

DEVELOPING BASIC CONCEPTS IN
NINTH GRADE ALGEBRA

BY
DARLE HEMMY

Submitted in partial fulfillment of the requirements
for the Master of Education degree
in the School of Education
Montana State College
August, 1956

P378
H376a
cop. 2

TABLE OF CONTENTS

Chapter	Page
I. INTRODUCTION	1
The Problem	1
The Need	4
Methods And Limitations	6
II. SIGNED NUMBERS	8
Use of Practical Application	8
Use of Charts	13
The Use of Tables	14
The Use of Geometric Relationships	16
Summary	22
III. EQUATIONS AND FORMULAS	23
The Use of Varied Solutions	23
The Use of Word Relationships	26
The Use of Models	32
Summary	36
IV. VERBAL PROBLEMS	37
Essential Elements of a Problem	37
Mathematical Language	38
Diagramming Problems	41
Summary	41
V. SUMMARY	43
To The Teacher	47
BIBLIOGRAPHY	49

LIST OF FIGURES

Figure	Page
1. Signed number scale	11
2. Have the student fill in the blank spaces	13
3. Some hay farmers make it a practice to fasten a tab on the wire of a bale of hay indicating the amount it weighs over or under 100 lbs. Have the student fill in the blank spaces	13
4. Multiplication table of signed numbers	15
5. On a straight line lay off equal units starting with 0, number each unit. Assign a and b a number then $a+b$ will be their sum and ab their product	16
6. Similar right triangles	17
7. Congruent triangles	18
8. Similar triangles in division of signed numbers .	19
9. Grove's Model	20
10. Archer's Device	34

CHAPTER I

INTRODUCTION

An important part of ninth grade algebra is the development of basic concepts in order to give the student the basic foundation for future study in mathematics and for use in everyday living. The ninth grade algebra teacher has an important task in helping the students develop such basic concepts as signed numbers, equations and formulas, and verbal problems. The writer contends that visualization can be helpful in developing these basic concepts in algebra. With this belief, the desire to provide the teacher with some of the methods and approaches that can be helpful in developing basic concepts in algebra through visualization has resulted in the making of this study.

The Problem

What are concepts and how are they developed? How does visualization affect the development of basic concepts? How can the teacher use visualization to help the ninth grade algebra student to develop the basic concepts? What are some methods and approaches for developing basic concepts in ninth grade algebra using visualization? The problem has resolved itself into answering these questions in a study of "Developing some basic concepts in ninth grade algebra through visualization."

The problem as presented needs clarification of the

meaning of developing basic concepts through visualization. The term "concept" while used frequently by people in the mathematics field, is rather hard to define in a few definite words. Van Engen had compiled some of the thinking of eminent mathematics writers on the meaning of the term, "concepts," as given in the following quotation:

In addition to being impressed by this ever recurring set of words one is conscious of a fuzzy use of words, at least in some instances. If there is no fuzziness there is no common agreement as to the precise use of some of the key words. For example, Woodworth, speaks of "induction or concept formation." This use of the word, "induction," may come as a surprise to mathematics teachers. Smoke, speaks of "concept formation, generalization, or concept learning." Other authors do not use the term, "generalization," in this way but subordinate to the term, "concept," Vinacke, in commenting on the fact that concept formation, as yet, is poorly understood says, "thus terms like, 'abstraction,' and, 'generalization' are still utilized without sufficient analysis of the behavioral and genetic processes involved."¹

The preceding description of the meaning of the term, "concept," or, "concept development," brings out such terms as, "induction," "generalization," and, "abstraction." Even this limited study of the development of concepts uncovers the existence of a closely knit set of terms which are somehow or other almost welded together. Engen² has maintained that there is a good reason for this, because these terms

¹Henry Van Engen, The Learning of Mathematics, Twenty-First Yearbook, 1953, p. 69.

²Ibid., p. 69.

refer to mental constructs which in themselves are concepts, and these concepts are essential for any intelligent discussion of the nature of concepts or the formation of concepts.

Fehr³ has referred to a concept as a mental image that can be related to other things through definition, laws of operation, application, or generalization. The term, "concept," in this paper then will have the meaning of a mental image or mental construct.

In developing these mental constructs or concepts, Fehr⁴ has emphasized the importance of the Law of Analysis, (similarity and dissimilarity), which says, "When a given response has been connected with many different situations which differ in all respects except one common element, the response becomes bound to that element." An example of the Law of Analysis is given by him in the following illustration:

Thus the S-R, $a^2 - b^2 = (a+b)(a-b)$, as a desired learning should be taught in many different situations involving a , b as a common element but always with the same response $(a+b)(a-b)$. Then when the right triangle occurs with hypotenuse r and one side x , the situation $r^2 - x^2$ should evoke $(r+x)(r-x)$ as the remaining side.⁵

The importance of similarity and dissimilarity in developing concepts is illustrated in the preceding example.

³Howard F. Fehr, The Learning of Mathematics, Twenty-First Yearbook, 1953, p. 8.

⁴Ibid., p. 8.

⁵Ibid., p. 9.

It may then be said that similarity and dissimilarity are an important part of developing basic concepts in ninth grade algebra. Jones⁶ believed that every teacher of mathematics discovers sooner or later that it is desirable to be able to explain problems and justify processes in more than one way as a student who is puzzled by one method of solution, may find another quite easy to follow.

From the preceding discussion, it seems likely that the student needs to compare and contrast in order to see or to visualize these concepts. In developing basic concepts through visualization, the term, "visualization," in this paper, mean the comparisions and contrasts necessary to develop these concepts or mental images.

The need for developing basic concepts in ninth grade algebra through visualization is brought out in the following section.

The Need

The importance of concepts in algebra, to both the students that continue in formal education and to those that go directly out into the community, has been brought out in the following statement by Brown:

The method and concepts of algebra are so much a part of scientific-refective thinking and writing that an individual who is ignorant of them will find the task of achieving any real understanding

⁶Phillip S. Jones, "Mathematics Miscellanecy," The Mathematics Teacher, December, 1951, p. 557.

of the modern world difficult, if not impossible. Unquestionable, it is possible for one to live a happy and useful life without ever having understood the concepts or mastered the techniques of algebra. But a knowledge of algebra can contribute so much to competence to cope with this modern world, and its concepts and techniques are so readily applicable to problems of daily living, that no individual who aspires to be other than, "a hewer of wood and a drawer of water," can afford to remain ignorant of the subject.⁷

The preceding statement points out that there is a need of these basic concepts for all of the students in ninth grade algebra regardless of their future work.

Breslich⁸ has expressed the belief that much of the difficulty the students experience in the study of mathematics is caused by the failure to understand the basic algebra concepts found at the ninth grade level, such as literal numbers, signed numbers, exponents, and equations. Brown has pointed out that the lack of these fundamental concepts in high school mathematics will limit the student's vocation as brought out in the following quotation:

Since success in a large and ever increasing segment of the professions and vocations of our civilization requires an extensive knowledge of science and mathematics, it would appear that the secondary school program, if it is to provide adequate opportunities for vocational exploratory experiences, must provide for extensive and serious work in these fields. Mathematics in particular constitutes a fundamental part of the basic preparation for so many occupations and professions that an individual who lacks mastery of its fundamental concepts will find his field of choice of

⁷E. R. Breslich, Models For Mathematics, p. 30.

vocation uncomfortably limited.⁹

The preceding discussion brings out how important basic concepts in ninth grade algebra are to the student. The students develop these basic concepts. Van Engen¹⁰ has maintained that the teacher often regards these basic concepts as simple, but that the student finds difficulty in developing basic concepts in algebra. The teacher then must understand the difficulties the student may have in developing these concepts.

