On the electromagnetic interaction of spin-1/2 particles
by James B. Calvert

A THESIS Submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of Master of Science in Physics
Montana State University

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The complete Hamiltonian to order v2 /c2 for two interacting Pauli particles is derived in this thesis. First, the wave equations for Dirac and Pauli particles are exhibited, together with certain properties pertinent to the discussion. Lorentz-covariant operators for spin, Dirac moment, and Pauli moment are formulated and the interaction with the electromagnetic field is explicitly stated in terms of these operators. The classical Hamiltonian method in quantum mechanics is applied to the problem of several-particle interactions, in which the limitations and deficiencies of the method, as well as its advantages, are exhibited. Approximate wave equations to order v2/c2 are derived. The description of collisions by the method of Miller is next outlined. This method is Lorentz-covariant to all orders, but is applicable only to the first power in the interaction. The Moller Hamiltonian for two Pauli particles is derived and compared with that obtained earlier. Free-particle solutions for use with the Miller Hamiltonian are given in an appendix, as is the explicit form of the Dirac matrices used in this paper.

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Approved:

Adam M. Johnson
Head, Major Department

Frank P. Woods
Chairman, Examining Committee

Leon H. Johnson
Dean, Graduate Division

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ABSTRACT

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I. INTRODUCTION

In this paper we will investigate the form of the relativistic wave equation for particles of spin 1/2 under the assumption that all forces acting are electromagnetic. This hypothesis on the nature of the forces is exactly correct for a single particle moving in an external electro-magnetic field, and is nearly precise in the presence of gravitational and weak-interaction forces, for these are quite small compared to the electric forces. The weak-interaction forces are, in particular, of almost unimaginable impotence even compared to gravitational forces. The principal neglect will be that of the nuclear, or meson, forces. These forces are of limited range (about 10^{-13} cm.), but within this range their effects predominate over those of the electric forces by many orders of magnitude. Satisfactory wave equations including the meson forces are not currently available, although intensive researches have been in progress for over two decades.

Qualitatively, and semi-quantitatively, the meson-force phenomena are well-known. Simply stated, the massive particles (nucleons and hyperons) possess a complicated inner structure that is intimately associated with the apparent or virtual presence of intermediate-mass particles (light mesons), which are the corpuscular manifestations of a new field, the meson field. Any precise theory must include this structure. Currently, nucleons are investigated theoretically by describing them with the Dirac equation, in accordance with the magnitude of their spin. In the presence of interaction this equation is modified by adding certain descriptive terms. Although these terms are rigorous in themselves, their introduction is empirical.

Electromagnetically, the important effect of the complex structure of the nucleon is the appearance of a magnetic moment in excess of the amount a simple charged particle would exhibit. This can be so simply stated by virtue of the rather complete knowledge of the electromagnetic field that has been attained. For the meson field, this degree of understanding does not currently exist; we cannot therefore state the effects of the nuclear interaction precisely. The best that has been done is to list all possible types of interactions allowed, and to test them, one by one. Since this empirical method has met with no great success, we are carried back to the fundamental problem: to find the precise nature of the meson field.

This paper does not treat these great and interesting problems; we deal here only with the electromagnetic field. This study however, has three important uses, and one distinct advantage: it is somewhat soluble. The uses are: first, the equations hold for particles that do not approach too closely; second, in the presence of other forces the equations can be used to subtract the electric effects, revealing the other forces for study; third, the various terms give clues to possible forms of interactions, particularly the velocity-dependent ones, and with appropriate modification may be used to describe nuclear forces.

Throughout this paper we shall use relativistic quantum mechanics.

Introductory material on the quantum mechanics may be found in the books of Dirac (17), Mott and Sneddon (47), and Pauli (53), in review articles by Hill and Landshoff (35) and by Feshbach and Villars (27), and in the discovery articles of Dirac (19) (20). The matrix notation is completely described by Good (33). Bade and Jehle (1) have written an excellent and accessbile article on the spinor analysis. Although we will not explicitly employ spinors, this work is mentioned because of its large bibliography on relativistic quantum mechanics. A list of selected references would also include Darwin (16), Klein (41), Sauter (57), Dirac (22) (23), Beck (2), Thomas (61), and Taub (60). The extension to non-electric forces is discussed by Kemmer (40), Inglis (37) (38), Furry (29), Wigner (65), and Wheeler (63). The textbooks of Møller (45) Panofsky and Phillips (49), and Goldstein (32) contain a good background in the theory of relativity and electrodynamics. Swann (59) has written an excellent review article on relativity and electrodynamics.

The more penetrating questions on the interaction of particles with the electromagnetic field belong to quantum electrodynamics. The most famous and readable introduction to this field was written by Fermi (26). For further references and recent work, see Dirac (18) (21), Eliezer (25), Pauli (50) (51) (52), Podolsky and Schwed (54), Weisskopf (62), Wheeler and Feynman (64), Heitler (34) and Jauch and Rohrlich (39).

II. WAVE EQUATIONS

A. Introduction. In the absence of an external field a Dirac particle is described by the Lorentz-invariant wave equation

(1)
$$\left(\frac{\partial}{\partial x_\mu}\gamma_\mu + \frac{mc}{\hbar}\right)\psi = 0$$

Each term of this equation has units cm^{-1} (energy divided by $\hbar c$), and the repeated greek index indicates a summation for $\mu = 1, 2, 3, 4$. The four coordinates are x_1, x_2, x_3, x_4 = ict, m is the proper mass, and the γ are hermitian Dirac matrices in the representation

(2)
$$\gamma_k = -i\beta\alpha_k \qquad k = 1, 2, 3$$
$$\gamma_4 = -\beta$$

These matrices are written explicitly in Appendix B. The fundamental algebraic property of the γ is their anticommutativity and idempotency:

(3)
$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}$$

The four components of the wave function ψ correspond to the four possible values of the spin coordinate; ψ has five degrees of freedom. The γ matrices are representations of the abstract γ operators in spin space, here with γ_4 diagonal. From the basic γ's, unity, and the products of the γ's a 16-element basis (i.e., linearly independent basis) of Dirac operator space can be constructed such that any Dirac operator has a unique expansion in terms of the elements of the basis. Any two

Dirac operator spaces are related by a similarity transformation.

The adjoint wave equation, analogous to the conjugate wave equation of nonrelativistic quantum mechanics, is written:

(4)
$$\left(\frac{\partial}{\partial x_\mu}\gamma_\mu - \frac{mc}{\hbar}\right)\psi^A = 0\,, \quad \psi^A = \psi^H\gamma_4 .$$

The "A" indicates adjoint, and the "H" indicates hermitian, or transposed, conjugate. Since the γ now operate on ψ from the right, an arrow is written under the γ to indicate this explicitly.

The four-vector ic $\psi^A\gamma_\mu\psi$ is interpreted as the particle current density. Its x_4-component divided by ic is cometimes called probability density. The wave function is normalized for a finite system according to the condition

(5)
$$\int\psi^A\gamma_4\psi\, d\tau = \int\psi^H\psi\, d\tau = 1 .$$

The integration extends over the entire configuration space. A beam of particles is normalized according to $\psi^A\ \gamma_4\ \psi = 1$ (i.e., one particle per unit volume). This normalization is not Lorentz-invariant; renormalization is necessary after a Lorentz transformation. The wave functions transform as ~~spinors, but~~ the representation of the γ is invariant.

The four-vector $j_\mu = -ie\psi A\ \gamma_\mu\ \psi$, where e is the electric charge of the particle, is the current density vector in e.m.u. Premultiplying eq. 1 by ie ψ^A, postmultiplying eq. 4 by ie ψ, and adding the two, it is easily shown that

(6) $$\frac{\partial j_\mu}{\partial x_\mu} = 0, \qquad j_\mu = -ie\psi^A\gamma_\mu\psi,$$

expressing the conservation of current and charge. Conservation of the particle density is exhibited in the same manner.

In the presence of an electromagnetic field, terms expressing the interaction must be added to eq. 1. These interaction terms, in addition to satisfying other requirements, must be Lorentz-covariant. The covariance is automatic if they are constructed from Lorentz tensors in such a manner that each term is the very same kind of tensor. An inspection of eq. 1 reveals that they must be scalars. In the next two sections the two pertinent examples are constructed.

B. Dirac Particles. Classical analogy immediately provides us with an interaction term expressible in quantum-mechanical language: $+A_\mu j_\mu$. $A_\mu = (\bar{A}, i\phi)$ is the potential of the external field; $\bar{A}$ is the vector potential and ϕ is the scalar potential. Replacing j_μ by its operator $-ie\gamma_\mu$, and dividing by $\hbar c$ to conform with the units in the rest of the equation, we find the wave equation

$$\left(\frac{\partial}{\partial x_\mu}\gamma_\mu - \frac{ie}{\hbar c}A_\mu\gamma_\mu + \frac{mc}{\hbar}\right)\psi = 0 .$$

Writing $\pi_\mu = \frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}A_\mu$, the equation assumes the form

(7) $$\left(\pi_\mu\gamma_\mu + \frac{mc}{\hbar}\right)\psi = 0 .$$

This is the wave equation of a simple charged particle of charge e. The automatic appearance of spin and intrinsic moment, which we shall show later, is a portentious result that has vastly extended the knowledge of the fundamental particles. Eq. 7 describes electrons and positrons simultaneously with great accuracy. Pending further knowledge in the field of quantum electrodynamics, this equation is believed to describe electrons and positrons exactly.

C. Pauli Particles. The term "Pauli particle" will be used to denote a particle of spin-1/2 that exhibits a magnetic moment in excess of the intrinsic moment of a simple charged particle. The interaction due to the charge of a Pauli particle is identical to that expressed in eq. 7. It is now necessary to add terms descriptive of the interaction of the excess magnetic moment with the field. The measure of the excess moment will be denoted by μ.

The correct relativistic expression of a magnetic moment density is an antisymmetric tensor of rank two, usually called a six-vector. A system which has a pure magnetic moment at rest will in general appear to have a certain electric moment when it moves with respect to the observer. The decomposition into the electric and magnetic parts is not unique, analogously to the decomposition of the electromagnetic field into its magnetic and electric parts. The magnetic moment is an axial vector, and the electric moment is a polar vector, in any given decomposition of the electromagnetic moment six-vector. Explicitly, this six-vector is

(8)

$$\mu_{\mu\nu} = \begin{bmatrix} 0 & M_3 & -M_2 & -iP_1 \\ -M_3 & 0 & M_1 & -iP_2 \\ M_2 & -M_1 & 0 & -iP_3 \\ iP_1 & iP_2 & iP_3 & 0 \end{bmatrix},$$

where the M_i are components of the magnetization and the P_i are components of the polarization. The electromagnetic field six-vector is

(9)

$$F_{\mu\nu} = \begin{bmatrix} 0 & H_3 & -H_2 & -iE_1 \\ -H_3 & 0 & H_1 & -iE_2 \\ H_2 & -H_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{bmatrix}$$

The H_i are components of the magnetic field, and the E_i are components of the electric intensity. Classically, the interaction energy is half the scalar product of the moment and field six-vectors:

(10)

$$-\frac{1}{2} F_{\mu\nu}\mu_{\mu\nu} .$$

To pass to quantum mechanics, we must find the Dirac operator for the moment. The requirement that it must form a six-vector leads unambigously to the Dirac six-vector $\mu_{\mu\nu} = \mu\frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$. It is convenient to set $\sigma_{\mu\nu} = \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$. The "modified Dirac equation" then follows:

(11)

$$\left(\frac{\partial}{\partial x_\mu}\gamma_\mu - \frac{1}{2c\hbar}F_{\mu\nu}\mu_{\mu\nu} + \frac{mc}{\hbar}\right)\psi = 0 .$$

This equation describes a neutral particle that, at rest, has a permanent magnetic moment μ. Without extensive modifications of the theory, there is no place for a particle of similar permanent electric moment. No such particle is known. Eq. 11 was used by Carlson and Oppenheimer (13) for the then-hypothetical neutron, just prior to the confirmation of its discovery.

