



A photoelastic investigation of three-dimensional contact stresses
by Douglas Craig Schafer

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of
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Abstract:

The object of this investigation was to produce an accurate analysis of the contact stress distribution on a three-dimensional body. The case presented in this thesis was a photoelastic study of an elastic body with surface discontinuities, loaded between rigid planes.

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A PHOTOELASTIC INVESTIGATION OF
THREE-DIMENSIONAL CONTACT STRESSES

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DOUGLAS CRAIG SCHAFER

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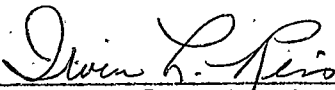
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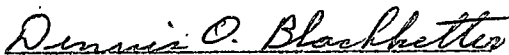
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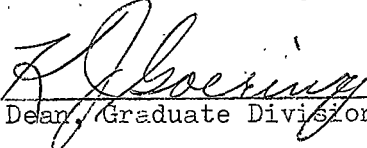
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Abstract

The object of this investigation was to produce an accurate analysis of the contact stress distribution on a three-dimensional body. The case presented in this thesis was a photoelastic study of an elastic body with surface discontinuities, loaded between rigid planes.

A model, shaped similar to a roller bearing, was machined from an epoxy resin and loaded under a constant weight. The "frozen stress" method was then used in an analysis of the strains (stress) in the body and in the contact region in particular. Contact stresses were calculated using a Fortran program of the shear-difference equations and source data obtained from photographs of the photoelastic stress patterns in model slices.

The results of this investigation were compared with theoretical predictions of contact stresses for a two-dimensional body of similar shape. The two-dimensional theory predicted higher contact stresses in the regions of discontinuities than were found actually to occur in the three-dimensional body.

CHAPTER I

INTRODUCTION AND PROBLEM STATEMENT

A rather special problem facing research and design people today, particularly in the bearing industry, is the determination of contact stresses between elastic bodies. Solutions have been found by both theoretical and experimental means for three-dimensional contact stresses (17,5),¹ but these solutions are currently restricted to a small number of configurations. Unfortunately there is not available a general analytical procedure for solving the contact stress problem. The theory of elasticity does provide a system of differential equations that can be solved for three-dimensional stresses, but because of their complexity a solution is usually possible only for very simple shapes.

In most cases an exact analytical solution to the contact stress problem is unavailable, and so efforts to solve any particular contact problem often turn to the methods of experimental stress analysis. Since there is little chance of obtaining an accurate solution to an unsolved configuration by extrapolating available experimental solutions, each new contact problem will usually require a separate investigation. As might be expected, the number of experimentally-solved problems is small, which makes the need for a relatively simple and general analytical method of determining contact stresses obvious.

¹ Numbers in parenthesis refer to the reference list at the end of the paper.

The methods of photoelasticity have proven to be the most satisfactory for experimentally determining the contact stress distribution in a three-dimensional body. The stresses are not usually found for the prototype or body of interest but must be analyzed in a model manufactured out of a special plastic. The stress distribution in the model can be shown to be identical to the stresses in the loaded body, subject to certain restrictions explained by Goodier (10).

This approach of using a model to determine the stresses in another loaded body raises important questions on how the model must be constructed to insure a similarity in the stress distributions. Obvious requirements are that the shape and loading of the model be dimensionally similar to that of the prototype. The exact requirements of model similarity have been developed in Appendix (A) by applying the principles of dimensional analysis; the results are expressed in equations which must be satisfied for the model and prototype to have proportional stress distributions. The proportionality of stress is expressed as

$$\frac{S_m}{S_p} = \frac{E_m}{E_p}$$

Dimensional analysis provides a justification of photoelasticity, (which will be the experimental method used in this paper), and illustrates the required condition model similarity. In general, an experimental solution for a contact stress problem cannot be extended to other configurations. For instance, experimental solutions

for contact stresses between spheres cannot be extended to find the contact stresses between cylinders. What is desired, then, is a general analytical method of finding a solution to the contact stress problem that is convenient and inexpensive.

An analytical technique for two-dimensional elasticity has been developed by Blackketter (1) that frees an analysis of contact stresses from the restraints imposed by the experimental method. Attempts are being made to expand this two-dimensional theory and to develop other analytical tools for accurately predicting three-dimensional contact stress distributions.

For the accuracy of any analytical scheme to be evaluated some reference is needed, some standard for comparison. Though the experimental method of determining stresses by photoelasticity has many shortcomings, it also has the outstanding strength of being quite accurate. Taking advantage of the accuracy of photoelasticity and using the "frozen stress" technique, this paper will present an investigation of contact stress on a three-dimensional body. The results will be used as a reference for an analytical solution using Blackketter's method for the stress distribution on a plane of the three-dimensional body. It is hoped that the comparison might substantiate Blackketter's theory for three-dimensions or at least suggest new approaches to an analytical solution.

CHAPTER II

THREE-DIMENSIONAL STRESS ANALYSIS BY PHOTOELASTICITY

2.1 An Outline of the Frozen Stress Method

In the nineteenth century, Maxwell, who was doing research on torsion, heated an isinglass cylinder and applied a torsional load to it. After permitting the cylinder to cool under load, he found that the strains remained after the load was removed and if the cylinder was placed in polarized light it exhibited a photoelastic effect. This effect was apparently dismissed until recently when M. Hetenyi developed many of the techniques associated with the frozen stress type of analysis.

The procedure of freezing stresses in a body for photoelastic purposes involves heating a plastic model to a certain temperature (referred to as the critical temperature), loading the model at this high temperature, and slowly cooling while maintaining the load. At room temperature the load is removed and the strains are found to be permanently locked or annealed in the model. It is known that these deformations represent an elastic distribution of stress if the yield point of the plastic was not exceeded at the elevated temperature. Slices removed from the model will not significantly disturb the original elastic distribution of stress, and when viewed in a polariscope, produce the same type of information as a two-dimensional photoelastic analysis. The isochromatic fringe patterns represent the difference in maximum and minimum normal stresses in the plane of the slice; these

stresses are referred to as secondary principal stresses and are usually different from the principal stresses. The isochromatics result from a relative retardation of light waves passing through the slice and so are an integral effect closely approximating the stresses on the central plane of the slice. The photoelastic fringe pattern more accurately represents the shearing stress on a central plane of a slice as the slice is made thinner. It is particularly desirable to make the slices as thin as possible in regions of high stress gradient.

2.2 Properties of Plastics

The physical behavior of the plastic used in three-dimensional photoelasticity provides the basis for the experimental procedure. The requirements for a material to be used in a "frozen stress" analysis are that the strains produced in the model under load at the high temperature must remain in the plastic after cooling and all strains must represent an elastic stress distribution. The plastic should also be machinable into thin flat plates without disturbing the original stress distribution, and the strains in the model should duplicate strains in the prototype (model similarity).

The behavior of plastics that exhibit this property of locking in the strains is explained by a diphasic theory. It is assumed that the plastic is structured of a completely polymerized internal skeleton of molecules and of a surrounding amorphous phase. The strength of the molecular skeleton does not change much with tempera-

ture, but at the critical temperature the amorphous phase becomes soft and carries only a minute portion of the total load. When the plastic model in this investigation was heated to its critical temperature of 280°F , the modulus of elasticity dropped from 500,000 psi to 2000 psi.

If the model is slowly cooled from the critical temperature under stress, the soft viscous component will solidify around the primary network holding or locking in the stress and displacements the model underwent when loaded at the high temperature. The cooled model has a stress system in the primary network balanced by stresses in the solidified viscous phase. Because these stresses are in equilibrium on a microscopic scale, sawing of the model will not appreciably disturb the displacements in the plastic. Any planes removed from such a model will exhibit the photoelastic effect, the only difference from the two-dimensional problem being that the birefringence is produced by the secondary principal stresses (principal stresses in the plane of the slice) instead of the principal stresses.

The properties of certain plastics that permit an elastic stress distribution to be frozen into a loaded model and slices to be cut from the model without disturbing the stress equilibrium make a three-dimensional analysis possible. Photoelastic data obtained from certain slices in conjunction with the shear-difference equations (see Chapter 4) make it possible to determine all six components of stress at any point in the body. In the case to be considered in this inves-

tigation, all that is required is one normal component -- the contact stress. Because it is not possible to determine this stress directly in the contact area, i.e., on the loaded surface, a somewhat indirect method as explained in Chapter 4 will be used.

2.3 Requirements of a "Frozen Stress" Analysis

Although the details of a photoelastic analysis will vary from problem to problem, the general procedure usually involves the construction of a model out of plastic, determining the optical properties of the plastic, loading and freezing a deformation into the plastic, slicing, recording the isoclinic and isochromatic fringe patterns, and the reduction of the photoelastic data into meaningful graphs. The basic equipment required for carrying out these steps in a three-dimensional analysis includes a polariscope, an annealing oven with programmed temperature control, loading frame, machine tools, polishing wheels and photographic equipment and darkroom.

CHAPTER III

MODEL CONFIGURATION AND FABRICATION

3.1 Model Configuration

The model configuration of a nearly right circular cylinder with end tapers that was used in this analysis is shown in Figure 1. The selection of a configuration took into account the limitations of the two-dimensional analytical solution that the results of this paper were to be compared with, optical and material properties of the plastic, and the difficulty of accurately machining the model.

Restraints on the two-dimensional theoretical development required that the maximum deviation from contact of the surface of the elastic body be of the order of the magnitude of the maximum strain displacement in the body. Also, the surface-defining equations must be continuous functions of position and their derivatives should be small. These limitations require that the slopes of all points on the surface be small but permit such complications as changes in curvature of the surface and transitions from a plane surface to a curved one.

3.2 Selection of a Plastic

Perhaps the most important performance feature of a photoelastic material is its optical sensitivity; this property is expressed as the amount of shearing stress required to produce a fringe. High optical sensitivity in a plastic is shown by a low fringe contrast. For instance, epoxy resin and plexiglass have material fringe con-

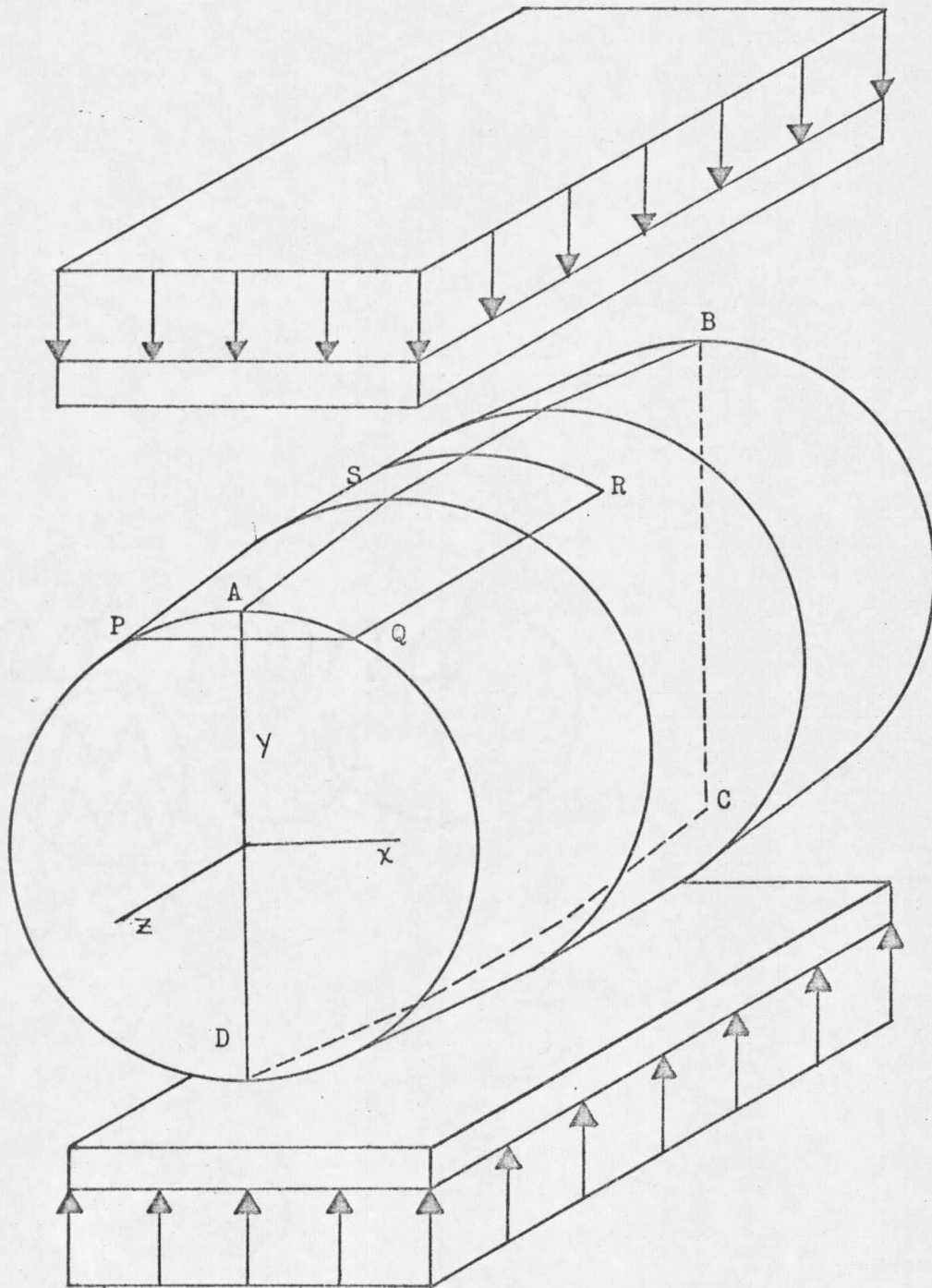


Fig. (1) Model Loading. Contact stresses were calculated by the two-dimensional method of ref. (1) for the central plane ABCD of the model. Plane PQRS is the longitudinal slice removed for the analysis.

stants, f , of 30 psi-in./fringe and 380 psi-in./fringe, respectively, showing that the epoxy is almost 13 times as sensitive to stress as plexiglass. The ratio of the material fringe constant to the modulus of elasticity of the material is a common way of evaluating the desirability of a three-dimensional photoelastic plastic. This ratio gives a term commonly referred to as the figure of merit.

$$Q = E/f$$

$$Q = \text{Figure of Merit}$$

Other qualities of a plastic being equal, the highest figure of merit indicates the most desirable model material.

An epoxy, Hysol 4290, was finally settled upon and ordered from the Hysol Corporation. The material, ordered in a cast cylinder 4 in. in diameter and 36 in. long, satisfied the above requirements in addition to other important qualities of a photoelastic material such as cost, availability, and a minimum of creep and time edge effects. The resin had a figure of merit of about $2000/.603 = 3320$ at the critical temperature, which made it more desirable from the optical standpoint than any other material considered.

