

THE MATHEMATICS IN
MATHEMATICAL MODELING

by

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ABSTRACT

The purpose of this research was to investigate how teachers interact with mathematics while teaching mathematical modeling to elementary students. To conduct this study, I used a case study approach with four elementary teachers. Each teacher participated in professional development on mathematical modeling prior to the study and incorporated mathematical modeling into their classroom. Modeling task lessons were observed and teachers participated in interviews before and after each lesson. I qualitatively explored what mathematical decisions teachers made while teaching mathematical modeling and how students' mathematical contributions influenced the modeling cycle. This analysis took place through three analytical lenses: the mathematics used, the teacher's interactions with their student's mathematical ideas, and as compared to components of the mathematical modeling cycle. Findings indicate that students engaged in meaningful mathematics to explore real-world problems. Across all cases, teachers prepared students to use mathematics by creating tasks with mathematical opportunities and by orienting the students towards using mathematics to investigate the problems. Each teacher allowed their students to introduce most of the mathematical ideas used to investigate the modeling questions. Each task became a mathematical modeling task by the way it was implemented, through teachers' and students' contributions to the activity.

CHAPTER ONE

THE PROBLEM

Introduction

In recent years, mathematical modeling has gained the attention of teachers and education researchers alike. The Common Core State Standards for Mathematics (CCSSM) have ignited this conversation by incorporating modeling as a “practice standard” for grades K-12 and a “content standard” for high school mathematics curriculum (National Governors Association Center for Best Practices, 2010), with near nationwide adoption of the Standards. While mathematical modeling across grades K-12 is a relatively new concept, long standing research on topics such as mathematical literacy and mathematical problem solving have supported and supplemented the modeling conversation (Schoenfeld, 2013). Due to the topic’s fledgling prominence, many questions regarding the role of modeling and the teaching of modeling stand unanswered.

Mathematical modeling is a process that uses mathematics as a tool to answer authentic real-world questions. This process comprises a non-linear sequence steps – form a mathematical question, define assumptions, formulate a model, determine results, and interpret and test the solution in context of the real-world scenario (Mooney & Swift, 1999). The Society for Industrial and Applied Mathematics (SIAM) defines a mathematical model as “a representation of a system or scenario that is used to gain qualitative or quantitative understanding of some real-world problems and to predict

future behavior” (Bliss, Fowler, & Galluzzo, 2014, p. 3). A model is a *product* and modeling is a *process*, an important distinction that will be elaborated on and discussed throughout this paper (Niss, Blum, & Galbraith, 2007).

Proponents of mathematical modeling believe that modeling is important to include across grades kindergarten through 12th grade (Asempapa, 2015) rather than only at the collegiate or professional level, where it has historically resided (Malkevitch, 2012). Mathematical modeling is important for students to engage in because it can benefit their understanding of mathematics. Modeling helps students develop mathematical literacy by building connections between the real world and mathematics (Steen, Turner, & Burkhardt, 2007). By seeing mathematics as useful and relatable, students develop productive dispositions toward mathematics (Lesh & Yoon, 2007). Finally, modeling supports deep, integrated understanding of mathematical content and practices (Lehrer & Schauble, 2007).

Proponents argue that students should be introduced to mathematical modeling early in their schooling (Kaiser & Maass, 2007). It is important that when students learn about mathematical tools, they also learn to ask mathematical questions of the world around them. Students do not automatically know how to ask a mathematical question or apply a mathematical tool just because they learn to use a mathematical skill (Darling-Hammond & Austin, 2014). Moskal and Skokan (2011) hypothesize that because many students are not taught authentic connections between mathematics and the real world and because middle grade students do not find mathematics relatable, students lose interest in the subject. “Students don’t see math in the real world; they enter school ready

to share mathematical ideas, but often lose interest when it does not seem to apply to their own lives” (Steen et al., 2007).

Definitions and Benefits of Modeling

Mathematical modeling can easily be confused with teaching concepts that share the term “model,” as well as mathematical activities that have similarities to mathematical modeling (Garfunkel & Montgomery, 2016). When teachers plan or implement a task this confusion may surface, so it is important to clarify what modeling is and is not.

Mathematical modeling is sometimes referred to as *modeling with mathematics*, and is a process of solving a problem using mathematics (Wolf, 2015). In contrast, *models of mathematics* use the real world to help students understand mathematics. Using a drawing model, like an area model for multiplication, or a concrete model, like number cubes, could easily be confused with mathematical modeling, particularly when the context of the problem is to add or multiply two real-world objects. Another point of confusion occurs when teachers hear the term “modeling” and associate the term with “demonstration” (Wolf, 2015). Once teachers learn what mathematical modeling is, these two uses of the terms are less likely to become confused.

It may be difficult to distinguish mathematical modeling from other classroom activities and problems based in the real world. Problems situated in the real world that necessitate mathematics as a tool to solve are categorized as *word problems*, *standard applications*, and sometimes *problem-solving* tasks. These problems, along with

modeling, vary by levels of authenticity, number of possible solutions, number of possible solution strategies, and whether a solver will need to make judgements (Jablonka, 2007) to interpret the real world scenario as a mathematical equation. There also exist types of activities that center around real-world problems with no obvious method of solution. These activities share goals with mathematical modeling in that both promote sense-making, curiosity, and questioning skills. However, these activities do not have the same content goals as mathematical modeling.

Learning and Teaching Modeling

In addition to the previously noted benefits gained by students who actively participate in the process of mathematical modeling, teachers may also find that incorporating modeling is beneficial in supporting the classroom environment they seek to create in all subjects. Differentiation and equitable teaching practices are often difficult to regularly incorporate in the classroom; mathematical modeling works well as an avenue to include these teaching techniques. Because modeling begins with an open-ended question, mathematical modeling lends itself to differentiation by allowing students multiple entry points. Teachers are also able to address equitable teaching practices in their classroom both by showing students that mathematics can solve important problems in their world and by valuing the individual and unique perspectives that students bring to the context of a modeling problem.

While there are many potential benefits to including modeling in the classroom, challenges also may arise. These difficulties likely stem from the fact that teachers lack

experience with modeling. As students, few teachers engaged in mathematical modeling tasks, and in teacher preparation, few teachers learned how to engage students in mathematical modeling tasks (Doerr, 2007).

Implementation of mathematical modeling in the classroom can be challenging (Niss et al., 2007). Teachers will most likely face additional challenges if their classroom is set up in a traditional lecture-and-practice mode or believe that mathematics is a static subject (Kaiser & Maass, 2007). Teachers whose classrooms are teacher-centered and operate around lecture will have to adapt many parts of their class; not only will their teaching methods need to change but the classroom culture in which they teach will also. They will have to teach students to have mathematical conversations with each other, to share ideas, to understand that there is not only one correct answer, and that the teacher will not direct their solution strategies. These teachers will face pedagogical demands such as listening to students' ideas and making connections between these ideas.

A Framework for Modeling

In 2014, the National Science Foundation (NSF) funded a project called IMMERSION to provide professional development in mathematical modeling to elementary grade teachers. The project's goals also encompass research on the effects of this professional development on a teacher's classroom practice. In preparation for providing professional development on mathematical modeling, Carlson, Wickstrom, Burroughs, and Fulton (2016) developed a *Teaching Framework for Modeling in the K-5 Setting* (also referred to as the Teaching Framework for Modeling) to support teachers in creating modeling opportunities for their students (Figure 1). The Teaching Framework

for Modeling considers how teachers might successfully prepare to engage their students in a modeling task and what teachers might do as their students work through the modeling process. Teachers spend time *developing* and *anticipating* a modeling task, they then *enact* the modeling lesson, and finally follow up with *revisiting* the task. In the enactment stage, the teacher and students both have roles. As students work through the modeler's cycle of *pose mathematical questions*, *build mathematical solutions*, and *validate conclusions* in the context of the task, the teacher moves through the steps of *organizing* the students to the modeling step, *monitoring* student work, and *regrouping* students to share ideas and work. The Teaching Framework for Modeling provides questions and points for teachers to consider and is described in more detail in Chapters 2 and 3. IMMERSION and the Teaching Framework for Modeling offer a setting for investigating the emerging research area of teaching modeling in grades K-5.

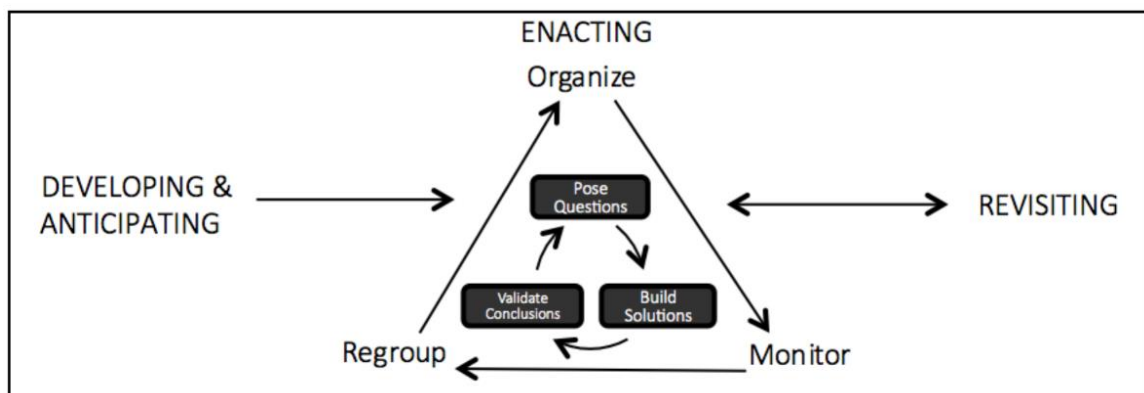


Figure 1. Illustration of the Teaching Framework for Modeling in the K-5 Setting (Carlson et al., 2016).

The Teaching Framework for Modeling is theoretically based and thus should be empirically studied to determine how it describes the realities of teaching mathematical modeling in the elementary grades.

Statement of the Problem

Few teachers and researchers know how the teaching of mathematical modeling at the elementary level is enacted, and it remains unknown how a teacher should best prepare and implement a mathematical modeling task. “Implementing, on a large scale, excellent mathematical teaching in the area of modeling and application is particularly challenging” (Tam, 2011, p. 32).

Part of the difficulty of teaching mathematical modeling is that it requires students to make assumptions and value judgments. This allows for students’ input and perspective to shape the direction of the modeling cycle; some teachers may be unfamiliar or uncomfortable with teaching lessons based on students’ input because they cannot plan the entire lesson ahead of time or know exactly how the task ought to culminate and close. It is important to know how successful teachers of mathematical modelers interact with and incorporate their students’ contributions and mathematical thinking into their teaching of mathematical modeling.

What does it look like to teach mathematical modeling in elementary grades? How does a teacher interact with mathematics while teaching mathematical modeling? What mathematical ideas emerge when an elementary teacher uses modeling to teach particular mathematical ideas? What is the interplay between the teacher’s choice of task or context, the teacher’s responses to student ideas, and his or her facilitation of classroom mathematical discourse? These questions motivated the research reported here.

Research Questions

This study investigated the following questions to examine how a teacher incorporates mathematics when teaching mathematical modeling to elementary students. The research addressed planning for the mathematics of a modeling task, implementation of mathematical modeling, and teachers' interaction with students' mathematical ideas and questions throughout the modeling cycle.

1. What mathematical decisions and choices do teachers make in the process of implementing mathematical modeling?
2. How do students' mathematical contributions and behaviors influence the implementation of a modeling lesson?

Importance of the Study

Researchers believe it is critical that students understand the usefulness of mathematics in order to become successful problem-solvers and critical thinkers (Middleton, Lesh, & Heger, 2003). While learning the skills taught in mathematics is important, students also must be taught how those skills are useful and be given opportunities to apply what they learn to situations around them (Verschaffel, De Corte, & Vierstraete, 1999). Modeling is an authentic way to address this need.

What really matters in mathematics education is learning and practicing the mathematical modeling process. The particular field of application, whether it is everyday life or being a good citizen or understanding some piece of science, is less important than the experience with this thinking process (Pollak, 2012, p. ix).

Researchers and teachers want to know how young students can authentically model, and how to teach in such a way that this is possible. NSF currently funds programs such as IMMERSION to study this topic. The National Council of Teachers of Mathematics (NCTM) *Annual Perspectives in Mathematics Education*, whose role is to move the field of mathematics education forward by considering current ideas, recently published a book on mathematical modeling (Hirsch & McDuff, 2016). The book, a compilation of articles that address questions and issues related to modeling, considers questions of how to teach mathematical modeling. One article presents the Teaching Framework for Modeling (M. A. Carlson et al., 2016) which might help teachers and researchers think about the work teachers must do to provide opportunities for their students to engage in mathematical modeling.

This study reported here provides an in-depth description of how teachers incorporate mathematics through teaching mathematical modeling to elementary students. It chronicles the mathematics that arises from mathematical modeling tasks, both intended mathematics and unexpected mathematics. It documents how teachers and students introduce mathematical ideas, and how teachers use students' ideas to develop the modeling lesson. The findings are instructive to other teachers who want to learn how teachers use mathematics in a modeling lesson and how they handle the uncertainty of students' mathematical contributions to a modeling lesson.

List of Definitions and Acronyms

The terms and acronyms used throughout this dissertation are defined below.

CCSSM	Common Core State Standards for Mathematics
Facilitator	A university professor (or graduate student) or teacher leader who leads a teacher study group in the IMMERSION professional development
IMMERSION	IMMERSION is a 3-year professional development program funded by NSF through the STEM-C Program. The acronym stands for Integrating Mathematical Modeling, Experiential learning and Research through a Sustainable Infrastructure and an Online Network for teachers in the elementary grades.
NSF	National Science Foundation
OECD	Organisation for Economic Co-operation and Development
SIAM	Society of Industrial and Applied Mathematics
PBL	Problem Based Learning
STEM	Science, Technology, Engineering, and Mathematics
TSG	Teacher Study Group

CHAPTER TWO

REVIEW OF THE LITERATURE

Introduction

This chapter is a review of the literature about mathematical modeling in elementary grades. The review begins by defining mathematical modeling from a mathematical modeler's perspective. To clarify issues surrounding the teaching of modeling and teachers' understandings of the topic, I will compare mathematical modeling to other views of modeling in mathematics as well as compare mathematical modeling to other problem-solving activities aimed at engaging students in "real world" mathematics. Following this is a discussion on the benefits of elementary students engaging in mathematical modeling. The review concludes with literature aimed at describing the challenges of teaching mathematical modeling as well as support for teachers.

Defining Mathematical Modeling

Mathematical modeling is a process that takes a real-life scenario and uses mathematics to pose and answer a question in context of the situation. The process of mathematical modeling involves identifying a problem in the real world, choosing the tools (e.g. specific equations, operations, and formulas) needed to solve the problem, and implementing these tools to find the solution (Swetz, 1991). This process also requires modelers to determine if the solution makes sense in context of the original scenario

(NGA, 2010). Mathematical modeling is cyclic in nature because the process allows for refinement and allows for choice in assumptions and variables since there is not one correct solution to problematic situations (Bliss et al., 2014). In general, researchers agree on the overall process of modeling, though slight differences in terminology or categorization distinguish one researcher's representation of the modeling cycle from others (e.g., Doerr & English, 2003; NGA, 2010; Swetz, 1991; Tran & Dougherty, 2014; Verschaffel & De Corte, 1997). Bliss et al. (2014) present their diagram of the modeling cycle (Figure 2); they emphasize that it is not important to follow the steps in a particular sequential order as modeling is a cyclic and iterative process. Steps in mathematical modeling laid out by Bliss et al. (2014) are as follows:

- Defining the Problem Statement – a mathematical problem is posed which indicates what the output of the model will be;
- Making Assumptions – a complex situation can be simplified by making assumptions and determining important factors;
- Defining Variables – determine the important factors and how they vary;
- Getting a Solution – use appropriate mathematical tools to solve the model;
- Analysis and Model Assessment – consider how well the model satisfies the original situation. Consider the strengths, weaknesses, assumptions needed, and possible improvements; and

- Reporting the Results – the reporting process depends on who the model is intended for (Bliss et al., 2014, p. 7).

Some advocates for modeling point out the opportunities to broaden students' mathematical thinking and beliefs through modeling. Unlike mathematical activities students are familiar with, true modeling activities require students to make judgments about the situation (Jablonka, 2007). The modeler may bounce back and forth between mathematics processes as well as earlier decisions and assumptions (OECD, 2012).

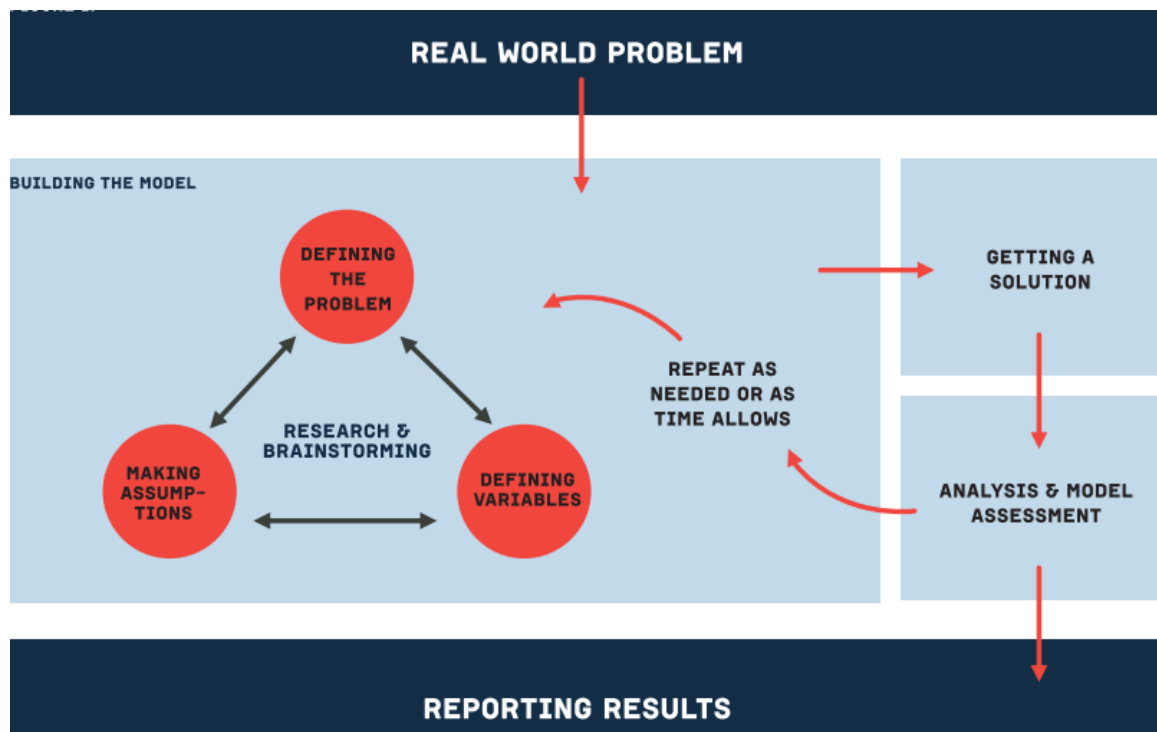


Figure 2. Modeling Cycle (Bliss et al., 2014, p. 6).

Mathematical modeling, as described, is work traditionally done by applied mathematicians, statisticians, and other scientists. The definition and perspective of modeling described above will be referred to as the *modeler's perspective*. Modeling

from this perspective is taught at the college level in preparation for professional work in fields using applied mathematics (Niss, Blum, & Galbraith, 2007).

Word problems and problem solving have long been a part of the K-12 mathematics curriculum, though modeling is a recent addition to the curriculum. With the introduction of CCSSM in 2011, modeling is now included both as a high school-level mathematical content standard and as a K-12 mathematics practice (NGA, 2010). However, implementation at the high school level may not be wide spread and inclusion in textbooks is limited (Meyer, 2015).

Recent attention has turned to modeling in elementary grades because the CCSSM includes “model with mathematics” as a K-12 practice. Introducing modeling at the elementary level is new to students and teachers alike. Therefore, it is important to study teaching modeling at the elementary level. Despite the fact that the CCSSM mathematical content standards for modeling is limited to high school, this is not necessarily because elementary students cannot model; there is some evidence that elementary students are indeed able to model (Doerr & English, 2003; English, 2006). Though elementary students have far fewer mathematical tools than secondary and college students, elementary students want to understand the world around them (Sarama & Clements, 2009). This inclination primes students to model; modeling at its core is observing, explaining, and understanding the real world through the lens of mathematics. “This realization suggests that modeling might begin as early as the very first years of primary school” (Greer, Verschaffel, & Mukhopadhyay, 2007, p. 90).

Other Interpretations of “Modeling”

Complicating the understanding of modeling is that there exist different definitions of the word “model”. Particularly in elementary grades, there are many uses of the terms “model” and “modeling” that differ from the modeler’s perspective of mathematical modeling. This may make learning to model difficult as teachers need to distinguish between different types of modeling. These other types of modeling are important teaching techniques and are not considered wrong or inappropriate uses of the term. Rather, the purpose of this section is to make distinctions between the various meanings and purposes of the terms *model* and *modeling*.

Models of mathematics are tools and techniques that help the learner understand and do mathematics. These models can help illustrate ways of thinking about mathematics, or they can be models that help in solving mathematics.

Some models of mathematics are called *concrete models*. These are typically physical objects that students can manipulate in order to understand mathematical concepts. They help students understand the connection of conceptual understanding and symbolic representations of mathematics (Learner, n.d.). For instance, in primary grades students add objects and learn to write mathematical sentences with the addition and equals signs. A scenario combining a group of 2 items with 3 items could be represented by combining 2 unifix cubes with 3 unifix cubes. This strategy is also called *direct modeling* (Carpenter, Fennema, Franke, Levi, & Empson, 2015).

Visual models, like area models, are often used by students developing reasoning for multi-digit multiplication and for fractions. An area model can help students

understand the issues of place values in multiplying multi-digit numbers (Fuson & Beckmann, 2013). Pictures showing groups can help students to reason that different fractions are equal to each other (Figure 3). A teacher may use an area model to introduce equivalent fractions and a student may use an area model to explain mathematical work. In both cases, the model is a tool for understanding the underlying mathematical content. These models are not mathematical models because they are not translating the real world into mathematics.



Figure 3. Visual model of equivalent fractions (Annenberg Learner, n.d.).

Another interpretation of the term “modeling” is *teacher modeling*, which is the act of demonstration. “Many teachers view mathematical modeling as a process of showing the students how to approach or solve a problem” (Wolf, 2015, p. 5). The modeling instructional strategy is to show students how to do a skill while explaining each step, giving students both visual and verbal examples to follow (Louisville, n.d.). This is very different from the modeler’s definition because mathematical modeling is an action done by students, not a teacher action that students copy. This instructional strategy is often used for teaching mathematical skills; it is not applying mathematical knowledge to solve an authentic question in the real world.

Mental models is another theory in education using the term modeling. A mental model is a “cognitive map,” a representation of the way individuals transform ideas mentally (Johnson-Laird, 2004). Mental models can refer to teachers’ view of students’ process of learning mathematics (Ernest, 1989). It also can be much more specific, a categorization of the different ways that students construct understanding of a particular type of problem (Chinnappan, 1998).

Mathematical Problem Solving

Various types of mathematical problem solving are concerned with solving real world problems. This is similar to mathematical modeling, but there are distinguishing factors among these types of problems: authenticity of the real-world situation, how the real-world situation affects the interpretation of a mathematical question, possible mathematical strategies, and possible mathematical solutions. Problem solving encompasses many types of problems; some consider the whole real world, some study only parts of the real world, and others delve mostly into the mathematical world. Below are three major types of problem sets that are considered problem solving that either authentically explore the real world or that examine only parts of the real world. In order to differentiate types of problems by how “real world” they are, researchers have made distinctions between *word problems*, *standard applications*, and *mathematical modeling* (Niss et al., 2007; Tran & Dougherty, 2014) (Figure 4).

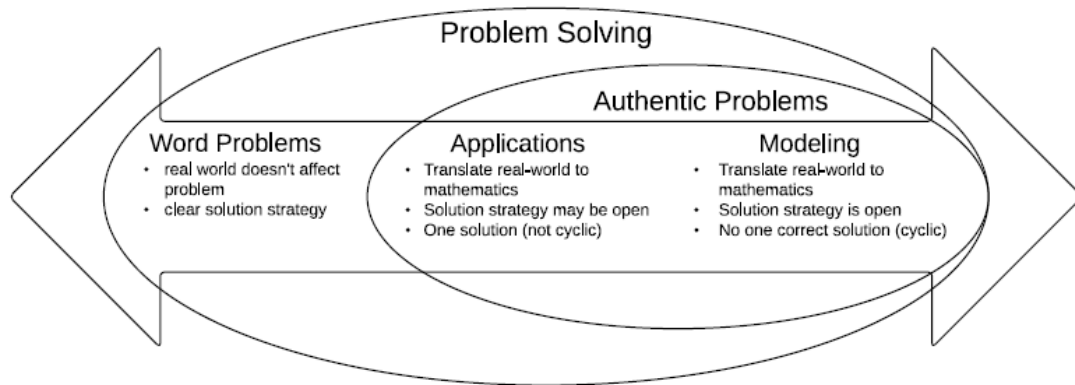


Figure 4. Problem Solving Continuum. Adapted from Tran, D., & Dougherty, B. J. (2014).

Since problem solving encompasses many types of problems, problem solving has had different definitions (Schoenfeld, 1992) ranging from intending to train students to think creatively and to develop problem solving strategies “to provide potential teachers with instruction in a narrow band of heuristic strategies...to learn standard techniques in particular domains, most frequently in mathematical modeling...to provide a new approach to remedial mathematics” (Schoenfeld, 1992, p. 337). In considering problem solving, some researchers consider *routine* and *non-routine* problem solving. Routine problem solving is learning a technique and applying it to similar problems. Non-routine problem solving aims to teach students to use their mathematical skills in conjunction with realistic sense-making. An example of a non-routine problem is “Steve has bought 4 planks of 2.5 m each. How many planks of 1 m can he saw out of these planks?” (Verschaffel et al., 1999, p. 266). This is considered a non-routine problem because in order to solve this correctly, the student must realize that a 10-meter continuous plank is not the same as four 2.5 meter planks; the student needs to use knowledge of the real world rather than solely reduce the problem to a set of numbers. In cases like this, one

goal of problem solving is to “stop elementary school children’s suspension of realistic sense-making during arithmetic word problem solving and help them to develop a disposition toward (more) realistic modeling of mathematical application problems” (Verschaffel & De Corte, 1997, p. 580).

In contrast, routine problem solving is taught as a unit where students learn certain techniques and then practice problems employing such techniques. For instance, Schoenfeld (1992) discusses the merits of the following problem-solving exercises from Milne’s *Mental Arithmetic*: “How much will it cost to plow 32 acres of land at \$3.75 per acre?” The problem was taught with a specific method of solution: 3.75 is $\frac{3}{8}$ of \$10, and \$10/acre is \$320, and $\frac{3}{8}$ of \$320 is \$120. This example illustrates how problem-solving problems are often written in order to practice the mathematical strategies from a particular lesson. Schoenfeld (1992) introduces concerns of not recognizing several valid solution strategies when teaching a problem such as this with one specific solution method when there are multiple ways to approach the problem.

There are other famous problem-solving questions that neither dictate a method of solution nor ask students to their use everyday real-world understanding. The famous Tower of Hanoi question (Figure 5) is an example of such a problem: *There are three posts, on the left most post is 64 golden disks whose diameter steadily decreases as the stack grows taller. The goal is to move all disks to the right-most post moving only one disk at a time while never stacking a larger disk atop a smaller disk. The story states that the end of world occurs when the last piece is moved – when will this happen?* Problems such as these are commonly introduced in mathematics classes as fun and challenging

tasks. The purpose is usually to stimulate students' thinking and introduce a new mathematical topic.

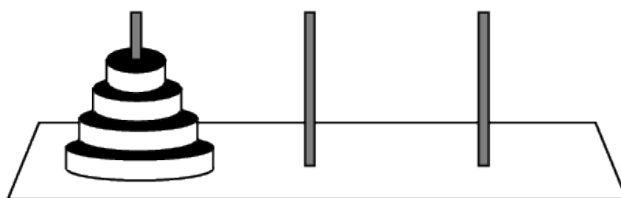


Figure 5. Tower of Hanoi disks and posts.

Mathematical Modeling versus Problem Solving

Given the wide spectrum of meanings for problem solving, it is reasonable to ask how mathematical modeling relates to problem solving.

You may be wondering how mathematical modeling differs from what you already teach, particularly, ‘problem solving.’ Problem solving may not refer to the outside world at all. Even when it does, problem solving usually begins with the idealized real-world situation in mathematical terms, and ends with a mathematical result. Mathematical modeling, on the other hand, begins in the “unedited” real world, requires problem formulating before problem solving, and once the problem is solved, moves back into the real world where the results are considered in their original context (Pollak, 2012, viii).

The following sections contrast word problems and other forms of problem-solving with the process of mathematical modeling.

Word Problems. Many word problems, such as Milne’s (on page 19), are set in the “real world” because they deal with everyday objects and units but are constrained by including only the crucial information. Changing the context in the problem would not change the problem at all. Milne’s problem could ask about selling sandwiches just as easily as any other widget – the interpretation of the problem and the mathematics would

not change; the unit's primary function is to show the relationship between the different variables. For this reason, the context is not important to the problem other than to indicate that multiplication is the necessary operation.

Unlike mathematical modeling and other authentic problem solving, word problems do not encourage students to use sense-making. Instead students see word problems as artificial, contrived, and unrelated to the real world (Verschaffel et al., 1999). Word problems typically only “‘dress up’ a purely mathematical problem in words referring to a segment of the real world” (Niss et al., 2007, p. 11). Palm (2009) draws from Boaler (1992), Nesher (1980), and Verschaffel (2000) in saying “many researchers (and teachers and students) have argued that too many word problems are ‘pseudo-realistic’ and require the students to think differently than in out-of-school task situations” (Palm, 2009). Not only do word problems ask students to problem solve differently than they problem solve in real life, they often are low cognitive demand tasks (Smith & Stein, 2011). The distinction between word problems and modeling problems often lies in what students are expected to do with the task rather than in context itself.

In the context of a *word problem* the situation is not important to the problem. Since word problems only contain the essential information, there is only one solution. This is a stark contrast to mathematical modeling where the situation is imperative because the context influences interpretation and choice of variables and constraints. The choices of modeling lead to several appropriate solutions.

Authentic Problems. The importance of a real-world context is undebated for modeling tasks. It is a distinguishing characteristic between modeling and some problem-

solving tasks and word problems. Thus, it is important to consider what makes an authentic real-world setting or problem. Having authentic tasks encourages students to make sense of the mathematics behind a real-life situation “rather than simply serving as cover stories for proceduralized and frequently irrelevant tasks” (Doerr & English, 2003, p. 112). Unfortunately, according to Niss et al., (2007) authenticity does not have one distinct definition – what makes an authentic problem is debatable. Qualities of authentic tasks, according to Palm (2008), are described in Table 1.

Table 1. Characteristics of Authentic Tasks (Palm, 2008).

	<i>The Six Criteria for Authentic Tasks</i>
<i>Characteristic</i>	<i>Description</i>
<i>Event</i>	<i>In a simulation of a real-life task, the event has taken place or has a fair potential of taking place.</i>
<i>Question</i>	<i>The question is one that actually might be posed in the event.</i>
<i>Purpose in the figurative context</i>	<i>The purpose of the task needs to be as clear to the students as it would be if they actually encountered it in context - either provided explicitly from the description or implicitly from the context.</i>
<i>Information or data</i>	<i>The task describes specific subjects, objects, and places, includes accessible data, and provides identical or close numbers and values as in the simulated situation.</i>
<i>Language use</i>	<i>The task does not, for example, include difficult terms that hinder the students in the solution process if the corresponding difficulties do not occur in the simulated situation.</i>
<i>Tool demands</i>	<i>The tools and guidance in using them have to be reasonable the same in the simulated problem and school situation; mathematical knowledge and skills that students need for solving the task are available.</i>

The Tower of Hanoi problem, described earlier, is a common problem-solving task. The task is not modeling from the modeler's perspective, in part because it lacks authenticity since it is not an *event* that might take place, the *question* is unlikely to be posed, and the *purpose* would not be encountered. The Tower of Hanoi represents a class of problem solving problems that are inauthentic and “whimsical problems, where mythical kingdoms and incredible professions and procedures may become the setting of some lovely mathematics. They make no pretense of being problems motivated by the real world” (Pollak, 2012, viii).

Authentic tasks are not sufficient to making a modeling task, though authenticity is necessary. Consider the following: *I am making a grocery list of ingredients I need to make an apple pie. I have 2 apples and need 5 to make an apple pie. How many apples do I need to buy at the grocery store?* This is a question that could easily be posed by a person interested in making an apple pie. The information, language, and tool demands all satisfy that of an authentic task. But if this question was posed in a class with the intention of students recognizing that subtraction or addition is needed to get the solution of three apples, students will not be asked to pose a mathematical question, and this would not be a modeling task. This problem could lead to a modeling task by considering that apples come in different sizes, but as this problem is typically approached, it remains an application problem for elementary students.

Standard Applications. Standard applications, sometimes referred to simply as applications, are identified as middle-ground questions between word problems and mathematical modeling. These problems typically require the student to interpret the

mathematics needed to solve the problem, though a solution strategy (or several strategies may be appropriate) is rather clear. Assumptions and value judgments are not needed in application problems (Niss et al., 2007). “Even though [the] context has the potential to occur in real life and there is an explicit purpose for solving the problem, placing the context within a particular... setting could enhance the problem” (Tran & Dougherty, 2014, p. 676). Applications do not use the cyclical process of mathematical modeling and there is one correct solution. An example of an application question is below.

Your family is looking to purchase a new cell phone plan. You research three providers in the area and get the following information:

- Plan A costs a basic fee of \$29.95 per month and 10 cents per text message
- Plan B costs a basic fee of \$90.20 per month and has unlimited text messages
- Plan C costs a basic fee of \$49.95 per month and 5 cents per text message
- All plans offer unlimited calling
- Calling on nights and weekends are free
- Long distance calls are included.

Which company should you choose? (Illustrative Mathematics, n.d.)

Asking this question with real information from local companies in the area, would make this an authentic question for someone shopping for a cell phone plane. This problem will take some work for students to translate the information to a mathematical problem.

There are also multiple solution strategies that a student could use to solve – graphs, tables, or equations.

When introducing applications of mathematics, teachers typically have taught a topic and want to show how the mathematical topic is useful in the real world. The intention is often to show “parts of the real world which are accessible to a mathematical

treatment and to which [a] corresponding mathematical model already exists” (Niss et al., 2007, p. 10).

Modeling Problems. Modeling problems are types of problem-solving problems; they are authentic problems and have resemblances to word problems and applications due to their real-world nature. For these reasons, mathematical modeling has similarities to types of problems that all teachers are familiar with. Modeling problems, like applications, require the modeler to understand the real-world context and information available to translate into mathematics. However, often in modeling the modeler must first ask a mathematical question.

In most [non-modeling] cases, the problem information has already been carefully mathematized for the student...with modeling tasks, the student’s goal is to make sense of the situation so that she or he can mathematize it in ways that are meaningful to her or him. This involves a cyclic process of selecting relevant quantities, creating meaningful representations, and defining operations that may lead to new quantities (Doerr & English, 2003, p. 113).

In modeling, modelers must consider the validity of their solutions, potentially engaging in the modeling cycle another time. This is different from application problems and other types of problem solving. With modeling, the purpose is to illustrate the process of taking a real-world problem and using mathematics as a tool to solve the problem (Niss et al., 2007). This differs from application problems, the purpose of which is often to show the usefulness of a particular mathematical procedure.

Modeling and Other Categories of Exploration-Based Classroom Activities

There are a variety of instructional strategies that have a purpose similar to mathematical modeling: to solve authentic problems using grade appropriate tools. In learning to teach mathematical modeling, teachers may confuse these strategies, which have fundamental differences in the content required to solve the problem. This section will briefly describe these instructional strategies and compare them to mathematical modeling.

