

THE BRANCH LOCUS FOR TWO DIMENSIONAL TILING SPACES

by

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A dissertation submitted in partial fulfillment  
of the requirements for the degree

of

Doctor of Philosophy

in

Mathematics

MONTANA STATE UNIVERSITY  
Bozeman, Montana

July, 2010

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July, 2010

## ACKNOWLEDGEMENTS

In no small way Dr. Marcy Barge has helped me to complete this dissertation. His seemingly endless understanding is matched only by his patience. These are strengths I am extremely grateful for. I thank my committee: Richard Swanson, Lukas Geyar, Russ Walker, and Jack Dockery. I am grateful to the faculty and staff of Montana State University mathematics Department who gave me the opportunity to succeed in this endeavor. Finally, I would like to thank my beautiful wife Sarah, whose love and support has made this journey possible.

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## ABSTRACT

We explore the asymptotic arc components made by the continuous  $\mathbb{R}^2$ -action of translation on two-dimensional nonperiodic substitution tiling spaces. As there is a strong connection between the topology of a tiling space and the tiling dynamics that it supports, the results in this dissertation represent a qualitative approach to the study of tiling dynamics. Our results are the establishment of techniques to isolate and visualize the asymptotic behavior.

In a recent paper, Barge, *et al*, showed the cohomology formed from the asymptotic structure in one-dimensional Pisot substitution tiling spaces is a topological invariant, [BDS]. However, in one dimension there exist only a finite number of asymptotic pairs, whereas there are infinitely many asymptotic leaves in two dimensions. By considering periodic tilings that are asymptotic in more than a half plane we are able to use the stable manifold under inflation and substitution to show there are a finite number of ‘directions’ of branching. This yields a description of the asymptotic structure in terms of an inverse limit of a branched set in the approximating collared Anderson-Putnam complex.

Using rigidity results from [JK], we show the cohomology formed from the asymptotic structure is a topological invariant.

## INTRODUCTION

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There is a single light of science,  
and to brighten it anywhere is to brighten it everywhere.

Isaac Asimov

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Tilings in Vivo

Tiling theory is a relatively new yet rich development in mathematics. A tiling of  $\mathbb{R}^2$  is a subdivision of  $\mathbb{R}^2$  into tiles that intersect only along their boundaries. These tiles are taken from a finite set of prototiles that we take to be compact subsets which are the closures of their interiors. A tiling  $T$  is nonperiodic if its translation group  $\{\mathbf{x} \in \mathbb{R}^2 : T - \mathbf{x} = T\}$  is trivial. There are prototile sets that form periodic and nonperiodic tilings. A set of prototiles that tile the plane only non-periodically is said to be an aperiodic set; the tiling formed is called an aperiodic tiling.

The study of nonperiodic tiling began in 1961 with the work of Wang [W], whose interest centered on decidability issues. That is, under what conditions does any finite set of prototiles tessellate the plane? His student, Berger, proved no such algorithm exists by constructing an example that did so aperiodically. Berger's set was extremely large, with more than 20,000 prototiles. Later, Robinson found a simpler example with 32 prototiles [Sol].

A major motivation for studying nonperiodic tilings came from the physics of quasi-crystals, which were discovered by Schechtman in 1984 [SBGC]. Such a material appeared to be very ordered, because its x-ray diffraction pattern showed clear peaks. However, it could not be periodic, as the diffraction pattern had symmetries which should not have occurred according to the classical periodic models of crystals.

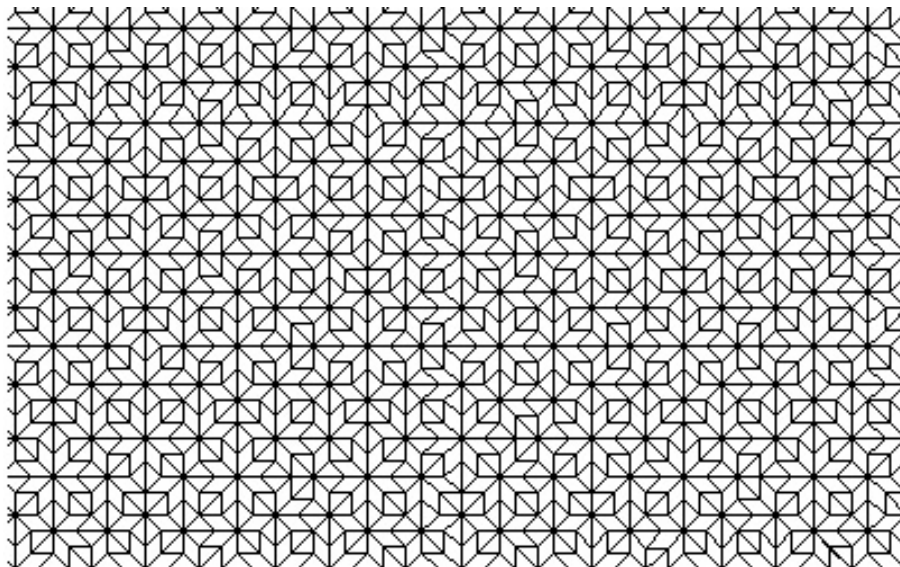


Figure 1: A part of an Octagonal Tiling

The appearance of quasi-crystals in physics created the need for new mathematical objects in order to model these materials. Focus was shifted then to the combinatorial structure of nonperiodic tilings. The combinatorics of the molecular bonds of a quasi-crystal can be seen in the diffraction. The combinatorics can be modeled with nonperiodic tilings and the diffraction can be studied through the spectral properties of the tilings. By passing to the space of all tilings locally indistinguishable from a given tiling, combinatorial and spectral properties of these tilings are translated into topological properties of the tiling space. Questions in solid state physics become purely topological in nature.

Thus, topological dynamical systems play a major role in the study of nonperiodic tilings. The group  $\mathbb{R}^2$  continuously acts on the tiling space by translation. This gives rise to a tiling dynamical system. One can then study the topological invariants of the associated topological space. Tiling spaces have infinity many path components, each of which is usually contractible, so fundamental groups or homotopy groups give

only trivial information. Čech cohomology<sup>1</sup> does better since it uses nerves of cofinal open covers to measure connected components.

In 1998, Anderson and Putnam [AP] showed how to describe a substitution tiling space as an inverse limit of branched manifolds. If the substitution has the property called ‘forcing the border,’ they constructed a CW complex  $K$  consisting of one copy of every kind of tile, with certain edge identifications. The substitution  $\Phi$  induces a map  $F$  which maps  $K$  to itself. The inverse limit  $\varprojlim(K, F)$  is homeomorphic to the tiling space  $\Omega_\Phi$ . By the continuity of Čech cohomology, the cohomology of the inverse limit is then the direct limit of the cohomology of the approximate  $K$  under the pullback map  $F^*$ .

Recent success in tiling theory has come from studying the structure of the asymptotic arc components in the tiling space. Two tilings are asymptotic, in the direction of a vector  $\mathbf{x}$ , if  $d(T - u\mathbf{x}, T' - u\mathbf{x}) \rightarrow 0$ , as  $u \rightarrow \infty$ . In their topological classification of 1-dimensional tiling spaces, ([BD]) Barge and Diamond made critical use of the asymptotic orbits of the tiling spaces. A homeomorphism carries a pair of asymptotic orbits to a pair of asymptotic orbits. This is true since in 1-dimensional tiling spaces there are finitely many asymptotic components. Thus the set of asymptotic tilings is periodic under  $\Phi$ . In higher dimensions there are infinity many of these asymptotic leaves, so the techniques used do not generalize easily.

It is possible that the asymptotic arc components contribute to the Čech cohomology of the tiling space, which can be seen in the existence of asymptotic cycles. In one dimension an asymptotic cycle is a collection of asymptotic arc components,  $\{C_1, \dots, C_{2n}\}$  such that  $C_1$  is forward asymptotic to  $C_2$ ,  $C_2$  is backward asymptotic

---

<sup>1</sup>It was Čech who first defined the nerve of a finite covering by open sets, and used such complexes as approximations of a space. By using inverse systems instead of sequences, he defined homology groups of arbitrary spaces[ES].

to  $C_3, C_3$  is forward asymptotic to  $C_4, \dots, C_{2n}$  is backward asymptotic to  $C_1$ . The higher dimensional analog is not as well defined, but we can find tilings that capture this behavior. For example, if we let  $A$  and  $B$  denote two unit square prototiles and define inflation and substitution to be

$$A \mapsto \begin{array}{ccc} B & A & A \\ A & A & A \end{array} \quad B \mapsto \begin{array}{ccc} B & B & A \\ A & B & A \end{array}$$

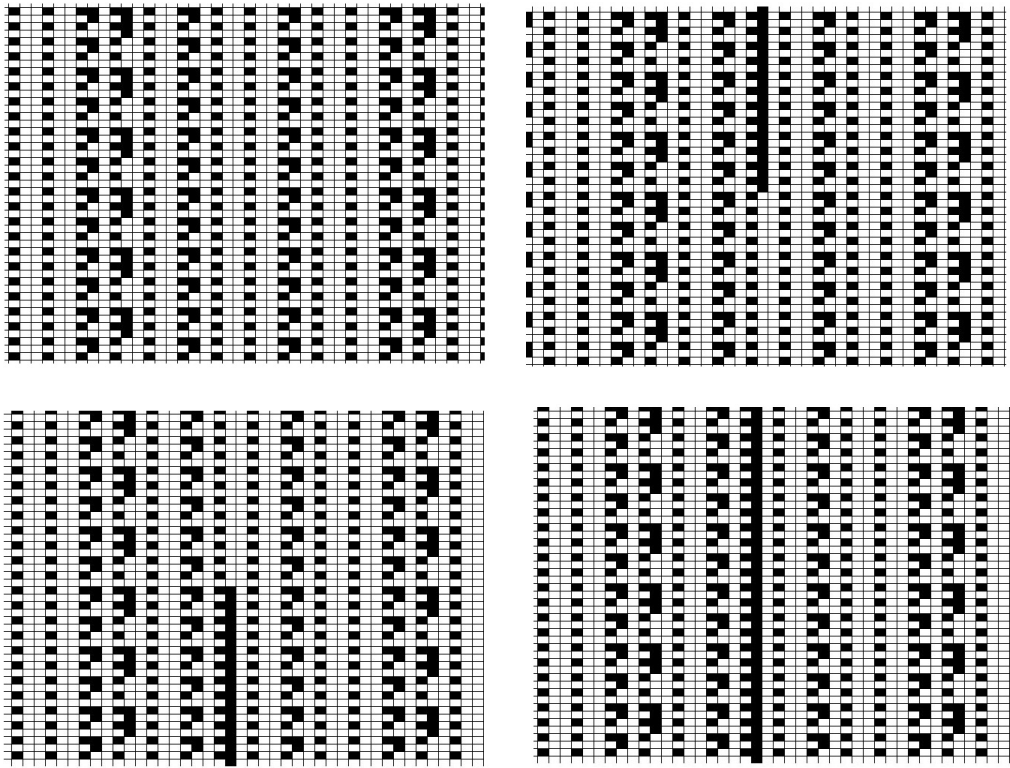


Figure 2: Four tilings that form an asymptotic cycle

then there are four tilings,  $T_1, T_2, T_3, T_4$  such that

$$d(T_* - ui, T_* - ui) \rightarrow 0 \text{ as } u \rightarrow \pm\infty$$

and

$$d(T_1 - u\mathbf{i}, T_3 - u\mathbf{j}) \rightarrow 0 \text{ as } u \rightarrow \infty$$

$$d(T_2 - u\mathbf{j}, T_4 - u\mathbf{j}) \rightarrow 0 \text{ as } u \rightarrow \infty$$

$$d(T_1 - u\mathbf{j}, T_2 - u\mathbf{j}) \rightarrow 0 \text{ as } u \rightarrow -\infty$$

$$d(T_3 - u\mathbf{j}, T_4 - u\mathbf{j}) \rightarrow 0 \text{ as } u \rightarrow -\infty$$

where  $\mathbf{i}, \mathbf{j}$  are the standard basis vectors of  $\mathbb{R}^2$ . In this case, these tilings form a 'bubble' in the tiling space that contributes a copy of  $\mathbb{Z}$  to the top dimensional Čech cohomology, see Figure 3.

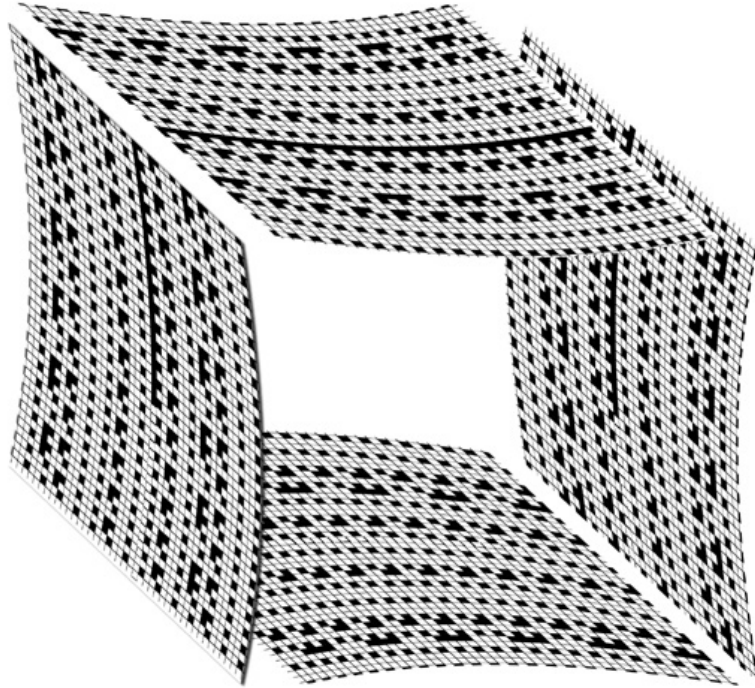


Figure 3: A 2-dimensional asymptotic cycle

As stated, the Anderson and Putnam collaring technique allows us to construct a complex whose inverse limit is homeomorphic to the tiling space and thus has the same Čech cohomology. However, in doing so, any evidence of the asymptotic structure can

be obscured. [BDHS] have developed methods of augmenting this construction so that elements of cohomology can be isolated. In the limit, the preimage of the branches can be observed in the existence of asymptotic components. By inflating the zero and one skeleton of this complex, one can isolate and visualize any contribution from asymptotic structure that is given by the substitution. However, if the substitution forces the border, the inflated skeleton will 'flatten' and no new information will be recovered.

The branch locus has been constructed for one-dimensional tilings spaces. Tilings that are asymptotic in 'half planes' and periodic under  $\Phi$  are labeled *special tilings* (see [BDS]) and are represented on the geometric realization of the tiling space. Since there are many directions in which tiling can be asymptotic, and, typically, infinity many asymptotic pairs in higher dimensions, our development of the branch locus differs from the one-dimensional approach. We aggregate the asymptotic arc components that are asymptotic in at least a half plane so that a closed set in the tiling space can be constructed. Moreover, this set is generated by a finite set of asymptotic pairs. Our construction is dependent on two facts: there are a finite number of "directions" in which branching can occur and a finite number of tilings that lie on two distinct "branches" simultaneously.

For self-similar tilings with expansion constant  $\lambda$ , the link between the  $\mathbb{R}^2$ -action of translation and the  $\mathbb{Z}$ -action of inflation and substitution is given by

$$\Phi^n(T - \mathbf{x}) = \Phi^n(T) - \lambda^n \mathbf{x}, \quad n \in \mathbb{Z}$$

This relation that allows us to discuss asymptotic pairs in terms of stable manifolds of  $\Phi$ . This will act as our primary tool as we investigate the asymptotic structure in the tiling space.

$\Phi$ -periodic tilings that are asymptotic in at least a half plane are called asymptotic pairs. Using the stable manifold of  $\Phi$  it can be shown there are three kinds of such pairs. That is, three different types of branching can occur in the tiling space. This allows us to define the branch locus which works as a skeleton of the asymptotic behavior in the tiling space. By using methods similar to those found in [BDHS], we inflate this skeleton and make identification based on the asymptotic behavior. The resulting branched space represents the asymptotic structure in the tiling space, which we denote  $\mathcal{A}_\Phi$ .

### Structure of Dissertation

Substitutive nonperiodic self-similar tiling spaces are constructed in Chapter 2 and properties of the stable manifold of the substitution  $\Phi$  are derived. We adopt a notion of ‘slippage’ in tiling spaces which plays an important role here: two distinct tilings in the same stable manifold cannot be arbitrarily close to each other. This result, proven in [Sol], is modified for use in our context. Combined with properties of the stable manifold we show if  $T - \mathbf{x} \in W^s(T' - \mathbf{x})$  for all  $\mathbf{x} \in \mathbb{R}^2$  then  $T = T'$ . (Theorem 2.0.26).

The core of this dissertation is in Chapter 3. We define asymptotic pairs ( $\Phi$ -periodic tilings that are asymptotic in at least a half plane) and prove their existence (Theorem 3.0.42). The only possible asymptotic pairs are asymptotic point, line and corner pairs. From these we define branch vertex tilings and show there are only finitely many in the tiling space (Theorem 3.0.49). This allows for construction of the Branch Locus (Definition 3.0.54) which is represented as an inverse limit of a subset of a Anderson-Putnam collared complex (Proposition 3.0.56).

To identify the asymptotic structure, we inflate the branch locus in the tiling space and make identifications, via an equivalence relation, based on asymptotic behavior. This new branched space represents the asymptotic structure in the tiling space. A link then is established to an inverse limit of a neighborhood of the branch locus projected into a CW complex (Theorem 3.0.60). Using rigidity results from [JK], we show this asymptotic structure is a topological invariant (Theorem 3.0.62).

Section 4 is dedicated to three examples which illustrate the kinds of branching that can be found. The half-hex has only asymptotic point pairs, whose cohomology persist into the cohomology of the tiling space. The Chair Tiling has both corner and line pairs whose cohomology is calculated. The Octagonal Tiling has only line pairs whose cohomology contains torsion.

Appendix A covers the use of the Smith Normal Form of a boundary map to compute homology. For convenience, many of the larger matrices have been moved to Appendix B.

TILING SPACES

”There is no wealth like knowledge, no poverty like ignorance.”

- Ali ibn Abi-Talib

Notation and Definitions

The terminology used throughout this document follows that given in [BD]. Let  $\mathcal{P} = \{p_1, \dots, p_n\}$  be a finite set of disjoint compact sets of  $\mathbb{R}^2$ , each the closure of its interior, called *prototiles*. A *tile* is a set obtained from a prototile by translation (note that we do not allow for arbitrary rotations or reflections). A *patch*  $P$  is a set of tiles with pairwise disjoint interiors. The support of a tile,  $\text{spt}(t)$ , is the union of points in  $\mathbb{R}^2$  that are contained in  $t$ , and more generally, the support of a patch  $P$  is the union of the supports of the tiles of  $P$ . A *tiling* of  $\mathbb{R}^2$  with prototiles  $\mathcal{P}$  is a patch with support  $\mathbb{R}^2$ . It will be convenient to talk about patches about the origin of certain radius, so let  $B_r(\mathbf{x})$  to be the closed  $r$ -ball of  $\mathbf{x} \in \mathbb{R}^2$ . Then given a tiling  $T$ , define an  $r$ -patch to be  $B_r[T] = \{t : t \in T \text{ and } t \cap B_r(0) \neq \emptyset\}$ . In the case  $r = 0$  we let  $B_0[T] = \{t \text{ (tile)} : t \text{ in } T \text{ and } 0 \in \text{spt}(t)\}$ . Two patches  $P_1$  and  $P_2$  are *translationally equivalent* if there is a  $\mathbf{x} \in \mathbb{R}^2$  so that  $P_1 + \mathbf{x} = P_2$ . A tiling has *finite local complexity* if, for each  $r > 0$ , the tiling contains only finitely many translational equivalence classes of patches of diameter less than  $r$ .

Two tilings are close if they almost agree on a large ball about the origin. Given tilings  $T_1, T_2$  define the tiling metric

$$d(T_1, T_2) := \min\{\epsilon, 2^{-1/2} : B_{\frac{1}{\epsilon}}[T_1] = B_{\frac{1}{\epsilon}}[T_2 - \mathbf{x}], \text{ where } |\mathbf{x}| < \epsilon\}$$

**Remark 2.0.1.** If  $d(T, T') < \epsilon$  then there exists an  $\mathbf{x} \in \mathbb{R}^2$ ,  $|\mathbf{x}| < \epsilon$ , such that  $B_1[T] = B_1[T' - \mathbf{x}]$ .

**Remark 2.0.2.**  $T_n \rightarrow T$  if and only if for all  $\epsilon, R > 0$ ,  $\exists$  an  $N > 0$  such that for each  $n > N$  there exist vectors  $\mathbf{x}_n$  with  $|\mathbf{x}_n| < \epsilon$  such that  $B_R[T_n - \mathbf{x}_n] = B_R[T]$ .

A substitution  $\Phi$  on  $\mathcal{P}$  is a map  $\Phi : \mathcal{P} \rightarrow \{P : P \text{ is a patch}\}$  such that for  $p \in \mathcal{P}$ , the support of  $\Phi(p)$  is  $\phi \text{spt}(p)$ , where  $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a fixed expansive linear map (that is, all eigenvalues of  $\phi$  have modulus larger than one). A substitution is *self-similar* if  $\phi$  is a similarity, in which case we denote the expansion constant as  $\lambda$  (note that  $\lambda > 1$ ). The substitution can be extended to patches, dilating the entire patch by a factor of  $\phi$  and replacing each tile  $\phi t = \phi(p + \mathbf{x})$  with  $\Phi(t) = \Phi(p) + \phi \mathbf{x}$ . A tiling  $T$  of  $\mathbb{R}^2$  is *admissible* for  $\Phi$  if, for every finite patch  $P$  of  $T$ , there is a prototile  $p$  and an integer  $n$  such that  $P$  is translationally equivalent to a subpatch of  $\Phi^n(p)$ . Define the *substitution tiling space*  $\Omega_\Phi$  to be the set of all admissible tilings for  $\Phi$  and  $\mathcal{P}$ .

A substitution  $\Phi$  is primitive if there exists an integer  $n$  such that for any two prototiles  $p_1$  and  $p_2$ ,  $\Phi^n(p_1)$  contains a copy of  $p_2$ . The *substitution matrix* or *incidence matrix*  $\mathbf{M}$  is defined such that the  $\mathbf{M}^{ij\text{th}}$  entry gives the number of  $i$ -tiles in a substituted  $j$ -tile.  $\mathbf{M}$  is primitive if there exists an  $n \in \mathbb{N}^+$  such that  $\mathbf{M}^n$  contains strictly positive entries.

**Remark 2.0.3.**  $(\Omega_\Phi, d)$  is a connected, compact topological space.

To show compactness take a sequence of tilings  $T_n$  and a fixed  $R > 0$ , and consider  $B_R[T_n]$ . By finite local complexity, there are only a finite number of patches we will see, and thus there exist infinitely many tilings in  $T_n$  that agree, up to translation, on  $B_R[T_n]$ . From this set we can find a convergent subsequence of tilings, showing that  $(\Omega_\Phi, d)$  is sequentially compact, and thus compact. The substitution  $\Phi$  has

*finite local complexity* if every tiling  $T \in \Omega_\Phi$  has finite local complexity. A tiling space is *nonperiodic* if it contains no tilings periodic under translation. The inflation and substitution map of a nonperiodic substitution with finite local complexity is a homeomorphism.