The chapters of this paper will deal with various methods and approaches in developing basic concepts in ninth grade algebra through visualization. The methods used in this paper and the limitations of the paper are discussed in the next section.

Methods And Limitations

The problem as stated, "developing some of the basic concepts in ninth grade algebra through visualization," has already limited the basic concepts found in ninth grade algebra to a limited number. It was not felt that in the time available, all of the concepts in ninth grade algebra could be covered. The paper has dealt with only some of the more important concepts. The material that is presented has been

⁹Claude H. Brown, op. cit., p. 136.

¹⁰Henry Van Engen, op. cit., p. 95.

limited to the publications in the Montana State College Library.

The paper has been divided into three remaining chapters based on three basic concepts found in ninth grade algebra. The subjects covered will be "signed numbers", "equations and formulas", and "verbal problems". In each chapter a number of methods and approaches will be given for the development of the concept through visualization. The teacher may find these helpful in the improvement of the much needed basic concepts in her students.

Chapter II deals with the basic concept of signed numbers which the students many times find difficult to develop.

CHAPTER II

SIGNED NUMBERS

One of the concepts it is necessary to develop in ninth grade algebra is that of signed numbers. Signed numbers or directed numbers, as called by some authors, often present difficulties for students in understanding. The teacher may find the use of visualization helpful in developing the basic concepts of signed numbers. This chapter presents some of the methods and approaches for the use of practical application, the use of tables, the use of geometric figures, and the use of models in developing the basic concepts of signed numbers through visualization.

Use of Practical Application

Meyers¹, with the help of Dr. Harold P. Fawcett, showed how application can be used in developing the basic concepts of signed numbers by the use of the thermometer, the application of addition, number scale, and charts.

Use of The Thermometer. Meyers has suggested the use of the following group of questions on the simple Fahrenheit Thermometer to start the student on the road of developing the concept of signed numbers:

1. What is the reading on the thermometer?
2. Suppose the thermometer read -23° . What would be the meaning of the reading? What does the reading in question one mean?

¹Sheldon S. Meyers, "Applications," The Mathematics Teacher, March, 1952, pp. 210-214.

3. Locate and label the following readings on the thermometer scaled at the blackboard.
a. 46° , b. -3° , c. 63° , d. $-8\frac{1}{2}^{\circ}$, etc.

4. What is the reading halfway between the following readings? a. 74° and 110° , b. 0° and 10° , c. -8° and 8° , etc.

5. Which number is farther from 0° ? a. 6° or -10° , b. 26° or 18° , etc.

6. Which number is farther from 12° ? a. 18° or 8° , b. 26° or -5° , etc.

7. Which number is the farther from -2° ? a. 7° or -10° , b. 2° or -8° , etc.

8. If the thermometer starts at 40° and moves to 47° , how far has it moved? What was the direction of the change, up or down?

9. If the thermometer starts at 40° and moves to 33° , how far has it moved? What was the direction of change?

10. If the thermometer stands at 10° and then drops 16° , what will the new reading be?

11. If the thermometer stands at -12° and rises 5° , what will the new reading be?

12. If the thermometer stands at 23° and falls 27° , what will the new reading be?

13. If the thermometer moves from 14° to -10° , how far and in what direction (up or down) did it move?

14. If the temperature at the present time is -10° and we are told that the temperature has dropped steadily at the rate of $\frac{1}{2}^{\circ}$ every hour, was the temperature higher or lower six hours ago and by how much?

15. At 8 A.M. the temperature was 40° . If the temperature rises at the rate of 2° every hour, what is the temperature at 2 P.M.?²

The preceding group of questions has been used without previous orientation to signed numbers. The students are led to develop their own concept of the operation of signed numbers. The emphasis is on the approach rather than a claim of originality for specific application. A second group of problems may follow to lead the student to his own

²Ibid., p. 210.

rules for addition of signed numbers.

Application of Addition. Meyers has presented a group of problems that the student may use to develop the concept of addition in signed numbers in the following set of problems:

1. At various times during the first quarter of a game against Michigan the Buckeyes made the following four series of downs. Find the net gain or loss of yardage on each series of downs. a. 5 yds. gain, 3 yds. gain, 10 yds. gain, 7 yds. gain, 4 yds. gain; b. use lose and gain, c. etc.

2. In forecasting weather the hourly change in temperature is often more helpful than the actual temperature. What is the net change in temperature if the hourly change has been reported to be as follows: a. up 2° , up 5° , up 1° , up 4° ; b. use down readings; c. use up and down readings, d. etc.

3. What is the total change in value for the past week of one share of A. T. and T. stock if the daily changes are as follows: a. down 1, down $1/2$, down $1/8$, down $3/8$, down $1/8$, down $1/4$; b. use up, c. use up and down changes, etc.

4. In the middle of the 15th Century, the European merchants developed the practice of marking barrels and boxes with a positive sign (+) if they were over regulation weight and with a negative sign (-) if they were under. Thus +7 on a barrel meant 7 pounds overweight and -5 meant 5 pounds under regulation weight. If a merchant bought the following sets of barrels and boxes with the indicated markings find the total amount that each set was over or under weight. a. +15, +4, +6, +8; b. use negative; c. use both positive and negative, etc.³

The actual mathematical operation of these problems may not be new to many students. With their reference to signed numbers, the student may be well on his way to understand the addition operation of signed numbers by use of practical

³Ibid., p. 211.

application. The approach here is to let the student develop his own rules for the addition operation rather than have the teacher start with the rule and continue the explanation of the rule. The student's rule may vary in wording from the common rule, but the concept of signed numbers should be the same. Davis⁴ states the common rule as, "Thus, to add two numbers with like signs, prefix the common sign to their numerical sum; otherwise prefix the sign of the numerically larger number to their numerical difference."

The use of a number scale may be used to help the student develop the concept of the addition operation as brought out in the next section.

Number Scale. If the students have completed the problems in addition of signed numbers, they have already been informally brought into contact with the meaning of signed numbers. Meyers⁵ believed that the student may then use the number scale illustrated in Figure 1, to become familiar with the addition operation.

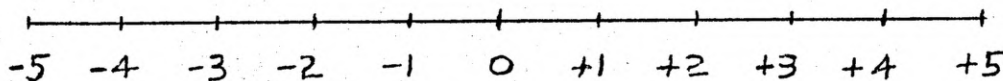


Figure 1. Signed number scale.

⁴David R. Davis, The Teacher of Mathematics, p. 198.

⁵Sheldon S. Meyers, op. cit., p. 212.

The suggestion is made by Meyers⁶ that the student make a story problem to fit one or more addition problems after examples of the addition operation has been presented. After the student understands the operation, he should present his own rule for addition.

These experiences stress the inductive, generalizing type of learning rather than the formal deductive type. Davis⁷ used an approach in developing the concept of signed numbers in much the same way. The number scale used by Davis is the same as Figure 1, with the following four associations made to the scale:

1. Kind of operation: + add; - subtract.
2. Direction: + rightward, upward; - leftward, downward (opposite direction).
3. Direction or signed numbers: + positive; - negative.
4. Effect on direction: + continuation in direction already established; - reversal of such direction.⁸

Although the first three interpretations are sufficient for explanatory puposes in algebra, the fourth one is also used.⁹ The rules are then developed jointly by the teacher and the student for future use in algebra. The subtraction operation in developing the concept of signed numbers in the preceding illustration.

⁶Ibid., p. 212.

⁷David R. Davis, op. cit., p. 198.

⁸Ibid., p. 198.

⁹Ibid., p. 198.

The following section presents the use of charts in developing the subtraction operation in signed numbers.

Use of Charts

Charts may be used as a way of introducing subtraction by practical application. Meyers¹⁰ suggested the charts in Figure 2 and Figure 3, as a means of using practical application to help develop the concept of signed numbers and the operation of subtraction.