The extension to a particle with both charge and excess magnetic moment is immediate:

(12)

$$\left(\pi_\mu\gamma_\mu - \frac{1}{2c\hbar}F_{\mu\nu}\mu_{\mu\nu} + \frac{mc}{\hbar}\right)\psi = 0 .$$

The proton is an example of a particle of this last type.

The excess, or anomalous, moment may arise from at least two sources, the interaction of a particle with its own field, or through a complex structure. The former leads to a very small moment, even for the very light electron, while the latter produces a moment of the same order as the intrinsic moment. Table I lists the electromagnetic properties of the common spin-1/2 particles.

Table I. Electromagnetic Properties of Spin-1/2 Particles

particle	charge	Dirac mom.	Anom. mom.
neutrino	0	0	probably 0
electron	$\pm e$	1	0.001145
μ-meson	$\pm e$	1	?
proton	$\pm e$	1	1.79275
neutron	0	0	-1.9128

(moments are given in magnetons for the usual charge state)

D. Discussion of the Operators. The Dirac interaction term is not hermitian:

(13)

$$\left(-\frac{ie}{\hbar c}A_\mu\gamma_\mu\right)^H = \frac{ie}{\hbar c}A_\mu^*\gamma_\mu^H = \frac{ie}{\hbar c}A_k\gamma_k - \frac{ie}{\hbar c}A_4\gamma_4 .$$

The behavior of this term is complementary to that of $\frac{\partial}{\partial x_\mu}\gamma_\mu$, where the imaginary x_4-coordinate behaves differently from the first three. At first glance this might be attributed to the choice of coordinates, but the metric properties of the relativistic (3 + 1)-space require a similar behavior for any choice of coordinates. The physical result of this property is the charge-conjugation operation and the consequent appearance of antiparticles in the theory.

Since $F_{\mu\nu}$ is not hermitian, the Pauli term is also non-hermitian. $\sigma_{\mu\nu}$ is, of course, hermitian. The sign of the Pauli term changes in charge conjugation, as it should. Therefore, the operation of charge conjugation produces the correct sign changes in both types of terms.

For a free particle, the moment six-vector can be expressed as an equivalent current density. First, premultiply eq. 1 by γ_ν:

$$\left(\frac{\partial}{\partial x_\mu}\gamma_\nu\gamma_\mu + \frac{mc}{\hbar}\gamma_\nu\right)\psi = 0 .$$

Interchanging γ_μ and γ_ν in the first term,

$$\left(-\frac{\partial}{\partial x_\mu}\gamma_\mu\gamma_\nu + \frac{mc}{\hbar}\gamma_\nu\right)\psi = 0 .$$

Adding and adjusting the signs,

(14)
$$\frac{\partial}{\partial x_\mu}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\psi = 2\frac{mc}{\hbar}\gamma_\nu\psi$$

$$\frac{\partial}{\partial x_\mu}\sigma_{\mu\nu}\psi = \frac{imc}{\hbar}\gamma_\nu\psi .$$

Treating eq. 4, the adjoint equation, in a similar manner we find that

(15)
$$\frac{\partial}{\partial x_\mu}\left(\psi^A\sigma_{\mu\nu}\right) = \frac{imc}{\hbar}\gamma_\nu\psi .$$

Finally, multiplying eq. 14 by ψ^A and eq. 15 by ψ and adding:

(16)
$$\frac{\partial}{\partial x_\mu}\left(\psi^A\sigma_{\mu\nu}\psi\right) = \frac{2imc}{\hbar}\psi^A\gamma_\nu\psi .$$

Multiplying by $e\hbar/2mc$, we obtain the relation between the Dirac current and the corresponding moment density:

(17)

$$\frac{\partial}{\partial x_\mu}(\psi^A \mu^D_{\mu\nu}\psi) = ie\psi^A \gamma_\nu \psi = -\dot{j}^D_\nu ,$$

where $\mu^D_{\mu\nu} = e\hbar/2mc$. The field due to the moment can now be found by Maxwell's equations:

(18)

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = 4\pi \dot{j}_\mu = 4\pi\mu \frac{\partial}{\partial x_\mu}(\psi^A \sigma_{\mu\nu}\psi).$$

In the general case, the current density due to the moment is no longer simply proportional to $i\psi^A\gamma_\mu\psi$, but we assert that the current is still given by the covariant derivative of the moment six-vector, as in the last term on the right of eq. 18. The divergence of this current is always zero, expressing the conservation of charge.

The spin angular momentum is represented in this formalism by the pseudovector (completely antisymmetric tensor of rank 3) $S_\mu = \frac{i\hbar}{2}\gamma_s\gamma_\mu$, where $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. Note that the magnetic moment is not simply proportional to this quantity.

Each operator mentioned to this point, and the wave equations themselves, are Lorentz-covariant and agree completely in their algebraic properties, including their behavior under space reflection, time reflection, and charge conjugation, with their classical analogues. The spin and magnetic moment commute, as required, with the centralfield Hamiltonian and have the expected proper values. We shall now assume that these operators are established, and proceed to examine the interactions with the field.

E. Interaction with the Field. In order to express the interactions in terms of the resolved electric and magnetic fields, the matrices j, defined by eq. 19, are introduced.

(19)

$$\gamma_j \gamma_k = i\epsilon_{jkl}\sigma_l + \delta_{jk} \qquad j,\ k,\ l = 1,\ 2,\ 3$$

Where ϵ_{ijk} is the usual completely antisymmetric three-index symbol that is zero if any two indices are alike, and +1 or −1 as i, j, k is an even or odd permutation of 1, 2, 3. Explicitly, the matrix is then:

(20)

$$\sigma_{\mu\nu} = \begin{bmatrix} 0 & -\sigma_3 & \sigma_2 & \alpha_1 \\ \sigma_3 & 0 & -\sigma_1 & \alpha_2 \\ -\sigma_2 & \sigma_1 & 0 & \alpha_3 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 & 0 \end{bmatrix}$$

It follows that:

(21)

$$-\frac{\mu}{2c\hbar} F_{\mu\nu}\sigma_{\mu\nu} = \frac{\mu}{c\hbar}\left(\bar{H}\cdot\bar{\sigma} + i\,\bar{E}\cdot\bar{\alpha}\right).$$

The Dirac interaction is expressed in terms of the vector and scalar potentials:

(22)

$$-\frac{ie}{c\hbar}A_\mu\gamma_\mu = -\frac{\beta e}{c\hbar}\bar{A}\cdot\bar{\alpha} - \beta\frac{e\phi}{c\hbar} .$$

$\bar{E}$, $\bar{H}$, $\bar{A}$, and ϕ are the fields and potentials measured by an observer at rest. From these expressions we obtain the operators for the properties of the moving particle as seen by an observer at rest:

(23) current: $\beta e\bar{\alpha}$ magnetic moment: $\mu\bar{\ }$

charge: βe electric moment: $i\mu\bar{\alpha}$

The Hamiltonian that we have used till now is the convenient one for discussions of general properties. Its terms are homogeneous and simple, and the units are especially convenient in the choice $\hbar=c=1$. For detailed calculations, however, it is usual to consider the Hamiltonian obtained from the one previously employed by postmultiplication by $\beta\hbar c$. In this case eq. 21 becomes:

(24)

$$\mu\beta\bar{H}\cdot\bar{\sigma} - i\beta\bar{E}\cdot\bar{\alpha} .$$

Eq. 22 becomes:

(25)

$$e\bar{A}\cdot\bar{\alpha} - e\phi .$$

The new operators are:

(26) current: $-e\bar{\alpha}$ magnetic moment: $\mu\beta\bar{\ }$

charge: e electric moment: $-i\mu\beta\bar{\alpha}$

F. Nonrelativistic Limit. In the notation just introduced, the wave equation is:

(27)

$$\left(i\hbar\frac{\partial}{\partial t} + ic\hbar\,\bar{\alpha}\cdot\nabla + \beta mc^2 + e\bar{A}\cdot\bar{\alpha} - e\phi + \mu\beta\bar{H}\cdot\bar{\sigma} - i\mu\beta\bar{E}\cdot\bar{\alpha}\right)\psi = 0$$

Making the substitution $\psi = (\psi_S, \psi_L)\exp\frac{mc^2t}{i\hbar}$, we get two coupled equations:

(28)

$$\left(2mc^2 + i\hbar\frac{\partial}{\partial t} - e\phi + \mu\bar{H}\cdot\bar{\sigma}\right)\psi_S + \left(ic\hbar\,\bar{\sigma}\cdot\nabla + i\mu\bar{\sigma}\cdot\bar{E} + e\bar{\sigma}\cdot\bar{A}\right)\psi_L = 0$$

$$\left(i\hbar\frac{\partial}{\partial t} - e\phi - \mu\bar{H}\cdot\bar{\sigma}\right)\psi_L + \left(ic\hbar\,\bar{\sigma}\cdot\nabla - i\mu\bar{\sigma}\cdot\bar{E} + e\bar{\sigma}\cdot\bar{A}\right)\psi_S = 0$$

In order to avoid a multitude of higher-order terms due to the simultaneous presence of charge and anomalous moment, the two cases will be treated separately. The purpose of the analysis is to attempt an approximate solution of the equations in the form of a two-component wave function which satisfies a Pauli-type Schrodinger equation. Such a solution will exhibit the effects of the electromagnetic interaction in a form that can be interpreted in terms of elementary considerations, and will verify that the interaction terms behave properly at zero velocity. It will be pointed out that certain difficulties arise in the application of the reduced equations. If possible, it is better to solve the system of eq. 27 in terms of a four-component wave function.

The substitution leading to eq. 28 causes one of the two-component wave functions, ψ_s, to become small compared to the other, ψ_L, when the energy of the particle is small compared to the proper energy mc^2. In the limit of zero velocity, the small components vanish.

For a Dirac particle, the equations become:

(29)
$$\left(2mc^2 + i\hbar\frac{\partial}{\partial t} - e\phi\right)\psi_S + \left(ic\hbar\bar{\sigma}\cdot\nabla + e\bar{\sigma}\cdot\bar{A}\right)\psi_L = 0$$
$$\left(i\hbar\frac{\partial}{\partial t} - e\phi\right)\psi_L + \left(ic\hbar\bar{\sigma}\cdot\nabla + e\bar{\sigma}\cdot\bar{A}\right)\psi_S = 0$$

Solving for ψ_s symbolically:

(30)
$$\psi_S = -\left(2mc^2 + i\hbar\frac{\partial}{\partial t} - e\phi\right)^{-1}\left(ic\hbar\bar{\sigma}\cdot\nabla + e\bar{\sigma}\cdot\bar{A}\right)\psi_L\,,$$

$$\psi_S = -\frac{1}{2mc^2}\left(1 + \frac{i\hbar\partial/\partial t - e\phi}{2mc^2}\right)^{-1}\left(ic\hbar\bar{\sigma}\cdot\nabla + e\bar{\sigma}\cdot\bar{A}\right)\psi_L\,,$$

$$\psi_S \simeq -\frac{1}{2mc^2}\left(1 - \frac{i\hbar\partial/\partial t - e\phi}{2mc^2}\right)\left(ic\hbar\bar{\sigma}\cdot\nabla + e\bar{\sigma}\cdot\bar{A}\right)\psi_L\,.$$

This approximation will give terms to order v^2/c^2. Substituting eq. 30 in the lower of eqs. 29:

$$\left\{i\hbar\frac{\partial}{\partial t} - e\phi + \frac{\left(ic\hbar\bar{\sigma}\cdot\nabla + e\bar{\sigma}\cdot\bar{A}\right)\left(1 - \frac{i\hbar\partial/\partial t - e\phi}{2mc^2}\right)\left(ic\hbar\bar{\sigma}\cdot\nabla + e\bar{\sigma}\cdot\bar{A}\right)}{-2mc^2}\right\}\psi_L = 0$$

Let $\Pi_k = -i\hbar \frac{\partial}{\partial x_k} - \frac{e}{c} A_k$. Then, $ic\hbar\bar{\sigma}\cdot\nabla + e\bar{\sigma}\cdot\bar{A} = -c\bar{\sigma}\cdot\bar{\Pi}$.