3.3 Model Fabrication

The model shape used in this investigation is shown in Figure 2. Ease in machining this model to close tolerances and a straightforward formulation of the analytical computations were deciding factors in choosing this model configuration. The contact

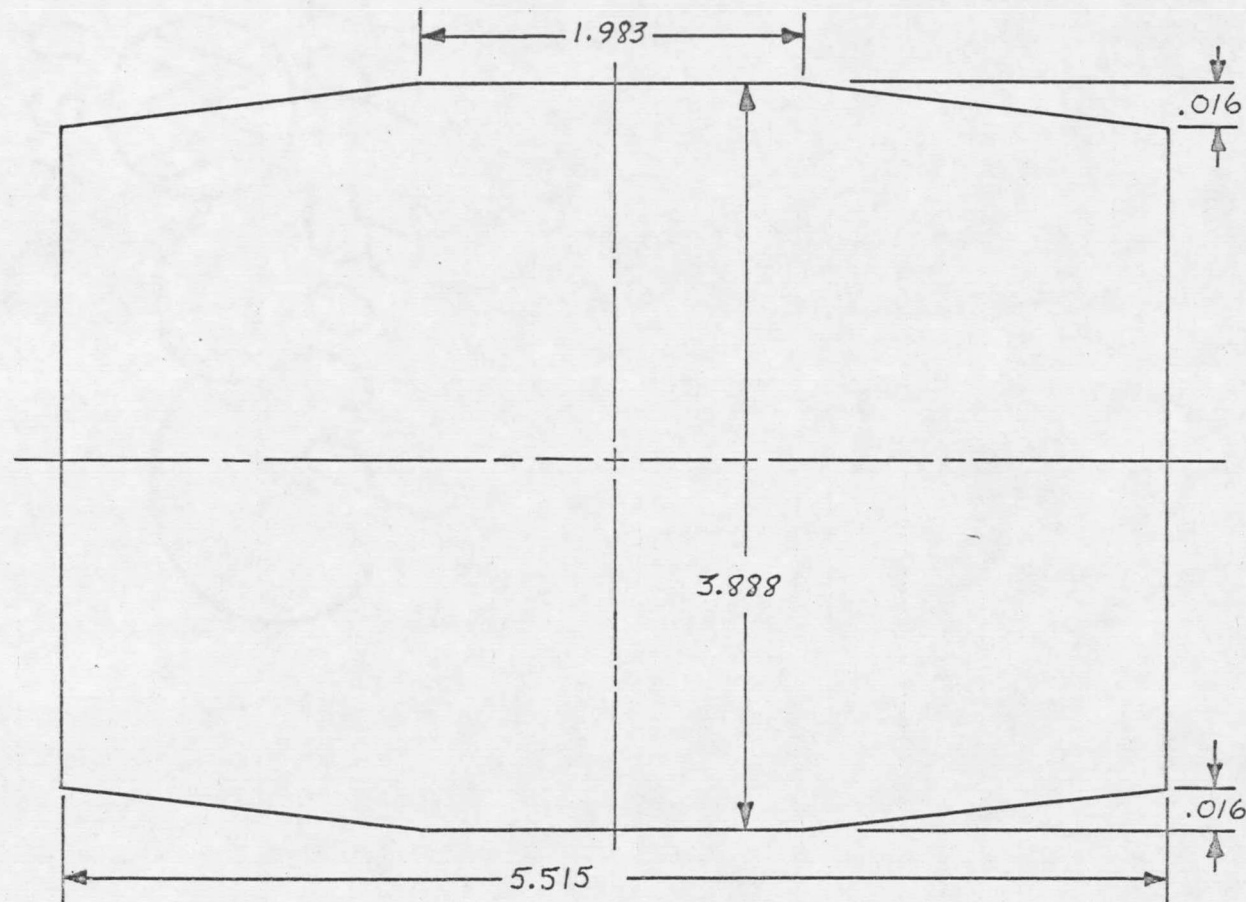


Fig. (2) Model Dimensions. Dimensions of experimental model and the configuration of the two-dimensional contact stress problem solved analytically.

problem was selected to be the tapered cylinder loaded between two rigid planes; this condition was approximated by loading the elastic body in diametral compression between two steel plates.

To obtain the necessary photoelastic information, a model had to be machined, loaded and annealed, and then sliced before the analysis could be made. Using a high cutting speed and an open skip-tooth blade on a bandsaw, a 7-in. length was first cut from the cast cylinder, providing enough material for both a model and the necessary calibration disk. After annealing to remove any residual stresses, the piece was turned to a right circular cylinder. Tapers were then cut on each end with the taper attachment, leaving enough material in the center section for it to be turned down to the design width. This material is highly abrasive, quickly dulling and even burning tool steel cutters. Since the surface finish deteriorated so quickly using tool steel, it was found to be a good practice to use only a new carbide tool reserved for work on these models. A cutting speed of 680 feet per minute was used along with a slow feed of .5 in. per minute to produce an excellent surface finish.

After machining, the actual taper runout and other physical dimensions were measured on an optical comparator that was accurate to within ± 0.0001 in. Because the taper runout was so hard to set accurately and machine on the lathe to the tolerance required, ± 0.0005 in., it was sometimes necessary to machine the surface configuration two or three times before symmetry was achieved.

3.4 Model Loading and Slicing

Loading the model while it was being annealed meant that a special loading frame had to be designed and built. A constant weight loading arrangement was designed and built and appears in Figure (3). Both the loading frame for the model and the calibration disk were fabricated from channel and angle iron. For the model loading frame the upper bar was constrained to move vertically by rollers on each end, horizontal movement was controlled by a set screw through the left vertical support. Ground steel plates were screwed to the channel for flat loading surfaces with pieces of plate glass placed between the steel plates and the model in an attempt to reduce surface friction.

The small calibration frame, seen to the left of the model loading frame in Figure (3), was made specifically for disks to be loaded in diametral compression. Scribe marks on the frame and disk in conjunction with the parallel flat clamped to the rear of the frame insured correct placement of the disk for symmetrical loading.

The model was loaded by placing weights across the upper channel and on the auxiliary loading flat. These weights were positioned prior to starting the annealing or freezing cycle and held off of the model by nuts on the threaded vertical supports. The model was heated slowly to the critical temperature of 280°F at a rate of 10°F per hour so that induced thermal stresses would be kept to a minimum. After the critical temperature had been reached and the model was in

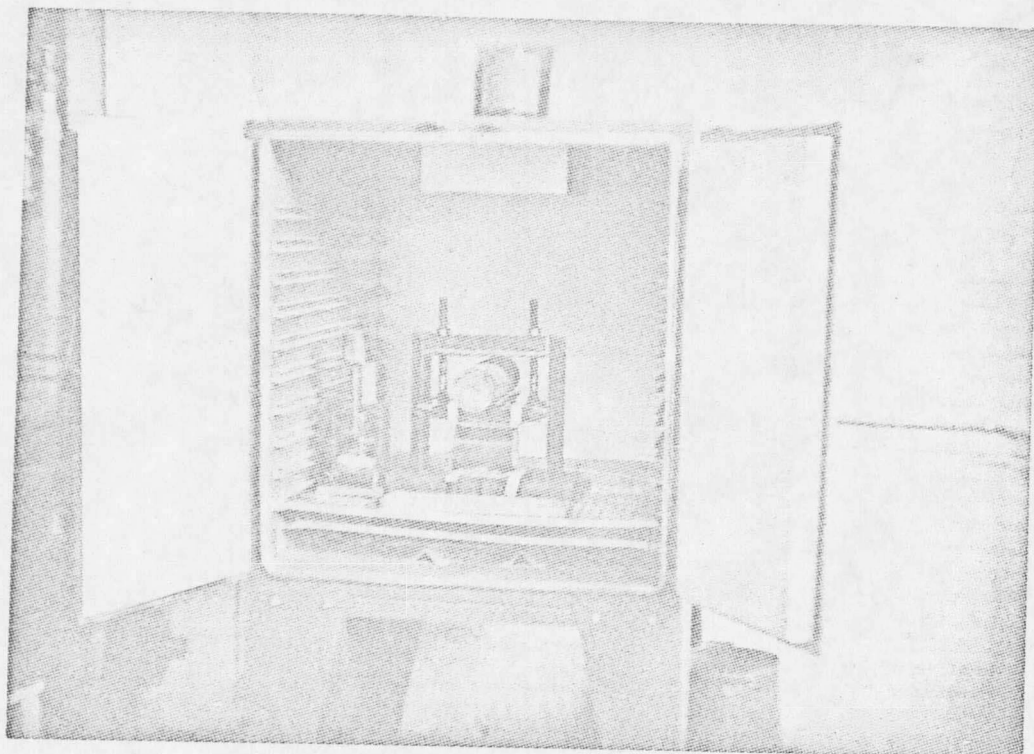


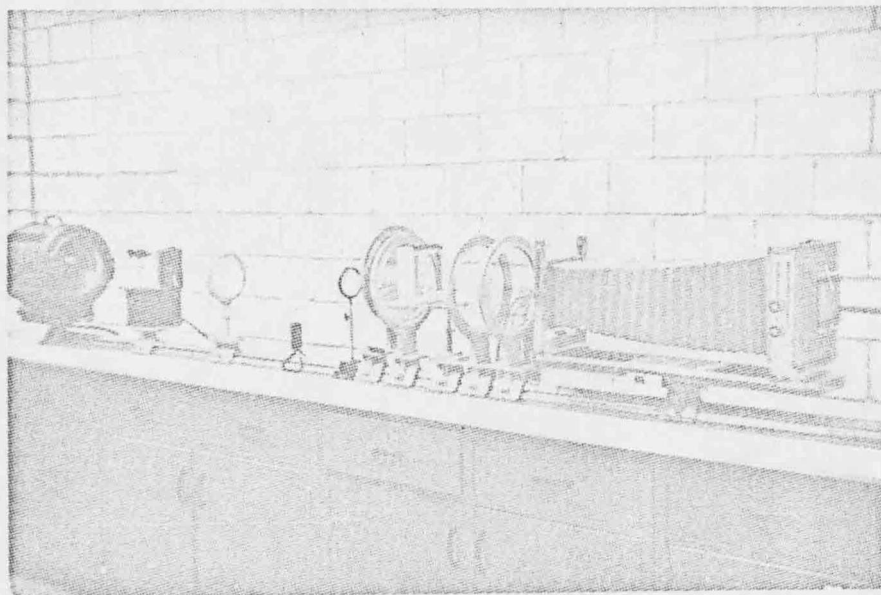
Fig. (3) Loading Frames and
Annealing Oven

thermal equilibrium, the oven was opened and the nuts loosened so that the upper channel rested on the model. Loading of the calibration specimen at the same time was handled by removing a support that kept all weight off of the disk; in this way, both model and disk had the same temperature and loading history. Two hours elapsed at a constant temperature of 280°C before load was applied; then the model was soaked for four hours before cooling below the critical temperature. The model was cooled at a rate of 5°F/hr. to room temperature. This same cycle was used as an annealing cycle to relieve residual stresses from the cylinder before machining.

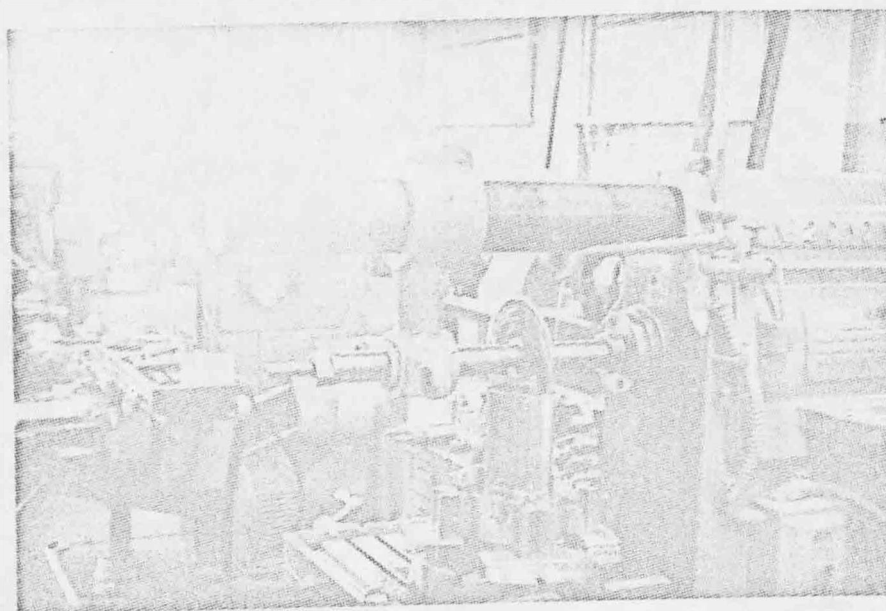
The final step after "locking in" the stresses prior to recording photoelastic data is the removal of certain planes from the model. The selection of these planes is explained in the section on reduction of data. The model was sliced in a milling machine using a $3/16$ -in. alternate-tooth side milling cutter, Figure (4-b). A four-jaw chuck was used to hold the work for the longitudinal slice. The slowest feed speed of the milling table of $1/2$ -inch per minute was used along with a cutter speed of 142 feet per minute. Higher cutter speeds induced chatter and resulted in a poor surface finish.

3.5 Recording Isoclinics and Isochromatics

The photoelastic data consisted of photographs of the isochromatic patterns and of sketches made on tracing paper of the isoclinic lines (Appendix (C)). To obtain this data the slicing plan shown



a. Polariscope used in the investigation



b. Removing slice in xz plane

Fig. (4) Polariscope and slicing method.

in Figure (5) was followed, each slice being analyzed in the polariscope in exactly the same manner as if it were a two-dimensional problem.

Isoclinics were recorded first to provide a reference in using Tardy's method of compensation for determining fractional fringe orders. Isoclinics were traced over the image thrown on the ground glass camera screen in a plane polariscope setup. Extreme care was taken in finding isoclinic curves so that later computations for shearing stress would have as small an error as possible. Isoclinic curves often appeared in the image as broad black bands, making it difficult to accurately find the actual center of the line. To aid in tracing these lines the paper was taped to the top of the screen so that it could be lifted and the image viewed directly. Points were then picked that lay in the center of the curve and their positions carefully noted on the screen; the point was then transferred to the tracing paper and its position rechecked. After several points were located, particularly in the contact region, a smooth curve was drawn and the final line again compared with the image.