**Theorem 2.0.4** ([Mos], [Sol]). *If  $\Phi$  is a primitive substitution and  $\Omega_\Phi$  contains at least one nonperiodic tiling, then every tiling in  $\Omega_\Phi$  is nonperiodic and  $\Phi : \Omega_\Phi \rightarrow \Omega_\Phi$  is a homeomorphism.*

It will be useful to talk about the largest circular disc seen in  $\mathcal{P}$ . Define  $\Delta := \sup\{r \in \mathbb{R} : \text{there exist a } p \in \mathcal{P} \text{ and a } \mathbf{y} \in \text{spt}(p) \text{ such that } B_r(\mathbf{y}) \subset \text{spt}(p)\}$ .

**Remark 2.0.5.** *Given a tile  $t$  in  $T \in \Omega_\Phi$ , if  $\mathbf{y} \in \mathbb{R}^2$  is such that  $|\mathbf{y}| < \Delta$  then  $t + \mathbf{y}$  is not a tile in  $T$ .*

### Tiling Dynamics

From now on assume  $\Phi$  is a primitive substitution and  $\Omega_\Phi$  the corresponding nonperiodic tiling space. Define the  $\mathbb{R}^2$  action  $\Gamma : \mathbb{R}^2 \times \Omega_\Phi \rightarrow \Omega_\Phi$ , given by  $\mathbf{x} \cdot T := T - \mathbf{x}$ . The resulting system  $(\Omega_\Phi, \Gamma)$  is called a *tiling dynamical system*.

#### Minimal Sets

A tiling  $T$  is called *repetitive* if for any patch  $P$  in  $T$  there is an  $r > 0$  so that a translation of  $P$  occurs in every  $r$ -ball in  $T$ . In topological dynamics such a point is called *almost periodic*. Denote the orbit of a tiling by  $\mathcal{O}(T) = \{T - \mathbf{x} : \mathbf{x} \in \mathbb{R}^2\}$ . The topological dynamical system  $(\Omega_\Phi, \Gamma)$  is minimal if there are no proper closed  $\Gamma$ -invariant subspaces of  $\Omega$ . The basic correspondence between almost periodic points and minimal sets is this: If  $T \in \Omega$  then  $T$  is almost periodic if and only if the orbit closure  $\overline{\mathcal{O}(T)}$  of  $T$  is minimal [Gott].

**Theorem 2.0.6.** *If  $\Omega$  is a tiling substitution space corresponding to a primitive tiling substitution  $\Phi$ , then any  $T \in \Omega$  is repetitive. Moreover, every tiling has a dense orbit.*

**Corollary 2.0.7.** *Substitution tiling dynamical systems are minimal and nonperiodic.*

### Topological Conjugacy

In topological dynamics the basic idea of equivalence is topological conjugacy. A *factor map* between tiling spaces is a map  $Q$  that commutes with the action of translation:  $Q(T - \mathbf{x}) = Q(T) - \mathbf{x}$ . A *topological conjugacy* between tiling spaces is a homeomorphism that is also a factor map.

A factor map is *local* if  $Q(T)$  depends only on  $B_1[T]$ , in which case we say  $Q(T)$  is locally derivable from  $T$ . Tiling spaces are *Mutually Locally Derivable* (MLD) (see [BSJ]) if there exists a topological conjugacy defined locally. That is,  $\exists R$  such that if  $T$  and  $T'$  agree on  $B_R(0)$ , then  $Q(T)$  and  $Q(T')$  agree on  $B_1(0)$ . A topological conjugacy preserves the dynamics and thus, two tiling spaces that are MLD share the same local dynamics [Sad].

In one-dimensional tiling spaces the underlying topology determines the dynamical system it supports. However, there exist two-dimensional topological spaces that are homeomorphic but the dynamics they support are not the conjugate. For example, consider the  $n$ -adic solenoids where  $n = 6$  and  $12$ . These spaces are homeomorphic, however the shift dynamics are not conjugate. In 1999, [Pet] and [RS], showed that topologically conjugate tiling spaces need not be MLD. In [BD1] it was proved this cannot happen in one dimensional tiling spaces and some results were extended to self similar tilings in higher dimensions in [JK]. Specifically, the existence of a homeomorphism of tiling spaces implies conjugacy of the translation actions up to a linear rescaling.

**Proposition 2.0.8.** (*Kwapisz*) Let  $\Gamma_{\mathbf{u}}$  and  $\tilde{\Gamma}_{\mathbf{u}}$  be the translation action on nonperiodic self similar substitutive tiling spaces  $\Omega_{\Phi}$  and  $\tilde{\Omega}_{\tilde{\Phi}}$ , respectively. If there exists a homeomorphism  $h_0 : \Omega_{\Phi} \rightarrow \tilde{\Omega}_{\tilde{\Phi}}$  such that  $h_0(T) = \tilde{T}$ , for tilings  $T, \tilde{T}$  fixed under  $\Phi$  and  $\tilde{\Phi}$ , then  $h_0$  is homotopic to some homeomorphism  $h_{\text{lin}} : \Omega_{\Phi} \rightarrow \tilde{\Omega}_{\tilde{\Phi}}$  such that  $h_{\text{lin}}(T) = \tilde{T}$ , there are  $n, m \in \mathbb{N}$  with  $h_{\text{lin}} \circ \Phi^n = \tilde{\Phi}^m \circ h_{\text{lin}}$ , and for some linear map  $L$ ,  $h_{\text{lin}}(T + \mathbf{x}) = h_{\text{lin}}(T) + L\mathbf{x}$ , for all  $\mathbf{x} \in \mathbb{R}^2$ .

**Remark 2.0.9.** The homeomorphism  $h_{\text{lin}} : \Omega_{\Phi} \rightarrow \tilde{\Omega}_{\tilde{\Phi}}$  preserves the local stable sets of  $\Phi$ .

This result is weaker than the one dimensional analog since the homeomorphism  $h$  has to be "pinned down" with the pointed homomorphism  $h_0(T) = \tilde{T}$ .

### Shift Equivalence and Homology

**Definition 2.0.10.** (*Shift Equivalence*) Two maps  $f : K \rightarrow K, g : Y \rightarrow Y$  are shift equivalent of lag  $m$ , provided there are maps  $r : K \rightarrow Y, r' : Y \rightarrow K$  and an integer  $m$  such that the following diagrams commute:

$$\begin{array}{ccc} K & \xrightarrow{f} & K \\ r \downarrow & & \downarrow r \\ Y & \xrightarrow{g} & Y \end{array} \quad \begin{array}{ccc} K & \xrightarrow{f} & K \\ \uparrow r' & & \uparrow r' \\ Y & \xrightarrow{g} & Y \end{array} \quad \begin{array}{ccc} K & \xrightarrow{f^m} & K \\ r \downarrow & \nearrow r' & \downarrow r \\ Y & \xrightarrow{g^m} & Y \end{array}$$

**Theorem 2.0.11.** (*Williams*) If  $f$  is shift equivalent to  $g$ , then the shift homomorphisms  $\hat{f}$  and  $\hat{g}$  defined on  $\varprojlim(K, f)$  and  $\varprojlim(Y, g)$ , respectively, are topologically conjugate.

**Remark 2.0.12.** Since a topological conjugacy induces an isomorphism in homology we have  $\check{H}^*(\varprojlim(K, f))$  is isomorphic to  $\check{H}^*(\varprojlim(Y, g))$ .

Properties of  $\Phi$

$\Phi$ -Periodic tilings

**Lemma 2.0.13.** *Let  $t$  be a tile containing the origin <sup>1</sup> such that  $t - \mathbf{x} \in \Phi(t)$  for some  $\mathbf{x} \in \mathbb{R}^2$ . Then there exists a vector  $\mathbf{y} \in \mathbb{R}^2$  such that  $t - \mathbf{y} \in \Phi(t - \mathbf{y})$ .*

**Proof:** Let  $\mathbf{y} = \frac{\mathbf{x}}{1-\lambda}$ . Then  $\Phi(t - \mathbf{y}) = \Phi(t) - \lambda\mathbf{y}$  which contains  $t - \mathbf{x} - \lambda\mathbf{y}$ , by assumption. Notice  $t - \mathbf{x} - \lambda\mathbf{y} = t - \mathbf{y}(1 - \lambda) - \lambda\mathbf{y} = t - \mathbf{y}$ . So  $t - \mathbf{y} \in \Phi(t - \mathbf{y})$  ■

**Proposition 2.0.14.** *Periodic points under  $\Phi$  are dense in  $\Omega_\Phi$ .*

**Proof:** Given  $\epsilon > 0$ , there is an  $R > 0$  such that, for all  $S$  and  $S' \in \Omega_\Phi$ , if  $S$  and  $S'$  have the same  $R$ -patch, then  $d(S, S') < \epsilon$ . Fix a tiling  $T$ , and let  $t$  be the tile in the  $R$ -patch of  $T$  that contains the origin in its interior. Since  $\Phi$  is primitive, the patch  $\Phi^N(t)$  contains a tile  $t - \mathbf{x}$ . Take  $N$  large enough to that  $|\mathbf{x}/(1 - \lambda^N)| < \epsilon$ . By Lemma 2.0.13, if  $\mathbf{y} = \mathbf{x}/(1 - \lambda^N)$ , the patch  $\Phi^N(t - \mathbf{y})$  contains  $t - \mathbf{y}$ . Since  $|\mathbf{y}| < \epsilon$  we have  $B_R[T] = B_R[T - \mathbf{y}]$ , so  $d(T, T - \mathbf{y}) < \epsilon$ . ■

**Proposition 2.0.15.** *For each  $n \in \mathbb{N}$ , there exist only a finite number of  $\Phi$ -periodic tilings that have period  $n$ .*

**Proof:** It suffices to show fixed points are isolated. Without loss of generality assume  $n = 1$  and suppose  $T$  is such that  $\Phi(T) = T$ . For all  $\epsilon > 0$ , if  $d(T, T') < \epsilon$ , then  $B_1[T] = B_1[T' - \mathbf{x}]$ , for some  $\mathbf{x}$  with  $|\mathbf{x}| < \epsilon$ . It follows that  $B_1[\Phi(T')] = B_1[\Phi(T - \mathbf{x})] = B_1[T - \lambda\mathbf{x}]$ . If  $T'$  is fixed then  $B_1[T - \mathbf{x}] = B_1[T - \lambda\mathbf{x}]$ . By remark 2.0.5,  $|(\lambda - 1)\mathbf{x}| > \Delta$ . Since we can make  $\epsilon$  as small as we like,  $T$  is isolated from other fixed points. ■

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<sup>1</sup>that is,  $0 \in \text{int}(\text{spt}(t))$

### Stable Manifolds of $\Phi$

A primary tool used in this dissertation is the relation between translation asymptoticity and the stable manifold under inflation and substitution. Our goal is to understand how the asymptotic structure of a tiling space contributes to its Čech cohomology. The key qualitative attribute of asymptotic pairs is their stable manifold under inflation and substitution. The local stable set for a tiling  $T$  consists of tilings that agree with  $T$  on a large ball about the origin, while the unstable set consists of tilings that are small translates of  $T$ . Inflation and substitution is hyperbolic on  $\Omega_\Phi$  [Kell]. Moreover, by proving  $\Phi$  to be topological mixing <sup>2</sup>, it was shown in [AP] that  $(\Omega_\Phi, d, \Phi)$  is a Smale space.

**Definition 2.0.16.**  $W^s(T) := \{T' \in \Omega_\Phi : d(\Phi^n(T), \Phi^n(T')) \rightarrow 0 \text{ as } n \rightarrow \infty\}$

**Definition 2.0.17.**  $W_\epsilon^s(T) = \{T' \in \Omega_\Phi : d(\Phi^n(T'), \Phi^n(T)) \leq \epsilon, \forall n \geq 0\}$

**Remark 2.0.18.** *If  $B_0[T] = B_0[T']$ , then  $T \in W^s(T')$*

The reasoning is that applying inflation and substitution to  $T$  and  $T'$ , results in agreement on an arbitrary large ball. A version of the converse is covered in

**Lemma 2.0.19.** *There exists an  $\epsilon > 0$  such that if  $T \in W_\epsilon^s(T')$ , there exists a  $N \geq 0$  such that  $B_0[\Phi^N(T)] = B_0[\Phi^N(T')]$ .*

**Proof:** Notice for some  $0 < \epsilon < 2^{-1/2}$ , if  $d(T, T') < \epsilon$  then there exists a vector  $\mathbf{x} \in \mathbb{R}^2$  such that  $|\mathbf{x}| < \epsilon$  and  $B_1[T' - \mathbf{x}] = B_1[T]$ . Choose  $0 < \delta < (2^{1/2}\lambda)^{-1}$ . Let  $\epsilon_1 = \inf\{d(T, T - \mathbf{x}) : B_1[T] = B_1[T'] \text{ and } \delta < |\mathbf{x}| \leq \lambda\delta\}$ . Notice  $\epsilon_1 > 0$ , since if  $\epsilon_1 = 0$  there exists a sequence of vectors  $\mathbf{x}_n$ , with  $|\mathbf{x}_n| > \delta$ , such that  $\lim_n d(T, T' - \mathbf{x}_n) = 0$ . Since  $d$  is continuous on the compact set  $\Omega_\Phi \times \Omega_\Phi$  we have that some subset of

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<sup>2</sup>[Sol] proved the dynamical system arising from a self-affine tiling is never strongly mixing.

$(T, T' - \mathbf{x}_n)$  converges to some  $(T, T' - \mathbf{x}_0)$  such that  $d(T, T' - \mathbf{x}_0) = 0$ ,  $\mathbf{x}_0 \neq 0$ . This implies  $T = T' - \mathbf{x}_0$  and so  $B_1[T'] = B_1[T] = B_1[T' - \mathbf{x}_0]$ , which cannot happen.

Let  $\epsilon = \min\{\epsilon_1, \delta\}$ . Now if  $d(T, T') < \epsilon$  there exists an  $\mathbf{x}$  with  $B_1[T' - \mathbf{x}] = B_1[T]$ , with  $|\mathbf{x}| < \delta$ . Now suppose that  $B_1[T'] \neq B_1[T]$ , so that  $\mathbf{x} \neq 0$ . Then there exists a  $k \in \mathbb{N}^+$  such that  $\delta \leq |\lambda^k \mathbf{x}| \leq \lambda \delta$ . Then

$$B_1[\Phi^k(T' - \mathbf{x})] = B_1[\Phi^k(T)]$$

so

$$B_1[\Phi^k(T') - \lambda^k \mathbf{x}] = B_1[\Phi^k(T)]$$

but then  $d(\Phi^k(T'), \Phi^k(T)) \geq \epsilon$ , a contradiction. ■

**Corollary 2.0.20.** *If  $T \in W^s(T')$  and  $r \geq 0$  then there exist  $N \geq 0$  such that  $B_r[\Phi^n(T)] = B_r[\Phi^n(T')]$ , for all  $n \geq N$ .*

**Lemma 2.0.21.** *If  $T \in W^s(T')$  then for all  $\epsilon > 0$  there exists  $N \in \mathbb{N}^+$ ,  $R \in \mathbb{R}^+$  so that  $\Phi^N(T - \mathbf{x}) \in W_\epsilon^s(\Phi^N(T' - \mathbf{x}))$  for all  $\mathbf{x} \in \mathbb{R}^2$  such that  $|\mathbf{x}| < R$ .*

**Proof:** By the previous corollary there exists a  $N > 0$  such that  $B_0[\Phi^N(T)] = B_0[\Phi^N(T')]$ . Define  $R = \sup\{R^* \in \mathbb{R} : B_{R^*}(0) \subset \text{spt}((B_0[\Phi^N(T)]))\}$ . Then for any vector  $\mathbf{x} \in \mathbb{R}^2$  with  $|\mathbf{x}| < R$  we have  $B_0[\Phi^N(T - \mathbf{x})] = B_0[\Phi^N(T' - \mathbf{x})]$ . Choose  $r = r(\epsilon) > 0$  so that  $B_{2r}[\Phi^N(T - \mathbf{x})] = B_{2r}[\Phi^N(T' - \mathbf{x})]$  implies  $d(\Phi^{k+N}(T - \mathbf{x}), \Phi^{k+N}(T' - \mathbf{x})) < \epsilon$ , for all  $k > 0$ . The claim follows. ■

We adapt a general result of 'slippage' in tiling spaces found in [Sol].

**Definition 2.0.22.** *We say that two patches  $P$  and  $P'$  are stably related on overlap if for all  $\mathbf{y} \in \text{int}(\text{spt}(P)) \cap \text{int}(\text{spt}(P'))$  and for all  $R > 0$  there exists  $n = n(\mathbf{y}, R) \in \mathbb{N}^+$  such that  $B_R[\Phi^n(P - \mathbf{y})] = B_R[\Phi^n(P' - \mathbf{y})]$ .*

**Lemma 2.0.23.** (*[Sol]*) *Let  $U$  be a nonperiodic self-similar tiling. There exists an  $N \geq 1$  such that for all  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ , if  $P$  and  $P - \mathbf{x}$  are patches in  $U$  and  $B_r(\mathbf{y}) \subset \text{spt}(P)$  then  $|\mathbf{x}| \leq r/N$  implies  $\mathbf{x} = 0$ .*

**Proposition 2.0.24.** *Given any prototile  $p$ , there exists an  $\epsilon > 0$  such that for any  $\mathbf{x} \in \mathbb{R}^2$  with  $0 < |\mathbf{x}| < \epsilon$ ,  $p$  and  $p - \mathbf{x}$  are not stably related on overlap.*

**Proof:** Suppose otherwise. That is, for all  $\epsilon > 0$ , there exists an  $\mathbf{x}$ ,  $0 < |\mathbf{x}| < \epsilon$ , so that  $p$  and  $p - \mathbf{x}$  are stably related on overlap. Let  $N$  be the constant given by the Lemma 2.0.20. Fix  $\mathbf{y} \in \text{int}(\text{spt}(p))$ . There exists an  $\epsilon_1 > 0$  and an  $r > 0$  such that if  $|\mathbf{x}_1| < \epsilon_1$ , then  $\overline{B_r(\mathbf{y})} \subset \text{int}(\text{spt}(p)) \cap \text{int}(\text{spt}(p - \mathbf{x}_1))$  and  $\overline{B_r(\mathbf{y})} - \mathbf{x}_1 \subset \text{int}(\text{spt}(p))$ .

Define  $\epsilon = \min\{\epsilon_1, \frac{r}{N}\}$  and let  $0 < |\mathbf{x}| < \epsilon$  be so that  $p$  and  $p - \mathbf{x}$  are stably related on overlap. For the compact ball  $\overline{B_r(\mathbf{y})}$  there exists an  $M > 0$  so that for all  $\mathbf{z} \in \overline{B_r(\mathbf{y})}$  we have  $B_0[\Phi^M(p - \mathbf{y} - \mathbf{z})] = B_0[\Phi^M(p - \mathbf{y} - \mathbf{z} - \mathbf{x})]$ . Let  $P$  denote the patch  $B_{\lambda^M r}[\Phi^M(p) - \lambda^M \mathbf{y}]$ . Then  $P$  and  $P - \lambda^M \mathbf{x}$  are found in the patch  $\Phi^M(p) - \lambda^M \mathbf{y}$ . Let  $U$  be any tiling containing  $\Phi^M(p) - \lambda^M \mathbf{y}$ . Since  $|\mathbf{x}| < \epsilon$  we have

$$\lambda^M |\mathbf{x}| < \frac{\lambda^M r}{N}$$

Let  $r_1 = \lambda^M r$ . Then  $B_{r_1}(\lambda^M \mathbf{y}) \subset \text{int}(\text{spt}(P))$ . By the previous lemma  $\lambda^M \mathbf{x} = 0$ . This forms a contradiction which proves the proposition. ■

**Corollary 2.0.25.** *Given patches  $P$  and  $P'$  with  $\text{int}(\text{spt}(P)) \cap \text{int}(\text{spt}(P')) \neq \emptyset$ , there does not exist a sequence of vectors  $\mathbf{x}_i \in \mathbb{R}^2$ ,  $\mathbf{x}_i \neq 0$ , with  $|\mathbf{x}_i| \rightarrow 0$  so that  $P$  and  $P' - \mathbf{x}_i$  are stably related on overlap.*

**Proof:** Assume otherwise. Let  $\mathbf{y} \in \text{int}(\text{spt}(P)) \cap \text{int}(\text{spt}(P'))$ . There exists an  $r > 0$  so that  $B_r(\mathbf{y}) \subset \text{int}(\text{spt}(P))$ . Let  $n$  be large enough so that  $\lambda^n r > 2\Delta$  and there exists tiles  $t, t' - \mathbf{x}_i$  in  $\Phi^n(P) \cap \Phi^n(P')$  such that  $\lambda^n \mathbf{y} \in \text{int}(\text{spt}(t)) \cap \text{int}(\text{spt}(t' - \mathbf{x}_i))$ . Notice the pairs  $t, t' - \mathbf{x}_i$  and  $t, t' - \mathbf{x}_{i+1}$  are each stably related on overlap. Thus,  $t'$  and  $t' - (\mathbf{x}_i + \mathbf{x}_{i+1})$  are stably related on overlap. Also, both are contained in  $\Phi^N(P)$ . As  $i \rightarrow \infty$  we have  $(\mathbf{x}_i + \mathbf{x}_{i+1}) \rightarrow 0$ , contradicting the previous theorem.  $\blacksquare$

**Proposition 2.0.26.** *If  $T - \mathbf{x} \in W^s(T' - \mathbf{x})$  for all  $\mathbf{x} \in \mathbb{R}^2$ , then  $T = T'$ .*

**Proof:** We will show that given  $\epsilon > 0$  there exists  $N$  such that  $\Phi^N(T - \mathbf{x}) \in W_\epsilon^s(\Phi^N(T' - \mathbf{x}))$  for all  $\mathbf{x} \in \mathbb{R}^2$ .