(31)

$$\left\{ i\hbar\frac{\partial}{\partial t} - e\phi - \frac{1}{2m}\left[\bar{\sigma}\cdot\bar{\Pi}\left(1 - \frac{i\hbar\partial/\partial t - e\phi}{2mc^2}\right)\bar{\sigma}\cdot\bar{\Pi}\right]\right\}\psi_L = 0$$

Making use of the equations:

(32)

$$\left(1 + \frac{e\phi - i\hbar\partial/\partial t}{2mc^2}\right)\bar{\sigma}\cdot\bar{\Pi} = \bar{\sigma}\cdot\bar{\Pi}\left(1 + \frac{e\phi - i\hbar\partial/\partial t}{2mc^2}\right) - \frac{ie\hbar}{2mc^2}\bar{\sigma}\cdot\bar{E},$$

(33)

$$\bar{\sigma}\cdot\bar{\Pi}\,\bar{\sigma}\cdot\bar{E} = -i\,\bar{\sigma}\cdot\bar{E}\times\bar{\Pi} - i\hbar\,\nabla\cdot\bar{E} + \frac{\hbar}{c}\bar{\sigma}\cdot\frac{\partial\bar{H}}{\partial t} + \bar{E}\cdot\bar{\Pi},$$

(34)

$$\bar{\sigma}\cdot\bar{\Pi}\,\bar{\sigma}\cdot\bar{\Pi} = -\frac{e\hbar}{c}\bar{\sigma}\cdot\bar{H} + \bar{\Pi}^2.$$

We now obtain from eq. 31 the wave equation:

(35)

$$\left(i\hbar\frac{\partial}{\partial t} - e\phi - \frac{\bar{\Pi}^2}{2m} + \frac{e\hbar}{2mc}\bar{\sigma}\cdot\bar{H} + \frac{e\hbar}{2mc}\bar{\sigma}\cdot\frac{\bar{E}\times\bar{\Pi}}{2mc} + \frac{e\hbar}{2mc}\frac{\hbar}{2mc}\nabla\cdot\bar{E} + i\frac{e\hbar}{2mc}\frac{\hbar}{2mc^2}\bar{\sigma}\cdot\frac{\partial\bar{H}}{\partial t} + i\frac{e\hbar}{2mc}\frac{\bar{E}\cdot\bar{\Pi}}{c}\right)\psi_L = 0.$$

Where, for the sake of simplicity, the factor $\left(1 + \frac{e\phi - i\hbar\partial/\partial t}{2mc^2}\right)$ that multiplies the third and fourth terms has been approximated by unity. This factor represents the relativistic "variation of mass."

Since $(e\phi - i\hbar\frac{\partial}{\partial t})$ is about equal to half the kinetic energy, this term can be written

(36)

$$w_R = \left(1 + \frac{e\phi - i\hbar\partial/\partial t}{2mc^2}\right) \sim 1 - \frac{T}{mc^2} = 1 - \frac{1}{2}\eta^2 - \frac{1}{8}\eta^4 + \cdots$$

Where $\eta = \frac{\hbar}{mc}\frac{\partial}{\partial x} = 3.9 \times 10^{-11}$ (this factor is dimensionless). For easier interpretation, we write eq. 35 in the usual Hamiltonian form:

(37)

$$i\hbar\frac{\partial\psi}{\partial t} = \left\{\frac{\bar{\Pi}^2}{2m} + e\phi - \frac{e\hbar}{2mc}\bar{\sigma}\cdot\bar{H} - \frac{e\hbar}{2mc}\bar{\sigma}\cdot\frac{\bar{E}\times\bar{\Pi}}{2mc} + i\frac{e\hbar}{2mc}\frac{\bar{E}\cdot\bar{\Pi}}{c}\right.$$

$$\left. + \frac{e\hbar}{2mc}\frac{\hbar}{2mc}\nabla\cdot\bar{E} + i\frac{e\hbar}{2mc}\frac{\hbar}{2mc^2}\bar{\sigma}\cdot\frac{\partial\bar{H}}{\partial t}\right\}\psi .$$

Note the appearance of non-hermitian terms (those containing i) in the approximation to this order. These terms indicate that we have not achieved a clean separation into small and large components, and that the negative-mass solutions are making themselves felt. To order v^2/c^2, the ψ_L alone do not describe a complete set of states, in the presence of an external field. In the absence of a field, however, the large and small components separate strictly.

The non-hermitian terms arise from odd operators in the Dirac Hamiltonian. Even operators are diagonal and odd operators are antidiagonal in the matrix representation. Any operator may be separated into an even and an odd part. Foldy and Wouthuysen (28) describe a method by which the Dirac Hamiltonian may be freed of odd operators to any desired order by means of a series of increasingly complex unitary transformations. The resulting two-component Hamiltonians are then free of non-hermitian operators to the same order. This procedure, particularly as extended and illuminated by Feshbach and Villars (27), provides an excellent interpretation of the nature of relativistic operators.

In the limit of zero velocity, only four terms of eq. 37 remain. The electrostatic and magnetic dipole potential energies are represented by the terms $e\phi$ and $-\frac{e\hbar}{2mc}\bar{\sigma}\cdot\bar{H}$. The intrinsic moment $\frac{e\hbar\bar{\sigma}}{2mc}$ appears as a fundamental property of a Dirac particle.

The Darwin term, $\frac{e\hbar}{2mc}\frac{\hbar}{2mc}\nabla\cdot E$, has no classical analogue. The factor $\frac{\hbar}{mc}$ is the approximate linear extent of the region swept by the rapid excursions (Zitterbewegung) of the relativistic particle. To this approximation we do not care to follow the particle in these jumps of velocity equal to that of light, and merely average the effect of the field over the region. This process is not to be confused with the "smearing" due to the ordinary, inaccurate, interpretation of probability density in nonrelativistic quantum mechanics. In a region where $\nabla\cdot\bar{E} = 0$ the Darwin term vanishes, but in the case of an s-electron which strongly penetrates an atomic nucleus (as most do) it produces a small, detectable shift in the energy levels (not the Lamb-Retherford shift). For further discussion, refer to Darwin (16).

The non-hermitian term remaining for zero velocity is the average energy of the electric dipole in the electric field in the direction of the dipole. The pertinent electric field is produced by a varying magnetic field; the effect of the static electric field averages to zero. A descriptive qualitative analysis of this effect is easily constructed:

$$\nabla \times \bar{E} = -\frac{1}{c}\frac{\partial \bar{H}}{\partial t} \qquad -\bar{p}\cdot\bar{E}_{av} = i\frac{e\hbar}{2mc}\frac{\hbar}{2mc^2}\bar{\sigma}\cdot\frac{\partial \bar{H}}{\partial t}$$

$$\oint \bar{E}\cdot d\bar{s} = -\frac{1}{c}\frac{\partial}{\partial t}\int \bar{H}\cdot d\bar{\sigma}$$

$$\bar{E}_{av} = -\frac{1}{c}\frac{\partial H}{\partial t}\cdot\frac{r}{2} \qquad r = \frac{\hbar}{mc}$$

This term is small if the magnetic field varies slowly.

The other three terms of eq. 37 appear for nonvanishing velocities. The first term, a familiar one, represents the kinetic energy. The other two represent the energy of the magnetic dipole in two types of magnetic field produced by the motion of the particle. The spin-orbit term is easily recognized: $-\frac{e\hbar}{2mc}\bar{\sigma}\cdot\frac{\bar{E}\times\bar{\Pi}}{2mc}$. The field $\bar{H}$ is produced by the average velocity of the particle, $\bar{\Pi}/mc$. The term is usually named after Thomas, who explained why the term was only half as large as was expected.

The non-hermitian term $\frac{ie\hbar}{2mc}\frac{\bar{E}\cdot\bar{\Pi}}{c}$, peculiar to the Dirac moment, has a similar interpretation. This is a sort of spin-orbit term for the electric moment, roughly speaking.

Many of the terms mentioned above will appear also for the anomalous moment, verifying our choice of the interaction term. It is instructive to compare the two sets of terms.

The reduction for a particle possessing an anomalous moment is analogous to that followed for a Dirac particle, and will be made largely without comment in what follows. The wave equation is:

(38)

$$\left(2mc^2 + i\hbar\frac{\partial}{\partial t} - e\phi - \mu\bar{H}\cdot\bar{\sigma}\right)\psi_S + \left(ic\hbar\bar{\sigma}\cdot\nabla - i\mu\bar{\sigma}\cdot\bar{E}\right)\psi_L = 0$$

$$\left(i\hbar\frac{\partial}{\partial t} - e\phi + \mu\bar{H}\cdot\bar{\sigma}\right)\psi_L + \left(ic\hbar\bar{\sigma}\cdot\nabla + i\mu\bar{\sigma}\cdot\bar{E}\right)\psi_S = 0.$$

We make a simpler approximation for ψ_S:

(39)

$$\psi_S = -\frac{ic\hbar\bar{\sigma}\cdot\nabla - i\mu\bar{\sigma}\cdot\bar{E}}{2mc^2}\psi_L ;$$

Therefore,

(40)

$$\left\{i\hbar\frac{\partial}{\partial t} - e\phi + \mu\bar{H}\cdot\bar{\sigma} - \frac{(ic\hbar\bar{\sigma}\cdot\nabla + i\mu\bar{\sigma}\cdot\bar{E})(ic\hbar\bar{\sigma}\cdot\nabla - i\mu\bar{\sigma}\cdot\bar{E})}{2mc^2}\right\}\psi_L = 0$$

And we obtain, in Hamiltonian form,

(41)

$$i\hbar\frac{\partial\psi_L}{\partial t} = \left\{\frac{\bar{P}^2}{2m} - \mu\bar{\sigma}\cdot\bar{H} - \mu\bar{\sigma}\cdot\frac{\bar{E}\times\bar{P}}{mc} + \mu\frac{\hbar}{2mc}\nabla\cdot\bar{E} + i\mu\frac{\hbar}{2mc^2}\bar{\sigma}\cdot\frac{\partial\bar{H}}{\partial t} + \frac{(\mu\bar{\sigma}\cdot\bar{E})^2}{2mc^2}\right\}\psi_L ,$$

(Cf. eq. 37), where $\bar{p} = -i\hbar\nabla$.

The terms are all familiar except for $\frac{(\mu\bar{\sigma}_1\bar{E})^2}{2mc^2}$, which is the electric dipole energy multiplied by its ratio to the proper energy. This term has become hermitian by the multiplication of two anti-hermitian terms.