Drawing the isoclinic lines for the top longitudinal slice was difficult because of the vague appearance of the curves. For an idea of the general positions of these lines, the slice was watched during several rapid rotations of polarizer and analyzer. This procedure gave an idea of the isoclinic development; each isoclinic line was then sketched on a separate sheet of paper and a final sketch was

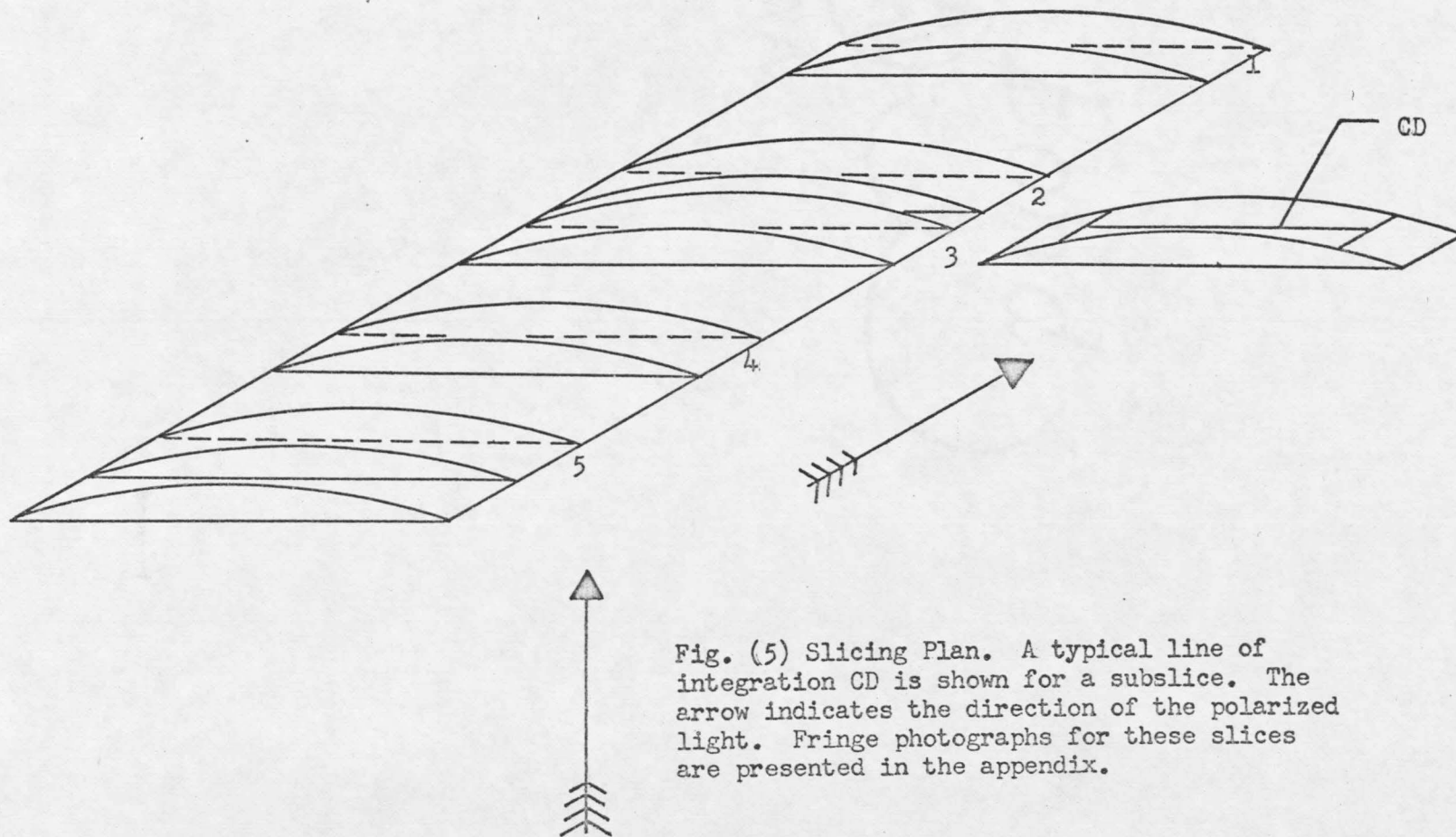


Fig. (5) Slicing Plan. A typical line of integration CD is shown for a subslice. The arrow indicates the direction of the polarized light. Fringe photographs for these slices are presented in the appendix.

made of all lines on one sheet. This last sketch was a valuable guide in sketching isoclinics under high magnification.

A considerable reduction in time and effort was effected by making the isoclinic diagrams quite large for the regions of interest and then making isochromatic prints to the same scale. The lines of integration were then drawn on the photographic prints as in Figure (6), and the isoclinic tracing fastened over the print.

Isochromatics were evaluated using a monochromatic light source, circular polariscope, and photographing the resulting fringe pattern. Tardy's method of compensation was used to determine fractional fringe orders to within $\pm 1/25$ fringe (11). This method depends upon the fact that if the axis of the polarizer and analyzer are aligned parallel to the directions of the principal stresses in the model slice, a rotation of the analyzer will give a proportional shifting of the fringes to a higher or lower order. If the polariscope is aligned to find the fringe order at a point and a rotation of the analyzer from a dark field setup moves the fringe through the point towards a higher order fringe, the fringe order is found from the relation

$$n = \text{lower order fringe} + \frac{\text{angle of rotation}}{180}$$

If the fringe moves from the higher order to the lower order fringe the following relationship is used.

$$n = \text{higher order fringe} - \frac{\text{angle of rotation}}{180}$$

The fringe order for a rotation of 90° of the analyzer

$$n = \text{fringe order} \frac{\pm 90}{180} = \text{fringe order} \pm 1/2$$

is just the relative retardation observed in a light field polariscope.

A limitation of the above method of compensation is that only fractional orders of fringes can be found. The absolute magnitude must be found from a knowledge of the fringe pattern, examination in a plane polariscope, or compensation by some other method.

3.6 Equipment

A great deal of effort was spent trying to get a parallel, uniform field of light and high fringe resolution. The original polariscope left much to be desired in these respects and many different setups were tried before a final arrangement was settled upon. The final polariscope setup is shown in Figure (4-a). For a uniform field of light a diffuser made of sanded Lucite was placed over the hole used to approximate a point source of light. The effects of parallel light needed for a sharp boundary and good fringe definition was approximated by making the distance between the condensor lens and model slice as long as possible and by stopping down the aperture of the camera lens. These techniques made possible a clear resolution of fringes in the region of high stress gradient (11).

Light and dark field isochromatic fringe patterns were recorded on photographs, two for each transverse slice and two for the longitudinal slices. Kodak contrast process panchromatic film was

developed in HC-110 developer for maximum contrast negatives. Prints were made on Kodak medalist F-4 photographic paper. Exposure time ran up to 30 seconds with a shutter opening of f-32.

CHAPTER IV

COMPUTATION OF CONTACT STRESSES

4.1 Shear-Difference Equations

A complete solution of the three-dimensional stress problem requires the determination of the six independent cartesian stress components from six equations; however, this investigation is concerned with finding only contact stresses. The orientation of the coordinate system, Figure (1), makes the problem one of finding σ_y along the contact surface. Of the methods available, the shear-difference method, which has been used extensively in two-dimensional elasticity (7), was chosen as being the most direct and least troublesome for the determination of the contact stress distribution.

The shear-difference method utilizes the equations of equilibrium from the theory of elasticity. Neglecting body forces, the equilibrium equations for a three-dimensional body are

$$1. \quad \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0$$

$$2. \quad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z} = 0$$

$$3. \quad \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0$$

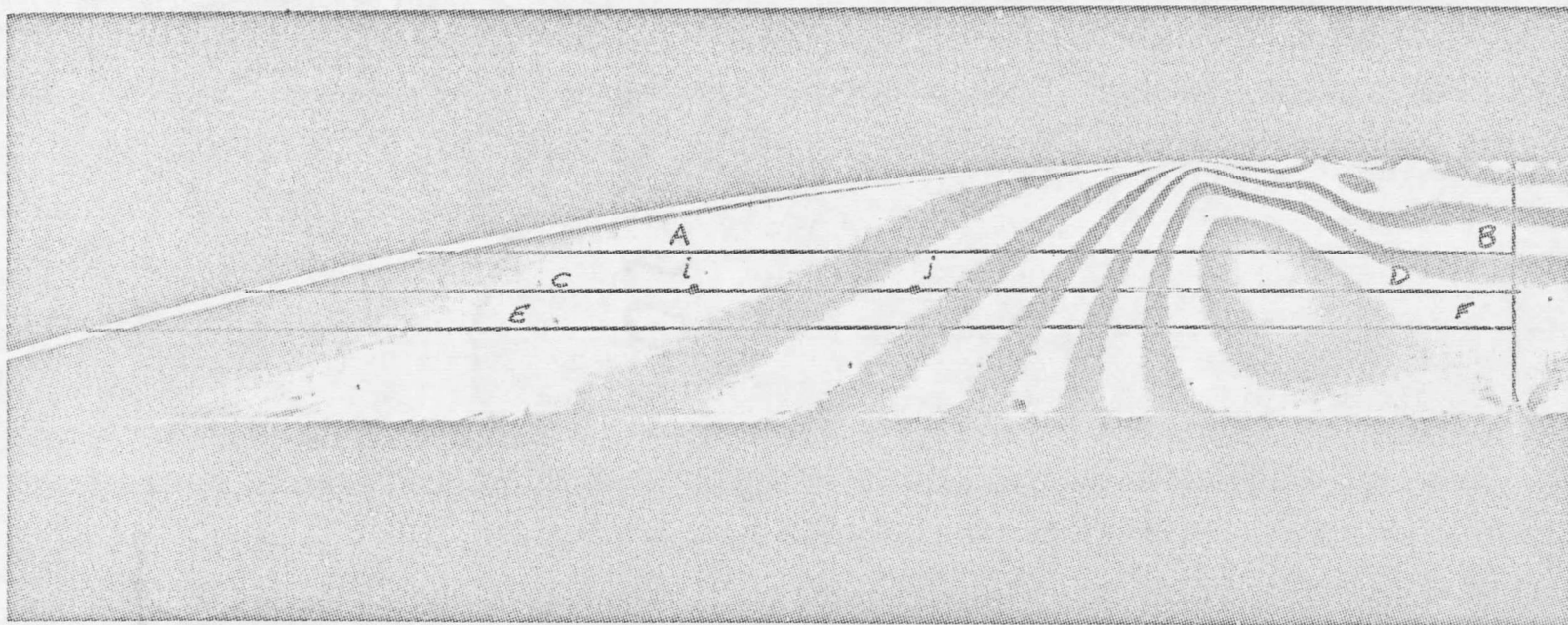


Fig. (6) Subslice No. 3. Dark Field photograph of transverse subslice no. 3 and the lines along which $(\tau_{xy})_{\max}$ is found.

Assume that the distribution of stress is known and let points i and j be general points in the body, Figure (5). To determine one normal component of stress at a point, only one of the equilibrium equations need be solved because the orientation of a coordinate system is arbitrary.

Thus, equation 3 can be written in the form

$$\int_i^j \frac{\partial \sigma_x}{\partial x} dx = - \int_i^j \frac{\partial \tau_{xy}}{\partial y} dx - \int_i^j \frac{\partial \tau_{xz}}{\partial z} dx$$

resulting in

$$4. (\sigma_x)_j = (\sigma_x)_i - \int_i^j \frac{\partial \tau_{xy}}{\partial y} dx - \int_i^j \frac{\partial \tau_{xz}}{\partial z} dx$$

Equation 4 is referred to as the shear-difference equation and by rewriting it in finite difference form it will be of most use in this analysis.

$$5. (\sigma_x)_j = (\sigma_x)_i - \sum_i^j \frac{\Delta \tau_{yx}}{\Delta y} \Delta x - \sum_i^j \frac{\Delta \tau_{zx}}{\Delta z} \Delta x$$

To determine σ_x at a point j , three terms must be known $(\sigma_x)_i$,

Calculating $(\sigma_x)_i$ usually requires an analysis of photo-elastic data at the free boundaries of the model in the xy or xz planes. An initial value of $(\sigma_x)_i$ was calculated in this analysis from a transverse slice of the model, see Figure (7).

On a slice in the xy plane

$$6. (\tau_{xy})_{\max} = \frac{p - q}{2} = nF$$

where p and q are secondary principal stresses and n is the fringe order at a particular point. At the boundary, p becomes zero and q is found experimentally to be compression; therefore

$$7. -q = 2nF$$

Fringe values n were taken from graphs of shearing stress distribution along the x axis, see Figure (8). To determine a value of σ_x an infinitesimal element was examined at the boundary of the slice, Figure (7). Summation of forces on the element gives

$$8. (\sigma_x)_0 = q \cos^2 \theta = 2nF \cos^2 \theta$$

If a curve were to be plotted of the rate of change of shearing stress in the xy plane with respect to y, the $\int_i^j \frac{\partial \tau_{xy}}{\partial y} dx$ term would represent the area under the $\frac{\partial \tau_{xy}}{\partial y}$ curve plotted as a

function of x . In finite-difference form the integral $\int_i^j \frac{\partial \tau_{xy}}{\partial y} dx$

may be expressed as $\sum_i^j \frac{\Delta \tau_{xy}}{\Delta y} \Delta x$. To best approximate the area

under a curve of $\frac{\partial \tau_{xy}}{\partial y}$ a mean value of $\Delta \tau_{xy}$ is used, a value located halfway between points i and j .

Graphs of shearing stress along auxiliary lines AB and EF (Figure (6)) were drawn, Figure (8), from photoelastic data and the $\Delta \tau_{xy}$ distribution was plotted as a difference in these curves. A value of $\Delta \tau_{xy}$ was taken from the graph of $\Delta \tau_{xy}$ halfway between neighboring points i and j . The Δx and Δy terms are the distances between i and j and the auxiliary lines, respectively. Similar computations in the xz plane yielded a value for the $\sum \frac{\Delta \tau_{xz}}{\Delta y} \Delta x$

term in equation 7. Complete information for determining $\Delta \tau_{xy}$ and $\Delta \tau_{xz}$ is given by the curves of Figure (8).

Care should be taken in using a sign convention for $\Delta \tau_{xy}$, Δx , Δy , and Δz . For the right-handed coordinate system used, Δx , Δy , and Δz will have a positive value when integrating from i to j . The sign of $\Delta \tau_{xy}$ and $\Delta \tau_{xz}$ depends on the magnitudes of shearing stress on the auxiliary lines.

