d_1}{d_1^2 + \omega^2 d_2^2} \tag{4.59}$$

$$b = \left[\frac{\rho\omega^2 d_1}{2\left(d_1^2 + \omega^2 d_2^2\right)}\left(\sqrt{1 + \left(\frac{\omega d_2}{d_1}\right)^2} - 1\right)\right]^{1/2} \tag{4.60}$$

$$q = \left[\frac{\rho\omega^2 d_1}{2\left(d_1^2 + \omega^2 d_2^2\right)}\left(\sqrt{1 + \left(\frac{\omega d_2}{d_1}\right)^2} + 1\right)\right]^{1/2} \tag{4.61}$$

If we set the damping constant, d_2, to zero, we get $b = 0$, and q reduces to the same value as for the undamped case. After applying the $x = 0$ boundary condition on u, we get the same condition on the g_i as before.

$$\begin{gathered} u\left(0, t\right) = 0 = \left(g_1 + g_3 + c_1 + c_3\right)\cos\left(\omega t\right) + \left(g_4 - g_2 + c_4 - c_2\right)\sin\left(\omega t\right) \\ \rightarrow \quad g_3 = -\left(g_1 + c_1 + c_3\right), \quad g_4 = g_2 + c_2 - c_4 \end{gathered} \tag{4.62}$$

Now we apply the second boundary condition.

$$d_1 \frac{\partial u}{\partial x}\bigg|_{x=L} + d_2 \frac{\partial^2 u}{\partial x \partial t}\bigg|_{x=L} + d_3 E_3(L,t) = 0 \tag{4.63}$$

$$\begin{aligned} &\gamma_1 f_1(L,t) + \gamma_2 f_2(L,t) + \gamma_3 f_3(L,t) + \gamma_4 f_4(L,t) \\ &+ \big[\zeta_2 g_1 + \zeta_1 g_2\big] h_1(L,t) + \big[-\zeta_1 g_1 + \zeta_2 g_2\big] h_2(L,t) \\ &+ \big[\zeta_2 g_1 + \zeta_1 g_2 + \gamma_5\big] h_3(L,t) + \big[\zeta_1 g_1 - \zeta_2 g_2 + \gamma_6\big] h_4(L,t) \\ &= 0 \end{aligned} \tag{4.64}$$

$$\zeta_1 = d_1 q + d_2 \omega b \tag{4.65a}$$

$$\zeta_2 = d_2 \omega q - d_1 b \tag{4.65b}$$

$$\gamma_1 = d_1 \alpha \left(c_2 - c_1\right) + d_2 \alpha \omega \left(c_1 + c_2\right) + d_3 a_1 \tag{4.66a}$$

$$\gamma_2 = -d_1 \alpha \left(c_1 + c_2\right) + d_2 \alpha \omega \left(c_2 - c_1\right) + d_3 a_2 \tag{4.66b}$$

$$\gamma_3 = d_1 \alpha \left(c_3 + c_4\right) + d_2 \alpha \omega \left(c_4 - c_3\right) + d_3 a_3 \tag{4.66c}$$

$$\gamma_4 = d_1 \alpha \left(c_4 - c_3\right) - d_2 \alpha \omega \left(c_3 + c_4\right) + d_3 a_4 \tag{4.66d}$$

$$\gamma_5 = \zeta_2 \left(c_1 + c_3\right) + \zeta_1 \left(c_2 - c_4\right) \tag{4.67a}$$

$$\gamma_6 = \zeta_1 \left(c_1 + c_3\right) - \zeta_2 \left(c_2 - c_4\right) \tag{4.67b}$$

We separate the $f_i(L,t)$ and the $h_i(L,t)$ into $\cos \omega t$ and $\sin \omega t$ terms, which provides two necessary conditions in order for equation (4.64) to be zero for all time.

$$h_1(L,t) = e^{-bL} \cos(qL) \cos(\omega t) + e^{-bL} \sin(qL) \sin(\omega t) \tag{4.68a}$$

$$h_2(L,t) = e^{-bL} \sin(qL) \cos(\omega t) - e^{-bL} \cos(qL) \sin(\omega t) \tag{4.68b}$$

$$h_3(L,t) = e^{bL} \cos(qL) \cos(\omega t) - e^{bL} \sin(qL) \sin(\omega t) \tag{4.68c}$$

$$h_4(L,t) = e^{bL} \sin(qL) \cos(\omega t) + e^{bL} \cos(qL) \sin(\omega t) \tag{4.68d}$$

$$\begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = - \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \tag{4.69}$$

$$\begin{aligned} K_1 = {} & \gamma_1 e^{-\alpha L} \cos \alpha L + \gamma_2 e^{-\alpha L} \sin \alpha L + \gamma_3 e^{\alpha L} \cos \alpha L \\ & + \gamma_4 e^{\alpha L} \sin \alpha L + \gamma_5 e^{bL} \cos qL + \gamma_6 e^{bL} \sin qL \end{aligned} \tag{4.70a}$$