New Temperature	Present Temperature	Difference	Direction
55°	45°	-	-
-10°	15°	-	-
etc.	etc.	-	-

Figure 2. Have the student fill in the blank spaces.

Friends Bale	Your Bale	Difference	More or Less Than Your's
8	5	-	-
6	-9	-	-
etc.	etc.	-	-

Figure 3. Some hay farmers make it a practice to fasten a tab on the wire of a bale of hay indicating the amount it weighs over or under 100 lbs. Have the student fill in the blank spaces.

The student may complete the column under difference

¹⁰Sheldon S. Meyers, op. cit., p. 213-214.

and direction in Figure 2, and under difference and more or less than your's in Figure 3. The use of charts similar to Figure 2 and 3, with different subject matter, could also be used. The previous experience and familiarity of the subject to the student should be taken into consideration.

The multiplication table for signed numbers may be another approach for developing the concept of signed numbers.

The Use of Tables

Jones¹¹ has maintained that the use of the multiplication table for signed numbers offers a good means of helping develop the basic concept of signed numbers in algebra. In the familiar multiplication table for positive integers and zero, when one of the factors is decreased by one, the product is decreased by the other factor. Why not have the student continue this table to apply when one or both of the factors is a negative integer? In Figure 4 a small part of such a table is illustrated.

¹¹Philip S. Jones, "Mathematical Miscellanea", The Mathematics Teacher, December, 1950, p. 420.

$(-2) \times (-5) = 10$	$(-1) \times (-5) = 5$	$0 \times (-5) = 0$	$1 \times (-5) = -5$	$2 \times (-5) = -10$
$(-2) \times (-4) = 8$	$(-1) \times (-4) = 4$	$0 \times (-4) = 0$	$1 \times (-4) = -4$	$2 \times (-4) = -8$
$(-2) \times (-3) = 6$	$(-1) \times (-3) = 3$	$0 \times (-3) = 0$	$1 \times (-3) = -3$	$2 \times (-3) = -6$
$(-2) \times (-2) = 4$	$(-1) \times (-2) = 2$	$0 \times (-2) = 0$	$1 \times (-2) = -2$	$2 \times (-2) = -4$
$(-2) \times (-1) = 2$	$(-1) \times (-1) = 1$	$0 \times (-1) = 0$	$1 \times (-1) = -1$	$2 \times (-1) = -2$
$(-2) \times 0 = 0$	$(-1) \times 0 = 0$	$0 \times 0 = 0$	$1 \times 0 = 0$	$2 \times 0 = 0$
$(-2) \times 1 = -2$	$(-1) \times 1 = -1$	$0 \times 1 = 0$	$1 \times 1 = 1$	$2 \times 1 = 2$
$(-2) \times 2 = -4$	$(-1) \times 2 = -2$	$0 \times 2 = 0$	$1 \times 2 = 2$	$2 \times 2 = 4$
$(-2) \times 3 = -6$	$(-1) \times 3 = -3$	$0 \times 3 = 0$	$1 \times 3 = 3$	$2 \times 3 = 6$

Figure 4. Multiplication table of signed numbers.

Jones used the table in Figure 4 to illustrate the following relationships:

In the 2-table, as the multiplier is decreased from zero to -1, the product will drop to -2, and as we move further up, lowering the multiplier by one, the product is diminished by 2 each time, to -4, -6, -8, etc. Similarly in the 3-table in the bottom row, as we lower the first factor from zero to -1, the product is diminished by the other factor (in the same case by 3) and becomes successively, -3, -6, -9, etc. Turning now to the top row where the "minus five" table is being formed, we are to obtain each product from its right-hand neighbor by subtracting minus five each time. These same results are also obtained as we move up the columns. As we keep on moving upward and to the left, fitting in rows and columns, we obtain a novel, understandable, unforgettable array.¹²

The student may gain more understanding of signed numbers by developing and analyzing such a multiplication table

¹²Ibid., p. 421.

and thus help set in the students mind the concept of signed numbers.

As mentioned in the introduction, concepts may be mental images and therefore figures may help develop these concepts. Geometric figures, as illustrated in the next section, when related to algebra may help give the student these mental images.

The Use of Geometric Relationships

The use of geometrical concepts may be somewhat limited in developing basic concepts in ninth grade algebra because of the limited previous use of geometry. However, the use of geometric concepts in developing concepts in algebra, as brought out in this paper, deal with the ninth grade algebraic concepts. Some of these concepts may be used in the actual teaching in ninth grade algebra depending upon the student's background.

Meserve¹³ showed that if given any two numbers a , b ; we define $a < b$ if a is to the left of b (Figure 5) $a = b$, if a and b coincide $a > b$ if a is to the right of b .

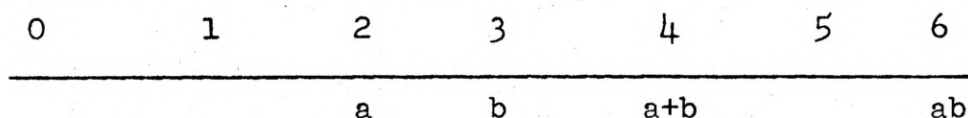


Figure 5. On a straight line lay off equal units starting with 0, number each unit. Assign a and b a number then $a+b$ will be their sum and ab their product.

¹³B. E. Meserve, "Using Geometry in Teaching Algebra," The Mathematics Teacher, December, 1952, p. 568.

In Figure 5 we associate $a+b$ with a point b units to the right of a and consider ab as the sum of b elements a . The number ab may also be determined using similar triangles as illustrated in Figure 6.¹⁴

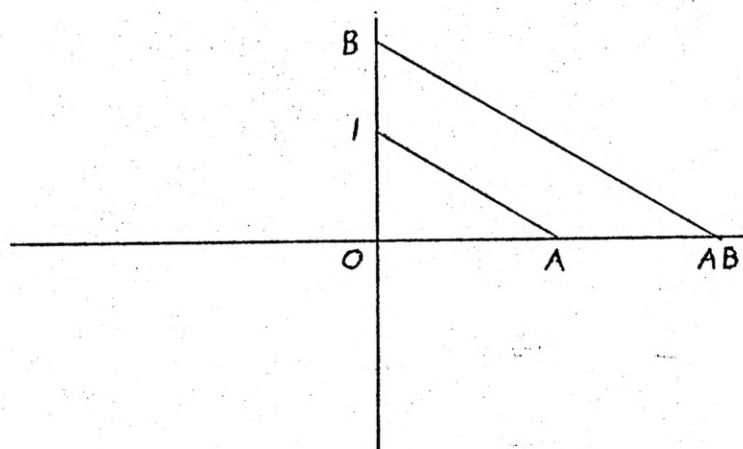


Figure 6. Similar right triangles.

Addition and subtraction are inverse operations in the sense that $(a+b)-b = a$ for any positive integers a, b . Since $a+b$ is b units to the right of a ; $(a+b)-b$ is b units to the left of $a+b$. (Figure 6). Accordingly we define $0-b$ as $-b$ and associate negative integers with units marked off to the left of the origin.¹⁵

Multiplication of signed integers may be interpreted

¹⁴Ibid., p. 569.

¹⁵Ibid., p. 570.

either as repeated addition or in terms of similar triangles. When similar triangles are used the order relation in geometry imply the usual order relations for the numbers. For example if a and b are any two positive integers, we may take the point of intersection as 0 on both axes, determine a on one axis; determine 1 and b on the other, draw the line joining 1 and a , and determine ab by drawing a parallel line through b as in Figure 6.¹⁶ We next determine $-b$ and $a(-b)$ as in Figure 7, and show by congruent triangles that under the postulates of our geometry $a(-b) = -ab$.