Computation of σ_x was done with the aid of the computer program in Appendix (D). The normal stress component σ_y at a general point j can be calculated from values of σ_x and photoelastic information in

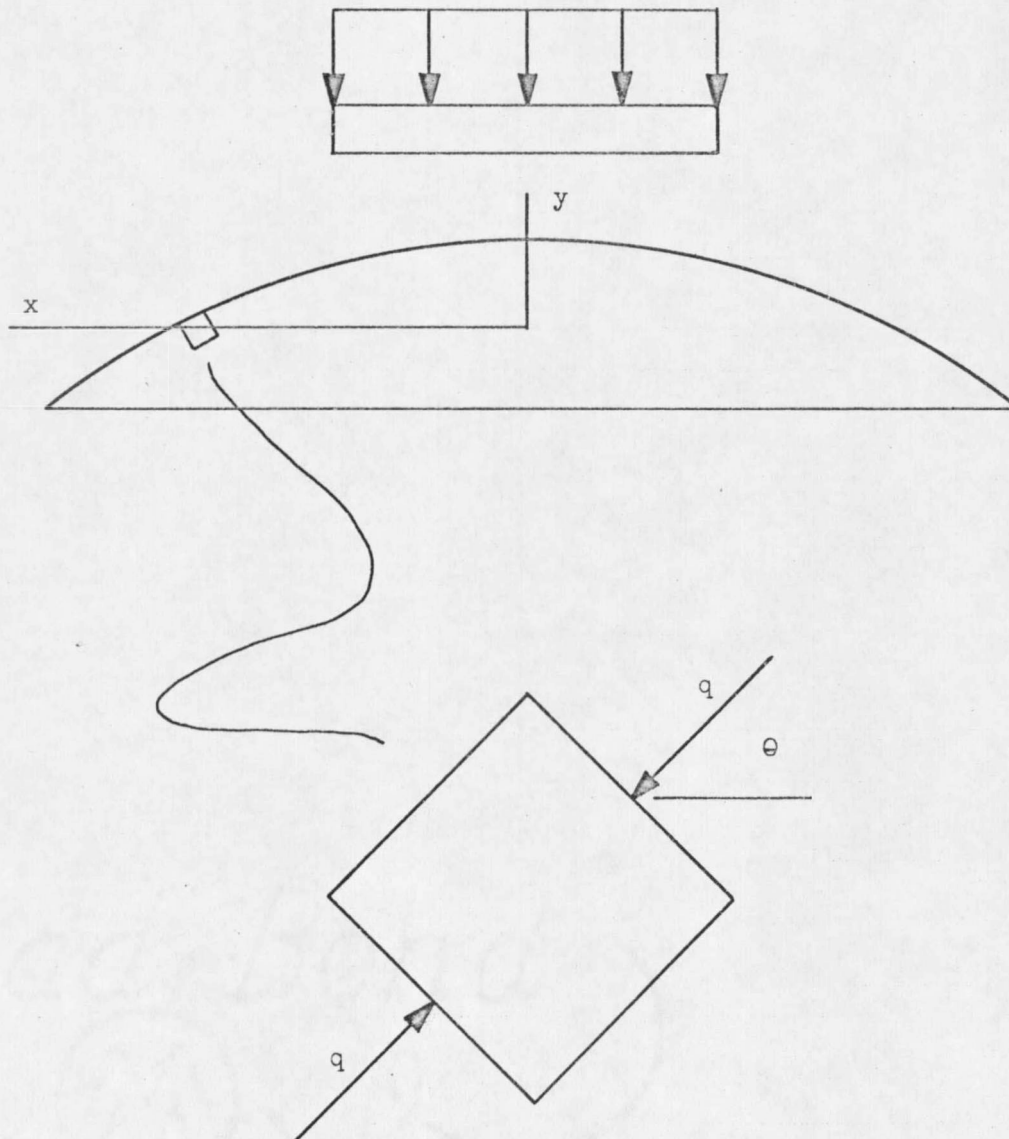
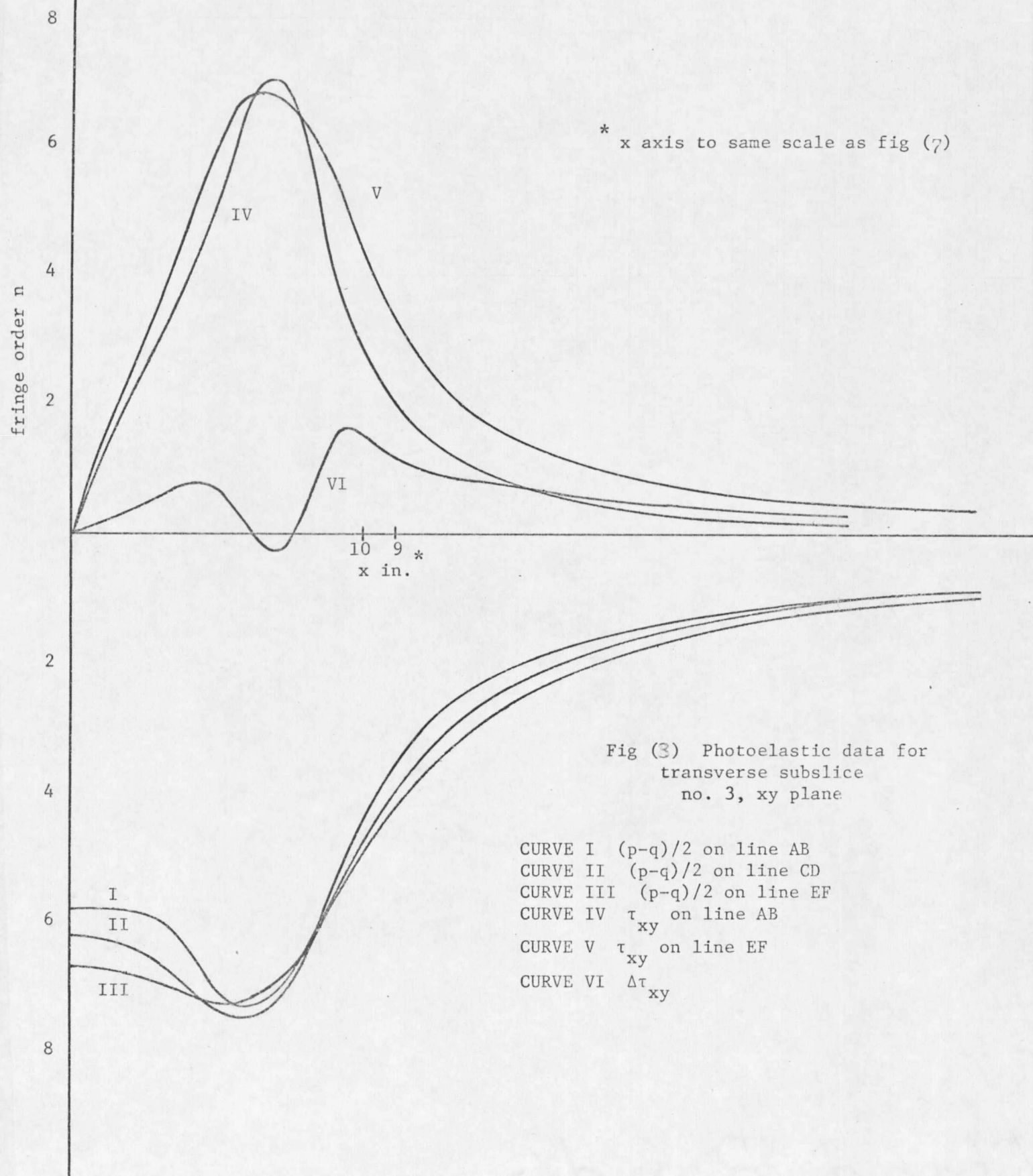


Fig. (7) Surface Element. Examination of an infinitesimal element at the free boundary of a transverse subslice to determine σ_x .



the xy plane from the equation

$$(\sigma_x - \sigma_y)_j = (p - q)_j \cos^2 \theta_j$$

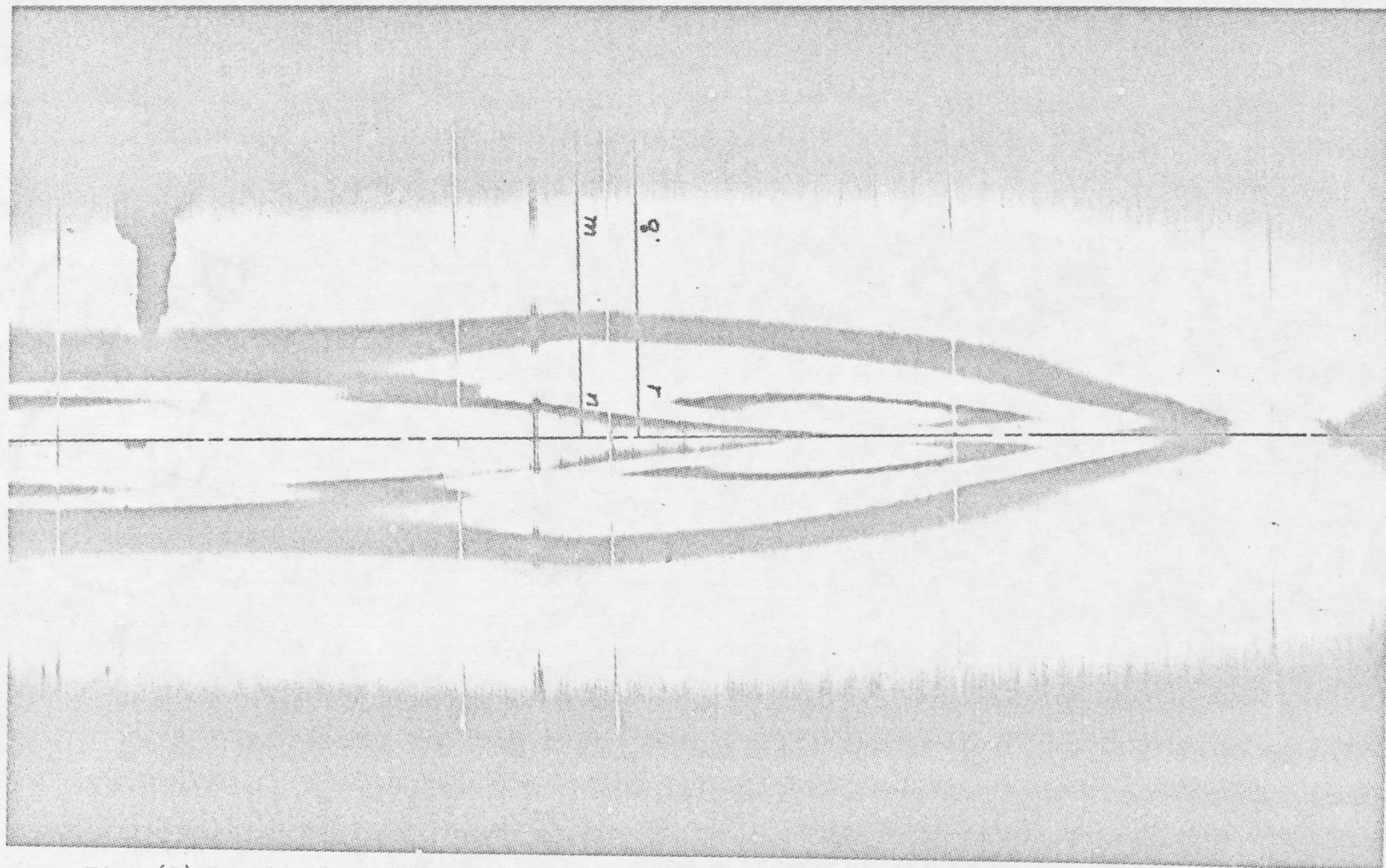
$$9. (\sigma_y)_j = (\sigma_x)_j - 2n_j F \cos^2 \theta_j$$

Values of σ_y had to be computed below the surface because the fringes photographed in the model slices represented an average stress assumed to exist at the center of each slice. If the longitudinal slices are approximately 0.1 in. thick, the shearing stresses computed from the fringes in Figure (9) will represent values of τ_{xz} on a plane of .05 in. below the contact surface and values calculated for σ_x in equation 5 can be no closer than this to the contact surface.

4.2 Sample Computation

As an example of the computation of σ_y at a point on the line of integration, consider slice #3 in Figure (5). Assume that σ_x has been found for a point on the line CD, that will be designated as station 9, and σ_x is to be calculated for station 10. Complete photo-elastic information is given in Figure (8) and Figure (10). Using shear-difference equation no. 5

$$(\sigma_x)_{10} = (\sigma_x)_9 - \Delta\tau_{xy} \cdot \frac{\Delta x}{\Delta y} - \Delta\tau_{xz} \cdot \frac{\Delta x}{\Delta z}$$



30

Fig. (9) Longitudinal Slice. Dark Field photograph showing lines gr and mn along which $(\tau_{xy})_{\max}$ is found.