Problem Based Learning. Problem Based Learning (PBL), also called project based learning and problem based instruction, is a way of teaching problem solving as well as learning a subject through the problem. PBL originated in medical school teaching, and is now used in elementary and secondary schools. PBL is oriented around a complex problem without a unique answer (Hmelo-Silver, 2004). Students are given the autonomy to work self-directed, individually, or as a group to determine what they need to learn to solve the problem, and to then investigate that context to solve the problem. Like mathematical modeling, PBL “address[es] simulations of an authentic, ill-structured problem” (Hung, Jonassen, & Liu, 2008, p. 488). In PBL, “the content and skills to be learned are organized around problems, rather than as a hierarchical list of topics, ... faculty cannot dictate learning” (Hung et al., 2008, p. 488). This is similar to modeling, which is taught to practice the thinking process of mathematical modeling rather than to learn the context of the situation or practice the mathematical skills used (Pollak, 2012). Teachers are viewed as facilitators of students’ work and learning – not “knowledge disseminators.” They “support and model [demonstrate] reasoning processes, facilitate

group processes and interpersonal dynamics, probe students' knowledge deeply, and never interject content or provide direct answers to questions" (Hung et al., 2008, p. 489).

PBL is not specifically about teaching mathematics. The premise is to use an authentic problem to let students direct their learning. Modeling is different because mathematics is the curricular goal and teachers can facilitate their students' learning. While in modeling students are the doers of mathematics, the teacher is guiding the classroom work. There may be times when a teacher needs to give an assignment or to interject content (Carlson et al., 2016). From this view, the teacher is more directive than the teacher in PBL. Secondly, the focus of mathematical modeling is to solve a real-world problem using mathematics while there is not a curricular goal with PBL.

Genius Hour. Genius Hour (GH) "is student driven passion based learning" (Juliani, 2015, p. 49). GH is also referred to as "20% time" and Passion Projects. PBL can be considered a type of GH. While in PBL, a class investigates a problem/project determined by the teacher, in GH students investigate a problem/project that each student is interested in. There is a loose format to GH, largely structured by the teacher. Teachers follow a series of steps:

- 1) Teacher plans out strong guidelines and rubrics at the beginning of the project.
- 2) Teacher prepares the class by giving reasons for GH and expectations.
- 3) Teacher plans classroom time and checkpoints along the way. Each class period has a purpose.
- 4) Teacher has 1 on 1 conferences with each student throughout project.
- 5) Teacher facilitates sharing of student projects with classmates.

- 6) Teacher has students reflect on their (ongoing) work.
- 7) Teacher facilitates success and failure.
- 8) Teacher has students present and teacher and classmates give feedback.

(Juliani, 2015)

An instructional coach from a school district that promotes Genius Hour and guides teachers in the practice states that GH projects tend to be focused around STEM issues. The tasks tend to be oriented more towards design than mathematics. Since the goal of GH is to answer the question at hand and teachers are not orienting problems toward any type of solution strategy, students may or may not explicitly use mathematics while investigating their problem. The purpose of GH is to promote creativity within students and to get them to find a purposeful project in something they are passionate about.

Model-Eliciting Activities. Model-Eliciting Activities (MEAs) are “problem solving activities that elicit a model. That is, their solutions require students to express their current ways of thinking in forms that are tested and refined multiple times” (Lesh & Yoon, 2007, p. 163). They are posed as realistic, open-ended questions requiring the solver to develop a model to find a solution. Students are given a situation where students “mathematize situations” by making mathematical interpretations of situations (Lesh, Hoover, Hole, Kelly, & Post, 2000, p. 592). There is no “right answer” to an MEA, instead there are different possible models and solutions. Lesh et. al. (2000) suggest that MEAs satisfy the following six principles:

1. The Reality Principle. Will students make sense of the situation by extending their own knowledge and experiences?
2. The Model Construction Principle. Does the task immerse students in a situation in which they are likely to confront the need to develop a mathematically significant construct?
3. The Self-Evaluation Principle. Does the activity promote self-evaluation on the part of the students?
4. The Construct Documentation Principle. Will the question require students to reveal their thinking about the situation?
5. The Construct Generalization Principle. Does the model provide a general model for analyzing this type of dynamic situation?
6. The Simplicity Principle. Is the situation simple? (Carlson, Larsen, & Lesh, 2003, p. 472)

MEAs are also designed to “help document student thinking” (Larson et al., 2010, p. 66). By revealing students’ thoughts, these activities are useful for instruction, assessment, and research (Lesh et al., 2000)

MEAs focus on inventing, documenting, and generalizing mathematical models (Larson et al., 2010). This focuses on the product of modeling, not on the process of modeling. Focusing on documenting and generalizing models requires skills that are not accessible to elementary, particularly primary, students. Model documentation is going to look very different in an elementary class as students are less advanced in writing and documenting their work. Similarly, higher levels of thinking are needed to share work.

While MEAs were originally developed as mathematical activities, other resources for teachers treat MEAs as interdisciplinary tasks that have much less focus on mathematics. For instance, CPALMS (Florida’s platform for educators to Collaborate, Plan, Align, Learn, Motivate, Share), the “official source for standards” for the state of Florida supported by the University of Florida, calls MEAs interdisciplinary tasks and likens them to engineering activities. The website states “MEAs were originally developed as a research tool in the K-12 mathematics education, MEAs have since been used successfully as teaching tools in a range of subject areas and grade levels” (About CPALMS, n.d.). MEAs can vary in purpose, thus researchers and teachers ought to closely examine MEAs to determine how mathematics is used to investigate the problems.

STEM Design Activities. Like mathematical modeling, STEM activities address real-world issues and problems. Additionally, there are several correct answers to a given activity (Jolly, 2014). However, STEM design activities are guided by the engineering design process, not guided by asking mathematical questions (Jolly, 2014). These activities are open-ended and focus on developing solutions. Since mathematical modeling specifically asks students to pose mathematical questions and use their mathematical tools, there are some differences between STEM activities and mathematical modeling. Burroughs et. al. (2015) propose an adjustment to the problem-solving continuum adding that STEM design activities overlap, but are not on, the problem-solving continuum because while they may use mathematics, mathematics is not required (Figure 6).

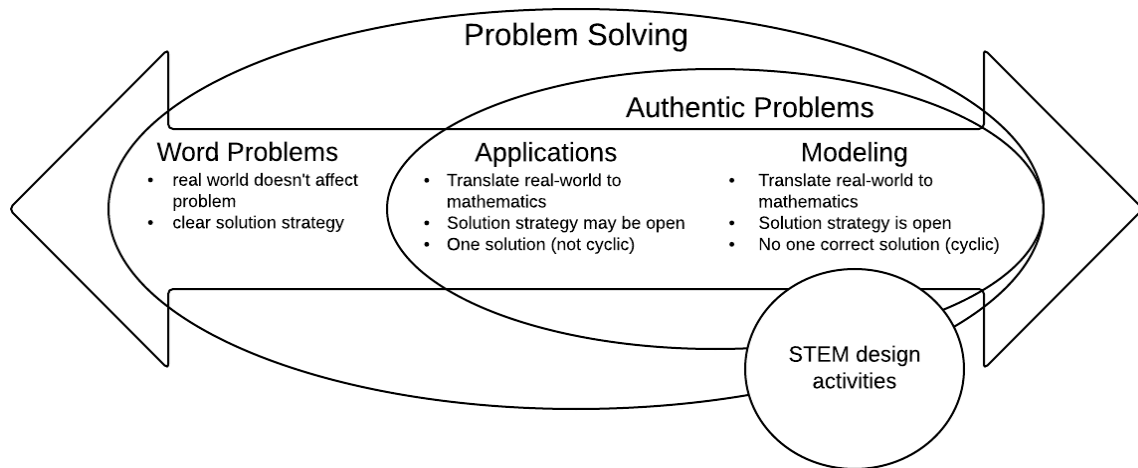


Figure 6. Problem solving continuum adaption. Adapted by Burroughs, Carlson, Wickstrom, & Fulton (2015) from Tran & Dougherty (2014).

Modeling in Curricula

Mathematical modeling is a part of mathematics standards for grade K -12 through documents such as Common Core State Standards and the GAISE framework.

Common Core State Standards for Mathematics. In the Common Core State Standards for Mathematics, modeling is addressed in two ways. First, modeling is a practice standard for grades K-12. Second, modeling is a content standard for high school. The practice standard for all grades states that “mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace... in early grades, this might be as simple as writing an addition equation to describe a situation.”(National Governors Association Center for Best Practices, 2010, p. 7) This description of modeling and the accompanying example can leave the reader with questions regarding exactly how modeling is defined. Tam (2011) suggests that the CCSSM typically uses the term *model* in the elementary context meaning models of mathematics or word problems.

There are various online platforms for teachers that provide examples of problems for teachers that align to the CCSSM standards. An example is Illustrative Mathematics, an independent non-profit group that writes and provides mathematics resources for teachers (Illustrative Mathematics, n.d.). The group is independent from the CCSSM, though it is committed to writing tasks that align with the CCSSM and is founded by a lead author of the CCSSM. Below is an example of a typical Illustrative Mathematics task addressing the modeling practice standard for a kindergarten class:

Students are given small bags of counting objects (the “goodies”). Each bag should contain a number of objects in the counting sequence students are working on, between 1 and 20. Students count the objects, record the number on the post-it note and stick the post-it note onto the outside of the bag. (Illustrative Mathematics, n.d.)

This task is an example of students using manipulatives to deepen their understanding of mathematics and is not an example of using mathematics to answer a question. Examples given to illustrate modeling in the elementary grades are often tasks that are models of mathematics or are word problems.

The content standard of modeling for high school discusses modeling as modelers describe it. “Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions” (NGA, 2010, p. 72). This content standard makes distinctions between modeling (the process) and the model (a product) (Tam, 2011). Incorporating modeling both as a practice standard and a content standard and addressing the standard

in both contexts, *models of mathematics* and *modeling with mathematics* “gives rise to confusion” (Tam, 2011, p. 30).

GAISE Framework. The Guidelines for Assessment and Instruction in Statistics Education Framework argues that statistical literacy is necessary for problem solving and understanding quantitative real-world situations (Franklin et al., 2007). Statistics is part of the mathematical curriculum for teaching mathematics at the elementary level (Usiskin, 2015). GAISE states that students, including elementary students should;

- Formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them;
- Select and use appropriate statistical methods to analyze data;
- Develop and evaluate inferences and predictions that are based on data; and
- Understand and apply basic concepts of probability. (Franklin et al., 2007, p. 5)

The work that the GAISE framework outlines for students is not described as modeling. “The modeling process is somewhat similar, but certainly not identical, to the four-step “investigative process” described in GAISE: formulate questions, collect data, analyze data, interpret results” (Usiskin, 2015, p. 5). The process is similar to the modeling cycle.

This section described the differences between mathematical modeling and both common mathematical classroom work and exploration classroom work. How mathematics is introduced and used is what distinguishes mathematical modeling from

other types of classroom work. The following section reviews literature describing proposed benefits of mathematical modeling in elementary grades.

Mathematical Modeling in Elementary Grades

Modeling may be beneficial for students because it may improve their mathematical competencies: modeling has the potential to promote mathematical literacy through which students see mathematics in the real world (Steen et al., 2007), to improve positive disposition towards mathematics (Lesh & Yoon, 2007), and to help students have a deeper understanding of mathematics (Lehrer & Schauble, 2007). Modeling provides an avenue for teachers to address pedagogically important aspects of teaching: differentiation, standards, student ownership, and equitable teaching practices. Mathematical modeling has the potential to offer benefits for students and teachers, but there are few empirical studies documenting modeling at the elementary level.

Students and Modeling

Proponents of mathematical modeling have made arguments regarding the benefits of mathematical modeling because through modeling, students can develop mathematical literacy, develop positive dispositions towards mathematics, deepen their understanding of mathematics, and have opportunities to use choice in mathematics. The following sections describe how modeling may allow students access to such benefits.

Mathematical Literacy. The OECD (1999) states that,

Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded

judgments, and to engage in mathematics in ways that meet the needs of that individual's current and future life as a constructive, concerned and reflective citizen.

Developing mathematical skills and tools is one necessary component to developing mathematical literacy. This alone is not sufficient to making students mathematically literate (Heath, 1983). Students need to learn the function of these mathematical skills outside of the mathematics classroom. "It is not uncommon that someone familiar with mathematical tools fails to recognize its usefulness in a real life situation" (de Lange, 2003, p. 77). By including modeling at the elementary level, students may develop mathematical literacy by making connections between the real world and mathematics (Steen et al., 2007).

Researchers such as Niss and de Lange have each identified fundamental capabilities necessary to the development of mathematical literacy. Niss' mathematical competencies for mathematical literacy, defined as "the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role" (Niss, 2003, p. 7), include mathematical modeling. de Lange (2003) also includes modeling in his list of competencies for literacy, defining modeling as both the ability to apply existing models to situations as well as modeling from the modeler's perspective.

Mathematical literacy is hard to teach because it "involves insight as well as algorithms" (de Lange, 2003, p. 78). Due to the difficult nature of teaching mathematical literacy, teachers need training and students need well-written tasks (Parker & Novak, 2012).

For most primary or lower secondary students it is difficult to motivate or learn mathematical concepts, methods, techniques, terminology, and results and to engage in mathematical activity, unless clear reference is being established to the use and relevance of mathematics to extra-mathematical contexts and situations, which are often also responsible for creating meaning and sense-making with regard to the mathematical entities at issue. (Niss et al. 2007, p. 5).

It is argued that students' motivation is increased by seeing the relevance of the mathematics they study to other subjects and to their world outside of the classroom (OECD, 2012).

Developing a Positive Disposition Towards Mathematics. The coupling of mathematical skills with the ability to apply those skills makes a mathematically literate student. de Lange (2003) further argues that students also need to be confident in their abilities. Modeling may not only be a skill to developing mathematical literacy, but potentially also encourages positive dispositions (Bonotto, 2007; Lesh & Yoon, 2007) as modelers learn to see mathematics as a tool to solve relevant problems. As argued above, real-world scenarios are incredibly important for allowing students to see the applications of mathematics and helping them to develop mathematical literacy. The context for modeling problems for elementary students ought to be more specific than just “real world,” but ought to be issues important to elementary students; topics they understand and care about in their everyday lives (Greer, Verschaffel, & Mukhopadhyay, 2007). To further support this notion, Zbiek and Conner (2006) argue that students are motivated by working with authentic situations for two reasons. Some students find excitement in exploring contexts that matter – their excitement has little to do with mathematics. On the

other hand, some students are excited by association, making the connections between significant issues and mathematics (Zbiek & Conner, 2006).

A central part of mathematical modeling is problem posing, where students consider a complex, authentic situation and ask a mathematical question. The step of asking a mathematical question can be beneficial for students for whom mathematics creates anxiety (Brown & Walter, 2005). Mathematical modeling has no “right or wrong” answers, so some students may find modeling less threatening than traditional mathematics problems. In addition, students who view themselves as being “bad at math” find success more often since modeling is different from typical tasks. “The positive attitude towards modeling examples evoked by the connection to reality and the unusual success of weaker students allows an affective access to mathematics and, from a long-term perspective, may positively improve the acquirement of mathematical competencies” (Kaiser & Maass, 2007, p. 104).

Deepening Understanding of Mathematics. Connecting mathematics to the real world creates many sense-making opportunities for students, which in turn can help to develop a deeper understanding of the mathematics at hand. This process of sense-making has the potential to produce powerful thinkers of mathematics (Schoenfeld, 2013). By being able to do mathematics in the context of something they care about and see mathematics as something useful, students’ overall understanding of mathematics may improve. Zbiek and Conner (2006) argue that through mathematical modeling, students may be motivated to study mathematics in general because they realize they need more mathematics in order to fully understand complex real-world phenomena.

Modeling allows students to deepen their understanding of mathematics through the application of and connections between mathematical ideas (Lehrer & Schauble, 2007). Therefore, in addition to students potentially seeing mathematics as more useful, students may become better mathematical thinkers as well (Bonotto, 2007). Modeling can encourage students to integrate and connect multiple mathematical topics to make sense of a larger problem. Modeling also acts as a gateway to other mathematical practices; when engaged in modeling, students make sense of complex problems, decontextualize and contextualize situations, and communicate their reasoning to others (NGA, 2010).

Another topic important in the process of developing a positive disposition towards mathematics lies in choice, which can be addressed by choosing the proper tools for the job. Choosing tools is an important step in modeling, a step that is not easy for students, but very important to their learning of mathematics (Muller & Burkhardt, 2007). When students choose tools, they must be making connections within mathematics and as a result, “the iterative self-correcting cycle of asking questions, using tools, producing answers, and then asking new questions helps *the cognitive connections* required to understand mathematics as a discipline” (Muller & Burkhardt, 2007, p. 269). Even when students are not modeling, but using models developed by others, they tend to have more appreciation for the models they are given when they have developed models themselves (Lehrer & Schauble, 2007).

Allowing Creativity and Choice. If students are able to use creativity and are presented with choice in mathematics, they will likely learn that mathematics is not a static subject good only for getting single number solutions. Arikan and Unal (2014) note

that the CCSSM highlights making choices as an important part of modeling (Arikan & Unal, 2014) stating, “Even such simple models involve making choices. It is up to us whether to model a coin as a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes.”

While there is little research on creativity and choice in mathematical modeling, researchers have considered it in the realm of problem posing. Researchers have found that problem posing develops creative thinking skills (Mestre, 2002). Creativity in mathematics generally comes with the fluency, flexibility, and originality that problem solving offers (Singer, Ellerton, & Cai, 2013). When students are given choice in problem solving activities, students of varying abilities can challenge themselves (Whitin, 2004).

Access to Elementary Level Mathematics Through Modeling

Mathematical modeling is included in the CCSSM for elementary grades, but often is described in terms of other types of modeling as illustrated in the section above. But that does not mean that modeling from the modeler’s perspective should not be included in elementary grades. This section will give proposed reasons to support why mathematical modeling ought to be taught in the K-8 grades.

As discussed earlier, the CCSSM document is clear that modeling from the modeler’s perspective should be taught at the high school level, but modeling at the elementary grades is often interpreted instead to mean using models of mathematics or word problems. Since mathematical modeling and models of mathematics are very different, one interpretation of the term does not necessarily prepare a student for the

other. Though students may have learned mathematical skills needed in modeling, modeling itself will not come naturally since students do not automatically transfer knowledge (Darling-Hammond & Austin, 2014). “There is no reason to expect that a student who has spent years doing problems in order to arrive at correct answers will suddenly view mathematics as an empowering tool that helps one make sense of the world” (Carlson et al., 2016, p. 122). For students to be proficient at mathematical modeling, students must be explicitly taught and given the opportunity to practice (Niss et al., 2007). While this can certainly be an attainable goal in the education of a high school student, teachers, researchers, and policy writers cannot expect that the elementary modeling practice (as many understand it) will prepare students to arrive to high school ready to model the world around them.

By incorporating modeling early in students’ education, students at an early age will ask questions about their world in their mathematics classes, see mathematics as useful, maintain a positive disposition towards mathematics and “from a long-term perspective, may positively improve the acquirement of mathematical competencies” (Kaiser & Maass, 2007, p. 140). Students tend to lose interest in mathematics in middle school (Moskal & Skokan, 2011). Moskal and Skokan (2011) argue this is due to a number of factors. The first factor is that students lack interest and find the subject unrelatable – simply learning elements of grammar in the language of mathematics. Second, mathematics teachers at this level may teach highly procedural mathematics, forgoing authentic applications and modeling. This teaching style contributes to students’ perception that mathematics is disconnected from their everyday life. “The intense

disapproval reaction of students with scheme and formalism oriented mathematical beliefs ... demonstrate how relevant it is to tackle applications and modelling examples as an integral part of mathematics teaching, starting at primary level” (Kaiser & Maass, 2007, p. 107).

Teaching and Modeling

As described in the previous section, including modeling in the elementary classroom may have benefits for students in terms of content and disposition. Teachers may find that there are additional reasons to include modeling as it allows them to address differentiation and equitable teaching practices.

Differentiation

Differentiation is known to improve students’ attitudes and performance (Tomlinson, 2005). Mathematical modeling encourages differentiation by allowing for various entry points, multiple solution methods, decision making, and revising of one’s work (Confrey, 2007). Modeling fosters multiple entry points because inherently “each modeler brings to the situation a unique set of knowledge, intuitions, and conceptions about mathematics and the real world” (Zbiek & Conner, 2006, p. 93). When determining mathematical tools, students generally have authentic choice because a true mathematical modeling task does not have one correct answer or solution method. The mathematics students use will depend on their knowledge and on how they believe the task relates to the mathematical tool (Zbiek & Conner, 2006). When students are engaged in the practice of reflection, explanation to classmates, and choice in groups, greater freedom

not only allows but encourages students to find and fix errors and develop more sophisticated approaches to problem solving (Moore, Doerr, Glancy, & Ntow, 2015). By viewing mathematics broadly, a wider range of abilities and understandings will be emphasized and more students will find success (Lesh & Yoon, 2007).

Equitable Teaching Practices

Research on reform approaches to teaching mathematics has found that some reform promote equity and achievement (Boaler, 2002). Boaler (2002) argues that equity is promoted not just through curriculum, but also through the teaching practices necessary to teaching reform curricula. When students are encouraged to ask questions about their world and community, students may ask questions about conflicts and problems that exist in their community. Through these investigations, students may realize that mathematics may help to alleviate struggles. In this way, modeling can lead to social issues and critical analysis (Greer, 2007).

Transfer

Just as students cannot automatically transfer their knowledge of mathematical tools and apply them to real-life events, teachers cannot automatically transfer their knowledge of teaching to teaching mathematical modeling.

Teachers do not become able to orchestrate environments, situations, and activities for applications and modelling as an automatic result of having been trained as mathematicians or mathematics teachers in traditional ways that focus entirely on purely mathematical subject matter...they need opportunities to develop that capacity (Niss et al., 2007, p. 7).

There is evidence that teachers new to teaching with authentic modeling tasks struggle to introduce them into their classrooms (Alsina, 2007).

Experience and Confidence

Most pre-service teacher education programs do not teach modeling and instead focus on realistic applications. As a result, teachers early in their careers typically have little to no experience and/or confidence in implementation of modeling tasks in the classroom. For teachers to implement mathematical modeling in the classroom, they first need to feel confident and qualified for the task. Biembengut and Hein (2010) found that through coursework that focuses specifically on mathematical modeling, teachers will gain experience and confidence and thus feel much more inclined to include mathematical modeling in their curriculum. A secondary result of doing mathematical modeling themselves in their training, rather than just receiving materials to digest on their own was that teachers were “more attentive to students’ difficulties” (pg. 486).

Writing and Finding Modeling Tasks

Teaching mathematical modeling is difficult because not only does a teacher need authentic real-world tasks, but the context around the task as well as the mathematics involved needs to be accessible and interesting to students. Finding or writing good tasks is hard, and there is not a one-size-fits-all repository of tasks. Modeling is not a type of problem or collection of problems; it is a process and should be taught as such (Swetz, 1991).

Some researchers and educators, such as Kang and Noh (2012), suggest that teachers take textbook problems and write them as modeling questions. The majority of textbook “modeling problems” consist of word problems; they may include a few application problems, but rarely contain any actual modeling questions. Typical mathematics curricula focus on students learning a particular skill and will present problems to encourage students to solve them in a certain way (Zbiek & Conner, 2006). Kang and Noh (2012) suggest that teachers consider the real-world nature of the textbook problem and consider what is inauthentic about the task. A teacher can open up the task by asking a question about the content that is likely to be asked in real life (Kang & Noh 2012). Writing a modeling task is more than just opening a problem however, and choosing or creating modeling tasks is an important skill for teachers to possess. This skill includes being able to anticipate how students might react to a task, what models they might develop, activities to develop students’ modeling abilities, and how to help students critically assess their models (Doerr, 2007).

Posing Mathematical Questions

Once teachers have a task to present to their students, the students must first pose a mathematical question. There is little research on problem posing in mathematical modeling, though research does exist on problem posing in the context of problem solving. Researchers consider problem posing to be an important part of problem solving (Silver & Cai, 2005). Assumptions are critical in some tasks (Zbiek & Conner, 2006). When problem posing, the assumptions that one makes has implications for the solution and as a result, having clear assumptions is an important step to the modeling process.

Though difficult, problem posing has a positive impact on students' problem solving skills, their attitude towards mathematics, and maintaining creativity (Arikan & Unal, 2014; Silver & Cai, 1996, 2005). Studies have shown that students are quite capable of learning how to pose mathematical questions and that this may be a skill that is good to start in young grades as it may become more difficult as students get older (Arikan & Unal, 2014).

Non-traditional Teaching Techniques

Teaching modeling requires teachers to use non-traditional teaching techniques, which may be new for many teachers. This may introduce pedagogical practices that teachers are not prepared to assume. Doerr (2007) lists several pedagogical techniques that teachers will need to implement modeling:

- to be able to listen for anticipated ambiguities;
- to offer useful representations of student ideas;
- to hear unexpected approaches; and
- to support students in making connections to other representations (p. 77).

High level tasks such as modeling do not lead all students to answer questions in the same way (Smith & Stein, 2011). Teachers need to be prepared not just to teach the one correct way to do a problem, but rather need to be able to anticipate multiple ways to approach a problem and connect the different ideas.

de Oliveira and Barbosa (2011) conducted a case study of a secondary teacher who taught a modeling task in his classroom. They found three main “tensions” for the

teacher: “Deciding what to do next, students’ involvement, and students’ domination of the mathematical content” (de Oliveira & Barbosa, 2010, p. 513). All three tensions arose from unanticipated student reactions, suggestions, and (lack of) mathematical skills. Since students’ development in the modeling activity did not follow the same progress that the teacher had planned, the teacher was uncertain of how to lead students and what to do next. When students responded with unanticipated ideas, though the teacher recognized that different ideas occur in modeling tasks, he was not sure how to move forward. The teacher was also unsure of how to teach skills when realizing that students did not have the mathematical skill he expected of them.

Mathematical Beliefs

Beliefs about mathematics may affect both teachers’ and students’ reactions to mathematical modeling. Kaiser and Maass (2007) found that some teachers and students have *static* views of mathematics, meaning they believe mathematics is a “formal science or a collection of rules and formulas” (p. 100). Others have *dynamic* beliefs about mathematics, believing that mathematics is a process of problem solving and that mathematics is relevant to life. The researchers’ study suggests that engaging in modeling will increase dynamic beliefs about mathematics for both students and teachers alike. Strong static beliefs, however, can impede the modeling process; strong “mathematical beliefs about mathematics and mathematics teaching control the pedagogical behavior of teachers” (Kaiser & Maass, 2007, p. 105). Teachers with static beliefs about mathematics will struggle with the pedagogical demands of modeling. Likewise, the researchers found that students with strong static beliefs did not believe that modeling was mathematics

because they were not doing calculations. “Students’ beliefs might even prevent a broad implementation of realistic tasks in everyday mathematics teaching” (Kaiser & Maass, 2007, p. 104).

Strategies for Teachers of Mathematical Modeling

There are teaching practices that may help teachers teach mathematical modeling, though the teaching practices are not specific to teaching mathematical modeling. Because teaching mathematical modeling may require non-traditional teaching techniques identified by Doerr (2007), strategies for High Cognitive Demand Tasks and for mathematical discourse may apply to teaching mathematical modeling. In addition, there is a framework for teaching that is specifically oriented towards teaching modeling, the Teaching Framework for Modeling in the K-5 Setting (M. A. Carlson et al., 2016).

Strategies for High Cognitive Demand Tasks

Mathematical tasks can be examined by the cognitive demand required of students. Stein and Smith (1998) categorize tasks into four levels: *memorization* and *procedures without connections* make up lower-level demands while *procedures with connections* and *doing mathematics* comprise the high-level demands. High cognitive demand tasks involve making connections, analyzing information, and drawing conclusions (Smith & Stein, 1998). Modeling tasks, as activities that require students to use complex, non-algorithmic thinking along with relevant knowledge and experiences, are tasks with high cognitive demand (Asempapa, 2015).

Engaging students in complex thinking tasks in conjunction with high level thinking requires teachers to start with a high cognitive demand task. “Selecting and setting up a high-level task well does not guarantee students’ engagement at a high level. Starting with a good task does, however, appear to be a necessary condition” (Smith & Stein, 1998, p. 344). Once a high cognitive demand task is selected, the demands of a task are frequently changed by other factors as a teacher sets up a task and implements the task (Stein, Smith, Henningsen, & Silver, 2009). Stein et al. (2009) describe these factors in their book *Implementing Standards-Based Mathematics Instruction*, to maintain high-level cognitive demands while implementing the task. Teachers preparing for mathematical modeling tasks may find the factors noted by Stein et al. (2009) to be beneficial to their enactment of any lesson.

Strategies for Mathematical Discourse

Part of the work of leading students to engage in modeling tasks is to guide discussion to orient students towards learning mathematics. The book *5 Practices for Orchestrating Productive Mathematics Discussion* by Smith and Stein (2011) provides advice and guidance for teachers and teacher educators for guiding student-centered mathematics classrooms. The authors encourage teachers to move classes from teacher-oriented learning to students sharing mathematical understanding and thinking. Many teachers have concerns about moving to this style of teaching because they give up much of the control of the classroom. The premise of this book is to help teachers use students’ responses to

advance the mathematical understanding of the class as a whole by providing teachers with some control over what is likely to happen in the discussion as well as more time to make instructional decisions by shifting some decision making to the planning phase (Smith & Stein, 2011, p. 7).

The authors suggest that teachers may work towards running a smooth, meaningful, and engaged classroom by practicing the following actions:

- *anticipating* likely student responses to challenging mathematical tasks;
- *monitoring* students' actual responses to the tasks;
- *selecting* particular students to present their mathematical work during the whole-class discussion;
- *sequencing* the student responses that will be displayed in a specific order; and
- *connecting* different students' responses and connecting the responses to key mathematical ideas (Smith & Stein, 2011, p. 8)

These practices relate to modeling by guiding preparation prior to implementing a task the anticipating practice. Monitoring, selecting, sequencing, and connecting could help teachers while students are working to lead discussions around students' ideas.

Teaching Framework for Modeling in the K-5 Setting

Carlson et. al. (2016) present a theoretical *Teaching Framework for Modeling in the K-5 Setting* (referred to as the Teaching Framework for Modeling) for teachers to follow while engaging their students in modeling (Figure 7). Like Smith and Stein, the authors suggest that teachers spend time selecting an appropriate task and anticipating students' thinking and choices. When presenting the task to students, teachers should

guide students through the modeler's steps of posing mathematical questions, building mathematical solutions, and validating conclusions. Teachers should structure students' work by first organizing the lessons' work with students, then monitoring students as they work, and finally regrouping students to share their ideas before engaging in the cycle again. Once the teacher has engaged students in the organize-monitor-regroup cycle enough times that the students have fully completed the modeler's cycle of posing questions, building solutions, and validating conclusions, the modeling task may be complete. The teacher can always revisit the task later once students have gained further mathematical tools or have learned another perspective on the task that may change how students approach the problem.

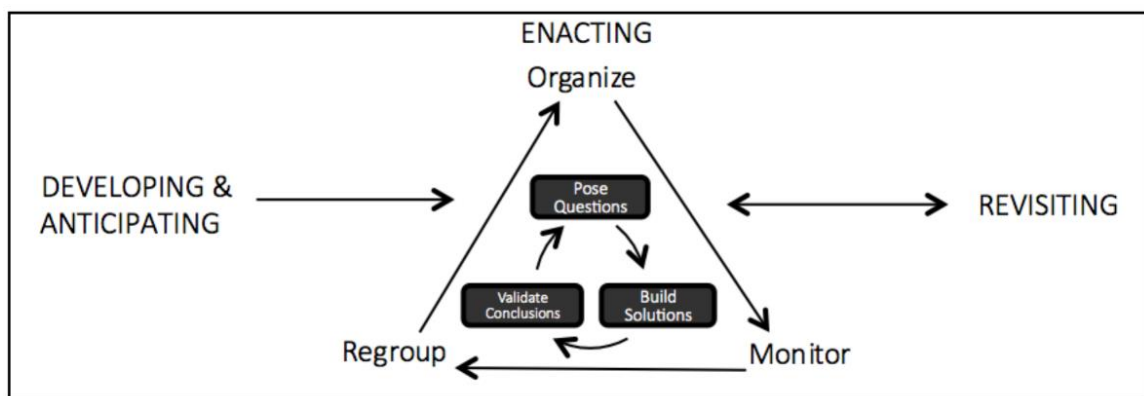


Figure 7. Illustration of the Teaching Framework for Modeling (Carlson et al., 2016).

Developing and Anticipating. In the Developing and Anticipating stage, the teacher prepares a task that is accessible to students and that will draw on their experiences and mathematical tools. In this stage, the teacher also anticipates what students might do. Because a modeling task is shaped by students' contribution in the beginning, middle, and end of a task, preparing for student involvement will allow the

teacher to develop responses ahead of the lesson. The Teaching Framework for Modeling suggests the following five guiding questions at this phase:

1. What mathematical content tools have students developed?
2. What mathematical process tools could students access and use as they engage in mathematical modeling?
3. What settings are interesting and accessible to all students?
4. What might students do as they engage in the modeling process?
5. What mathematical understandings and insights might emerge as students engage in the modeling process? (Carlson et al., 2016)

Enacting. During the Enacting stage, there are two processes occurring. First, the students are mathematical modelers traversing through the modeler's cycle. The cycle has been simplified to three steps: *Pose Questions*, where the modeler asks a mathematical question from the situation given by making value judgments and assumptions; *Build Solutions*, where the modeler determines variables and appropriate mathematical tools to solve the mathematical question; and *Validate Conclusions*, where the modeler assesses the solution within the context of the real-world situation. While the students work through the modeler's cycle, the teacher progresses through the proposed teaching cycle: *Organize*, where the teacher organizes the students as they work through the modeling cycle, orienting them toward steps or ideas to consider; *Monitor*, where the teacher observes students' work and considers how to focus attention on one group's method or how to connect students' ideas; and *Regroup*, where the teacher brings the class together to share ideas. This teaching cycle will occur several times throughout the modeling cycle

as the teacher sees appropriate. The Teaching Framework for Modeling suggests that teachers consider the following questions (shown in Table 2) as students work through the modeling cycle.

Table 2. Questions for Teachers from the Teaching Framework for Modeling in the K-5 Setting (Carlson et al., 2016).

Pose Question	Build Solutions	Validate Conclusions
<i>What aspects of the task are subjective and need to be defined within the community?</i>	<i>What are we learning about the situation?</i>	<i>What assumptions are students making?</i>
<i>What contextual factors inherent to the situation are fixed and what factors vary?</i>	<i>What can the whole class agree on and how do our approaches to the problem differ?</i>	<i>What are the strengths of their work? What are the limitations?</i>
<i>How does adjusting the constraints and variables alter the mathematical demands of the task?</i>	<i>How will we move forward?</i>	<i>What questions can I ask to help students see new possibilities for their models?</i>

Revisiting. Revisiting is a stage that allows teachers to take advantage of the shared knowledge and student interest created through engaging in the modeling cycle. The Teaching Framework for Modeling suggests that teachers consider the following questions when deciding whether to revisit a task:

1. How might students' emerging mathematical knowledge lead them to revise existing solutions?
2. What new, real-world understandings of the context might allow reentry into the task from a new perspective? (Carlson et al., 2016)

This Teaching Framework for Modeling is a theoretical framework based on literature on modeling and teaching practice. The authors give guiding questions for teachers to

consider while planning and while teaching. As of yet, there is no evidence whether this Teaching Framework for Modeling reflects teachers' work in modeling or if the guiding questions help teachers' planning and implementation as intended.

Conclusions

This review of literature investigated the many ways that the term *modeling* is used by elementary teachers, and how modeling from a modeler's perspective compares with problem solving and activity-based lessons. It describes possible benefits for students and teachers as described by literature to support why modeling should be included at the elementary level. There are still many questions surrounding "how to teach mathematical modeling?" as well as "how do we teach mathematics for mathematical literacy?" (de Lange, 2003, p. 88). These questions need to address both mathematical content as well as pedagogical moves. In this study, the theoretical Teaching Framework for Modeling by Carlson et al. (2016) is used to build knowledge toward the teaching practices necessary in implementing mathematical modeling.

CHAPTER THREE

METHODOLOGY

Introduction

The purpose of this research is to investigate *how* teachers implement mathematical modeling and *what* decisions they make in the midst of teaching modeling and interacting with their students.