Case 1: There exists finite a collection of patches  $\{P_1, \dots, P_n\}$  such that for all  $\mathbf{y}$  there exists  $\mathbf{s} = \mathbf{s}(\mathbf{y})$  such that  $(B_1[T - \mathbf{y}], B_1[T' - \mathbf{y}]) = (P_i, P_j) - \mathbf{s}$  for some  $i = i(\mathbf{y}), j = j(\mathbf{y})$ . Let  $\epsilon > 0$  be given. Given a tiling  $S$ , let  $cB_1[S] := \{t \in S : \text{spt}(t) \cap \text{spt}(t') \neq \emptyset \text{ for some } t' \in B_1[S]\}$ . For each  $\mathbf{y}, i, j$ , and  $\mathbf{s}$  let  $cP_i, cP_j$  denote the patches  $cP_i := cB_1[T - \mathbf{y}] + \mathbf{s}, cP_j := cB_1[T' - \mathbf{y}] + \mathbf{s}$  so that  $cP_i$  and  $cP_j$  are collarings of  $P_i$  and  $P_j$ . (Note that  $cP_i$  depends on  $\mathbf{y}, \mathbf{s}, j$ , as well as  $i$  but there are still only finitely many pairs  $(cP_i, cP_j)$ ). Since  $cP_i$  and  $cP_j$  are stably related on overlap, given  $R$  and  $\mathbf{w} \in \text{spt}(P_i) \cap \text{spt}(P_j)$ , there exists  $N = N(\mathbf{w}, R)$  so that  $B_{2R}[\Phi^n(cP_i - \mathbf{w})] = B_{2R}[\Phi^n(cP_j - \mathbf{w})]$  for all  $n \geq N$ . There is then  $\delta_{\mathbf{w}} > 0$  such that  $B_R[\Phi^n(cP_i - \mathbf{w}')] = B_R[\Phi^n(cP_j - \mathbf{w}')] for all  $\mathbf{w}' \in B_{\delta_{\mathbf{w}}}(\mathbf{w})$  and all  $n \geq N(\mathbf{w})$ . Now  $\text{spt}(P_i) \cap \text{spt}(P_j)$  is compact so is covered by finitely many of these balls, say  $\text{spt}P_i \cap \text{spt}P_j \subset \bigcup_{k=1}^K B_{\delta_{\mathbf{s}_k}}(\mathbf{s}_k)$ . Let  $N = N(i, j) = \max\{N(\mathbf{s}_k, R) : k = 1, \dots, K\}$ . Let  $R$  be large enough that  $B_R[T] = B_R[T']$  implies that  $d(T, T') < \epsilon$ . Now, since there are only finitely many pairs  $(cP_i, cP_j)$  there is a  $M$  such that  $d(\Phi^n(T - \mathbf{y}), \Phi^n(T' - \mathbf{y})) < \epsilon$  for all  $n \geq M$ , for all  $\mathbf{y} \in \mathbb{R}^2$ . That is,  $\Phi^M(T - \mathbf{y}) \in W_\epsilon^s(\Phi^M(T' - \mathbf{y}))$  for all  $\mathbf{y}$ .$

Case 2: Assume case 1 to be false. Then, since there is only a finite number of  $B_1[*]$  patches up to translations, there must be  $\mathbf{y}_n \in \mathbb{R}^2$ , two patches  $P, P'$ , and vectors  $\mathbf{s}_n \neq 0$ ,  $|\mathbf{s}_n| \rightarrow 0$ , with  $(B_1[T - \mathbf{y}_n], B_1[T' - \mathbf{y}_n]) = (P, P' - \mathbf{s}_n)$ . Then  $\{P, P' - \mathbf{s}_n\}$  are stably related on the overlap. This contradicts Lemma 2.0.25, so we need only consider case 1.

Therefore there exist a  $N$ , independent of  $\mathbf{y}$ , such that  $\Phi^N(T - \mathbf{y}) \in W_\epsilon^s(\Phi^N(T' - \mathbf{y}))$ , for all  $\mathbf{y}$ . Using Lemma 2.0.18 there exists  $k > 0$ , independent of  $\mathbf{y}$ , such that  $B_0[\Phi^{N+k}(T - \mathbf{y})] = B_0[\Phi^{N+k}(T' - \mathbf{y})]$ , for all  $\mathbf{y}$ . It follows that  $\Phi^{N+k}(T) = \Phi^{N+k}(T')$ . Since  $\Phi$  is a homeomorphism,  $T = T'$ . ■

### The Anderson-Putnam Complex

[AP] defined an inverse limit construction of the tiling space which provided a computational approach to calculating cohomology. In the paper, a compact Hausdorff topological space  $K$  was constructed by gluing together prototiles in all ways in which the substitution rule allow them to be adjacent. That is, let  $T$  be any tiling in  $\Omega_\Phi$ . Define  $\sim$  on  $\mathbb{R}^2$  to be the smallest equivalence relation such that  $\mathbf{x} \sim \mathbf{y}$  if there exists a tile  $t \in T$  so that  $t - (\mathbf{x} - \mathbf{y}) \in T$ .

**Theorem 2.0.27.** *Inflation and substitution  $\Phi$  induces a continuous surjection  $F : K \rightarrow K$  defined by  $F([\mathbf{x}]_\sim) = [\lambda\mathbf{x}]_\sim$ ,  $x \in \mathbb{R}^2$*

Later on, it will be useful to discuss the zero and one skeleton of  $K$ . Let  $q$  denote the quotient map  $q : \mathbb{R}^2 \rightarrow \mathbb{R}^2 / \sim$ . Define  $K^1 = \{q(x) \in K : x \in \partial t, \text{ for some } t \in T\}$  and  $K^0 = \{q(x) \in K : x \text{ lies on a vertex of some tile } t \in T\}$ .

[Kell] introduced the concept of a substitution *forcing its border*. Basically,  $\Phi$  forces the border if there exists a positive integer  $N$  such that for any tile  $t$  and any

two tilings  $T_1, T_2$  containing  $t$ ,  $\Phi^N(T_1)$  and  $\Phi^N(T_2)$  coincide, not just on  $\Phi^N(t)$ , but on all tiles that meet  $\Phi^N(t)$ . Define  $\Xi = \lim_{\leftarrow} K$  be the inverse limit of  $K$  with bonding map  $F$ .

**Theorem 2.0.28.** (*[AP]*) *If  $\Phi$  forces its border  $\Xi$  is homeomorphic to  $\Omega_\Phi$ .*

## VISUALIZING ASYMPTOTIC STRUCTURE

Tilings spaces are inhomogeneous. To understand these inhomogeneities as embodied by the asymptotic tilings <sup>1</sup> we will investigate the relationship between the stable manifold of inflation and substitution and asymptotic pairs. As noted in [BD], there is no actual branching in a tiling space. Every point has a neighborhood that is homeomorphic with the product of a disk and a cantor set. However, an inverse limit description of the tiling spaces gives a sequence of approximating branched manifolds such that, in the limit, the preimage of the branches can be observed in the existence of asymptotic arc components. The Anderson and Putnam collaring technique allows us to construct a complex whose inverse limit is homeomorphic to the tiling space and thus has the same Čech cohomology. However, in doing so all evidence of the asymptotic structure is obscured.

If the substitution does not force the border, it was shown in [BDHS] how to visualize elements of cohomology of the tiling space using an inflated skeleton of the Anderson-Putnam complex. It is possible to identify asymptotic behavior with this technique. However, there may be asymptotic arc components not given by the combinatorics of the substitution that are missed.

Our goal in this chapter is to link a closed subset of the tiling space with asymptotic behavior. This allows us to construct an inverse limit representation of the asymptotic structure found in the tiling space.

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<sup>1</sup>Regional proximality is another type of inhomogeneity found in tiling spaces. See [Aus].

The Branch Locus

**Definition 3.0.29.** Let  $T$  and  $T'$  be tilings in  $\Omega_\Phi$ , with  $T \neq T'$ . We say  $(T, T')$  is an asymptotic pair if  $T$  and  $T'$  are periodic under  $\Phi$  and are asymptotic (with respect to translation) in at least a half plane. That is, there exist a vector  $\mathbf{v} \neq 0$  so that  $d(T - u\mathbf{y}, T' - u\mathbf{y}) \rightarrow 0$  as  $u \rightarrow \infty$ , for all  $\mathbf{y}$  such that  $\langle \mathbf{y}, \mathbf{v} \rangle > 0$ . Let  $\mathcal{AP}$  be the set of asymptotic pairs in  $\Omega_\Phi$ .

For each asymptotic pair  $(T, T')$  there is an open arc  $\alpha(T, T')$  on the unit circle with the properties:

1.  $\mathbf{v}/|\mathbf{v}| \in \alpha(T, T')$
2.  $d(T - s\mathbf{y}, T' - s\mathbf{y}) \rightarrow 0$  as  $s \rightarrow \infty$ , for all  $\mathbf{y} \in \alpha(T, T')$ .
3. For  $\mathbf{y}_0 \in \partial\alpha(T, T')$ ,  $d(T - s\mathbf{y}_0, T' - s\mathbf{y}_0) \not\rightarrow 0$  as  $s \rightarrow \infty$

For the asymptotic pair  $(T, T')$  we define the sector of asymptoticity to be

$$S_{(T, T')} := \left\{ \mathbf{x} : \frac{\mathbf{x}}{|\mathbf{x}|} \in \alpha(T, T') \right\}$$

**Proposition 3.0.30.** If  $(T, T') \in \mathcal{AP}$  then  $T \notin W^s(T')$ .

**Proof:** Since  $T$  and  $T'$  are  $\Phi$ -periodic, there exist  $k \in \mathbb{N}$ , such that  $\Phi^k(T) = T$  and  $\Phi^k(T') = T'$ . Then  $d(\Phi^{nk}(T), \Phi^{nk}(T')) = d(T, T') \not\rightarrow 0$  as  $n \rightarrow \infty$ , so  $T \notin W^s(T')$ .

■

Existence of Asymptotic Pairs

To show the existence of asymptotic pairs we first show (Proposition 3.0.36) there exist tilings that are in different stable manifolds, yet agree on a half plane. The proof requires the following lemmas.

**Lemma 3.0.31.** *Suppose  $B_0[T_1] = B_0[T_2]$ ,  $B_0[T'_1] = B_0[T'_2]$  then  $T_1 \in W^s(T'_1)$  if and only if  $T_2 \in W^s(T'_2)$ .*

**Lemma 3.0.32.** *Suppose  $T_n \rightarrow T$  and there exists a tile  $t$  such that  $t \in T_n$ , for all  $n$ . Then for all  $r > 0$  there exists an  $N(r)$  so that  $B_r[T_n] = B_r[T]$ , for all  $n \geq N(r)$ .*

**Proof:** Without loss of generality assume  $r$  is such that  $\text{spt}(t) \subset B_r(0)$ . There exist a sequence  $\mathbf{x}_n$ , with  $|\mathbf{x}_n| \rightarrow 0$ , such that  $B_r[T_n - \mathbf{x}_n] = B_r[T]$ . Let  $N$  be large enough so that  $\text{spt}(t - \mathbf{x}_n) \subset B_r(0)$  and  $|\mathbf{x}_n| \leq \Delta/2$  for all  $n \geq N$ . We have  $t - \mathbf{x}_n, t - \mathbf{x}_n \in T$  for all  $n \geq N$ . That is,  $t, t - (\mathbf{x}_n - \mathbf{x}_N) \in T + \mathbf{x}_N$ , for all  $n \geq N$ . Since  $|\mathbf{x} - \mathbf{x}_N| < \Delta$ ,  $\text{int}(\text{spt}(t)) \cap \text{int}(\text{spt}(t - (\mathbf{x}_n - \mathbf{x}_N))) \neq \emptyset$ . Thus  $t = t - (\mathbf{x}_n - \mathbf{x}_N)$ . So,  $\mathbf{x}_n = \mathbf{x}_N$ , for all  $n \geq N$ . Since  $|\mathbf{x}_n| \rightarrow 0$ , it must be the case that  $\mathbf{x}_n = 0$ , for all  $n \geq N$ . Thus,  $B_r[T_n] = B_r[T]$ , for all  $n \geq N$ . ■

**Lemma 3.0.33.** *If  $(t, t')$  is a pair of tiles,  $t \in T$ ,  $t' \in T'$ , with  $T - \mathbf{y} \in W^s(T' - \mathbf{y})$ , for all  $\mathbf{y} \in \text{spt}(t) \cap \text{spt}(t')$ , then there exist an  $N > 0$  such that  $B_0[\Phi^N(T - \mathbf{y})] = B_0[\Phi^N(T' - \mathbf{y})]$  for all  $\mathbf{y} \in \text{spt}(t) \cap \text{spt}(t')$ .*

**Proof:** By Corollary 2.0.20, for each  $\mathbf{y} \in \text{spt}(t) \cap \text{spt}(t')$ , there exists  $N(\mathbf{y})$  so that  $B_1[\Phi^n(T - \mathbf{y})] = B_1[\Phi^n(T' - \mathbf{y})]$ , for all  $n \geq N(\mathbf{y})$ . There is then an  $\epsilon(\mathbf{y}) > 0$  so that

$$B_0[\Phi^n(T - \mathbf{y} - \mathbf{s})] = B_0[\Phi^n(T' - \mathbf{y} - \mathbf{s})]$$

for all  $\mathbf{s} \in B_{\epsilon(\mathbf{y})}$  and all  $n \geq N(\mathbf{y})$ . Since  $\text{spt}(t) \cap \text{spt}(t')$  is a closed bounded subset of  $\mathbb{R}^2$ , it is compact. So there are vectors  $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k \in \text{spt}(t) \cap \text{spt}(t')$  so that  $\bigcup_{i=1}^k B_{\epsilon(\mathbf{y}_i)}(\mathbf{y}_i)$  forms a cover of  $\text{spt}(t) \cap \text{spt}(t')$ . Setting  $N = \max\{N(\mathbf{y}_i)\}_{i=1}^k$ , gives the desired result. ■

**Lemma 3.0.34.** *There is a finite collection of pairs  $(p_i, p_j + \mathbf{v}_{ij})$ ,  $p_i, p_j \in \mathcal{P}$ , so that if there exist tiles  $t$  and  $t'$  with  $\text{int}(\text{spt}(t)) \cap \text{int}(\text{spt}(t')) \neq \emptyset$ , that are stably related on overlap then  $(t, t') = (p_i, p_j + \mathbf{v}_{ij}) + \mathbf{w}$  for some  $i, j$  and  $\mathbf{w} \in \mathbb{R}^2$ .*

**Proof** Assume otherwise. Then there exist a sequence  $(p_{i_n}, p_{j_n} + \mathbf{v}_n)$  with  $p_{i_n}, p_{j_n} + \mathbf{v}_n$  stably related on overlap and with  $(p_{i_n}, p_{j_n} + \mathbf{v}_n) \neq (p_{i_k}, p_{j_k} + \mathbf{v}_k)$ , for any  $n \neq k$ . Since there are only a finite number of prototiles, without loss of generality, set  $p_{i_n} = p$  and  $p_{j_n} = p'$ , for all  $n$ . Then  $\text{spt}(p) \cap \text{spt}(p' + \mathbf{v}_n) \neq \emptyset$ ,  $p$  and  $p' + \mathbf{v}_n$  are stably related on overlap, and  $(p, p' + \mathbf{v}_n) \neq (p, p' + \mathbf{v}_k)$ , for  $n \neq k$ . The  $\mathbf{v}_n$ 's are bounded so there exist a convergent subsequence  $\mathbf{v}_{n_l} \rightarrow \mathbf{v}$ , so  $|\mathbf{v} - \mathbf{v}_{n_l}| \rightarrow 0$  as  $n_l \rightarrow \infty$ . Being stably related on overlap is translation invariant, so  $p - \mathbf{v}$  and  $p' - (\mathbf{v} - \mathbf{v}_n)$  are stably related on overlap. This forms a contradiction with the slippage result in Corollary 2.0.25, proving the lemma. ■

**Lemma 3.0.35.** *There exists an  $N > 0$  such that if  $(t, t')$  is a pair of tiles,  $t \in T$ ,  $t' \in T'$ , with  $\text{spt}(t) \cap \text{spt}(t') \neq \emptyset$  and  $t, t'$  stably related on overlap, then for all  $\mathbf{y} \in \text{spt}(t) \cap \text{spt}(t')$*

$$B_0[\Phi^N(T - \mathbf{y})] = B_0[\Phi^N(T' - \mathbf{y})]$$

**Proof:** It follows from Lemma 3.0.33 that for each pair  $(t, t')$  there exist a  $N$  such that  $B_0[\Phi^N(T - \mathbf{y})] = B_0[\Phi^N(T' - \mathbf{y})]$  for  $\mathbf{y} \in \text{spt}(t) \cap \text{spt}(t')$ . By Lemma 3.0.34 we have, up to translation, only finitely many such pairs. ■

**Proposition 3.0.36.** *(Existence of tilings that are asymptotic in a half plane) Given a substitution tiling space  $\Omega_\Phi$ , there exist tilings  $T \neq T' \in \Omega_\Phi$ ,  $T \neq T'$ , and a vector  $\mathbf{v} \neq 0$ , such that  $B_0[T - \mathbf{y}] = B_0[T' - \mathbf{y}]$ , for all  $\mathbf{y}$  with  $\langle \mathbf{y}, \mathbf{v} \rangle > 0$ .*

**Proof:** Let  $T_0, T'_0$  be tilings that share a tile  $t_0$  such that  $0 \in \text{int}(\text{spt}(t_0))$ . By Remark 2.0.18,  $T_0 \in W^s(T'_0)$ . Let  $R = \sup\{r > 0 : T_0 - \mathbf{y} \in W^s(T'_0 - \mathbf{y}), \text{ for all } \mathbf{y}$

with  $|\mathbf{y}| < r$ . By Lemma 2.0.21 the set  $\{\mathbf{y} \in \mathbb{R}^2 : T_0 - \mathbf{y} \in W^s(T'_0 - \mathbf{y})\}$  is open. Thus, there is a  $\bar{\mathbf{x}} \in \mathbb{R}^2$ , with  $|\bar{\mathbf{x}}| = R$  such that  $T_0 - \bar{\mathbf{x}} \notin W^s(T'_0 - \bar{\mathbf{x}})$ . Let  $T = T_0 - \bar{\mathbf{x}}$  and  $T' = T'_0 - \bar{\mathbf{x}}$ .

For any  $n$  such that  $\lambda^n R \geq \Delta$ , consider the ball  $B_{\lambda^n R}(\lambda^n \bar{\mathbf{x}})$ . By Lemma 3.0.34, there are up to translation only a finite number of pairs  $(t, t')$ , with  $t \in \Phi^n(T)$ ,  $t' \in \Phi^n(T')$ , and  $\mathbf{y} \in \text{spt}(t) \cap \text{spt}(t') \subset B_{\lambda^n R - \Delta}(\lambda^n \bar{\mathbf{x}})$ . So, by Lemma 3.0.33, there exist an  $N > 0$  such that  $B_0[\Phi^N(\Phi^n(T) - \mathbf{y})] = B_0[\Phi^N(\Phi^n(T') - \mathbf{y})]$ , for all  $\mathbf{y} \in B_{\lambda^n R - \Delta}(\lambda^n \bar{\mathbf{x}})$ . So,  $B_0[\Phi^{Nn}(T) - \lambda^N \mathbf{y}] = B_0[\Phi^{Nn}(T') - \lambda^N \mathbf{y}]$  for all  $\mathbf{y} \in B_{\lambda^n R - \Delta}(\lambda^n \bar{\mathbf{x}})$ . Which is to say  $B_0[\Phi^{Nn}(T) - \mathbf{y}] = B_0[\Phi^{Nn}(T') - \mathbf{y}]$  for all  $\mathbf{y} \in B_{\lambda^N(\lambda^n R - \Delta)}(\lambda^{Nn} \bar{\mathbf{x}})$ .

Let  $T_n = (\Phi^N)^n(T)$  and  $T'_n = (\Phi^N)^n(T')$ . Since  $T \notin W^s(T')$ , for some  $\delta > 0$  there exist a subsequence  $(T_{n_i}, T'_{n_i})$  such that  $d(T_{n_i}, T'_{n_i}) \geq \delta$ , for all  $n_i$ . From this choose a convergent subsequence  $T_{n_{i_j}} \rightarrow \bar{T}$  and  $T'_{n_{i_j}} \rightarrow \bar{T}'$ . Note  $d(\bar{T}, \bar{T}') \geq \delta$ , so  $T \neq T'$ .

Claim:  $B_0[\bar{T} - \mathbf{y}] = B_0[\bar{T}' - \mathbf{y}]$  for all  $\mathbf{y}$  with  $\langle \mathbf{y}, \bar{\mathbf{x}}/|\bar{\mathbf{x}}| \rangle > \lambda^N \Delta$ .

See that  $T_{n_{i_j}} \rightarrow \bar{T}$  implies  $T_{n_{i_j}} - \mathbf{y} \rightarrow \bar{T} - \mathbf{y}$ . Let  $\mathbf{y}$  be any vector such that  $\langle \mathbf{y}, \bar{\mathbf{x}}/|\bar{\mathbf{x}}| \rangle > \lambda^N \Delta$ . There exist  $I$  so that for all  $j > I$ ,  $\mathbf{y} \in B_{\lambda^{Nn_{i_j}} R - \lambda^N \Delta}(\lambda^{Nn_{i_j}} \bar{\mathbf{x}})$ . Then  $B_0[T_{n_{i_j}} - \mathbf{y}] = B_0[T'_{n_{i_j}} - \mathbf{y}]$  for all  $j > I$ . Thus  $B_0[\bar{T} - \mathbf{y}] = B_0[\bar{T}' - \mathbf{y}]$ , proving the claim.

Now  $\bar{T}$  and  $\bar{T}'$  are such that  $B_0[\bar{T} - \mathbf{y}] = B_0[\bar{T}' - \mathbf{y}]$  for all  $\mathbf{y} \in \{\mathbf{x} : \langle \mathbf{s}, \bar{\mathbf{x}}/|\bar{\mathbf{x}}| \rangle > \lambda^N \Delta\}$ . Let  $\mathbf{z} \in \partial\{\mathbf{x} : \langle \mathbf{s}, \bar{\mathbf{x}}/|\bar{\mathbf{x}}| \rangle > \lambda^N \Delta\}$ . Then  $\bar{T} - \mathbf{z}$ ,  $\bar{T}' - \mathbf{z}$  are the desired tilings. ■

**Proposition 3.0.37.** *Let  $(T, T')$  be an asymptotic pair with sector  $S_{(T, T')}$ . If  $\mathbf{x} \in S_{(T, T')}$  then  $T - \mathbf{x} \in W^s(T - \mathbf{x})$ .*

**Proof:** Since  $T$  and  $T'$  are periodic under  $\Phi$ , there exists some  $k \in \mathbb{N}$  such that  $\Phi^k(T) = T$  and  $\Phi^k(T') = T'$ . Assume  $d(T - u\mathbf{x}, T' - u\mathbf{x}) \rightarrow 0$  as  $u \rightarrow \infty$ . Then  $\Phi^k(T - \mathbf{x}) = \Phi^k(T) - \lambda^k \mathbf{x} = T - \lambda^k \mathbf{x}$  and  $\Phi^k(T' - \mathbf{x}) = \Phi^k(T') - \lambda^k \mathbf{x} = T' - \lambda^k \mathbf{x}$ . So  $d(\Phi^{kn}(T - \mathbf{x}), \Phi^{kn}(T' - \mathbf{x})) = d(T - \lambda^{kn} \mathbf{x}, T' - \lambda^{kn} \mathbf{x}) \rightarrow 0$  as  $n \rightarrow \infty$ . ■

**Proposition 3.0.38.** *If  $T, T' \in \Omega_\Phi$  and  $\mathbf{x} \in \mathbb{R}^2$  are such that  $d(T - u\mathbf{x}, T' - u\mathbf{x}) \rightarrow 0$  as  $u \rightarrow \infty$ , then  $\forall r \in \mathbb{R}, \exists R$  such that if  $u > R$  then  $B_r[T - u\mathbf{x}] = B_r[T' - u\mathbf{x}]$ .*

**Proof:** Since  $d(T - u\mathbf{x}, T' - u\mathbf{x}) \rightarrow 0$  as  $u \rightarrow \infty$  there are  $\mathbf{y}(n)$ , with  $|\mathbf{y}(n)| \rightarrow 0$ , so that  $B_r[T - u\mathbf{x}] = B_r[T' - u\mathbf{x} - \mathbf{y}(n)]$ . If there is no  $R > 0$  so that  $B_r[T - u\mathbf{x}] = B_r[T' - u\mathbf{x}]$  for all  $u > R$  then  $\mathbf{y}(n)$  is not eventually constant. Let  $\delta > 0$  be small enough so that if  $t$  is any tile, then  $\text{int}(\text{spt}(t)) \cap \text{int}(\text{spt}(t - \mathbf{y})) \neq \emptyset$  for any  $\mathbf{y}$  with  $|\mathbf{y}| < \delta$ . There is then  $U$  large enough so that  $|\mathbf{y}(n_2) - \mathbf{y}(n_1)| < \delta$  for any  $n_1, n_2$  in  $U$  and choose  $n_1 > n + 1$  so that  $\mathbf{y}$  is not constant on any neighborhood of  $n_1$ .