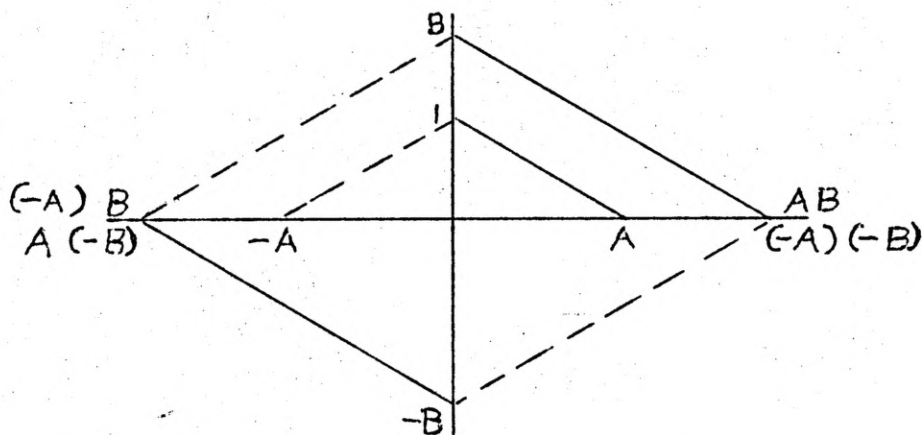


Figure 7. Congruent triangles.

Similarly in using $-a$ we may show that $(-a)b < 0$, $(-a)b = -ab = a(-b)$, $(-a)(-b) > 0$, and $(-a)(-b) = ab$.

Division may be defined as the inverse of multiplication

¹⁶Ibid., p. 571.

in the sense that $(ab) \div b = a$ when $b \neq 0$. Under the assumption that $b \neq 0$, we may construct a/b for any integer a as follows; determine 0 and 1 on the number line axis, draw a second axis intersecting the number line at 0, on the second axis take 0 coinciding with the 0 on the first axis and determine 1, a and b ; draw a parallel line through 1 to obtain $1/b$ and in general through a to obtain a/b .¹⁷ (Illustrated in Figure 8)

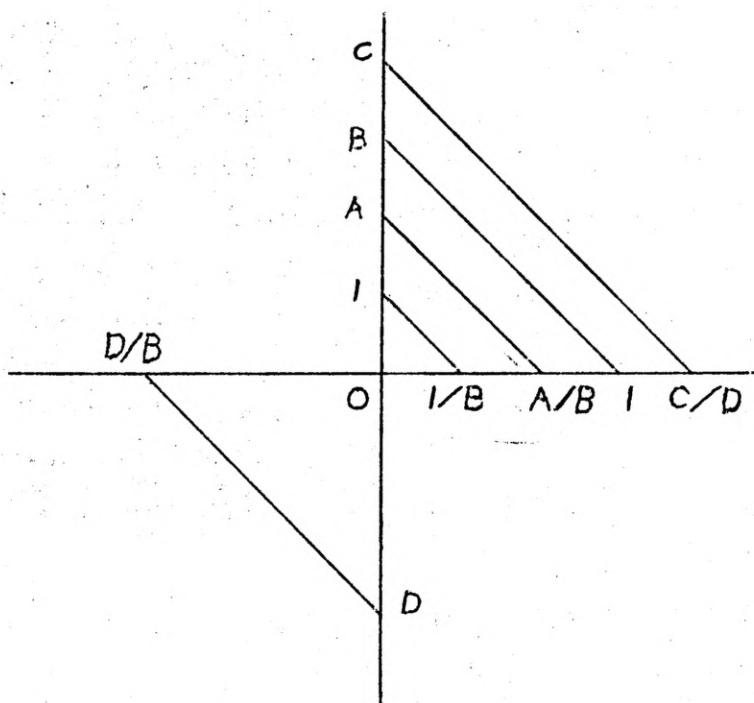


Figure 8. Similar triangles in division of signed numbers.

When $b \neq 0$, $1/b \neq 0$. If $0 < a < b$ then $0 < a/b < 1$. If $0 < b < c$ then $1 < c/b$. If $d \neq 0$ and $0 < b$ then $d/b \neq 0$.

Use of Models

The student may find the use of models helpful in

¹⁷Ibid., p. 572.

developing basic concepts in ninth grade algebra through visualization. Grove¹⁸ suggested one type of model that is designed to help develop the basic concept of signed numbers. The model is constructed with one strip of stiff cardboard approximately 5" x 10", two dozen spring-type clothespins, red and black ink, and pen or crayons. On the upper edge of the cardboard strip, starting $\frac{1}{4}$ " from the left hand end, mark off 26 three-quarter-inch spaces and number these division points consecutively from -12 to 14. Set numbers back $\frac{1}{2}$ " from the edge and use red ink for negative numbers and black ink for positive values. Color one clothespin black or some other convenient color. An illustration of the model is pictured in Figure 9.

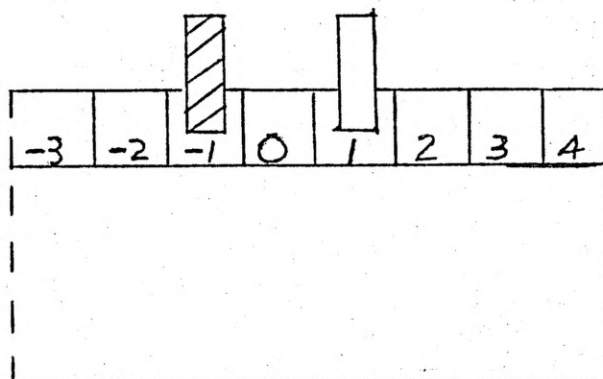


Figure 9. Grove's Model.

¹⁸Ethel L. Grove, Ewart L. Grove, "Number Scale For Illustrating All Four Fundamental Process With Directed Numbers," School Science And Mathematics, June 1951, pp. 456-460.

In Grove's Model,¹⁹ one addend indicates the beginning quantity and its position right or left of zero. The other addend shows the number of units to be combined with the beginning quantity and the direction of this operation. In all cases the starting point for the first addend is zero, and the values to the right of zero or changes in value from left to right are considered positive and values to the left of zero or changes in values from right to left are considered negative.

This model may also be used to show the operation of subtraction. The difference between two algebra quantities is usually defined to be the number of units that must be combined with the subtrahend to produce the minuend together with the sign indicating the direction of this operation. Hence, since the starting point for the difference is the subtrahend and differs in each case, mark the point with a black or other colored pen which will not be considered as a part of the result.²⁰

Grove's Model²¹ may be used for the operations of multiplication and division. Multiplication by a positive multiplier is usually considered to mean the repeated addition of the multiplicand the number of times indicated by the multiplier, starting from 0. Since $(-a)(b) = (b)(-a)$,

¹⁹Ibid., p. 457.

²⁰Ibid., p. 458.

²¹Ibid., p. 459.

multiplication by a negative multiplier may be considered as negative additions or repeated subtractions of the multiplier and the number of times indicated by the multiplier, starting from 0. Division is the reverse of multiplication or a system of repeated subtraction as addition was in multiplication.

Grove's Model, as shown in Figure 9, illustrated a very easy and workable model for use in signed numbers. It provides a method for developing the concept of signed numbers through visualization.

In the next chapter, methods and approaches are given for the development of the concepts in the area of equations and formulas.

Summary

In developing concepts in signed numbers, inductive reasoning may be used with comparisons, contrasts, and mental pictures. Practical applications, such as the thermometer, offers comparisons and contrasts along with a mental picture. The number scale shows the operations of addition, subtraction, multiplication, and division in a way that should help develop concepts. "Grove's Model" as an elaborate number scale will do the same. Charts and tables may also be used to help the student develop the concepts of signed numbers through visualization.