Teachers may incorporate mathematical modeling into their classrooms for a variety of reasons. These include addressing the real-world relevance of mathematics, providing motivation, and addressing mathematics content standards. This process of incorporating modeling into the classroom is a difficult task and requires additional support and professional development to be provided for teachers. Due to the current lack of resources available to elementary teachers in the category of mathematical modeling, teachers and students would benefit greatly from research on successful methods of implementation in the classroom. This case study of teachers investigates the following research questions:

1. What mathematical decisions and choices do teachers make in the process of implementing mathematical modeling?
2. How do students' mathematical contributions and behaviors influence the implementation of a modeling lesson?

Background and Setting

This study was situated within a 3-year research and professional development project on mathematical modeling for elementary teachers, called IMMERSION. IMMERSION was an NSF-funded project designed the purpose to provide professional development on mathematical modeling for elementary teachers and to research the effects of the professional development on teachers' classroom practice. There were three sites, each consisting of a partnership between a school district and a university; one site on the West Coast, one on the East Coast, and one in the Rocky Mountain West of the USA. This section will describe the research project, the professional development design, and my involvement in the research and professional development.

The IMMERSION Professional Development Process

The IMMERSION project was a three-year project, with a new cohort of teachers each year. The first cohort began in summer 2015 and the last cohort in 2017. Approximately 24 teachers from each site participate in professional development, beginning with a one-week summer workshop and continuing into the fall semester, concluding with a symposium in November (Table 3).

Table 3. Timeline of IMMERSION's Professional Development.

Summer workshop	Fall semester	November
Professional Development	Professional Development	Symposium
Professional Development Course	Teacher study group (TSG) meets 7 times; teachers implement modeling in classroom.	Teachers share projects. PD is officially finished.

Summer Professional Development. The teachers met for an intensive week of professional development in the summer, during which they took on two roles: first, as mathematical modelers learning how to model with mathematics; and second, as teachers learning to implement modeling tasks. The summer professional development focused on the following topics displayed in Table 4. These roles and topics are discussed in greater detail below.

Table 4. Summer Professional Development Topics.

	Monday	Tuesday	Wednesday	Thursday	Friday
Emphasis of Modeling Task	Open-endedness	Mathematical Problem Posing	Representing Qualitative Situations	Revising	Teacher Study Groups
Emphasis for Teaching Modeling	Defining Mathematical Modeling	Developing Tasks	Enacting	Revisiting	Differentiation

Emphasis of Modeling Task. In the mornings, the teachers were presented with a mathematical modeling task that they engaged in as mathematical modelers. Each modeling task was developed to address features of modeling: open-endedness, mathematical problem posing, using mathematics in qualitative situations, and revision. The professional development facilitators led conversations during or after the modeling task addressing these features.

The first day's task provided an opportunity to discuss open-endedness in mathematical modeling in the beginning, middle, and end of the modeling process. In the beginning of a modeling task, open-endedness means there are several ways to pose a mathematical question which allows students to make decisions about the mathematical

question. Open-endedness in the middle of the modeling cycle allows choices for students regarding solution methods, which ultimately, in the final portion of the process, allows for students to produce reasonable, yet different solutions.

Problem posing was another feature of modeling emphasized through a modeling task and activities. Teachers considered problematic situations and how asking different questions about the situation could orient the solver towards a solution method. Activities engaged teachers in posing problems that asked well-defined mathematical questions about a situation. The teachers then considered the effects of assumptions about the situation on the problems posed and the mathematical tools used.

To investigate using mathematics to address qualitative situations, teachers were given a task to rank several items based on different perspectives. In this activity, teachers needed to quantify a situation that was inherently qualitative. In discussion, teachers addressed how their choices led to different solutions.

Revision can occur throughout the modeling process when the modeler feels the need to adjust the mathematical problem to better suit the real-world situation. A goal of the professional development was for teachers to realize that revision is not a teacher-directed action but should occur as the modeler recognizes a need to improve the mathematical model. Teachers also discussed the differences between correction and revision; teachers concluded that one corrects because something is wrong but revises to make something better.

Emphasis for Teaching Modeling. In the afternoons, teachers learned about their roles in preparing for and implementing modeling. They spent time thinking about anticipating and developing modeling tasks. Teachers considered problems in their curriculum and discussed how to “open them up.” They also were introduced to mathematical modeling from the modeler’s perspective (as described in Chapter 2). The teachers considered how open-endedness related to mathematical modeling as well as what information to provide to students and who gathers additional needed information. For instance, first-grade students do not necessarily have the skill set to use the internet as a resource, but instead could ask their teacher questions about information they want to know.

The developing, enacting, and revisiting modeling stages of teaching modeling were addressed on separate days, as teachers learned how to best prepare their students to engage in the modeling cycle. Teachers met in groups and prepared situations they thought students would find interesting but that also addressed appropriate mathematics. The teachers then began to work through the tasks while anticipating student reactions and developed lesson plans. They also considered how the task led to differentiation for students with different learning needs.

Teacher Study Groups. In the fall, the teachers worked in Teacher Study Groups (TSG), consisting of three to six teachers and one facilitator. The facilitators, who were university professors, graduate students, and teacher leaders in the district, were identified in the preceding school year and helped to develop the professional development and led the summer workshop. In the fall, the teachers finalized a modeling

task started in the summer and implemented the task with their students. The TSG met seven times throughout the fall; there were not separate goals for each meeting, but one shared goal to develop the modeling task and to discuss implementation of the task. During these meetings, teachers worked with each other discussing how to move forward with the modeling task and how to respond to student ideas, and considered student thinking elicited from the modeling task. In November, as a completion of teacher study groups, all groups made a presentation of their project and student work. The groups presented at a fall symposium and afterwards the facilitators led round-table discussions asking teachers about their understanding and experience in implementing mathematical modeling with their students. Teachers were encouraged to continue meeting and modeling after this point, but formal teacher study groups and professional development concluded with the symposium.

Involvement in IMMERSION Research

The IMMERSION project had its own research agenda with qualitative and quantitative measurements and methods of analysis. The research goal was to study the effect of targeted professional development in mathematical modeling on teacher practice. The quantitative portion of the research did not influence this case study research and therefore will not be addressed here. The qualitative component of the IMMERSION research agenda investigated the ways that modeling is exhibited in elementary grades classrooms and the challenging aspects of teaching modeling. Two instrumental case studies (Stake, 1995) were conducted at each site for the first two cohorts in the spring semester. Data collection consisted of pre-observation interviews,

interviews, post-observations interviews, post-observation memos, and photographs of student work.

The research study reported here was conducted during the second year of IMMERSION. While I was a graduate student working with, and partially funded by, the IMMERSION project. My role has been threefold: developing and implementing the summer professional development; facilitating a Teacher Study Group; and supporting grant activities as a graduate research assistant. My graduate advisor was a Co-PI on the IMMERSION grant. I helped develop the outline of the summer professional development and the daily activities. In addition, I led sessions during the summer professional development and provided support during other sessions. In the fall, I worked with a Teacher Study Group as their facilitator. The teachers in my cohort invited me into their classrooms where I observed them working with their students on mathematical modeling tasks. As a graduate research assistant helped to develop the observation and interview protocol used for the case studies as well as the quantitative measurements. In the spring semesters, I observed and interviewed five teachers for the IMMERSION case study. This exposure gave me additional experience and helped prepare me for the observations and interviews in this study.

Theoretical Framework and Analytical Lens

My experience as a mathematics student, a mathematics high school and college teacher, a mathematics education graduate student, and a mathematics teacher educator

has shaped my belief that learning occurs in social contexts. My epistemological stance is that of a social constructivist.

My research is guided with this underlying philosophy of social constructivism; the notion that knowledge is actively constructed by the learner in a social setting (Vygotsky, 1978; Smith & Stein, 2011). Teachers provide materials, opportunities, and environments to best facilitate learning, but it is the students who create their own knowledge (Schneider & Stern, 2010). Through the process of learning in a social setting, learners take their prior knowledge and experiences and integrate new information, resources, motivation, and ideas. This philosophical stance defines mathematical modeling in the classroom as a discursive activity that requires time and effort (Schneider & Stern, 2010; Zbiek & Conner, 2006). I believe that teachers build their understanding of modeling through their work and collaboration in teacher study groups as well as in their work of implementing modeling with their students.

The theoretical framework guiding this study relies on prior research about three aspects of mathematical modeling; the *mathematics* in mathematical modeling, the *teaching* of mathematical modeling, and the mathematical modeling *cycle*. Each of the three lenses of analysis has a place within social constructivism.

Because mathematics is fundamental to the mathematical modeling process, I must use *mathematics* as a lens of analysis. This lens examines the mathematics specific to mathematical modeling and influenced data collection and data analysis by focusing both topics on mathematics. In data collection, I asked interview questions about the mathematics that teachers expected students to use and the mathematics that came out of

the task upon reflection of the lesson. I collected teachers' lesson plans, photographs of teachers' board work exhibiting mathematical ideas, and students' mathematical work. I video recorded and audio recorded the class to listen to the mathematical conversations between students and teachers. In analysis, I identified mathematical ideas in classroom conversations as well as curricular standards addressed in those mathematical ideas. I analyzed who introduced mathematical ideas and how the mathematics contributed to the mathematical modeling task. This lens is informed by social constructivism by considering not just what mathematical ideas were considered, but *who* introduced the ideas and the relationship of the students and teacher with the mathematical ideas.

The second lens of analysis is informed by teaching techniques to teach mathematical modeling or other high level mathematical tasks (Doerr, 2007; Smith & Stein, 2011). These techniques, along with others, include strategies such as listening for anticipated questions, listening for unexpected approaches, anticipating multiple approaches, and connecting different ideas. The Teaching Framework for Modeling is one framework that is specifically designed to address the teachers' work in teaching mathematical modeling. This body of research offers suggestions for how teachers may make decisions and work with their students while teaching mathematical modeling.

Because the process of teaching modeling relies on an interaction between teacher and students, the second lens of analysis examines mathematical modeling through *teaching*, including teachers' interactions with students' mathematical ideas. The interactions are valuable because I am a social constructivist; the students and teachers are working together throughout the modeling process for the students to create their

understanding of the modeling task. The teachers may guide their students experience, but it is the students who are the modelers in a modeling task. This perspective influenced data collection through specific interview questions pertaining to teachers' expectations of their students and their varied approaches. Through this lens I examined the teachers' intentions on how to guide students through the modeling process. I also collected audio recordings of teacher and students' conversations to enable analysis of interactions between the teacher and students during group work. During analysis, I considered the teachers' actions during each interaction with a mathematical idea. First, I analyzed their actions generally, then I analyzed their actions from the perspective of the Teaching Framework for Modeling.

The third lens of analysis is through the mathematical modeling cycle. Chapter Two defines mathematical modeling from the modelers' perspective; in order to model with mathematics, one considers an authentic task and asks a mathematical question to help understand the situation. The modeler then uses their mathematical tools to solve the question and finally interprets the solution in context of the real world. Recognizing the importance of the steps of a modeling cycle impacted the interview questions in this study. Before the teachers taught the lesson, I asked them what made their lesson a mathematical modeling lesson. After the task concluded, I asked the teachers how they believed that the implemented task was a modeling task. I also asked the teachers about specific steps of the modeling task, such as the problem posing step. To analyze the data, I compared how the implemented task related to the description in the literature of mathematical modeling tasks.

Research Design

I wished to understand *how* teachers implement mathematical modeling and *how* they make decisions in the midst of teaching modeling and interacting with their students. I used a qualitative approach in this study because qualitative work intends to give meaning and understanding to a topic (Merriam, 2009). Because little is known about teaching mathematical modeling to elementary students, description and documentation of such occurrences will contribute to understanding of mathematical modeling at the elementary level. An appropriate method of answering these “how and why” questions is through a case study (Yin, 2005). Case studies produce “richly descriptive” work (Merriam, 2009, p. 39). My research thoroughly describes the mathematics involved in the four cases of teaching mathematical modeling; the mathematical ideas introduced by teachers and students, the way teachers and students interact with mathematical ideas, and how the mathematical work integrates into a mathematical modeling task.

A particular type of case study is an *instrumental case study*, where the case “is examined mainly to provide insight into an issue ... The case is of secondary interest...it facilitates our understanding of something else” (Stake, 1995, p. 437). This study places primary interest on the teaching of mathematical modeling and secondary interest on the teachers themselves. The participants in this study were purposefully selected because they were likely to provide an opportunity to observe and study mathematical modeling in an elementary classroom.

Preliminary Data

Data for this research study were collected in the 2016-2017 school year, in the second year of the IMMERSION project. I was involved in IMMERSION during the first year as a teacher study group facilitator and collected preliminary data that allowed me to develop research questions and methodology for the current study. The following section describes how observations and interviews from the 2015-2016 school year impacted this research study.

Fall 2015 Teacher Study Group Observations. In the Fall of 2015, I visited the classrooms of the three teachers in the Teacher Study Group I facilitated. I observed their lessons following an observation protocol and afterward reflected on the mathematics of the task and how the task satisfied a modeling cycle. This process allowed me to consider the steps of modeling (as identified in the Teaching Framework for Modeling) in an elementary classroom, understand the role of mathematics in mathematical modeling, and identify reasons that teaching modeling is difficult in ways that go beyond understanding of modeling. Along with the researchers conducting the IMMERSION case study, I adapted the observation protocol in order to address the project's research questions

My experience in Fall 2015 led me to question if the modeling cycle described for secondary students and applied mathematicians also describes mathematical modeling for elementary students. Since mathematical modeling is an important process for applied mathematicians, applied statisticians, and scientists, many modeling cycles used by researchers are developed and oriented toward advanced mathematicians with complex understandings of mathematics (e.g., Doerr & English, 2003; NGA, 2010; Swetz, 1991;

Tran & Dougherty, 2014; Verschaffel & De Corte, 1997). It is reasonable to ask if the modeling cycle also applies to elementary students and if all the steps of the modeling process are equally pertinent. Posing a mathematical question is a critical step of mathematical modeling (Bliss et al., 2014); but many elementary students may not regularly pose mathematical questions in their mathematics classes. Elementary students are more likely to generate mathematical solutions and verify their work.

I also came to question how mathematics interacts with modeling. One teacher I observed during my preliminary study expertly demonstrated the “organize, monitor, regroup” cycle of the Teaching Framework for Modeling. She was skilled at preparing students to work in groups to investigate a topic and then grouping students back together to share ideas. However, the student cycle of “pose, build, interpret” was missing because of the teacher’s view of the role of mathematics in the modeling task. She believed that because students were doing mathematics in their explorations they were also modeling. While the activity may have been valuable, the students were working on a problem that did not meet the criteria of modeling because they never posed a mathematical question or used mathematics to understand their world.

Observing this discordance helped me understand the role of mathematics in two ways. Firstly, a teacher’s understanding of the role of mathematics in mathematical modeling influences their ability to implement modeling in their classroom. A teacher may demonstrate important teaching practices such as classroom discourse and valuing student ideas but may not reach the goal of modeling because the students did not engage in the full process of mathematical modeling. Secondly, mathematics can be used for

different purposes. Asking a mathematical question is different from asking a design question where students use mathematics to answer it. Teachers implementing design tasks may believe they are engaging in modeling because of the students' level of engagement, their use of many grade-appropriate skills and vocabulary, and their use of mathematics to solve problems. While there is value in this work, it is different from the goals of mathematical modeling.

Finally, the culture and teaching practices of a mathematics class appear to influence the implementation of modeling tasks. I witnessed difficulties in classes where the teacher intended to model but the activity fell short, not because of how teachers viewed modeling. It may have been due to the teacher's change in teaching practices; one teacher did not regularly have students discuss their mathematical thinking or work in groups in mathematics class. Some teachers had their students work in groups and then come together as a class to share their products and ideas. One of the teachers I worked with identified her struggle to be the "regroup" step of modeling. Her struggle may be attributed to her inexperience in leading mathematical discussions with her students. She tried to get students to interact with each other through conversations, to listen to each other's ideas, and to give each student an opportunity to participate. Despite her efforts, the "regroup" sessions often seemed to go much longer than the students or the teacher could tolerate. Students would get off task or became impatient with listening to others' ideas, wanting instead to. The culture and teaching practices of a mathematics class appear to influence the implementation of modeling tasks.

Spring 2016 IMMERSION Data Collection. In the Spring of 2016, while conducting case studies as a part of the IMMERSION research team, I interviewed five teachers before and after their modeling lessons, and also observed their lessons using protocols developed for the IMMERSION research agenda. From this experience, I was able to consider how the interview questions and observation protocol elicited data that addressed my research questions. A few questions gave information that did not inform my research questions, such as asking teachers about materials. Other questions were necessary to elicit more information, particularly regarding the modeling cycle and how students use mathematics to investigate the situation. I then adapted that IMMERSION interview and observation protocol for the purposes of my research study. Data collection occurred during the second year of IMMERSION, from the summer of 2016 through the spring of 2017. Teachers were selected to participate using the procedures described in the next section.

Participant Selection

All teachers in this study had completed the full course of IMMERSION professional development in the first cohort during the 2015-2016 school year. I intended to select teachers who understood the full process of mathematical modeling and who wanted to continue to incorporate modeling into their classrooms. Therefore, I purposefully selected teachers who demonstrated strong understanding of modeling. This evidence of understanding of modeling is based on the teachers' contributions and participation in the Fall 2015 teacher study groups (TSG) and the modeling symposium. In these sessions, participants discussed:

- How to implement a selected modeling activity with their classes,
- How to further develop the activity,
- How to incorporate their students' ideas,
- How to understand their students' mathematical ideas and thinking, and
- How the work was (or was not) mathematical modeling.

Teachers then indicated at the Fall 2015 symposium if they wanted to continue having their students model with mathematics. Based on reports from each Teacher Study Group facilitator, I was able to identify which teachers indicated that they embraced modeling and understood the role of mathematics in mathematical modeling. While the IMMERSION research team had not observed all the teachers' mathematical modeling tasks implemented in their classrooms nor had we conducted individual in-depth interviews, the research team identified a group of teachers in the 2015-2016 cohort who implemented modeling tasks, pushed their students to ask mathematical questions of real-world situations, and gave good descriptions of the process of mathematical modeling. This group of identified teachers were the most-likely elementary teachers to articulate their decision-making process and use of mathematics while teaching modeling.

This cohort of teachers all worked in the same school district but taught at different schools. Five of the cohort teachers made up the full team from one school that participated in IMMERSION. I invited all five of these teachers to participate in this research study because the participants taught in the same school culture, other IMMERSION participants would not feel singled out, and for managing logistics of data

collection. Of the five teachers invited, four teachers, Lindsay, Rebecca, Erica, and Amy, agreed to participate.

Data Collection

In this case study, I collected video and audio recordings of modeling tasks carried out in classrooms, conducted classroom observations, conducted interviews before and after each observation, and collected teacher lesson plans as well as student work samples. I used multiple sources of data and multiple methods of data collections to triangulate the data and support validity (Merriam, 2009). In what follows, I describe data collection as illustrated in the timeline displayed in Figure 8.

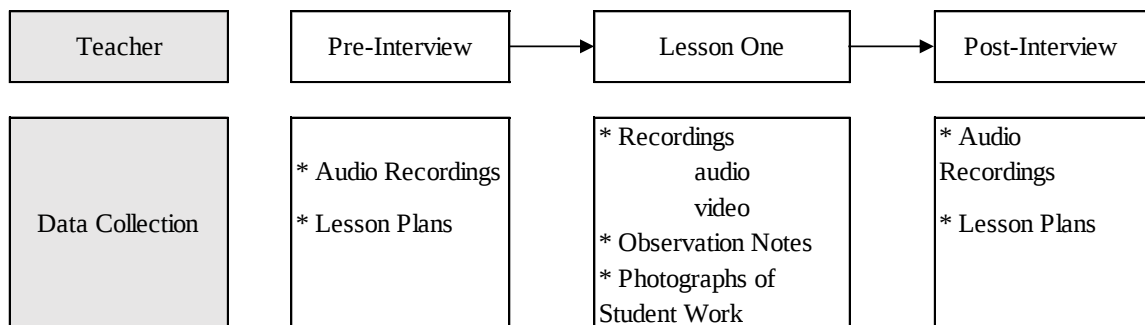


Figure 8. Timeline of Data Collection surrounding a lesson.

Interviews

Interviews are given in qualitative research for the purpose of obtaining information that cannot be obtained in other ways. For instance, interviews are necessary when the researcher wants to know specifically what a participant is thinking, what they understand, and how they view the world (Merriam, 2009). I conducted a series of two

interviews on either side of each observation of a modeling lesson. One interview occurred before the observation and was based on a set of questions designed to understand what the teacher planned for the lesson and what mathematics were anticipated. These interviews were often conducted the day before or the day of the lesson. The second interview occurred after each lesson, either the day of the lesson or the next day; interview questions asked the teacher how the implemented lesson compared to her planned lessons and addressed the mathematics of the lesson. Several interviews combined a post-interview from a previous lesson with a pre-interview from an upcoming lesson. For both types of interviews, a semi-structured interview protocol was followed (see Appendix B). Semi-structured interviews are “a mix of more and less structured questions” where the majority “of the interview is guided by a list of questions” (Merriam, 2009, p. 89). I asked the structured, pre-planned questions written in the interview protocol and incorporated additional, spontaneous questions as needed. The additional questions were always follow-up questions to teachers’ responses or addressed instances from the observation. Each interview was recorded with an audio recorder.

Observations

In contrast to interviews, which are designed to document what a teacher knows and thinks, classroom lesson observations in this study were used to document how the process of modeling unfolds in the classroom. Observations occur in their natural settings and are firsthand accounts of a phenomenon (Merriam, 2009). With classroom

observations, I directly witnessed the implementation of modeling lessons and observed teacher interactions with their students while modeling.

The teachers in the study designed their modeling tasks and invited me to watch the lessons they described as modeling lessons. The teachers taught modeling lessons as it fit into their teaching schedule. I observed each lesson the teachers invited me to observe. I recorded each lesson with a video-camera and a small microphone worn by the teachers. The video captured the whole-class discussion and some group activity. The video was unable to record all of the teachers' actions and conversations during group work, so to supplement the recording I used an audio recorder to record the conversations between teachers and individuals. During the lesson, I took notes on a computer following an observation protocol (See Table 5). I recorded observations on student and teacher roles, teacher and student questions and ideas, interactions between students, and student-teacher interactions. My focus was on the mathematical activity of the class and all interactions surrounding the mathematical activity. During whole-class discussion, I sat to the side and took notes. During group work, I sometimes walked around the classroom taking notes and photographing students work. Two of the teachers had student teachers and all teachers occasionally had teacher aides. Often, there were several adults in the classroom walking around and helping students. Because students were familiar with several adults in the classroom, my presence and activity was minimally disruptive to the students and their learning environment.

Table 5. Observation Protocol

Time	Lesson Design & Organization	Student Interaction & Classroom Culture	Other
	[roles, planning, management, student backgrounds, specific teacher questions, student ideas]	[student interactions, respect, student-teacher collaboration, climate, absence of interaction]	[interruption, schedule, incidents]

While taking observations using my observation protocol (Table 5), I intentionally did not interpret any portion of my observations but focused on objectively documenting the activity. I wrote clear descriptions of what the teacher said, her actions, and her conversations with students. After the observations, I summarized the lesson formally and informally. I formally summarized the lesson by completing an observation memo (Appendix A) within a day of the lesson. In the memo, I wrote an overview the lesson, how it differed from what the teacher described in the interview, and the mathematics of the lesson. Informally, I discussed the lessons with the IMMERSION research team and my advisor.

Data Collection Matrix

Table 6 illustrates how the collected data informed the two research questions. The primary data sources for analysis came from the observation notes and observation recordings. These data allowed me to analyze the mathematics introduced and used by both teachers and students. I made use of pre-interview data to understand the mathematics that teachers expected students to use. Post-interview data gave insight into some of the decisions made by teachers. Few teachers had formal lesson plans, so this source did not provide much data. In some cases, however, lesson plans did show

questions that some teachers had planned in advance and some mathematical ideas they expected students to struggle with. Student work samples illustrated different mathematical strategies used by students.

Table 6. Data Collection Matrix

Research Questions	Data Sources					
	Pre-interview	Observation Notes	Observation Recordings	Post-interview	Teacher Plans	Student Work
What mathematical decisions and choices do teachers make in the process of implementing mathematical modeling?	X	X	X	X	X	
How do students' mathematical contributions and behaviors influence the implementation of a modeling lesson?		X	X	X		X

Data Analysis

In this section, I describe the timeline of data collection and analysis and discuss the analysis. I analyzed the data from three different lenses; the mathematics used to investigate the task, the teachers' interactions with students' mathematical ideas, and ways in which the task was a modeling task.

Timeline of Analysis

Analysis was an ongoing process, beginning with the first modeling lesson in Fall 2016 and continuing concurrently with data collection (Maxwell, 2013). I asked the teachers to invite me to observe their classrooms when they taught mathematical modeling. They made the choice when to implement modeling tasks and what modeling tasks to implement. Two teachers taught an “introductory” modeling task before teaching a full modeling task. The other two teachers’ first tasks were intended to be complete mathematical modeling tasks, so I did not observe subsequent tasks.

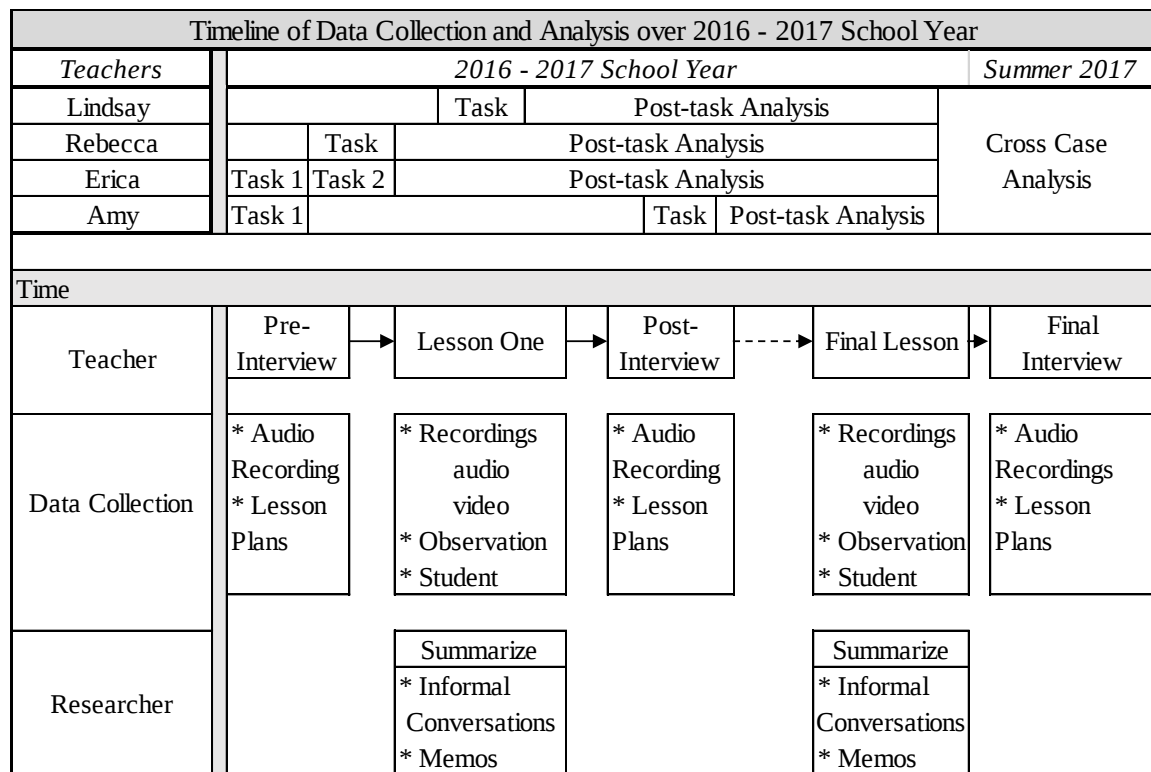


Figure 9. Timeline of Data Collection and Data Analysis.

Following completion of each task, I wrote a summary of the task. The summary described how the teacher created the modeling task, how the task was introduced narrative, generally how the class worked on the task, and the final result. I also wrote a for each teacher which described their experience in IMMERSION, their views on modeling, and their typical mathematics classes. I sent the summaries and narratives to each teacher and asked them to check for accuracy and any mischaracterizations.

Figure 9 illustrates the research timeline throughout the 2016-2017 academic school year. The upper section of the figure demonstrates when teachers taught their modeling tasks. Post-task analysis and cross-case analysis follow implementation of the modeling tasks. The figure also illustrates the timeline of data collection and analysis within a task. After each lesson was taught, I formally summarized the lesson using a post-observation memo protocol. The memo protocol segmented each summary into topics of primary interest: evidence of planned lesson, organization of the lesson, mathematics, student and teacher interactions, student interactions, and modeling cycle.

After each live lesson observation, I reviewed video recordings of the class. I added details to the observation notes, recording dialogue and details that I missed during the observation. I then listened to the audio recording of teachers' conversations during the group work, and. I added this dialogue to the observation notes. I used these supplemented observation notes as the primary data source for analysis.

I analyze the data in layers, using the three lenses of analysis described earlier; first through the lens of mathematics, and second through the lens of teachers' interactions with students' mathematical ideas. During these phases of analysis, I

considered the task from the modelers' perspective. Finally, the third lens focused on the components of a mathematical modeling task. The following sections describes this analysis process.

The Mathematics in Mathematical Modeling

To begin analysis through the lens of mathematics, I read my supplemented observation notes, marking each mathematical occurrence. A mathematical occurrence, adapted from Leatham, Peterson, Stockero, and Van Zoest (2015), is defined as actions, or a collection of actions, in which students or the teacher introduce or discuss a mathematical strategy or idea. I documented the time of the occurrence (either time of day or time on the video/audio recording), if the comment was made in a whole-class or group setting, and who made the comment(s). I also analyzed the actions that resulted because of the mathematical occurrence; if the teacher asked further questions about the idea, if the class used the idea, or if the idea was ignored. I identified curricular mathematical standards that addressed the mathematics of the occurrence. For curricular mathematical standards, I used the Common Core State Standards if applicable and if not, I referred to the GAISE report. The resulting data were recorded in a Mathematical Occurrences Table. A sample of the analysis is shown in Table 7, which shows all data related to the occurrence, with the exception of the setting of the occurrence.

Table 7: Excerpt of an entry in the Mathematical Occurrences Table. All three rows in this excerpt occurred during whole class discussion.

Mathematical Occurrence	Who	Mathematical Ideas	What was done with idea?	Curricular Mathematics
Count to check if there are enough crackers	Student (in response to question)	Count to find the total amount	T: how do we know if we have enough for everybody even if we count them?	K.CC.B.5 counting -student suggests to count to answer question.
Need the number of crackers to be the same as students in the class	Student (in response to question)	Compare numbers, more, less, equal	Later the class did count the crackers and said that because there were 12 packets, that was not enough because there are 21 students.	K.CC.C.6 compare numbers – in response to “how do we know if we have enough?” a student says we need the packages to be the same as students in the class. Need to compare the number of students and number of packages. Pre- identifying important variables. (number of students and packages of Ritz)
Count by 10’s	Students (in response to how to count)	A counting strategy is to count by tens.	T said that they were counting by tens in class. Re-asked the question of “how do we know if we have enough?”	K.CC.A.1 count by tens – students have been learning methods to count in class.

The process of identifying mathematical ideas and standards associated with each mathematical occurrence sometimes focuses on small details. To generate a big picture understanding of the mathematical ideas used over the course of the task, I also created a timeline of overarching mathematical ideas (adapted from Remillard, 2016). This timeline helped to illustrate how mathematical ideas were connected and how time was spent investigating ideas. Figure 10 gives an example of a timeline. This example shows the individual ideas and connected mathematical ideas discussed in a whole class discussion. The arcs in the graphic illustrate mathematical ideas discussed by the whole

class. Some mathematical ideas are related to one another; these are illustrated by embedded arcs. Individual mathematical ideas are singular arcs. Many mathematical ideas arise during group work, but because ideas may be present in some groups, but not others, these are not illustrated. Instead, group work is illustrated with a rectangle. Two mathematical ideas of a similar topic are illustrated with two overlapping arcs. The timeline of modeling task is represented by a horizontal arrow.

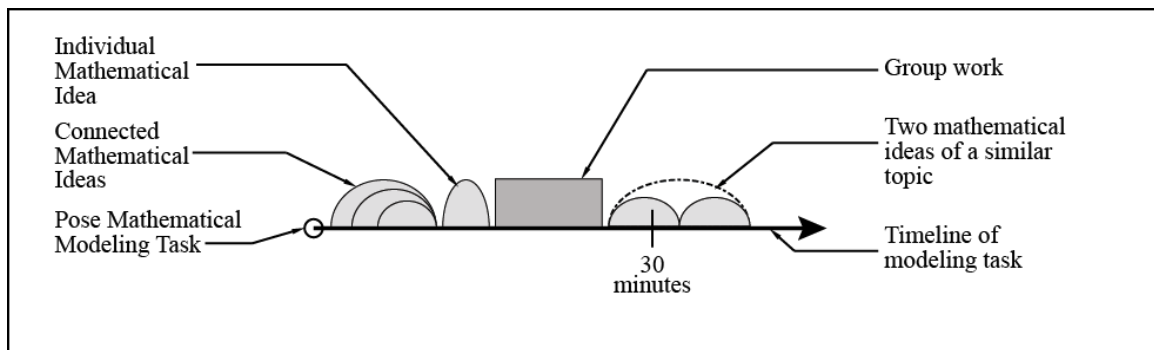


Figure 11. Example of a mathematical timeline created to illustrate how mathematical ideas are connected to investigate the mathematical modeling task. Adapted from (Remillard, 2016).

I used the mathematical occurrences table (Table 7) to understand each instance of mathematics shared and the timeline (Figure 10) to illustrate the overall mathematical ideas. From this information, I wrote a description of the mathematics used over the course of the modeling task. To describe the mathematics of each task, I looked for common strands of standards, grade levels of the standards, and which questions in a task elicited certain types of standards.

After analyzing the mathematics within each task, I then examined the use of mathematics across the four cases. I compared the types of questions asked and the strand of mathematics used to determine whether certain types of questions appeared to elicit

strands of mathematics. Finally, I compared who, students or teachers, introduced mathematical ideas across the entire range of mathematical modeling tasks.

Teachers' Interaction with Students' Mathematical Ideas

After analyzing of the mathematics within the lessons, I returned to the data using the lens of teachers' interactions. I began analysis by building on the table of mathematical occurrences. I included a column for teacher actions; for each mathematical occurrence, I considered teachers' actions related to the occurrence. Table 8 demonstrates how teacher actions were added to the Mathematical Occurrences Table.

Table 8. An example illustrating Teacher Action column added to Mathematical Occurrences Table. For ease of readability, several columns have been removed.

Mathematical Occurrence	Who	Mathematical Ideas	Teacher Action
Count to check if there are enough crackers	Student (in response if enough?)	Count to find the total amount	T asked questions to have students justify.
Need the number of crackers to be the same as students in the class	Student (in response to question)	Compare numbers, more, less, equal	T repeat student idea.
Count by 10's	Students (in response on how to count)	A way to count is to count by tens.	T connected students' idea to classwork T re-asked question

I began with the list of each Teacher Actions and looked for overarching categories. Within the categories I found identified codes for teacher actions to describe teachers' interactions with students' mathematical ideas. Findings are described in chapter four. I then added teacher actions in whole-class conversations to the mathematical timeline, and created a full graphic analysis tool (Figure 11). The arcs

above the timeline indicate the primary mathematical ideas explored, and the letters below the timeline represent different types of teacher interactions. For instance, in this example, Q represents questions that the teacher asked about mathematical ideas; R represents explaining mathematical reasoning; C represents connecting mathematical ideas.

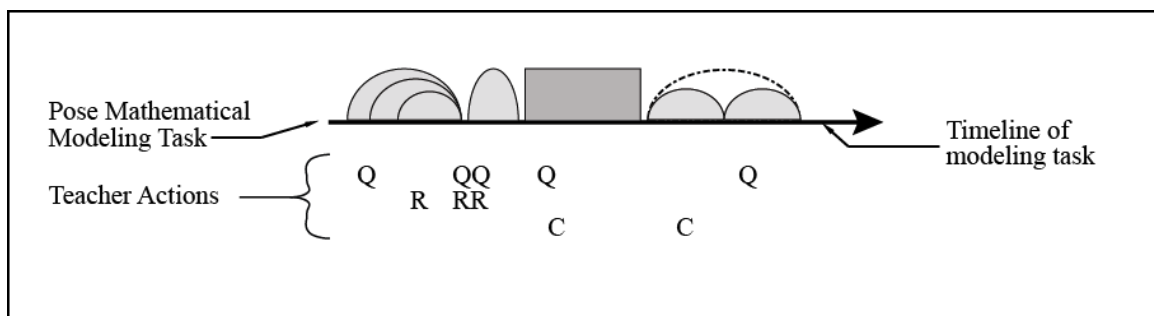


Figure 11. Example of a fully developed graphic analysis tool.

