Integrated calculus

by Michel Helfgott

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Education
Montana State University

Abstract:
This research addressed the question of whether there were differences in achievement between students that followed an integrated approach to calculus that integrated mathematics and physics, compared to students that followed a non-integrated approach.

The subjects in the study were the 151 students that completed Calculus II (second semester calculus intended mainly for engineering students) at Montana State University-Bozeman during the fall of 1996. There were a total of five sections with five different instructors. All the sections used the Harvard Calculus book and took common exams. Three sections were assigned to the experimental group, which followed the integrated method, while two sections acted as the control group.

Both groups covered the main topics of chapters 6 -10 of the Harvard Calculus book. The instructors in the experimental group stressed problems about applications to physics, as well as the conceptual and computational aspects of calculus. In addition, students in this group received enrichment notes that supplemented the textbook. The instructors in the control group also stressed the conceptual and computational aspects of calculus as well as applications to physics. However, the control group did not delve as deeply into these applications and did not have the support of the enrichment notes.

Analysis of Covariance (ANCOVA), with Calculus I scores and SAT - math scores acting as covariates, was the technique of choice to compare methods with regard to Calculus II and Physics I scores. Physics I is the first semester calculus-based physics course. ANCOVAS were also used with gender as a factor, and when students take Physics I as a factor (not yet, concurrently with Calculus II, or before Calculus II). For interaction analyses, two-way analyses of variance were employed once students were categorized into three groups according to their scores in Calculus I, SAT- math, and Calculus II.

Students in the integrated group did significantly better in Calculus II. Interaction was found when Physics I scores were analyzed, with method and SAT-math groups as factors. Students with high mathematical aptitude in the integrated group scored significantly better than students with high mathematical aptitude in the non-integrated group, when Physics I scores were analyzed. No other interactions were detected. Furthermore, there were no differences in Calculus II achievement according to when students took Physics I. No differences in achievement according to gender were found either.

On the basis of the findings of this study, an integrated approach to the teaching of second semester calculus is recommended.

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APPROVAL

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Michel Helfgott

This thesis has been read by each member of the graduate committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

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ABSTRACT

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Students in the integrated group did significantly better in Calculus II. Interaction was found when Physics I scores were analyzed, with method and SAT-math groups as factors. Students with high mathematical aptitude in the integrated group scored significantly better than students with high mathematical aptitude in the non-integrated group, when Physics I scores were analyzed. No other interactions were detected. Furthermore, there were no differences in Calculus II achievement according to when students took Physics I. No differences in achievement according to gender were found either.

On the basis of the findings of this study, an integrated approach to the teaching of second semester calculus is recommended.

CHAPTER 1

PROBLEM STATEMENT AND REVIEW OF LITERATURE

Introduction

Since calculus was created in the late seventeenth century, the teaching of it has been a matter of concern to mathematicians and educators. Probably, in no other basic mathematical subject can one find so many pedagogical difficulties. This should not be a surprise; for two hundred years (1670 - 1870) the best mathematicians of each generation struggled trying to understand the underlying structure that held calculus together as a coherent whole.

Since Newton's time, calculus has played a central role in mathematics. In its beginnings, it was intimately linked to physics, and it remained so for a long time. Problems in physics led to new mathematical theories that originated from calculus, for instance, differential equations or the calculus of variations. These mathematical theories helped to explain innumerable aspects of physics. Well into the 19th century there was almost no gap between mathematicians and physicists; quite often, their concerns were similar.

Euler's "Introductio in Analysin Infinitorum" (1748) can be considered to be the first textbook on calculus in a modern sense. One way or another, every book on this branch of mathematics published since then stems from Euler's work. The books by Leonhard Euler epitomize a whole epoch concerned with the accelerated development of the subject and its multiple applications.

At the end of the 18th century and the beginning of the 19th century, mathematicians started to analyze with care the foundations of calculus. Joseph Lagrange, and especially Louis Cauchy in the first decades of the past century, started a movement towards rigorization that culminated around 1870 with the works of Karl Weierstrass. Calculus was built around the real line, and depended solely on it.