In the next chapter, methods and approaches are given for the development of concepts in the area of equations and formulas.

CHAPTER III

EQUATIONS AND FORMULAS

Developing concepts of equations and formulas are necessary for the ninth grade algebra student. Some ninth grade algebra students find it difficult to develop these concepts of equations and formulas. Therefore, it may be necessary for the teacher to devise varied methods and approaches to enable the student to develop the mental images or concepts of equations and formulas. This chapter presents some of the methods and approaches for the use of varied solutions, the use of word relationships, the use of graphs, and the use of models in developing the basic concept of equations and formulas through visualization.

The Use of Varied Solutions

The introduction of this paper brought out the importance of varied methods and approaches because all students do not respond equally to the same method or approach. Jones used this same idea in proposing eighteen solutions for any equation. If the student does not respond to one, he may try another. The eighteen methods are listed in the following quotation:

1. By factoring by inspection.
2. By factoring after a substitution $z=ax$, which leads to $z^2+bz+ac = 0$.
3. By factoring in pairs by splitting bx into two terms.
4. By completing the square when a is 1 and b is even.

5. By completing the square as usual after dividing through by a .
6. By completing the square by the Hindu method by multiplying through by $4a$ and adding b^2 to both sides.
7. By completing square as given, adding $b^2/4a$.
8. By the formula.
9. By trigonometric methods.
10. By slide rule.
11. By graphing for real roots.
12. By graph, extended for complex roots.
13. Real roots by Lill circle.
14. By extension of Lill circle to include complex roots.
15. Using the graph of $y=x^2$ and $y=-bx$ to find real roots.
16. By extending (15) to include complex roots.
17. By use of a table of quarter squares. This is a practical method of handling an equation having large constants, as we already have the table in print.
18. By use of "Form Factors".¹

The preceding list of solutions show a great variation to methods of solving equations. It certainly would not be necessary to use them all for one class of students, but it may provide the teacher with varied methods that she may use when needed.

Fehr² has stated that, "The analysis of an equation is an examination within the mind of the student, of the situation in which there is dissatisfaction. He discovers why he is dissatisfied and clarifies the goal that would give satisfaction." This is related directly to the trial and error solution of an equation or sometimes called the analyzing

¹Howard F. Fehr, The Learning of Mathematics, Twenty-First Yearbook, 1953, p. 34.

²Ibid., p. 35.

relationship. Fehr brought this out in the following example:

Recalling the solution of linear equations, a student may try putting x terms on one side of a quadratic equation, and the constant on the other side. If the equation is $x^2 = 25$, the hypothesis may prove fruitful, but if the equation is $x^2 + x = 25$, the trial fails. Then a new response is made. It may be suggested that the equation can be written in the form $x^2 - 4x + 3 = 0$ and the student asked to focus attention, for the time being, on the left hand member. Here is a recall of identical elements and perhaps with it the response $x^2 - 4x + 3 = (x-3)(x-1)$. This pattern suggests two linear equations so let each factor be zero. This gives $x = 3$ or $x = 1$ and either answer satisfies.³

The analyzing relationship may also be thought of in a different manner than the illustration mentioned in the preceding discussion. This is the solving of equations by mental substitution. Jones⁴ believed that the method of solving by mental substitution may help develop the basic concept of equations. He suggests that the solution of simple equations by mental substitution may be extended to more difficult types of equations as illustrated in the following example:

In the study of equations, for example, the method of solving by mental substitution may not only be used to introduce the topic, but also may well be extended to more difficult types of equations. Left to his own devices, the better student in this age group will solve equations of the type, $3x+2 = 8$, $5y-1 = 14$, $x/3+2 = 17$, and $n/2-5 = 10$ by mental substitution. His thought process in

³Phillip S. Jones, "Mathematical Miscellanea," The Mathematics Teacher, March, 1951, p. 194.

⁴Ibid., p. 195.

solving the first equation is about as follows:
 "3x plus 2 equals 8. Since 6 plus 2 equals 8, 3x must be equal to 6 and x equals 2," or, in the last equation above: "n/2 minus 5 equals 10, so n must be equal to 30."⁵

The preceding examples of analyzing relationships may be helpful in developing the concepts of equations. The equation can also be closely related to words as discussed in the following section.

The Use of Word Relationships

Brown⁶ stated that, "It is desirable that the pupils' concept of an equation shall involve three elements. First, the equation is a sentence; second, when an unknown is involved it is an interrogative sentence; and third, it represents an equality or a balanced situation." This statement puts great importance on word relationships in developing the concept of equations.

Schaaf suggested the use of business formulas in developing the basic concept of equations by word relationship in the following example problems:

1. The cost of goods sold during a given period:

$$C = I_1 + P - I_2$$

where I_1 = initial inventory, I_2 = final inventory, and P = purchases during the period.

2. Average inventory:

$$I = \frac{I_1 + I_2}{2}$$

⁵Ibid., p. 196.

⁶Claude H. Brown, The Teaching of Secondary Mathematics, p. 196.

3. Annual depreciation (straight line method):

$$D = \frac{C-S}{n}$$

where C = initial cost, S = scrap value, and n = estimated life of asset in years.

4. Book value (straight line depreciation) after k years:

$$B_k = \frac{C-k(C-S)}{n}$$

5. Amount paid on a fire insurance claim under an 80% coinsurance clause:

$$A = \frac{PL}{.8V}$$

where A = amount paid on claim, P = face value of policy, V = value of the property, and L = amount of loss as determined by the insurance adjuster.⁷

Schaaf illustrated some additional word relationships that may be used to develop the concept of equations in the following examples:

If we use this relationship:

P principal or present value.

S principal interest, or future value.

I interest.

D discount.

i interest rate.

d discount rate.

n time in years.

then these formulas may be obtained:

1. $I = Pni$

2. $S = P + I$

3. $I = S - P$

4. $S = P(1 + ni)$

5. $P = \frac{S}{1 + ni}$

1a. $D = Snd$

2a. $S = P + D$

3a. $D = S - P$

4a. $S = \frac{P}{1 - nd}$

5a. $P = S(1 - nd)$

Now by equating the right-hand members of equations

⁷William L. Schaaf, "The Use of Simple Algebra in Business," The Mathematics Teacher, January, 1951, p. 25.

(5) and (5a), it follows that

$$i = \frac{d}{1 - nd}$$

and

$$d = \frac{i}{1 - ni}$$

In other words, the interest is the future value of the discount, and the discount is the present value of the interest. Other relationships may also be made.⁸

Meyers⁹ suggested the use of suitable physical laws for developing the concept of equations and formulas through word relationships. The laws that may be used are the law of levers, the law of falling objects under gravitational pull, and the relation between Fahrenheit and Centigrade Temperature.¹⁰ This activity is very appropriate in beginning algebra where the phenomena investigated serve to illustrate various types of functions in algebra. The slope-intercept form of linear equations can also be used to derive from the graph of the data the formula

$$F = 9/5C + 32.11$$

Jones¹⁰ also mentioned the use of graphs in developing the concept of equations in the same manner. The use of graphs will be discussed more fully in the next section.

The Use of Graphs

Dodes listed a number of questions that should be used

⁸Ibid., p. 26.

⁹Sheldon S. Meyers, Laboratory Techniques With Specific Practice, The Mathematics Teacher, February, 1952, p.107.