Combined the graphic analysis tool and Mathematical Occurrences Table provide a detailed history of mathematical interactions and when they occurred during each modeling task. For each teacher, I considered the best way to categorize their types of interactions to most clearly present the interactions that occurred over the course of the task.

For cross-case analysis, I used the Teaching Framework for Modeling. Examining the graphic analysis of each task, I questioned if the lesson could be broken into the stages of Organize, Monitor, and Regroup. I also questioned if there was an additional teaching stage that the Organize, Monitor, Regroup cycle did not capture. I considered each teacher's interactions and questioned whether there were types of interactions that were common across the four teachers, though not frequent for each teacher thus not

described in any individual teacher's summary. I also looked for examples of teachers' actions that contradicted the generalization of teacher actions.

Components of a Mathematical Modeling Task

Finally, I analyzed each task through the lens of the modelers' perspective to determine whether the task fit the established criteria for a modeling task. In conversations with the IMMERSION research team, I described how students engaged in the task as related to the modeling cycle. I used descriptions of mathematical modeling from the literature to describe in what ways the task was a modeling task, in what ways the task was not a modeling task, and in what ways the elementary tasks differed from descriptions of secondary modeling.

Issues of Validity and Reliability

As a means of validation, Creswell recommends "prolonged engagement" in the research setting which includes building trust with participants (Creswell, 2013, p. 250). Through the IMMERSION project, I directly or indirectly worked with the participants in this study over the course of two years. While I did not conduct regular observations of the teachers, I maintained long-term contact with the teachers of the study through professional development and professional interactions. Through emails and meetings with the teachers, I established communication and expectations, which contributed to building trust.

Triangulation is a strategy that "uses multiple methods, multiple sources of data, multiple investigators, or multiple theories to confirm emerging findings" (Merriam,

2009, p. 215). I used multiple sources of data in this study: pre- and post-observation interviews, lesson observations, lesson plans, and student work samples. I have multiple ways of collecting data for the observations: observation notes taken during the lesson, whole-class video recording of the entire lesson, and audio recordings of the teachers' conversations throughout the lesson. I also used multiple methods of analysis, examining the same data through three unique lenses. Observing and studying teachers' work and ideas from these many viewpoints helps ensure valid and confirmable interpretation of the data.

“Debriefing provides an external check of the research process” (Creswell, 2013, p. 251). I had weekly debriefing meetings with my research advisor throughout research design, data collection, and data analysis. I also periodically debriefed with other members of the IMMERSION project regarding analysis of the data.

Negative cases are instances of discrepant or disconfirming evidence to a developing theory or conclusion (Maxwell, 2013). For each lens of analysis, I made generalizations about teachers' decisions or students' contributions during the modeling process, and then looked for negative cases that might disconfirm or expand on these generalizations.

Member checking is the act of taking data, analysis, interpretations, and conclusions to the participants of the study to be vetted for accuracy (Merriam, 2009). I shared my overviews of the tasks and the descriptions of the teachers with the participants and asked them to correct any errors or misinterpretations. They have not, however, reviewed my analysis of their teaching.

Transferability is often attended to by using rich, thick descriptions, which is a thorough account of the setting and participants and a description of findings with strong evidence from a variety of sources (Merriam, 2009). I described the participants and the setting in detail, attending to factors that may influence the teachers' perceptions and experience with modeling. Additionally, analyzing the data from three lenses enriched the descriptions with interconnected details about teaching from different perspectives.

Merriam (2009) states the importance of addressing the researcher's bias so readers understand how bias may affect the study. I was a high school mathematics teacher and I am interested in ways to intrinsically motivate students in mathematics and ways to convince students of the importance of mathematics. I view modeling as an authentic way to engage students in mathematics, that by its own nature, illustrates the usefulness of mathematics. I am interested in modeling's potential as an engaging mathematical activity; this interest did not hinder my research but instead advanced and strengthened my research. Additionally, my research is enhanced by accurately detailing how teachers enact modeling in their classrooms and fully describing their decision-making so that future teachers may benefit from such work. I acknowledge my bias and to combat my bias I attended to the various methods to assure validity and reliability as described in this section.

CHAPTER FOUR

RESULTS

Introduction

This study examines the aspects of mathematics that teachers interacted with while teaching mathematical modeling to elementary students. Teachers planned and introduced the mathematical ideas and responded to mathematical ideas posed by their students. The purpose of the study is to contribute to the description of teaching mathematical modeling in the elementary schools. These practices were investigated through analysis of interviews and observations surrounding the implementation of mathematical modeling tasks of four elementary teachers. Data were collected from pre-observation and post-observation interviews, observations and recording of implementation of the modeling task, as well as teacher lesson plans and student work. These data are used to describe the modeling task, the mathematics of the task, and how teachers interact with students' mathematical ideas. The research questions are:

1. What mathematical decisions and choices do teachers make in the process of implementing mathematical modeling?
2. How do students' mathematical contributions and behaviors influence the implementation of a modeling lesson?

In this study, I analyzed the data from three perspectives:

- What mathematics is used and explored throughout the task and who introduced the mathematical ideas?

- How do the teachers interact with their students' mathematical ideas?
- In what ways is the implemented task a mathematical modeling task?

In addressing the research questions that guide this study, it is important to examine the data from the perspectives of mathematics, teaching practices, and qualities of a modeling task. While each of the three lenses are quite different, each lens contributes to answer the two research questions.

The mathematics of the task allows me to understand the role of mathematics in a modeling task and to argue that mathematical modeling is worthwhile. In order to understand the first research question, which considers the mathematical decisions teachers make, the mathematics of the task must be addressed. As described in chapter two, mathematics is a key feature that distinguishes mathematical modeling from other open-ended, real-world exploration tasks. The mathematics that students used in each case was not incidental or auxiliary; the mathematics was fundamental to the students' solutions to the real-world problem. Additionally, mathematical modeling tasks require teachers' time, which is difficult to include in an already demanding elementary curriculum. To recommend that teachers also include mathematical modeling into their mathematics curriculum, researchers need to know that students will address important grade-level standards, will be deepening their mathematical understandings, or make connections between mathematical ideas. In this case study, students worked on tasks relevant to many curricular mathematical standards and accessed and used previously-learned mathematics. Students learned new mathematics or used mathematics in new ways. Each teacher indicated that from a mathematical perspective, the time they spent

on the task was a valuable use of time. In this case study, students introduced most of the mathematical ideas; considering the mathematics and who introduces the mathematics helps to understand students' mathematical contributions, the focus of my second research question.

Analyzing the data through the lens of teachers' interactions with students' mathematical ideas is important because through this analysis, I address both research questions. Students' participation in mathematical modeling is a necessary component of the modeling cycle and teachers must make teaching decisions based on their students' mathematical contributions. Research shows that teaching mathematical modeling may require new pedagogical techniques from teachers to successfully implement (de Oliveira & Barbosa, 2010; Doerr, 2007; Kaiser & Maass, 2007). In this case study, teachers listened for anticipated and unanticipated mathematical ideas, made connections between ideas, held high standards for mathematical work, and focused students on using mathematics as a primary means to answer the modeling questions. Additionally, the teaching of mathematical modeling presents opportunities for teachers to apply best practices of mathematics teaching (Smith & Stein, 2011).

To answer the research questions that examine teachers' and students' roles in a mathematical modeling activity, I need to justify that the task is indeed a mathematical modeling task. Therefore, the third lens of analysis is the modeling cycle of a task. Each teacher developed her own version of a mathematical modeling task. Because there are many questions surrounding mathematical modeling at the elementary level it is important to analyze if and how the teachers' tasks were mathematical modeling tasks.

Since many descriptions of modeling focus on secondary-level tasks, I compare how the modeling tasks in this study were implemented in elementary grades to descriptions from the literature. This research will add to the literature on modeling in elementary grades. Additionally, it is valuable to study the components that make a task a modeling task, identifying which components are inherent to the design of the task and which are a part of the implementation of the task. This consideration illuminates teachers' decisions in the preparation and implementation of a modeling task

In this chapter, I discuss the qualitative data that address the two research questions. I begin by reviewing the setting and the participants of the study. I then analyze the data through three separate lenses: the mathematics used to investigate the task, the teachers' interactions with students' mathematical ideas throughout implementation of the task, and components that made the task a mathematical modeling task. In each section, I describe analysis for each participant and conclude with analysis across all four cases. I conclude the chapter with a summary of the critical components of teaching mathematical modeling.

The Participants

To give full descriptions of the participants, I describe the setting, the teachers' teaching experience, their experience with the IMMERSION project, their description of mathematical modeling, and their typical mathematics class.

Setting

I chose participants for this study from the participants of IMMERSION. The teachers I considered stated to me or their teacher study group facilitators that they were interested in continuing to teach mathematical modeling in their classrooms. Based on facilitators recommendations, I narrowed the list of participants to a list of teachers who had a strong understanding of modeling and were likely to implement modeling in their elementary grades classrooms.

From this list, I identified five teachers who were from one school and I decided to invite these teachers as case study participants. This limited the variability among participants, since they all taught within the same school culture. It had the added benefit of logistical efficiency for data collection. It also eliminated the risk that IMMERSION participants would feel they were being treated differently, since the clear boundary for the group was all IMMERSION teachers at one school.

The four participants in the study teach at Jefferson Elementary in Valley Public Schools, a district with about 6,000 students and eight elementary schools. Jefferson Elementary is located in a small city in the Rocky Mountain West. Classroom demographics in each class were similar to the district and school demographics though Jefferson has a higher rate of students eligible for free and reduced lunch than the district overall. Demographics of the district and school are in Table 9.

Table 9: Valley Public Schools and Jefferson Elementary Demographics (Institute of Education Sciences, 2016)

	<u>Valley Public Schools</u>	<u>Jefferson Elementary</u>
Total Population	6,292 students	467 students
White	88.8%	87.4%
Black	1.1%	0.1%
Native American	1.7%	1.9%
Asian American	1.9%	1.1%
Hispanic	4.3%	5.8%
Two or More Races	2.3%	3.4%
Free and Reduced Lunch	22.5%	36.8%

Each teacher has a degree in Elementary Education and two participants hold master's degrees. Details of the teacher's experience, current grade level, and number of students are below in Table 10.

Table 10: Demographic Information for the Four Participants.

<u>Participant</u>	<u>Lindsay</u>	<u>Rebecca</u>	<u>Erica</u>	<u>Amy</u>
Grade	1 st	1 st	3 rd	5 th
Experience (years)	9	6	1 in 3 rd grade 10 at Jefferson 12 total	5 at Jefferson 20 total
Number of Students	21	20	29	28

The participants all had similar layouts of their classrooms. The first-grade teachers had four tables of 4 to 5 students each and the third and fifth grade teachers had groups of 4 to 5 desks clustered together. All four classrooms had a carpet area where the class gathered to have group discussions or mini-lessons. During my observations, each

teacher and her students used a carpeted area for class discussion – a practice that both teacher and students were accustomed to for number talks and other tasks.

Descriptions of the Teachers

Each of the teachers who participated in this study work at Jefferson Elementary School. Lindsay is a first-grade teacher who has taught her full nine-year teaching career at Jefferson Elementary. Lindsay participated in IMMERSION in the 2015-2016 and 2016-2017 school years. Rebecca has taught first grade for six years and she also has only taught at Jefferson Elementary. She participated in IMMERSION in the 2015-2016 school year. Lindsay and Rebecca are two of three or four first grade teachers, depending on the year. Erica has taught at Jefferson Elementary School for the past ten years with 12 years of experience teaching elementary school. The first nine years at Jefferson Elementary she taught first grade and moved to third grade for the 2016-2017 school year. She is one of three third grade teachers. Erica participated in IMMERSION both in the 2015-2016 and 2016-2017 school year. Amy is a fifth-grade teacher who has been teaching at Jefferson Elementary School for the past five years with 20 total years of experience teaching elementary school. She is one of two fifth grade teachers; Amy teaches mathematics and science to both classes and the other teacher teaches English and history to both classes. Amy participated in IMMERSION in the 2015-2016 school year and informally worked with members of her teacher study group in the 2016-2017 school year.

Teachers' Experience with IMMERSION

In the 2015-2016 year, Lindsay, Rebecca, and Erica participated in a teacher study group (TSG) made up of three first grade teachers from Jefferson Elementary. Working together, the TSG developed three modeling tasks. The first task was enacted after the class participated in the school's Fall Festival, a field day for students. The teachers asked, "What game was the best?" Students surveyed classmates to answer the question and found the most popular game. The teachers found that this task was not as engaging as they expected. Perhaps, they later reflected, this was because the students were too removed from the activity as it occurred one week prior to the activity.

For the second task, the first-grade teachers brought in one bag of candy. They told the students that they purchased the candy to hand out at Halloween at their home and wanted to buy the same candy for the class for Halloween. They asked the class how many bags of candy they should purchase to share with the class. Students made decisions about how many candies would be fair to give each student, determined the number of candies needed for the whole class, and counted the number of candies in the bag. The teachers were pleased with the task; they found that context was authentic for their students, that having the candy was helpful to keep students motivated, and that the question was appropriate for their level of mathematics.

For the third task, later in the spring, the first-grade teachers worked with Amy, the fifth-grade teacher, to develop a cross-grades task. Amy asked her fifth-grade students to determine the best restaurant to order pizza for an end-of-year celebration between fifth and first grades. Amy's fifth-grade students asked the first-grade class to determine

how many pizzas they needed to order for their class to have the combined pizza party. Students counted desired slices of pizza and determined the number of pizzas needed. Again, the teachers found the task to be engaging and mathematically appropriate for their students.

In the 2016-2017 year, Lindsay again participated in IMMERSION. Because she was one of two returning teachers, she acted as a co-leader to a group of first-grade teachers from another school. Lindsay attended TSG meetings and helped to develop a modeling task that the group created. The context of the modeling task was only relevant at the other school however, and Lindsay did not teach the task to her students. As a result, the tasks Lindsay used were ones she developed herself or were tasks from the previous year.

Rebecca was not involved with IMMERSION in its second year. She independently developed the Ritz cracker task, which is described in detail in this chapter.

In the 2016-2017 school year, Erica participated in IMMERSION for a second year. Instead of acting as a co-leader as Lindsay did, Erica was a participant working with a fourth- and fifth-grade teacher because she switched from teaching first grade to teaching third grade. She was the only third-grade teacher in her TSG group. Erica's TSG developed the Community Lunch task (described in a later section of this chapter). The teachers developed the task together, but tailored the task to their individual classes. Prior to teaching the Community Lunch task, Erica implemented a Classroom Game task. This task was based on her experience working as a first-grade teacher. The first week of

school, Erica asked her class what their class game should be; the game should be one the whole class would like. The students surveyed their class and ultimately voted as a class to select the best game. It was a short task that Erica hoped would give students some experience with modeling before attempting a “big” modeling task. Erica also developed several other modeling tasks over the course of the year and worked with her TSG facilitator to develop and co-teach a Thanksgiving Table task, an Earth Day task, and a Pizza Party task. These three tasks were implemented after the Community Lunch task.

In the 2015-2016 year, Amy participated in a Teacher Study Group (TSG) comprised of six fifth-grade teachers from elementary schools within the school district. The TSG developed one main modeling task, the Field Trip task. The teachers had the district’s Deputy Superintendent of Instruction ask the fifth-grade students for a way to evaluate field trip proposals. She told the students she wanted a mathematical method to determine if a field trip was good for fifth graders in the district. Students identified and wrote rubrics to score four categories deemed important by the students to consider when evaluating a field trip. The students then created a formula for a field trip score based on weighting each category and adding the values together. Later in the school year, Amy and several teachers in her TSG developed the Pringles Challenge modeling activity, which is described in the next section. The Pringles Challenge task is one of the four tasks I analyze in this study.

In the 2016-2017 year, Amy continued to work with members of her TSG to develop new tasks and used the Pringles Challenge task again. They recruited new fifth grade teachers to participate in the Pringles Challenge. Amy had a student teacher in the

fall semester and again in the spring semester. She explained that she was not able to implement as many modeling activities as she wanted because she needed to allow her student teacher time to teach independently.

Table 11 displays the modeling tasks that teachers taught in 2015-16 and 2016-17 as they relate to their participation with the IMMERSION project.

Table 11: IMMERSION experience and modeling tasks for each participant.

Participant	Lindsay	Rebecca	Erica	Amy
Experience	2015 participant 2016 co-leader	2015 participant	2015 participant 2016 participant	2015 participant 2016 informal participation
2015 – 2016 Modeling Tasks	Fall Festival Task Candy Task Pizza Task	Fall Festival Task Candy Task Pizza Task	Fall Festival Task Candy Task Pizza Task	Field Trip Task Pringles Challenge Task Pizza Task
2016 – 2017 Modeling Tasks	Book Nook Task Pizza Task	Ritz Cracker Task	Classroom Game Task Community Lunch Task Thanksgiving Table Task Earth Day Task Pizza Party Task	Cookie Task Pringles Challenge Task

Teachers' Definition of Mathematical Modeling

In the first interview, I asked teachers to describe what modeling means to them and why they decided to continue teaching modeling in their classrooms. Below are their responses to this question.

Lindsay described modeling as solving a real problem with mathematics. She stated that the problem is open at the beginning, but then often closes, either through the

class making a choice or getting restrictions from a client. To solve the modeling task, she said, there may be many strategies that are appropriate to use and there may also be multiple answers. Though she has an idea of what students will do, Lindsay said that she neither knows exactly which strategies her students will use to solve the problem, nor does she know exactly what the solution will be; Lindsay said that is what makes modeling different from other mathematics. Lindsay continues to teach modeling because “This is real life, this is how people use math to solve problems.” She believes that modeling is empowering for her students because they learn that they “can solve real problems in real life using math. We need math because it helps us solve problems.” Lindsay appreciates that modeling problems are naturally differentiated and that students have real choice in their use of mathematics.

Rebecca described mathematical modeling as beginning with a real-world problem that is relevant to the students. Rebecca contrasted modeling to many school mathematics problems which only have one right answer, whereas modeling is very open-ended. She finds that the process of modeling is directed by students who have input on the question asked, the tools used, and strategies applied. Rebecca stated that she has been able to learn what skills her students have mastered and the skills that some students need more support in learning. Rebecca finds that modeling is valuable not only because she is able to learn about her students and their mastery of skills, but she also values modeling because her students are engaged and enjoy the tasks. The students have fun and they actively collaborate with each other.

Erica described modeling as a process which solves real-life problems using mathematics. She said that the problems are open-ended and do not have right or wrong answers. Erica finds that her students are all engaged when doing modeling tasks and that most students have fun with the tasks. Even if students don't find the tasks fun, they learn perseverance and they remember what they learn in the task. Connecting her students' learning experience to her own, Erica feels better about teaching mathematics through mathematical modeling because modeling makes mathematics connected to the real world. Erica was successful as a student because she memorized procedures, but never felt confident because she did not conceptually understand the mathematics. Modeling, Erica implied, helps students understand mathematical connections and see the usefulness of mathematics.

Amy is excited to teach mathematical modeling to her students. When asked why she chooses to teach modeling, Amy said, "Because it's fun! They [the students] love it." Amy explained that modeling is challenging for her students, but that the students buy into the problems and are engaged. Parents of her students have reported that their children are finding mathematics in many places and want to use modeling to make decisions at home. She found that even her "reluctant" students become involved in the problems. Amy works to build her students' anticipation for modeling projects. For instance, one week prior to introducing the Pringles Challenge, Amy set out cans of Pringles with a sign asking, "Where's the math?" The day before introducing the task, Amy posted a Fifth Grade Pringles Challenge poster. Without knowing what the Pringles

Challenge was, students were excited for the task. Amy acknowledges that the big modeling tasks she teaches do “take a lot of time, but it’s worthy time.”

Teachers’ Typical Mathematics Class

Lindsay and Rebecca use a “math stations workshop” model in their mathematics class, the first-grade teachers acquired and created the stations from a variety of resources and have developed the stations over the past five years. The teachers begin class with the number-of-the-day, which is the number of days the students have been in school.

Rebecca described this portion of the lesson as the “meat of foundational skills” of her lesson. Students represent the number in as many ways as possible and discuss how the number relates to other numbers. Then the teachers teach a mini-lesson or introduce a new math station. The whole class will work on this lesson in pairs or individually, and the stations are usually an activity or game. After explaining the new station, partners are assigned two review stations to work on for 15 minutes each. These stations have already been introduced and taught. Lindsay said that the stations are grouped into units and are meant to both build conceptual understanding and fact fluency.

Erica did not have a consistent typical mathematics class throughout the school year. For the past four years, Erica taught using a math stations workshop model as part of the first-grade team with Lindsay and Rebecca. Erica liked this model and felt it worked well for her first-grade students. The third-grade teachers at Jefferson Elementary had not developed a workshop model or a collection of math stations as the first-grade teachers had. Since Erica had such good experiences teaching the workshop model, she tried to develop math stations that fit with the current third grade curriculum. For the

beginning of the school year, Erica tried to split class time between doing math stations and doing the worksheets that the other third grade teachers used. She explained that it was difficult to stay on pace with the other teachers, to develop math stations, and to manage 29 students in third grade (compared to 20 in first grade). During the middle of the school year, Erica invited Amy, the fifth-grade teacher, into her classroom to help her adapt a workshop model to third grade. Erica learned about ways to use number talks in her class and decided to use math stations to help her students develop their fact fluency. Erica was also able to watch Amy teach her version of a workshop model in third grade. By the end of the school year, Erica would begin class with a number talk centered around a word problem or equation. After the number talk, she would teach a mini lesson and give students time work on an activity related to the lesson. Erica would often conclude her lessons with an exit ticket, which she said allowed her to assess how students understood the day's lesson. Erica worked to develop her "typical math class" over the course of the year, and reported that she was looking forward to teaching mathematics in the next school year.

Amy reported that modeling fits with her teaching style. In her typical mathematics classroom, Amy encourages multiple strategies in mathematical work, which she described as her favorite part about modeling. Amy's typical mathematics lesson begins with a number talk to encourage fact fluency and mental math. She then uses a workshop model for her lessons. She starts by giving a problem and time for her students to work individually and silently, calling this time "think time." After think time, Amy has students collaborate in partners or groups. The class then has a conversation

about the task and Amy “teaches her mini-lesson.” She then gives students more work time to finish the task. The class concludes with students reflecting on the lesson and what they learned. This is the first year that she has begun the class with silent think time and group work; prior to this year she began each class with the mini-lesson. Though she makes small adaptations and changes each school year, Amy has been teaching with this general format and curriculum for the past four years.

The next three sections report analysis of data collected from four specific tasks, one from each teacher. The teachers invited me into their classrooms when they taught modeling tasks; the timing or placement of the task was decided by them rather than scheduled because of a logistical request on my part. The first tasks implemented by Lindsay and Rebecca are the tasks analyzed for this case study. Erica and Amy each considered their first modeling tasks “introductory” tasks. Their second tasks, which they each described as full modeling tasks, are the tasks I studied. Lindsay and Erica continued to implement modeling tasks later in the school year. Because each teacher taught a modeling task, I chose to analyze each teacher’s first full modeling task. There are three sections of analysis, one for each of the three distinct analytical lenses that were used to examine data from the four tasks. The first focuses on the mathematics embedded within, or drawn out from, each modeling task. The second studies the teachers’ interactions with students’ mathematical ideas. The third concentrates on ways in which the task exhibited components of a modeling task and important components of a modeling task in elementary grades. A complete analysis of the four tasks, separately and combined, is presented for each lens.

Mathematics within the Modeling Tasks

In this section, I summarize the implementation of each modeling task and subsequently analyze from the first lens of analysis, the mathematics within, or drawn out from, the task. To analyze each task's mathematical activities, I reviewed observation notes and identified mathematical occurrences. Mathematical occurrences are defined as actions, or a collection of actions, in which students or the teacher introduce or discuss a mathematical strategy or idea as adapted from Leatham, Peterson, Stockero, & Van Zoest (2015). As described in chapter three, in a table as illustrated by table 7, I documented the exchange capturing the mathematical occurrences and summarized the mathematical ideas in each mathematical occurrence. A "mathematical idea" means the mathematics embedded in the mathematical occurrence (Leatham, Peterson, Stockero, & Van Zoest, 2015). For this analysis, the mathematical idea is the unit of analysis. In the table, I also noted the time of occurrence, and identified curricular standards associated with the mathematical ideas. Using the table, I diagrammed a timeline illustrating how the mathematical ideas were addressed through the lesson.

To describe analysis of the mathematics in the modeling tasks, I consider the distinct questions that each lesson addresses. Lindsay's and Rebecca's lessons only considered one question, while Erica's lesson addressed two and Amy's addressed three questions. Within each question, I describe the mathematics by CCSSM grade level, by components of the GAISE framework, or by content strand. For each question, I also describe who introduced most of the mathematical ideas.

Overview of Lindsay's Book Nook Task

Lindsay implemented the Book Nook Task the week of Thanksgiving. This was the first modeling task her first-grade class had encountered. Lindsay got the ideas for the task from seeing an anchor chart made by another teacher about places to read. In an interview, Lindsay said that when she saw the chart, “I thought, ‘The best reading nook! That could be a [modeling problem].’” Lindsay had used modeling tasks the previous year that asked students to determine “the best carnival game” and she looked for opportunities to ask her students what they thought was best of something.

Lindsay invited me into her classroom for the lesson when students considered what reading nook was best for the class. They had spent two earlier days talking about what makes reading nooks good and what reading nooks they liked as individuals. Lindsay described the following activities about those two days to me. Lindsay started by asking students “What is your favorite place to read?” and “What makes those places good places to read?” The class brainstormed a list of qualities of reading areas that were important. Then Lindsay asked the class, “Which of these ideas is really important? And how [can we] figure it out using math?” A student suggested voting and writing tallies. Through voting, the class determined that *soft*, *quiet*, *laying down*, and *dim* were important qualities for a good reading space. Lindsay told the students that the model they created was “Best Reading Spot = Soft + Quiet + Laying Down + Dim.” The second day the students brainstormed ideas of items that they could buy for a new book nook. Then the class “ran each idea through the model” they had created the day before. The four items that satisfied all criteria were a tent, a body pillow, a bean bag chair, and a

lounger. Each student picked the item that they preferred and drew a picture of their preferred item at the end of the lesson.

The lesson I observed began with Lindsay telling the class that she knew what each student wanted, but now she needed to know what the *whole class* wanted (Lindsay's emphasis). Lindsay asked, "What is best for us to buy for everyone?" Lindsay offered paper, graph paper, and a class roster as tools for students to use. Lindsay spent about five minutes describing the question to the students before letting them work in groups. In groups, students decided to survey each other to learn what their peers wanted. They recorded their information in different ways, with some students recording the names of classmates and their votes and others just recording the votes without recording who had voted. Lindsay gave students almost 20 minutes to work gathering votes before she called the class back to the carpet. While students worked to collect votes, Lindsay watched students work and observed their collection method and recording method. She asked students questions about their recording methods, what data they were collecting, and what their notes meant.

At the carpet, one student identified a problem their group had noticed: they found a different solution from other groups; some groups had that a tent was most popular, some found that a bean bag was most popular, and some had a tie between a bean bag and a tent. The student presenting this problem suggested that the reason for different solutions was that students were changing their answers. The class discussed ways to vote and after a few minutes of conversation Lindsay re-launched the problem. Students continued surveying, some using new methods. Lindsay continued to watch what

methods groups were using, questioned and helped groups with faulty methods, and questioned groups with good methods. After the students worked for another 20 minutes, Lindsay collected each group's work.

I was not present for the next class session in this lesson. However, Lindsay told me groups found that the tent or the beanbag had the most votes, but there was not a clear winner.

[The class then] narrowed it down to just those two choices. [The class] used [their] model and assigned a point value to each part according to the most to least important aspect of "best." [The class] voted to determine the ranking. Then our model became (and I wrote all of it on the board for them to see)--

Soft (4 pts) + Quiet (3 pts) + Laying Down (2 pts) + Dim (1 pt)

Tent = $0 + 3 + 2 + 1 = 6$ pts total

Bean bag = $4 + 3 + 2 + 0 = 9$ pts total

So [the class] determined that the one that fit our model best was bean bag. Then I asked if the results would be consistent if [they] voted, so [the class] voted. Bean bag had nine votes, tent had eight votes, so bean bag won again. Thus [the class] determined that using [their] model and using voting to determine was everyone thought was the best book nook, [the class] could say with assurance that bean bag really is the best.

Mathematics of Lindsay's Book Nook Task

When asked what mathematical standards the students would use, Lindsay said that her students would likely use surveying and potentially use graphing. Interestingly, the mathematical work of students followed the framework set by the GAISE report, a report unfamiliar to Lindsay. The students and Lindsay attended to components of Level A of the GAISE framework, which is the beginning level for developing statistical literacy (Franklin et al., 2007). The students are in first grade and have little experience

with statistics, so Level A is an appropriate level. The GAISE framework is structured into four stages: formulate question, collect data, analyze data, and interpret results.

Lindsay's students addressed all four stages throughout the class I observed.

The first stage of the GAISE framework, formulate question, is described as "beginning awareness of the statistics question distinction" (Franklin et al., 2007, p. 14), where teachers likely pose the question of interest. In the Book Nook task, Lindsay began the lesson by asking her students what reading nook item the whole class wanted and not just what individuals wanted. Students did pose the question asked in their surveys, but the overall question of "What is the best book nook for the class?" was posed by the teacher.

The majority of the observed lesson was spent on the data collection stage of the GAISE framework. Students determined that they could answer the posed question by asking their classmates what book nook they wanted. From there, groups conducted a census, a data collection method described in Level A. Lindsay provided tools (graph paper, class roster, plain paper) for students to use to aid their data collection process, but she did not tell them which tool to use, or how to use it. Students used their chosen tool to collect data from their classmates; most students recorded their data by drawing the item, writing the first letter of the item, or writing the item that their classmates wanted. As students worked, Lindsay noticed that several students were asking their classmates what they wanted, but were not recording the data. She asked these students what they were learning about their classmates. Through her questioning, the students realized that they needed to be recording the data they collected.

Once students collected their data, without prompting by Lindsay, they began to analyze their data, the next stage described by the GAISE framework. Students counted the total number of votes for each item. Figure 12 is an example of one student's approach to display the results through graphing. Students also compared the count of each item to determine what was most popular. In this stage, Lindsay did not direct students in what to do to analyze their work. She did ask students questions about what they were doing, but did not suggest that they analyze in any specific way. This stage of GAISE overlaps with several CCSSM standards, such as counting to find the total votes for each item, a kindergarten standard. Students also “organized, represented, and interpreted data with up to three categories”(National Governors Association Center for Best Practices, 2010), a first-grade CCSSM standard.

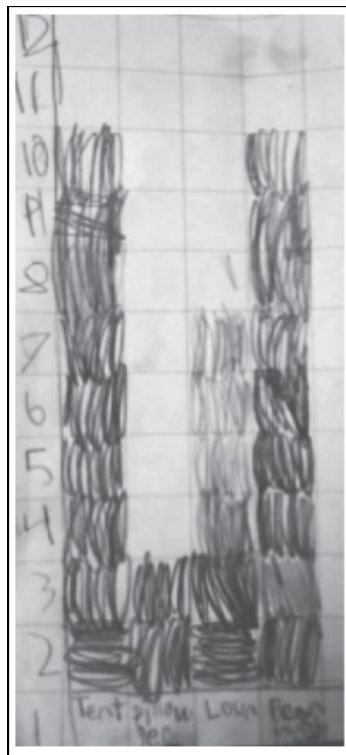


Figure 12: Graph showing votes for book nook items

To interpret their results, the final stage suggested in the GAISE framework, students compared the number of votes that each item received. Some groups noted a discrepancy of solutions between the groups. Some students determined that they did not get the votes of students who were absent for the day. They corrected their mistake by looking at the students' drawings from the day before to determine their vote. Other groups speculated that students were changing their votes, and that was causing groups to have different solutions.

During the observed lesson, students worked through each of the stages of the GAISE framework, using skills described in level A. One goal of GAISE is that students realize that "data are not just numbers, they are numbers with a context" (Franklin et al., 2007, p. 7). Through analysis and interpretation, students referred the meaning of the numbers and their relationship back to the question. It was clear that the numbers had meaning within the context for all students in Lindsay's class.

Aside from asking her initial question, Lindsay did not initiate the mathematics used in the lesson. She observed students' choices and processes and asked them to explain and justify their thinking.

Lindsay's description of the next day's lesson indicated that the students voted to determine the most important attributes of a book nook and ranked them (soft, quiet, laying down, and dim). Lindsay then weighted the linear combination to get the formula "Soft (4 pts) + Quiet (3 pts) + Laying Down (2 pts) + Dim (1 pt)." This mathematics is not a first-grade standard, nor did Lindsay expect that students would create the model themselves. Instead, Lindsay validated that students wanted to give additional value to

attributes they believed to be important and laid the foundation for learning about linear combinations in the future. The students could compare the outputs of the equation Lindsay had created to determine the best book nook.

Overview of Rebecca's Ritz Cracker Task

Rebecca implemented the Ritz Cracker Task in the fourth week of school. This was the first modeling task her first-grade class encountered. In an interview Rebecca said that she was not intentionally seeking out a modeling task to do with her students but was aware of looking for real-world problems that use mathematics as a means to a solution. She recognized an opportunity for a modeling task when she asked herself how she should hand out the class snack with a box of Ritz Crackers.

At Jefferson Elementary School, families bring in large containers of snacks at the beginning of the school year for the teachers to distribute at snack time. Rebecca received a box of short tubes of Ritz crackers. Because of the non-traditional size of the packaging, Rebecca found herself questioning how many tubes she would need to distribute snack. In an interview, Rebecca stated that she was aware that answering this question used mathematics, and it was appropriate mathematics for her students early in the school year because it could be solved with counting.

Rebecca scheduled the task during the class's snack time. With her students seated at the carpet, she described that she had a problem that they could use mathematics to help her solve. She showed her class the box of Ritz crackers (Figure 13) and said we have "all of these things of crackers, and what I wonder is how are we going to do snack today with all of these different containers of snack."



Figure 13: The box of crackers with small packaging that was the focus of the Ritz Cracker Task

One student immediately suggested giving each student a pack of crackers. The first three minutes of class were spent with the teacher describing the problem and the class considering the idea of giving each student one package of crackers. When the class realized that the suggestion would not work, Rebecca asked partners to brainstorm other ways of solving the problem. After students talked with the person sitting next to them, Rebecca described to the class that one student had an idea, but could not yet prove the idea. Rebecca suggested that students might want to use a white board to show that their ideas would work. Students worked in groups for about ten minutes investigating how they would distribute the crackers. Many students got into groups on their own. For those who did not, Rebecca organized the students into groups. Then she spent her time moving from group to group asking students questions about their ideas.

After the group work time, the students met back at the carpet to share their ideas. Groups shared their ideas one at a time, and Rebecca asked individual groups questions, rephrased their ideas, and compared ideas among the groups. After the groups presented,

Rebecca summarized for the class that there were three ideas: each student gets two crackers, three crackers, or six and one-half crackers. She told them that they would take a vote on how many crackers they would like for their snack for that day. Students raised their hands and there was a clear majority for six and one-half crackers. The class agreed, by looking at the hands, that six and a half won. Rebecca then gave pairs of students one packet of crackers and told them that they needed to figure out “how to give six crackers to each person and how to split the remaining cracker.” The students then ate their snack after making sure that each partner had the same amount.

Mathematics of Rebecca’s Ritz Cracker Task

Over the course of the modeling task, students investigated mathematics that are described by CCSSM written for kindergarten, first grade, and second grade.

The kindergarten standards that students focused on addressed counting and comparing numbers. The Ritz Cracker task was completed in the first month of school, and Rebecca’s students had been learning and practicing counting and the number system. As a result, it is neither surprising nor unexpected that students would employ a kindergarten mathematics standard in first grade. Rebecca did not plan the lesson by first considering standards that related to the problem, but did expect that most students would count and sort the crackers, or count and use repeated addition. The kindergarten standards were used to count the number of packs in a box, the number of crackers in a pack, and the number of crackers distributed. These mathematical ideas were initiated by the students. The teacher had students explain their thinking to the class and then compared student ideas.