Now pick  $\epsilon > 0$  small enough so that if  $|\mathbf{w}| < \epsilon$  then there is a tile  $t \in B_r[T - n_1\mathbf{x}]$  with  $t - \mathbf{w} \in B_r[T - n_1\mathbf{x} - \mathbf{w}]$ . Pick  $n_2$  so that:  $|n_2 - n_1||\mathbf{x}| < \epsilon$ ,  $n_2 \in U$ , and  $\mathbf{y}(n_2) \neq \mathbf{y}(n_1)$ . Let  $t \in B_r[T - n_1\mathbf{x}]$  be such that  $t - (n_2 - n_1)\mathbf{x} \in B_r[T' - n_2\mathbf{x} + \mathbf{y}(n_1)]$ . Then  $t + n_1\mathbf{x} + \mathbf{y}(n_1) \in B_r[T']$  and  $t + n_1\mathbf{x} + \mathbf{y}(n_2) \in B_r[T']$ . But  $|\mathbf{y}(n_2) - \mathbf{y}(n_1)| < \delta$  means that  $\text{int}(t + n_1\mathbf{x} + \mathbf{y}(n_1)) \cap \text{int}(t + n_1\mathbf{x} + \mathbf{y}(n_2)) = \text{int}(t + n_1\mathbf{x} + \mathbf{y}(n_2)) \cap \text{int}(t + n_1\mathbf{x} + \mathbf{y}(n_2) - (\mathbf{y}(n_2) - \mathbf{y}(n_1))) \neq \emptyset$ . Thus  $t + n_1\mathbf{x} + \mathbf{y}(n_2) = t + n_1\mathbf{x} + \mathbf{y}(n_1)$ ; ie,  $\mathbf{y}(n_2) = \mathbf{y}(n_1)$ , contrary to the choice of  $n_2$ . ■

**Lemma 3.0.39.** *There exists a finite number of patches  $\{P_1, \dots, P_m\}$  such that if  $T$  and  $T'$  are any two tilings with some tile  $t \in T \cap T'$ , then there exists  $\mathbf{x} = \mathbf{x}(i, j)$  such that  $(B_1[T], B_1[T']) = (P_i, P_j) + \mathbf{x}$ .*

**Proof:** Let  $r$  be such that  $\text{spt}(t) \subset B_r(0)$ . By finite local complexity, up to translation there are only a finite number of  $B_{2r}[*]$  patches. From these consider the 1-patches,  $P_i = B_1[*]$ ,  $1 \leq i \leq m$ . Since  $\text{spt}(t) \subset B_{2r}(0)$  there exist patches  $P_i, P_j$  such that  $t \in (P_i - \mathbf{x}) \cap (P_j - \mathbf{x})$ . Thus for that  $i$  and  $j$  we have  $(B_1[T], B_1[T']) = (P_i, P_j) + \mathbf{x}$ . ■

**Proposition 3.0.40.** *There exists a  $R > 0$  such that if there exists an unit vector  $\mathbf{v}$  such that  $T - \mathbf{y} \in W^s(T' - \mathbf{y})$  for all  $\langle \mathbf{y}, \mathbf{v} \rangle > 0$ , then  $B_0[T - \mathbf{y}] = B_0[T' - \mathbf{y}]$  for all  $\mathbf{y}$  such that  $\langle \mathbf{y}, \mathbf{v} \rangle > R$ .*

**Proof:** By Lemma 3.0.35 there exists an  $N \in \mathbb{N}$  such that for any pair of tiles  $(t, t')$  with  $t \in T$ ,  $t' \in T'$ ,  $\text{spt}(t) \cap \text{spt}(t') \neq \emptyset$  and  $\text{spt}(t), \text{spt}(t') \subset \{\mathbf{y} : \langle \mathbf{y}, \mathbf{v} \rangle > \Delta\}$ , then  $B_0[\Phi^N(T - \mathbf{y})] = B_0[\Phi^N(T' - \mathbf{y})]$ , for all  $\mathbf{y} \in \text{spt}(t) \cap \text{spt}(t')$ .

Suppose  $\langle \mathbf{y}, \mathbf{v} \rangle > \lambda^N \Delta := R$ . Then  $\Phi^{-N}(T) - \frac{\mathbf{y}}{\lambda^N} \in W^s(\Phi^{-N}(T') - \frac{\mathbf{y}}{\lambda^N})$ . If  $t \in \Phi^{-N}(T)$ ,  $t' \in \Phi^{-N}(T')$  and  $\frac{\mathbf{y}}{\lambda^N} \in \text{spt}(t) \cap \text{spt}(t')$ , then  $B_0[\Phi^N(\Phi^{-N}(T) - \frac{\mathbf{y}}{\lambda^N})] = B_0[\Phi^N(\Phi^{-N}(T') - \frac{\mathbf{y}}{\lambda^N})]$ , which is equivalent to  $B_0[T - \mathbf{y}] = B_0[T' - \mathbf{y}]$  for all  $\mathbf{y}$  such that  $\langle \mathbf{y}, \mathbf{v} \rangle > 0$ .

**Corollary 3.0.41.** *There exists an  $R > 0$  such that if  $(T, T') \in \mathcal{AP}$  with sector  $S_{(T, T')}$ , and  $\mathbf{x} \in S_{(T, T')}$  with  $\mathbf{d}(\mathbf{x}, \mathbb{R}^2 \setminus S_{(T, T')}) \geq R$ , then  $B_0[T - \mathbf{x}] = B_0[T' - \mathbf{x}]$ .*

Given a nonperiodic substitution tiling space we have shown that there exist tilings that are asymptotic in at least a half plane. We have yet to show there exist  $\Phi$ -periodic tilings that are asymptotic in at least a half plane.

**Theorem 3.0.42.** *(Existence of asymptotic pairs) Given a primitive self-similar substitution  $\Phi$  with corresponding tiling space  $\Omega_\Phi$ , the set  $\mathcal{AP}$  is nonempty.*

**Proof:** By Proposition 3.0.36 there exist tilings  $T_0, T'_0$ , with  $T_0 \neq T'_0$ , and an unit vector  $\mathbf{v}$  such that  $B_0[T_0 - \mathbf{y}] = B_0[T'_0 - \mathbf{y}]$ , for all  $\mathbf{y}$  with  $\langle \mathbf{y}, \mathbf{v} \rangle > 0$ . There must exist a vector  $\bar{\mathbf{x}} \in \mathbb{R}^2$  such that  $T_0 - \bar{\mathbf{x}} \notin W^s(T'_0 - \bar{\mathbf{x}})$ , since if not we have  $T_0 - \mathbf{y} \in W^s(T'_0 - \mathbf{y})$  for all  $\mathbf{y} \in \mathbb{R}^2$ . By Proposition 2.0.26,  $T_0 = T'_0$ , contradicting our hypothesis. Let  $T = T_0 - \bar{\mathbf{x}}$  and  $T' = T'_0 - \bar{\mathbf{x}}$  and note  $B_0[T - \mathbf{y}] = B_0[T' - \mathbf{y}]$ , for all  $\mathbf{y}$  with  $\langle \mathbf{y}, \mathbf{v} \rangle > |\mathbf{x}| := R$ .

Let  $T_n = \Phi^{-n}(T)$  and  $T'_n = \Phi^{-n}(T')$  and note for each  $n$  the pair  $(T_n, T'_n)$  have the property:  $T_n - \mathbf{y} \in W^s(T'_n - \mathbf{y})$ , for all  $\mathbf{y}$  with  $\langle \mathbf{y}, \mathbf{v} \rangle > R/\lambda^n$ . Also,  $T_n \notin W^s(T'_n)$ , for

all  $n$ . By Lemma 3.0.39, there exists a finite number of patches  $P_1, \dots, P_m$  such that for each  $n$ , there exist an  $i = i(n)$ , a  $j = j(n)$ , and a  $\mathbf{x}_n$  such that  $(B_1[T_n], B_1[T'_n]) = (P_i, P'_j) - \mathbf{x}_n$ . There is then a subsequence  $n_i$  and patches  $P, P' \in \{P_1, \dots, P_m\}$ ,  $P \neq P'$ , and  $k \in \mathbb{N}$ , with

$$(B_1[T_{n_i}], B_1[T'_{n_i}]) = (P, P') - \mathbf{x}_{n_i},$$

and such that, for infinity many  $i$ ,  $n_{i+1} = n_i + k$ . Note that the condition  $n_{i+1} = n_i + k$  gives us

$$\begin{aligned} B_1[\Phi^k(T_{n_{i+1}})] &= B_1[\Phi^k(T_{n_i+k})] = B_1[T_{n_i}], \\ B_1[\Phi^k(T'_{n_{i+1}})] &= B_1[\Phi^k(T'_{n_i+k})] = B_1[T'_{n_i}]. \end{aligned} \tag{1}$$

Let  $A_n = \{\mathbf{y} : \langle \mathbf{y}, \mathbf{v} \rangle > (1/\lambda^n)R\}$ ,  $l_n = \partial A_n$ ,  $A = \bigcup_i (A_{n_i} + \mathbf{x}_{n_i})$ , and  $l = \partial A$ . Note  $A$  is not a nested union. If  $\mathbf{y} \in A \cap \text{int}(\text{spt}(P)) \cap \text{int}(\text{spt}(P'))$  then there exists an  $M$  such that  $B_0[\Phi^{Mk}(P - \mathbf{y})] = B_0[\Phi^{Mk}(P' - \mathbf{y})]$ . Choose a subsequence  $n_{i_j}$ , such that  $\mathbf{x}_{n_{i_j+1}} \rightarrow \mathbf{x}$ ,  $\mathbf{x}_{n_{i_j}} \rightarrow \mathbf{y}$  and  $n_{i_j+1} = n_{i_j} + k$  for infinitely many  $j$ . From the equations in (1) we have

$$\begin{aligned} B_1[\Phi^k(P - \mathbf{x})] &\supset B_1[P - \mathbf{y}], \\ B_1[\Phi^k(P' - \mathbf{x})] &\supset B_1[P' - \mathbf{y}]. \end{aligned}$$

So

$$(B_0[\Phi^k(P - \mathbf{x})], B_0[\Phi^k(P' - \mathbf{x})]) = (B_0[P - \mathbf{y}], B_0[P' - \mathbf{y}]).$$

Therefore  $0 \in \text{int}(\text{spt}(P - \mathbf{x})) \cap \text{int}(\text{spt}(P' - \mathbf{y}))$ . Now  $\mathbf{d}(\mathbf{x}_{n_i}, l) \leq \mathbf{d}(0, l_{n_i}) \rightarrow 0$  as  $n_i \rightarrow \infty$ . So  $\mathbf{x}, \mathbf{y} \in l$ . Thus  $\mathbf{x} - \mathbf{y}$  is parallel to  $l$ .

By Lemma 2.0.13 we can find a  $\mathbf{z} = (1 - s)\mathbf{x} + s\mathbf{y}$ , for some  $s \in [0, 1]$ , with

$$(B_0[\Phi^k(P - \mathbf{z})], B_0[\Phi^k(P' - \mathbf{z})]) = (B_0[P - \mathbf{z}], B_0[P' - \mathbf{z}]).$$

We see that  $0 \in \text{int}(\text{spt}(P - \mathbf{z})) \cap \text{int}(\text{spt}(P' - \mathbf{z}))$ .

Let  $\bar{T} = \bigcup_{n \in \mathbb{N}^+} \Phi^{kn}(P - \mathbf{z})$  and  $\bar{T}' = \bigcup_{n \in \mathbb{N}^+} \Phi^{kn}(P' - \mathbf{z})$ . Since  $0 \in \text{int}(\text{spt}(P - \mathbf{z}))$  and  $0 \in \text{int}(\text{spt}(P' - \mathbf{z}))$ ,  $\bar{T}$  and  $\bar{T}'$  are tilings of the plane.

Claim:  $(\bar{T}, \bar{T}')$  is an asymptotic pair.

Proof of Claim: By construction,  $\Phi^k(\bar{T}) = \bar{T}$  and  $\Phi^k(\bar{T}') = \bar{T}'$ . Let  $\mathbf{y} \in \{\mathbf{x} : \langle \mathbf{x}, \mathbf{v} \rangle > 0\}$ . There is an  $n > 0$  such that  $\lambda^{-kn}\mathbf{y} \in \text{int}(\text{spt}(B_1[\bar{T}])) \cap \text{int}(\text{spt}(B_1[\bar{T}']))$ . So  $B_1[\bar{T} - \lambda^{-kn}\mathbf{y}] = P - \mathbf{z} - \lambda^{-kn}\mathbf{y}$  and  $B_1[\bar{T}' - \lambda^{-kn}\mathbf{y}] = P' - \mathbf{z} - \lambda^{-kn}\mathbf{y}$ .

Now  $\mathbf{z} - \lambda^{-kn}\mathbf{y} \in A \cap \text{int}(\text{spt}(P)) \cap \text{int}(\text{spt}(P'))$ , so there exists an  $M > 0$  such that

$$B_0[\Phi^{kM}(P - \mathbf{z} - \lambda^{-kn}\mathbf{y})] = B_0[\Phi^{kM}(P' - \mathbf{z} - \lambda^{-kn}\mathbf{y})]$$

Since  $\bar{T}, \bar{T}'$  are  $\Phi^k$ -periodic we have  $B_0[\bar{T} - \lambda^{k(M-n)}\mathbf{y}] = B_0[(\bar{T}' - \lambda^{k(M-n)}\mathbf{y})]$ . It follows from Proposition 2.0.18 that  $\bar{T} - \mathbf{y} \in W^s(\bar{T}' - \mathbf{y})$ , for all  $\mathbf{y}$  such that  $\langle \mathbf{y}, \mathbf{v} \rangle > 0$ . By Proposition 3.0.40, there exists an  $R > 0$  such that  $B_0[\bar{T} - \mathbf{y}] = B_0[\bar{T}' - \mathbf{y}]$ , for all  $\mathbf{y}$  such that  $\langle \mathbf{y}, \mathbf{v} \rangle > R$ . It follows that, for such  $\mathbf{y}$ ,  $\mathbf{d}(\bar{T} - u\mathbf{y}, \bar{T}' - u\mathbf{y}) \rightarrow 0$  as  $u \rightarrow \infty$ . ■

### Types of branching

There are three ways tilings can be asymptotic in more than a half-plane, each defines a different type of sector of asymptoticity,  $S_{(T, T')}$ . We say an asymptotic pair  $(T, T')$  is an:

- asymptotic point pair, denoted  $(T, T')_{\{\mathbf{0}\}}$ , if  $\mathbf{d}(T - u\mathbf{x}, T' - u\mathbf{x}) \rightarrow 0$  as  $u \rightarrow \infty$ , for all vectors  $\mathbf{x} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ . In this case we take  $\partial\alpha(T, T') := \{\mathbf{0}\}$ .
- asymptotic line pair, denoted  $(T, T')_{\{\mathbf{v}\}}$ , if there exists a unit vector  $\mathbf{v} \in \mathbb{R}^2$  such that  $\mathbf{d}(T - u\mathbf{x}, T' - u\mathbf{x}) \rightarrow 0$  as  $u \rightarrow \infty$ , for all vectors  $\mathbf{x} \in \{\mathbf{y} : \langle \mathbf{v}, \mathbf{y} \rangle > 0\}$  and  $\mathbf{d}(T - u\mathbf{v}^\perp, T' - u\mathbf{v}^\perp) \not\rightarrow 0$  as  $u \rightarrow \pm\infty$ .

**Lemma 3.0.43.** *Let  $(T, T')$  be an asymptotic pair that is neither an asymptotic point or line pair. Then there exist nonparallel nonzero unit vectors  $\mathbf{v}_1, \mathbf{v}_2$  such that  $d(T - u\mathbf{x}, T' - u\mathbf{x}) \rightarrow 0$  as  $u \rightarrow \infty$  for all vectors  $\mathbf{x}$  with  $\langle \mathbf{v}_1, \mathbf{x} \rangle > 0$  or  $\langle \mathbf{v}_2, \mathbf{x} \rangle > 0$  and  $d(T - u\mathbf{v}_i^\perp, T' - u\mathbf{v}_i^\perp) \not\rightarrow 0$  as  $u \rightarrow \infty$  for  $i = 1$  or  $2$ .*

**Proof** Since  $(T, T')$  is an asymptotic pair there exists a nonzero vector  $\mathbf{v}$  such that  $d(T - u\mathbf{s}, T' - u\mathbf{s}) \rightarrow 0$  as  $u \rightarrow \infty$ , for all  $\mathbf{s} \in \mathbb{R}^2$  such that  $\langle \mathbf{v}, \mathbf{s} \rangle > 0$ . We are assuming  $(T, T')$  is not an asymptotic line pair, so either  $d(T - u\mathbf{v}^\perp, T' - u\mathbf{v}^\perp) \rightarrow 0$  as  $u \rightarrow \infty$  or  $d(T - u\mathbf{v}^\perp, T' - u\mathbf{v}^\perp) \rightarrow 0$  as  $u \rightarrow -\infty$ . Define  $\beta$  to be an open arc on  $S^1$  containing  $\mathbf{v}$  such that  $d(T - u\mathbf{s}, T' - u\mathbf{s}) \rightarrow 0$  as  $u \rightarrow \infty$ , for all  $\mathbf{s} \in \beta$ . Let  $\sigma = \bigcup \beta$  and note that  $\sigma \neq S^1$  since  $(T, T')$  is not an asymptotic point pair. Denote the endpoints of  $\sigma$  to be  $\mathbf{s}_1$  and  $\mathbf{s}_2$ .

Claim:  $d(T - u\mathbf{s}_i, T' - u\mathbf{s}_i) \not\rightarrow 0$  as  $u \rightarrow \infty$ , for  $i = 1$  or  $2$ .

Proof of claim: Assume otherwise. By Proposition 3.0.37,  $T - \mathbf{s}_1 \in W^s(T' - \mathbf{s}_1)$ . Define the line segment  $L := u\mathbf{s}_1$ ,  $u \in [1, \lambda]$ . For each  $\mathbf{x} \in L$ ,  $T - \mathbf{x} \in W^s(T' - \mathbf{x})$ . So there is  $n(\mathbf{x})$  so that  $B_0[\Phi^{n(\mathbf{x})}(T - \mathbf{x})] = B_0[\Phi^{n(\mathbf{x})}(T' - \mathbf{x})]$ . Then there is  $\epsilon = \epsilon(\mathbf{x}) > 0$ , so that if  $\mathbf{y} \in B_{\epsilon(\mathbf{x})}(\mathbf{x})$ , then  $B_0[\Phi^{n(\mathbf{x})}(T - \mathbf{y})] = B_0[\Phi^{n(\mathbf{x})}(T' - \mathbf{y})]$ . Then, by compactness of  $L$ , there exist  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k \in L$  such that  $\mathcal{U} := \bigcup_{i=1}^k B_{\epsilon(\mathbf{x}_i)}(\mathbf{x}_i)$  cover  $L$ . Let  $N = \max\{n(\mathbf{x}_1), \dots, n(\mathbf{x}_k)\}$ . So if  $\mathbf{y} \in \mathcal{U}$ , then  $B_0[\Phi^N(T - \mathbf{y})] = B_0[\Phi^N(T' - \mathbf{y})]$ . Now let  $\epsilon > 0$  be small enough so that if  $\mathbf{x}' \in B_\epsilon(\mathbf{s}_1)$  then  $u\mathbf{x}' \in \mathcal{U}$  for  $u \in [1, \lambda]$ . Now let  $\mathbf{x} \in B_\epsilon(\mathbf{s}_1) \cap S^1$ . To see that  $d(T - u\mathbf{x}, T' - u\mathbf{x}) \rightarrow 0$  as  $u \rightarrow \infty$  let  $\delta > 0$  be given. Let  $M$  be large enough so that if  $S, S' \in \Omega_\Phi$  are such that  $B_0[S] = B_0[S']$  then  $d(\Phi^n(S), \Phi^n(S')) < \delta$ , for all  $n \geq M$ . Now take  $u_0 = \lambda^{M+N}$ . Then, if  $u \geq u_0$ ,  $u\mathbf{x} = \lambda^m u'\mathbf{x}$ , for some  $u' \in [1, \lambda]$  and  $m \geq M + N$ . Then  $T - u\mathbf{x} = \Phi^n(\Phi^N(T - u'\mathbf{x}))$  and  $T' - u\mathbf{x} = \Phi^n(\Phi^N(T' - u'\mathbf{x}))$  with  $n \geq m$ . Since  $B_0[\Phi^N(T - u'\mathbf{x})] = B_0[\Phi^N(T' - u'\mathbf{x})]$  we have  $d(T - u\mathbf{x}, T' - u\mathbf{x}) < \delta$ . ■

In light of the last lemma, we say  $(T, T')$  is an:

- asymptotic corner pair, denoted  $(T, T')_{\{\mathbf{v}_1, \mathbf{v}_2\}}$ , if there exist nonparallel unit vectors  $\mathbf{v}_1, \mathbf{v}_2$ , such that  $d(T - u\mathbf{x}, T' - u\mathbf{x}) \rightarrow 0$  as  $u \rightarrow \infty$ , for all  $\mathbf{x} \in \{\mathbf{y} : \langle \mathbf{v}_1, \mathbf{y} \rangle > 0 \text{ or } \langle \mathbf{v}_2, \mathbf{y} \rangle > 0\}$  and  $d(T - u\mathbf{v}_i^\perp, T' - u\mathbf{v}_i^\perp) \not\rightarrow 0$  as  $u \rightarrow \infty$  for  $i = 1$  or  $2$ , where the perpendicular vectors  $\mathbf{v}_1^\perp, \mathbf{v}_2^\perp$  are chosen such that  $\langle \mathbf{v}_1^\perp, \mathbf{v}_2 \rangle < 0$  and  $\langle \mathbf{v}_2^\perp, \mathbf{v}_1 \rangle < 0$ .

**Corollary 3.0.44.** *Given an asymptotic corner pair  $(T, T')_{\{\mathbf{v}_1, \mathbf{v}_2\}}$ ,  $\mathbf{y} \in \partial\alpha(T, T')$  there exists a  $u \in \mathbb{R}^+$  such that*

$$T - u\mathbf{y} \notin W^s(T' - u\mathbf{y}).$$

**Lemma 3.0.45.** *Suppose  $(T_n, T'_n)$  is a sequence of asymptotic pairs converging to tilings  $(T, T')$ . There exist an  $N \in \mathbb{N}$ , vectors  $\mathbf{x}_n$ , with  $|\mathbf{x}_n| \rightarrow 0$ , such that for all  $n \geq N$ ,*

$$B_1[T_n - \mathbf{x}_n] = B_1[T]$$

$$B_1[T'_n - \mathbf{x}_n] = B_1[T']$$

**Proof:** Choose  $\mathbf{y}$  such that  $\langle \mathbf{y}, \mathbf{v}_n \rangle > R$ , for all  $n$ . Then  $B_0[T_n - \mathbf{y}] = B_0[T'_n - \mathbf{y}]$ , for all  $n$ . Let  $t_n \in B_0[T_n - \mathbf{y}]$ . There are then a  $t$ , a sequence  $\{n_k\}$ , and vectors  $\mathbf{x}_k$ ,  $|\mathbf{x}_k| < \Delta$ , such that  $t_{n_k} = t - \mathbf{x}_k$ . To ease notation we assume that  $n_k = k$ . (That is, replace  $(T_n, T'_n)$  by  $(T_{n_k}, T'_{n_k})$ ). Notice  $|\mathbf{x}_n| \rightarrow 0$  as  $n \rightarrow \infty$  since, if not, we would have a contraction with Remark 2.0.2.