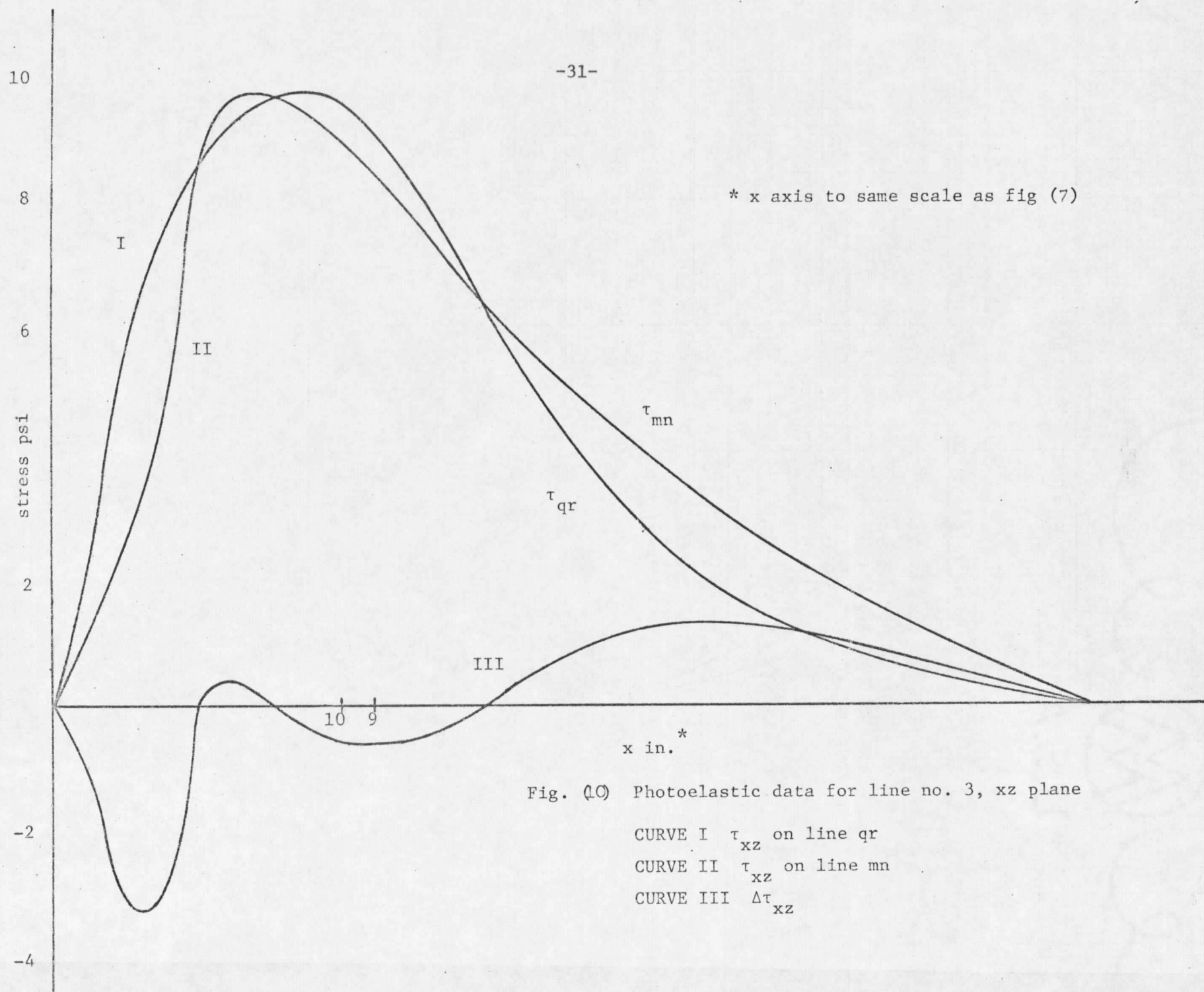


Fig. (10) Photoelastic data for line no. 3, xz plane

CURVE I τ_{xz} on line qr

CURVE II τ_{xz} on line mn

CURVE III $\Delta\tau_{xz}$

As has been mentioned, mean values of $\Delta\tau$ must be used. Taking values from the graphs of $\Delta\tau_{xy}$ and $\Delta\tau_{xz}$ in Figures (8) and (10), respectively

$$\begin{aligned}\Delta\tau_{xy} &= 1.30 \text{ fringes} = (1.30 \text{ fringes}) (2.87 \frac{\text{psi}}{\text{fringe}}) \\ &= 3.73 \text{ psi}\end{aligned}$$

$$\Delta\tau_{xz} = -0.53 \text{ psi}$$

Moving from point 9 to point 10 along the x axis gives a Δx of 1/4 inch. Values of $\frac{\Delta x}{\Delta y}$ and $\frac{\Delta x}{\Delta z}$ are found to be

$$\frac{\Delta x}{\Delta y} = \frac{.250}{.400} = .625$$

$$\frac{\Delta x}{\Delta z} = \frac{.052}{.375} = .138$$

and σ_x at point 10 is

$$\begin{aligned}(\sigma_x)_{10} &= -17.98 - (3.73)(.625) - (-.53)(.138) \\ &= 20.23 \text{ psi}\end{aligned}$$

Equation 13 gives $(\sigma_y)_{10} = (\sigma_x)_{10} - 2n_{10} F \cos 2\theta_{10}$

The fringe order at point 10 on line CD is found from Figure (8) to be

$$n_{10} = 4.68$$

For slice no. 3 the model fringe value is

$$F = 2.87$$

and the isoclinic parameter

$$\theta = 63^\circ$$

The isoclinic parameter is found from Appendix (C). The model fringe value F is calculated from the material fringe value f and the thickness of each subslice, using the equation

$$F = \frac{f}{t}$$

For slice no. 4

$$F = \frac{.603 \frac{\text{psi} \cdot \text{in}}{\text{fringe}}}{.210 \text{ in.}}$$
$$= 2.87 \frac{\text{psi}}{\text{fringe}}$$

Now calculating σ_y at point 10

$$(\sigma_y)_{10} = -4.43 \text{ psi}$$

This sample calculation is a summary of the computing procedure.

A printout of values for σ_y located on a plane .05 in. below the contact surface of the model is found in Appendix (D).

CHAPTER V

EXPERIMENTAL AND ANALYTICAL RESULTS

5.1 A Comparison of Experimental Results with a Two-Dimensional Analytical Solution

A solution for the contact stress distribution on the central yz plane of the model, using a two-dimensional analytical method, ref. (1), is compared with the three-dimensional experimental results of this investigation in Figure (11). In the graph of experimental contact stresses the values plotted for the surface were taken from the experimental results on a plane 0.05 in. below the contact surface. It is seen that the theoretical solution contact length was computed to be the same length as that found experimentally. The area under the two-dimensional curve was made equal to the area under the experimental curve, so that the actual loading of the model and the assumed loading in the theoretical case were equal. By dividing these areas by the common contact length a mean stress can be found. Comparing the maximum theoretical stress to the mean stress and the maximum experimental stress to the mean stress gives

$$\begin{aligned} 1. \text{ Theoretical comparison } \quad \frac{\sigma_{\max}}{\sigma_{\text{mean}}} &= \frac{\frac{1b}{86.2 \text{ in}^2}}{\frac{1b}{43.3 \text{ in}^2}} \\ &= 1.99 \\ 2. \text{ Experimental comparison } \quad \frac{\sigma_{\max}}{\sigma_{\text{mean}}} &= \frac{\frac{1b}{63.0 \text{ in}^2}}{\frac{1b}{43.3 \text{ in}^2}} \\ &= 1.46 \end{aligned}$$

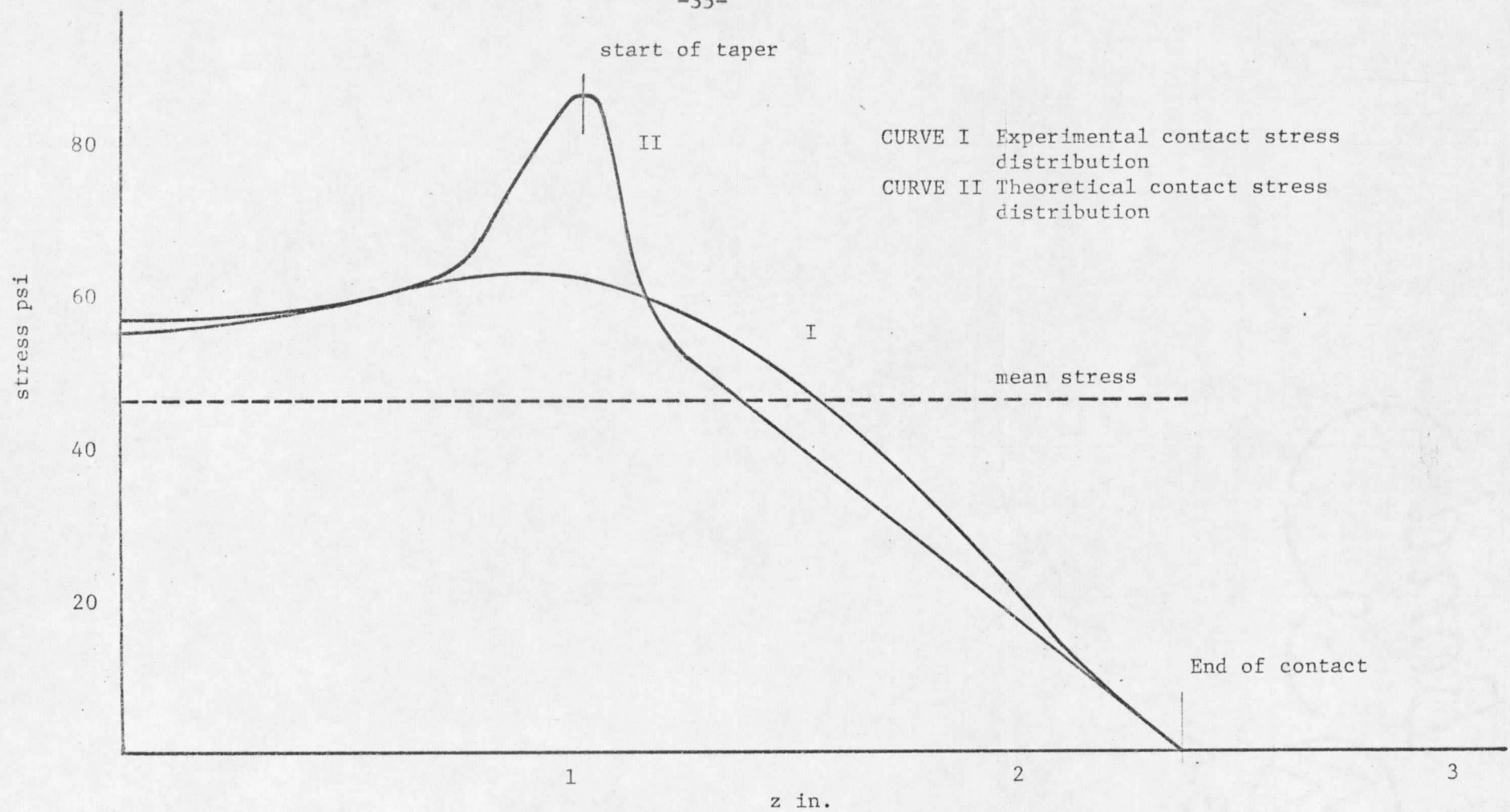


Fig (II) Comparison of theoretical two-dimensional contact stress distribution for the surface configuration of the central plane of the model and the experimental results on that plane.

The ratios of maximum stress to mean stress and the curves of contact stresses in Figure (11) show the two-dimensional solution to give a much higher stress concentration than was found to exist in the three-dimensional model. Referring to Figure (11), it appears that the plane stress problem is more sensitive to changes in the surface slope than is the three-dimensional case. The three-dimensional experimental results indicate a more even distribution of stress over the contact surface than is predicted theoretically by the two-dimensional solution.

When observed in an optical comparator, it appeared that the maximum width of the contact area in the xy plane occurred about .1 in. before the taper began in the central portion of the slice. In Figure (11), the two-dimensional theoretical solution predicts the maximum stress at the taper or shortly after the taper began, which does not quite agree with the experimental results.

The contact stresses on the surface were approximated by extending the experimental stresses found on the plane of integration to the surface. (The plane of integration on which all experimental values were calculated is located .05 in. below the contact surface.) Three properties of the stress distribution on a transverse subslice of unit thickness were used to make this extension of normal stress components σ_y to the surface.

By carefully measuring the contact width in an optical comparator it was possible to plot the point where the contact stress distribution

goes to zero. In this investigation the relative magnitudes of the contact stresses in the central portion of the contact width assured a fairly accurate curve definition. It was assumed that the contact stress distribution had essentially the same shape as the normal stresses on the plane of integration in the region where the values of σ_y were large.

The third consideration was a requirement that the stresses on the plane of integration be in equilibrium with stresses on the contact surface. Integration of the subsurface stress curve yielded a value required for the area of the contact stress curve.

The contact stress distribution in the xz plane was obtained by using the shape of the curve on the plane of integration in regions of high normal stress, requiring that this curve go through the point where contact ends, and scaling the curve so that its area as measured with a planimeter was equal to the net area of the normal stresses on the plane of integration. This resulted in the contact stress distributions pictured in Figures (12-16). It is thought that these distributions closely approximate the true contact stresses.

A static equilibrium check was made on the experimental results by computing the approximate total volume under the predicted contact stress distribution and comparing it to the known loading.