The first-grade standards that students applied were counting and subtracting within 20, comparing amounts as more or less, and relating counting to addition. Rebecca had not yet taught lessons around these standards, though the ideas may have been addressed indirectly in class. Students brought up the mathematical ideas of adding and subtracting within 20 to determine the number of total packs in the box and to determine the number of packs needed if each student at their table received three crackers. Students suggested that they count the number of packs in a box and a student told the class that there were 12 because there were two rows of six. Rebecca then told the class to check to see if it was 12 by counting, using the standard of comparing addition to counting.

The students applied some second-grade standards and ideas related to higher level grades when trying to determine how to share packets of crackers. When thinking about how to split one pack of crackers between two people, one student explained that 13 was an odd number and how that affected sharing the crackers. Odd and even numbers are part of the second-grade mathematics standards. When Rebecca asked students what they would do with the remaining cracker, students said they would break the cracker and each get one half. Though ideas about fractions were not formally addressed, students thought about fractions in ways that added background and context when they would formally learn fractions in future grades.

There were several mathematical ideas brought up by Rebecca or her students that foreshadowed future mathematics the students would encounter, such as division and proportions. One student considered how she could divide the number of crackers into equal groups. Her drawing (as shown in Figure 14) showed that she first made five equal

groups of one, then five equal groups of two. Her table had five students, so to equally share one pack of Ritz crackers each student could get two crackers. Going over two would have used more than 13 crackers, the number of crackers in one pack. This work was not division in a formal sense, but she used repeated subtraction and took away groups of equal size – concepts essential to division.

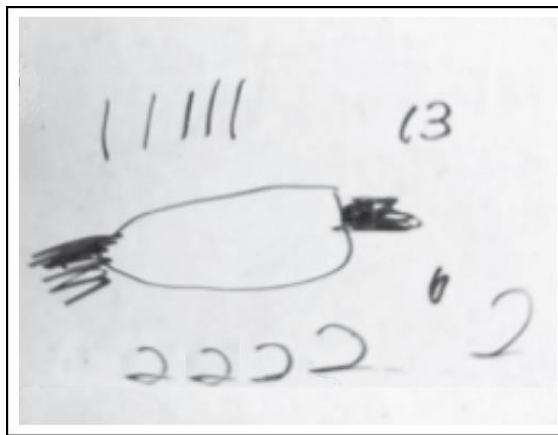


Figure 14: A drawing of one pack of Ritz crackers. The ones above the packet indicate giving each of the five people at a table one cracker. The twos below the packet indicate giving each person two crackers.

Overall, most of the mathematical ideas were introduced by students. The teacher did not introduce mathematical ideas to the whole class, but helped to connect students' mathematical ideas, asked students questions about their mathematics, and encouraged students to justify their mathematical thinking.

Overview of Erica's Community Lunch Task

Erica began the community lunch task in the second week of school. This was her third-grade class's second modeling task, having done a mini-modeling task the first week of school. Erica developed the task collaboratively with her Teacher Study Group; a

fourth-grade teacher, a fifth-grade teacher, and a university professor. Erica's class spent six periods (almost five hours) solving the task and one lunch period for their community lunch.

Erica began the lesson by briefly reviewing mathematical modeling and their previous task. Erica played a video of their client, a university professor who explained that she would fund a community lunch for the class. The professor said the class had a budget of \$100, and that she needed the class to tell her what to buy for the lunch. After showing the video twice, Erica asked the class what they were being asked and reviewed vocabulary such as "client," "budget," and "constraints." The class then spent time asking each other questions about what food they liked, allergies they had, and food they did not like.

On the second day, Erica led the class in a discussion about the group's results from the prior day. The conversation focused on how they were building their surveys, and what they should do with the survey results. After a 20-minute discussion, the class spent the rest of the lesson working in groups to develop a survey and then asked their classmates to vote. In the middle of the groupwork, Erica paused the groupwork to let the class know that one student wanted to use a class roster to help keep track of who had voted and who had not. She offered class rosters to the class as a tool to use.

The third lesson began with a brief review of the previous day where Erica defined the budget and the class discussed using a roster and a menu. The class then worked in groups to finish collecting data from their classmates on what food was best

for them to order. Erica called her students to the carpet for the last 15 minutes of class and groups shared their results.

The fourth lesson began by looking at the groups' results. Students shared ideas on how they should move forward in picking a type of food to order and if they could order food from multiple restaurants. The class agreed that they should narrow the choices down by looking at their data from the day before and vote as a class. Erica organized the students and they voted – Chinese food had the most votes. Erica then pulled up a menu from a Chinese restaurant near the school and organized the students to vote on two types of chicken entrees and two types of sides. After the class determined the number of students wanting each of the two types of entrees and sides, Erica asked them, “Can we order this food for under our budget of \$100?” She gave the students menus and they started working in groups for the last 10 minutes of class. During this time students asked Erica questions about how many students the sizes could feed and worked to understand how the menu could help them order food. Erica expected that students would choose pizza. Not anticipating that students would choose Chinese food, Erica was not confident on children's servings per order of items on the menu and decided to wait to answer the students' questions the following day.

Erica began the fifth lesson by presenting the class with information regarding: serving sizes available, the number of students each size would feed, the price of each size, and the number of students wanting each entrée and side dish. The class realized that the number of student orders did not match the number of students in the class, and they discussed ways to resolve the problem. The class then spent the remaining 35

minutes of the fifth lesson and all but the last few minutes of the sixth lesson working in groups to determine what their order should be and how much the order would cost. In groups, students decided which sizes to order to feed their classmates. Once the number of different sizes were determined, they used various addition strategies to add the prices, which included cents. After repeating this for each of the two entrées and two sides, students added each component to determine the final price. Erica helped her students label and organize their work, explain their serving size selection, and add the large numbers to find a total price. Erica concluded the lesson by stating the range of costs that groups found (there was a range for different combinations of orders that could feed the class). She then told the class that the restaurant offered a party pack, and that the price was very similar to their solutions.

Mathematics of Erica's Community Lunch Task

Erica's community lunch task naturally separated into two phases where students addressed two different questions. In the first phase, students determined the best food – which food the class wanted for their community lunch. The second phase considered the budget for lunch, and students determined the food order and total cost. The mathematics in each phase were quite different and are described separately.

Mathematics to Determine the Best Food. The mathematics used to determine the food for the community lunch are best described by the GAISE reports. Like Lindsay, Erica identified surveying and graphing to be the mathematics that students would use to address the question. Though she was also unfamiliar with the GAISE report, the class

progressed through the four stages of the framework. Student's work attended to Level A as described in the GAISE report, which is the introductory level for developing statistical literacy (Franklin et al., 2007). The four GAISE stages were all addressed by the students: formulate question, collect data, analyze data, and interpret results. Students answered the question, "What is the best food to order for our community lunch?" over the course of three days, refining their statistical work and questions.

The first component of the GAISE framework, form a statistical question, required the students to ask themselves how they would decide what food they wanted to order. Initially, the client posed the question, asking for the students to develop a food order for their community lunch; however it was up to the students to form the specific questions they needed to arrive to a solution. Ultimately, with the help of Erica, they arrived at the question, "What food was best to order that the whole class would like?" At first, many students asked their classmates what food they liked and if classmates had any allergies. After refining their questions, groups developed a list of foods, and asked classmates which food they would prefer for lunch from a defined set of options.

Collecting data is the second component of GAISE. Students collected data several times over the three days. The first day many students asked their classmates what food they liked and if people had any allergies. This information helped students to determine what foods were popular and if there were foods that they could not eat. Students used this information to develop a list that they used in the next round of surveying. Another initial strategy was to list foods that they knew were popular lunch items. Alternatively, a few students knew what their favorite food was and asked

classmates if they also liked that food. Erica addressed this idea and suggested that surveys typically need more than one food option. On the second day, students surveyed their classmates and realized that they needed help in keeping their information organized. Erica suggested that students use a grid roster with the names of each student in the class. On the final day of data collection, students used the roster to survey their classmates to make sure they had collected data for each classmate.

Students recorded their data in different ways: bar graphs where the x-axis is the food and the y-axis is the number of votes; or lists of foods with X's or tallies indicating votes. Once Erica distributed grid rosters, most students put marks in a food's column next to the student's name to indicate a vote. Erica asked her students what their X's and tallies meant. Students stated that "X's are how many people like something." With this understanding, students could analyze their data to find the most popular foods. Students counted the votes and compared groups. Other students looked at their graphs and said that the tallest graph had the most votes.

Erica brought students together and asked groups to tell her what food they found to be the most popular. Groups then looked at their data and reported what they found. Erica collected each groups' best food and used this to have a class discussion to pick which food they should choose for their community lunch. This launched another cycle of modeling following the GAISE components. Erica led a whole-class discussion to take a census of the class on the best food from the list of popular foods. Led by Erica, the class went through the same voting process applied to the menu at the Chinese restaurant

to determine which two entrees and two sides they should order that the whole class would approve of.

Mathematics to Determine the Food Order and Total Cost. Students used a wide variety of standards addressed in the CCSSM to determine an appropriate food order to feed the class so that each student received the entrée and side dish they wanted and to find the price for the order. Students used several standards written for grades preceding, including, and proceeding third grade.

The primary first and second grade standards that Erica's students applied were to compare numbers and to add and subtract within 100 to solve one- and two-step word problems. Students compared numbers by comparing the number of entrée orders and side orders to the number of students. Additionally, when students created a food order, they compared the number of students that the order fed to the number of students wanting the food item. Within the task, students found many problems that require addition and subtraction within 100 to solve. Students needed to add the number of orders for chicken and the number of orders for sides. They needed to add the number of people per serving to find the number of people that their food order would feed (see Figure 15). Students also added and subtracted the price of each order to determine the final order. To add and subtract, students used several different tool and strategies, such as drawing, calculators, and bar graphs to help manage their data. This variety of standards was met because of the open nature of this part of the task that allowed students to instigate any necessary tool to help solve the task.

Handwritten student work on grid paper. The work consists of three rows of equations and one row of items:

$8 + 8 = 16$	
$16 + 4 = 20$	2 Large
$20 + 2 = 22$	1 Medium
	1 Small

Figure 15: Student work to find the number of people per size.

The third-grade standards used in the task related to using multiplication: finding the product of whole numbers, multiplying with arrays, and using the distributive property. Opportunities to use multiplication presented themselves when students calculated the number of people that two or three large entrees would feed and when they multiplied the price of 13 small orders (after rounding the price to a whole number). The student who rounded the price to three dollars calculated 3×13 by drawing an array. As seen in Figure 16, the student wrote $13 + 3$ and $12 + 3$, but the student explained using the term “multiply.” The third-grade standard that Erica formally introduced was the distributive property. She was working with a group of students who needed to calculate the price of both side dishes, which resulted in the expressions 2.70×13 and 2.70×12 . Erica showed the students that this was the same as 2.70×25 .

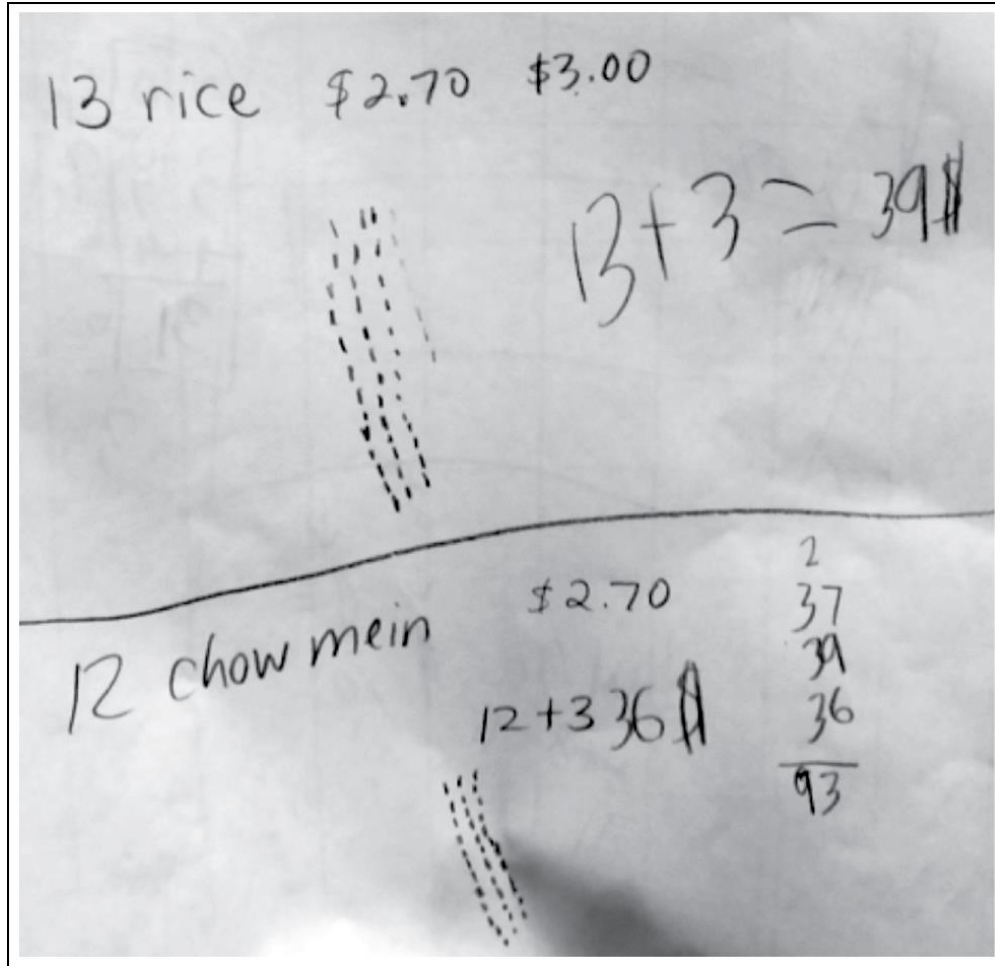


Figure 16. Student work rounding the price to whole dollars and creating an array to add.

Students used fourth- and fifth-grade standards to compute the price of their orders because the food prices were given to the hundredths of a dollar. Students added decimals to the hundredth by adding prices, compared decimals to the hundredth by comparing prices, and rounded decimals by rounding prices to the whole dollar. Adding prices was initiated by students, but many students struggled initially and required varying levels of scaffolding to complete the process. Either Erica, the student teacher, or aides in the class provided help. For instance, in figure 16 shown above, \$2.70 was rounded to \$3, but the rounding looks like it may not be in the student's handwriting.

With or without assistance, students used many different strategies to add decimals. In an example of alternative techniques of adding decimals (Figure 17), two different students added the decimal numbers and whole numbers separately and then added the sums. Although incorrect, Figure 18 shows how one student tried to use a doubling strategy to add 2.70 eight times.

$$3.70 + 3.70 = 7.50$$

1.50

$$9.70 + 9.70 = 19.50$$

1.50

18

$$3.70 + 3.70 + 3.70 = 11.20$$

1.50 + .70 = 2.20

$$9.70 + 9.70 + 9.70 = 27 + 21 = 29.10$$

240

Figure 17. Student strategies of adding cents separately from whole dollars, then finding the sum to calculate a total price.

The image shows handwritten mathematical work on a piece of paper. At the top, there are two numbers, 1 and 2, written above the first two terms of an equation. The first equation is $2.78 + 2.78 = 5.40$. Below this, the second equation is $5.40 + 5.40 = 10.80$, with a small '2' written under each of the 5.40 terms. To the right of this, the number 20.60 is written. Below these, the equations $10.80 = 4$ and $20.60 = 8$ are written, with the 8 underlined.

Figure 18. Student's doubling strategy. The 4 and 8 indicate that the price for four people is \$10.80 and the price for eight people is \$20.60.

The class of 25 students voted for two side dishes between four available side dishes. Erica asked the class what they noticed about 10 students choosing one side dish and 11 students choosing another side dish. The class realized that the total votes did not add to the number of students in the class and another student suggested that they re-vote. Erica introduced proportional reasoning, a sixth-grade content domain, by stating that the votes were almost equal, differing by only one. She suggested that by adding two votes to each order, 12 and 13, the orders would remain almost equal.

Overall, for both parts of the task, students introduced many of the mathematical ideas used: using surveys, voting, comparing, adding, and subtracting whole and decimal numbers. Erica also introduced mathematical ideas such as using the distributive property, how to make linear combinations that add up to 22, and to use multiplication instead of addition to calculate costs. A description of how Erica interacted with her student's mathematical ideas and used them to introduce her own is described in a later section of this chapter.

The eight mathematical practice standards of the CCSSM were not a focus of the study; they are, however, present in each teacher's task. Of particular interest is the standard "appropriate use of tools," because the use of calculators influenced the mathematics of this task. Erica's task provides an instance where the teacher needed to make a choice about using technology that was not an issue in the other three classes. This was Erica's first encounter with calculator use in third grade and she believed the students had limited experience in second grade. She chose not to introduce calculators but to provide them if students requested calculators. A student did request a calculator and Erica then offered calculators to the whole class. Many students tried using the calculators, but Erica found that some students did not use them appropriately. Erica decided that students needed to first work on the problem without a calculator but could use the calculator to later check their work. During the second day, perhaps because the lesson was short on time, Erica let students use calculators to add decimals rather than to only check their work. While Erica "sees the value in showing the numbers and the math," using the calculators indicates that Erica placed more value on understanding how the prices related to the problem rather than practicing the skill of adding many decimals.

Overview of Amy's Pringles Challenge Task

Amy began the Pringles Challenge in February and finished the task just before spring break began. This was the class's second modeling task; the first task was intended to be an introduction of modeling at the beginning of the school year. Amy had implemented the Pringles Challenge task the previous year, and worked collaboratively with other 5th grade teachers in the district. The challenge was to design a box that was

small and light and protected a single Pringle potato chip in travel via inner-district mail to another elementary school; the seven 5th grade classes competed against each other to design the best box.

Amy began the task with an introductory lesson about modeling in which the class brainstormed the meaning of mathematical modeling, what kind of mathematical questions United Parcel Service (UPS) might ask, and how a mathematical modeler might solve problems. After defining mathematical modeling, the class did a mini-modeling activity in which they watched a video of a fish tank being filled with water. The class then posed mathematical questions about the video, talked about the information they would need to solve their mathematical questions, and discussed how to solve their questions.

Having defined mathematical modeling on the first day, Amy then introduced the Pringles Challenge to her students on the second day. The class spent 15 minutes reading the rules, asking questions, and determining the important variables of the challenge. The class finished the second day's lesson working individually to design a box, and thinking about the mass, dimensions, and materials of their boxes.

For the third lesson, Amy asked her students to consider the next step of modeling: mathematizing the problem. The class discussed how to mathematize the variables of mass, volume, and chip intactness. The class had the mathematical tools to easily mathematize the mass and volume, but mathematizing the chip intactness was difficult. Amy led a discussion to consider and describe the best-case scenario and worst-case scenario. After describing the situation, the class got into groups and worked to

assign values to describe how their chips might arrive. Amy paused group work twice to share students' thinking and scoring systems. Groups submitted their ideas at the end of class, which Amy said she would present to the other fifth-grade teachers to determine an "official" scoring system based on students' ideas. Amy began the next day's lesson by sharing students' work, and the class compared several scoring systems. Then Amy presented the official scoring system chosen by the teachers and students noted similarities between the official scoring system and the systems they created.

Next, Amy asked her class how they should combine their variables of volume, mass, and chip intactness to create a relationship that used the four operations to produce a final score. Students worked in groups to develop equations to find a final score. Amy selected one group's equation, $V-M+C$, for the class to discuss at the carpet. Amy shared the equation and showed the class how they could test the equation. To test the equation, Amy substituted test values – reasonable numbers for mass, volume, and chip intactness that were also easy to work with. They tried three examples with the equation, changing only one variable at a time. The class discussed if a high score or low score was a better result, and the meaning of the final score in relationship to the type of box they were testing. After discussing this equation, groups continued working on their own equations, again substituting values to test their equation. Students submitted their equations at the end of class, with the understanding that Amy would take their proposals to the other fifth-grade teachers for the teachers to determine the final equations. The next day, Amy shared a few groups' equations and the class discussed why the equations might work. Comparing these equations, Amy introduced the final equation that the teachers decided

to use for the Pringles Challenge, $\frac{I}{MV} = \text{Final Score}$, and students noted similarities with the equations their class produced. The class used the equation to score a test package to understand how the equation worked.

Once the final equation was determined, each group built a box prototype and scored their box. Amy directed her students to revise their packages based on the equation. Most groups made their boxes smaller while some took out material to lower mass; the groups then predicted the score of their revised boxes. Amy used intra-district mail to send the boxes to another fifth-grade class. Later the class received boxes from other schools. They scored the mailed boxes with Amy checking their work. The 5th grade teachers then made a presentation of each group's box and score that was shared with all participating classes.

Mathematics of Amy's Pringles Challenge Task

Students used a variety of mathematics throughout the Pringles Challenge Task. Students used mathematics to learn and understand the process of mathematical modeling and how certain quantitative measures may help answer mathematical questions. Students considered how to give quantitative values to variables inherent to the Pringles Challenge. Students created equations to evaluate boxes that carried a Pringles chip, used the equation to create boxes, and used the final equation to evaluate. These skills are part of the CCSSM fifth grade standards.

Introduction to Mathematical Modeling. Before introducing the Pringles Challenge, Amy taught about mathematical modeling by asking students how people use

mathematics to address questions they might have. Amy had students ask mathematical questions, consider the variables associated with their mathematical question, then determine the mathematical tools needed to solve the mathematical question. Students repeated this process for two scenarios: mathematical questions UPS might ask and mathematical questions associated with filing a fish tank.

For example, in considering the question, “How can we use math to make predictions on which driver will get in a wreck?” students suggested that UPS consider the following variables: vision, weight of vehicle, number of past wrecks, age of driver, speed of truck, volume of music, how close the vehicle gets to other cars, and how fast the vehicle turns. Asking mathematical questions, considering variables necessary to the question, and identifying mathematical tools to solve help students address the CCSSM practice standard of making sense of problems by thinking about mathematics inherent to a problem. This process helps students understand and identify variables; understanding variable is necessary for many 6th – 12th grade CCSSM standards. Additionally, this process helps to prepare students for the high school statistics standard to “define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space” (National Governors Association Center for Best Practices, 2010).

After these activities, Amy defined mathematical modeling as a five step process (as shown in Figure 19), which students then followed in the Pringles Challenge Task.

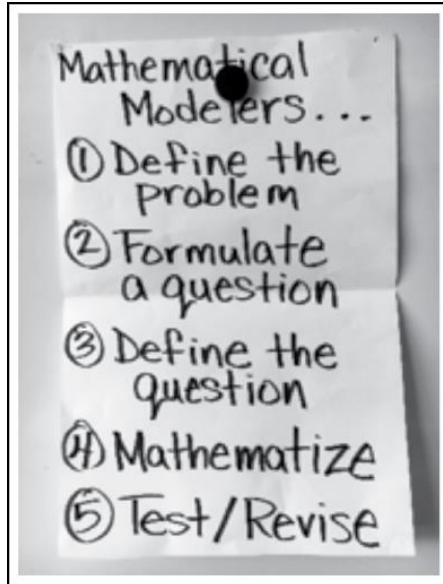


Figure 19. Five steps of mathematical modeling as displayed in Amy's class.

Quantitative Variables in the Pringles Challenge. Amy and her students identified the three variables for the Pringles Challenge as (1) mass of the box, (2) volume of the box, and (3) intactness of the Pringles chip. To get a quantitative value of the mass of the box, students weighed the box with a scale and converted the mass in grams to kilograms; this process used conversion and unit standards for third grade and fifth grade. To determine a quantitative value of the volume of the box, students measured their box, and used these measurements to find the volume of a rectangular prism, a skill that Amy stated was quite important in fifth grade. The students multiplied, by hand, with decimals to find the volume, another important fifth-grade standard.

To generate a quantitative value for the intactness of a Pringles chip, the students and Amy used mathematical ideas that are necessary to understand how to measure and quantitatively describe the world. They created numerical values associated with the best- and worst-case scenarios and then determined values and descriptions for chips between

best and worst. These steps addressed skills important to modeling by quantifying qualitative data (English, 2006). Figures 20 and 21 show different ways that students quantified the quality of Pringles chips.

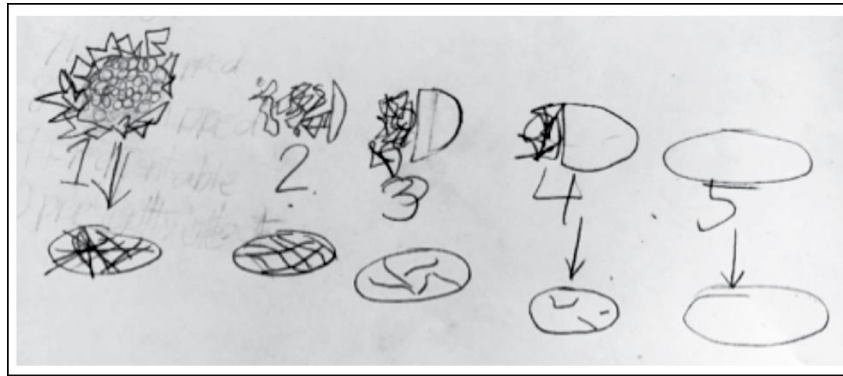


Figure 20. Student's work showing that a perfect Pringle receives 5 points, a very broken Pringle receives 1 point, and that different, imperfect chips can receive the same number of points.

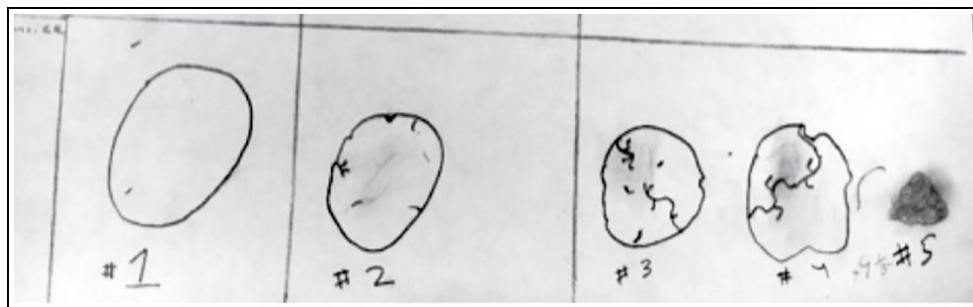


Figure 21. Student's work showing that a perfect Pringle receives 1 point, that a pile of Pringle dust receives 5 points, examples in between the best and worst.

Equations for evaluation of the Pringles Challenge. The fifth-grade teachers told students that they needed to use the smallest, lightest box they could to safely mail one Pringles chip. Once students knew how to find the mass of the box, the volume of the box, and the intactness of the chip, Amy instructed her students to create an equation that would score the box using mass (M), volume (V), and intactness (I). Amy and her class

never called this equation a mathematical model, though from the perspective of mathematical modeling, this equation was the model.

Students used mathematics that they were familiar with to answer complex questions. Students applied their understanding of the meaning of division, a third- and fourth-grade standard, to complex problems with variables to explain how division affects the score in an equation. One student explained the final score of $\frac{I}{MV}$ saying, “the better the chip, the higher the score. The smaller the box (MV) then less is taken away, and take away more times... [to get] a bigger score.” Students also used fourth grade standards to compare addition and subtraction equations to multiplication and division equations, again applying their understanding to complex equations with variables. Students explained that multiplication is repeated addition, and division is repeated subtraction, explaining how $M + V - I$ is similar to $\frac{MV}{I}$. One student explained to Amy that they confused properties of addition with properties of multiplication. Some students also made errors in their work and were able to identify the reason for their confusion.

Students used many fifth-grade standards in this task, as is expected since the teachers developed the task to address fifth grade standards. The students addressed measurement standards such as converting units, using cubic units, and finding volume. The Pringles Challenge rules require weight units in kilograms, but the classroom scale measured in grams. As a result, students needed to calculate conversions. Many students reasoned to determine if converting from grams to kilograms made the number larger or smaller. Amy had many conversations with students about how to divide by 1000, that dividing by 1000 is the same as dividing by 10 three times, and how to keep track of the

decimal place value. Students also kept track of the unit they were measuring (centimeters and millimeters) and wrote the measurement as centimeters with a decimal value. Once students multiplied the measurements, Amy encouraged the students to label their units as cubic centimeters. Throughout the activity, students calculated volume five times; once in the fish tank task, three times to create, score, and improve their boxes, and finally to evaluate the box that arrived from another school.

Students also used fifth grade standards relating to number and operation in the base ten system throughout the task. Since students measured boxes precisely, the length measurements had mostly values with decimals. Similarly, the weight in kilograms of most boxes had decimal values. To find volume, and mass $\times$ volume, students needed to multiply multi-digit numbers with decimals. Students' equations and the teacher's equation contained division, $\frac{I}{MV}$ and $\frac{MV}{I}$, respectively. This prompted several conversations with regards to division, such as how division affected the final score, how to divide with decimals, how fraction notation relates to division notation, and to consider order of operations when typing in values using a calculator.

When students created their own models to determine a final score and when they used the final equation to evaluate a final score, they addressed many sixth-grade standards for expressions and equations. The students used variables to represent numbers and wrote equations. When trying to determine if their equations were useful, students used trial values for the variables to see if the equations made sense. Once introduced to the final model, they again used test values for the variables to determine if a good score was represented by a large or small value. One group wanted to know if

they would get a better score by crushing their chip, resulting in a very small volume and mass for the box, but sacrificing their chip score. The students also estimated the values for their crushed-chip scenario and intact-chip scenario. After testing the values in the equation, they determined that crushing their chip would have been valuable for their model ($I - V - M$) but not for the final model ($\frac{I}{VM}$).

Due to the intentional structure of the task and requirements of the Pringles Challenge created by the teachers, students instigated implementation of multiple mathematical standards and ideas independently rather than in response to direct teacher instruction. Amy anticipated that students would apply standards to convert units, multiply and divide decimals, and calculate volume. In addition, students introduced and considered other standards that Amy had not articulated as an expectation, such as rounding decimals, explaining the meaning of division in context, and noting how cubes relate to rectangular prisms. Throughout the lesson it was clear to Amy that students needed additional explanation, scaffolding, or practice of the mathematical ideas they introduced. In response to the student needs that arose, Amy brought her students to the classroom carpet for conversation and short instructional periods similar to the mini-lessons that she taught in her typical mathematics lessons. For instance, Amy used the mini-lesson format to discuss multiplication with decimals. During these mini-lessons, Amy would help her students understand and explain mathematical concepts that they had introduced.

Analysis across Cases: Mathematics within the Modeling Task

In this section, I identify questions that students addressed in each task to answer their overall question and describe the teachers' classroom experiences in teaching their modeling task. Then, I compare the primary mathematical ideas used to address each task. Finally, in comparing across all cases, I describe how students generally introduced the mathematics in the tasks and explain the negative cases. Table 12 summarizes when the task was taught, the standards addressed, and the experience the teachers had with their respective tasks.

Table 12: Task information for each teacher.

Teacher Task	Number of Days	Number of Questions	Month Implemented	Experience Teaching Task	Major mathematics standards
Lindsay	4	1	November	Taught similar task	GAISE framework
Rebecca	1	1	September, late	Taught similar task	Counting Repeated addition Comparing numbers
Erica	6	2	September, mid	New task	GAISE framework Repeated addition Decimals Multiplication
Amy	7	3 + 1 (Intro)	February	2 nd year with task	Calculate volume Multiplication with decimals Write and evaluate equations Quantitatively describe variables Convert units

Recall that the first-grade modeling tasks addressed one question. The third and fifth-grade tasks grew to be more complex, asking two or four questions. Table 13 summarizes the overarching and supporting questions for each task examined in this study.

Table 13. Overarching and supporting questions for each teachers' task

Teacher	Overarching Question	Supporting Questions
Lindsay	What is the best book nook for our class?	-
Rebecca	How many crackers does each student get?	-
Erica	What should the university professor purchase for the class community lunch?	What is the best food to order? What is the food order and total cost?
Amy	How do we evaluate the Pringles Challenge and create a winning box?	How to get a quantitative variable describing the intactness of a Pringles chip? How to get an equation to determine a final score using mass, volume, and chip intactness as variables? How to design, refine, and evaluate a box using the equation? + Introductory Questions

Teacher's Experience with the Task. During this study, Lindsay and Rebecca implemented new modeling tasks in their classroom; however, they had implemented similar tasks the previous year. Lindsay's Book Nook task, which asked students to determine the most popular book nook object, was similar to the Fall Festival task, where students determined the most popular game at the school's fall festival. In both tasks students used surveys to find the class's favorite item. Rebecca's Ritz Cracker task was similar to the Candy task, where students were asked to determine how many candies should be given to each student.

Amy had taught the Pringles Challenge task the year prior, and because she taught two fifth-grade mathematics classes, she had experience with two groups addressing the task. To describe anticipated students' strategies to approach problems, questions, or difficulties they might encounter, Amy would describe students' approaches from the previous year.

While Lindsay, Rebecca, and Amy had taught modeling tasks similar to the tasks in this study, Erica was in a different situation in that the task was new, she was teaching a new grade, and the timing of her task was very early in the school year. She was familiar with asking a "What is the most popular item?" question to her first-grade class the year before, which likely prepared her to anticipate how students might address the first question of her task. But she didn't have experience with the second question, "What is the food order and total cost?" Additionally, because Erica was new to teaching third grade, her experience with her students' mathematical understanding, skills, and experiences was limited. Lastly, Erica began the task in the second week of school, so her students had less experience having mathematical conversations and classroom norms had not been fully established. This variation in Erica's experience is valuable to this case study because it presents an opportunity to examine how a teacher addresses uncertainty due to lack of experience and limited knowledge of her grade and class.

Curricular Mathematics of the Tasks. Some questions addressed in the modeling task were mathematical in nature and others were statistical in nature. The questions that led to statistical investigation were questions that asked, "What is the most popular item?" The popular item scenarios were addressed in Lindsay and Erica's classes. These

questions (Book Nook and lunch order in the Community Lunch) were statistical in nature and students addressed standards outlined in the GAISE framework: formulate questions, collect data, analyze data, and interpret results. Neither Lindsay nor Erica discussed the GAISE report when asked what standards their tasks addressed. Instead, they stated that students would learn to create surveys.

For both tasks, students learned to carefully collect and record information – in both classes students found that they needed to record what each classmate wanted. Students in both classes learned that if they only recorded a vote for an item (either the best place to read or the best food for lunch) then they did not know whom they had surveyed. Some students in Lindsay’s class found that if they only asked each student on the class roster for their favorite book nook without recording the results then they didn’t know the number of votes. For the analyze data component of the GAISE, both classes worked to make sense of results. Groups in Lindsay’s class had different results but the students recognized that because every student was choosing one of four items, each group should have the same results. Erica’s class had totals of votes for the entrees and sides that did not sum to the number of the students in the class. When students in both classes interpreted results, they did so within the context of the problem.

While students in both Lindsay’s and Erica’s class worked through the components of the GAISE framework, there were several differences in how the teachers taught the tasks. In first grade, Lindsay separated the task into two parts; first, finding the book nook options and second, surveying students about the same options. In third grade, Erica let her students find the food options and survey students at the same time. As a

result, each group in Erica's class had a different survey and the class needed to decide how to work with each group's results. Also, Lindsay provided her students with many tools (graph paper, paper, grid rosters, etc.) to begin the task. Erica waited to give her students tools until they identified the need for a tool, like the class roster. Erica's approach allowed more independent work for her students, and as a result, this process took longer. Her students went through the surveying process three times; first to find foods people liked or were allergic to, then to survey without a class roster, and finally to survey with a class roster.

Introduction of Mathematical Ideas. Students introduced most of the mathematical ideas used to investigate the mathematical modeling tasks. In the following section, I describe how the students in each class introduced mathematical ideas and the few cases where teachers introduced mathematical ideas.

In the Book Nook task, when Lindsay instructed her whole class, she used her time primarily to direct her students to the problem (that they needed to determine what was best for the class, not just for an individual), but she did not discuss mathematical ideas. It was her students who decided to survey the class. The students decided what question to ask their classmates, how to use the data to answer the question, and to compare their results with other groups in the class. Working with groups, Lindsay did talk with her students about mathematical ideas, but she only *responded* to the mathematical ideas that they had. The analysis of how Lindsay responded to student's mathematical ideas is described later. In the last lesson, a lesson I did not directly observe, Lindsay used mathematics that was much beyond the scope of first grade in

creating an equation using linear combinations, which mathematically expressed her students' ideas of valuing (or weighting) certain qualities of a book nook more than others. Lindsay's students then added the points together to determine the best book nook item.