¹⁰Ibid., p. 108.

in the approach for using graphs to develop concepts in equations in the following quotation:

1. How do we represent a pair of numbers on the graph?
2. How many numbers are attached to each point on the graph?
3. How do we represent a linear equation like $y = x$?
4. What do you call the straight lines that are drawn before the graph is drawn?
5. How many pairs of numbers do we need to have to draw a straight line graph?
6. What is the third pair of numbers used for?
7. Suppose we took any pair of numbers whatsoever. Would the point necessarily be on this straight line?
8. When would it be on the line?
9. Suppose I chose any point on the line $y = 2x$. What would be the relationship between the two numbers?
10. Suppose I chose a point not on $y = 2x$, what could you say about the relationship between the two numbers?
11. Why do you suppose that the straight line $y = 2x$ is often called a picture of the equation $y = 2x$?¹²

The preceding list of questions should be answered by illustration on the blackboard.¹³

Brown¹⁴ believed that in algebra the fundamental concept to be considered is the function in relation to the solution of real situations and the functional relationship may be expressed by equation, the formula (which is only a specialized form of the equation), and the graph. Therefore,

¹²Irving Allen Dodes, "Planned Instruction," The Mathematics Teacher, 1953, p. 331.

¹³Ibid., p. 330.

¹⁴Claude H. Brown, op. cit., p. 209.

these three should become the fundamental concepts of a course in algebra and the mechanics included in the junior high school algebra should be those needed for the solution of problems by means of these concepts.¹⁵

Reeve¹⁶ believed the graph shows best the functional relationship that one quantity depends upon another. The graph is an algebra picture to be used for comparing one quantity to another, as illustrated in the following example.

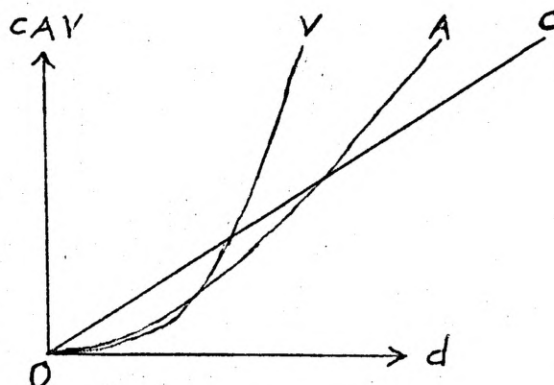
The graph helps to show one quantity depends upon another as in the following example.
If we take the three formulas

$$\begin{aligned} C &= d \\ A &= \frac{1}{4} d^2 \\ V &= \frac{1}{6} d^3 \end{aligned}$$

this table follows,

If	d is	0	1	2	3	4	5	etc.
then	C is	0	3.1	6.3	9.4	12.6	15.7	
and	A is	0	.8	3.1	7.1	12.6	19.6	
and	V is	0	.5	4.2	14.1	33.5	65.4	

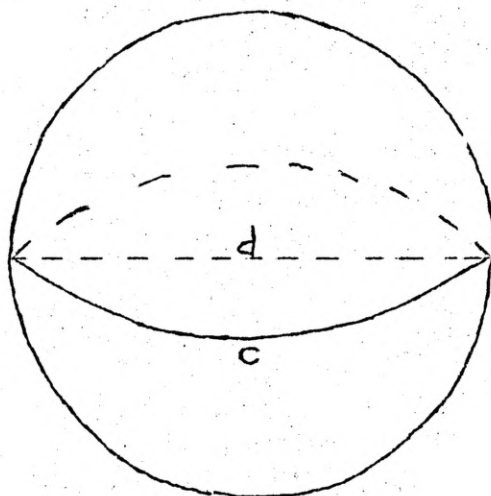
and this graph follows,



¹⁵Ibid., p. 210.

¹⁶William D. Reeve, Mathematics For Secondary Schools, p. 124.

The three formulas and their respective graphs which appeared in the preceding figure are related to a sphere of diameter as shown in the following figure.



The circumference of the circle whose plane passes through the center of the sphere is represented by C , A is the area of this circle, and V represents the volume.¹⁷

The preceding illustration shows a method of developing the mental image or concept by giving the student a visual picture of the algebra operation.

The solution of the quadratic equation $x^2+bx+c = 0$ can be obtained graphically by finding the points of intersection of the parabola $y = x^2$ and the straight line $y = -bx-c$. The abscissa of these points are the desired roots of the given quadratic equation. Berger¹⁸ has designed a special graph

¹⁷Ibid., pp. 125-126.

¹⁸Emil J. Berger, "Devices For a Mathematics Laboratory", The Mathematics Teacher, October, 1952, p. 439.

on which it is possible to find these roots mechanically by painting a permanent graph of $y = x^2$ on a piece of plywood and attaching a rotating and sliding bow to the back of the board in such a way that it will carry a string across the front of the graph. By adjusting the bow the string can be made to represent any line with a slope of $-b$ and y -intercept of $-c$.¹⁹ In this way it is possible to represent any line of the form $y = -bx - c$. This special graph should help the student form a picture of the functional relationships of the quadratic $x^2 + bx + c = 0$.

The special graph just mentioned could very well be called an algebraic model. In the next section more algebraic models that may help the student develop the concepts of equations through visualization will be illustrated.

The Use of Models

There is evidence that the wheel, the lever, and the gear employ and embody the concept of the variable, the linear function and the four fundamental operations of an equation as shown by Denbow.²⁰ The relationship between mathematics and machinery is developing a set of devices, called variators, whose purpose is to help pupils to see and understand algebraic concepts.²¹

¹⁹Ibid., p. 440.

²⁰Carl H. Denbow, "The Use of Variators in the Teaching of Algebra," School Science and Mathematics, January, 1954, p. 65.

²¹Ibid., p. 65.

The simplest type of variator is the flexible rubber band, on which are evenly spaced marks representing zero, x , $2x$, etc.²² When the zero point of this band is placed on the zero of a ruler or scale, and the band stretched until the x point is on 2, say, then the next mark will be on 4, the next on 6, etc. The student can use it to evaluate $3x$ when $x = 2$ for example and conversely to solve $3x = 6$ and find x .²³ The rubber band is limited because of inaccuracy and clumsiness.

The principle of the wheel makes a more useful variator.²⁴ That is we can attach chains to a sprocket wheel, at different distances from its center, so that turning the wheel will move the chains along scale pointers which are marked x , $2x$, $3x$, and are attached to three such chains.²⁵ When the x point is at 2.5, the $2x$ pointer is at 5, $3x = 7.5$, and etc.

Denbow²⁶ has suggested the use of a bar linkage, resembling a set of pantographs joined end to end, or similar to the extensible porch gates used to constrain children, mounted on a scale, can be used to demonstrate algebraic concepts. A circular slide rule, with several concentric

²²Ibid., p. 66.

²³Ibid., p. 66.

²⁴Ibid., p. 66.

²⁵Ibid., p. 66.

²⁶Ibid., p. 66.

uniform scales, embodying the principle of the wheel, may be used.²⁷ Also, the means of pointers moving on a scale can be used to consider the relation between variators and graphical methods of solving equations.²⁸

The graph variator consists of certain lines drawn on an xy coordinate system plus a sliding y axis which can be moved to the right or the left, remaining always parallel to the y axis.²⁹ To solve $1/3x = 2$, for example, on the graph variator, a movable marker is placed at 2 on the sliding y axis, which is then moved, to the right or the left, until this marker meets the line $y = 1/3x$. The answer, $x = 6$ is read off on the x axis.

Archer designed a device for picturing the operation of the Pythagorean Theorem as shown in the following figure:

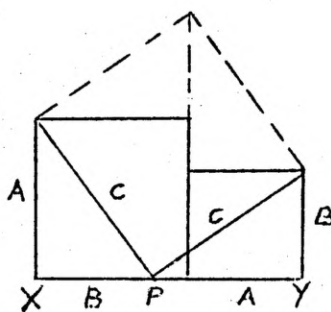


Figure 10. Archer's Device.³⁰

²⁷Ibid., p. 67.