In general, the reduction of Hamiltonians follows the plan outlined above. In any case of average complexity, the result is a multiplicity of terms, some of doubtful significance. The expansion is never neatly in powers of v/c, but the relative magnitude of the terms depends upon the field strengths and their rate of variation in space and time.

III. THE ELECTROMAGNETIC HAMILTONIAN

A. Introduction. The interaction of two static electric charges is elementary: the force between them is obtained by differentiation of the potential $\frac{e_1 e_2}{r}$. Should the charges be in motion, the potential of the interaction is no longer elementary. In general, such a potential cannot be written in finite form. The reason for this complication is that the electromagnetic interaction is a field interaction, which propagates at a finite speed: the speed of light. The forces in the field depend on more than the instantaneous positions of the interacting particles, since the forces must also depend on the previous dynamical motions.

If the particles move slowly compared to the velocity of light, and with small accelerations, the past action of the system may well be described by the derivatives of the positions of the particles at any desired instant. In this case it may be possible to write a potential from which the forces are obtained by differentiation, but this potential function will generally depend on the time derivatives of the positions of the particles. A potential involving the velocities is termed a velocity-dependent potential, especially if it does not contain higher derivatives.

It is difficult to write potentials involving the accelerations and higher derivatives. The familiar Lagrangian formulation of classical mechanics employs certain types of velocity-dependent potentials with ease, a common example being the description of the motion of a charged particle. The inclusion of higher derivatives involves a reworking of the entire theory, so that the higher differentiations may be included in the equations of motion. Quantum mechanics is generally based on the Lagrangian or Hamiltonian formulations of classical mechanics, so that difficulties in the classical theory are usually translated into difficulties in the quantum theory. Indeed, a fundamental difficulty is experienced with higher time derivatives, as in the case of the early development of the Klein-Gordon equation.

One difficulty that carries over in slightly modified form into quantum mechanics is the problem of radiation of an accelerated charge. The classical and quantum viewpoints on radiation are widely divergent, the viewpoint of classical mechanics not being in accord with the properties of matter, and the quantum viewpoint lacking a satisfactory formulation. It is doubtful that allowing for the classical reaction of radiation on the motion of a particle would lead to any sort of satisfactory wave equation.

The usual method of obtaining a wave equation, or quantum Hamiltonian, from a classical Hamiltonian is the substitution of the proper operators for the dynamical variables in the classical expression. Two difficulties may arise at once. First, the order of the generally noncommuting quantum-mechanical operators is quite important. The classical dynamical variables, however, commute. The difficulty here is then the interpretation of the order of the factors. Second, there exist dynamical variables that have no satisfactory operator. A good example of this is the acceleration. If an acceleration should appear in a classical Hamiltonian, the quantum-mechanical expression of the Hamiltonian would be in doubt. From the two preceding considerations, we see that our results will be of dubious value unless the classical Hamiltonian for several charged particles neither gives rise to an ambiguous order of factors, nor includes the accelerations and higher derivatives.

As pointed out by Breit (6), the usual properties of the quantum operators and wave equations must not be modified or destroyed. This means that the conservation of particles and charge must hold, that the wave function must retain its interpretation, and that the quantum system must have the proper classical limit, among other things.

It must be conceded from the preceding arguments that under the present formulation of quantum mechanics it is impossible to derive a wave equation from the classical Hamiltonian that will include retarded interactions rigorously. However, it will be seen that approximate equations of some utility and convenience are easily constructed.

As a starting point, we consider the non-retarded interaction of two charges. The pertinent potential function is then:

(42) $$\frac{e_1 e_2}{r_{12}}\left(1 - \frac{\bar{v}_1 \cdot \bar{v}_2}{c^2}\right).$$

The first term is the electrostatic interaction. The second term is the energy of the current $(\frac{e\bar{v}}{c})$ of one particle in the vector potential $(-\frac{e\bar{v}}{c})$ produced by the other. This equation is symmetrical in the two particles, as is required by a very general argument: it should make no difference which particle is called "1" and which "2". Gaunt and Eddington (30) (31) (24) used this expression for the interaction of two electrons, adding the sum of the free-particle Hamiltonians to complete the wave equation. It should be noted that it is not correct to insert the potentials caused by one charge into the wave equation of the other. As is easily seen, this is a procedure that radically modifies the interpretation of the operators, and leads to incorrect conclusions. In a two-particle system, the interaction must appear in a term symmetrical in the particles which includes all information about the interaction.

B. Dirac Particles to Order v^2/c^2.

Darwin (15), before the advent of wave mechanics, found the approximate classical lagrangian for two charged particles to order v^2/c^2. This expression was derived on the basis of the Lienard retarded potentials for point charges, and its correctness to the order indicated was shown. Any higher approximation involved both the effects of radiation and the appearance of higher derivatives.

Darwin used the expression to study the problem of the Helium atom in the old quantum theory.

In applying Darwin's equation to relativistic quantum mechanics, we begin with the lagrangian:

(43) $$L = -m_1c^2\sqrt{1-v_1^2/c^2} - m_2c^2\sqrt{1-v_2^2/c^2} - e_1\phi_1 - e_2\phi_2$$
$$+\frac{e_1}{c}\bar{A}_1\cdot\bar{v}_1 + \frac{e_2}{c}\bar{A}_2\cdot\bar{v}_2 - \frac{e_1e_2}{r}\left[1-\frac{1}{2}\left(\frac{\bar{v}_1\cdot\bar{v}_2}{c^2}+\frac{\bar{v}_1\cdot\bar{r}\;\bar{v}_2\cdot\bar{r}}{r^2}\right)\right].$$

The proper masses, charges, and velocities of the particles are represented by m_1, m_2, e_1, e_2, v_1, and v_2. $\bar{A}_1$, $\bar{A}_2$, Φ_1, and Φ_2 are the vector and scalar potentials at the positions of the two particles. The distance between the particles is r, and c denotes the speed of light.

The canonical moments are obtained from eq. 43 by differentiation:

(44)
$$\bar{p}_1 = \frac{\partial L}{\partial \bar{v}_1} = \frac{m_1v_1}{\sqrt{1-v_1^2/c^2}} - \frac{e_1}{c}\bar{A}_1 + \frac{e_1e_2}{2rc^2}\left(\bar{v}_2 + \frac{\bar{r}\;\bar{v}_2\cdot\bar{r}}{r^2}\right)$$

Proceed in a similar manner for $\bar{p}_2$. The classical Hamiltonian is now written after performing the usual Lagrange transformation:

(45)
$$H = \sum_{1,2}\bar{p}\cdot\bar{v} - L = \frac{m_1c^2}{\sqrt{1-v_1^2/c^2}} + \frac{m_2c^2}{\sqrt{1-v_2^2/c^2}} + e_1\phi_1 + e_2\phi_2 +$$
$$\frac{e_1e_2}{r}\left[1+\frac{1}{2}\left(\frac{\bar{v}_1\cdot\bar{v}_2}{c^2}+\frac{\bar{v}_1\cdot\bar{r}\;\bar{v}_2\cdot\bar{r}}{c^2r^2}\right)\right].$$

Now the Hamiltonian is expressed as a function of the p's and q's after condensing all the vector potential in eq. 44 into a single term.

First,

(46)
$$\bar{p} - \frac{e}{c}\bar{a} = \frac{m\bar{v}}{\sqrt{1 - v^2/c^2}}$$

Then it follows that:

(47)
$$H = c\left[m_1^2c^2 + (\bar{p}_1 - \frac{e_1}{c}\bar{a}_1)^2\right]^{\frac{1}{2}} + c\left[m_2^2c^2 + (\bar{p}_2 - \frac{e_2}{c}\bar{a}_2)^2\right]^{\frac{1}{2}}$$
$$+ e_1\phi_1 + e_2\phi_2 + \frac{e_1e_2}{r}\left[1 + \frac{1}{2}\left(\frac{\bar{v}_1\cdot\bar{v}_2}{c^2} + \frac{\bar{v}_1\cdot\bar{r}\;\bar{v}_2\cdot\bar{r}}{c^2r^2}\right)\right].$$

However, since the interaction terms are given only to order v^2/c^2, it is permitted to expand the radicals to the same approximation. This will allow placing all the interaction in a single term, as desired. The expansion, after a little algebra, leads to the result:

(48)
$$H = m_1c^2 + \frac{1}{2m_1}(\bar{p}_1 - \frac{e_1}{c}\bar{A}_1)^2 - \frac{1}{8m_1^3c^2}(\bar{p}_1 - \frac{e_1}{c}\bar{A}_1)^4$$
$$+ m_2c^2 + \frac{1}{2m_2}(\bar{p}_2 - \frac{e_2}{c}\bar{A}_2)^2 - \frac{1}{8m_2^3c^2}(\bar{p}_2 - \frac{e_2}{c}\bar{A}_2)^4$$
$$- \frac{e_1e_2}{r}\left(\frac{\bar{v}_1\cdot\bar{v}_2}{c^2} + \frac{\bar{v}_1\cdot\bar{r}\;\bar{v}_2\cdot\bar{r}}{c^2r^2}\right) + e_1\phi_1 + e_2\phi_2$$
$$+ \frac{e_1e_2}{r}\left[1 + \frac{1}{2}\left(\frac{\bar{v}_1\cdot\bar{v}_2}{c^2} + \frac{\bar{v}_1\cdot\bar{r}\;\bar{v}_2\cdot\bar{r}}{c^2r^2}\right)\right] + O(v^3/c^3).$$

The first two lines of eq. 48 are terms of the expansion of the familiar free-particle Hamiltonian $c[m^2c^2 + (\bar{p} - \frac{e}{c}\bar{A})^2]^{1/2}$, so that to order v^2/c^2 we may write:

(49) $$H = c\left[m_1^2c^2 + (\bar{p}_1 - \frac{e_1}{c}\bar{A}_1)^2\right]^{\frac{1}{2}} + c\left[m_2^2c^2 + (\bar{p}_2 - \frac{e_2}{c}\bar{A}_2)^2\right]^{1/2}$$

$$+ e_1\phi_1 + e_2\phi_2 + \frac{e_1e_2}{r}\left[1 - \frac{1}{2}\left(\frac{\bar{v}_1\cdot\bar{v}_2}{c^2} + \frac{\bar{v}_1\cdot\bar{r}\ \bar{v}_2\cdot\bar{r}}{c^2r^2}\right)\right].$$

This expression has the desired form, since it is the sum of the Hamiltonians for two non-interacting particles plus an expression yielding their interaction. In order to make the passage to quantum mechanics, we note that the velocities occur similar to current densities, and replace $\frac{\bar{v}}{c}$ by $-\bar{\alpha}$. Breit (7) was the inventor of this substitution. The free-particle Hamiltonians, of course, have well known quantum-mechanical equivalents. These substitutions lead to the Dirac Hamiltonian for two Dirac particles:

(50) $$H_D = -c\,\bar{\alpha}^{I}\cdot\bar{\pi}^{I} - \beta^{I}m^{I}c^2 + e^{I}\phi^{I} - c\,\bar{\alpha}^{II}\cdot\bar{\pi}^{II} - \beta^{II}m^{II}c^2$$

$$+ e^{II}\phi^{II} + \frac{e^{I}e^{II}}{r}\left[1 - \frac{1}{2}\left(\bar{\alpha}^{I}\cdot\bar{\alpha}^{II} + \frac{\bar{\alpha}^{I}\cdot\bar{r}\ \bar{\alpha}^{II}\cdot\bar{r}}{r^2}\right)\right].$$

This equation was first derived and used by Breit (6), in the study of Helium fine structure. It agrees better with experiment than the equation of Gaunt and Eddington.