Rebecca introduced very little mathematics in her class. She began the Ritz Cracker task by asking a question in the real world and asking her students if they could use mathematics to help her answer the question. One student offered a solution and other students used mathematics to explain how to determine if that solution worked. Students then worked in groups using more mathematical ideas. Afterwards, groups presented their solutions to the class with mathematical explanation and justification. During these three periods, Rebecca talked about the mathematical ideas with her students. The only instance where Rebecca introduced a mathematical idea was in suggesting that students use a doubling strategy to find the number of crackers needed for all students in the class. Rebecca said to a student, "One pack is two people, so two packs is what?" The student responded, "Four people. Or we can count by twos! 2, 4, 6, 8, ...". Even in this instance where Rebecca introduced a mathematical idea to a student, the student chose to use a different idea that made better sense to him.

For parts of the Community Lunch task, Erica gave her students quite a bit of freedom and time to address the problem and introduce mathematical ideas. Her students developed surveys and found the need to revise their process. Erica helped her students solve problems they found. For instance, one group identified that they did not know if they had surveyed every student in the class or if they asked some students twice and

Erica pressed the group to think about a tool that would help them. She tried to get them to think about a class roster, but ultimately introduce it herself as a tool that would help. When students began calculating the food order and its cost, many began by using their own mathematical ideas of counting, adding, and multiplying. While students did introduce many mathematical ideas, Erica also introduced mathematical ideas and directed the mathematics for some of her students. An example of directing the mathematics occurred after students had surveyed the class for a third time and in group discussion one student suggested that they vote as a class. Erica led the process of voting for the favorite food. Once Chinese food was picked, Erica told the class they would vote in the same manner as they voted earlier and led the process of voting for the types of entrees and sides to order. On the second day, when the students worked to calculate the food order and its' cost, Erica gave groups a lot more support than she had the previous day. She specifically told some groups what to type into their calculator.

Amy's lessons were structured such that each day had a clear objective. For instance, on the second day, students knew that they needed to draw a box design, determine the materials they would use, and estimate the volume and the mass of the box (see Figure 23). While students worked, Amy did not introduce mathematical ideas. Students introduced many mathematical ideas, leading Amy to conduct mini-lessons where she helped to explain mathematics or give students some guided practice, as she did with using the given equation to practice multiplying decimals and dividing decimals - but only after students realized that they needed the mathematical skills. The only time that Amy introduced mathematical ideas was when she introduced the task of needing to

mathematize the Pringles chip. She told her students that to mathematize the chip, they needed to describe the best chip and the worst chip. Using her students' descriptions, she wrote qualities of the best Pringles chip and the worst Pringles chip on the board (See figure 22). Then she told her students to get into groups and to mathematize the chip intactness score.

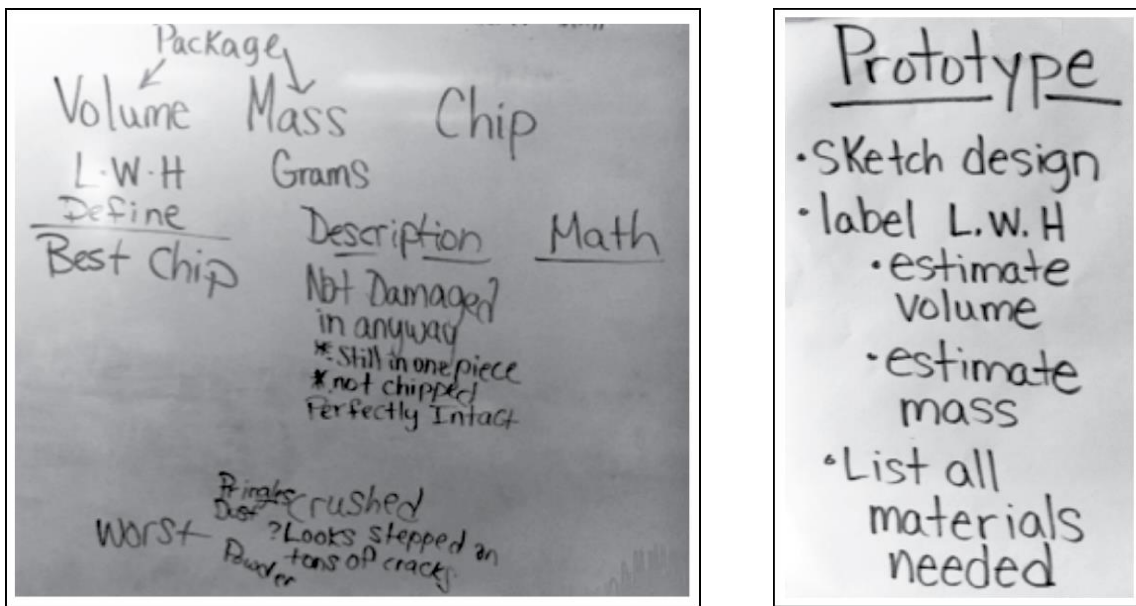


Figure 22: Amy's directions for her class.

Teacher's Interactions with Student's Mathematical Ideas

Analysis through the second lens, teachers' interactions with students' mathematical ideas, is described in this section. The Teaching Framework for Modeling proposes a teaching cycle for teachers that runs concurrently while students engage in the modeling cycle and gives a general description of how teachers interact with their students through the modeling process (M. A. Carlson et al., 2016). I intentionally did not use this framework as an initial analytical tool because the primary interest of my

analysis is the *mathematical* interactions. Instead I looked at teachers' practices from a mathematical perspective. But I used it as a secondary tool because the teacher's enacted lessons followed the steps proposed by the framework.

For this lens of analysis, the unit of analysis is the mathematical occurrence. Mathematical occurrences are defined as actions, or a collection of actions, in which students or the teacher introduce or discuss a mathematical strategy or idea (Leatham et al., 2015). As described in chapter three, I documented teacher actions related to each mathematical occurrence. Finding common themes for types of interactions, I created a graphic analysis tool by adding teacher actions to the timeline, using symbols to indicate specific teaching practices. These included how teachers responded to students' written mathematical work, such as prompting students to share ideas to the whole class, and how teachers questioned students. Also notated were actions teachers took to encourage, generate, and connect students' mathematical work.

Lindsay – Book Nook Task

The analysis of Lindsay's interaction with students' mathematical ideas is based only on the lesson I observed (the third lesson of a four-lesson series). In this lesson, there was little whole-class discussion of mathematical ideas – most mathematical ideas were discussed in small groups. Lindsay did report that she used the work they turned in to begin class the next day.

Lindsay regrouped the class part way through the lesson after a few groups explained that they found a problem; groups had different results. The presenting student explained the problem and proposed that the reason for the survey variation was due to

people changing their votes while tallies were being collected. The student suggested that they re-survey and that their classmates choose their favorite item and not change their vote. While this student presented, Lindsay repeated and rephrased the question and idea and asked the rest of the class what they thought. She asked the class if they had ideas about how to revise their work and compared their ideas to other voting activities students had done. After summarizing the problem Lindsay sent them to continue their group work.

Much of the class period was spent with students working collaboratively in groups. Lindsay used some of the independent group work time to observe her students and take mental note of their mathematical work – five times during the lesson she walked over to me and pointed out examples of data collection that her students were doing. “He is walking around with his paper asking people and crossing them off (the class roster), but not recording it...” was one observation that she made to me regarding one student’s process. From her comments, it was clear that Lindsay noted how students used their materials, how they surveyed and recorded information, and errors that might occur.

When Lindsay was not observing students working in groups, she interacted with the students by asking questions about their mathematical ideas. She asked many “How do you know?” questions such as, “How do you know you have everyone’s vote?” and “How do you know what they wanted?” Students explained their data collection process and referred to their data to explain which item won as the most popular book nook.

To prompt students to revise their work Lindsay asked questions that they had difficulty answering. An example of Lindsay's questioning tactics occurred when Lindsay noticed that one group was not recording their classmates' votes but were simply asking other students for their preference of type of Book Nook. Without explicitly telling the students their error, she posed questions so that students would hopefully discover a problem in their proposed solution. When asked, "How do you know what they wanted?" one group could not answer her questions so Lindsay then asked, "How can we keep track of what people want, so we can remember?" This prompted the group to draw the best book nook item next to each name of the class roster as their means of recording data.

Another questioning technique Lindsay used was in response to students' questions; she acknowledged their questions and responded with a question of her own, usually asking them what they thought. In one instance, students questioned why two groups had different solutions. To this she responded, "Oh boy, you do have different numbers..." and later, "What can we do about this?" As a result of her questioning, the groups hypothesized that other students were changing their votes. Students gave several different ideas about how to solve the problem and ultimately decided to resurvey the class.

Most of Lindsay's interactions with students' mathematical ideas occurred through interactions with students working in groups. For this lesson, Lindsay made very few ideas public to the class. From her reports, however, a whole-class conversation ensued in the lessons leading to this lesson and the concluding lesson.

Rebecca – Ritz Cracker Task

The analysis of Rebecca's interaction with students examines the way she interacted with mathematical ideas during three specific stages of the class's modeling task – first during the whole-class introductory discussion, then in group work, and finally in whole-class final presentations.

In the whole-class introductory discussion, Rebecca focused students' mathematical ideas on investigating one student's suggestion: that the teacher give one pack of crackers to each student. Early in the lesson Rebecca asked open-ended questions with more than one solution for which the students also needed to justify their answers. This question allowed different students to propose various mathematical ideas. One open-ended question she asked was, "How can we figure out if we have enough to give one to each person?" In response to this question, students gave several different mathematical suggestions. Rebecca also asked less open-ended questions which helped focus the students' attention towards specific mathematical ideas. For example, Rebecca asked "How do we know if we have enough for everyone even if we count [the crackers]?" This spectrum of narrow to open-ended questions helped students to focus on proposed mathematical ideas.

In addition to using questions to focus students on a proposed solution, Rebecca helped to navigate towards or away from students' mathematical suggestions. After a discussion around a student's idea of counting, Rebecca said, "Let's count to check," and the students started counting "1, 2, 3...". In this case, counting by ones helped the class compare the number of students in the class with the number of packs of crackers, which

one student suggested. Another student proposed that the class count by tens. When the student proposed this idea, Rebecca stated to the class that in their mathematics class they were learning to count by tens. Rebecca then re-posed her initial question, directing the students away from the idea of counting by tens.

During group work, Rebecca moved from group to group; while she engaged with student ideas in each group, she did not work with any one group for a lengthy period of time. Her interactions with groups were largely in asking questions. Some questions appeared to be posed so that Rebecca could understand what approach groups were taking; Rebecca asked one group, “Now, is that a cracker or a pack of crackers?” when looking at their picture. She also asked the students questions that caused them to justify their work. When students shared their ideas, Rebecca commonly responded with, “How do you know?” One student had an idea for how to share the crackers but struggled to explain why her idea worked. Rebecca asked the student, “Can you draw a picture to explain your idea?” Rebecca also posed questions prompting students to clarify their ideas. Examples of her questions of this type are “How many people will share the 13 crackers?” and “How many crackers should each person get?”

When the class re-convened at the carpet to present their solutions, Rebecca had groups present one at a time. Rebecca seemed to try to have groups present in order of similar ideas; for instance, groups that suggested splitting a pack between two students presented one after the other, building upon each other’s ideas. After the groups presented, Rebecca re-phrased the groups’ idea, asked questions, and compared with

other group's ideas. With every group that presented, Rebecca repeated or re-phrased the idea to the whole class.

Rebecca used similar questioning in group presentation as she did while students worked; her questions prompted groups to clarify and justify their ideas. When one student, referring to two students sharing one pack of crackers, said that 13 is odd so they need to "take down one," Rebecca asked the student to explain what they meant. The student then explained that 12 was even, so two students could evenly share the crackers, but 13 is odd, so there will be one leftover. Another student shared, "We each get two..." Rebecca then asked, "Two what?" Rebecca continually asked students to clarify and complete their thoughts in nearly all aspects of the modeling process.

Rebecca asked the class to make connections between mathematical ideas and also compared mathematical ideas herself. After one group presented, Rebecca asked the class, "Did anyone else do it this way?" This question prompted another group to come forward with a similar method. While another group presented, Rebecca asked the group, "[Student's] idea was to use four packs, if we do it your way do we need all of the packs or most of the packs?" Questions like this oriented students to other groups' ideas and asked students to think about how two strategies relate. Rebecca also compared students' ideas while they shared. After all the groups presented, Rebecca stated "[we have] a lot of similar ideas... we can have two [crackers] and then there are more [crackers] for later, or three crackers or six and a half."

Erica – Community Lunch Task

Analysis of Erica’s interaction with student’s mathematical ideas is separated into two sections. First, I address the whole class discussion, then I analyze Erica’s interactions with students’ mathematical ideas during group work.

Whole-Class Discussions. Erica’s class had two questions for their modeling task: “What is the best food to order?” and “What is the food order and total cost?” Whole-class discussions that occurred following these two questions looked different from each other. For the first question, time was spent rotating back and forth between whole-class discussion and group work, while the second question largely was addressed in group work with a final class discussion to close the task. When asked about this, Erica gave two reasons for the different approaches; students had fewer questions and Erica was constricted by time. Erica said that she found regrouping to have a whole-class discussion beneficial when students had questions and the class needed guidance. She would pause group work to share student strategies when students were struggling or when groups had similar questions. Erica explained that with calculating the budget, “enough groups were on track,” so she did not need to interject the way she did when students surveyed. The second reason for not having more whole-class discussion was that Erica was constrained by time. Because of this, she chose to end the task with a short class discussion without students sharing their mathematical work. She said that ideally, she would have the students share their solutions and strategies to conclude the task. “In a perfect world,” she said, “I think it would have been great to have them share this [work] in some way. It does take a really long time to have them share as groups.”

In early class conversations Erica encouraged students to share their strategies and initial solutions. She wrote lists on the board making students' work and results public knowledge for the whole class to see (Figure 23). After the class generated a comprehensive list of their ideas, Erica asked the students about their process. "How did you know to include these foods?" she asked. "Did you pick school-type lunches because you knew – ?" she cut her question short when a student started explaining. "Is it okay to just go with the food that one group found?" These questions encouraged the students to explain their thinking and encouraged the environment of an open dialogue in the classroom.

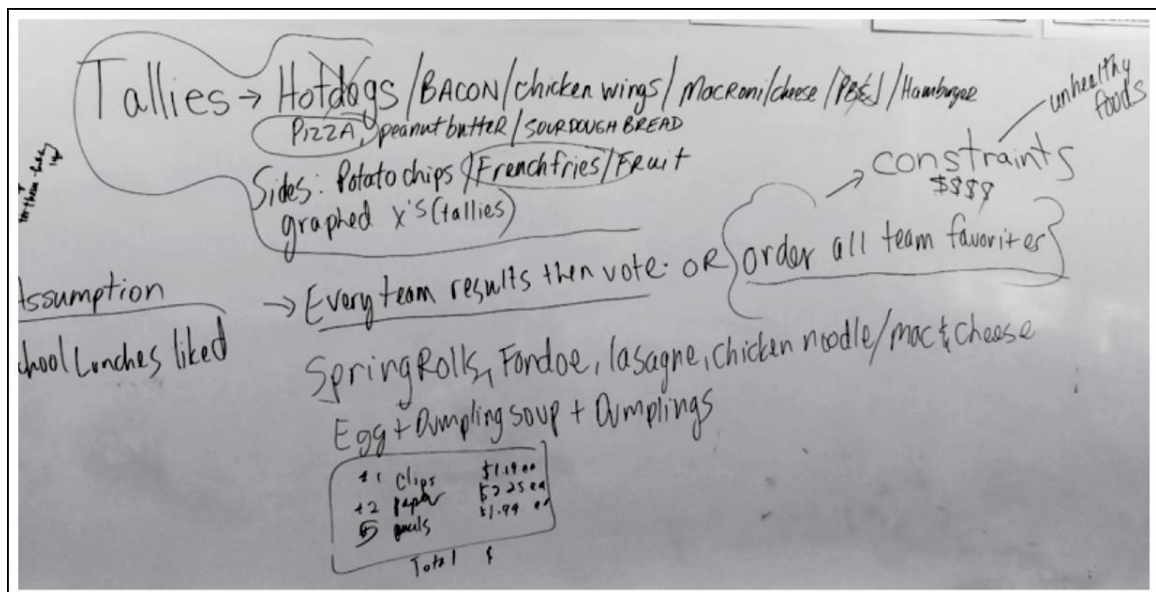


Figure 23: Comprehensive list of foods suggested by students.

Students had the freedom to create and execute their own surveys; Erica's questions asked students to describe their survey creation process and to consider how to move forward given other groups' surveys as well. In conversations towards the end of

three-day process, Erica began to direct ideas towards arriving to one strategy. One student suggested that they narrow the public list of favorite foods on the board and another student suggested that they then vote on that narrowed list to choose the class favorite. A third student offered, “Maybe pick a theme and vote on them? Like pick a restaurant and then choose food at the restaurant.” Once students had shared their ideas, Erica focused students’ work towards these three strategies. When one student suggested creating a fourth survey, Erica responded, “Like what we did before? All over again?... Won’t that take a long time?” In instances such as this, Erica did not pursue all ideas but instead redirected their attention and towards another student’s idea.

At the very end of the class conversation around determining the best food, once the class agreed to vote by raising hands, Erica transitioned to a direct teaching style and led the class through a voting process. When the class voted on a type a food, she chose the restaurant and stated that the class would vote on entrées and side dishes, again leading the voting process in selection of the foods. For this lesson, Erica said she was constrained by time. This is reflected in how she gave students freedom to design the process, but then acted to save time by leading the voting.

In the whole-class discussion regarding “What is the food order and the total cost?”, Erica gave students information so they could calculate the food order. This discussion was not about students’ mathematical ideas; instead it prepared the students to do mathematical work. To conclude, Erica did not have students share ideas, instead she shared that the groups’ results ranged from \$90-\$107 and that she found a party pack for \$99.

Group Work. As her students worked in groups, Erica used three distinct types of interaction with her students' mathematical ideas. First, she asked questions to understand their work and to elicit justification and explanation. Second, Erica helped students organize their written work. In the third type of interaction, which was much less prevalent, she directed mathematical ideas and work.

When students worked in groups creating, implementing, and analyzing surveys, Erica observed the work and asked questions. Erica often asked her students to describe and think critically about their data collection and their next steps. When students were giving surveys before they had a class list, she asked groups, "How many people are in the class?" and "How can you be sure that you had every kid in the class?" This line of questioning helped students consider organizing their work so that they knew they had a complete census of the class. After a similar line of questioning, one group told Erica that a list of names would be helpful. Erica used the moment as an opportunity to pause all students and share the group's request with the whole class. She then offered the class roster to all groups.

Erica asked her students to explain their mathematical work regarding the cost of the order. She sometimes responded with questions that helped them to clarify their work or allowed students to think about their own mathematical process. In the following dialogue, Erica asks a student to explain their thought process and the meaning of the numbers on a page of addition.

Erica: Tell me what you are doing. I see all these twos. What do the twos mean?

Student: I just did twos, I thought about twos in my brain and I added them

...

Erica: You have all those twos. What serving size are you thinking about?

Student: Small

Erica: How many smalls do you need? ... Did you count? ... I guess we need to count those up. How many twos do you have?

Student: Thirteen

In this exchange, Erica focused on having her student explain their mathematical work. In conversation with a different student, Erica talked through her student's written work and tried to help the student explain their method. In the following dialogue, Erica asked a group to share their progress. One student explained their work, the student's work is shown in Figure 24.

Student: \$10.80 for six.

...

Erica: Is that your work? So you have 12 2.70 and 13 2.70?

Student: Oh, that's white rice.

Erica: How many do you have? You have two 2.70 is 5.40. So, you have two and two, which is four all together, which is 20.60. So 20.60 equals four? I don't know what the eight is.

Student: That was 10.80. And I did 10.80 and 10.80.

Erica: So, 10.80

Student: That's 12, 'cause $6 + 6$ is 12.

Erica: This is two orders and two orders, so 10.80 equals four? Double that and 20.60 is eight. I see what you are doing... You are up to eight. How many do you need?

In this conversation, Erica tried to understand the student's method and realized that the student was doubling the price. It is interesting to note that Erica focused on having the student explain the method rather than on checking the correctness of the student's addition.

$$\begin{array}{r} 1 \\ 2.78 + 2.40 = 5.40 \\ 5.40 + 5.40 = 10.80 \\ 2 \quad 2 \end{array}$$

$$10.80 = 4$$

$$\underline{20.60} = 8$$

Figure 24: Student work for doubling.

Erica's second type of interaction with students' mathematical ideas was to help organize their mathematical work. In considering the food order and its cost, students had to consider three types of numbers: the number of sizes of an order, the number of people an order feeds, and the cost of the order. Many students needed help organizing their work because the students did not typically work with three categories of numbers and did not always label their numbers. Erica often asked students what numbers meant and then asked them to label the numbers. Figure 25 corresponds to the following dialogue in which Erica labels her students' work.

Student: Two larges is, one large plus one large equals 18.40.

Erica: What's the 6.90?

Student: That's a medium.

Erica: Can we write that down, can I just put L?

Student: Yeah.

Erica: So, two larges plus one?

Student: Medium is 23.30. Just one dollar over.

Erica: And how many kids will that feed?

Student: Twenty.

Erica: Can we write the numbers? How many kids per large?

Student: Eight.

Erica: Eight plus?

Student: Eight is 16.

Erica: 16 plus, how many does the medium feed?

Student: Four. And I should add a small.

Erica: 16 plus 4 is?

Student: 20. So I just need to add a small.

Erica: To this amount? Ok, you keep going.

Student: So, 23.30.

Erica: Is this just the orange chicken?

Student: Yeah.

Erica: Write that there so we don't get confused. You are going to add that small amount, keep going.

In this conversation, the student had written the prices for the food. Erica asked the student for the size of the order and the number of people the order feeds. The written work shows the student's addition of prices and the teacher's labels of order size and number of students. It is worth noting that Erica kept the focus on the student's strategy and on organizing the work rather than correcting the students' addition error.

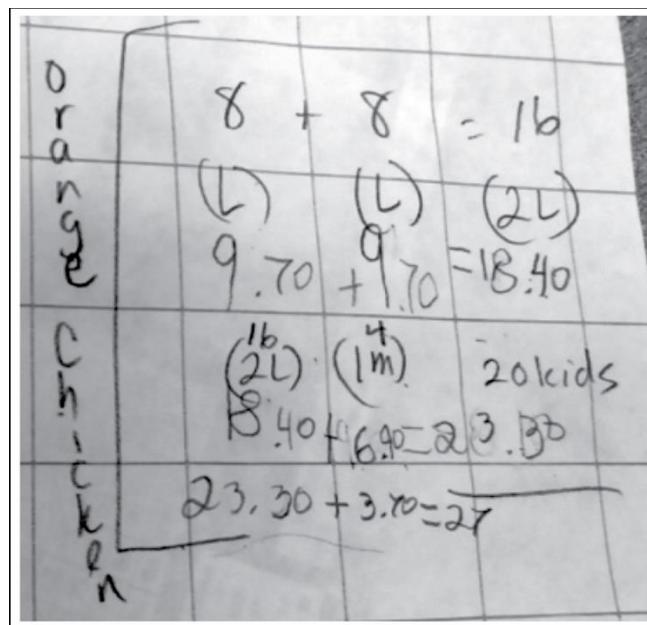


Figure 25: Student work for adding prices. Erica's handwriting is for number of people and sizes.

Initially, Erica asked her students many questions. She stated that she was trying to understand students' strategies and routes. Once she understood what the group was trying to do, she would ask guiding questions to help. At the end of the task, however, she began to more directly guide students' work and she directed mathematics in many of interactions with her students. These actions were distinct from her open-questioning because Erica either told her students what mathematics to do or asked specific one-solution questions rather than working with students' mathematical ideas. In the following dialogue, one group proposes that they order three large dishes for an entrée. Erica then tells the students that they will be ordering too much and tells them one way to add orders to feed exactly 22 students.

Erica: How many 8's could you get without going over?

Student: Eight plus eight plus eight.

Erica: How many 8's can we write without going over 24?

Student: Three.

Erica: Let's write them down.

Student: Eight plus eight.

Erica: What would that be?

Student: 12, wait, no 16.

Erica: And if we added another 8, would that give us...16, 17, 18, 19, 20, 21, 22, 23, 24. Is that more than 22?

Student: Yes.

Erica: Hmm, not going to work, hmm, how about a medium, how many does that feed?

Student: Four.

Erica: So that would give us 20. Should we add that? Then what would serve two kids?

Student: That'd be small.

Erica: So, let's write a small, put a 2.

Student: We only need 22?

In this conversation, the student did not contribute mathematical ideas for how to solve the problem, but simply added numbers per Erica's suggestions and stated the number of students an order would feed.

Erica also directed work by telling students explicitly what mathematical work they should do next. For example, when Erica saw that one group had found the price for each entrée and sides, she told the group to add the components they found and to compare that number to their budgeted amount of \$100.

Amy – the Pringles Challenge Task

Amy's teaching of the Pringles Challenge allowed students to introduce many mathematical ideas. The seven-day lesson addressed three different topics:

1. Introduction to modeling;
2. Creating and understanding how to evaluate a package which included introducing the Pringles challenge, creating a chip scoring system, and creating an evaluation equation; and
3. Building and evaluating packages.

Amy's class alternated many times between group work and whole-class discussion. Often, whole-class discussion involves sharing one or several groups' ideas. For this section, I analyzed how Amy interacted with students' mathematical ideas during group work, how Amy chose to make individual work public knowledge, and her interactions during whole-class work.

Group Work. While students worked in groups, Amy circulated the classroom, sometimes talking with the students and other times only listening to group work without inserting herself into the conversation. Amy asked her students many questions during group work; some questions were pre-determined by her and her Teacher Study Group

while others were in response to student's work. Figure 26 shows questions from the fourth day that Amy and her Teacher Study Group prepared in anticipation of the lesson.

Group Work

Monitor each group. Use questioning to guide their thinking if they are stuck.

Example Questions:

"What do you think is most important in this case? (the intactness of the chip), "then set that aside first and focus on what to do with the size."

"What will you do to the size of the package and why (add because it goes together?)?"

"Now that you have mathematized the size what will you do with that score and the intactness score?"

"How does the size of the package relate to the intactness score of the Pringle?"

"Should a bigger package have the same value as a smaller package?"

"How could you reward a smaller package based on its intactness score?"

"Does a higher or lower score determine a good package?"

Figure 26: Part of Amy's lesson plan showing questions that her TSG prepared.

Amy's questions appeared to ask students to justify their mathematical ideas, to clarify their ideas, or to guide students. An example of a line of questioning that prompted students to justify their work occurred while groups were revising their boxes. Groups had constructed boxes to hold a Pringles chip and tested the box with the evaluation equation. Then groups were instructed to revise the box and evaluate again. In the dialogue below, one group mathematically justifies why they revised their box to make the sides shorter.

Amy: Why would you cut those parts off?

Student: To make it even.

Amy: Okay, mathematically why?

Student: To make it small.

Amy: What do you want to make smaller?

Student: You want the smallest volume and smallest mass possible.

Amy often asked her students, “Mathematically, why?” in response to a posed mathematical idea or solution; this common occurrence usually elicited mathematical justification from her students.

At times during their interactions with Amy, students needed to clarify their mathematical work. She regularly asked her students what numbers on the page meant, prompting students to explain their scoring systems, equations, measurements, and conversions. When students created equations, they tested their equations with various mass, volume, and chip intactness scores. Amy suggested that the groups only vary one variable at a time and then asked groups to tell her what their numbers meant and what the change in score meant. Sometimes students could not answer these questions but after Amy’s questioning, the students were prompted to clarify their own ideas.

Student: I did another one [test of the equation with one changed variable], I made the chip 100. I just changed the chip.

Amy: Do we want the lower score or the higher score?

Student: Bigger.

Amy: Do the better packages have higher scores? Can you tell yet?

The students then indicated that they could not yet tell and continued working.

Amy also asked questions with the intent of guiding her students. At times the guiding was expected, as the lesson plan in Figure 27 shows, and other times the guiding was based off of questions that students had about their own work. The following dialogue occurred when students were measuring boxes and used the teacher-created equation to evaluate their projects. This required students to multiply to find the volume, and quite often the measurements were in centimeters with decimals.

Student A: I just don’t know how to do half.

Amy: Do you know how to multiply by half?

Student B: When we started we used this.

Amy: Is that multiplying or adding?

Student B: Adding.

Student A: I just don't know how to multiply with a half.

Amy: Do you know how to multiply with decimals?

Student A: Yes.

Amy: Can you turn this into a decimal?

Student B: So, 6.5?

Amy: What do you do with the decimal when you multiply? You multiply whole numbers.

Student A: Decide where to put the decimal.

In this case, Amy gave her students the mathematical idea of converting a fraction to a decimal, thus enabling the students to continue using their original mathematical ideas. When Amy guided her students, it was often because the students identified that they needed mathematical help with a mathematical problem.

Choose Individual Work to Make Public. While students worked in groups or individually, Amy interacted with the students by helping and asking questions. She also observed their work and looked for mathematical ideas and questions to share with the whole class. I observed Amy use different tactics towards choosing mathematical ideas to share with the class. Some of her tactics included the following:

- Find several different ideas;
- Use one rich idea to explore; and
- Have many students volunteer while asking a few individuals to volunteer.

For one re-grouping period, Amy decided to use only one group's idea to make public with the class. The class was working on creating and testing evaluation equations for the Pringles Challenge and one group created an equation, $S = V - M + I$ (volume – mass + chip intactness). Amy realized that due to the number of students needing help

with testing an equation, it was time to consider this topic as a class. Amy said that she wanted to use this equation because there were problems with the equation that the class could consider. Amy thought the equation, $S = V - M + I$, was a good attempt and the group would not mind being wrong in front of the class. In this regrouping strategy, Amy looked for an example that provided rich mathematics that the class could investigate in depth.

When students mathematized the Pringle problem, several groups struggled to define a scoring system. If a group was stuck considering only the best- and worst-case scenario, Amy would ask them to consider in-between scenarios. If they had several scenarios, Amy asked the groups how they could assign values to the numbers. One group found a scoring system much earlier than other groups, but Amy waited until three groups had an idea to share. Once she saw three different ideas, she had the three groups present to the whole class. After each group presented, she asked the class to compare differences between ideas. Amy stated that she did not want to have students share earlier because she wanted her students to know that there was not one right way to mathematize the intactness of a Pringle chip. Amy's tactic was to select a few different ideas to make public to the class to help students get ideas, but not to direct class work towards one specific idea.

After individual work time, Amy often had students share out their ideas. Individual work sometimes consisted of reflection on the modeling task, watching and reflecting on a video clip, or generating mathematical questions. For this type of work, Amy primarily observed her students' written work and did not talk with individuals

during this time. She would, however, quietly ask a few students to share their work during share-out time. Then Amy asked for volunteers to share their reflections or ideas. Students not asked to volunteer would also share their reflections. This tactic brought forth a wide variety of ideas.

Whole-Class Discussions and Mini-lessons. Each day that the class worked on the Pringles Challenge task, Amy led a mini-lesson or a whole-class discussion. Amy interacted with students' mathematical ideas similarly, whether the purpose was to teach a mathematical idea or to discuss the class's work on the modeling task. It was not clear to me whether the interactions Amy had with students was entirely pre-planned or in response to current classroom conditions. Therefore, I describe interactions in the whole-class settings together.

On several occasions, Amy displayed students' scoring systems, equations, or the teacher's scoring system or equation, and asked the students, "What do you notice?" Each time Amy began a conversation this way, she gave students time to look at the displayed work before having students share. Students responded with, "I notice..." and always commented on mathematical features of the work. Amy would ask students, "What do you notice?" only after the class spent a lesson creating their own scoring systems and equations. As a result, the features that students commented on were often differences or similarities with their own work, and many students in the class contributed in the subsequent conversations. By beginning the class discussion asking for students' mathematical observations, Amy saw their mathematical connections and questions; many ideas were brought up and Amy could focus discussion towards

particular mathematical ideas. In the instance below, Amy presented one student's solution of $\frac{VM}{I}$ and later offered $V + M - I$.

Amy: What do you notice (about $V + M - I$)?

Student A: The operations are like similar.

Student B: Inverse.

Student C: Not inverse, but close to inverse, maybe inverse.

Amy: Inverse? These are inverse. (Amy points to $\times$ and $\div$ symbols on the board.)

Student D: Division is taking away, and subtraction is taking away, and addition is adding and multiplication is adding.

Amy: Agree or disagree? Multiplication is repeated addition and division is repeated subtraction. [The students wrote], "We want a small V and M, and when we add them, if we designed a small package, we'll still get a small amount :)" And that's what some of you were saying. So, if we have a small V and M that means we have a small package. So then "If we protected the chip, we'll take more away." Is that what we are doing with this (points to division), in a sense? Are we taking more away by having a better chip?

Amy began the class conversation wanting to share one group's explanation of their equation. By sharing students' work and asking the class what they noticed, Amy was able to have students introduce mathematical ideas themselves and address the topic she hoped to address.

Throughout the modeling task, Amy asked her students to justify and clarify their work and mathematical ideas. In response to Amy's question, "Why?", students either successfully explained their mathematical thinking, continued to work so that they could answer "Why?", or revised their thinking. During the introduction to modeling, students created an equation to find the time required to fill a fish tank. One student suggested the equation $V \div 14 \times 10$:

Amy: Why times by 10?

Student A: Because the width is 10.

Student B: No, because it's 10 seconds.

Amy: 10 seconds is how long it took [in the video]. We'll find the volume of the tank, then do what?

Student C: Divide by 14.

Amy: 14?

Student C: It doesn't have to be 14, it could be 14.5.

In this exchange, the students justified why they multiplied by 10 and likely thought about the meaning of 14 or 14.5 because the original number Amy gave the class was 14.437 and the students rounded to a whole number. Later in the conversation, the equation had been modified to $V/10 \times 14$, and one student suggested simplifying the equation:

Student E: Instead of dividing by ten and multiplying by 14 you should just divide the volume by four.

Amy: Why would you do that?

Student E: You can take the 10 away because you multiply by 10.

Amy: You'll just take 10 away from 14?

Student E: Yeah.

Student F: I don't think you can do that.

After more discussion –

Amy: Can you try both ways and see what you get? Show me [your work].

I want you to try and we'll see if it works.

The student who suggested simplifying $V \div 10 \times 14$ to $V \times 4$ could not explain why the two expressions were equal Amy gave the student more time to clarify their thinking, asking the student to return later with their work.

Overall, Amy's teaching brought students' mathematical ideas to the forefront of the classroom discussions. Amy constantly repeated, re-phrased, or summarized her students' ideas. Occasionally after repeating an idea she would ask the class if they agreed or disagreed. Other times she compared groups ideas after repeating. Repeating students' ideas was a constant practice for Amy.

Analysis across Cases: Teachers' Interactions with Students' Mathematical Ideas

To analyze all four teachers' interactions with students' mathematical ideas, I used the Teaching Framework for Modeling. I use the framework for two reasons. First, the Teaching Framework for Modeling is the only document that describes the teacher's role in teaching mathematical modeling to elementary students. Second, the framework informed the development of the IMMERSION professional development, which is where the teachers learned about mathematical modeling.

In examining individual teachers' interactions, I created a graphic analysis tool which showed the mathematical ideas discussed in the whole class setting and during group work time as well as teacher actions for each teacher. To compare teachers, I made a simplified graphic analysis tool, as seen in Figure 28. The arcs represent mathematical ideas discussed with the whole class and the rectangles indicate group work time.

The Teaching Framework for Modeling states, "elementary teachers balance the progress of individual students with keeping the entire class moving forward. The teacher organizes students for engagement in the modeling cycle, monitors students as they work, and regroups the class to share developing ideas"(M. A. Carlson et al., 2016). For analysis, I considered how teachers interacted with mathematics and their students' mathematical ideas during the Organize, Monitor, and Regrouping stages as described in the Teaching Framework for Modeling. As seen in the simplified graphic analysis tool in Figure 27, most of the class time was spent either discussing mathematical ideas or in group work. This indicates that most teaching time in the classes I observed was in the

Monitor or Regrouping stage of the Teaching Framework for Modeling because very few mathematical ideas were introduced in the Organize stage.