$$\begin{aligned} K_2 = {} & \gamma_1 e^{-\alpha L} \sin \alpha L - \gamma_2 e^{-\alpha L} \cos \alpha L - \gamma_3 e^{\alpha L} \sin \alpha L \\ & + \gamma_4 e^{\alpha L} \cos \alpha L - \gamma_5 e^{bL} \sin qL + \gamma_6 e^{bL} \cos qL \end{aligned} \tag{4.70b}$$

$$M_1 = -\zeta_1 \left(e^{-bL} - e^{bL} \right) \sin qL + \zeta_2 \left(e^{-bL} + e^{bL} \right) \cos qL \tag{4.71a}$$

$$M_2 = \zeta_1 \left(e^{-bL} + e^{bL} \right) \cos qL + \zeta_2 \left(e^{-bL} - e^{bL} \right) \sin qL \tag{4.71b}$$

$$M_3 = \zeta_1 \left(e^{-bL} + e^{bL} \right) \cos qL + \zeta_2 \left(e^{-bL} - e^{bL} \right) \sin qL = M_2 \tag{4.71c}$$

$$M_4 = \zeta_1 \left(e^{bL} - e^{-bL} \right) \sin qL - \zeta_2 \left(e^{-bL} + e^{bL} \right) \cos qL = -M_1 \tag{4.71d}$$

$$\begin{bmatrix} M_1 & M_2 \\ M_2 & -M_1 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = - \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \tag{4.72}$$

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = - \frac{1}{M_1^1 + M_2^2} \begin{bmatrix} M_1 & M_2 \\ M_2 & -M_1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \tag{4.73}$$

$$g_1 = - \frac{M_1 K_1 + M_2 K_2}{M_1^1 + M_2^2} \tag{4.74a}$$

$$g_2 = \frac{M_1 K_2 - M_2 K_1}{M_1^1 + M_2^2} \tag{4.74b}$$

$$g_3 = \frac{M_1 K_1 + M_2 K_2}{M_1^1 + M_2^2} - c_1 - c_3 \tag{4.74c}$$

$$g_4 = \frac{M_1 K_2 - M_2 K_1}{M_1^1 + M_2^2} + c_2 - c_4 \tag{4.74d}$$

Although the coefficients are a little bit messier, the solution for the damped case has pretty much the same form as for the undamped case. Once again we can find the amplitude and phase of u:

$$u = |u| \sin (\omega t + \delta) = |u| \sin \delta \cos \omega t + |u| \cos \delta \sin \omega t \tag{4.75}$$

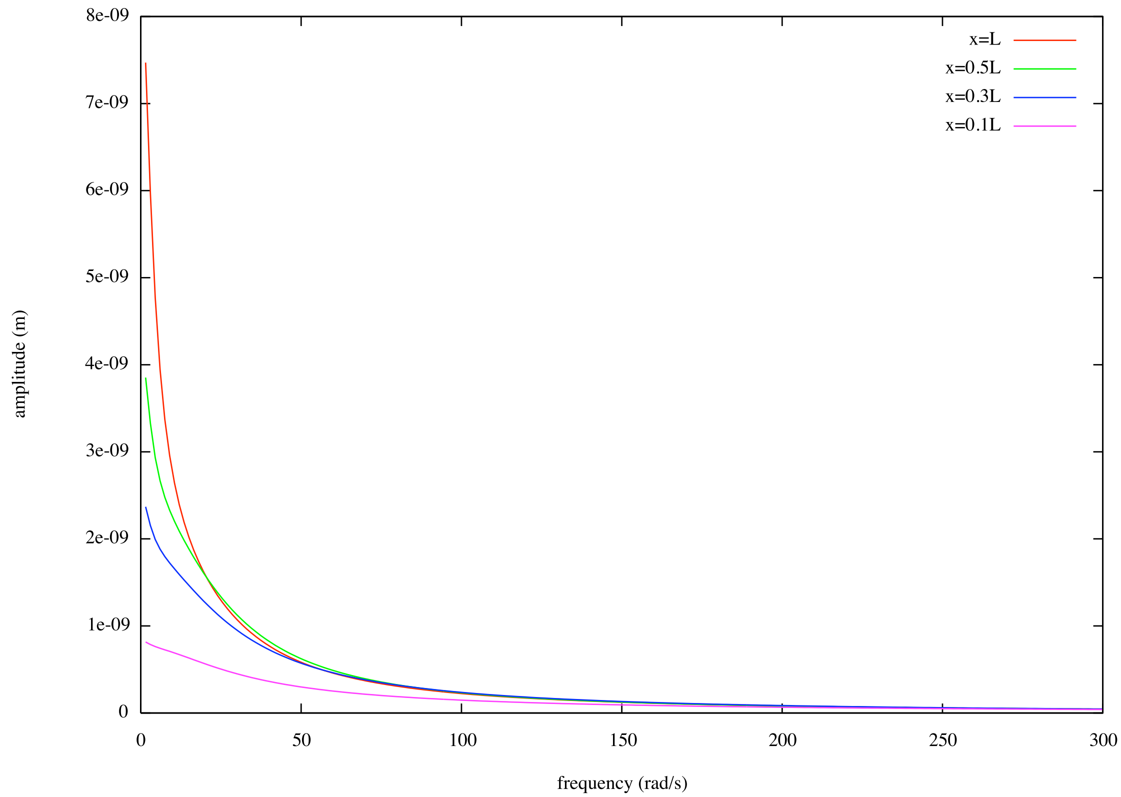

Figure 4.2: Displacement amplitude vs. frequency.