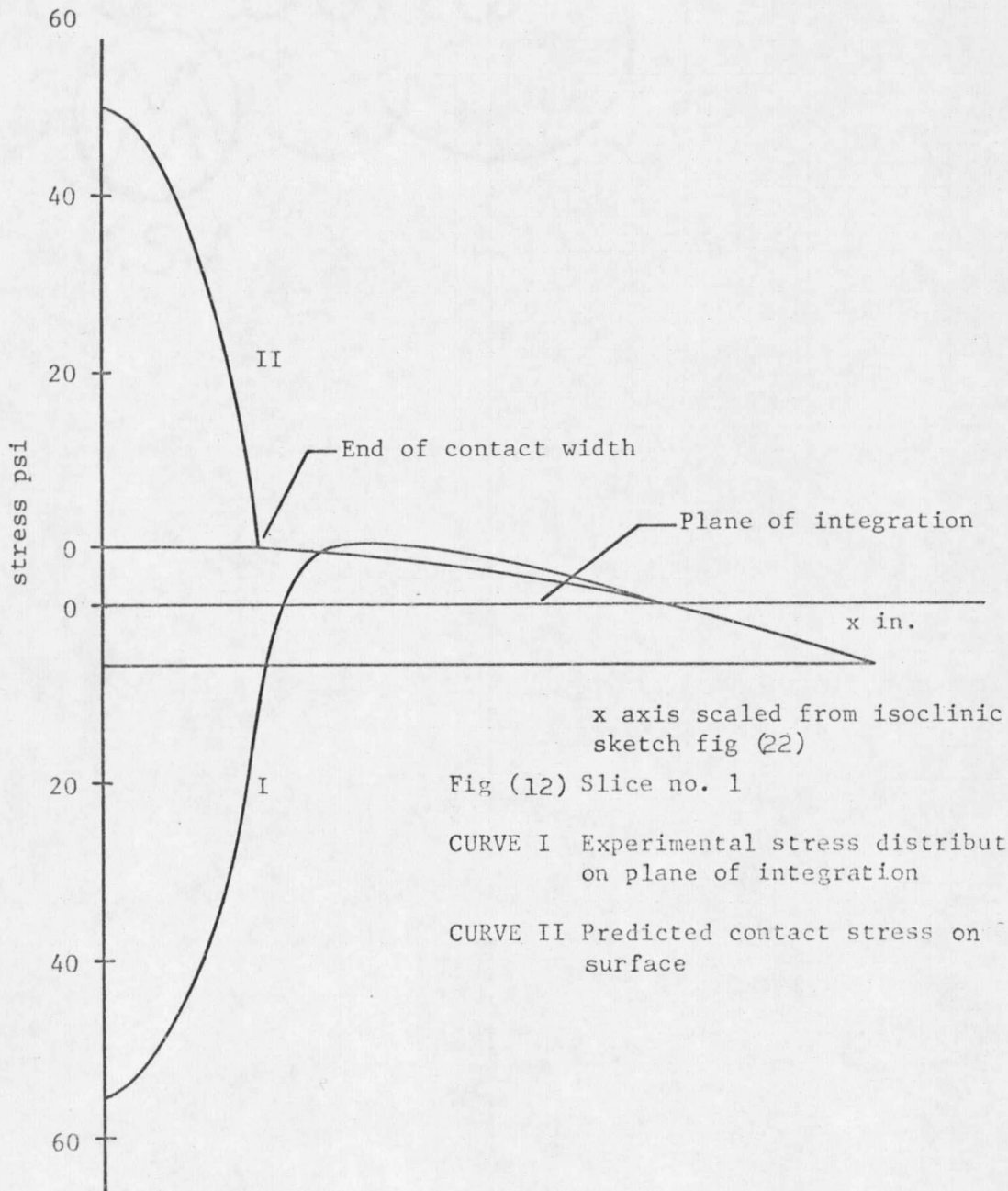


Fig (12) Slice no. 1

CURVE I Experimental stress distribution on plane of integration

CURVE II Predicted contact stress on surface

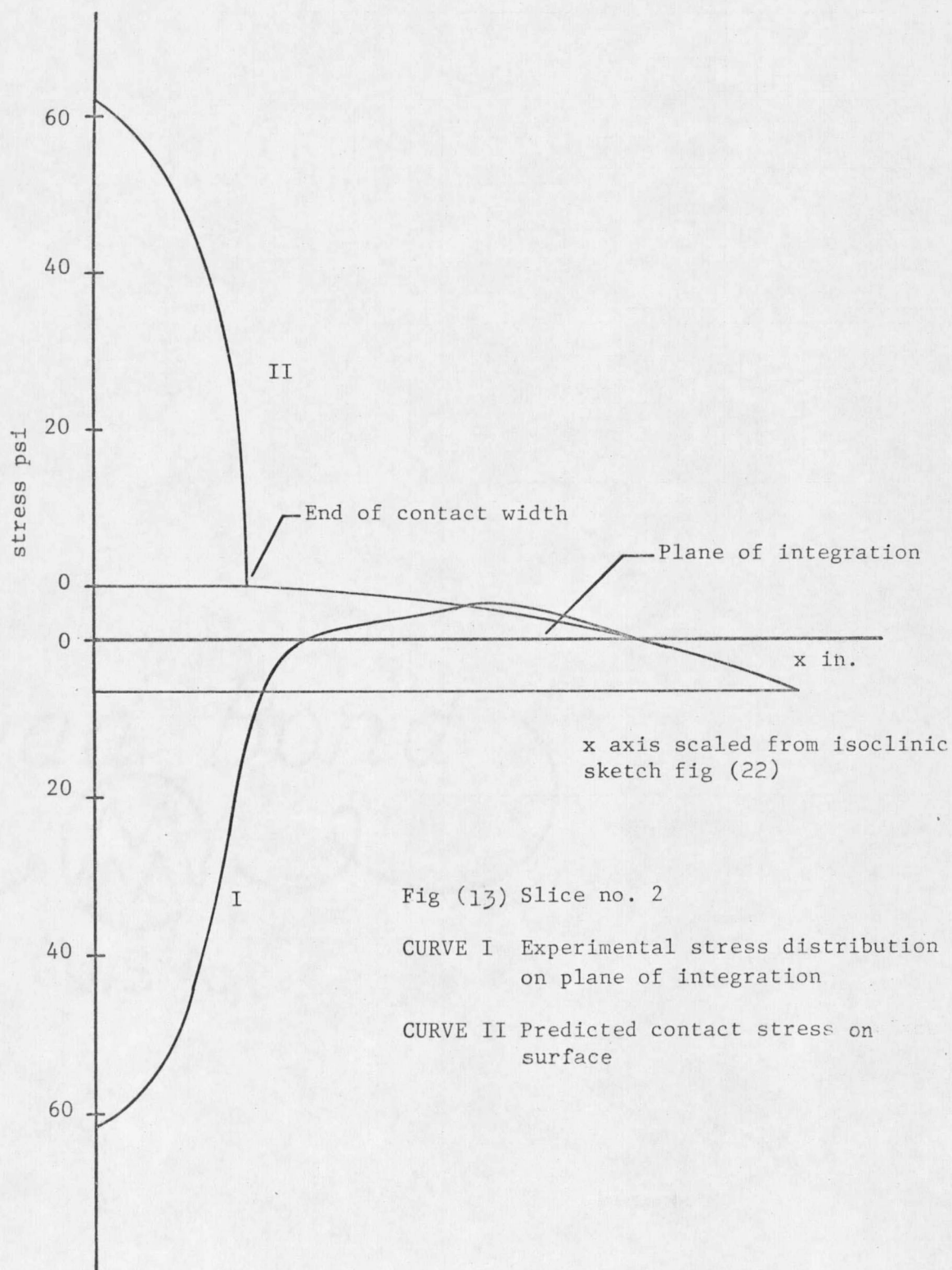
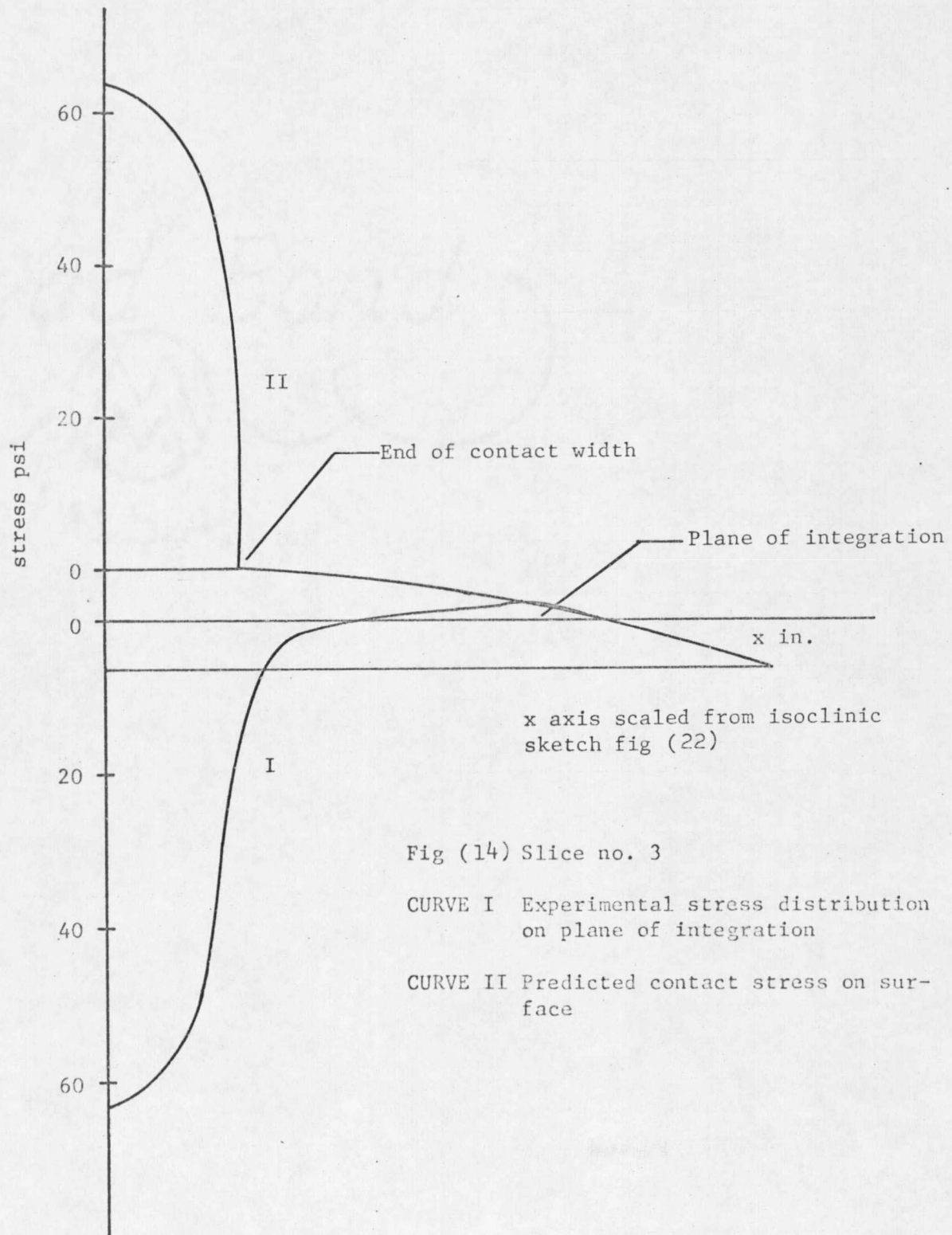
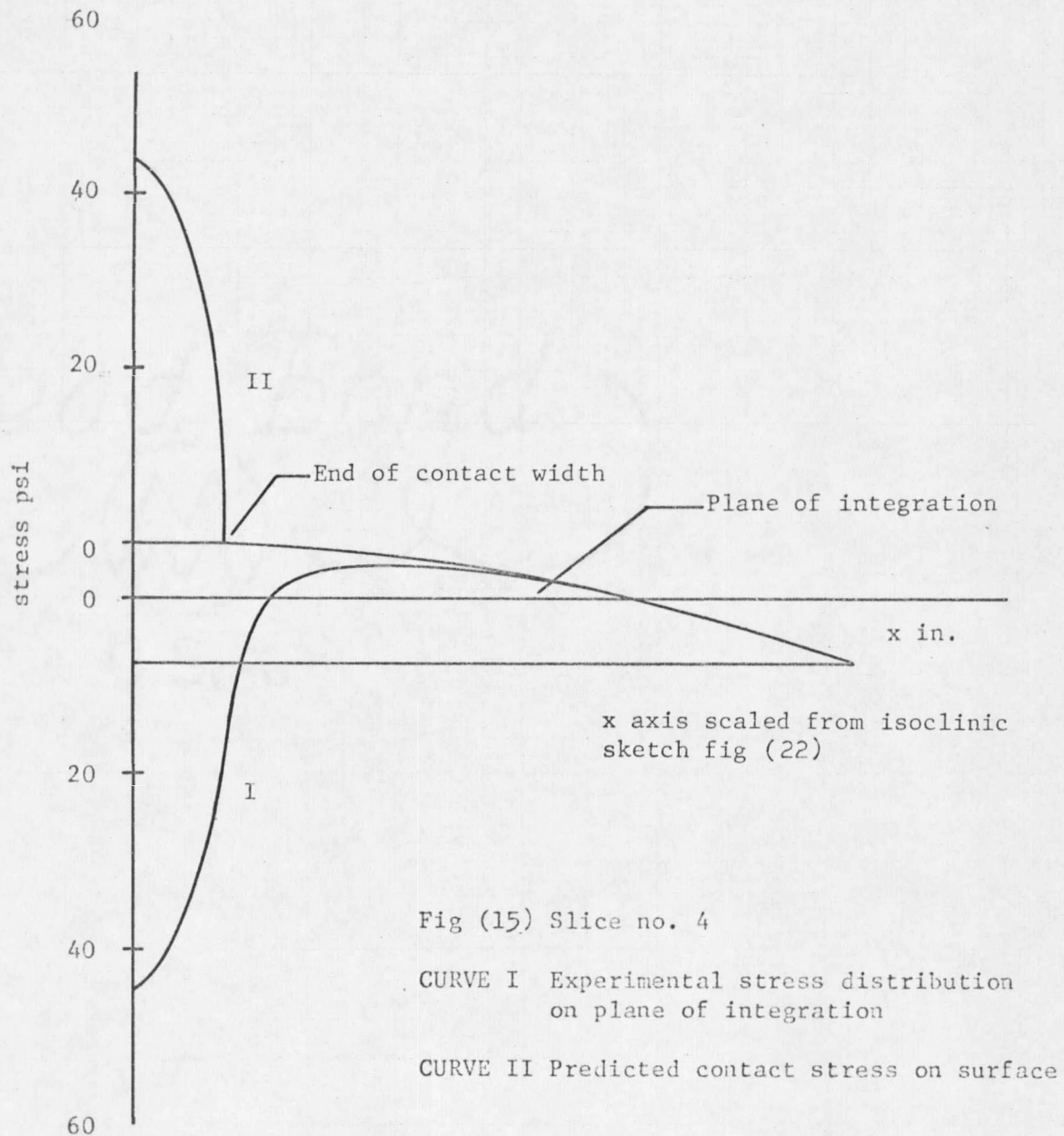


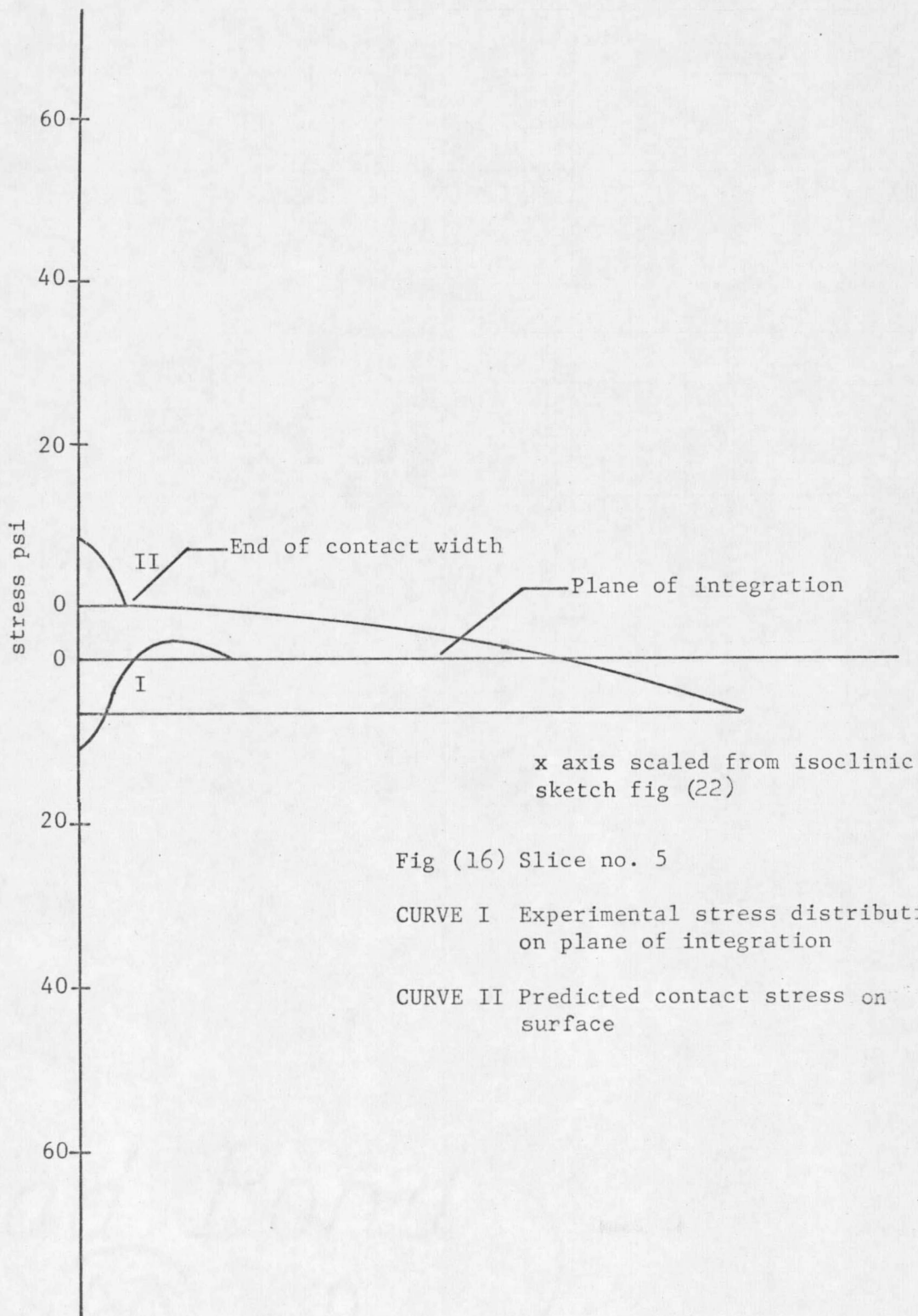
Fig (13) Slice no. 2

CURVE I Experimental stress distribution on plane of integration

CURVE II Predicted contact stress on surface







4.2. Conclusions

The three-dimensional experimental results of this paper cannot be predicted accurately by the two-dimensional theory of reference (1). In general, the contact stress distribution in three-dimensional problems differs significantly from two-dimensional solutions. However clever and ingenious the assumptions, it does not seem possible to accurately predict the contact stresses on a three-dimensional body using two-dimensional solutions.