When the operator substitution for the velocity was made, the interaction between the intrinsic moments was automatically included, as in eq. 7. This important point should not be overlooked. To find the interaction of particles possessing an anomalous moment, it is only necessary to add terms descriptive of the additional moment.

C. Pauli Particles to Order v^2/c^2. There are four principal contributions to the mutual energy of the anomalous moments:

1. The energy of the magnetic dipoles in their mutual field.

2. The energy of the electric dipoles in their mutual field.

3. The energy of the magnetic dipoles in the magnetic field of the moving charges.

4. The energy of the electric dipoles in the electric field of the charges.

The electric dipole mentioned is again the apparent electric dipole of the moving particle. This apparent moment is in the "direction" of the vector $\bar{\alpha}$, and its magnitude is v/c times the magnitude of the magnetic moment. Of the energies listed above, only the first remains at zero velocity. Interactions 3 and 4 are likely to be the strongest at moderate velocities, while interaction 2 will be about v/c times the first contribution.

The higher-order effects, even without retardation, become quite numerous. For example, the moving magnetic dipole will establish an electric field that will interact with the electric dipole of the other particle. The interaction of this electric field with the charge of the other particle is, of course, interaction 3 from a different point of view. These terms will be very small compared to the listed terms.

The effect of retardation deserves a closer look. In order to assess its importance, it is necessary to consider the absolute size of the terms as well as their v/c dependence. Due to the size of the basic term, the retardation correction to the interaction of the charges predominates. For moderate velocities, it can be expected to be of the same order of magnitude as the magnetic dipole energy. At low velocities, however, the magnetic dipole interaction will be more important. It is evident that the importance of the various terms depends sensitively on the problem under consideration.

For this reason we shall not attempt to consider retarded interactions of the anomalous moments. The major effect of these terms will be felt at the lower velocities, and at higher velocities the neglected part of the retarded interaction of the charges will mask any correction to the anomalous moment terms. The basic expression for each type of important interaction will be included, however. Under certain conditions, the mere presence or absence of a small term has a deciding influence on the conclusions that may be drawn.

The interaction terms are constructed from classical analogy, using the operators exhibited in part II, eq. 26. The interaction of the electric dipole and the electric field is then:

(51)

$$(-i\mu^{I}\beta^{I}\alpha^{I})\cdot\frac{e^{II}\bar{r}}{r^3}, \quad \bar{r}=\bar{r}^{I}-\bar{r}^{II}.$$

The magnetic moment interacts with the magnetic field of the moving charge:

(52)
$$(\mu^{I}\beta^{I}\bar{\sigma}^{I})\cdot e^{II}\frac{c\bar{\alpha}^{II}\times\bar{r}}{r^{3}}.$$

Symmetrizing the terms of eq. 51 and eq. 52, we now have:

(53)
$$i e^{I}\mu^{II}\beta^{II}\frac{\bar{\alpha}^{II}\cdot\bar{r}}{r^{3}} + i e^{II}\mu^{I}\beta^{I}\frac{\bar{\alpha}^{I}\cdot(-\bar{r})}{r^{3}}$$
$$+ e^{I}\mu^{II}\beta^{II}\bar{\alpha}^{I}\cdot\frac{\bar{r}\times\bar{\sigma}^{II}}{r^{3}} + e^{II}\mu^{I}\beta^{I}\alpha^{II}\cdot\frac{(-\bar{r})\times\bar{\sigma}^{I}}{r^{3}}.$$

Classically, we obtain the field due to a dipole by differentiation of the scalar potential for an electric dipole, and of the vector potential for a magnetic dipole. The basic result for a dipole field is:

(54)
$$\bar{p}\cdot\nabla\nabla\frac{1}{r} = \frac{1}{r^{3}}\left(3\frac{\bar{r}\,\bar{p}\cdot\bar{r}}{r^{2}} - \bar{p}\right).$$

The electric-dipole energy is then:

(55)
$$\frac{(-i\mu^{I}\beta^{I})(-i\mu^{II}\beta^{II})}{r^{3}}\left(3\frac{\bar{\alpha}^{I}\cdot\bar{r}\;\bar{\alpha}^{II}\cdot\bar{r}}{r^{2}} - \bar{\alpha}^{I}\cdot\bar{\alpha}^{II}\right)$$
$$= -\frac{\mu^{I}\mu^{II}\beta^{I}\beta^{II}}{r^{3}}\left(3\frac{\bar{\alpha}^{I}\cdot\bar{r}\;\bar{\alpha}^{II}\cdot\bar{r}}{r^{2}} - \bar{\alpha}^{I}\cdot\bar{\alpha}^{II}\right).$$

In the magnetic case, we get an additional term containing the Dirac delta function. This arises from the interaction of the dipole with the magnetization producing the field. The magnetic dipole energy is written in eq. 56, where the magnetization can be taken as

$$(56)\quad \frac{\mu^{I}\mu^{II}\beta^{I}\beta^{II}}{r^3}\left(3\,\frac{\bar{\sigma}^{I}\cdot\bar{r}\;\bar{\sigma}^{II}\cdot\bar{r}}{r^2}-\bar{\sigma}^{I}\cdot\bar{\sigma}^{II}\right)$$

$$+\,4\pi\mu^{I}\mu^{II}\beta^{I}\beta^{II}\,\bar{\sigma}^{I}\bar{\sigma}^{II}\,\delta(\bar{r}).$$

Collecting all the terms in one equation, we have the following interaction Hamiltonian for two Pauli particles:

(57)

$$H_p = ie^{I}\mu^{II}\beta^{II}\,\frac{\bar{\alpha}^{II}\cdot\bar{r}}{r^3}+ie^{II}\mu^{I}\beta^{I}\,\frac{\bar{\alpha}^{I}\cdot(-\bar{r})}{r^3}+e^{I}\mu^{II}\beta^{II}\bar{\alpha}^{I}\cdot\frac{\bar{r}\times\bar{\sigma}^{II}}{r^3}$$

$$+\,e^{II}\mu^{I}\beta^{I}\bar{\alpha}^{II}\cdot\frac{(-\bar{r})\times\bar{\sigma}^{I}}{r^3}-\frac{\mu^{I}\mu^{II}\beta^{I}\beta^{II}}{r^3}\left(3\,\frac{\bar{\alpha}^{I}\cdot\bar{r}\;\bar{\alpha}^{II}\cdot\bar{r}}{r^2}-\bar{\alpha}^{I}\cdot\bar{\alpha}^{II}\right)$$

$$+\,\frac{\mu^{I}\mu^{II}\beta^{I}\beta^{II}}{r^3}\left(3\,\frac{\bar{\sigma}^{I}\cdot\bar{r}\;\bar{\sigma}^{II}\cdot\bar{r}}{r^2}-\bar{\sigma}^{I}\cdot\bar{\sigma}^{II}\right)+4\pi\mu^{I}\mu^{II}\beta^{I}\beta^{II}\bar{\sigma}^{I}\cdot\bar{\sigma}^{II}\,\delta(\bar{r})$$

$$+\,\frac{e^{I}e^{II}}{r}\left[1-\frac{1}{2}\left(\bar{\alpha}^{I}\cdot\bar{\alpha}^{II}+\frac{\bar{\alpha}^{I}\cdot\bar{r}\;\bar{\alpha}^{II}\cdot\bar{r}}{r^2}\right)\right].$$

Extension of the results to the case of several particles is made by considering the interactions pair by pair. For N particles, there will be $4N(N - 1)$ terms of the above types; three particles will give rise to twenty-four interaction terms. As an example, the Dirac interaction of three particles is:

(58)

$$\frac{e^{I}e^{II}}{r_{12}}\left[1 - \frac{1}{2}\left(\bar{\alpha}^{I}\cdot\bar{\alpha}^{II} + \frac{\bar{\alpha}^{I}\cdot\bar{r}_{12}\,\bar{\alpha}^{II}\cdot\bar{r}_{12}}{r_{12}^{2}}\right)\right] + \frac{e^{I}e^{III}}{r_{13}}\left[1 - \frac{1}{2}\left(\bar{\alpha}^{I}\cdot\bar{\alpha}^{III} + \frac{\bar{\alpha}^{I}\cdot\bar{r}_{13}\,\bar{\alpha}^{III}\cdot\bar{r}_{13}}{r_{13}^{2}}\right)\right] + \frac{e^{II}e^{III}}{r_{23}}\left[1 - \frac{1}{2}\left(\bar{\alpha}^{II}\cdot\bar{\alpha}^{III} + \frac{\bar{\alpha}^{II}\cdot\bar{r}_{23}\,\bar{\alpha}^{III}\cdot\bar{r}_{23}}{r_{23}^{2}}\right)\right].$$

D. <u>Reduction of the Hamiltonian</u>. Solution of the wave equation in terms of wave functions requires that the equations for the components of the wave function be written explicitly. In the case of two particles, there is a system of sixteen simultaneous first-order partial differential equations in seven variables and sixteen unknown functions. In general, if there are n particles, one has 4^n unknown functions. It is usually convenient to distinguish the wave function components by a set of n subscripts, each running from 1 to 4. Of these, the components containing only 3's and 4's in the subscripts will be large components. Thus, for two particles, there are four large components and twelve small. The general procedure is to approximate the small components and then solve for the large ones.

Dirac operators for two particles should actually involve a system of 16 x 16 matrices, but in this case it is possible to continue the use of the much more convenient 4 x 4 matrix space through a few alterations in the algebra. The wave function is written as a 4 x 4, with the indices corresponding to the two particles increasing, respectively, across the top and down the front edge of the matrix. The operators corresponding to one particle are then transposed and operate from the other side of the wave function. For example, if

(59)

$$\psi = \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} \\ \psi_{21} & \psi_{22} & \psi_{23} & \psi_{24} \\ \psi_{31} & \psi_{32} & \psi_{33} & \psi_{34} \\ \psi_{41} & \psi_{42} & \psi_{43} & \psi_{44} \end{bmatrix} \qquad \text{and} \qquad \beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

then

(60)

$$\beta^{I}\beta^{II}\psi = \begin{bmatrix} \psi_{11} & \psi_{12} & -\psi_{13} & -\psi_{14} \\ \psi_{21} & \psi_{22} & -\psi_{23} & -\psi_{24} \\ -\psi_{31} & -\psi_{32} & \psi_{33} & \psi_{34} \\ -\psi_{41} & -\psi_{42} & \psi_{43} & \psi_{44} \end{bmatrix}$$

For more than two particles, the easiest route is to forsake the matrix algebra and proceed from definition, operating directly on the indices of the wave-function components.

IV. THE MØLLER CORRESPONDENCE METHOD

A. Introduction. It is possible, under certain restrictions, to treat collisions in a strictly relativistic way by a method due to Møller (42). A collision may be defined as the interaction of two particles such that before and after the collision they are widely separated in space, but during the collision they are close together. For instance, let an initially free primary particle in the state 0 approach a target particle in state A from infinity, and suppose it is scattered into state 1 while the target particle jumps to state B as a result of their interaction. Knowing the initial and final states of the primary, the transition current between the states can be written. The transition current is the current due to an incident beam of state 0 which is subsequently scattered into state 1. The notion itself is purely quantum-mechanical, but the classical analogue is easily visualized. Analytically, the transition current is written $\psi_0^A j_\mu \psi_1$. Since the state 0 and 1 are free-particle states with wave functions of the form $u e^{-i/\hbar E t}$, the space and the time dependency of the transition current separates easily: $u_0^A j_\mu e^{i/\hbar (E_0 - E_1)t}$. It must be remembered, of course, that the states 0 and 1 are the states of a beam of particles with specified vector momentum. With the restriction that the result must be interpreted wholly quantum-mechanically, the precise computation of the retarded potentials due to the transition current is now possible.