It took a long time to put calculus on a solid foundation. No wonder that the solution to the problem of rigorization happened to be a sophisticated structure that was out of reach of most beginning college students. While students in Europe benefited from the famous collection of "Cours' d' Analyse" by Cauchy, Picard, Jordan, and other great 19th century mathematicians, teaching of calculus in America lagged far behind. With few exceptions, well into the 1950's, most calculus textbooks lacked rigor and motivation. Applications were few and scattered across the textbooks; little or no connection between mathematics and the real world could be found. Calculus was presented as a series of clever tricks and procedures.

Starting in 1957, with the launching of the Sputnik, mathematics and the natural sciences received great impetus in the United States of America. In the next two decades, new books on the calculus were published, stressing the theoretical aspects of the subject but paying little attention to the links between calculus and physics or chemistry. The pendulum swung completely in the other direction, from a lack of rigor to excessive rigor, somehow blurring the distinction between calculus and real analysis. Quite often during their first year in college, students had to deal with techniques that baffled them. Moreover, applications were relegated to the end of each chapter, many times as optional material. The net result of this way of presenting calculus was a high degree of student failure. Voices of discontent were raised among mathematicians, natural scientists, educators, and the community in general.

A conference was convened at Tulane University in 1986 to address the problem of calculus teaching. This conference is considered to be the beginning of what is now called "Reform Calculus." Radical steps were taken towards promoting a "lean and lively calculus," wherein a balance could be reached between conceptual developments and applications, and where modern technologies would be used in a pervasive way.

Several projects came to light under the inspiration of the Tulane conference. The best known is the one started by a Consortium based at Harvard University. Montana State University-Bozeman has adopted the Harvard Consortium textbook (Hughes-Hallet et al. 1994) for its two semester first year calculus sequence, intended mostly for engineering majors. This book is a radical departure from the traditional presentation since it focuses on enhancing student's understanding, and applications are more numerous than in the past.

It is an open question whether or not the introduction of substantial applications from the natural sciences, within the calculus course, has a pedagogical impact (Ferrini-Mundy and Geuther Graham 1991, p. 633). The purpose of this work is to investigate whether there is a difference in achievement between students that follow the Harvard Calculus textbook, and those students who, besides using the same textbook, receive supplemental materials and study several mathematical aspects of physics related to the course they are taking concurrently in physics.

Statement of the Problem

Is there a difference in achievement between students that follow an integrated Harvard Calculus approach, integrating mathematics and physics, compared to students that follow a Harvard Calculus non-integrated approach?

The Importance of the Study

There is ample evidence that an integrated approach to mathematics teaching, integrating the natural sciences and mathematics, is an advisable path to follow at all levels. However, most of this evidence is anecdotal. There is a need for carefully conducted research to address the question of whether or not integration of academic areas (e.g., mathematics and physics) benefits students as their proponents assert. Some research on the subject has been done at the middle school level, but little at the high school level and almost none at the college level. In particular, calculus is a subject taken by more than half a million college students every year in the USA. Thus, it is important to carefully determine whether steps toward integrating calculus with the natural sciences is a sound option or not. This study purports to give an answer to this question.

Definitions of Terms

For the purpose of this study, the researcher used the following definitions:

Calculus II: Second semester calculus, using the Harvard Consortium book. This course is intended primarily for engineering, mathematics, and the natural sciences students.

Achievement in Calculus II: Achievement in Calculus II was measured by three one-hour examinations and a comprehensive final examination.

Physics I: First semester of a three semester sequence, primarily for engineering and physical sciences students. Covers topics in mechanics.

Achievement in Physics I: Achievement in Physics I was measured by a midterm examination and a comprehensive final examination.

Calculus I: First semester calculus, using the Harvard Consortium book. Intended primarily for engineering, mathematics, and the natural sciences students.

Previous knowledge of Calculus: Previous knowledge of calculus was measured by three one-hour examinations and the final comprehensive examination in Calculus I taken at Montana State University-Bozeman the previous semester.

Harvard Calculus: A reform calculus approach developed by a consortium based at Harvard University.

Integrated Harvard Calculus Approach: This is the approach that follows the content of the Harvard Calculus textbook, and supplements it with enrichment notes that cover several mathematical aspects of mechanics and chemical kinetics that go beyond the textbook. Additionally, application problems from the book are discussed thoroughly.