It is believed that the results of this paper provide an accurate reference for comparison with analytical methods for predicting contact stresses on three-dimensional bodies.

APPENDIX A

MODEL SIMILARITY BY DIMENSIONAL ANALYSIS

A list of the variables that uniquely define the stress in a body is necessary before a dimensional analysis can be made. A list of defining parameters for the externally loaded elastic body in this investigation would be expressed as

$(s, x, y, z, E, u, a, r_1, r_2, \dots, P, r_1', r_2', \dots, r_1'', r_2'', \dots)$

a = characteristic length

P = a reference load

E = Young's modulus

u = Poisson's ratio

x, y, z = coordinates

s = stress

$r_1, r_2, \dots$ = ratios of all lengths to a

$r_1', r_2', \dots$ = ratios of all other loads to P

$r_1'', r_2'', \dots$ = ratios defining directions

All of the quantities in this list can be defined by a force P and length a , two fundamental units in the standard engineering system of units.

The idea in a dimensional analysis of a physical system is to arrange all of the defining parameters into groups so that the units in the numerator of each group cancel the units of the denominator. Such a grouping for the above list would be

$$\frac{sa^2}{P}, \frac{x}{a}, \frac{x}{a}, \frac{x}{a}, u, \frac{P}{Ea^2}, r_1, r_2, \dots, r_1', r_2', \dots, r_1'', r_2'', \dots$$

These groupings are now made into equations that show the relationship between variables regardless of the specific case under consideration. Buckingham, one of the original investigators into this type of analysis, came to the conclusion that the number of independent dimensionless groupings was equal to the number of system parameters less the number of fundamental units.² The resulting stress formula for the system considered in this paper, expressed as a function of the independent dimensionless numbers, is

$$\frac{sa^2}{P} = f \left(\frac{x}{a}, \frac{x}{a}, \frac{x}{a}, u, \frac{P}{Ea^2}, r_1, r_2, \dots, r_1', r_2', \dots, r_1'', r_2'', \dots \right)$$

All stress formulas for elastic bodies must take the above form. For a model to have exactly the same stress distribution, all of the above terms on the right must be the same for both the model and the prototype. Dimensional analysis indicates that the ratio of model stress to prototype stress is directly proportional to the ratio of Young's modulus.

$$\frac{S_m}{S_p} = \frac{E_m}{E_p}$$

² This is the Buckingham - π Theorem.

APPENDIX B

DETERMINATION OF MATERIAL FRINGE VALUE

In the usual statement of the stress-optic law written in terms of fringe order

$$N_{\max} = \frac{n}{2Ct} = nF$$

where F is known as the model stress fringe value, one of the basic photoelastic constants. However, F is an expression for the optical properties of a particular model, or in this case, a particular slice from the "frozen" model. This model fringe value can be related to a material fringe value

$$F = \frac{f}{t}$$

where f is the fringe value of a model of unit thickness.

The material fringe value is found by solving some simple stress problem theoretically such as a beam in pure bending or a standard tension specimen, and relating the theoretical solution to the fringe pattern in a photoelastic model. In "frozen stress" work, the calibration piece should have exactly the same loading and temperature history as the model, insuring that optical effects of the plastic such as creep and relaxation are automatically accounted for when slices from the model are analyzed. The material stress fringe value should be calculated at the same time that slices from the model are photographed.

Since most of the model machining was done on the lathe, it was a simple job to turn a disk off of one end. The stress distribution for a disk in diametral compression has a theoretical solution that is easily correlated to experimental results. Durelli (4), makes such a correlation and expresses the material fringe value f as

$$f = \frac{4P}{Dn} g\left(\frac{x}{D}\right)$$

In this investigation these variables had the following values:

$g\left(\frac{x}{D}\right)$ = a function usually evaluated at the center
of a disk $g(0) = 1$ or midway between the
center and boundary $g(.25) = 0.480$

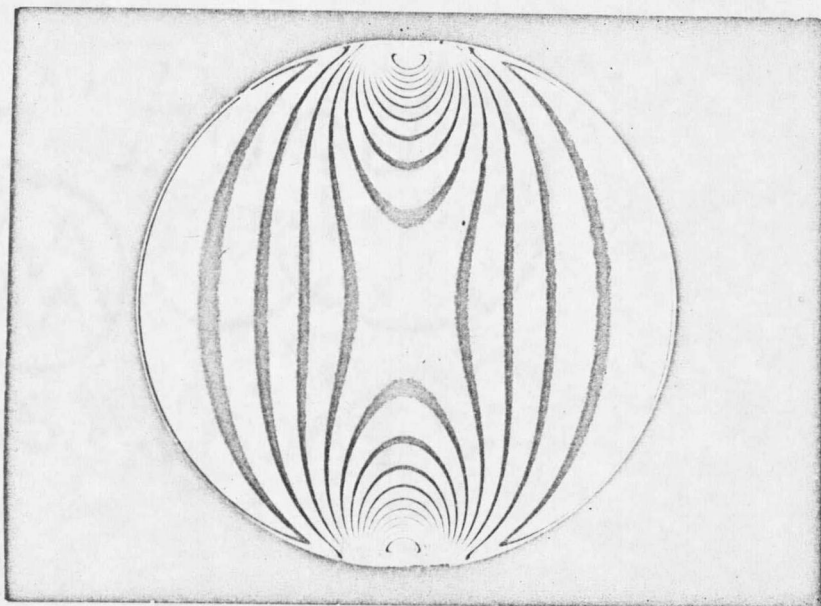
P = load = 4.73 lbs.

D = diameter = 2.08 in.

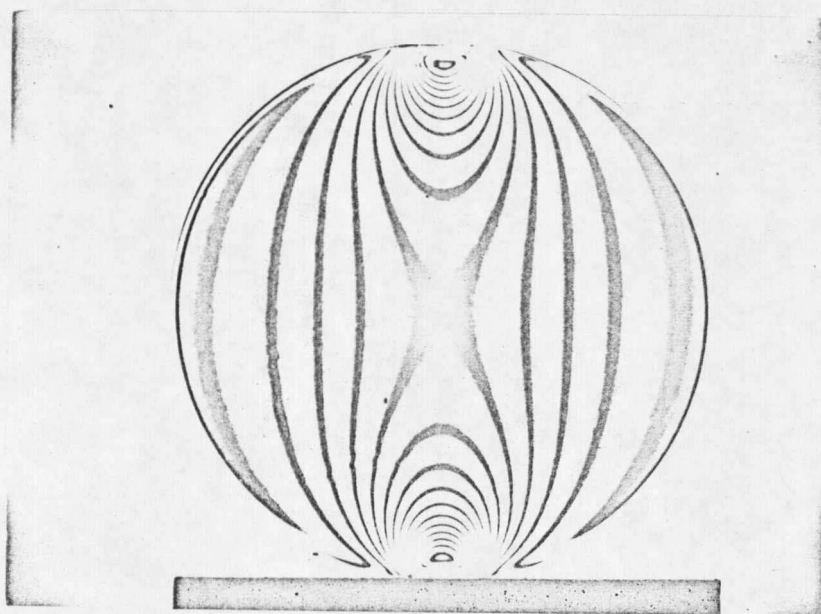
n = fringe order along a horizontal diameter taken
from the graph in Figure (18) = 2.30

The material shear stress fringe value is then

$$\begin{aligned} f &= \frac{(4)(4.73 \text{ lbs.})}{\pi (2.00 \text{ in.})(2.4)} (.480) \\ &= .603 \text{ psi} - \text{in./fringe} \end{aligned}$$



a.



b.

Fig (17) Dark (a) and light (b) field
photographs of calibration
disk

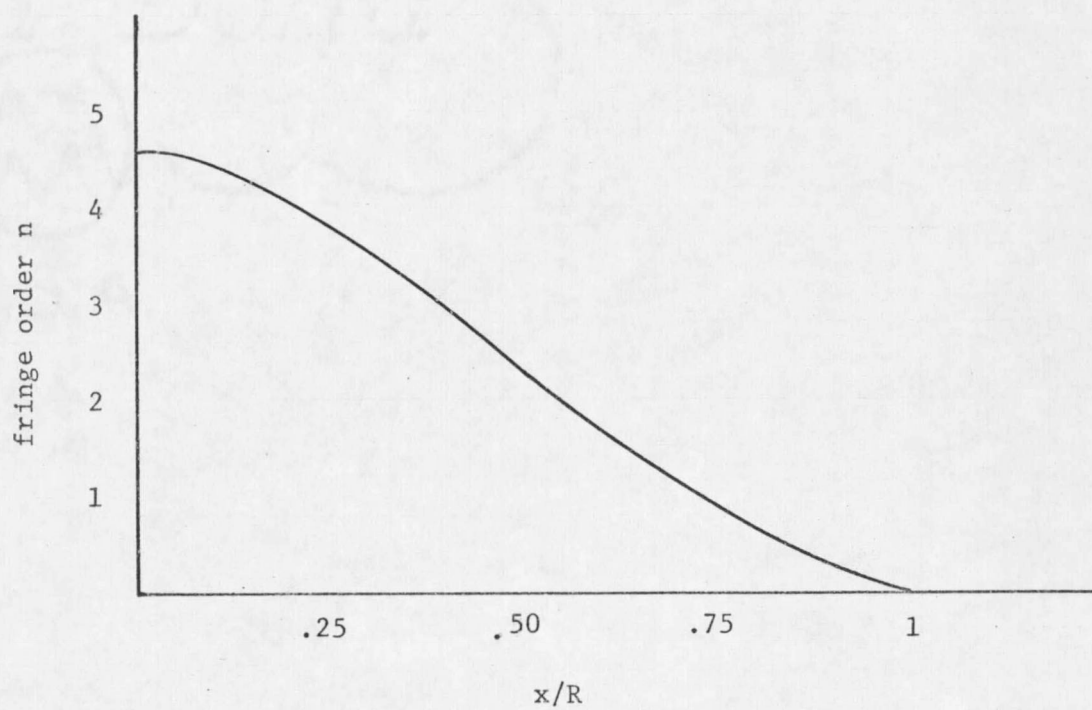
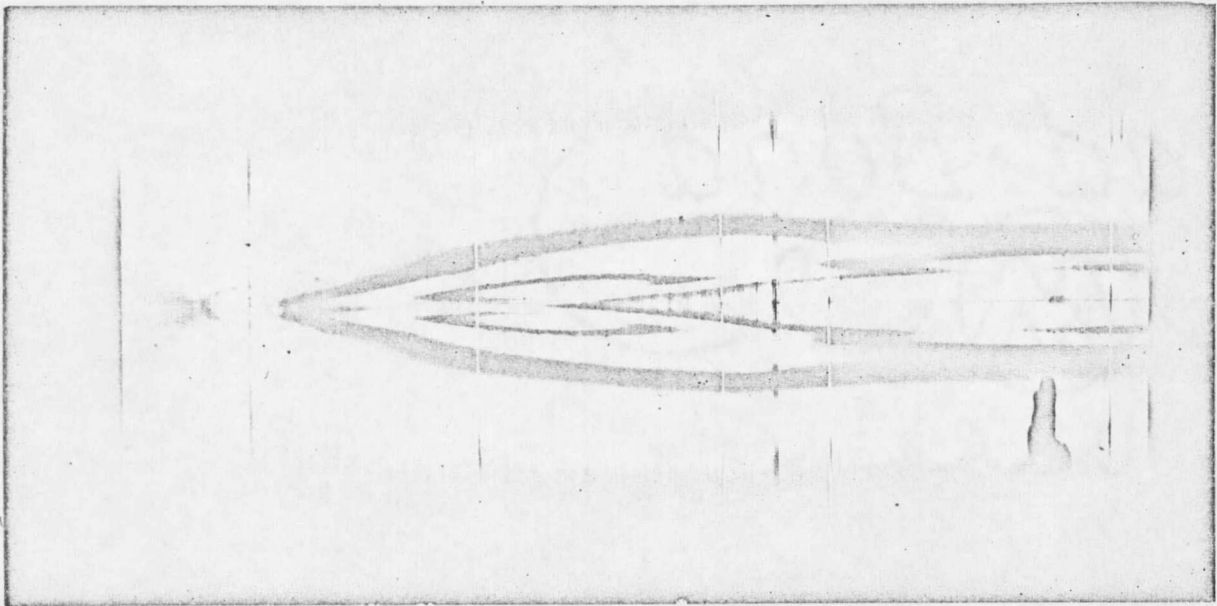


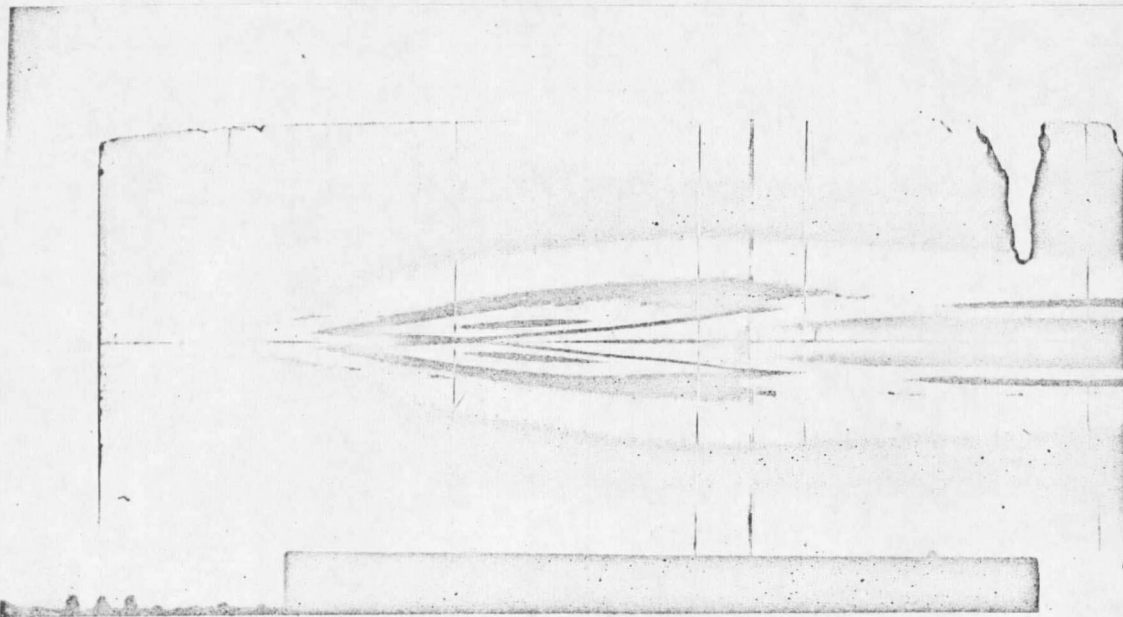
Fig (18) Fringe order n on a horizontal radius of the calibration disk, photoelastic information was taken from the photographs in fig (17).