Now  $t \in T_n - \mathbf{y} - \mathbf{x}_n \cap T'_n - \mathbf{y} - \mathbf{x}_n$  for all  $n$ . Thus by Lemma 3.0.32, there exist a  $N$  such that, for all  $n \geq N$

$$B_{R+1}[T_n - \mathbf{y} - \mathbf{x}_n] = B_{R+1}[T - \mathbf{y}],$$

$$B_{R+1}[T'_n - \mathbf{y} - \mathbf{x}_n] = B_{R+1}[T' - \mathbf{y}].$$

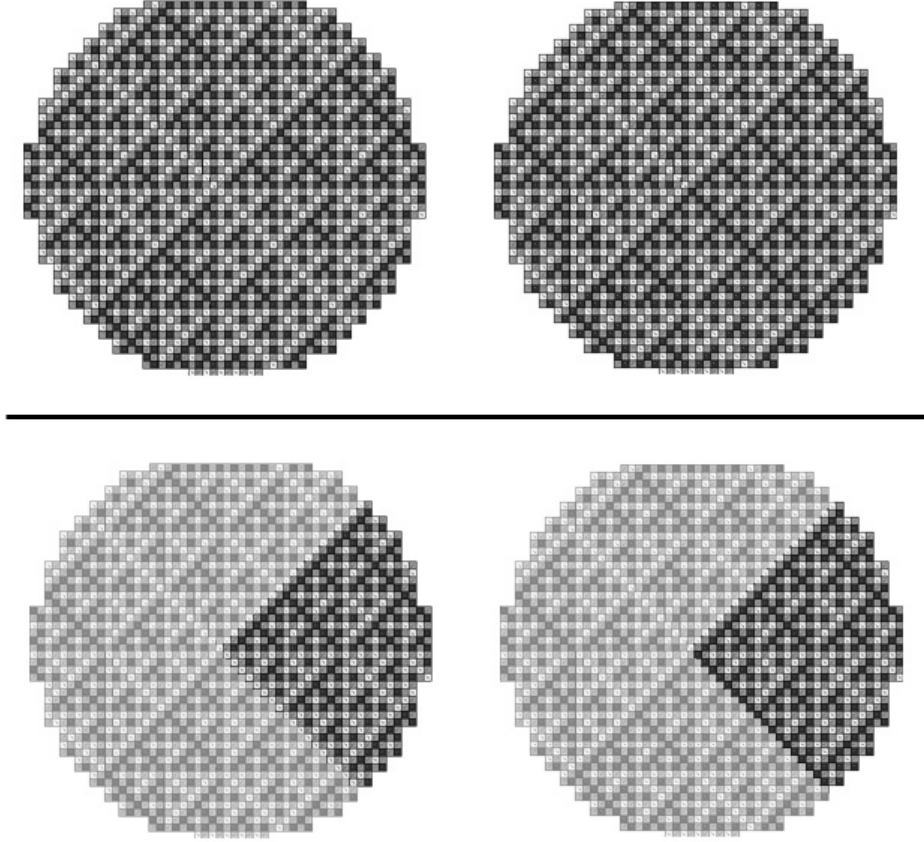


Figure 4: An asymptotic corner pair found in the Chair Tiling space.  $S_{(T,T')}$  has been highlighted.

In particular,  $B_1[T_n - \mathbf{x}_n] = B_1[T]$  and  $B_1[T'_n - \mathbf{x}_n] = B_1[T']$ , for all  $n \geq N$ . ■

For  $\mathbf{y} \in \partial\alpha(T, T')$ , Corollary 3.0.44 gives us a point  $u\mathbf{y}$  such that  $T - u\mathbf{y} \notin W^s(T' - u\mathbf{y})$ . For our construction we require this to be true for all  $u \in \mathbb{R}$ .

**Definition 3.0.46.** We say  $(T, T') \in \mathcal{AP}$  is non-collapsing if  $\forall u \in \mathbb{R}^+$  and  $\mathbf{y} \in \partial\alpha(T, T')$ ,  $T - u\mathbf{y} \notin W^s(T - u\mathbf{y})$ .

**Definition 3.0.47.** We say  $T$  is a branch vertex tiling if there exist a non-collapsing asymptotic pair  $(T, T') \in \mathcal{AP}$  such that:

- $(T, T') = (T, T')_{\{0\}}$  is a point pair, or

- $(T, T') = (T, T')_{\{\mathbf{v}_1, \mathbf{v}_2\}}$  is a corner pair, or
- $(T, T') = (T, T')_{\{\mathbf{v}_1\}}$  is a line pair and there exists  $T'' \in \Omega_\Phi$  such that  $(T, T'')_{\{\mathbf{v}_2\}}$  is a non-collapsing asymptotic line pair with  $\mathbf{v}_1 \neq \pm \mathbf{v}_2$ .

Denote by  $\mathcal{BV}$  the set of branch vertex tilings.

**Theorem 3.0.48.** (Barge) *For two-dimensional nonperiodic substitution tiling spaces the set of branch vertex tilings is nonempty.*

**Proposition 3.0.49.** *There exist a finite number of branch vertex tilings.*

**Proof:** Suppose  $(T_n, T'_n)$  are distinct point asymptotic pairs with  $(T_n, T'_n) \rightarrow (T, T')$  as  $n \rightarrow \infty$ . By Lemma 3.0.45, there exist vectors  $\mathbf{x}_n$ , with  $|\mathbf{x}_n| \rightarrow 0$  such that  $B_1[T_n - \mathbf{x}_n] = B_1[T]$  and  $B_1[T'_n - \mathbf{x}_n] = B_1[T']$ , for all  $n \geq N$ . So,  $B_1[T_N - \mathbf{x}_N] = B_1[T_n - \mathbf{x}_n]$  and  $B_1[T'_N - \mathbf{x}_N] = B_1[T'_n - \mathbf{x}_n]$ , for all  $n \geq N$ . By taking  $N$  even larger (so that  $|\mathbf{x}_n| < 1/2\Delta$ , for all  $n \geq N$ ) we have  $B_0[T_N] = B_0[T_n - (\mathbf{x}_n - \mathbf{x}_N)]$  and  $B_0[T'_N] = B_0[T'_n - (\mathbf{x}_n - \mathbf{x}_N)]$ , for all  $n \geq N$ . Since  $T_N \notin W^s(T'_N)$ , by Lemma 3.0.31,  $T_n - (\mathbf{x}_n - \mathbf{x}_N) \notin W^s(T'_n - (\mathbf{x}_n - \mathbf{x}_N))$ .

For asymptotic point pairs  $S_{(T, T')} = \mathbb{R}^2 \setminus \mathbf{0}$ . So, by Lemma 3.0.37,  $T_n - \mathbf{y} \in W^s(T'_n - \mathbf{y})$ , for all  $\mathbf{y} \neq \mathbf{0}$ . So  $\mathbf{x}_n - \mathbf{x}_N = \mathbf{0}$ , for all  $n \geq N$ . That is, the  $(T_n, T'_n)$  are not all distinct.

Now assume  $(T_n, T'_n)_{\{\mathbf{v}_n, \mathbf{v}'_n\}}$  are asymptotic corner pairs with  $\mathbf{v}_n \rightarrow \mathbf{v}$ , and  $\mathbf{v}'_n \rightarrow \mathbf{v}'$ . Consider the line  $l_n$  given by the vector  $(\mathbf{v}_n + \mathbf{v}'_n)^\perp$  containing the point  $\mathbf{x}_n$  and the line  $l_N$  given by the vector  $(\mathbf{v}_N + \mathbf{v}'_N)^\perp$  containing the point  $\mathbf{x}_N$ , see Figure 5. If  $l_n$  is parallel to  $l_N$  then either  $\mathbf{x}_n - \mathbf{x}_N \in W^s(T_n - \mathbf{x}_n)$  or  $\mathbf{x}_n - \mathbf{x}_N \in W^s(T'_N - \mathbf{x}_N)$ .

If the lines intersect there are scalars  $\alpha_n, \beta_n \in \mathbb{R}$  such that

$$\mathbf{z}_n = \mathbf{x}_n + \alpha_n(\mathbf{v}_n + \mathbf{v}'_n)^\perp = \mathbf{x}_N - \beta_n(\mathbf{v}_N + \mathbf{v}'_N)^\perp.$$

Rewrite  $\mathbf{x}_n - \mathbf{x}_N = -\alpha_n(\mathbf{v}_n + \mathbf{v}'_n)^\perp - \beta_n(\mathbf{v}_N + \mathbf{v}'_N)^\perp$ . Now  $|\mathbf{x}_n - \mathbf{x}_N| \rightarrow 0$  as  $n \rightarrow \infty$  implies  $\alpha_n, \beta_n \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore we can make  $d(\mathbf{z}_n, \mathbf{x}_n)$  as small as we wish. Let  $\epsilon > 0$  be small enough so that  $\mathbf{x}_n, \mathbf{x}_N, \mathbf{z}_n \in B_\epsilon(0) \subset \text{spt}(B_0[T_n - \mathbf{x}_n]) \cap \text{spt}(B_0[T_N - \mathbf{x}_N])$ .

Consider the line segment  $s\mathbf{z}_n + (s-1)\mathbf{x}_N$ ,  $s \in [0, 1]$ . If there is an  $s$  and some  $c \in \mathbb{R}$ , so that  $T - \mathbf{x} - c\mathbf{v}_n^\perp = s\mathbf{z}_n + (s-1)\mathbf{x}_N$  then  $\mathbf{x}_n - \mathbf{x}_N \in W^s(T_N, T'_N)$ . If not,  $\mathbf{x}_n - \mathbf{x}_N \in W^s(T_n, T'_n)$ .

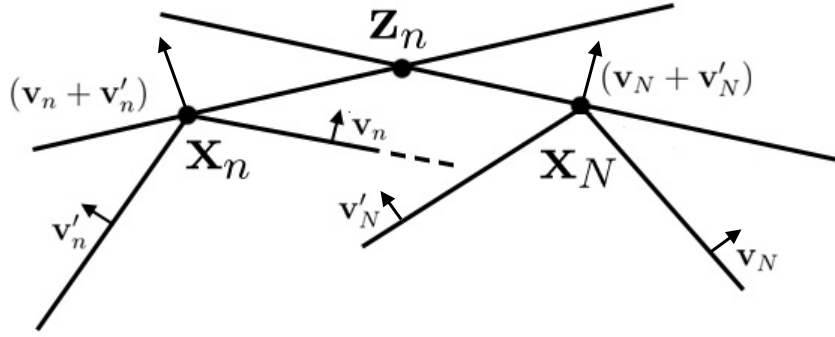


Figure 5:  $T_n - \mathbf{x}_n$  and  $T_N - \mathbf{x}_N$  found in the proof of Proposition 3.0.49

So we have either  $\mathbf{x}_n - \mathbf{x}_N \in S_{(T_N, T'_N)}$  or  $\mathbf{x}_n - \mathbf{x}_N \in S_{(T_n, T'_n)}$ . If  $\mathbf{x}_n - \mathbf{x}_N \in S_{(T_N, T'_N)}$  then  $T_N - (\mathbf{x}_n - \mathbf{x}_N) \in W^s(T'_N - (\mathbf{x}_n - \mathbf{x}_N))$ . So

$$B_0[T_N - (\mathbf{x}_n - \mathbf{x}_N)] = B_0[T_n],$$

$$B_0[T'_N - (\mathbf{x}_n - \mathbf{x}_N)] = B_0[T'_n].$$

and Lemma 3.0.31, give  $T_n \in W^s(T'_n)$ , a contradiction.

Now if  $\mathbf{x}_n - \mathbf{x}_N \in S_{(T_n, T'_n)}$  a similar contradiction can be reached. So there exist only a finite number of asymptotic corner pairs.

Assume  $(T_n, T'_n)$  are neither asymptotic point or corner pairs. Then, for each  $n$ ,  $(T_n, T'_n)_{\{\mathbf{v}'_n\}}$  is an asymptotic line pair, such that there also exist  $T''_n \in \Omega_\Phi$  with

$(T_n, T_n'')$  $_{\{\mathbf{v}'_n\}}$  an asymptotic line pair, and  $\pm \mathbf{v}_n \neq \pm \mathbf{v}'_n$ . So we assume we have a convergent sequence of distinct triples  $(T_n, T_n', T_n'') \rightarrow (T, T', T'')$ , with  $\mathbf{v}_n \rightarrow \mathbf{v}$ ,  $\mathbf{v}'_n \rightarrow \mathbf{v}'$ . By the same reasoning as above, for  $(T_n, T_n')$  there exists an  $N_1 \in \mathbb{N}$  such that for all  $n > N_1$ , there exist vectors  $\mathbf{x}_n$  with

$$\begin{aligned} B_1[T_N] &= B_1[T_n - \mathbf{x}_n], \\ B_0[T_N'] &= B_1[T_n' - \mathbf{x}_n]. \end{aligned}$$

Also, for  $(T_n, T_n'')$ , there exist a  $N_2 \in \mathbb{N}$  such that for all  $n \geq N_2$  there exist vectors  $\mathbf{y}_n$  with

$$\begin{aligned} B_1[T_N] &= B_1[T_n - \mathbf{y}_n], \\ B_0[T_N''] &= B_1[T_n'' - \mathbf{y}_n]. \end{aligned}$$

Since  $B_1[T_n - \mathbf{x}_n] = B_1[T_n - \mathbf{y}_n]$  for all  $n \geq \max\{N_1, N_2\}$ , it must be the case that  $\mathbf{x}_n = \mathbf{y}_n$ . Thus,

$$\begin{aligned} B_0[T_n - \mathbf{x}_n] &= B_1[T] = B_1[T_N - \mathbf{x}_n], \\ B_0[T_n' - \mathbf{x}_n] &= B_1[T'] = B_1[T_N' - \mathbf{x}_n], \\ B_0[T_n'' - \mathbf{x}_n] &= B_1[T''] = B_1[T_N'' - \mathbf{x}_n]. \end{aligned}$$

Again, for  $N$  large we have

$$\begin{aligned} B_0[T_N] &= B_0[T_n - (\mathbf{x}_n - \mathbf{x}_N)], \\ B_0[T_N'] &= B_0[T_n' - (\mathbf{x}_n - \mathbf{x}_N)], \\ B_0[T_N''] &= B_0[T_n'' - (\mathbf{x}_n - \mathbf{x}_N)]. \end{aligned}$$

Now if  $\mathbf{x}_n - \mathbf{x}_N \in S_{(T_n, T_n')}$  then  $\langle \mathbf{x}_n - \mathbf{x}_N, \mathbf{v}_n \rangle > 0$ . So  $T_n - (\mathbf{x}_n - \mathbf{x}_N) \in W^s(T_n' - (\mathbf{x}_n - \mathbf{x}_N))$  which implies, by Lemma 3.0.31, that  $T_N \in W^s(T_N')$ . If  $\mathbf{x}_n - \mathbf{x}_N \notin S_{(T_n, T_n')}$

then  $\langle \mathbf{x}_n - \mathbf{x}_N, \mathbf{v}_n \rangle < 0$  and either  $\langle \mathbf{x}_n - \mathbf{x}_N, \mathbf{v}'_n \rangle > 0$  or  $\langle \mathbf{x}_n - \mathbf{x}_N, \mathbf{v}'_n \rangle \leq 0$ . If  $\langle \mathbf{x}_n - \mathbf{x}_N, \mathbf{v}'_n \rangle > 0$  then  $\mathbf{x}_n - \mathbf{x}_N \in S_{(T_n, T''_n)}$ . So  $T_n - (\mathbf{x}_n - \mathbf{x}_N) \in W^s(T''_n - (\mathbf{x}_n - \mathbf{x}_N))$  which implies that  $T_N \in W^s(T''_N)$ .

Now assuming  $\mathbf{x}_n - \mathbf{x}_N \notin S_{(T_n, T'_n)}$  and  $\langle \mathbf{x}_n - \mathbf{x}_N, \mathbf{v}'_n \rangle \leq 0$  we have that  $\langle -(\mathbf{x}_n - \mathbf{x}_N), \mathbf{v}_N \rangle > 0$ . Thus  $T_n - (\mathbf{x}_N - \mathbf{x}_n) \in W^s(T'_n - (\mathbf{x}_N - \mathbf{x}_n))$  then finally,  $T_N \in W^s(T'_N)$ .

By Proposition 3.0.30,  $T_n \notin W^s(T'_n)$  and  $T_n \notin W^s(T''_n)$  for all  $n$ , thus each case leads to a contradiction.  $\blacksquare$

**Theorem 3.0.50.** *There are only a finite number of directions of branching in the tiling space. That is, the set  $\bigcup_{(T, T') \in \mathcal{AP}} \partial\alpha(T, T')$  is finite.*

**Proof:** Suppose otherwise. Then there exist  $(T_n, T'_n)_{\{\mathbf{v}_n\}} \in \mathcal{AP}$  such that  $(T_n, T'_n) \rightarrow (T, T')$  as  $n \rightarrow \infty$ . We may assume  $\mathbf{v}_n \rightarrow \mathbf{v}$ , with  $|\mathbf{v}_n| = 1$  and  $\mathbf{v}_n \neq \mathbf{v}_m$ , for  $n \neq m$ . Note if  $\langle \mathbf{v}_n, \mathbf{y} \rangle > 0$  then  $T_n - \mathbf{y} \in W^s(T'_n - \mathbf{y})$ . Let  $(P, P') = (B_1[T], B_1[T'])$ . We may assume (after passing to a subsequence) that  $(B_1[T_n], B_1[T'_n]) = (P, P') - \mathbf{x}_n$  with  $\mathbf{x}_n < 1$ ,  $|\mathbf{x}_n| \rightarrow 0$ , and  $\mathbf{x}_n \neq \mathbf{x}_m$  for  $n \neq m$ .

If  $T \in W^s(T')$ , then for  $n$  large,  $(P, P') = (B_1[T_n], B_1[T'_n])$  implies  $T_n \in W^s(T_n)$ , which cannot happen. So,  $T \notin W^s(T')$ .

Now for  $n$  large,  $\langle \mathbf{v}, \mathbf{v}_n \rangle > 0$ . For such  $n$ , if  $\langle \mathbf{v}^\perp, \mathbf{x}_n \rangle = 0$  then either  $\mathbf{x}_n \in S_{(T, T')}$  or  $\mathbf{x}_n \in S_{(T_n, T'_n)}$ . So, either  $T_n \in W^s(T'_n)$  or  $T \in W^s(T')$  which cannot happen. Thus, we have  $\langle \mathbf{v}^\perp, \mathbf{x}_n \rangle > 0$ .

Now let  $0 \leq k_1 < k_2$  be such that  $(Q, Q') = (B_1[\Phi^{k_1}(T)], B_1[\Phi^{k_1}(T')]) = (B_1[\Phi^{k_2}(T)], B_1[\Phi^{k_2}(T')]) - \mathbf{x}$ , for some  $\mathbf{x}$  with  $|\mathbf{x}| < 1/2$ . Replace  $T$  by  $\Phi^{k_1}(T)$ ,  $T'$  by  $\Phi^{k_1}(T')$ ,  $T_n$  by  $\Phi^{k_1}(T_n)$ ,  $T'_n$  by  $\Phi^{k_1}(T'_n)$  and let  $m = k_2 - k_1 > 0$ . (Now  $(Q, Q')$  is replaced by  $(P, P')$ ). Then  $(B_1[T], B_1[T']) = (B_1[\Phi^m(T)], B_1[\Phi^m(T')]) - \mathbf{x} = (P, P')$ . We may now assume that  $\lambda^m |\mathbf{x}_n| < 1$ , for all  $n$ .

We will show that  $\mathbf{x} = \mathbf{0}$ . Let  $\mathbf{y}$  be such that  $\langle \mathbf{y}, \mathbf{v} \rangle > 0$  and  $|\mathbf{y}| < 1$ . Then  $T - \mathbf{y} \in W^s(T' - \mathbf{y})$  and  $\Phi^m(T) - \mathbf{y} \in W^s(\Phi^m(T') - \mathbf{y})$ . So,  $\langle \mathbf{x}, \mathbf{v} \rangle = 0$ . Without loss of generality, assume  $\langle \mathbf{x}, \mathbf{v}^\perp \rangle \geq 0$ . Now since  $\langle \mathbf{x}_n, \mathbf{v} \rangle \leq 0$ ,  $\langle \mathbf{x}_n, \mathbf{v}^\perp \rangle > 0$  and  $\langle \mathbf{x}_n, \mathbf{v}_n^\perp \rangle > 0$ , there must be scalars  $\alpha_n, \beta_n \geq 0$  so that  $\alpha_n \mathbf{v}^\perp = -\beta_n \mathbf{v}_n^\perp := \mathbf{z}_n$ ;  $\alpha_n, \beta_n \rightarrow 0$  as  $n \rightarrow \infty$  (see Figure 6).

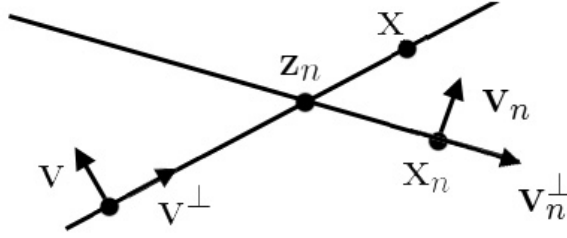


Figure 6: Vectors used in the proof of Proposition 3.0.50

Now suppose that  $\mathbf{x} \neq \mathbf{0}$ . Then, for large enough  $n$ ,  $\alpha_n < |\mathbf{x}|$ . But then  $\langle \mathbf{x} - \mathbf{x}_n, \mathbf{v}^\perp \rangle > 0$  so that  $\Phi^m(T) \in W^s(\Phi^m(T'))$ , which cannot happen. Thus  $\mathbf{x} = \mathbf{0}$ .

From this it follows that there is an asymptotic pair  $(\bar{T}, \bar{T}')$  with  $(B_1[\bar{T}], B_1[\bar{T}']) = (P, P')$ ,  $\Phi^m(\bar{T}) = \bar{T}$ ,  $\Phi^m(\bar{T}') = \bar{T}'$ , and  $\bar{T} - \mathbf{y} \in W^s(\bar{T}' - \mathbf{y})$ , for all  $\mathbf{y}$  such that  $\langle \mathbf{y}, \mathbf{v} \rangle > 0$ .

We may argue in the same manner as above to show  $\bar{T} - \mathbf{x} \in W^s(\bar{T}' - \mathbf{x})$ , for all  $\mathbf{x}$  with  $\mathbf{x} = \alpha \mathbf{v}^\perp$ ,  $0 < \alpha < 1$ . Thus  $(\bar{T}, \bar{T}')$  is a asymptotic point or corner pair. It is clearly not a point pair (then  $T_n \in W^s(T'_n)$ ). Say  $\partial\alpha(\bar{T}, \bar{T}') = \{\mathbf{y}_1, \mathbf{y}_2\}$ , with  $\mathbf{y}_1^\perp, \mathbf{y}_2^\perp$  chosen such that  $\langle \mathbf{y}_1^\perp, \mathbf{y}_2 \rangle < 0$  and  $\langle \mathbf{y}_2^\perp, \mathbf{y}_1 \rangle < 0$ . We must have that  $\langle \mathbf{x}_n, \mathbf{y}_i^\perp \rangle \leq 0$  (see Figure 7).