²⁸Ibid., p. 67.

²⁹Ibid., p. 68.

Allene Archer, Emerging Practices in Mathematics, Twenty-Second Yearbook, 1954, p. 140.

As shown in Figure 1, draw two squares side by side on 14-ply cardboard. Label the side of one square a and the side of the other square b . Cut out the square, but leave them joined together. Mark off a point P which is at a distance a from y and a distance b from x . With a as the altitude and b as the base, draw a line c which will be the hypotenuse of a right triangle. Cut off these triangles. To a piece of 14-ply cardboard, paste the parts of the two squares which are left, then fasten the triangle to the board to form the two original squares. Finally, move the squares and place them as shown in the figure where one square is formed with c , the hypotenuse of the right triangle, as its side.

Jones³¹ mentioned a method of making a model to picture equations by making a template of transparent plastics, very slightly smaller and parallel to the unit parabola $y = x^2$, so that the pencil or ruling pen will draw the true curve. Scratch coordinates on it for accurate placing, and for any equation move the template to a position parallel to the y axis and with its vertex at the point

$$\frac{-b}{2a}, \frac{4ac-b^2}{4c}.$$

While this can't be called a different method of picturing an equation, it should be included as a very handy and practical variation of the approach in one unknown.

³¹Jones, op. cit., p. 194.

Summary

The use of varied methods and approaches are necessary in developing concepts in equations and formulas. The analyzing relationships may be used to develop concepts through visualization as well as word relationships. The graph gives the student a mental picture of the functional relationships of equations for developing these concepts. The many types of variators provide excellent ways to help develop concepts through visualization.

Chapter IV deals with the basic concepts in verbal problems in ninth grade algebra.

CHAPTER IV

VERBAL PROBLEMS

The ninth grade algebra student often finds trouble in developing the basic concepts necessary for solving verbal problems. This chapter presents some of the methods and approaches for developing the basic concepts of verbal problems through visualization.

Essential Elements of a Problem

Henderson¹ stated, "The primary function of the verbal problem is to give meaning to and provide practice in applying mathematical generalizations and concepts. The use of verbal problems broadens the meaning of the concepts dealing with the method of solving problems."

O'Brien mentioned a number of ways teachers may help the students develop these concepts in the following list:

1. Encourage the student to estimate the size of the answer before he begins.
2. This suggestion is a don't. Don't do the problem for the student, let him poke around in it himself. We teachers may give him hints and clues, ask him leading questions, but we shouldn't do the problem for him.
3. The student should be led to acquire the habit of reading the problem again as soon as he has arrived at the value of x .
4. Has he labeled the answers correctly?
5. It is good to surprise a class occasionally with a problem that results in an indeterminate equation or a problem whose solution is

¹ Kenneth Henderson, Emerging Practices in Mathematics, Twenty-Second Yearbook, 1954, p. 235.

impossible under the conditions given.²

The preceding suggestions should help the teacher provide a method for developing the basic concepts needed for verbal problems.

Intelligent living requires that we transmit thought and that we communicate by means of language. The same is true in mathematics which is brought out in the next section.

Mathematical Language

Mathematical language facilitates thinking by complementing ordinary language and it also suggest solutions to problems.³ Let us look at an example. Measurements have shown that gold loses about one-nineteenth of its weight if it is weighed in water rather than in air.⁴ Similarly, silver weighed in water loses about one-tenth of its weight.⁵ If, then, a quantity of an alloy of gold and silver weighs 12 ounces in air and 11.16 ounces in water, how much of the alloy, weighed in air, is gold and how much is silver?⁶ The power of mathematical symbolism appears when we let n represent the number of ounces of gold, weighed in air, in

²Katharine O'Brien, "Problem-solving," The Mathematics Teacher, February, 1956, p. 79.

³Irvin Brune, The Learning of Mathematics, Twenty-First Yearbook, 1953, p. 158.

⁴Ibid., p. 159.

⁵Ibid., p. 159.

⁶Ibid., p. 159.

the alloy.⁷ Then $12-n$ represents the number of ounces of silver in the alloy and the equation, $18/19n \times 9/10(12-n) = 11.16$, both states concisely the condition of the problem and suggests clearly how to solve the problem.⁸ The results $n = 7.6$ and $12-n = 4.4$, represents ounces of gold and silver in the alloy.⁹

Obviously faulty language can confuse the process of reasoning. Brune gave a few examples that indicate how language confuses conclusions:

1. "Like signs give plus." Pupils glibly say these words, and reach wrong conclusions in algebra.

2. Mr. Parfit joined a cult and testified that "the new order changed the course of my life 360°." Somewhere, somehow language led Mr. Parfit to a ridiculous conclusion.

3. A junior high school pupil stated that "he could solve a problem about ages, by mathematics but not by algebra." Language apparently threw him for a loss here.

4. Usually no fewer than 90 per cent of a high school class will choose an income of \$4000 annually with an annual increase of \$200 as preferable to an income of \$2000 semiannually with a semiannual raise of \$150. Language makes the poorest proposition seem to be the better.¹⁰

One way of introducing mathematical language is by considering word meaning. O'Brien suggested that certain word meanings can be considered as derivations from Latin, as

⁷Ibid., p. 160

⁸Ibid., p. 160.

⁹Ibid., p. 160.

¹⁰Ibid., p. 161.

listed in the following examples:

1. tangent: tangere, touch.
2. secant: secare, cut.
3. quotient: quat, how many.
4. subtend: tendere, stretch; sub-, under.
5. adjacent: jacere, lie; ad-, at, to.
6. exponent: ponere, place; ex-, out.
7. opposite: ponere, place; ob-, before.
8. transversal: vertere, turn; trans-, across.
9. circumscribe: scribere, write; circum-,
around.
10. concurrent: currere, run; con-, together.
11. consecutive: sequi, follow; con-, to-
gether.
12. commutative: mutare, change; com-, to-
gether.
13. equivalent: valere, be worth; aequi-, of
equal.
14. contradiction: dicere, speak; contra-,
against.
15. interpolate: polere, polish; inter-, be-
tween.
16. quadrilateral: later, side; quadi-,
having 4.
17. perpendicular: pendere, hang; per-,
thoroughly or perfectly.¹¹

To be able to translate from one language to another, a person must understand what is said in the first language and the vocabulary, syntax, and idiomatic structure of the second. It is common experience that it is harder to translate into a foreign language than from it.¹² This may be true of algebraic problems so why not give more practice in translating from algebra into words as a preliminary to translating words into algebra?¹³

¹¹O'Brien, op. cit., p. 80.

¹²Ibid., p. 81.

¹³Henderson, op. cit., pp. 253-255.

Diagramming Problems

Henderson has maintained that the teacher can help the student develop problem concepts through visualization by diagramming a problem. He has also shown that diagramming a problem situation is a very helpful experience to many students. The diagram may be a sketch of the situation or it may be a symbolic diagram. The following exercise is a good example, "A ship is steaming toward Philadelphia at an average rate of 30 miles per hour. It radios that a person is ill aboard and must be picked up. When a plane leaves Philadelphia for the ship, the ship is 270 miles away. The plane averages 180 miles per hour. How long after the plane leaves Philadelphia does it reach the ship?" The diagram may be just a crude sketch to help the student keep the facts in mind or the diagram can be drawn to scale to give the student a check on his answer. The student may also want to use a graphic solution.

Summary

The use of verbal problems broadens the meaning of the concepts dealing with the method of solving problems. Mathematical language is essential in developing concepts in problem solving. It is obvious that faulty language can confuse the process of reasoning. Mathematical language may be introduced by considering word meaning and by word relationships

to algebraic symbols. The use of diagrams and graphs are also helpful in developing concepts in verbal problems through visualization.