APPENDIX C

PHOTOELASTIC DATA



a.



b.

Fig (19) Dark (a) and light (b) field photographs of longitudinal slice



a. Subslice no. 5



b. Subslice no. 4



c. Subslice no. 3



d. Subslice no. 2



e. Subslice no. 1

Fig (4) Light field photographs of transverse subslices



a. Subslice no. 5



b. Subslice no. 4



c. Subslice no. 3



d. Subslice no. 2



e. Subslice no. 1

Fig (21) Dark field photographs of transverse subslices

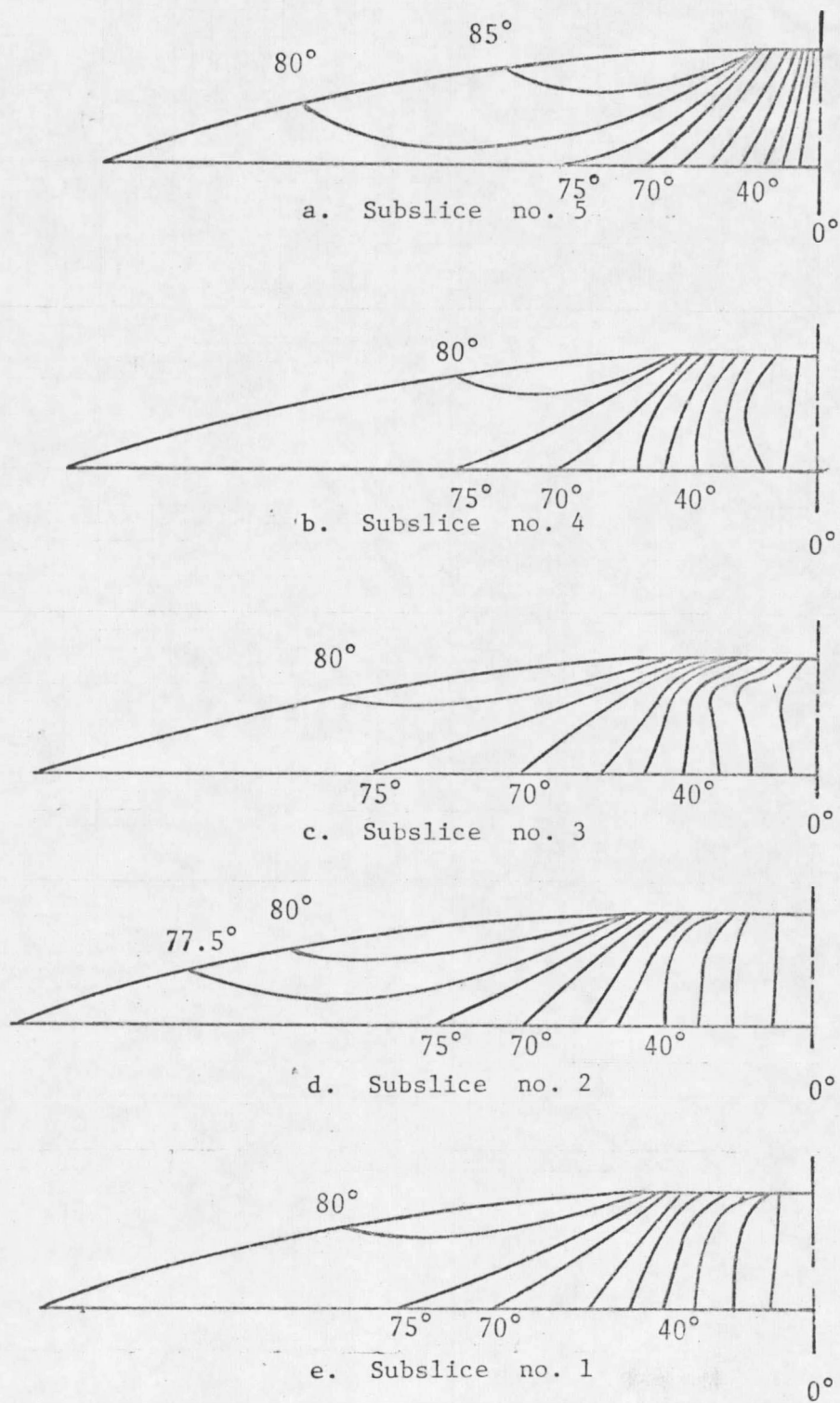


Fig (22) Isoclinics in transverse subslices

APPENDIX D

COMPUTER PROGRAM FOR CALCULATING σ_y AND RESULTS

*LIST PRINTER
**DOUGLAS SCHAFER

Fortran program for computing
 σ_x (SIGMAX) and σ_y (SIGMAY)

DOUGLAS SCHAFER

```
      DIMENSION SIGMAX(20),DELXY(20),DELXZ(20),TAUXY(20),TAUXZ(20)
      DIMENSION PHI(20),FRINGE(20)
12  READ 1,SIGMAO,F
      IF(ABSF(SIGMAO)+ABSF(F))14,11,14
      1  FORMAT(2F10.3)
14  I=1
      2  READ 3,DELXY(I),TAUXY(I),DELXZ(I),TAUXZ(I),PHI(I),FRINGE(I)
      3  FORMAT(6F10.3)
      X=ABSF(DELXY(I))+ABSF(DELXZ(I))+ABSF(TAUXY(I))+ABSF(TAUXZ(I))
      IF(X)6,4,6
      4  PRINT 5
      5  FORMAT(1H1)
      GO TO 12
      6  DELXY(I)=DELXY(I)*F
      SIGMAX (I)=SIGMAO-DELXY(I)*TAUXY(I)-DELXZ(I)*TAUXZ(I)
      SIGMAO=SIGMAX(I)
      IF(PHI(I))7,9,7
      7  PHI(I)=PHI(I)*3.14159265/180.
      SIGMAY=SIGMAX(I)-2.*FRINGE(I)*F*COSF(2.*PHI(I))
      PRINT 8,I,SIGMAX(I),I,SIGMAY
      8  FORMAT(8H SIGMAX(I2,4H) = F8.4,10X,7HSIGMAY(I2,4H) = F8.4//)
      I=I+1
      GO TO 2
      9  PRINT 10,I,SIGMAX(I),I
      10 FORMAT(8H SIGMAX(I2,4H) = F8.4,10X,7HSIGMAY(I2,16H) IS NOT DEFINED
1//)
      I=I+1
      GO TO 2
11  CALL EXIT
      END
```

σ_x	σ_y
SIGMAX(1) = -.2490	SIGMAY(1) = 5.6664
SIGMAX(2) = -.2490	SIGMAY(2) = 6.6321
SIGMAX(3) = -1.2658	SIGMAY(3) = 7.0220
SIGMAX(4) = -2.8608	SIGMAY(4) = 6.6914
SIGMAX(5) = -5.3729	SIGMAY(5) = 6.1005
SIGMAX(6) = -9.3604	SIGMAY(6) = -2.4600
SIGMAX(7) = -15.3416	SIGMAY(7) = -21.2134
SIGMAX(8) = -16.7373	SIGMAY(8) = -32.8261
SIGMAX(9) = -16.5379	SIGMAY(9) = -41.6634
SIGMAX(10) = -16.0793	SIGMAY(10) = -46.1398
SIGMAX(11) = -16.2787	SIGMAY(11) = -47.7640
SIGMAX(12) = -16.7572	SIGMAY(12) = -50.5564

Computer printout of σ_x and σ_y for line no. 1

σ_x

σ_y

SIGMAX(13) = -17.1759

SIGMAY(13) = -53.1473

SIGMAX(14) = -17.4351

SIGMAY(14) = -55.1335

SIGMAX(15) = -17.4351

SIGMAY(15) = -55.7151

Computer printout of σ_x and σ_y for line no. 1

σ_x	σ_y
SIGMAX(1) = -2.4450	SIGMAY(1) = 3.2664
SIGMAX(2) = -3.6385	SIGMAY(2) = 4.7573
SIGMAX(3) = -5.0331	SIGMAY(3) = 2.3951
SIGMAX(4) = -6.7173	SIGMAY(4) = 2.2865
SIGMAX(5) = -8.9991	SIGMAY(5) = 2.0740
SIGMAX(6) = -11.9825	SIGMAY(6) = 1.0774
SIGMAX(7) = -15.9186	SIGMAY(7) = -.4982
SIGMAX(8) = -18.5215	SIGMAY(8) = -2.9523
SIGMAX(9) = -21.4706	SIGMAY(9) = -10.0936
SIGMAX(10) = -22.3780	SIGMAY(10) = -19.6709
SIGMAX(11) = -22.2468	SIGMAY(11) = -32.5294
SIGMAX(12) = -22.0238	SIGMAY(12) = -42.1228
SIGMAX(13) = -22.4116	SIGMAY(13) = -50.2257

Computer printout of σ_x and σ_y for line no. 2

σ_x	σ_y
SIGMAX(14) = -23.1081	SIGMAY(14) = -54.7707
SIGMAX(15) = -23.7703	SIGMAY(15) = -58.0977
SIGMAX(16) = -24.2432	SIGMAY(16) = -60.8775
SIGMAX(17) = -24.3851	SIGMAY(17) = -61.8379

Computer printout of σ_x and σ_y for line no. 2

σ_x	σ_y
SIGMAX(1) = -1.1359	SIGMAY(1) = 3.8456
SIGMAX(2) = -2.3805	SIGMAY(2) = 3.0204
SIGMAX(3) = -3.8542	SIGMAY(3) = 2.3725
SIGMAX(4) = -5.5349	SIGMAY(4) = 1.3837
SIGMAX(5) = -7.3949	SIGMAY(5) = .6279
SIGMAX(6) = -9.4646	SIGMAY(6) = -.1007
SIGMAX(7) = -11.7551	SIGMAY(7) = -.7196
SIGMAX(8) = -14.5975	SIGMAY(8) = -1.5021
SIGMAX(9) = -17.9753	SIGMAY(9) = -2.4247
SIGMAX(10) = -20.2340	SIGMAY(10) = -4.4442
SIGMAX(11) = -23.1302	SIGMAY(11) = -10.1142
SIGMAX(12) = -24.8067	SIGMAY(12) = -23.5046
SIGMAX(13) = -24.5363	SIGMAY(13) = -34.7566

Computer printout of σ_x and σ_y for line no. 3

σ_x	σ_y
SIGMAX(14) = -24.5859	SIGMAY(14) = -46.1109
SIGMAX(15) = -25.9133	SIGMAY(15) = -54.6222
SIGMAX(16) = -26.9757	SIGMAY(16) = -58.8588
SIGMAX(17) = -27.4669	SIGMAY(17) = -61.0269
SIGMAX(18) = -27.7428	SIGMAY(18) = -62.4408
SIGMAX(19) = -27.7855	SIGMAY(19) = -63.2587

Computer printout of σ_x and σ_y for line no. 3

σ_x	σ_y
SIGMAX(1) = -.2220	SIGMAY(1) = 3.7245
SIGMAX(2) = -1.0608	SIGMAY(2) = 3.7926
SIGMAX(3) = -2.6628	SIGMAY(3) = 4.1076
SIGMAX(4) = -4.4027	SIGMAY(4) = 4.0692
SIGMAX(5) = -7.0812	SIGMAY(5) = .2443
SIGMAX(6) = -10.9049	SIGMAY(6) = -5.8100
SIGMAX(7) = -12.4635	SIGMAY(7) = -14.2716
SIGMAX(8) = -12.9094	SIGMAY(8) = -24.9895
SIGMAX(9) = -12.7357	SIGMAY(9) = -35.6539
SIGMAX(10) = -12.0956	SIGMAY(10) = -40.2951
SIGMAX(11) = -11.8132	SIGMAY(11) = -42.6275
SIGMAX(12) = -12.2520	SIGMAY(12) = -44.6520

Computer printout of σ_x and σ_y for line no. 4

	σ_x		σ_y
SIGMAX(1) =	.1370	SIGMAY(1) =	1.5885
SIGMAX(2) =	.2397	SIGMAY(2) =	2.2315
SIGMAX(3) =	.1198	SIGMAY(3) =	1.4793
SIGMAX(4) =	-.2911	SIGMAY(4) =	-.2911
SIGMAX(5) =	-1.0103	SIGMAY(5) =	-3.8843
SIGMAX(6) =	-1.6440	SIGMAY(6) =	-9.0796
SIGMAX(7) =	-1.7810	SIGMAY(7) =	-11.0970

Computer printout of σ_x and σ_y for line no. 5

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