The form of the results obtained in expressing the retarded potentials in terms of the transition current suggests that the method be used in computing the matrix element for the associated transition as in the ordinary first-order perturbation theory of quantum mechanics. The expression "to the first order in the interaction" refers to this application of the method.

The transition probability is, to the first order in the interactin, given by the expression:

(61)

$$P_{AB}^{01} = \frac{2\pi}{\hbar} \left| Q_{AB}^{01} \right|^2 \delta\left(E_0 + E_A - E_1 - E_B \right).$$

The delta-function expresses the conservation of energy with regard to the initial and final energies of the translatory motion of the colliding particles. Radiationless collisions without change in the internal states of the particles, or of systems to which they may belong, are here under consideration. Q_{AB}^{01} is the matrix element of the perturbing Hamiltonian between the initial states 0, A and the final states 1, B. Explicitly,

(62)

$$Q_{AB}^{01} = \iint \psi_0^{A} \psi_A^{A} \, Q \, \psi_1 \psi_B \, d\tau \, d\tau'$$

Since we wish to calculate Q from a known transition current, the matrix element for the transition induced on the target may be written

(63)

$$Q^{01}_{AB} = \int \psi_A^A Q^{01} \psi_B \, d\tau = \int u_A^A Q^{01} u_B e^{\frac{i}{\hbar}(E_A - E_B)t} d\tau.$$

Q^{01} is expressed now as a Fourier integral:

(64)

$$Q^{01} = \int_{-\infty}^{\infty} e^{-2\pi i \nu t} d\nu \int_{-\infty}^{\infty} Q^{01}(x,t') e^{2\pi i \nu t'} dt',$$

remembering that Q is time-dependent. But $Q^{01} = \int u_0^A q \, u_1 e^{\frac{i}{\hbar}(E_0 - E_1)t} d\tau$, and we can retard the contributions to the Fourier integral by the substitution $t' = t'' + r/c$.

Then we have:

(65)

$$Q^{01} = \int u_0^A \int_{-\infty}^{\infty} e^{-2\pi i \nu t} d\nu \int_{-\infty}^{\infty} q \, e^{\frac{i}{\hbar}(E_0 - E_1)t''} e^{2\pi i \nu (t'' + r/c)} dt'' u_1 \, d\tau.$$

The operator q may include functions of the positions of the particles and operators for both particles; we assume only that it is not explicitly time-dependent. With $i/\hbar \, (E_0 - E_1) = -2\pi i \nu_0$, we obtain:

(66)

$$Q^{01} = \int u_0^A q \int_{-\infty}^{\infty} e^{-2\pi i \nu t} e^{2\pi i \nu r/c} \int_{-\infty}^{\infty} e^{2\pi i (\nu - \nu_0) t''} dt'' u_1 \, d\tau.$$

Remembering that

(67)
$$\int_{-\infty}^{\infty} e^{2\pi i(\nu-\nu_0)t''}dt'' = \delta(\nu-\nu_0)$$

we can perform the integration of ν and obtain:

(68)
$$Q^{01} = \int \upsilon_0{}^A q\, e^{-2\pi i\nu_0(t+r/c)}\upsilon_1\, d\tau.$$

Substitution in eq. 63 gives the matrix element:

(69)
$$Q^{01}_{AB} = \iint \upsilon_0{}^A \upsilon_A{}^A q\, e^{-2\pi i\nu_0(t-r/c)}\upsilon_1\upsilon_B\, d\tau\, d\tau'.$$

The effect of retardation is to multiply the operator describing the nonretarded interaction by $\exp(-2\pi i\nu_0\left[t-r/c\right])$. Any differentiations with respect to time or position must operate on this factor as well as on the part of the expression describing the nonretarded interaction. The procedure will now be illustrated by applying it to the interaction of two Dirac particles.

B. <u>Interaction of the Dirac Currents</u>. The contribution of the Dirac current to the vector potential (four-potential) is:

(70)
$$A^D_\mu = \int \frac{\left[\psi_0{}^A j_\mu \psi_1\right]_{t-r/c}}{r}\, d\tau \qquad \mu = 1, 2, 3, 4.$$

Using free-particle wave functions $\psi_0{}^A = u_0{}^A e^{\frac{iE_0 t}{\hbar}}$, $\psi_1 = u_1 e^{-\frac{iE_1 t}{\hbar}}$:

(71)

$$\bar{A}_\mu^d = \int u_0{}^A \frac{\left[j_\mu e^{-2\pi i \nu_0 t}\right]_{t-r/c}}{r} u_1 \, d\tau .$$

$\bar{A}_\mu^d$ is expressed as a Fourier integral:

(72)

$$\bar{A}_\mu^D = \int u_0^A \int_{-\infty}^{\infty} e^{-2\pi i \nu t} d\nu \int_{-\infty}^{\infty} \frac{\left[j_\mu e^{-2\pi i \nu_0 \alpha}\right]_{\alpha+r/c}}{r} e^{2\pi i \nu \alpha} d\alpha \, u_1 \, d\tau$$

$$= \int u_0^A \int_{-\infty}^{\infty} e^{-2\pi i \nu t} d\nu \int_{-\infty}^{\infty} \frac{j_\mu}{r} e^{2\pi i (\nu - \nu_0)\alpha'} e^{2\pi i \nu r/c} d\alpha' \, u_1 d\tau$$

$$= \int u_0^A \int_{-\infty}^{\infty} \frac{j_\mu}{r} e^{2\pi i \nu \frac{r}{c}} e^{-2\pi i \nu t} \int_{-\infty}^{\infty} e^{2\pi i (\nu - \nu_0)\alpha'} d\alpha' \, u_1 d\tau$$

(73)

$$\bar{A}_\mu^D = \int u_0{}^A \frac{j_\mu}{r} e^{-2\pi i \nu_0 (t - r/c)} u_1 \, d\tau .$$

With the Dirac interaction $-\bar{A}_\mu j_\mu$, the perturbing "Hamiltonian" is:

(74)

$$Q_{AB} = -\int u_A{}^A A_\mu^{01\,\mathrm{II}} j_\mu^{\mathrm{I}} e^{i\omega t} u_B \, d\tau ,$$

where $\omega = 2\pi\nu_0 = \frac{E_0 - E_1}{h} = \frac{E_B - E_A}{h}$. Superscript II refers to the primary, and superscript I to the target. The matrix element, using eq. 73, becomes:

(75)

$$Q_{AB}^{01} = \iint u_A{}^A u_0{}^A \frac{-j_\mu^{\mathrm{I}} j_\mu^{\mathrm{II}}}{r} e^{-i\omega r/c} u_B u_1 \, d\tau \, d\tau' .$$

The interaction term becomes

(76) $$\frac{e^{I}e^{II}}{r}\left(1-\bar{\alpha}^{I}\cdot\bar{\alpha}^{II}\right)e^{-i\omega r/c},$$

after the substitution $j_\mu = ie\gamma_\mu$ and multiplication from the right by $\beta^{I}\,\beta^{II}$. This is the simple nonretarded interaction of eq. 42 if the exponential is neglected. Compare eq. 76 with the approximate retarded interaction of eq. 50.

C. Interaction of Pauli Particles. The interaction terms are:

(77) $$H_I = e\bar{\alpha}\cdot\bar{A} - e\phi + \mu\beta\bar{\sigma}\cdot\bar{B} - i\mu\beta\bar{\alpha}\cdot\bar{E},$$

where $\bar{B}$ is written for $\bar{H}$ since we now consider a density of magnetization. We have already found the Dirac potentials:

(78) $$\bar{A}^{D} = -\int U_0^{A}\,\frac{e\bar{\alpha}}{r}\,e^{i\omega(t-r/c)}\,U_1\,d\tau,$$

$$\phi^{D} = \int U_0^{A}\,\frac{e}{r}\,e^{i\omega(t-r/c)}\,U_1\,d\tau.$$

The total potential is split into a Dirac part and a Pauli part:

(79) $$\bar{A} = \bar{A}^{D} + \bar{A}^{P}, \qquad \phi = \phi^{D} + \phi^{P}.$$

Fields are obtained by the usual differentiations:

(80) $$\bar{B} = \nabla\times\bar{A}, \qquad \bar{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\bar{A}}{\partial t}.$$

Finally, eq. 77 is expressed in terms of a Dirac interaction and an anomalous interaction:

(81)
$$H_I = \left(e\bar{\alpha}\cdot\bar{A}^D - e\phi^D\right) + e\bar{\alpha}\cdot\bar{A}^P - e\phi^P + \mu\beta\bar{\sigma}\cdot\bar{B} - i\mu\beta\bar{\alpha}\cdot\bar{E}\,.$$

The first term of eq. 81 is already known (eq. 76).

The vector and scalar potentials of the anomalous moment can be expressed in terms of the polarization, magnetization, and their derivatives. Let $\bar{M}$ and $\bar{P}$ denote the vector magnetization and polarization, respectively, both in the general sense and in the operator sense. The brackets denote a retarded quantity.

(82)
$$\phi^P = \int \frac{[\nabla\cdot\bar{P}]}{r}\,d\tau = \int\left\{[\bar{P}]\cdot\nabla_1\frac{1}{r} + \frac{r}{c}\left[\frac{\partial\bar{P}}{\partial t}\right]\cdot\nabla_1\frac{1}{r}\right\}d\tau\,,$$

$$\bar{A}^P = \int\left\{\frac{[\nabla_1\times\bar{M}]}{r} - \frac{1}{cr}\left[\frac{\partial\bar{P}}{\partial t}\right]\right\}d\tau = \int\left\{[\bar{M}]\times\nabla_1\frac{1}{r} + \frac{r}{c}\left[\frac{\partial\bar{M}}{\partial t}\right]\times\nabla_1\frac{1}{r} - \frac{1}{cr}\left[\frac{\partial\bar{P}}{\partial t}\right]\right\}d\tau.$$

(83)
$$\bar{P} = \psi_0^A\bar{P}\psi_1 = u_0^A\,\bar{P}e^{i\omega t}u_1\,, \qquad \bar{M} = u_0^A\,\bar{M}e^{i\omega t}u_1\,,$$

$$\frac{\partial\bar{P}}{\partial t} = u_0^A\,i\omega\bar{P}e^{i\omega t}u_1\,, \qquad \frac{\partial\bar{M}}{\partial t} = u_0^A\,i\omega\bar{M}e^{i\omega t}u_1\,.$$

(84)
$$\bar{P} + \frac{r}{c}\frac{\partial \bar{P}}{\partial t} = \upsilon_0^A \bar{P}\left(1 + i\omega\frac{r}{c}\right)e^{i\omega t}\,\upsilon_1$$

$$\bar{M} + \frac{r}{c}\frac{\partial \bar{M}}{\partial t} = \upsilon_0^A \bar{M}\left(1 + i\omega\frac{r}{c}\right)e^{i\omega t}\,\upsilon_1$$

(85)
$$\Phi^P = \int \upsilon_0^A \left(\bar{P}\cdot\nabla_1\frac{1}{r}\right)\left(1+i\omega\frac{r}{c}\right)\exp i\omega(t-r/c)\,\upsilon_1\,d\tau$$

$$\bar{A}^P = \int \upsilon_0^A\left[\left(\bar{M}\times\nabla_1\frac{1}{r}\right)\left(1+i\omega\frac{r}{c}\right) - \frac{i\omega}{rc}\bar{P}\right]\exp i\omega\left(t-\frac{r}{c}\right)\upsilon_1\,d\tau$$

Replace $\bar{M}$ and $\bar{P}$ by their operator equivalents (eq. 86).