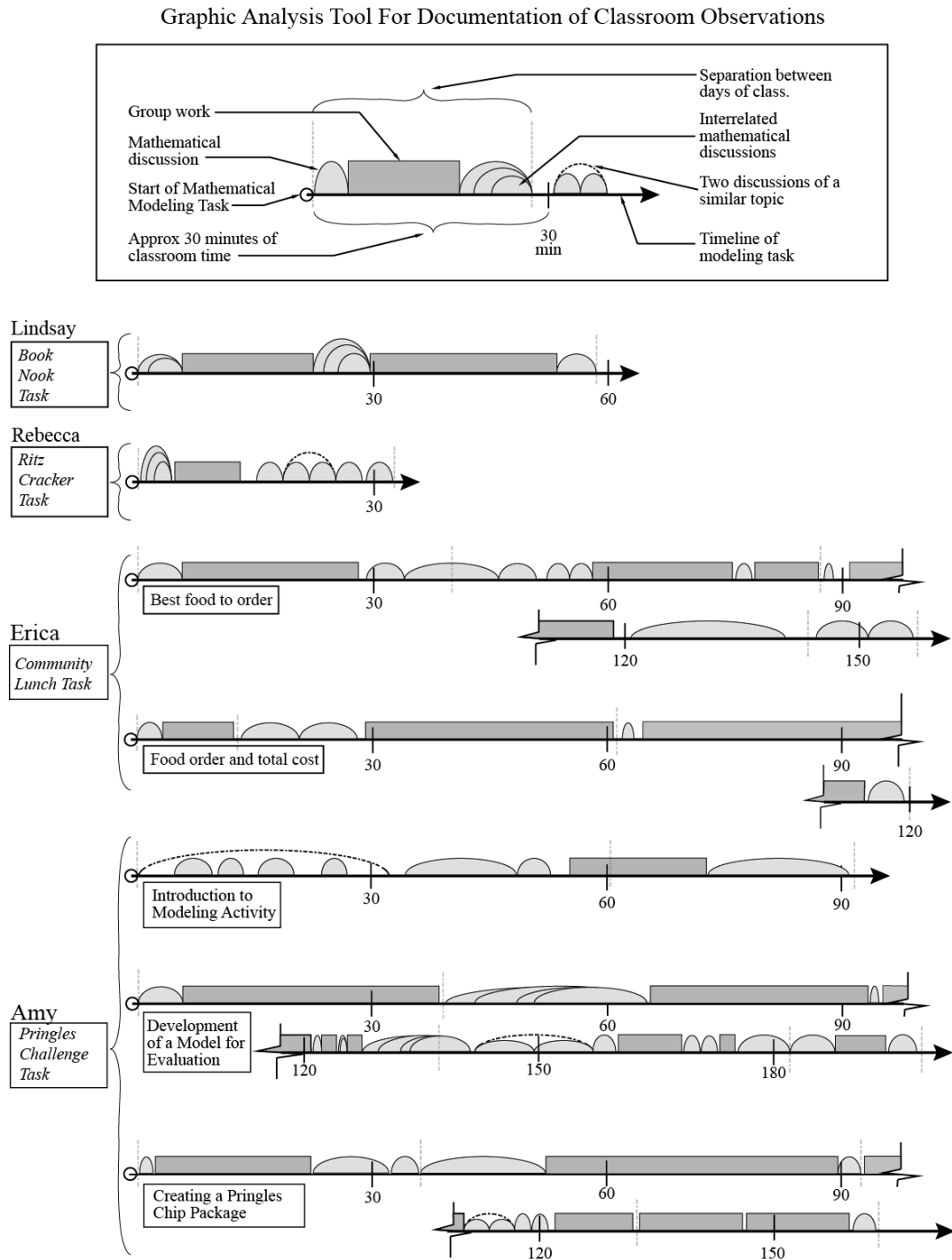


Figure 27. Graphic analysis tool illustrating the Monitor and Regroup stages for the four teachers.

In the following sections, I first analyze how teachers organized their lesson to encourage students to do mathematical work. Second, I describe how teachers interacted with students' ideas during individual or group work time and how teachers selected ideas to share with the class. Third, I share how teachers used students' ideas during whole-class discussions when the teachers regrouped students after groups worked on a topic.

Organizing Student Mathematical Work. Each teacher began the mathematical modeling task with describing the real-world problem and orienting the students towards using mathematics to solve the problem. Though the teachers prepared their students to do mathematical work, the teachers rarely introduced mathematical ideas themselves. This section contains a description of how each teacher oriented their students to use mathematical ideas and how the teachers introduced their own mathematical ideas.

Lindsay told her students that she needed to know what was best for the whole class for a book nook rather than what one individual wanted. Lindsay repeated and rephrased this question many times before the class began to solve the problem. The class immediately began surveying and using their own mathematical ideas to address the problem.

Rebecca also explained the problem regarding Ritz crackers and asked the students to use mathematics find a solution. Like Lindsay's students, their initial suggestions were mathematical in nature. Rebecca did organize the class's first discussion, guiding students to investigate the first proposal presented. Though all mathematical ideas introduced were generated by students, Rebecca's work led students

toward using their mathematics to consider one student's idea of one pack of crackers per student. By doing so, she demonstrated to the students a method of investigating one mathematical idea. Though she did not tell the students to investigate their own idea in the same manner that the class did, many students were ready to use mathematics to determine if their own idea worked after the initial class discussion.

Erica also started her lesson with thoroughly describing the community lunch task and giving information that the students wanted to answer both questions. For example, before students began determining the food order and the cost of the food order, Erica carefully organized the information that students needed. She provided the information the students requested the previous day: the number of people a size feeds, the cost of the size, and the number of students wanting each item. Once she fully described the information, students were ready to begin solving the problem with mathematics. There were other times when Erica organized student work or introduced her own mathematical ideas to the class. For example, when it was time for the class to vote on the best food, Erica directed the class voting. It was the students, however, who decided that they wanted to vote and as a class described the voting process. Erica directed students' actions to vote for not only the favorite type of food, but also for the entrees and side dishes at a restaurant of her choosing. Before the lesson began, Erica explained that she was concerned about the amount of time the lesson was taking. To handle time constraints, it appeared that Erica chose to spend class time having her students determine how to vote and then to save time by directing the voting herself.

Amy organized several conversations throughout the Pringles Challenge task around mathematical ideas that she wanted the class to investigate. She had students consider mathematizing the Pringles chip intactness by first considering the best- and worst-case scenarios before having students create scoring systems. Amy described a need for a common evaluation equation and discussed the necessary variables before the students created their own evaluation equations. Amy led the class through evaluating another school's package using the evaluation equation before having students evaluate and revise their own packages. For each of these conversations, Amy had mathematical ideas that she wanted to make public to the whole class but did not just tell the class how to use her idea. Instead Amy led a class conversation, asking guiding questions and facilitating the discussion in the general direction that she wanted. This allowed students to create and share their own mathematical ideas.

Before students worked in groups, each teacher held whole class discussions in which they set expectations of the students. This discussion did not always contain teachers' mathematical ideas; notable exceptions for Amy and Erica are described above. The expectations were clear to the students because in each case the students began investigating the problems generating and using their own mathematical ideas. These organizing conversations allowed students to work toward a common goal using mathematics to investigate their questions.

Monitoring Student Work. After teachers asked their classes to work in groups, they circulated the classroom asking their students about their work. Every teacher asked

students to share their work, to justify their mathematical ideas, to clarify their work, or asked questions to guide mathematical ideas.

In addition to questioning students, teachers also observed the mathematical work their students were doing and questions they were forming. At times, the teachers used the mathematical ideas and questions to lead a class discussion after the class regrouped. All four teachers stopped group work to share an idea from one group or student. Lindsay regrouped the class when one group found a problem with the surveying; Rebecca interrupted the class to share that a student thought that drawing would help her explain her work; Erica interrupted the class to share that a group requested a class roster to help them keep track of who had voted; Amy regrouped the class to investigate one groups' suggested equation. For each of these occurrences, teachers saw an idea that they chose to make public. Sometimes, the teacher pre-planned sharing an idea and they just needed to find a group to identify the idea. Examples of this occurred when Erica shared the class roster and when Amy shared one group's evaluation equation. Other ideas that teachers shared were not written in lesson plans and may have been decisions made in the moment.

Lindsay and Amy chose to share several selected mathematical ideas generated by groups. For a few lessons, both teachers ended class the day before with no regrouping and only asked students to turn in their work. Before the next class period, the teachers looked over groups' work and selected a few to share with the whole class. I was not present for Lindsay's sharing, but I was present when students turned in their work and Lindsay told me that she would present and compare work from a few groups to begin the

next lesson. Amy selected student work to share that had features she wanted to address as a class. Often, these were features that were present in the teachers' scoring system and equations that she wanted students to be familiar with. Selecting group work allowed Amy and Lindsay to control particular ideas made public to the class while also letting the students have ownership over the mathematics presented.

Another sharing strategy that teachers used was to not select groups to share, but to have students volunteer their work. I observed Rebecca, Erica, and Amy use this strategy. Because the teachers had already questioned and observed the groups, the volunteered ideas were not surprises.

Regrouping to Share Mathematical Ideas. Except for Lindsay, whose regrouping discussion occurred when I was not in the classroom, I observed each teacher regroup the class to share mathematical ideas and to discuss the class's next steps. For the classroom conversation, teachers interacted with students' ideas by asking questions, connecting ideas, or by choosing to not pursue all ideas shared. Teachers asked groups to justify and clarify their work; they also compared ideas that different groups introduced, sometimes asking students to compare their ideas to other groups.

Students generated many mathematical ideas throughout the modeling tasks. A common feature in all three cases was that each teacher made choices about which ideas to bring to the class in the regrouping periods. Because many mathematical ideas were generated, this meant that some mathematical ideas introduced by students in groups went unpursued. In the whole group discussion, students continued to share mathematical ideas, and the teachers helped guide the students towards investigating certain

mathematical ideas and guided students away from other mathematical ideas. Rebecca did this by acknowledging the student's idea and then asking the class a question about another student's idea, as she did when a student suggested counting by tens. During the share-out of Ritz cracker sharing ideas, one student suggested handing out goldfish crackers. Rebecca responded by saying, "Okay, so another solution is to get another snack." Once all groups presented and the class voted on the best option, Rebecca did not include goldfish as an option.

Erica's approach to redirecting mathematical ideas was to focus the conversation on other students' ideas or commenting on why the class would not pursue an idea. When the class was describing what X's and tally marks meant for their survey collection, one student said, "I love pizza so I put five tally marks!" Immediately after this comment, another student made a suggestion to which Erica asked a follow-up question. As a result, there was no conversation regarding giving extra votes to items a person likes more than other items. While Erica did not acknowledge this student's comment, she did acknowledge other ideas that she chose not to follow, such as the idea of surveying a fourth time.

Like Rebecca and Erica, Amy redirected students' mathematical ideas when guiding mathematical discussion to pursue other ideas. Several times Amy asked her students a question about an idea, then would ask to come back to their idea later. In the following dialogue, students explained why the equation $\frac{VM}{I}$ makes sense.

Student: We want the smallest amount to divide so we have to multiply small numbers, like 5×2 , mass 5 and volume 2, then it would be 10, If you have a high chip score like 50, then you'd have $10 \div 50$ would go in the negatives, it'd be really low.

Student: No, just a remainder.

Student: It's ok to have remainder if you have low number and a super good chip, you'll get a remainder.

Amy: What did you do with the remainder?

Student: It's just with the number, like if you have 2 left over then –

Amy: So, are you thinking fractions as the remainder, or decimals, or? – I want to come back to that later. Do you have it written down?

Student: Yes.

Amy initially wanted to discuss remainders and how the student calculated and represented the remainder. Then she decided to keep the focus of the conversation on why the equation $\frac{VM}{I}$ makes sense. I do not know if Amy and the student later discussed remainders, but it was not discussed as a class.

Each teacher guided the class to investigate mathematical ideas in their regrouping period. During class discussion, teachers pushed their students to justify, to clarify, and to make connections between ideas. Teachers also chose to not pursue some mathematical ideas. Because students engaged in modeling, it appears that not pursuing all ideas does not interfere with the progress of modeling.

Modeling Cycle of the Task

The first analysis section describes and explains how the students and teachers engaged with the mathematics of their task. The second analysis section documents the way that teachers interacted with their students' mathematical ideas throughout implementation of the task. This final section analyzes the specific mathematical modeling components within each task. The unit of analysis for this lens is the enacted task.

As described in the literature review, several components are fundamental to mathematical modeling. Researchers agree that the overall process of modeling starts with a real-world problem, interprets and solves the problem using mathematics, and interprets the solution back into the real world. However, researchers use different terminology and identify different steps in their unique attempts to characterize the process of mathematical modeling. From reviewing the literature, there are many distinct components that distinguish mathematical modeling from other mathematical activities and from other real-world exploration activities. Because some components, such as Determine Variables, were identified for more mathematically advanced students, and are not appropriate for elementary students. I identified eight components from the literature that were potentially appropriate for elementary grades, and used these components to assess if the activities were mathematical modeling. To analyzing the data, I used the following eight components described in the literature that were present in each of the four teachers' enacted tasks. The common modeling components for elementary grades identified in the process of modeling in the classroom are as follows:

1. Realistic context: Modeling problems are realistic and complex situations where there is a need or purpose for a product (English, 2006).
2. Problem posing: A mathematical problem is posed which indicates what the output of the model will be (Bliss, 2014). "Modeling problems require children to make sense of the situation so that they can mathematize it themselves in ways that are meaningful to them" (English, 2006, p. 305).

3. Student choice: Modeling activities require students to make judgements about the situation, these much be genuine choices (Garfunkel & Montgomery, 2016; Jablonka, 2007).
4. Mathematically informed decisions: Modeling tasks ask students to make a decision or a prediction. Students need to make decisions using mathematics to get a solution (Garfunkel & Montgomery, 2016).
5. Contextual solutions: The reporting process depends on who the model is intended for (Bliss, 2014).
6. Multiple solutions: “People who look at the same modeling problem may have different perspectives into its resolution and can certainly come up with different, valid alternative solutions” (Bliss et al., 2014, p. 5)
7. Cyclic process: Mathematical modeling is an iterative process, not a product or a checklist of steps (Bliss et al., 2014).
8. Creation of a model: Children engage with modeling situations and “develop significant mathematical constructs. [They] then extend, explore, and refine those constructs in other problem situations, leading to a generalizable model that could be used in a range of contexts” (English, 2006, p. 306).

While analyzing the mathematics used by the teachers and the students as well as the interactions between teachers and their students’ mathematical ideas, I looked for components of mathematical modeling as described in the literature. I questioned *if* and *how* a component was addressed and considered who addressed the component. If it was

not only the students who considered the component, I described how the teacher and students addressed the component or how the teacher addressed the component. Since most people doing mathematical modeling are not young children, but secondary students or adults (English, 2006), I considered how the modeling I observed compared to the modeling described in the literature. From this analysis, I determined that the eight components listed above were present in all four cases of modeling. I also determined that there was an additional component that was present in every case but not addressed in the literature:

9. Clarification Questions: Students ask mathematical questions that clarify the situation. These questions differ from the fundamental mathematical question addressed in problem posing and instead allow students to understand how the problem is quantified and to address the variables and assumptions to the problem.

The following analysis considers these nine components for each task. Descriptions of each component is presented in the appendix C. The section contains a discussion of which components were explicitly met, meaning that they were present as defined by literature, and a description of components that differ from literature. In analysis across cases I compare each component for the four modeling tasks as taught by Lindsay, Rebecca, Erica, and Amy.

Lindsay's Book Nook Task

For the modeling cycle to consider the question, "What is the best book nook?", seven of the nine components of modeling were explicitly met: Realistic Context, Ask

Clarifying Questions, Student Choice, Mathematically Informed Decisions, Contextual Solutions, and Cyclic Process. SIAM proposed guidelines for mathematical modeling for the secondary level, in which they state that the goal of problem posing “is a concise statement that explains what the model will predict” (Bliss et al., 2014, p. 12). Problem posing did occur, though not as described in the literature. Through Lindsay’s introduction of the problem the class implicitly understood the question to ask, “What is the most popular item?” It was clear that class understood a common mathematical question because once Lindsay introduced the problem, students began working toward a common goal.

For the Creation of a Model portion of the modeling process, students did create a model by using class rosters to survey, adding total votes, and identifying the item with the most votes. This is a model that can be repeated with different items and in different contexts. The other models created, $\text{dim} + \text{soft} + \text{laying down} + \text{quiet and soft}$ (4 points) + quiet (3 points) + laying down (2 points) + dim (1 point), were likely not created by the students. The equations come from teacher-reported data, and therefore the students’ role in the creation of these models is not clear.

Rebecca’s Ritz Cracker Task

Of the nine components of modeling, all were met. The Problem Posing was observed, but not in the way that some literature describes problem posing because students did not explicitly define a mathematical question. Rebecca asked, “How are we going to do snack today with all of these different containers of snack?” She told her students that she wanted them to use their mathematics to solve the problem, but

otherwise did not interpret the question as mathematical for them. Regardless, the students proceeded to answer the unstated question, “How can one equally distribute crackers from this box of Ritz crackers?” They also implicitly understood the question, “Of the presented options, which is the most popular?” because students suggested that they vote to choose from the presented options.

Erica- Community Lunch Task

Erica’s task required that students address two questions: “What is the best food to order?” and “What is the food order and the total cost?” These two questions produced two modeling cycles because each question could be considered without the investigating the other. While each component of modeling may not have been met for each question, each component was met over the course of the whole task. The components for each question are analyzed separately and presented in two separate tables (see Appendix C).

For the question “What is the best food to order?” all components were met. For the Student Choice component, students had choice and made decisions at times and at other times Erica made the decisions based on individual students’ choices. Erica decided which restaurant to order from and decided to determine the entrees and side dishes by voting to determine the best food. Students decided to vote, and it was Erica who led them through the voting process. Erica stated that she felt constrained by time that day and she likely made these choices to save time. Though Erica made these decisions and led the voting process, students still engaged in each component of the modeling process.

For the question “What is the food order and the total cost?” three of the nine components were explicitly met: Realistic Context, Multiple Solutions, and Creation of a

Model. For Problem Posing, Erica and the students worked together to pose the problem. Erica asked if they could order enough food to feed everyone and have the food order be within budget. One student said they needed to find the cost for each dish, add it together, and compare it to the budget. It was not stated that students needed to find the number of dishes for each entrée and side, but it was implied because the students began their work by finding the number of sizes required for each dish.

For the Student Choice component of modeling, both the students and the teacher made choices. Similarly, for Use Math to Make Decisions and Contextual Solutions, students could decide if their food order was within budget. Erica wanted to have her students present their solutions and to compare strategies. However, because she was constrained by time Erica told her students that their solutions ranged from \$90 -107 and the “party pack” option she found was \$99. Ultimately Erica decided for the class that they would order the party pack.

Amy’s Pringles Challenge Task

Amy’s class went through three modeling cycles or partial modeling cycles. For the first cycle, Amy wanted her students to have an introduction to mathematical modeling to prepare them to engage in the modeling cycles of the Pringles Challenge. Though this introduction was separate from the Pringles Challenge, it is analyzed because Amy considered it part of the instruction for the Pringles Challenge and stated that the Introduction to modeling helped her students to . Within the Pringles Challenge, students completed a modeling cycle to create a model that could be used as an evaluation tool and second to create a box that they would use for the Pringles Challenge. These three

cycles are analyzed separately and described in separate charts in Appendix C. While each component of modeling may not have been met for each question, each component was met over the course of the whole task.

For the Introduction to Modeling cycle, students did not fully engage in the process of mathematical modeling. While filling a fish tank with water is a real-world activity, there was no authentic problem or no purpose to consider within the activity. Because there was no purpose or context, there were no choices or decisions for students to make other than in determining the equation to calculate the time required to fill the tank. The students did not solve the problem or engage in revision. However, the activity enabled students to engage in other important aspects of modeling; they posed a mathematical problem, asked mathematical questions of a situation, considered variables important to the situation, and wrote a mathematical model to solve the question.

To create the evaluation model, each of the nine components of mathematical modeling were met. The teacher and the students worked together to Problem Pose; the students identified from the rules of the Pringles Challenge that they needed a box with a small mass, small volume, and intact chip. Amy held a discussion with her students about how to get numbers for these variables and how to “mathematize” their questions, and explained that they needed one evaluation equation. On different days, Amy told her students that their jobs for the day were to mathematize the chip or to write an equation using the four operations.

Prior to implementation of the task, the teachers implementing the Pringles Challenge determined the scoring system for chip intactness and the evaluation equation.

Amy did not communicate this to her students, choosing instead to have her students develop their own scoring formula. Instead, Amy told her students that she was meeting with the teachers and they would use the students' ideas to determine the final scoring system and evaluation equation. The students in Amy's class spent two days creating their own scoring systems and equations. Amy selected groups to share their ideas and the class considered important qualities of their scoring systems and equations that were present in the teachers' scoring system and equation. Although the students did not create the final system and equation or choose the system and equation, by the way that Amy compared student work to the final system, it was believable that the teachers considered their ideas.

When Amy's class created packages for the Pringles Challenge, this modeling cycle met all nine components of modeling. Considering the Mathematically Informed Decisions component, it is possible that students did not consider the formula to create and revise their boxes for this activity. However, Amy focused her students' attention on the equation by giving a mini-lesson on how to use and test the equation on the students' designs. When groups explained their revisions to her, she would ask, "Mathematically, why?" She directed her students to evaluate their boxes and then re-evaluate after revising their box. Groups had conversations about revising their box to lower the "V", "M", or raise "I" (volume, mass, or chip intactness) because of the equation. Through these actions, students showed they used mathematics when designing and revising their packages.

Analysis Across Cases: Components of Modeling

To analyze the components of modeling that were evident across cases, I consider each component of modeling and compare how each was addressed within each teacher's task.

Realistic Context. Each task was authentic for the students because the results of the task affected them personally, at least in theory. Lindsay's students received a book nook for reading time, Rebecca's and Erica's students ate their snack or community lunch, and Amy's students competed against other fifth-grade classes. Each task had a client who presented a purpose for their question.

Some tasks required time for students to explore the context. Lindsay's class needed to think about where they liked to read and important qualities of those places. Lindsay thought of the first two days of her lesson as an introduction, but found that students, with her help, were able to create the model, "soft + dim + quiet + laying down" to describe a good book nook. Amy's class spent over 20 minutes reading the rules of the Pringles Challenge and discussing those rules. Amy also began the task using 90 minutes to consider mathematical modeling and problem posing. The two activities that involved snacks and lunch required less time to introduce.

Problem Posing. Posing a mathematical question was the one component of mathematical modeling that was not necessarily exhibited in a way described by the literature. SIAM's handbook for mathematical modeling, intended for secondary teachers states, "You should have a concise statement that explains what the model will measure or predict" (Bliss et al., 2014, p. 14). The GAIMME report states for elementary grades,

pre-kindergarten through eighth grade, “Students (perhaps with facilitation from their teacher) [decide] what specific problem to approach” (Garfunkel & Montgomery, 2016, p. 28). The younger students in this case study did not explicitly state a mathematical problem or explain what the model would measure. However, problem posing did occur via conversation between the teacher and the students.

Lindsay and Rebecca did not pose a mathematical question for their students, but they clearly stated the real-world problem emphasizing that the students needed to use mathematics to solve the problem. English (2006) states that elementary students make sense of a situation in order to mathematize the real-world problem in a meaningful way; students in each class worked towards that goal, using mathematics to solve the problem. Amy talked with her students about the requirements of the task and ultimately gave directions and specific goals for each day in class; this style of problem posing fits the GAIMME report’s description of investigating a problem with facilitation from the teacher. In all four cases, problem posing was evidenced as an agreement about working together to solve a mathematical problem, even if the mathematical problem was not explicitly stated.

Student Choice. All students made choices for each task, even if they did not make choices for each question addressed in the modeling task. In the Pringles Challenge task, Amy’s students made meaningful choices when creating an evaluation equation and when creating a package. In the Community Lunch task, Erica made some choices for her students by directing how some groups choose their food order and what mathematics to

use, but students also made choices at that stage and clearly made choices when determining the best food.

Students made various kinds of choices; some were non-mathematical, others were semi-mathematical and some were fully mathematical. An example of non-mathematical choice was the type of food to order; students chose their preferences, but the choice still affected the outcome of the task. A semi-mathematical choice occurred when choosing tools and materials such as graph paper or class rosters. This example is semi-mathematical because the choice influenced how students collected and represented data. An example of a fully mathematical choice was using addition, counting, or partitioning packages to distribute crackers. This decision included students choosing which mathematical strategy they would use to investigate their problem. Sometimes, as with the example of choosing multiplication strategies to find the volume of a box, the mathematical choices did not affect the outcome of the task. Other mathematical choices did affect the outcome of the solution, such as deciding to use averages to calculate the height of an irregular box.

Mathematically Informed Decisions. In each task, students made meaningful decisions based on mathematical reasoning. The only occurrence where the teachers made decisions for the students was in Erica's class when determining the food order. Though Erica's students were not a part of the decision-making process for the food order, they did determine if their food order choice was within the budget and also used mathematics to make decisions when considering "What is the most popular food?"

Contextual Solutions. Students gave their answers in the real-world context: telling their teachers what food to order, how to distribute a snack, and how to design, mail and score packages. Throughout the process, students understood how their mathematics and numbers related back to the context and the problem. The first-grade students described their numbers within the context of the problem, either in terms of book nook items or in terms of crackers or tables or students. The few times they did not, the students were immediately able to define the meaning of a number when asked.

It appeared from observation of students' conversations that the students in both first-grade classes did not de-contextualize the problem such that they were working only in the mathematical world. In Erica's and Amy's classes, students' written work did not always indicate units, but when asked, the students could explain what their numbers and work meant and referred back to the original context to interpret the meaning of their work. Students may have been working in the mathematical world when calculating prices or evaluating a box, but could easily translate their work back to the real world during conversations with groups and the class.

Multiple Solutions. Every task satisfied this component of modeling. For three of the tasks the teacher did not know what the solution would be. Lindsay and Erica expected that their students would select the tent and pizza for their reading nook and lunch, respectively, but allowed for other choices. Rebecca did not know how many crackers her students would choose for their snack. Amy knew which mathematical model the teachers would choose for the Pringles scoring and evaluation, but she did not know what the winning package would look like. While there were different solutions

possible, the differences did not meaningfully affect implementation of the task for Lindsay, Rebecca, and Amy. The choice of Chinese food did impact the mathematics involved for the second part of Erica's task in choosing the food items and calculating the cost. Erica expected her students to choose pizza, which would involve finding how many slices students wanted and then determining the number of pizzas they would need to order. The students' choice of Chinese food presented additional complexity by way of different sizes for entrées and a variety of dishes.

Cyclic Process. The presence of a cycle was consistent across all cases. In Rebecca's class, the entire class considered one possible solution. Once the students determined that this option was not possible, students found different options and determined that the options were possible, and voted for the final solution. In Lindsay's class made adjustments to their data collection methods when they found problems, either from questioning from their teacher or in comparing their work to classmates'. Erica's students also made adjustments to their data collection methods to incorporate information that they learned and to correct problems that they encountered. Amy's students engaged in a cyclic process by considering classmates approaches and revising their work to incorporate new information and new ideas.

Creation of a Model. Students created a model in each class. Students in Amy's class wrote mathematical equations, though the final equation was written by the teachers. Students in Erica's class articulated a mathematical procedure that could be simplified to a mathematical equation. Since Erica's class was just beginning third grade,

it is not reasonable to expect that they would abstract this process to write an equation. Younger students followed procedures using mathematical ideas to solve their problems. The students would be able to use the same procedure if the task was altered to include additional options or a slightly different context. Erica articulated that a goal of hers was for her students to learn to conduct surveys, and that they could apply that model to other situations in the future. Each teacher presented the real-world problem in such a way that answering the question, not the creation of a model, drove the process. Though creating a model was not the focus, students could likely apply their model to a similar situation.

Ask Clarification Questions. Students in each class asked many mathematical questions throughout the task; many of the questions were related to the problem but were not fundamental questions that mathematically described the real-world question. These mathematical questions often asked for clarification, asked to quantify the situation, or asked to narrow the problem. Some of the questions that students asked considered the important variables inherent to the problem. For example, in Rebecca's class, students asked questions about the number of crackers in a pack and the number of packs in a box. Erica's students asked how many servings were in a large, medium, and small order. In their introductory task, Amy's students asked for the dimensions of the fish tank and for the number of cups in a cubic inch. These questions all identified important variables and pieces of information that were essential to solving the problems.

Literature describing mathematical modeling in secondary grades specifically includes "making assumptions" and "defining variables" as steps in the modeling process (Bliss et al., 2014). Because elementary students are not prepared to work with variables,

it is not reasonable to expect them to define the variables inherent in a modeling problem. However, through the types of questions they asked, students did address the topics of assumptions and variables. In this sense, they applied the CCSSM practice of “modeling with mathematics,” which states that “[students] are able to identify important quantities in a practical situation” (National Governors Association Center for Best Practices, 2010, p. 7). This data suggests that elementary students may identify important quantities through asking questions.

Some questions that students asked led to further revision. In Lindsay’s class, students asked, “Why are we getting different answers?” which led students to revise the process by re-surveying the class after discouraging vote changing. In Amy’s class one group asked, “Will we get a better score if we crush the chip or leave it intact?” This group found that crushing the chip led to a worse score and decided to make their box large enough to hold an intact Pringles chip.

From considering each case of mathematical modeling, this study offers empirical evidence that elementary students can engage in mathematical modeling. Each class addressed eight components of mathematical modeling that are identified in the literature. However, elementary students may not address each component as older, more mathematically mature modelers do. Finally, this study shows that elementary students ask clarification questions which may help students gain information about important variables and assumptions in the task.

The purpose of this research study was to investigate the ways that teachers and students interact with mathematics through mathematical modeling. I analyzed the data

through three distinct lenses to describe the complex interaction of teachers' and students' mathematical work throughout the process of mathematical modeling. In the following chapter, I consider the implications of this study and how the study lends insight to future work in mathematical modeling.

CHAPTER FIVE

CONCLUSIONS

Introduction

Mathematical modeling is a K-12 standard of mathematical practice in the Common Core State Standards. Proponents of mathematical modeling argue that mathematical modeling should occur in elementary school (Kaiser & Maass, 2007) because modeling helps students learn to think mathematically (Lesh & Doerr, 2003; Pollak, 2012; Zbiek & Conner, 2006). However, little research and few descriptions of teachers implementing mathematical modeling in elementary school exist (English, 2006). The distinguishing element of mathematical modeling from other mathematical activities or real-world exploration activities lies in the manner mathematics is introduced and used. Word problems and applications in mathematics curricula have idealized settings in mathematical terms, whereas modeling tasks are described in the real world and the modeler must formulate a mathematical problem (Pollak, 2012). Real-world exploration activities are often set in complex, real-world situations, but the curricular purpose of the activity is not necessarily mathematical. Because how mathematics is used distinguished modeling from other mathematical and real-world exploration activities, it is important to study the role of mathematics in cases of elementary mathematical modeling.

In response to the need for better information about modeling in elementary grades, this research examined the mathematical decisions that teachers make as well as

their interactions with students' mathematical ideas. I used a case study design to investigate four elementary teachers' implementation of mathematical modeling. The data collected from classroom observations and interviews before and after the observations focus on the mathematical decisions that teachers made and students' mathematical contributions during mathematical modeling. The analysis process consisted of studying the data from three perspectives: the mathematics of the task, teachers' interactions with students' mathematical ideas, and the components of the modeling cycle. In this chapter, I review the key results, discuss implications for teachers and researchers, and offer ideas for future research.

Purpose of the Three Lenses of Analysis

My research, given the above context, aims to answer two research questions. The first research question is supported by the second research question in asking how students' mathematical contributions and behaviors influence the implementation of a modeling lesson. Students contributions to the modeling task are strongly influenced by the nature of the modeling process and by the nature of the teachers' choices in implementation of the modeling task.

1. What mathematical decisions and choices do teachers make in the process of implementing mathematical modeling?
2. How do students' mathematical contributions and behaviors influence the implementation of a modeling lesson?

To answer the two research questions, I analyzed data through three lenses. Each of the three lenses adds a unique perspective regarding teachers' decisions and choices made in the process of implementing modeling.

The first lens addresses the mathematics of the task. Understanding the mathematics of a task is necessary to describe teachers' *mathematical* decisions and choices. The second lens of analysis examines teachers' interactions with students' mathematical ideas. This lens allows me to describe different types of mathematical decisions made by teachers while teaching mathematical modeling. The teachers' interactions with their students is important because students are the modelers in a modeling lesson, thus student contributions are necessary in mathematical modeling. Students' participation in tasks generated many mathematical ideas; this required teachers to make decisions on how to address students' mathematical ideas. Finally, analysis through the third lens considering the components of a modeling cycle ensures that the mathematical decisions teachers made in this study indeed occurred within the context of mathematical modeling.

Summary of Results with Respect to Research Questions

In this section, I summarize the key results of the study with respect to the research questions.

Research Question One: What mathematical decisions and choices do teachers make in the process of implementing mathematical modeling?

Teachers made many decisions and choices that were critical to the teaching of mathematical modeling. Each teacher chose to let students explore their task and introduce mathematical ideas. Teachers intentionally observed their students' mathematical strategies and used modeling as an opportunity to learn about their students' mathematical understanding and skills. When questions and ideas were shared in group work, teachers decided when and which student ideas to make public to the whole class. In whole class discussion, the teachers chose to pursue some mathematical ideas over other ideas. When students asked questions, the teachers made different decisions; at times, they answered student questions with guiding questions, other times they referred students to classmates who might help, and sometimes teachers chose to give mini-lessons to teach a mathematical concept. In both group conversation and whole-class discussion, teachers frequently questioned students to have them justify and clarify their work and to make connections between ideas.

The tasks are not the focus of this research study, yet the details of each task are critical to the implementation of modeling. The teachers in this study created tasks that addressed or enabled each component of the modeling process. Teachers made many decisions during the implementation of the modeling tasks including some decisions in developing the task that affected implementation of the task. Teachers chose contexts that were relevant and engaging to students. The real-world questions that teachers asked were mathematical in nature, which allowed for rich mathematical investigation. In preparing for the tasks, teachers recognized that individual students would make use of

different mathematical strategies, but knew the general mathematical opportunities afforded by the modeling tasks.

Research Question Two: How do students' mathematical contributions and behaviors influence the implementation of a modeling lesson?

Students' contributions strongly affected the implementation of the modeling task. It is critical to note that students were able to do so because the teachers in the case study encouraged student input throughout the modeling cycle. Students contributed by introducing most of the mathematical strategies that were used to investigate the modeling tasks. Students asked questions that led the teachers to clarify information and to refine their strategies and models.

The tasks became authentic modeling tasks due to students' engagement with the tasks. Again, it is important to note that this was set up by the teachers who allowed for such student participation in the modeling task. The tasks were modeling tasks because students made sense of the real-world problem and with facilitation of their teachers, mathematized the problems. Students made choices throughout the task: non-mathematical choices regarding their favorite items, semi-mathematical choices about tools and materials to use, and mathematical choices concerning the mathematical strategies. Ultimately, students made decisions based on mathematics to arrive at solutions to the tasks.

Findings

This discussion of findings summarizes the key findings, organized by analytical lenses.

The Mathematics in Mathematical Modeling

Elementary students can engage in meaningful and relevant mathematics throughout modeling tasks. In this study, students use mathematical skills and knowledge anticipated by their teachers as well as ideas or skills that teachers do not expect. Throughout the modeling tasks in this case study, students use mathematical skills they have been taught, skills they are learning in class, and skills that have not been formally introduced.

Modeling allows for variation and differentiation in mathematical approaches and skills by individual modelers while at the same time investigating a common mathematical concept. Teachers can expect individual students to use different mathematical strategies and have different mathematical ideas, but have clear expectations of the general mathematics that will be used. In this study, teachers let students explore problems and introduce their own mathematical ideas. As a result, most of the mathematical ideas were introduced by students. Though teachers may give students the freedom to explore their own mathematical ideas, they guide the task by regrouping to have students share particular mathematical ideas and questions. Teachers can emphasize certain ideas shared by students and redirect other mathematical ideas.