As argued before, there are points  $\mathbf{w}_n = \alpha_n \mathbf{y}_2 = -\beta_n \mathbf{v}_n^\perp$ ;  $\alpha_n, \beta_n \geq 0$  and  $\alpha_n, \beta_n \rightarrow 0$  as  $n \rightarrow \infty$ .

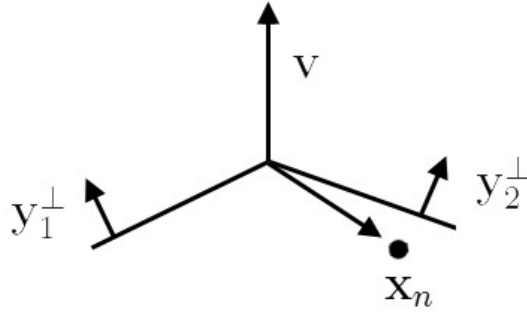


Figure 7: Vectors used in the proof of Proposition 3.0.50

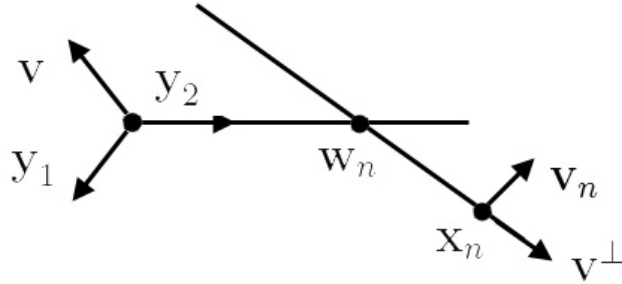


Figure 8: Vectors used in the proof of Proposition 3.0.50

And, as in the prior argument,  $\bar{T} - \mathbf{x} \in W^s(\bar{T} - \mathbf{x})$ , for all  $\mathbf{x}$  with  $\mathbf{x} = \alpha \mathbf{y}_2, 0 < \alpha < 1$ . But then  $\mathbf{y}_2 \notin \partial\alpha(\bar{T}, \bar{T}')$ , by Lemma 3.0.44. ■

**Definition 3.0.51.** *Given a non-collapsing asymptotic pair  $(T, T') \in \mathcal{AP}$  define a branch line to be*

$$\mathcal{L}_{(T, T')} = cl \left( \bigcup_{\substack{u \in \mathbb{R}^+ \cup \{0\} \\ \mathbf{y} \in \partial\alpha(T, T')}} \{T - u\mathbf{y}, T' - u\mathbf{y}\} \right)$$

**Proposition 3.0.52.** *There exists a finite collection of patches  $\{P_1, \dots, P_n\}$  such that given any non-collapsing asymptotic pair  $(T, T') \in \mathcal{AP}$ ,  $\mathbf{y} \in \partial\alpha(T, T')$ , and  $u \in \mathbb{R}^+$  there exists a patch  $P_i$  and vector  $\mathbf{s} = \mathbf{s}(\mathbf{y}, u)$  parallel to  $\mathbf{y}$  such that  $B_0[T - u\mathbf{y}] = P_i - \mathbf{s}$*

**Proof:** By Lemma 3.0.39, for any  $y = u\mathbf{v}$ ,  $u \in \mathbb{R}^+$ ,  $\mathbf{v} \in \partial\alpha(T, T')$ , there exist  $i = i(\mathbf{y})$ ,  $j = j(\mathbf{y})$ ,  $\mathbf{x} = \mathbf{x}(\mathbf{y})$ , and a finite number of patches  $P_1, \dots, P_m$  such that

$$(B_0[T - \mathbf{y}], B_0[T' - \mathbf{y}]) = (P_i, P_j) - \mathbf{x}.$$

Fix  $\mathbf{v}$ . Suppose there are vectors  $\mathbf{y}_1, \mathbf{y}_2$ , and patches  $P, P' \in \{P_1, \dots, P_m\}$  such that

$$\begin{aligned} (B_0[T - \mathbf{y}_1], B_0[T' - \mathbf{y}_1]) &= (P, P') - \mathbf{x}_1, \\ (B_0[T - \mathbf{y}_2], B_0[T' - \mathbf{y}_2]) &= (P, P') - \mathbf{x}_2. \end{aligned}$$

Claim:  $\mathbf{y}_2 - \mathbf{y}_1$  is parallel to  $\mathbf{x}_2 - \mathbf{x}_1$ .

Proof of claim: Suppose otherwise. Let  $A_i = S_{(T, T')} + \mathbf{x}_i$ ,  $i = 1, 2$ . Then either  $\mathbf{x}_1 \in A_2$  or  $\mathbf{x}_2 \in A_1$ . Let  $\mathbf{w} = \mathbf{x}_1 - \mathbf{x}_2$ . Now if  $\mathbf{x}_1 \in A_2$  then  $\mathbf{y}_2 - \mathbf{w} \in S_{(T, T')}$ , thus  $T - \mathbf{y}_2 - \mathbf{w} \in W^s(T' - \mathbf{y}_2 - \mathbf{w})$ . However,  $B_0[T - \mathbf{y}_2 - \mathbf{w}] = P - \mathbf{x}_2 - \mathbf{w} = P - \mathbf{x}_1$ . Likewise,  $B_0[T' - \mathbf{y}_2 - \mathbf{w}] = P' - \mathbf{x}_1$ . So,

$$\begin{aligned} B_0[T - \mathbf{y}_2 - \mathbf{w}] &= B_0[T - \mathbf{y}_1], \\ B_0[T' - \mathbf{y}_2 - \mathbf{w}] &= B_0[T' - \mathbf{y}_1]. \end{aligned}$$

Therefore, by Lemma 3.0.31,  $T - \mathbf{y}_1 \in W^s(T' - \mathbf{y}_1)$ , a contradiction. Similarly, if  $\mathbf{x}_2 \in A_1$  we will have  $T - \mathbf{y}_2 \in W^s(T' - \mathbf{y}_2)$ .  $\blacksquare$

**Lemma 3.0.53.** *Suppose  $T$  and  $T'$  are periodic tilings under  $\Phi$  such that  $B_1[T] = B_1[T'] - \mathbf{x}$ . Then*

$$cl(\{T - s\mathbf{y} : s \in \mathbb{R}\}) = cl(\{T' - s\mathbf{y} : s \in \mathbb{R}\})$$

where  $\mathbf{y}$  is a unit vector parallel to  $\mathbf{x}$ .

**Proof:** Let  $\epsilon > 0$  be small enough so that  $B_\epsilon(T) \subset \text{spt}(B_0[T])$ . For some  $u \in \mathbb{R}$ ,  $u\mathbf{x} = \mathbf{y}$ . Suppose  $S \in cl(\{T - s\mathbf{y} : s \in \mathbb{R}\})$ . Then  $S = \lim_{k \rightarrow \infty} T - x_k\mathbf{y}$ . Choose  $M_k$

large enough so that  $\frac{x_k}{\lambda^{M_k}} < \epsilon$ . There exist  $n_i$  and  $s_i$ , with  $|s_i| < \epsilon$  such that

$$\begin{aligned} S &= \lim_{i \rightarrow \infty} \Phi^{n_i}(T - s_i \mathbf{y}) \\ &= \lim_{i \rightarrow \infty} \Phi^{n_i}(T' - u \mathbf{y} - s_i \mathbf{y}) \\ &= \lim_{i \rightarrow \infty} T' - \lambda^{n_i}(u + s_i) \mathbf{y} \end{aligned}$$

So  $S \in cl(\{T' - s \mathbf{y} : s \in \mathbb{R}\})$ . ■

**Definition 3.0.54.** Define the branch locus to be the set

$$\mathcal{BL} = \bigcup_{(T, T') \in \mathcal{AP}} \mathcal{L}_{(T, T')}.$$

**Proposition 3.0.55.**  $\mathcal{BL}$  is a closed subset of  $\Omega_\Phi$ .

**Proof:** By Corollary 3.0.52 combined with Lemma 3.0.53, there are only a finite number of branch lines. Thus  $\bigcup_{(T, T') \in \mathcal{AP}} \mathcal{L}_{(T, T')}$  is a finite union of closed sets and hence closed. ■

### Inverse Limit Spaces

We wish to show the branch locus can be realized as an inverse limit of a subset of a collared Anderson-Putnam complex  $K$ . Let  $F : K \rightarrow K$  denote the induced map of inflation and substitution on  $K$ , and  $\pi : \Omega_\Phi \rightarrow K$  projection so that the following diagram commutes:

$$\begin{array}{ccc} \Omega_\Phi & \xrightarrow{\Phi} & \Omega_\Phi \\ \pi \downarrow & & \pi \downarrow \\ K & \xrightarrow{F} & K \end{array}$$

Without loss of generality we may assume  $\Phi$  forces its border. Consider the projection of  $\mathcal{BL}$  into  $K$ . Define

$$B := \{x \in K : \pi(T) = x \text{ for some } T \in \mathcal{BL}\}$$

For any  $T \in \mathcal{BL}$  it is clear that  $\Phi(T) \in \mathcal{BL}$ , so  $B$  is invariant under the induced map  $F$ . Since  $\Phi$  forces its border the map  $\hat{\pi} : \Omega_\Phi \rightarrow \varprojlim(K, F)$  is a homeomorphism.

$$\begin{array}{ccccc} \mathcal{BL} & & \dots & \xrightarrow{\Phi} & \mathcal{BL} & \xrightarrow{\Phi} & \mathcal{BL} \\ \hat{\pi} \downarrow & & & & \pi \downarrow & & \pi \downarrow \\ \varprojlim(B, F) & & \dots & \xrightarrow{F} & B & \xrightarrow{F} & B \end{array}$$

So  $\hat{\pi}$  restricted to the branch locus is automatically injective. To show  $\hat{\pi}$  is surjective let  $(x_0, x_1, \dots) \in \varprojlim(B, F)$ . For each  $k \in \mathbb{N}$ , let  $Y_k = \pi^{-1}(\{x_k\}) \cap \mathcal{BL}$ . Then  $J_k = \bigcap_{j>0} \Phi^j(Y_{k+j})$  is a nested intersection of nonempty compact sets and thus is nonempty. Let  $(T_0, T_1, \dots) \in \varprojlim(\mathcal{BL}, \Phi)$  be such that  $T_k \in J_k$  for each  $k$ . Then  $\hat{\pi}(T_0, T_1, \dots) = (x_0, x_1, \dots)$ . From this we have the following proposition:

**Proposition 3.0.56.**  $\mathcal{BL}$  is homeomorphic to  $\varprojlim(B, F)$ .

### The Asymptotic Structure $\mathcal{A}_\Phi$

Let  $\mathcal{F} := \{T \in \mathcal{BL} \setminus \mathcal{BV} : \exists (T_1, T'_1), (T_2, T'_2) \in \mathcal{AP}, \partial\alpha(T_1, T'_1) \neq \partial\alpha(T_2, T'_2), T \in \mathcal{L}_{(T_1, T'_1)} \text{ and } \exists T' \in \mathcal{L}_{(T_2, T'_2)}, T' \notin \mathcal{BV} \text{ such that } \pi(T) = \pi(T')\}$ . These represent the branch points in  $B$  that do not arise from the projection of branch vertex tilings into  $K$ .  $\pi(\mathcal{F})$  is a finite subset of  $K$  since there are only a finite number of directions of branching. Recall  $K^0$  and  $K^1$  are the zero and one skeleton of the Anderson-Putnam complex.

Let  $\epsilon > 0$  be small enough so that:

- If  $T, T' \in \mathcal{BV} \cup \mathcal{F}$  and  $\pi(T) \neq \pi(T')$  then  $d(\pi(T), \pi(T')) > 2\epsilon$
- If  $T \in \mathcal{BV} \cup \mathcal{F}$  and  $\pi(T) \notin K^1$  then  $d(\pi(T), K^1) > \epsilon$

- If  $T \in \mathcal{BV} \cup \mathcal{F}$  and  $\pi(T) \notin K^0$  then  $d(\pi(T), K^0) > \epsilon$

where  $d$  in  $K$  is so that if  $x, y \in P$ ,  $P$  a face of  $K$  (also a prototile) then  $d(x, y) = |x - y|$ . Let  $\delta = \epsilon/(2\lambda)$ .

**Definition 3.0.57.** Let  $(T, T') \in \mathcal{AP}$ . Define  $\mathcal{N}(T) = \{T - u\mathbf{y} - s\mathbf{v} : u \in \mathbb{R}^+ \cup \{0\}, \mathbf{v} \in \alpha(T, T'), \mathbf{y} \in \partial\alpha(T, T'), s \in [0, \delta]\}$

A closed set containing  $\mathcal{BL}$  is then

$$\mathcal{NBL} = cl \left( \bigcup_{T \in \mathcal{AP}} \mathcal{N}(T) \right)$$

To isolate the asymptotic structure we define an equivalence relation which glues together certain points in  $\mathcal{NBL}$ . For any  $(T, T') \in \mathcal{AP}$  define  $\sim_0$  on  $\mathcal{N}(T)$  as follows: if  $(T, T')_{\{\mathbf{v}\}}$  is an asymptotic line pair,  $T - u\mathbf{v}^\perp - \delta\mathbf{v} \sim_0 T' - u\mathbf{v}^\perp - \delta\mathbf{v}$ ,  $u \in \mathbb{R}$ , if  $(T, T')_{\{\mathbf{0}\}}$  is an asymptotic point pair,  $T - \delta\mathbf{w} \sim_0 T' - \delta\mathbf{w}$ ,  $\mathbf{w} \in S^1$ , if  $(T, T')_{\{\mathbf{v}_1, \mathbf{v}_2\}}$  is an asymptotic corner pair,  $T - u\mathbf{v}_i^\perp - \delta\mathbf{v}_i \sim_0 T' - u\mathbf{v}_i^\perp - \delta\mathbf{v}_i$ ,  $u \in \mathbb{R}^+$ ,  $i = 1, 2$ , and  $T - \delta\mathbf{w} \sim_0 T' - \delta\mathbf{w}$ , if  $\mathbf{w} \in \alpha(T, T')$  and  $\langle \mathbf{w}, \mathbf{y} \rangle \leq 0$  for some  $\mathbf{y} \in \partial\alpha(T, T')$ . Define  $\sim$  on  $\mathcal{NBL}$  to be the smallest closed equivalence relation containing  $\sim_0$ . The asymptotic structure in the tiling space  $\Omega_\Phi$  is represented by

$$\mathcal{A}_\Phi = \mathcal{NBL} / \sim$$

Our goal is to express  $\mathcal{A}_\Phi$  as an inverse limit of a subset of  $K$ . As noted, the projection of the branch locus into  $K$  is invariant under the induced map of inflation and substitution,  $F$ . Define

$$BV := \{x \in K : \pi(T) = x \text{ for some } T \in \mathcal{BV}\},$$

$$NBV := \{x \in K : \pi(T) = x \text{ for some } T \in \mathcal{N}(\bar{T}), \bar{T} \in \mathcal{BV}\},$$

and

$$NB := \{x \in K : \pi(T) = x \text{ for some } T \in \mathcal{NB}\mathcal{L}\}$$

By Proposition 3.0.49,  $BV$  is a finite point set in  $B$ . In order to express  $\mathcal{NB}\mathcal{L}$  as an inverse limit we will define maps  $G : \mathcal{NB}\mathcal{L} \rightarrow \mathcal{NB}\mathcal{L}$ , and  $g : NB \rightarrow NB$ , with  $g \circ \pi = \pi \circ G$  such that  $G|_{\mathcal{B}\mathcal{L}}$  is homotopic to  $\Phi|_{\mathcal{B}\mathcal{L}}$ . For any  $T \in \mathcal{NB}\mathcal{L}$  there is an asymptotic pair  $(T_1, T'_1)$  and vector  $\mathbf{v}$  such that  $T = T_1 - \mathbf{v}$ . If we let  $G : \mathcal{NB}\mathcal{L} \rightarrow \mathcal{NB}\mathcal{L}$  be given by  $G(T) = \Phi(T) - \mathbf{v}$ , the map is not well defined near  $\mathcal{B}\mathcal{V}$  (since there may be more than one asymptotic pair associated with  $T$ ).

Let  $\rho : [0, 1] \rightarrow [0, 1]$  be given by

$$\rho(s) = \begin{cases} (1/\lambda)s, & 0 \leq s \leq 1/2 \\ \frac{2\lambda-1}{\lambda}(s-1) + 1, & 1/2 < s < 1 \end{cases}$$

and  $\tilde{H}(T) : \mathcal{B}\mathcal{L} \rightarrow \mathcal{B}\mathcal{L}$  by

$$\tilde{H}(T) = \begin{cases} T_1 - \rho(s)\mathbf{w}, & \text{if there exist a } T_1 \in \pi^{-1}(BV) \cup \mathcal{F} \text{ such that} \\ & T = T_1 - s\mathbf{w} \text{ for some } \mathbf{w} \text{ with } |\mathbf{w}| = \epsilon \text{ and } s \in [0, 1] \\ T & \text{otherwise.} \end{cases}$$

Define  $H$  on  $K$  so that  $H \circ \pi = \pi \circ \tilde{H}$ .

$$\begin{array}{ccc} \Omega_\Phi & \xrightarrow{\tilde{H}} & \Omega_\Phi \\ \pi \downarrow & & \pi \downarrow \\ K & \xrightarrow{H} & K \end{array}$$

Define  $\bar{G} = \tilde{H} \circ \Phi$  and  $\bar{g} = H \circ F$ . If  $\pi(T)$  is not close to  $BV \cup \mathcal{F}$  then  $\tilde{H}(T) = T$ , and we have  $\pi \circ \tilde{H} \circ \Phi(T) = \pi \circ \Phi(T) = F \circ \pi(T)$ , since  $F \circ \pi = \pi \circ \Phi$ . So  $\pi \circ \bar{G} = \bar{g} \circ \pi$ . Now if there exist a  $T_1 \in \pi^{-1}(BV) \cup \mathcal{F}$  such that  $T = T_1 - s\mathbf{w}$  then  $\pi \circ \tilde{H} \circ \Phi(T) = \pi(T_1 - \rho(s)\mathbf{w})$ . Also,  $\bar{g} \circ \pi(T) = \bar{g} \circ H \circ F(T) = H(\pi(T_1 - s\mathbf{w})) = \pi \circ \tilde{H}(T_1 - s\mathbf{w}) = \pi(T_1 - \rho(s)\mathbf{w})$ . So, again,  $\pi \circ \bar{G} = \bar{g} \circ \pi$ .

For any  $T \in \mathcal{NB}\mathcal{L}$  there is a  $u \in \mathbb{R}^+$ , vector  $\mathbf{v}$ , and tiling  $\bar{T} \in \mathcal{AP}$  such that  $T = \bar{T} - \mathbf{v}$ . Define  $G(T) = G(\bar{T} - u\mathbf{v}) := \bar{G}(\bar{T}) - \mathbf{v}$ .

**Proposition 3.0.58.**  $G : \mathcal{NB}\mathcal{L} \rightarrow \mathcal{NB}\mathcal{L}$  is a homeomorphism.

**Proof:** Since  $\Phi$  and  $\tilde{H}$  are homeomorphisms on  $\mathcal{BL}$  it suffices to show  $G$  is well-defined. The only problems may occur near  $\mathcal{BV} \cup \mathcal{F}$ . Assume there exists  $T_0 \in \mathcal{BV}$ ,  $T_1, T_2 \in \mathcal{BL}$ ,  $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$  such that  $T_1 - \mathbf{v}_1 = T = T_2 - \mathbf{v}_2$  and  $T_1 = T_0 - s_1\mathbf{w}$ ,  $T_2 = T_0 - s_2\mathbf{w}$ , with  $|s_1|, |s_2| < \delta$ ,  $|\mathbf{w}| < \epsilon$ .

Then, there exists a  $\bar{T}_0 \in \mathcal{BV} \cup \mathcal{F}$  such that  $\Phi(T_0) = \bar{T}_0$ . Now  $\Phi(T_1) = \Phi(T_0 - s_1\mathbf{w}) = \bar{T}_0 - \lambda s_1\mathbf{w}$  and  $\Phi(T_2) = \Phi(T_0 - s_2\mathbf{w}) = \bar{T}_0 - \lambda s_2\mathbf{w}$ . By the choice of  $\delta$ ,  $\lambda s_i < 1/2$ ,  $i = 1, 2$ . Applying  $\tilde{H}$  gives

$$\bar{G}(T_1) = \tilde{H}(\Phi(T_1)) = \tilde{H}(\bar{T}_0 - \lambda s_1\mathbf{w}) = \bar{T}_0 - s_1\mathbf{w}$$

$$\bar{G}(T_2) = \tilde{H}(\Phi(T_2)) = \tilde{H}(\bar{T}_0 - \lambda s_2\mathbf{w}) = \bar{T}_0 - s_2\mathbf{w}$$

Thus  $G(T_1 - \mathbf{v}_1) = \bar{G}(T_1) - \mathbf{v}_1 = \bar{T}_0 - s_1\mathbf{w} - \mathbf{v}_1 = \bar{T}_0 - s_2\mathbf{w} - \mathbf{v}_2 = \bar{G}(T_2) - \mathbf{v}_2 = G(T_2 - \mathbf{v}_2)$  ■

Define  $g$  on  $NB$  so that  $g \circ \pi = \pi \circ G$ . We claim  $\varprojlim(NB, g)$  is homeomorphic to  $\mathcal{NB}\mathcal{L}$ . Define  $\hat{\pi} : \mathcal{NB}\mathcal{L} \rightarrow \Xi_{NB}$  by  $\hat{\pi}(T) = x = \{x_i\}_{i=0}^\infty$ , where  $x_i = \pi(\Phi^{-i}(T))$ .

$$\begin{array}{ccccc} \mathcal{NB}\mathcal{L} & \dots & \xrightarrow{G} & \mathcal{NB}\mathcal{L} & \xrightarrow{G} & \mathcal{NB}\mathcal{L} \\ \hat{\pi} \downarrow & & & \pi \downarrow & & \pi \downarrow \\ \varprojlim(NB, g) \dots & \xrightarrow{g} & NB & \xrightarrow{g} & NB \end{array}$$

$\mathcal{NB}\mathcal{L}$  is closed in  $\Omega_\Phi$  so  $\hat{\pi}$  is a surjection. To show that  $\hat{\pi}$  is injective let  $\hat{\pi}(T_1) = \hat{\pi}(T_2)$ . Then for all  $i$ ,  $\pi(\Phi^{-i}(T_1)) = \pi(\Phi^{-i}(T_2))$ . So for any  $R > 0$ , there is a  $i$  such that  $B_R[\Phi^i(T_1)] = B_R[\Phi^i(T_2)]$ . Since  $R$  is arbitrary,  $T_1 = T_2$ , and we have  $\hat{\pi}$  injective. It is clear that  $\hat{\pi}$  is continuous with continuous inverse. So we have the following

**Proposition 3.0.59.**  $\mathcal{NB}\mathcal{L} \simeq \varprojlim(NB, g)$

Now define an equivalence relation on  $NB$  that mimics the identifications found in  $\mathcal{NB}\mathcal{L}/\sim$ . For  $x, y \in NB$  we define  $x \sim_{NB} y$  if there exist tilings  $T, T' \in \mathcal{NB}\mathcal{L}$ , such that  $\pi(T) = x$ ,  $\pi(T') = y$ , and  $T \sim T'$ . For  $\bar{x} = (x_i)_{i=1}^\infty$ ,  $\bar{x}' = (x'_i)_{i=1}^\infty \in \varprojlim(NB, g)$ , define  $\sim_{\varprojlim NB}$  on  $\varprojlim(NB, g)$  by  $\bar{x} \sim_{\varprojlim NB} \bar{x}'$  if  $x_i \sim_{NB} x'_i$  for all  $i$ .