CHAPTER V

SUMMARY

The concept is a generalization in the form of a mental image which enables a student to respond to different situations. It is necessary that the student develop these concepts to achieve understanding in life situations.

In ninth grade algebra, the student should develop certain basic concepts to form a basic foundation for later study and for life situations. The teacher can help the student develop these basic concepts through visualization. There are many methods and approaches that are available to the teacher for developing basic concepts in ninth grade algebra through visualization.

The paper limited the study of the methods and approaches of developing basic concepts through visualization to only the basic concepts found in the areas of signed numbers, equations and formulas, and verbal problems.

The study of the uses of visualization in developing the basic concepts of signed numbers was made in four areas. They were the use of practical applications, the use of tables, the use of geometric figures, and the use of models.

The use of practical applications in developing basic concepts in ninth grade algebra through visualization offers varied methods and approaches. The thermometer offers many uses as does the application of addition and the number scale.

The thermometer with its positive and negative readings provides a method of visualizing the operation of signed numbers. The thermometer may be used to introduce signed numbers by letting the students develop their own concept of the operation of signed numbers through a group of specially prepared questions.

Through the use of a group of everyday problems, involving the application of addition, the student should be able to visualize the addition operation and form his own rule for that operation.

The number scale provides a means by which the student may visualize any of the operations of signed numbers. By actually seeing the operations carried on by a number scale should help the student develop his needed concepts in the operations of signed numbers. These experiences stress the inductive, generalizing type of learning rather than the formal deductive type.

Charts may be used as a way of introducing operations of signed numbers by practical application. By having the student complete information that is already familiar to them in a chart may provide a way for the student to visualize the concept of the operation.

Tables provide another means of developing concepts through visualization. The multiplication table is the most familiar and the most commonly used. The student may gain more understanding of signed numbers by analyzing the multiplication table.

The use of geometric concepts in developing algebraic concepts may be somewhat limited because of the lack of background in geometry for ninth grade students, however, the straight line layed off in equal units, similar right triangles, and congruent triangles are examples that may be possible to use in ninth grade algebra depending on the student's background.

The student may find the use of models helpful in developing basic concepts in signed numbers through visualization. The Groves Model is an example of a simple model that may be used for all operations of signed numbers. It operates on the principle of a number scale with the use of markers to indicate the operations carried on in addition, subtraction, multiplication, and division.

The study of the use of visualization in developing the basic concepts in equations and formulas was done in four areas. They were the use of varied solutions, the use of word relationships, the use of graphs, and the use of models.

The use of varied solutions is necessary because if the student does not respond to one, he may to another. It should be kept in mind that all students do not develop a concept by the same stimulus. Such things as analysis and mental substitutions help the student develop concepts through visualization.

The word relationship is important in developing concepts in algebra. The use of simple business problems

familiar to the student (as well as physical laws and graphs) offers a way of developing word relationship.

The graph offers many ways of letting the student see the solution of equations and formulas in picture form. The function in relation to the solution of real situations and the functional relationship may well be illustrated by the graph.

The use of models help students develop concepts through visualization by giving the student a mental picture. The many relationships between mathematics and machinery may be developed by the use of variators. The simplest type being the flexible rubber band. Many others such as the wheel, bar linkage, and the graph may be used successfully.

The primary function of the verbal problem is to give meaning to and provide practice in applying mathematical generalizations or concepts. It broadens the meaning of the concepts dealing with the method of solving problems. The verbal problem involves the changing of words to symbol and pictures. To help the student understand a problem he must first understand the mathematical words and the mathematical language. Obviously faulty language can confuse the process of reasoning. One way of introducing mathematical language is by word meaning and derivation. To be able to translate from one language to another, a person must understand what is said in the first language and the vocabulary, syntax, and idiomatic structure of the second language. The same is true

in verbal problems in mathematics. The diagramming of a verbal problem brings out the needed concept for solving the problem through visualization. The diagram may be a sketch of the situation or it may be a symbolic diagram.

To The Teacher

The study suggests the use of comparisons, contrasts, mental pictures, and the inductive reasoning in developing basic concepts in ninth grade algebra through visualization. The teaching of signed numbers may provide an example for the use of comparisons, contrasts, mental pictures, and inductive reasoning in developing concepts through visualization. In the introduction of signed numbers practical application may be used such as the thermometer. Here the student is already familiar with the use of a number scale with negative numbers. From the thermometer the student can easily draw a number scale. The student may then be given a mental picture of the relationships of the symbols - and + to a number scale along with the words negative and positive. Now by the use of verbal problems such as gains and losses in values, or gains and loses on a football field, or temperature changes that the student has had some experience in solving, the student may relate these problems to their number scale. Also from these problems and the number scale the student should be led to originate his own rules for the operations of addition, subtraction, multiplication, and division. "Grove's Model" will now be useful as a device

for demonstrating these operations. The concepts in signed numbers are developed in this example by the use of comparisons, contrasts, mental pictures, and inductive reasoning.

BIBLIOGRAPHY

- Berger, Emil J., "Devices for a Mathematics Laboratory," The Mathematics Teacher, October, 1952, 5 pp.
- Berger, Emil J., "Devices for a Mathematics Laboratory," The Mathematics Teacher, October, 1953, 6 pp.
- Brown, Claude H., The Teaching of Secondary Mathematics, Harper and Brothers, New York, 1953, 63 pp.
- Davis, David R., The Teaching of Mathematics, Addison-Wesley Press, Inc., Cambridge 42, Mass., 65 pp.
- Denbow, Carl H., "The Use of Variators in the Teaching of Algebra," School Science and Mathematics, January, 1954, 4 pp.
- Grove, Ethel L.; Grove, Ewart L., "Number Scale for Illustrating All Four Fundamental Processes With Directed Numbers," School Science and Mathematics, June, 1951, 5 pp.
- Jerbert, A. R., "The Linear Equation," The Mathematics Teachers, March, 1951, 2 pp.
- Jones, Phillip S., "Mathematical Miscellanea," The Mathematics Teacher, December, 1950, 6 pp.
- Jones, Phillip S., "Mathematical Miscellanea," The Mathematics Teacher, March, 1951, 3 pp.
- Jones, Phillip S., "Mathematical Miscellanea," The Mathematics Teacher, December, 1951, 4 pp.
- Meserve, B. E., "Using Geometry in Teaching Algebra," The Mathematics Teacher, December, 1952, 4 pp.
- Meyers, Sheldon S., "Applications," The Mathematics Teacher, March, 1952, 5 pp.
- Moser, Harold E., "Can We Teach Pupils to Distinguish the Measurement and Partition Ideas in Division," The Mathematics Teacher, February, 1952, 4 pp.
- The National Council of Teachers of Mathematics, Emerging Practices in Mathematics Education, Twenty-Second Yearbook, The National Council of Teachers of Mathematics, Washington D.C., 1953, 400 pp.

The National Council of Teachers of Mathematics, Muli-Sensory Aids in the Teaching of Mathematics, Eighteenth Year-book, Columbia Teachers College Press, New York, 1945, 350 pp.

O'Brien, Katharine E., "Problem-Solving," The Mathematics Teacher, February, 1956, 8 pp.

Reeve, William David, Mathematics for the Secondary School, Its Content and Methods of Teaching and Learning, Henry Holt and Co., New York, 1954, 75 pp.

Reeve, William David, "What Price Factoring," The Mathematics Teacher, November, 1952, 6 pp.

Schaaf, William L., "The Use of Simple Algebra in Business," The Mathematics Teacher, January, 1951, 5 pp.

Whetstone, George A., "High School Mathematics in College Engineering," School Science and Mathematics, December, 1950, 6 pp.