(86)
$$\bar{M} = \mu\beta\bar{\sigma}, \qquad \bar{P} = -i\mu\beta\bar{\alpha};$$

(87)
$$\phi^P = -\int \upsilon_0^A\left(i\mu\beta\bar{\alpha}\cdot\nabla_1\frac{1}{r}\right)\left(1+i\omega\frac{r}{c}\right)\exp i\omega\left(t-\frac{r}{c}\right)\upsilon_1\,d\tau,$$

$$\bar{A}^P = \int \upsilon_0^A\left[\left(\mu\beta\bar{\sigma}\times\nabla_1\frac{1}{r}\right)\left(1+i\omega\frac{r}{c}\right) - \frac{\omega\mu}{rc}\beta\bar{\alpha}\right]\exp i\omega\left(t-\frac{r}{c}\right)\upsilon_1\,d\tau.$$

(A careful observer will note the vector potential due to a moving electric dipole in the last of eqs. 87.)

In order to find the fields, we avail ourselves of the following differentiation formulae and vector identities, which are easily proved by elementary means:

(88) $\nabla f(r)\exp i\omega(t-\frac{r}{c}) = \left(\nabla f(r) - \frac{i\omega}{cr}\bar{r}f(r)\right)\exp i\omega(t-r/c)$

(89) $\nabla\times\bar{v}(r)\exp i\omega(t-\frac{r}{c}) = \left(\nabla\times\bar{v}(r) - \frac{i\omega}{cr}\bar{r}\times\bar{v}(r)\right)\exp i\omega(t-r/c)$

(90) $\frac{\partial}{\partial t}\bar{v}f\exp i\omega(t-\frac{r}{c}) = i\omega\bar{v}f\exp i\omega(t-r/c)$

(91) $\nabla f(r)(1+i\omega\frac{r}{c})\exp i\omega(t-r/c) = \left[(1+i\omega\frac{r}{c})\nabla f + \frac{\omega^2}{c^2}\bar{r}f\right]\exp i\omega(t-r/c)$

(92) $\nabla\times\bar{v}\left(1+i\omega\frac{r}{c}\right)\exp i\omega(t-r/c) = \left[(1+i\omega\frac{r}{c})\nabla\times\bar{v} + \frac{\omega^2}{c^2}\bar{r}\times\bar{v}\right]\exp i\omega(t-r/c)$

(93) $\bar{\nabla}_1 = -\nabla$

(94) $\nabla\frac{1}{r} = -\frac{\bar{r}}{r^3}$

(95) $\nabla\left(\bar{\alpha}\cdot\nabla_1\frac{1}{r}\right) = -\frac{1}{r^3}\left(3\frac{\bar{r}\,\bar{\alpha}\cdot\bar{r}}{r^2} - \bar{\alpha}\right)$

(96) $\nabla\times\frac{\bar{\alpha}}{r} = \bar{\alpha}\times\frac{\bar{r}}{r^3}$

(97) $\nabla\times\left(\bar{\sigma}\times\nabla_1\frac{1}{r}\right) = \frac{1}{r^3}\left(3\frac{\bar{r}\,\bar{\sigma}\cdot\bar{r}}{r^2} - \bar{\sigma}\right) + 4\pi\bar{\sigma}\delta(\bar{r})$

(98) $\bar{r}\times(\bar{\sigma}\times\bar{r}) = r^2\bar{\sigma} - \bar{r}\,\bar{r}\cdot\bar{\sigma}$

The fields may now be written by inspection from eqs. 78 and 87:

Dirac fields:

$$\nabla\phi^D = \left(-\frac{e\bar{r}}{r^3} - \frac{i\omega e\bar{r}}{cr^2}\right)\exp i\omega(t-r/c)$$

$$\frac{1}{c}\frac{\partial\bar{A}^D}{\partial t} = -\frac{i\omega}{cr}e\bar{\alpha}\exp i\omega(t-r/c)$$

$$\nabla\times\bar{A}^D = \left(-e\frac{\bar{\alpha}\times\bar{r}}{r^3} - i\frac{\omega}{cr}e\frac{\bar{\alpha}\times\bar{r}}{r}\right)\exp i\omega(t-r/c)$$

(99)

$$\bar{E}^D = \left[\frac{e\bar{r}}{r^3} + \frac{i\omega}{c}\left(\frac{e\bar{r}}{r^2} + \frac{e\bar{\alpha}}{r}\right)\right] \exp i\omega(t - r/c),$$

(100)

$$\bar{B}^D = e\,\frac{\bar{r}\times\bar{\alpha}}{r^2}\left(1 + i\frac{\omega}{c}\right)\exp i\omega(t - r/c).$$

Pauli (anomalous) fields:

(101)

$$\nabla\phi^P = \left[\frac{i\mu\beta}{r^3}\left(3\frac{\bar{r}\,\bar{\alpha}\cdot\bar{r}}{r^2} - \bar{\alpha}\right)\left(1 + i\frac{\omega}{c}\right) - i\mu\beta\,\frac{\omega^2}{c^2}\,\frac{\bar{r}\,\bar{\alpha}\cdot\bar{r}}{r^2}\right]\exp i\omega\left(t - \frac{r}{c}\right)$$

$$\frac{1}{c}\frac{\partial\bar{A}^P}{\partial t} = i\frac{\omega}{c}\left[-\mu\beta\bar{\sigma}\times\nabla\frac{1}{r}\left(1 + i\frac{\omega r}{c}\right) - \frac{\omega\mu}{rc}\beta\alpha\right]\exp i\omega\left(t - \frac{r}{c}\right)$$

$$\nabla\times\bar{A}^P = \left\{\mu\beta\left[\frac{1}{r^3}\left(3\frac{\bar{r}\,\bar{\sigma}\cdot\bar{r}}{r^2} - \bar{\sigma}\right) + 4\pi\delta(\bar{r})\right]\left(1 + i\frac{\omega r}{c}\right)\right.$$

$$\left. - \frac{\omega\mu\beta}{c}\bar{\alpha}\times\frac{\bar{r}}{r^3} + \frac{i\omega^2\mu\beta}{c^2}\,\bar{r}\times\frac{\bar{\alpha}}{r^2}\right\}\exp i\omega(t - r/c)$$

$$\bar{E}^P = \left\{-\frac{i\mu\beta}{r^3}\left(3\frac{\bar{r}\,\bar{\alpha}\cdot\bar{r}}{r^2} - \bar{\alpha}\right) - i\frac{\omega}{c}\left[\frac{i\mu\beta}{r^3}\left(3\frac{\bar{r}\,\bar{\alpha}\cdot\bar{r}}{r^2} - \alpha\right)\right.\right.$$

$$\left.\left. - \frac{\mu\beta}{r^2}\bar{\sigma}\times\bar{r}\right] + \frac{\omega^2}{c^2}\left[i\mu\beta\,\frac{\bar{r}\,\bar{\alpha}\cdot\bar{r}}{r^2} - \mu\beta\,\frac{\bar{\sigma}\times\bar{r}}{r} + i\mu\beta\,\frac{\bar{\alpha}}{r}\right]\right\}\exp i\omega\left(t - \frac{r}{c}\right)$$

(102)

$$\bar{B}^P = \left\{ \frac{\mu\beta}{r^3}\left(3\frac{\bar{r}\,\bar{\sigma}\cdot\bar{r}}{r^2} - \bar{\sigma}\right) + 4\pi\mu\beta\bar{\sigma}\,\delta(\bar{r}) + i\frac{\omega}{c}\frac{\mu\beta}{r^2}\left(3\frac{\bar{r}\,\bar{\sigma}\cdot\bar{r}}{r^2} - \sigma\right) - \frac{\omega}{c}\mu\beta\bar{\alpha}\times\frac{\bar{r}}{r^2} - \frac{\omega^2}{c^2}\mu\beta\left(\frac{\bar{r}\,\bar{\sigma}\cdot\bar{r}}{r^2} - \bar{\sigma}\right) + i\frac{\omega^2\mu\beta}{c^2}\frac{\bar{r}\times\bar{\alpha}}{r} \right\} \exp i\omega(t - r/c).$$

The terms containing ω will be known as retarded terms. We now write the nonretarded fields:

(103)

$$\bar{E}^D = \frac{e\bar{r}}{r^3}\exp i\omega\left(t - \frac{r}{c}\right), \quad \bar{B}^D = e\frac{\bar{r}\times\bar{\alpha}}{r^3}\exp i\omega\left(t - r/c\right),$$

(104)

$$\bar{E}^P = -\frac{i\mu\beta}{r^3}\left(3\frac{\bar{r}\,\bar{\alpha}\cdot\bar{r}}{r^2} - \bar{\alpha}\right)\exp i\omega\left(t - r/c\right),$$

$$\bar{B}^P = \left\{\frac{\mu\beta}{r^3}\left(3\frac{\bar{r}\,\bar{\sigma}\cdot\bar{r}}{r^2} - \bar{\sigma}\right) + 4\pi\mu\beta\bar{\sigma}\,\delta(\bar{r})\right\}\exp i\omega\left(t - r/c\right).$$

Using eq. 81, we find the following additional interaction terms (nonretarded).

$$H_I = -i\mu^{II}\beta^{II}\bar{\alpha}^{II}\cdot\frac{e^I\bar{r}}{r^3} - \frac{\mu^I\mu^{II}\beta^I\beta^{II}}{r^3}\left(3\frac{\bar{\alpha}^I\cdot\bar{r}\,\bar{\alpha}^{II}\cdot\bar{r}}{r^2} - \bar{\alpha}^I\cdot\bar{\alpha}^{II}\right) + \mu^{II}\beta^{II}\bar{\sigma}^{II}\frac{e^I\bar{r}\times\bar{\alpha}^I}{r^3}$$

$$+ \frac{\mu^I\mu^{II}\beta^I\beta^{II}}{r^3}\left(3\frac{\bar{\sigma}^I\cdot\bar{r}\,\bar{\sigma}^{II}\cdot\bar{r}}{r^2} - \bar{\sigma}^I\cdot\bar{\sigma}^{II}\right) + 4\pi\mu^I\mu^{II}\beta^I\beta^{II}\bar{\sigma}^I\cdot\bar{\sigma}^{II}\,\delta(\bar{r})$$

$$- e^{II}\alpha^{II}\cdot\mu^I\beta^I\frac{\bar{\sigma}^I\times\bar{r}}{r^3} + i e^{II}\mu^I\beta^I\alpha^I\cdot\frac{\bar{r}}{r^3}.$$

Writing this symmetrically,

(105)

$$H_I = ie^{II}\mu^{I}\beta^{I}\frac{\bar{\alpha}^{I}\cdot\bar{r}}{r^3} + ie^{I}\mu^{II}\beta^{II}\frac{\bar{\alpha}^{II}\cdot(-\bar{r})}{r^3} + e^{II}\mu^{I}\beta^{II}\bar{\alpha}^{II}\cdot\frac{\bar{r}\times\bar{\sigma}^{I}}{r^3}$$

$$+ e^{I}\mu^{II}\beta^{II}\bar{\alpha}^{I}\cdot\frac{(-\bar{r})\times\bar{\sigma}^{II}}{r^3} - \frac{\mu^{I}\mu^{II}\beta^{I}\beta^{II}}{r^3}\left(3\frac{\bar{\alpha}^{I}\cdot\bar{r}\,\bar{\alpha}^{II}\cdot\bar{r}}{r^2} - \bar{\alpha}^{I}\cdot\bar{\alpha}^{II}\right)$$

$$+ \frac{\mu^{I}\mu^{II}\beta^{I}\beta^{II}}{r^3}\left(3\frac{\bar{\sigma}^{I}\cdot\bar{r}\,\bar{\sigma}^{II}\cdot\bar{r}}{r^2} - \bar{\sigma}^{I}\cdot\bar{\sigma}^{II}\right) + 4\pi\mu^{I}\mu^{II}\beta^{I}\beta^{II}\,\bar{\sigma}^{I}\cdot\bar{\sigma}^{II}\,\delta(\bar{r}).$$

Compare eqs. 105 and 57.