**Theorem 3.0.60.**  $\mathcal{A}_\Phi \simeq \varprojlim(NB/\sim_{NB}, g)$

**Proof:** We have that  $\mathcal{NB}\mathcal{L} \simeq \varprojlim(NB, g)$  and  $(\mathcal{NB}\mathcal{L}/\sim) \simeq \mathcal{A}_\Phi$ . Also,  $\varprojlim(NB/\sim_{NB}, g) \simeq \varprojlim(NB, g)/\sim_{\varprojlim NB}$ . Thus  $\mathcal{NB}\mathcal{L}/\sim \simeq \varprojlim(NB/\sim_{NB}, g)$ . ■

### A Topological Invariant

Suppose  $\Phi$  and  $\Psi$  are primitive, nonperiodic substitutions and that  $h_0 : \Omega_\Phi \rightarrow \Omega_\Psi$  is a homeomorphism of the corresponding tiling spaces. Define  $\mathcal{BL}_\Phi$  and  $\mathcal{BL}_\Psi$  to be the corresponding Branch Loci. The following is obtained by adapting a pair of results of [JK].

**Theorem 3.0.61.** *[B] Suppose that  $\Omega_\Phi$  and  $\Omega_\Psi$  are homeomorphic 2-dimensional self-similar substitution tiling spaces. Suppose further that either  $\Phi$  has a point or corner asymptotic pair or that  $\lambda_\Phi$  is a Pisot number. There are then  $n, m \in \mathbb{N}$ , a linear isomorphism  $L$ , and a homeomorphism  $h : \Omega_\Phi \rightarrow \Omega_\Psi$  such that  $h \circ \Phi^n = \Psi^m \circ h$ , and  $h(T - x) = h(T) - Lx$  for all  $T \in \Omega_\Phi$  and all  $x \in \mathbb{R}^2$ .*

**Corollary 3.0.62.** *Let  $\Omega_\Phi$  and  $\Omega_\Psi$  be as in the preceding theorem. Then there is a homeomorphism  $h : \Omega_\Phi \rightarrow \Omega_\Psi$  and a linear isomorphism  $L$  so that if  $(T, T')$  is an asymptotic point, corner, or line pair for  $\Phi$  then  $(h(T), h(T'))$  is an asymptotic point, corner, resp, line pair for  $\Psi$ . Moreover, if  $\alpha(T, T')$  is the arc of asymptotic directions*

for an asymptotic pair  $(T, T')$  for  $\Phi$ , then  $\{L\mathbf{y}/|L\mathbf{y}| : \mathbf{y} \in \alpha(T, T')\}$  is the arc of asymptotic directions for  $(h(T), h(T'))$ .

**Proof:** Let  $(T, T') \in \mathcal{AP}$ . We claim  $(h(T), h(T'))$  is a asymptotic pair in  $\Omega_\Psi$ . It follows immediately from  $h \circ \Phi^n = \Psi^m \circ h$  that  $h(T)$ , and  $h(T')$  are periodic under  $\Phi$ . Since  $(T, T')$  is an asymptotic pair there is a vector  $\mathbf{v}$  such that  $T - \mathbf{y} \in W^s(T - \mathbf{y})$ , for all  $\mathbf{y}$  such that  $\langle \mathbf{y}, \mathbf{v} \rangle > 0$ . Applying  $h$ , we have  $h(T) - \mathbf{y} \in W^s(h(T) - \mathbf{y})$  for any vector  $\mathbf{y}$  such that  $\langle \mathbf{y}, L\mathbf{v} \rangle > 0$ . Regarding  $\Omega_\Psi$ , let  $R$  be given by Proposition 3.0.40. So,  $B_0[(h(T) - \mathbf{y})] = B_0[(h(T') - \mathbf{y})]$ , for all  $\mathbf{y}$  such that  $\langle \mathbf{y}, L\mathbf{v} \rangle > R$ . Then  $h(T) - u\mathbf{y}$  and  $h(T') - u\mathbf{y}$  agree on arbitrary large balls as  $u \rightarrow \infty$ . It follows that  $d(h(T), h(T')) \rightarrow 0$  as  $u \rightarrow \infty$ . So  $(h(T), h(T'))$  are an asymptotic pair.

It remains to be shown that  $\{L\mathbf{y}/|L\mathbf{y}| : \mathbf{y} \in \alpha(T, T')\}$  is the arc of asymptotic directions for  $(h(T), h(T'))$ . To this end let  $\mathbf{y} \in \alpha(T, T')$ . Then  $T - u\mathbf{y} \in W^s(T - u\mathbf{y})$  for all  $u \in \mathbb{R}^+$ . Since  $h \circ \Phi^n = \Psi^m \circ h$ , we have that

$$\begin{aligned} h(T - u\mathbf{y}) &\in W^s(h(T) - u\mathbf{y}), \\ h(T) - L(u\mathbf{y}) &\in W^s(h(T) - L(u\mathbf{y})), \\ h(T) - uL\mathbf{y} &\in W^s(h(T) - uL\mathbf{y}), \end{aligned}$$

for all  $u \in \mathbb{R}^+$ . Then, by definition,  $\frac{L\mathbf{y}}{|L\mathbf{y}|} \in \alpha(h(T), h(T'))$ . ■

**Remark 3.0.63.** *The fact that  $L$  is linear forces line pairs to map to line pairs. Let  $\mathbf{y}_1 \in \partial\alpha(T, T')_{\{\mathbf{v}\}}$  so that  $\mathbf{y}_2 = -\mathbf{y}_1$ , then both  $L\mathbf{y}_1$ , and  $-L\mathbf{y}_1 \in \partial\alpha(h(T), h(T'))_{\{L\mathbf{v}\}}$ .*

**Theorem 3.0.64.** *Suppose that  $\Omega_\Phi$  and  $\Omega_\Psi$  are homeomorphic 2-dimensional self-similar substitution tiling spaces and either  $\Phi$  has a point or corner asymptotic pair or that  $\lambda_\Phi$  is a Pisot number. There exists a homeomorphism  $h : \Omega_\Phi \rightarrow \Omega_\Psi$  such that  $h(\mathcal{A}_\Phi) = \mathcal{A}_\Psi$ .*

**Proof:** By Corollary 3.0.62 there exists a  $h : \Omega_\Phi \rightarrow \Omega_\Psi$  such that  $h \circ \Phi^n = \Psi^m \circ h$ , coupled with a linear isomorphism  $L$  such that  $h(T - \mathbf{x}) = h(T) - L\mathbf{x}$  for all  $T \in \Omega_\Phi$  and all  $\mathbf{x} \in \mathbb{R}^2$ . Further, if  $(T, T')$  is an asymptotic point, corner, or line pair for  $\Phi$  then  $(h(T), h(T'))$  is an asymptotic point, corner, resp, line pair for  $\Psi$ . So  $h$  takes asymptotic pairs to asymptotic pairs. We need to show branch lines go to branch lines.

Let  $(T, T') \in \mathcal{AP}$ ,  $\mathbf{y} \in \partial\alpha(T, T')$ ,  $u \in \mathbb{R}$ . For any  $T - u\mathbf{y} \in \mathcal{L}_{(T, T')}$  we have  $h(T - u\mathbf{y}) = h(T) - L(u\mathbf{y}) = h(T) - u(L\mathbf{y})$  and  $\frac{uL\mathbf{y}}{|uL\mathbf{y}|} \in \partial\alpha(h(T), h(T'))$ . So  $h(T - u\mathbf{y}) \in \mathcal{L}_{(h(T), h(T'))}$ .

Now if  $(T, T')$  is a non-collapsing asymptotic pair then for all  $u \in \mathbb{R}$ ,  $T - u\mathbf{y} \notin W^s(T' - u\mathbf{y})$ . Since  $h$  is a conjugacy between  $\Phi$  and  $\Psi$  we have  $h(T - u\mathbf{y}) \notin W^s(h(T' - u\mathbf{y}))$ . Thus the asymptotic pair  $(h(T), h(T'))$  is non-collapsing.

Take  $(T, T')_{\{\mathbf{v}\}} \in \mathcal{AP}$ . Since  $h$  is a homeomorphism

$$\begin{aligned} h(\mathcal{L}_{(T, T')}) &= h\left(\text{cl}\left(\bigcup_{u \in \mathbb{R}} \{T - u\mathbf{v}^\perp, T' - u\mathbf{v}^\perp\}\right)\right) \\ &= \text{cl}\left(h\left(\bigcup_{u \in \mathbb{R}} \{T - u\mathbf{v}^\perp, T' - u\mathbf{v}^\perp\}\right)\right) \\ &= \text{cl}\left(\bigcup_{u \in \mathbb{R}} \{h(T) - u(L\mathbf{v}^\perp), h(T') - u(L\mathbf{v}^\perp)\}\right) \\ &= \mathcal{L}_{(h(T), h(T'))} \end{aligned}$$

In the same manner,  $h^{-1}$  takes non-collapsing asymptotic pairs to non-collapsing asymptotic pairs. Therefore, the Branch Locus is a topological invariant.

For any  $\bar{T} \in \mathcal{NBL}$ , there is an asymptotic pair  $(T, T') \in \mathcal{AP}$ , vector  $\mathbf{v}$ , and scalar  $u \in \mathbb{R}$  such that  $\bar{T} = T - u\mathbf{v}$  and  $\mathbf{v}^\perp \in \partial\alpha(T, T')$ . Now  $h(\bar{T}) = h(T) - u(L\mathbf{v})$ . However,  $L$  need not be conformal, so  $L\mathbf{v}$  may not be perpendicular to  $L\mathbf{v}^\perp$ . Let  $\mathbf{w}$  be a unit vector such that:  $\mathbf{w}^\perp \in \partial\alpha(h(T), h(T'))$  and  $\langle \mathbf{w}, L\mathbf{v} \rangle > 0$ . Then define

$H : \mathcal{NB}\mathcal{L}_\Phi \rightarrow \mathcal{NB}\mathcal{L}_\Psi$  by  $H(\bar{T}) = h(T) - u\mathbf{w}$ . Since  $h$  is a homeomorphism it follows that  $H$  is a homeomorphism. Moreover,  $H$  respects the equivalence relation  $\sim$  on  $\mathcal{NB}\mathcal{L}$ . To see this let  $T \sim T'$ . Associated with  $T$  and  $T'$  is an asymptotic pair  $(\bar{T}, \bar{T}')$ . There is a vector  $\mathbf{v}$  such that  $T = \bar{T} - \mathbf{v}$  and  $T' = \bar{T}' - \mathbf{v}$ . Then  $H(T) = h(\bar{T}) - \mathbf{w}$  and  $H(T') = h(\bar{T}') - \mathbf{w}$ . By definition of  $\sim$  on  $\mathcal{NB}\mathcal{L}_\Psi$ ,  $h(\bar{T}) - \mathbf{w} \sim h(\bar{T}') - \mathbf{w}$ . So  $H(T) \sim H(T')$ . ■

## EXAMPLES

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”Few things are harder to put up with than a good example.”

- Mark Twain

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The following examples afford us the opportunity to explain the details of our construction and results. These examples illustrate different characteristics of  $\mathcal{A}_\Phi$  and its relation to the Čech cohomology of the corresponding tiling space. The Half Hex tiling space contains only asymptotic point pairs in the corresponding Branch Locus and the cohomology of these pairs persists into the cohomology of the tiling space. Both the Chair Tiling and Octagonal Tiling contain torsion elements which are not seen in  $\check{H}^2(\Omega)$

In each example we begin by computing the Čech cohomology of the tiling space via a direct limit of homology groups of the Anderson-Putnam complex  $K$ . This complex gives rise to a chain complex

$$0 \longleftarrow C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \longleftarrow 0$$

where  $C_k$  is the  $\mathbb{Z}$ -module generated by the  $k$ -cells of  $K$  and  $\partial_k$  the boundary operator.  $\partial_k$  applied to a generator corresponding to a  $k$ -cell is a signed sum of the generators corresponding to the  $k - 1$  cells which make up its boundary. Each  $C_k$  is torsion free, so the dual chain complex is given by

$$0 \longrightarrow C^0 \xrightarrow{\delta_1} C^1 \xrightarrow{\delta_2} C^2 \longrightarrow 0$$

where  $\delta_k = \partial_k^T$ . The cohomology of the complex is then given by

$$H^i(K) = \ker \delta_i / \text{Im} \delta_{i-1}$$

As done earlier, we denote the induced map of inflation and substitution on  $K$  as  $F$ . If the substitution forces the border then  $\Omega_\Phi$  and  $\varprojlim(K, F)$  are homeomorphic. By the continuity of Čech cohomology we have

$$\check{H}^*(\Omega) = \check{H}^*(\varprojlim(K, F)) = \varinjlim(H^*(K), F^*)$$

The inverse limit construction of  $\mathcal{A}_\Phi$  allows us to use Ad-hoc techniques to compute elements of cohomology of asymptotic structure for these examples. Define  $K_{\mathcal{A}_0} = \{B_1[T] : T \in \mathcal{BV}\} / \sim$ . There exists a one-to-one correspondence with tilings in  $\mathcal{BV}$  and connected components of  $NBV$ . So,  $\check{H}^*(\mathcal{BV}) = \check{H}^*(\varprojlim(NBV / \sim, g) = H^*(NBV / \sim) \simeq H^*(K_{\mathcal{A}_0})$ . In some cases we can associate the cohomology of  $\mathcal{A}_\Phi$  with the inverse limit of a complex  $K_{\mathcal{A}}$ , by considering one-patches found in  $\mathcal{BL}$ .

### Half Hex Tiling

The substitution  $\Phi$  is shown in Figure 8.

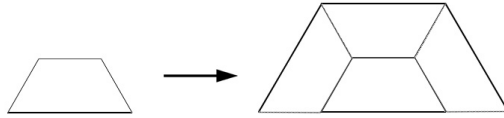


Figure 9: Half Hex Substitution

The tile on the left hand side appears in the tiling with all its rotations around  $\frac{n\pi}{6}$ , on which we extend the substitution by symmetry. A Half Hex tiling has 6 prototiles, all congruent to the half hex polygon. The substitution is primitive, recognizable and forces its border.

After making identifications, the Anderson Putnam complex becomes three tori connected along their one skeleton. The cochain complex is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\delta_1} \mathbb{Z}^3 \xrightarrow{\delta_2} \mathbb{Z}^6 \longrightarrow 0$$

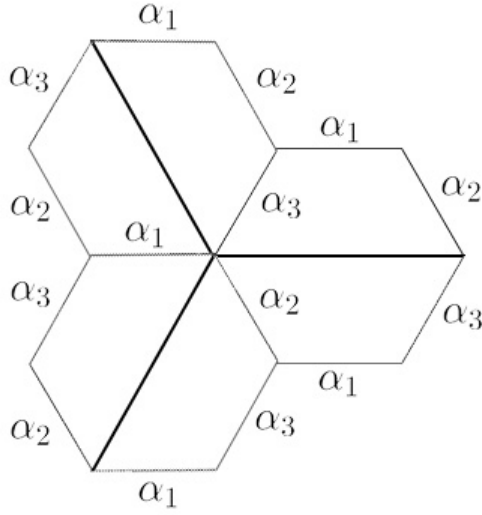


Figure 10: Anderson Putnam complex for the Half Hex tiling

$\delta_1$  has rank 1 and  $\delta_2$  has full rank. So, we have  $H_1(K) \simeq \mathbb{Z}^2$  and  $H_2(K) \simeq \mathbb{Z}^3$ . Inflation and substitution acts by multiplication by 2 on  $H_1(K)$  and is given by the incidence matrix  $\mathbf{A}$  on  $H_2(K)$ .

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

The eigenvalues of  $\mathbf{A}$  are  $\{1, 1, 4\}$ , so there is a two dimensional eigenspace corresponding to the span of the eigenvectors associated with the double root 1, and a one dimensional eigenspace on which everything is being multiplied by 4. Unfortunately, the direct sum of the eigenspaces are not all of  $\mathbb{Z}^3$ . However, it turns out the Čech cohomology of the half hex is isomorphic to this direct sum [Sad]. Thus  $\check{H}^1(\text{Half Hex}) \simeq (\mathbb{Z}[1/2])^2$  and  $\check{H}^2(\text{Half Hex}) = \mathbb{Z}^2 \oplus \mathbb{Z}[1/2]$

There are three asymptotic point pairs in this tiling space, corresponding to the origin placed at the center of the three patches shown in Figure 11.

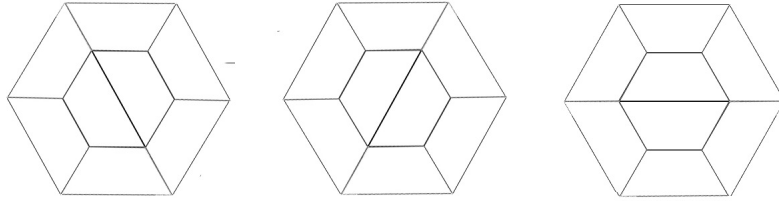


Figure 11: 1-patches of three asymptotic vertex tilings found in the Half Hex tiling space.

As done in Chapter 3, making the appropriate identifications leads to the complex  $K_{\mathcal{A}_0}$ , as shown in Figure 12.

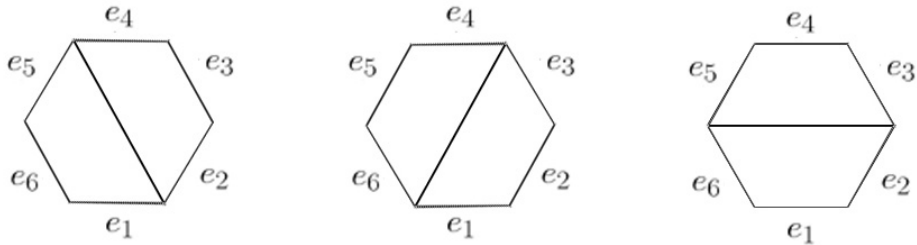


Figure 12: The complex  $K_{\mathcal{A}_0}$  for the Half Hex tiling space.

$K_{\mathcal{A}_0}$  is the wedge of two spheres. So  $H^1(K_{\mathcal{A}_0}) \simeq \emptyset$ ,  $H^2(K_{\mathcal{A}_0}) \simeq \mathbb{Z}^2$ . Moreover, inflation and substitution has determinant one so  $\check{H}^2(\varprojlim(K_{\mathcal{A}_0}), g) \simeq \mathbb{Z}^2$ .

Chair Tiling

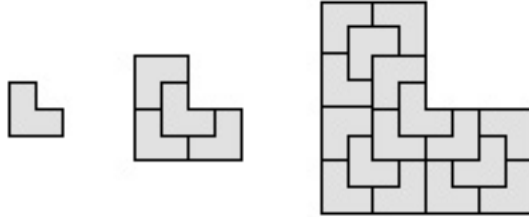


Figure 13: The Chair Substitution

The Chair substitution is shown in Figure 13. It is mutually locally derivable to the arrow substitution shown in figure 14.

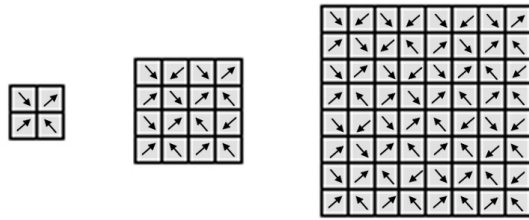


Figure 14: The Chair Arrow Substitution

An Arrow Chair tiling has 4 prototiles, each congruent to a square. The substitution is primitive, recognizable, but does not force its border. However, the cohomology of the inverse limit of the Anderson-Putnam complex is homeomorphic to the cohomology of the tiling space. That is,  $\varinjlim (H^*(K), F^*) \simeq \check{H}^*(\Omega)$ , even though the natural map from  $\Omega$  to  $\varinjlim (K, F)$  is not a homeomorphism [Sad]. This is not a general feature in tiling spaces.

After identifications the Anderson-Putnam complex has 2 vertices, 4 edges, and 4 faces. Which gives an Euler characteristic of 2. The cochain complex is given by

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\delta_1} \mathbb{Z}^4 \xrightarrow{\delta_2} \mathbb{Z}^4 \longrightarrow 0$$

Both maps  $\delta_1$  and  $\delta_2$  have rank 1. Thus

$$H^2(K) \simeq \ker \delta_3 / \text{img } \delta_2 \simeq \mathbb{Z}^3$$

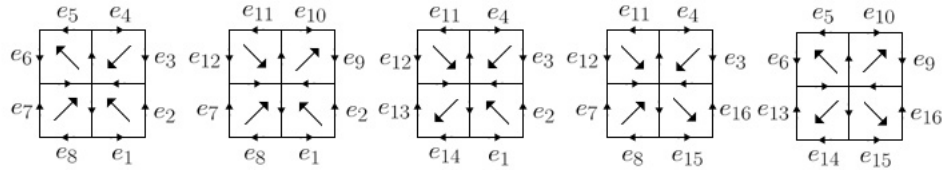
$$H^1(K) \simeq \ker \delta_2 / \text{img } \delta_1 \simeq \mathbb{Z}^2$$

Inflation and substitution is given by a matrix  $\mathbf{A}_1$  on  $H_1(K)$  with eigenvalues  $\{2, 2, 0, 0\}$  and  $\mathbf{A}_2$  on  $H_2(K)$  with eigenvalues  $\{2, 2, 4, 0\}$ . In general, the eigenvalues do not determine the direct limit. However, one can show  $\check{H}^*(Chair)$  and  $\varinjlim \{H^*(K), \mathbf{A}_*\}$  are isomorphic. So

$$\check{H}^2(Chair) = \varinjlim \{H^2(K), \mathbf{A}_2\} = (\mathbb{Z}[1/2])^2$$

$$\check{H}^1(Chair) = \varinjlim \{H^1(K), \mathbf{A}_1\} = H^1(K) = (\mathbb{Z}[1/2])^2 \oplus \mathbb{Z}[1/4]$$

The one dimensional cohomology comes from the the substitution, in which there are two sides, one horizontal, one vertical which are being stretch by 2. The  $\mathbb{Z}[1/4]$  term in  $H^2$  results from a prototile being divided into four additional prototiles. The other terms can be attributed to the asymptotic structure in the tiling space. To see this we compute  $\check{H}^*(\mathcal{A}_\Phi)$ .



There are five patches that carry the cohomology of  $K_{\mathcal{A}_0}$ , shown in figure 4. After identifications based on the asymptotic behavior of the corresponding tilings, we have 12 vertices, 16 edges, and 5 faces.