$$\begin{aligned} |u| \sin \delta =& c_1 e^{-\alpha x} \cos \alpha x + c_2 e^{-\alpha x} \sin \alpha x + c_3 e^{\alpha x} \cos \alpha x + c_4 e^{\alpha x} \sin \alpha x \\ & + g_1 e^{-bx} \cos qx + g_2 e^{-bx} \sin qx + g_3 e^{bx} \cos qx + g_4 e^{bx} \sin qx \end{aligned} \tag{4.76}$$

$$\begin{aligned} |u| \cos \delta =& c_1 e^{-\alpha x} \sin \alpha x - c_2 e^{-\alpha x} \cos \alpha x - c_3 e^{\alpha x} \sin \alpha x + c_4 e^{\alpha x} \cos \alpha x \\ & + g_1 e^{-bx} \sin qx - g_2 e^{-bx} \cos qx - g_3 e^{bx} \sin qx + g_4 e^{bx} \cos qx \end{aligned} \tag{4.77}$$

The amplitude of the displacement versus angular frequency is shown if figure 4.2.

CHAPTER 5

CONCLUSIONS

The voltage solution for the capacitor solution matches the form of the experimental measurements very well. Figure 5.1 shows an experimental measurement at 200 Hz [3]. In that plot the blue line is the voltage in the top electrode at $x = L$, and the red line is the voltage for the bottom electrode at $x = L$. Figure 5.2 shows the calculation from the capacitor model. In that plot the solid line represents the input voltage, and the dotted line represents the voltage in the top electrode at $x = L$.

The results of the mechanical deformation calculations have not yet been verified experimentally, but the form seems reasonable.

For future work I endeavor to solve the full coupled problem. Many people have solved the coupled piezoelectric problem for ideal electrodes (a uniform surface potential), but the problem of the coupled problem with an unkown electrode voltage distribution has yet to be solved. A method that has been widely employed to solve this type of problem is the finite element method, and that seems to me to be the best approach.

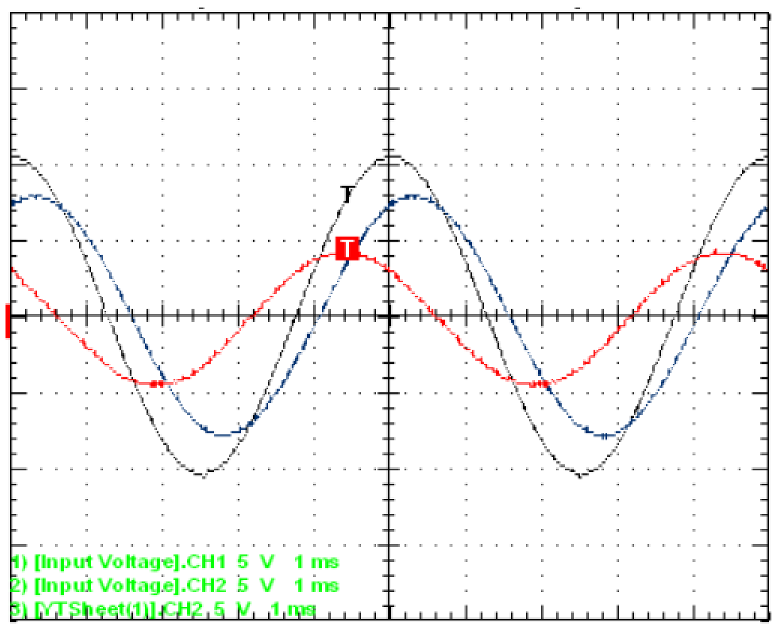

Figure 5.1: Experimental voltage measurement.

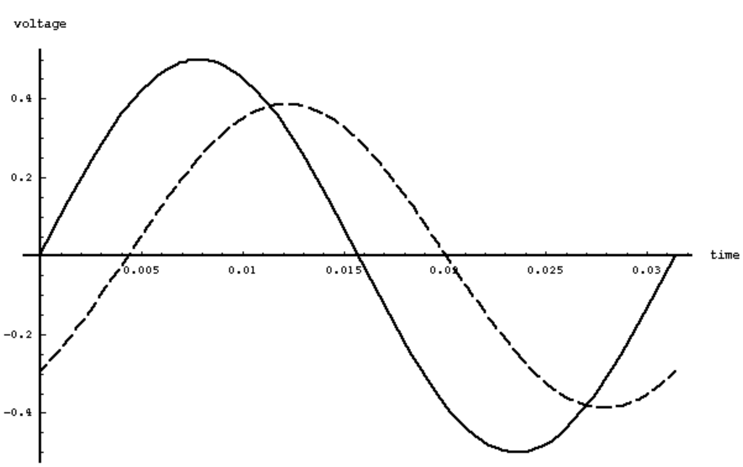

Figure 5.2: Capacitor voltage solution

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