The retarded terms may be written just as easily. Eq. 105 verifies the analysis leading to eq. 57 for the nonretarded terms. Retardation is introduced in a different manner in the two Hamiltonians.

Møller's correspondence method has been extended to the case of radiative collisions by Møller (32) (35), and to bound systems by Hulme (36). Carlson and Oppenheimer (13) treat several problems by this method, including the collisions of electrons with neutrons. Refer also to Rosenfeld (55). The method has not as yet been extensively used for nucleon scattering.

The general correctness of the method is evidenced by its early success in the description of electron-electron scattering and the passage of fast electrons through matter.

Further information about the general theory of scattering can be obtained from the works of Mott and Massey (48), Condon and Morse (14), Morse (46), and Williams (65). Nucleon scattering is introduced in any textbook of theoretical nuclear physics; for example, see Bethe and Morrison (3) or Sachs (56). More advanced information, and an introduction to the literature, can be obtained from Breit (12), Breit, Condon and Present (8), and Blatt and Jackson (5).

V. CONCLUSION

We have met with a degree of success in the theoretical description of the electromagnetic interaction of spin-1/2 particles that have an anomalous moment. The approximate wave equation of Part III is correct to order v^2/c^2 in its principal term, and is at least correct to order v/c in the less important terms. In addition, it contains all first order effects. This equation will describe the actions of nucleons at moderate energies, where our neglect of their complex structure is not too serious. At higher energies the basic hypothesis of the applicability of the modified Dirac equation is in doubt. At moderate energies particularly simple descriptions of the nuclear interaction are available. In a series of articles, Breit (9) (10) (11) has developed approximately relativistic equations for nucleons, and has applied his results to the deuteron and to scattering.

A particular example of the application of the results of Part III is given by Schwinger (58) in the article which led to the initiation of the present study. The thesis of the article is that the difference between the proton-proton and neutron-proton interactions is just the difference between their electromagnetic interactions. This demonstration would prove the valuable property of the charge independence of nuclear force. Our interest is in the Hamiltonian that is exhibited as a starting point for his discussion. This expression is equivalent to our eq. 57, except that the term involving the electric dipoles is absent. We believe that this term ought to be included. The reasoning in support of this assertion has already been given. Because of its size, its inclusion should not materially affect the conclusions drawn in its absence, since the particular problem under consideration was not sensitive to its presence. As a rough estimate, its magnitude should be no more than v/c times the magnitude of the magnetic dipole term ($v/c = 0.2$ for 20-Mev protons).

Turning to the Møller method of Part IV, we remark that much the same use can be made of this method in subtracting the electromagnetic effects in the analysis of scattering data. Here the results are correct to all orders of v/c, and may therefore be used at high energies. Since we used only the basic interaction term, and not the entire wave equation, the results should apply up to energies at which the electric forces become negligible by comparison to the nuclear forces—perhaps at several hundred Mev.

The two approaches give information over complementary energy ranges. Problems involving explicit wave functions, such as the theory of the deuteron and some low-energy scattering, may use the wave equation of Part III to advantage. High-energy scattering may be treated by the Møller method, and explicit wave functions are not needed.

In conclusion, I would thank Professor Frank Woods for his suggestions, encouragement, and untiring interest in this project, which would not have been undertaken without his influence and help.

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APPENDIX A. FREE PARTICLE SOLUTION

From the wave equation for a free particle,

$$\left(\frac{\partial}{\partial x_\mu}\gamma_\mu + \frac{mc}{\hbar}\right)\psi = 0,$$

the following system of four equations results:

$$-i\frac{\partial}{\partial x_1}\psi_4 - \frac{\partial}{\partial x_2}\psi_4 - i\frac{\partial}{\partial x_3}\psi_3 - \frac{1}{ic}\frac{\partial}{\partial t}\psi_1 + \frac{mc}{\hbar}\psi_1 = 0$$

$$-i\frac{\partial}{\partial x_1}\psi_3 + \frac{\partial}{\partial x_2}\psi_3 + i\frac{\partial}{\partial x_3}\psi_4 - \frac{1}{ic}\frac{\partial}{\partial t}\psi_2 + \frac{mc}{\hbar}\psi_2 = 0$$

$$i\frac{\partial}{\partial x_1}\psi_2 + \frac{\partial}{\partial x_2}\psi_2 + i\frac{\partial}{\partial x_3}\psi_1 + \frac{1}{ic}\frac{\partial}{\partial t}\psi_3 + \frac{mc}{\hbar}\psi_3 = 0$$

$$i\frac{\partial}{\partial x_1}\psi_1 - \frac{\partial}{\partial x_2}\psi_1 - i\frac{\partial}{\partial x_3}\psi_2 + \frac{1}{ic}\frac{\partial}{\partial t}\psi_4 + \frac{mc}{\hbar}\psi_4 = 0$$

Multiplying each equation by i and substituting $\psi_k = C_k e^{\frac{i}{\hbar}(\bar{p}\cdot\bar{x} - Et)}$:

$$C_4\frac{i}{\hbar}(p_1 - ip_2) + C_3\frac{i}{\hbar}p_3 - C_1\left(-\frac{iE}{c\hbar} - i\frac{mc}{\hbar}\right) = 0$$

$$C_3\frac{i}{\hbar}(p_1 + ip_2) - C_4\frac{i}{\hbar}p_3 - C_2\left(-\frac{iE}{c\hbar} - i\frac{mc}{\hbar}\right) = 0$$

$$-C_2\frac{i}{\hbar}(p_1 - ip_2) - C_1\frac{i}{\hbar}p_3 + C_3\left(-\frac{iE}{c\hbar} + i\frac{mc}{\hbar}\right) = 0$$

$$-C_1\frac{i}{\hbar}(p_1 + ip_2) + C_2\frac{i}{\hbar}p_3 + C_4\left(-\frac{iE}{c\hbar} + i\frac{mc}{\hbar}\right) = 0$$

Factoring $i/\hbar$ from each equation, we obtain the following equations for the four constants:

$$(p_1 - ip_2)C_4 + p_3 C_3 + (E/c + mc)C_1 = 0$$
$$(p_1 + ip_2)C_3 - p_3 C_4 + (E/c + mc)C_2 = 0$$
$$-(p_1 - ip_2)C_2 - p_3 C_1 + (mc - E/c)C_3 = 0$$
$$-(p_1 + ip_2)C_1 + p_3 C_2 + (mc - E/c)C_4 = 0$$

This system of homogeneous equations possesses nontrivial solutions only if the determinant of the coefficients vanishes:

$$\begin{vmatrix} mc + E/c & 0 & p_3 & p_1 - ip_2 \\ 0 & mc + E/c & p_1 + p_2 & -p_3 \\ -p_3 & -(p_1 - ip_2) & mc - E/c & 0 \\ -(p_1 + ip_2) & p_3 & 0 & mc - E/c \end{vmatrix} = 0$$

This may be written:

$$\begin{vmatrix} mc + E/c & 0 & p_3 & p_1 - ip_2 \\ -p_3 & -(p_1 - ip_2) & mc - E/c & 0 \\ 0 & mc + E/c & p_1 + ip_2 & -p_3 \\ -(p_1 + ip_2) & p_3 & 0 & mc - E/c \end{vmatrix} = 0$$

By Laplace's expansion in 2 x 2 determinants, this yields:

$$\left[(m^2c^2 - E^2/c^2) + p_1^2 + p_2^2 + p_3^2 \right]^2 = 0$$

or,

$$E = \pm c(p^2 + m^2c^2)^{\frac{1}{2}}, \text{ both roots double.}$$

Because of the multiplicity of the roots, two of the four constants are arbitrary. Let $C_3 = A$ and $C_4 = B$ be the arbitrary constants. Solutions for C_1 and C_2 that satisfy all four equations are then easily found to be:

$$C_1 = -c\,\frac{p_3A + (p_1 - ip_2)B}{E + mc^2} \qquad C_2 = -c\,\frac{-p_3B + (p_1 + ip_2)A}{E + mc^2}$$

If we had chosen as a solution $\psi_k = C_k e^{i/\hbar(\bar{p}\cdot\bar{x} + Et)}$, the equations would develop similarly. This is the case of a beam of particles moving to the left (for positive p) instead of to the right. If the primes denote the constants in this case, the solution is

$$C'_3 = -C_1 \qquad C'_4 = -C_2 \qquad C'_1 = C_3 \qquad C'_2 = C_4.$$

In summary, the free-particle solutions of Dirac's equation are:

$$\Psi_A = \begin{bmatrix} -\dfrac{p_3A + (p_1 - ip_2)B}{E/c + mc} \\ \dfrac{-p_3B + (p_1 + ip_2)A}{E/c + mc} \\ A \\ B \end{bmatrix} e^{\frac{i}{\hbar}(\bar{p}\cdot\bar{x} - Et)} \qquad \Psi_B = \begin{bmatrix} A \\ B \\ \dfrac{p_3A + (p_1 - ip_2)B}{E/c + mc} \\ \dfrac{-p_3B + (p_1 + ip_2)A}{E/c + mc} \end{bmatrix} e^{\frac{i}{\hbar}(\bar{p}\cdot\bar{x} + Et)}$$

Normalization to unit flux in the beam leads to a condition similar to

$$2(A^*A + B^*B) = 1 + mc^2/E.$$

APPENDIX B. DIRAC MATRICES

The 2 x 2 Pauli spin matrices are the basic elements. These three matrices and the unit matrix are a representation of a quaternion algebra.

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In terms of the spin matrices, the Dirac matrices α_i and β are:

$$\alpha_1 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} \quad \alpha_2 = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix} \quad \alpha_3 = \begin{bmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{bmatrix} \quad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The γ matrices are:

$$\gamma_1 = \begin{bmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{bmatrix} \quad \gamma_2 = \begin{bmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{bmatrix} \quad \gamma_3 = \begin{bmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{bmatrix} \quad \gamma_4 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

The following matrices, together with the γ, form a basis of Dirac operator space:

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad i\gamma_1\gamma_2\gamma_3 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

$$-\sigma_1 = i\gamma_2\gamma_3 = \begin{bmatrix} -\sigma_1 & 0 \\ 0 & -\sigma_1 \end{bmatrix} \quad -\sigma_2 = i\gamma_3\gamma_1 = \begin{bmatrix} -\sigma_2 & 0 \\ 0 & -\sigma_2 \end{bmatrix} \quad -\sigma_3 = i\gamma_1\gamma_2 = \begin{bmatrix} -\sigma_3 & 0 \\ 0 & -\sigma_3 \end{bmatrix}$$

$$\alpha_1 = i\gamma_1\gamma_4 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} \quad \alpha_2 = i\gamma_2\gamma_4 = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix} \quad \alpha_3 = i\gamma_3\gamma_4 = \begin{bmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{bmatrix}$$

$$i\gamma_2\gamma_3\gamma_4 = \begin{bmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{bmatrix} \quad i\gamma_3\gamma_1\gamma_4 = \begin{bmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{bmatrix} \quad i\gamma_1\gamma_2\gamma_4 = \begin{bmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{bmatrix}$$

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