Teachers' Interaction with Students' Mathematical Ideas

Working within the Teaching Framework for Modeling, teachers interact with their students' mathematical ideas in different ways as they navigate the “organize, monitor, and regroup” cycle. Teachers can organize a task to have students investigate a modeling question or a portion of a modeling question using mathematics. Teachers have students work in groups to investigate and develop mathematical ideas and solutions. Noting important ideas or questions, teachers can periodically regroup the class to share an idea or question and to advance the progress of the entire class. This section describes findings for the organize, monitor, and regroup stages.

In the Organize stage, the teacher orients the students to the problem and encourages students towards thinking about using mathematics to investigate the problem. Teachers usually do not introduce mathematical ideas; instead they set clear expectations so that once group work has started, while students pursue their own mathematical ideas, each group can work towards a common goal. Though the teachers' organizational language may not be mathematical, the approach sets the students up to do mathematical work.

Teachers' work during the Monitor stage is not distinct from a teacher's work in teaching for other ambitious mathematics practices. While students work in groups, teachers intentionally observe the mathematics used by students. The teachers ask students to explain their work, push them to clarify and develop their mathematical ideas, and help to organize their work. Teachers may respond to students' questions by asking questions, directing students to another groups' work, or guiding student's work.

Teachers often use mathematical ideas and questions that students have initiated the regrouping period. In the monitoring phase, most of the mathematics comes from students.

Each Regrouping stage helps the class to move forward in a similar direction. Regrouping periods have different purposes, and thus are implemented differently at various times. Sometimes regrouping can be short, with students remaining in their groups while the teacher shares just one student's idea. Other times a class moves to a designated learning area to have a class discussion where several groups' ideas are presented or one group's idea is investigated in detail. Teachers can regroup in the middle of class, at the end of class, or have students turn in their work at the end of the day and regroup to begin the next class period. While there is diversity in how a regrouping period looks, regrouping consistently allows for students to share their ideas and questions, teachers to question the mathematical ideas that provoke students to think deeply, and teachers to focus the class's attention on important mathematical points.

Components of a Mathematical Modeling Task

Not all components of modeling can exist in a lesson plan; some components are satisfied through the implementation of modeling in student and teacher contributions. Of the components important to a mathematical modeling task, several components are inherent to the design of the task and can be determined prior to implementation:

- The task is set in a real-world context with a client that asks an authentic question,
- There are mathematical opportunities embedded in the question, and

- Solutions can be given in the real-world setting.

Other components cannot be pre-determined, but can be considered prior to implementation:

- Posing the mathematical problem,
- Choices students can make,
- How decisions can be made based on mathematics, and
- What mathematical models can be created.

These components are often considered by teachers, but cannot always be satisfied in planning. While teachers can make sure that there are choices and mathematical opportunities in the preparation phase, these components are met only through implementation of the task. Even though students are the modelers who make choices and decisions based on mathematics, teachers play an integral role in orienting their students towards considering a problem with mathematics and focusing on important mathematical ideas.

Implications

The results from this study have implications for both teachers and researchers working to implement and understand mathematical modeling in the elementary grades. This assumes that teachers and researchers value modeling and believe that it is important that students learn to model. Based on each case in this study, mathematical modeling was valuable both to the teachers and to the students. This section describes why mathematical modeling is valuable in elementary grades, describes factors that support

teachers in mathematical modeling with elementary students, and offers implications for teachers and researchers.

Why Mathematical Modeling is Valuable in Elementary Grades

Modeling enables formative assessment. Teachers in the case study learned about their students from modeling – they learned the skills students have mastered and the skills they are still learning. Because the teachers went into the tasks expecting that individual students would approach the problems differently or use different strategies, they took time to observe the mathematics that their students used. From learning about their students' mathematical understanding, some teachers stated that they tailored later mathematics lessons based on formative assessment from the modeling task.

Mathematical modeling connects classroom mathematics to the real world. The teachers confirmed that modeling helps students appreciate how mathematics helps solve problems outside of class. In Amy's class, students reflected that modeling changed how they thought about mathematics. One student wrote, "I used to think math problems had one answer but the Pringles challenge has many different solutions, some might work better than others." Erica stated that teaching modeling makes her feel like she is a better mathematics teacher because she was giving her students opportunities to learn how mathematics relates to the real world.

Students have high engagement in mathematics through mathematical modeling. Each teacher told me that her students were fully engaged in modeling and enjoyed the process. Though it was not the focus of my observations, I too observed that students were almost always working on the task and seemed excited to begin modeling each day.

Through modeling, students investigate mathematics in new ways, which allows them to make connections. A student from Amy's class wrote, "The Pringles Challenge has made me have more deeper questions about math. Because this is the first year I have constructed in math, I used to think math was just a bunch of numbers." Students' attitudes were not the focus of this study, but it was apparent that students enjoyed the modeling tasks and meaningfully developed mathematical understanding.

Factors that Support Teachers

Each teacher implemented a mathematical modeling task that engaged students in meaningful mathematics. The teachers stated that the tasks achieved their teaching objectives; they were happy with their lessons and felt that the time was well spent. There were three contributing factors for why the tasks met the teachers' goals:

- Teachers collaborated with other teachers on modeling,
- Teachers had experience teaching modeling, and
- Teachers had classroom norms around mathematical discourse.

The teachers participated in Teacher Study Groups in the 2015-2016 and/or in the 2016-2017 school year. Teachers credit their TSG with helping them develop tasks and helping to see the task from the perspective of the students. They were able to talk with their TSG about questions that arose, which was helpful particularly in multi-day tasks. These experienced IMMERSION teachers highly recommended that new teachers use their TSG as a support system.

The teachers in this study all had experience with similar lessons leading into their tasks. Amy had implemented her task previously, Lindsay and Rebecca had

implemented similar task, and the first part of Erica's task was very similar to a task she had done before. Experience appeared to help teachers anticipate students' mathematical strategies and questions, appropriate mathematical models, and questioning strategies. This knowledge helped the teachers guide their students throughout the task. Their experience helped the teachers trust that their students would develop mathematical ideas; the teachers talked about the importance of "letting go" and "trusting" that their students would use mathematics to productively explore a modeling task.

The teachers set classroom norms in their mathematics class where students were expected to communicate their mathematical ideas. Each teacher engaged their students in nearly-daily math talks. Teachers' descriptions of their typical mathematics class indicate that students talked about mathematics in partners and in whole-class discussions regularly. This is relevant because in the modeling tasks teachers expect students to work together in groups and as a class around mathematical ideas. While mathematical modeling was new to students, the mathematical discourse was not new.

Implications for Teachers

Solving the modeling task before teaching and focusing on the mathematics used to solve the task can help teachers prepare for the mathematical demands they will face in teaching the task. The teachers in this study thought carefully about the mathematics of the task prior to implementation in their class. This helped teachers support their students in using mathematics to investigate the modeling tasks. Doing the task and considering the mathematics in advance will help teachers recognize the types of choices their students can make, how different choices may affect the outcome of the task, how

mathematics allows their students to make decisions about the situation, potential questions students might have, and mathematical ideas that the teacher will want to focus on or minimize. Teachers make many decisions during the implementation of modeling; they must also take opportunities to consider decisions needed before working with students.

Since experience appears to help teachers teach mathematical modeling, teachers new to modeling ought to be patient with modeling. Lindsey said that the Book Nook task was her best modeling experience yet, indicating that a teacher's first modeling task will likely not be as successful as subsequent tasks.

Collaborating with other teachers will help teachers to develop and anticipate a mathematical modeling task. The teachers in this task attributed their success to having support from other teachers. They worked with their colleagues to brainstorm mathematical ideas in a task, anticipate student strategies and questions, and prepared guiding questions.

Teachers in this case study made different choices about partitioning their task into segments. For example, Lindsay had students first consider what qualities make a good book nook, then consider items that satisfy those qualities, then survey on an agreed set of items. In contrast, Erica had a similar task but she did not break her question into three segments; students considered lunch qualities, lunch items, and survey techniques at the same time. Though the teachers made different choices about the set-up of the task, students in each class engaged in modeling and explored mathematics. The implications of this are that teachers may make different choices that allow for modeling to occur. The

choices placed different demands on the teachers for regrouping and making sure that students were working in a similar direction. Regardless of how the teachers partitioned their tasks, they all made sure that their students thought about the variables, constraints, and choices that were a part of the task.

Implications for Researchers

Mathematical modeling in elementary grades looks different from modeling for older students or experienced mathematical modelers. This does not mean that young students do not engage in modeling or that modeling is any less valuable. Some components of mathematical modeling, such as problem posing, defining variables, asking clarification questions, or validation may look different depending on grade level and experience with modeling. The reasons that make mathematical modeling valuable are the same at all grade levels. All students can successfully learn to understand and interpret real-world problems through mathematics, all students can use their mathematical skills to address the modeling task, and all students make decisions based on mathematics. This affects researchers work by indicating that researchers should not use the exact descriptions modeling for young students as they do for mathematically mature modelers.

Researchers may have different interpretations on what defines modeling than the teachers with whom they are working. This can affect research if researchers rely on teachers to identify when they teach modeling in their classes. I asked the teachers in my case study to identify and design the modeling tasks that they would implement and allow me to observe those lessons. Amy invited me to her “introduction to modeling” lesson,

which was not fundamental to the Pringles Challenge Task, but was a part of her introduction. Lindsay, on the other hand, did not invite me to her introduction lessons for her Book Nook Task. Both teachers viewed these lessons as introductions, but one assumed that I would want to watch and the other did not. Data from this study reveals that the introduction is an important step to modeling because it builds the context and develops an agreed-upon group understanding of the problem, which may contribute to the problem posing step. Observing Lindsay's introduction would have benefited this research.

This research provides evidence that a task becomes a modeling activity by the way that it is implemented, through teachers' and students' contributions to the activity. From individual conversations with teachers I have had and reactions from teachers and researchers at national conferences, I know that both teachers and researchers want repositories of modeling activities. Researchers should be careful about what is included in a repository of modeling tasks because modeling tasks are different from many other mathematical activities. The contexts developed for the tasks in this study have authenticity or relevance that may not apply to other classrooms. Students made mathematical choices and decisions by the way that teachers set up their tasks. These qualities of the task, the context as it relates to students and students' mathematical contribution, are not typically included in a lesson plan, but that ought to be described in a repository.

Future Research

Researchers state that mathematical modeling involves moving back and forth from the real world to the mathematical world to investigate a problem (See Figure 28). I saw little evidence in the first-grade classes that students left the real world. Students applied mathematics to the real world but their problem was never abstracted or decontextualized to be in the mathematical world because they consistently described their work within the context of their problem. In fifth grade, the students were constantly asking how their work related to the real-world question, but their mathematics was more abstract. This indicated that the fifth-grade students were transitioning from the real world to the mathematical world. Is it realistic to think about modeling as moving back and forth from the real world to the mathematical world for young students? How do elementary students decontextualize modeling problems? What is the role of context for elementary students and how do teachers use context in teaching mathematical modeling? I hypothesize that in mathematical modeling, young students use mathematics within the real world, and that as students mature, they begin to decontextualize the mathematics. I propose a changing interplay between the real world and mathematics in mathematical modeling as students mature as mathematical modelers (Figure 29).

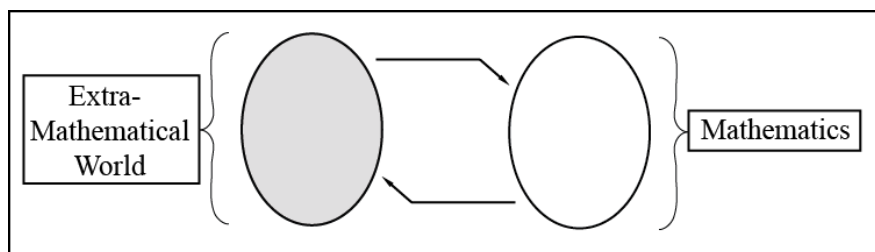


Figure 28. The interplay of mathematics and the real world in mathematical modeling (Niss et al., 2007, p. 4).

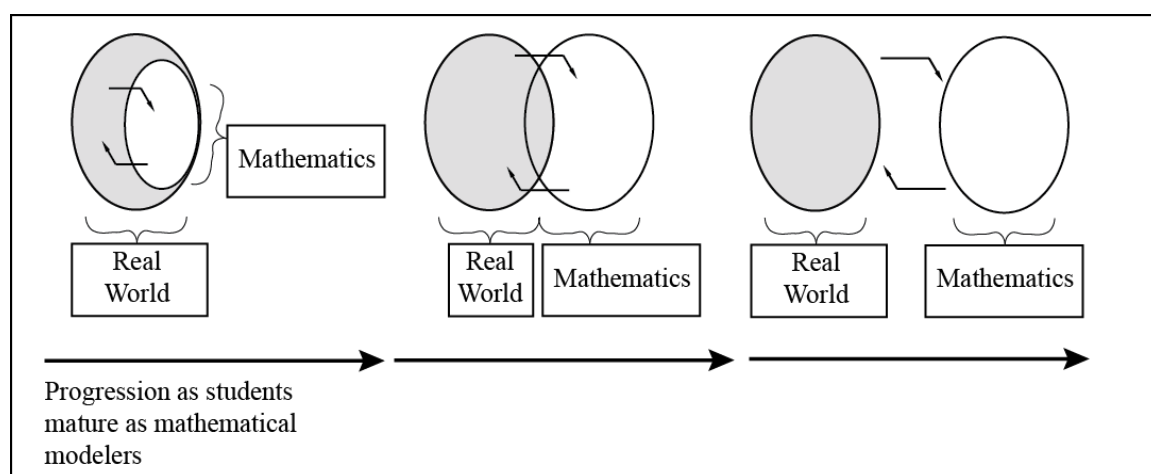


Figure 29. Proposed interplay of mathematics and the real world in mathematical modeling as students mature as mathematical modelers. Adapted from Niss, et. al (2007).

This study examined how *teachers* made mathematical decisions and interacted with students' mathematical contributions. Future research could target students. The Framework for Teaching Modeling states that modeling is important for students for three reasons: develop mathematical literacy, promote productive dispositions towards mathematics, and support deep, integrated understanding of mathematical content (M. A. Carlson et al., 2016). Though data collection did not focus on students, I observed instances of each of the above outcomes by way of student and teacher comments. One fifth-grade student wrote, "I used to think math was all separate, but now I [think] it can

be together and you have to make a lot of connections for multi-step problems.” Future research could study students with respect to any of these three reasons.

Limitations

This research studied four teachers with many similarities. Each teacher participated in the IMMERSION project, thus had a similar understanding of mathematical modeling and how to teach modeling. Each teacher spent 50 to 100 hours in professional development engaging in modeling and learning about teaching mathematical modeling. The teachers then spent additional time enacting at least three mathematical modeling tasks in their classrooms. This gave the teachers many more hours of experience thinking about modeling than their non-modeling colleagues. Finally, despite teaching different grade levels, the four teachers taught within the same school and under the same supportive administrator.

The teachers in this study learned about mathematical modeling from the IMMERSION project, which I contributed to in a variety of ways. I helped develop activities in the professional development mathematical modeling course. I also contributed to the Framework for Teaching Modeling. Because my understanding of mathematical modeling aligns with the same materials taught to the IMMERSION participants, a limitation to this study is that I, the researcher, have a very similar view of mathematical modeling as what the teachers were trying to implement.

Conclusion

Teaching mathematical modeling is an ambitious but important teaching practice. From introduction to conclusion, by its open nature, mathematical modeling presents many choices in planning and teaching. Teachers need to facilitate the initial step of mathematical interpretation of a real-world problem given its open nature. Students use multiple and varied mathematical strategies while investigating their modeling task and thus teachers need to be prepared for productive mathematical strategies and potential questions students generate in the middle of the task. Multiple reasonable solutions exist that students may present in modeling tasks.

Recognizing that teaching mathematical modeling is an ambitious and challenging endeavor, teachers can navigate through these challenges with planning and support. Teachers can work to understand how mathematics can be used to address a real-world problem. Teachers can also prepare themselves for modeling by planning and anticipating students' mathematical questions. In class, teachers can expect to pause their lessons to facilitate discussions regarding students' mathematical questions or ideas. Also, teachers can anticipate pausing their lessons to teach a mathematical skill that students have identified a need for.

Mathematical modeling in elementary grades provides deeply meaningful learning opportunities for students. Students display high levels of engagement and develop strong mathematical understandings and connections. Through the process of mathematical modeling students apply their mathematical skills to understand authentic and relevant problems tied to the real world. Mathematical modeling should be

incorporated in elementary mathematics classes because of the rich, sense-making opportunities that it provides.

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APPENDICES

APPENDIX A

OBSERVATION MEMO

Date:**Teacher:****Start Time:****End Time:****Evidence of Planned Lesson**

Describe the indicators of the teachers' planning, particularly with regards to mathematics.

How did enacted lesson reflect the teachers' description of the lesson from the pre-observation interview? How was it different?

Organization of the Lesson

How did teacher record and use students' mathematical ideas?

How did the teacher attend to students' experience, interests, and prior knowledge?

How did the teacher organize class and the lesson – by time, role, and spatially?

Mathematics

How did students interact with the mathematics in the lesson?

How did the teacher interact with the mathematics in the lesson?

How did (or did not) the mathematics in the lesson specifically relate to mathematical modeling?

Student and Teacher Interactions

How do students and teachers collaborate with each other?

How does the teacher encourage rigor, constructive criticism, and challenging of ideas?

Students Interactions

How do students interact with each other?

How do students respect their classmates' ideas, questions, and contributions?

Modeling Cycle

Where in the cycle were the students and teacher?

How did the teacher move students through steps of the cycle?

APPENDIX B

INTERVIEW PROTOCOL

Pre-Observation Interview Protocol

Interview 1

Researcher Introductions –

Have a list of the questions in front of you.

Record the interview. Give the participant control of the recorder.

The interview should last up to, but not more than 30 minutes.

Introductory comments:

- *Hello. Thanks for agreeing to participate in this interview.*
- *This interview is recorded, but you are free to ask that we stop the interview/stop recording at any time. I'm going to give you the recorder, if you want to stop the interview at any time, you are welcome to do so.*

Interview Questions

- (1) Tell me about your class this year.

- (2) Tell me about math with this group of students.
 - a. What does a typical math lesson look like?

- (3) You've been working on mathematical modeling over the past year. How might you explain mathematical modeling to an interested teacher who is unfamiliar with it? (What is your current understanding of mathematical modeling?)

- (4) Why did you choose to continue to teach mathematical modeling this year?

- (5) Let's talk about the modeling task you are planning to do with your students. Could you describe the task to me?
 - a. What is it about this task that made you choose it as a modeling task?
 - b. Tell me about how you came up with this task.
 - c. How has planning this modeling task been similar to or different from other planning you do?
 - d. What mathematical question do you hope students will investigate?
 - e. What mathematics (standards or practices) do you expect or hope students will use to investigate the question?
 - f. Are there any mathematical ideas that *you* plan bring up or teach through this task?
 - g. Talk about how the math of the task relate to material students have encountered or will encounter?

- (6) What makes this a good task for your students?
 - a. What do you think your students already know about this topic/task?
 - b. What do you expect will motivate and engage student throughout the lesson?

- c. What might students find easy? What might students find challenging?
 - d. How do you anticipate moving your students through the modeling process? (through your anticipated plans of the modeling task)
- (7) Let's get into the specifics about what might happen tomorrow when I come back to observe. What are your plans for tomorrow's lesson?
- a. What mathematical ideas do you expect to come up?
 - b. What are some of the different ways students might engage with the activities you have planned? (What different ideas or perspective might arise?) What are some different ways students may use mathematics in tomorrow's lesson?
 - c. Are there any parts of tomorrow's lesson that you feel unsure about?
- (8) Do you have any lesson plans, handouts, or other documents you have prepared for this task? Would you be willing to share them with me?
- (9) Is there anything else you want me to know about tomorrow's lesson?
- (10) Do you have any questions for me?

Subsequent Pre-Observation Interviews

Introductory comments:

Hello. Thanks again for agreeing to let me observe your class and for this interview.

With your permission I will start recording this interview. [Start recording].

This interview is recorded, but you are free to ask that we stop the interview/stop recording at any time. I'm going to give you the recorder, if you want to stop the interview at any time, you can with this button [Show how to stop the recorder].

For purposes of this recording, please state your name and class.

1. Tell me about how things are going in your class.
2. What have you been working on lately in math?
3. Let's talk what might happen tomorrow when I come back to observe. What are your plans for tomorrow's lesson?
 - a. What mathematical ideas do you expect to come up? Will you bring these ideas up if students do not?
 - b. What are some of the different ways students might engage with the activities you have planned? What are some different ways students may use mathematics in tomorrow's lesson?
 - c. Are there any parts of tomorrow's lesson that you feel unsure about?
4. Do you have any lesson plans, handouts, or other documents you have prepared for this task? Would you be willing to share them with me?
5. Is there anything else you want me to know about tomorrow's lesson?

Post-Observation Interview Protocol

Researcher –

Have list of questions in front of you and paper to take notes.

Record the interview. Give the participant control of the recorder.

The interview should last up to, but not more than, 30 minutes.

Introductory comments:

Hello. Thanks again for agreeing to let me observe your class and for this interview.

With your permission, I will start recording this interview. [Start recording]. This interview is recorded, but you are free to ask that we stop the interview/stop recording at any time. I'm going to give you the recorder, if you want to stop the interview at any time, you can with this button [Show how to stop the recorder].

For purposes of this recording, please state your name and class.

- (1) Tell me your thoughts about the lesson.
- (2) In what ways did the lesson go as planned? (How did the lesson reflect your plans and what you had anticipated?)
- (3) In what ways was the lesson different from what you had planned?
- (4) What did you learn about students' experience, interests, or prior knowledge in today's lesson?
- (5) I want to talk specifically about the mathematics of the lesson
 - a. How did students use mathematics in ways that you expected?
 - b. How did students use mathematics in ways you didn't expect? (If so) How did you respond to the idea? Will you incorporate this idea?
 - c. Are there mathematical ideas that you expected, but that weren't used? (How will the absence of the idea influence your teaching?)
- (6) I noticed the students spent time working in (pairs/groups). What did you think about how they worked together today?
 - a. Was today's group work typical of group work in your classroom? Atypical? How so?
 - b. How did students share their mathematical ideas with each other and with you?
- (7) Where there any instances where you were unsure of how to respond? Or an instance where you had to adapt your plans? How did you decide on how to proceed?
- (8) How might today's lesson influence your plans for your next modeling lesson?
- (9) Is there anything else you want me to know about today's lesson?

After the completion of the task:

1. Reflecting on the task as a whole, did your students investigate the mathematical question you hoped?
2. Did students use the mathematics you expected or hoped they would use to investigate the question.
3. What big ideas do you think that students learned through this task? How does what students did in this task relate to classroom activities?

APPENDIX C

COMPONENTS OF THE MODELING CYCLE

Modeling Components in Lindsay's Task

Lindsay taught the Book Nook task over the course of four days. I was only present for one day; The tables include teacher reported data as well as observed data. Teacher reported data is indicated by “in previous lesson” and “in the next lesson”.

Component	Description
Realistic Context	The teacher will purchase a “book nook item” for the class. The client, though not explicitly stated, is the class and the teacher. The class needed time to explore “What makes a good book nook?”, but using book nooks were a daily occurrence for students.
Problem Posing	To launch the task, the teacher told the students “I don’t just want to know what you want, I want to know what everyone wants. I need you to figure out what everyone wants.” The students and teacher do not articulate the mathematical question, though the whole class works towards answering the question, “What is the most popular book nook item?” (In the next lesson) the teacher reported that students voted to rank the importance of the reading nook qualities; dim, soft, laying down, quiet. This work answers the question, “Can we write a mathematical equation to determine the best reading nook?”
Student Choice	• (In previous lesson) choose qualities that are important to a reading nook. • Individual students choose what reading nook they want. • Groups choose which tools they will use to solve the problem. • Groups choose how to survey and record votes. • (In the next lesson) students vote on the importance of the 4 categories (dim, soft, quiet, laying down).
Mathematically Informed Decisions	• (In the previous lesson) the students/teacher used the model to determine what were good book nook items. • Groups used their survey results (counts from the survey) to determine which item they believed was the winner. • (In the next lesson) the class used the groups’ results (counts from the survey) to narrow the options to bean bag and tent. • (In the next lesson) the class voted between bean bag and tent. Bean bag received the most votes, so the class determined that it was the best item. • (In the next lesson) the class used the model created with weights “soft (4 points) + quiet (3 points) + laying down (2 points) + dim (1 point)” to determine which item was best. Bean bag got 9 points

	and tent received 6 points, so the mathematical model determined that bean bag won.
Contextual Solutions	Students determined that the bean bag was the item that the teacher should purchase.
Multiple Solutions	Any of the four items could have won. The teacher expected that the tent would win, but the class choose the bean bag.
Cyclic Process	(In the previous lesson) the class discussed qualities of a good book nook. Using a model created, the class determined four items they could purchase. Some students realized there was a problem because groups were getting different results. Groups proposed ideas to fix the problem. Groups then continued working on the same problem, but altered their methods. (In the next lesson) the class evaluated the bean bag and the tent using two different models, finding that the models produced consistent results.
Creation of a Model	In surveying the class, groups used a class roster to get every persons' vote. Students added the votes and the item with the most votes won. (In previous class) the model for a good book nook that the class should consider buying is dim + soft + laying down + quiet. If items had the four qualities then the item would be considered. (In the next lesson) the class created the model: soft (4 points) + quiet (3 points) + laying down (2 points) + dim (1 point). Students determined if each item satisfied each of the four categories. The item would receive indicated points if the quality was present. Students added the points together and the item with the most points won. They also voted between the two most popular items.
Ask Clarification Questions	Students asked the following questions: - Do I have everyone's vote? - Why do groups get different answers? - What do we need to change with the surveying process so groups get the same answer?

Modeling Components in Rebecca's Task

Component	Description
Realistic Context	Every day the teacher gives her students a snack. This day, the teacher asks students how she should hand out the snack that day. The client is the teacher and the class, but is not explicitly stated.
Problem Posing	The class did not ask an explicit mathematical question at the start of the activity. To launch the task, the teacher said, "There are these things of crackers. What I am wondering is, 'How are we going to do snack today with all of these different containers of snack?' What questions do you have about how to solve, or information that we need first? Any questions you have?"
Student Choice	• Groups choose what options to consider. For instance, some groups investigated if it worked to share a pack of crackers between two students. Another group considered how many crackers students would get if a pack was evenly distributed around their table. • Students voted, choosing their favorite option. • Students choose the mathematics to solve their problem (counting, adding, etc.).
Mathematically Informed Decisions	• Class determined that the option of one pack per student would not work because counting showed there are more students than packs. • Groups determined if their option works (if there are enough crackers) by counting and drawing crackers, students, packs, and/or tables. • Class used results from a class vote to determine which option they will use.
Contextual Solutions	• The class decided that each pair of students sharing one pack of crackers is the most popular option. • Students eat their crackers and find that there are enough crackers for everyone.
Multiple Solutions	• Students investigated four possible solutions: One pack per student, one pack per two students, two crackers per student, and three crackers per student. More solutions were possible. • Once students considered and presented, the class voted. Any of the solutions could have been chosen.
Cyclic Process	• One student proposed that each student get a pack of crackers for their snack. The class discussed if this option was possible. After

	determining that it was not possible, the teacher asked the class the question once again and the class, in groups, considered different options of how to distribute the snack.
Creation of a Model	• Students used drawing and counting to determine if their idea would feed the class. Students could use the same process to consider a different option. To pick the best option, the class voted.
Ask Clarification Questions	Students asked: • How many students in the class? • How many packs in the box? • How many crackers in a pack? • How many packs should a table get?

Modeling Components in Erica's Task

The nine components of the mathematical modeling cycle as addressed by Erica and her students when considering “What is the best food order?”

Component	Description
Realistic Context	Task introduced via video of a university professor telling the class she would purchase a community lunch for the class, and needed them to tell her what to buy and that she had a budget of \$100. The professor was the client, which the teacher stated. The teacher also emphasized that the lunch was for the class, implying that the class was also the client because they needed to be happy with their food choice because they were the ones partaking in the community lunch.
Problem Posing	Students determined that to find the best food for the class, they needed to survey the class. Groups surveyed the class using the following questions: • What foods are students allergic to or that students do not like? • What foods do students like? Once students learned about their classmates food preferences and allergies, they asked: • Of the options found, what food is most popular for our community lunch?
Student Choice	• Groups chose what to survey (favorite foods, foods people do not like, foods people prefer) for their first survey. • Groups chose what food options to put on their survey for their second and third surveys. • Groups choose how to survey and record their information. • Students choose what food they want. • Class narrows down the choices from what each group asked about to the popular items. • Teacher chose to use voting to determine what entrees and side dishes to order.
Mathematically Informed Decisions	• Students used the number of votes to choose what was the most popular food, for their survey and for the whole class vote.
Contextual Solutions	Students decided that Chinese food was the most popular food and determined which two entrée dishes they wanted and which two sides they wanted to order.

Multiple Solutions	Students considered many types of food. They could have chosen food that was not Chinese food. The teacher expected that they would choose pizza. The teacher chose what restaurant to order the food from, knowing that students wanted Chinese food.
Cyclic Process	For this cycle, the class engaged in surveying their classmates three times. First, students asked: • What types of foods their classmates like, • What their classmates did not like, • If any classmates had allergies. Second, the groups made surveys and asked students what food they wanted. Third, the teacher gave students a class roster, so groups could record who had voted. Once students finished surveying, the class took the data from each group and combined the data. The class narrowed the list of options down to the most popular items and voted on their narrowed list. The class then voted on what type of Chinese food they wanted.
Creation of a Model	Survey with a class roster to get popular foods. Then the class shared their results, narrowed the many options to the popular foods, and voted as a class.
Ask Clarifications Questions	• What food received the most votes? • What food won for the most groups? • Is the number of votes the same as the number of students?

The nine components of the mathematical modeling cycle as addressed by Erica and her students when considering “What is the food order and what does it cost?”

Component	Description
Realistic Context	The task was introduced via video of a university professor telling the class she would purchase a community lunch for the class, and needed them to tell her what to buy, and that she had a budget of \$100. The professor was the client.

Problem Posing	Erica launched the task by asking the students, “Can we order everything? Enough of everything for the budget? If I give you menus, do you have a suggested strategy of how to take the menu and get \$100?” From her question, students suggested finding the cost of each entrée and side dish, adding the costs together, and comparing the cost to \$100.
Student Choice	- Students chose what sizes to order to feed 22 students – there were different combinations of orders that feed at least 22 students. - Students chose what tools they would use and how to make calculations. One student chose to round the prices to whole numbers while the rest of the groups used exact prices that included decimals. Teacher Choices - Some groups chose sizes that feed 24 students. The teacher pushed some of these groups to choose sizes that feed exactly 22 students. - The teacher told some groups what they needed to do to add the prices.
Mathematically Informed Decisions	The students used addition to determine if there was enough food for the class and if the food they ordered fit within their budget of \$100. The teacher found a party pack after the students did their work. She compared students’ work and total price to the cost for a party pack and decided to order the party pack.
Contextual Solutions	Students determined how many orders of each size they needed and determined the total cost of their order. The teacher chose to order the party pack.
Multiple Solutions	- There were different sized combinations that would feed the whole class. - The different sized orders led to different prices. - Rounding led to different solutions.
Cyclic Process	The teacher asked groups questions about their work. For some groups, she directed their work and told them what to do next. The class could have considered the different orders that groups developed. Instead, the teacher told them about the party pack and decided that the party pack would work best for the classes food order.
Creation of a Model	The model was the cost of entrees plus the cost of sides. $Cost = 9.7 \times L + 6.7 \times M + 3.7 \times S + 25 \times 3.7$ where L is the number of large dishes, M is the number of medium dishes, and S is the number of small dishes. The equation was not written explicitly, but it was the model that students described and used.

Ask Clarifications Questions	Students asks: • How much should we order? • How many people does a large/medium/small feed? • What is the price? When some students finished determining the food order, the teacher told them that they then needed to calculate the total cost of the classroom order.
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Modeling Components in Amy's Task

The nine components of the mathematical modeling cycle as addressed by Amy and her students when considering the introduction to modeling

Component	Description
Realistic Context	Students watched a video of a fish tank being filled with water for 10 seconds. Task lacks a purpose to finding the solution and a client.
Problem Posing	Students are given time to think about mathematical questions to ask about the fish tank. They ask a variety of mathematical questions such as, "How long will it take to fill the tank?" and "How much water will it take to fill the tank?"
Student Choice	Students do not need to make judgements about the task as there is not context surrounding the answer.
Mathematically Informed Decisions	There is no decision to make in the task as there is not context surrounding the answer.
Contextual Solutions	The teacher ended the task before finding a solution.
Multiple Solutions	There was one solution. Students were allowed to round numbers. Differences in answers were due to rounding.
Cyclic Process	None
Creation of a Model	Students determined the equation to solve: $T = L \times W \times H \div 14.437 \times 10$
Ask Clarifications Questions	- How many cups are in a cubic inch? - What are the dimensions of the tank? - How fast is the water flowing into the tank?

The nine components of the mathematical modeling cycle as addressed by Amy and her students when considering the evaluation model.

Component	Present
Realistic Context	The task was introduced as a competition between fifth-grade classes in the district. The clients were the fifth-grade teachers because as a group they decided the rules and determined the scoring system and evaluation equation. The students knew they were presenting their ideas to the group of teachers who made the final decision.
Problem Posing	The Pringles Challenge rules told the students that they needed the "smallest, lightest" box to successfully mail one Pringles chip. The class interpreted this to mean the smallest mass and volume of the package and a completely intact chip post shipping.

	The teacher told the students: • Determine how to “mathematize” the chip. • Find a relationship between volume, mass, and intactness using addition, subtraction, multiplication, and/or division.
Student Choice	In creating a chip intactness scoring system, students chose: • The number of options within a scale. • The range of scale (e.g. 1 to 10, 1 to 450, etc.) • If a low number or a high number was ‘good’ In creating an equation to evaluate the box, students chose: • Mathematical operations to use. • Fake numbers to test their equation. • students chose which variable to vary as they changed one variable at a time. • Ultimately, the teachers chose the scoring system and equation. However, because students created their own and because the teacher’s system and equation shared qualities of shared student systems and equations, the chosen scoring system and equation felt authentic to the students.
Mathematically Informed Decisions	• Students evaluated different Pringle chips with their scoring system to determine if the scoring system worked and made sense. Using the results from testing reasonable variables, the students determined if their equation works and if a high score or low score determines a good box.
Contextual Solutions	• Students translated what the formulas $\frac{VM}{I}$, $\frac{I}{VM}$, and $V + M - I$ mean regarding potential boxes and Pringles chips.
Multiple Solutions	• Groups created and shared multiple scoring systems that were reasonable. • Groups created and shared multiple equations that were reasonable solutions.
Cyclic Process	• The teacher periodically had students share their ongoing work with the class. This sharing often prompted groups to re-approach the problem with a different strategy or to revise their work.
Creation of a Model	The final evaluation equation was $\frac{I}{VM}$ where I (intactness) was determined with a scoring system. The equation and scoring system was written by the teachers.
Ask Clarifications Questions	Students asked questions such as: • Why does the equation work? • What do the numbers from the equation mean? • Is the scoring system reasonable?

	Does a low score or high score predict a good box?
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The nine components of the mathematical modeling cycle as addressed by Amy and her students when creating the package.

Component	Present
Realistic Context	The task was introduced as a competition between classes. Students were motivated to create the best possible box as a shipping container for their Pringle chip because they wanted to win the competition.
Problem Posing	Students identified that they needed to create a box with a small mass and small volume and that would also reasonably protect a Pringles chip. Students asked: - Will they get a better score if they crush the chip to fit into a box with a tiny volume? - How should they revise the box to get a better score? Students were told by the teacher to calculate their predicted score. Students were then told to revise their boxes.
Student Choice	Students chose how to design their boxes, the sizes and the materials.
Mathematically Informed Decisions	- One group used the equation to determine that they should not crush the chip. - Students revised their boxes by taking away volume or by decreasing mass. - Take the average of the sides to find the height for the box.
Contextual Solutions	- Each group's solution was their box that they mailed. - The group with the box that received the highest evaluation score won the competition.
Multiple Solutions	There were many boxes created (no two were the same). Only one was a winner.
Cyclic Process	Students created a box, then after finding a predicted score then revised their box to improve their score.
Creation of a Model	A small, light box that would protect a chip. Some students created their boxes by choosing dimensions just larger than a Pringles chip. Other students began by choosing materials they believed would protect a Pringles chip.
Ask Clarifications Questions	Students asked: - What is the height of the box if one side is slightly higher than the other? - What is the best way to round?