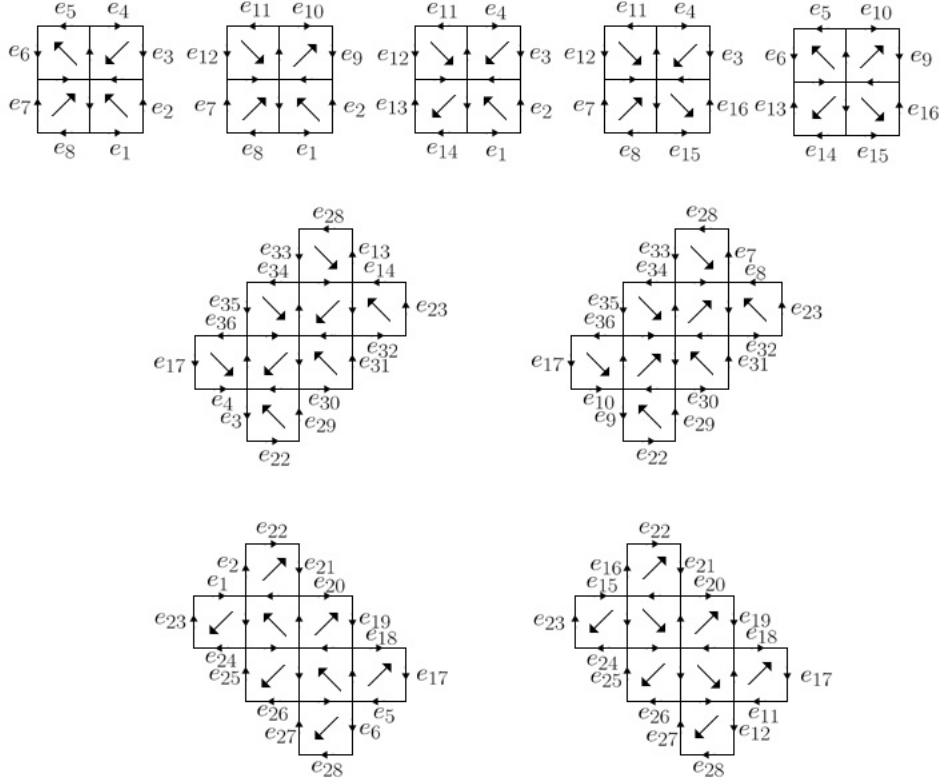


Figure 15:  $K_{\mathcal{A}}$  complex for the Chair Tiling

$$0 \longleftarrow \mathbb{Z}^{28} \xleftarrow{\partial_1} \mathbb{Z}^{36} \xleftarrow{\partial_2} \mathbb{Z}^9 \longleftarrow 0$$

$\partial_1$  has rank 27 while  $\partial_2$  has full rank. The matrices can be found in Appendix B.

Using the Smith Normal Form of  $\partial_2$  we have

$$H^2(K_{\mathcal{A}}) \simeq \mathbb{Z}^2 \oplus \mathbb{Z}_3,$$

$$H^1(K_{\mathcal{A}}) \simeq \mathbb{Z}^2.$$



The substitution is show in Figure 4. The two triangles and the rhombus appear in the tiling along with all  $\frac{\pi}{4}$  rotations. Take  $r^n := \frac{n\pi}{4}$ ,  $n = 1, 2, \dots, 7$ . An Octagonal tiling has 20 prototiles: 4 of them congruent to the rhombus, the remaining 16 congruent to the triangle.<sup>1</sup> The substitution is primitive, recognizable and forces its border.

After identifications the Anderson Putnam complex has 1 vertex, 16 edges, and 20 faces, see Figure 4. The cochain complex is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\delta_1} \mathbb{Z}^{16} \xrightarrow{\delta_2} \mathbb{Z}^{20} \longrightarrow 0$$

Since there is only one vertex the boundary map  $\partial_1 : C_1 \rightarrow C_0$  is the zero map. The coboundary map  $\delta_1 : C^2 \rightarrow C^1$  is the  $20 \times 16$  matrix shown in the Appendix B, which has rank eleven. So we have

$$H^2(K) \simeq \ker \delta_2 / \text{img } \delta_1 \simeq \mathbb{Z}^{20} / \mathbb{Z}^{11} \simeq \mathbb{Z}^9$$

$$H^1(K) \simeq \ker \delta_1 / \text{img } \delta_0 \simeq \ker \delta_1 \simeq \mathbb{Z}^5$$

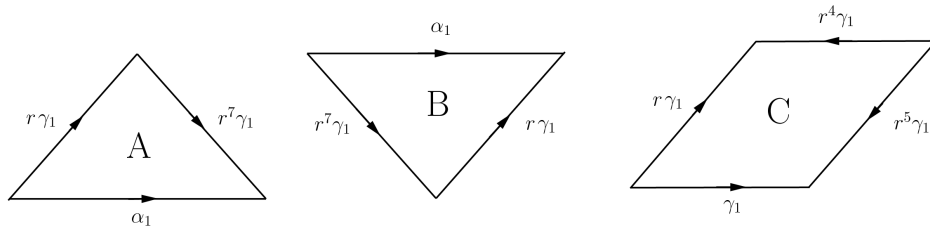


Figure 17: Anderson Putnam complex for the Octagonal tiling.

Denote inflation and substitution on the complex by  $\mathbf{A}_2 : C^2 \rightarrow C^2$  and  $\mathbf{A}_1 : C^1 \rightarrow C^1$ . Both have determinant one so the direct limit does not change the cohomology.

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<sup>1</sup>Decorating the rhombus breaks its 4-fold symmetry. This creates the famous Ammann-Beenker Tilings. Also, the new substitution on the decorated tiles no longer forces the border.

Therefore the Čech cohomology of the tiling space ( $OCT$ ) is:

$$\check{H}^2(Oct) = \lim\{H^2(K), \mathbf{A}_2^*\} = H^2(K) = \mathbb{Z}^9$$

$$\check{H}^1(Oct) = \lim\{H^1(K), \mathbf{A}_1^*\} = H^1(K) = \mathbb{Z}^5$$

The generators of  $\check{H}^2$  may be assigned to the eight decorated "squares" and the congruence class under rotation of the rhombus [Kell].

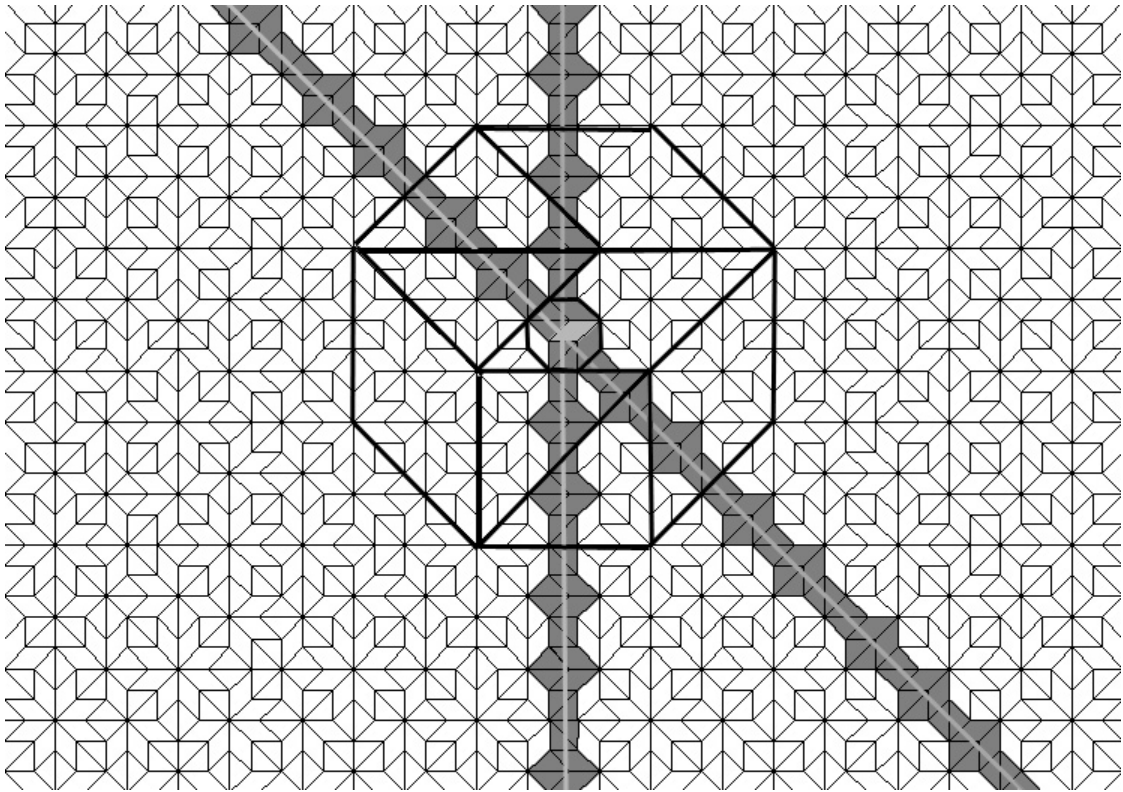


Figure 18: A branch vertex tiling found in the Octagonal Tiling

Figure 17 shows a branch vertex tiling in the tiling space which is asymptotic to two distinct tilings in distinct half planes. The boundary of the sector of asymptoticity is shown by the grey lines and the zero patches along these lines have been shaded in. The tiling is fixed under  $\Phi^2$ . Also, the rhombus tile containing the origin has been collared, (small octagon) and the image of the collar is shown (large octagon).

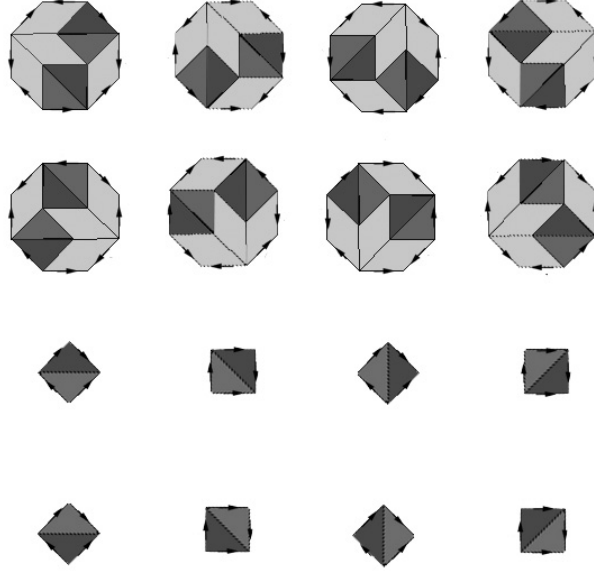


Figure 19: branch vertex tiles found in the Octagonal Tiling

The set  $\{B_1[T] : T \in \mathcal{BL}\}$  is shown in Figure 19. The identifications form the complex  $K_{\mathcal{A}_0}$ , which has three connected components; each corresponding to zero patches that are either squares, diamonds, or octagons.

After identifications the square and diamond components has 8 vertices, 16 edges and 8 faces. The cochain group is given by

$$0 \longrightarrow \mathbb{Z}^8 \xrightarrow{\delta_1} \mathbb{Z}^{16} \xrightarrow{\delta_2} \mathbb{Z}^8 \longrightarrow 0.$$

The boundary maps are given by the matrices:

$$\delta_1 = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\delta_2 = \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Both  $\delta_1$  and  $\delta_2$  have rank 6. So,

$$H^0(\text{Squares}) \simeq \mathbb{Z}^2, \quad H^1(\text{Squares}) \simeq \mathbb{Z}^4 \quad H^2(\text{Squares}) \simeq \mathbb{Z}^2$$

After identifications the octagonal component has 1 vertices, 16 edges and 8 faces.

Notice  $\delta_1$  is the zero map.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\delta_1 \equiv 0} \mathbb{Z}^{16} \xrightarrow{\delta_2} \mathbb{Z}^8 \longrightarrow 0$$

$$\delta_2 = \begin{pmatrix} 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 0 & 1 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ -1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & 1 & 1 & -1 & -1 \end{pmatrix}$$

$\delta_2$  has rank 8 and its Smith Normal Form is shown in Appendix B. From these maps we have the following cohomology groups.

$$H^0(\text{Octagons}) \simeq \mathbb{Z}, \quad H^1(\text{Octagons}) \simeq \mathbb{Z}^8 \quad H^2(\text{Octagons}) \simeq (\mathbb{Z}_2)^2 \oplus \mathbb{Z}_4$$

These patches are fixed under inflation and substitution, so the direct limit does not change the cohomology groups. Therefore,

$$H^0(K_{\mathcal{A}_0}) \simeq \mathbb{Z}^3, \quad H^1(K_{\mathcal{A}_0}) \simeq \mathbb{Z}^{12} \quad H^2(K_{\mathcal{A}_0}) \simeq \mathbb{Z}^2 \oplus (\mathbb{Z}_2)^2 \oplus \mathbb{Z}_4$$

## CONCLUSION

In this dissertation we developed methods to isolate the asymptotic structure found in tiling spaces. This represents a substantial step in our understanding of the inhomogeneities found in tiling spaces. In our efforts to generalize [BDS], we restrict our attention to periodic tilings under  $\Phi$  that are asymptotic in at least a half plane. This criteria leads to a well defined inverse limit representation of the asymptotic structure where cohomology can be calculated. Our primary tool was the relationship between the stable manifold of inflation and substitution with translation asymptotic behavior. Using results of [JK], we showed  $\mathcal{A}_\Phi$  is a topological invariant for self-similar tiling spaces.

The examples illustrate how this asymptotic structure can contribute to the cohomology of the tiling space. Using Ad-Hoc techniques we calculated the Čech cohomology of elements  $\mathcal{A}_\Phi$ . Further, torsion was found in the vertex structure of the projection of the branch locus onto a collared Anderson-Putnam complex.

We remain interested in the relation  $\mathcal{A}_\Phi$  has with  $\Omega_\Phi$ . The connections, if any, between the Čech cohomology groups remains to be investigated. We would like to find sufficient conditions for which torsion elements of  $\check{H}^*(\mathcal{A}_\Phi)$  would persist into  $\check{H}^*(\Omega_\Phi)$ .

## NOTATION INDEX

$B_r(0)$	<i>ball of radius <math>r</math> centered at zero</i>	$B_r[T]$	<i><math>r</math>-patch of a tiling <math>T</math></i>
$\mathcal{P}$	<i>set of prototiles</i>	$\mathcal{P}^*$	<i>set of allowed patches</i>
$\Phi$	<i>inflation and substitution</i>	$\Omega_\Phi$	<i>substitution tiling space</i>
$\mathbf{A}$	<i>incidence matrix of <math>\Phi</math></i>	$\lambda$	<i>expansion constant of <math>\Phi</math></i>
$(T, T')$	<i>asymptotic pair</i>	$(T, T')_{\{0\}}$	<i>asymptotic point pair</i>
$(T, T')_{\{\mathbf{v}\}}$	<i>asymptotic line pair</i>	$(T, T')_{\{\mathbf{v}_1, \mathbf{v}_2\}}$	<i>asymptotic corner pair</i>
$\mathcal{AP}$	<i>set of asymptotic pairs</i>	$S_{(T, T')}$	<i>sector of asymptoticity</i>
$\mathcal{L}_{(T, T')}$	<i>Branch Line</i>	$K$	<i>Anderson-Putnam Complex</i>
$K^1$	<i>One-skeleton of <math>K</math></i>	$K^0$	<i>Zero-skeleton of <math>K</math></i>
$\mathcal{BL}$	<i>Branch locus</i>	$\mathcal{NBL}$	<i>neighborhood of <math>\mathcal{BL}</math></i>
$B$	<i>projection of <math>\mathcal{BL}</math></i>	$NB$	<i>neighborhood of <math>B</math></i>
$BV$	<i>projection of <math>\mathcal{BV}</math></i>	$NBV$	<i>neighborhood of <math>BV</math></i>
$\Xi$	$\lim_{\leftarrow}(K, F)$	$\Xi_{NB}$	$\lim_{\leftarrow}(NB, g)$

APPENDICES

APPENDIX A

COMPUTABILITY OF HOMOLOGY GROUPS USING THE SMITH NORMAL  
FORM OF A MATRIX



**Proof:** To simplify matters we illustrate the idea for  $2 \times 2$  matrices. Start with a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Let  $e = \gcd(a, c)$ , and write  $e = ax + cy$  for some  $x, y \in \mathbb{Z}$ . Write  $a = e\alpha$  and  $c = e\beta$  for some  $\alpha, \beta \in \mathbb{Z}$ . Then  $1 = \alpha x + \beta y$ . We have

$$\begin{pmatrix} x & y \\ -\beta & \alpha \end{pmatrix}^{-1} = \begin{pmatrix} a & -y \\ \beta & x \end{pmatrix}$$

Thus, the matrix

$$\begin{pmatrix} x & y \\ -\beta & \alpha \end{pmatrix}$$

is invertible. Moreover,

$$\begin{pmatrix} x & y \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e & bx + dy \\ -a\beta + c\alpha & -b\beta + d\alpha \end{pmatrix}$$

Since  $e$  divides  $-a\beta + c\alpha$ , a row operation then reduces this matrix to one of the form

$$\begin{pmatrix} e & u \\ 0 & v \end{pmatrix}$$

A similar argument, applied to the first row instead of the first column, allows us to multiply on the right by an invertible matrix and obtain a matrix of the form

$$\begin{pmatrix} e_1 & 0 \\ * & * \end{pmatrix}$$

where  $e_1 = \gcd(e, u)$ . Continuing this process, alternating between the first row and the first column, will produce a sequence of elements  $e, e_1, \dots$  such that  $e_1$  divides

$e$ ,  $e_2$  divides  $e_1$ , and so on. In terms of ideals, this says  $(e) \subseteq (e_1) \subseteq \dots$ . Because any increasing sequence of principal ideals stabilizes in a principal ideal domain<sup>1</sup>, we must arrive, in finitely many steps, with a matrix of the form

$$\begin{pmatrix} f & 0 \\ g & h \end{pmatrix} \text{ or } \begin{pmatrix} f & g \\ 0 & h \end{pmatrix}$$

in which  $f$  divides  $g$ . One more row or column operation will then yield a matrix of the form

$$\begin{pmatrix} f & 0 \\ 0 & k \end{pmatrix}$$

Thus, by multiplying on the left and right by invertible matrices, we obtain a diagonal matrix. To get the Smith normal form let  $d = \gcd(f, k)$ . We may write  $d = fx + gy$  for some  $x, y \in \mathbb{Z}$ . Moreover, write  $a = d\alpha$  and  $b = d\beta$  for some  $\alpha, \beta \in \mathbb{Z}$ . We then perform the following row and column operations, yielding

$$\begin{aligned} \begin{pmatrix} f & 0 \\ 0 & k \end{pmatrix} &\rightarrow \begin{pmatrix} f & 0 \\ af & k \end{pmatrix} \rightarrow \begin{pmatrix} f & 0 \\ fx + ky & k \end{pmatrix} \\ &\rightarrow \begin{pmatrix} 0 & -k\alpha \\ d & k \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -k\alpha \\ d & 0 \end{pmatrix} \rightarrow \begin{pmatrix} d & 0 \\ 0 & -k\alpha \end{pmatrix} \end{aligned}$$

a diagonal matrix in Smith Normal form since  $d$  divides  $-k\alpha$ . ■

To get some feel for the importance of this theorem we recall the connection between row and column operations and matrix multiplication. Consider the two types of row or column operations:

1. interchanging two rows or columns

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<sup>1</sup>see [Hug] p137

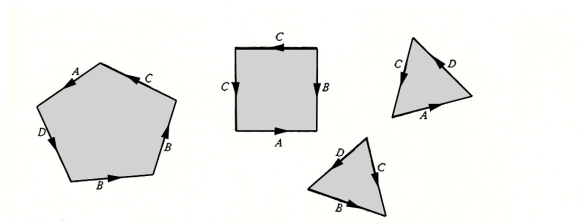
## 2. adding a multiple of one row or column to another

Each of these operations has an inverse operation that undoes the given operation. For example, if we add  $\alpha$  times row  $i$  to row  $j$  to convert a matrix  $A$  to a new matrix  $B$ , then we can undo this by adding  $-\alpha$  times row  $i$  to row  $j$  of  $B$  to recover  $A$ . Notice that we do not include the elementary operation of multiplying a row or column by an element of  $\mathbb{Z}$  since the only invertible elements are  $\pm 1$ . As a consequence of these operations, if we start with a matrix  $A$  and perform a series of row and column operations, the resulting matrix will have the form  $UAV$  for some invertible matrices  $U$  and  $V$ ; the matrix  $U$  will be a product of matrices corresponding to elementary row operations, and  $V$  will correspond to elementary column operations. Moreover,  $U$  will represent a change of basis in the range of  $A$  and  $V$  a change of basis in the domain.

If we write the boundary map  $\delta_i$  in the Smith Normal form such that  $S = U\delta_iV$  then  $U^{-1}$  is a change of basis for the range of  $\delta_1$  and the columns of  $U^{-1}$  give a basis of the image relative to a basis of the kernel of  $\delta_{i-1}$ . Thus the columns of the product  $U^{-1}S$  give the basis elements in the quotient  $\ker\delta_i/\text{img}\delta_{i-1}$ .



**Example A.0.68.**<sup>2</sup> Consider the cohomology groups of the quotient space  $K$  indicated in the following figure.



The identifications imply there exists only one vertex, thus  $C^0 \simeq \mathbb{Z}$ . There are four distinct edges and four polygons, so  $C^1 \simeq C^2 \simeq \mathbb{Z}^4$ . Let  $\alpha, \beta, \gamma$ , and  $\delta$  be a basis for  $C^2$ . Since there is only one vertex the boundary map in homology  $\partial_1 : C_1 \rightarrow C_0$  is the zero map. Thus  $\delta_0 = \partial_1^T \equiv 0$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\delta_0} \mathbb{Z}^4 \xrightarrow{\delta_1} \mathbb{Z}^4 \longrightarrow 0.$$

The coboundary map  $\delta_1$  is found by taking the transpose of the boundary map  $\partial_2 : C_2 \rightarrow C_1$

$$\delta_1 = \partial_2^T = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

This matrix has full rank, therefore the kernel of  $\delta_1$  is trivial.

$$H^1(K) = \ker \delta_1 / \text{img } \delta_0 = \ker \delta_1 \simeq \emptyset$$

by theorem A.0.66,  $\delta_1$  is similar to a matrix in Smith Normal form. That is, there exists invertible, over  $\mathbb{Z}$ , matrices  $U$  and  $V$  such that  $U\delta_1V = S$ .

<sup>2</sup>Example found in Munkres *Elements of Algebraic Topology* p.61

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

and

$$U\delta_1V = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 11 & -1 \end{pmatrix}$$

also

$$U^{-1} = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Therefore  $H^2(K)$  is the direct sum of two cyclic subgroups of order two with generators  $\alpha$  and  $\beta$ .

$$H^2(K) \simeq \mathbb{Z}^4 / (1\mathbb{Z} \oplus 1\mathbb{Z} \oplus 2\mathbb{Z} \oplus 2\mathbb{Z}) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

It is interesting to note that  $H_1(K) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$  and  $H_2(K) \simeq \emptyset$ .

APPENDIX B

MATRICES

$$\begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Octagonal Tiling: Boundary map  $\delta_2 : \mathbb{Z}^{16} \rightarrow \mathbb{Z}^{20}$



$$\mathbf{A}_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}^{16 \times 16}$$

Octagonal Tiling: Inflation and Substitution on  $C^1$



Chair Tiling:  $\partial_1 : \mathbb{Z}^{36} \rightarrow \mathbb{Z}^{28}$





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