Non-integrated Harvard Calculus Approach: This is the approach that follows the content of the Harvard Calculus textbook, without the enrichment notes. The non-integrated approach does not delve as deeply into the applications.

Scholastic Aptitude in Mathematics: Scholastic aptitude was measured by the mathematics portion of the SAT or ACT tests.

Interaction: "An interaction between two factors is said to exist if the mean differences among levels of factor A are not constant across levels (categories) of factor B" (Glass and Hopkins, 1996, p. 483).

Questions to be Answered

This study has attempted to answer the following questions:

1. Is there a difference in the adjusted Calculus II achievement means, between the integrated and non-integrated groups, when statistically equated with respect to previous knowledge of calculus and scholastic aptitude in mathematics?

2. In the analysis of Calculus II achievement, does method of teaching (integrated or non-integrated) interact with previous knowledge of calculus?

3. In the analysis of Calculus II achievement, does method of teaching (integrated or non-integrated) interact with scholastic aptitude in mathematics?

4. Is there a difference in the adjusted Calculus II achievement means, between students that have not yet taken Physics I, are taking Physics I concurrently with Calculus II, and took Physics I before Calculus II, when statistically equated with respect to previous knowledge of calculus and scholastic aptitude in mathematics?

5. In the analysis of Calculus II achievement, does method of teaching (integrated or non-integrated) interact with the time when students take Physics I (not yet, concurrently with calculus II, or before Calculus II)?

6. Is there a difference in the adjusted Physics I achievement means, between the integrated and non-integrated groups, when statistically equated with respect to previous knowledge of calculus and scholastic aptitude in mathematics?

7. In the analysis of Physics I achievement, does method of teaching (integrated or non-integrated) interact with previous knowledge of calculus?

8. In the analysis of Physics I achievement, does method of teaching (integrated or non-integrated) interact with scholastic aptitude in mathematics?

9. In the analysis of Physics I achievement, does method of teaching (integrated or non-integrated) interact with achievement in Calculus II?

10. Is there a difference in the adjusted Calculus II achievement means between female and male students, when statistically equated with respect to previous knowledge of calculus and scholastic aptitude in mathematics?

11. Is there a difference in the adjusted Physics I achievement means between female and male students, when statistically equated with respect to previous knowledge of calculus and scholastic aptitude in mathematics?

12. In the analysis of Calculus II achievement, does method of teaching (integrated or non-integrated) interact with gender?

13. In the analysis of Physics I achievement, does method of teaching (integrated or non-integrated) interact with gender?

Conceptual Framework

Mathematics has been used in physics for a long time, especially since Galileo Galilei established the basis of the scientific method at the beginning of the 17th century. Mathematics became "the language of science," and its use, both in science research and science instruction, steadily won ground. A natural science was considered to have achieved full maturity once it lost a merely descriptive stage, and a mathematical approach had been established. Chemistry followed the path of physics, and in our century biology started to use a mathematical approach to some degree too.

In contrast, mathematics research and instruction has drifted away, little by little, from the natural sciences. Despite the fact that physics played a very important role in the development of mathematics (calculus is a well-known example of this assertion), contemporary mathematicians - with few and scattered exceptions - have become isolated in their discipline, and have ceased to use physics in their research and instruction almost completely. It is common knowledge that the scientific method, with its stress on exploration, gathering of data, and prediction, is a tool that can be used with great profit in the learning of mathematics.

When we talk about integrating mathematics and the natural sciences in the classroom, we do not mean integration of content but of methodologies. Lynn Steen expresses this idea very clearly, when he analyzes one of the possibilities for integrating mathematics and science (Steen 1992, p. 8):

> [one such possibility is to employ] mathematical methods thoroughly in science, and scientific methods thoroughly in mathematics, coordinating both subjects sufficiently to make this feasible. This is, I submit, an ideal situation. Each discipline, science and mathematics, would accrue benefits from an infusion of methods of the other, but neither would lose its identity or distinguishing features in an artificial effort at union. There are, after all, important differences between science and mathematics, both philosophical, methodological, and historical. These should not be lost in a misguided effort at homogenization.

Further on (p. 12) the same author writes:

> Similarly, the compelling logic of inference and deduction can help the students experience the special power of science. Absent the rigorous logic of inference that is typical of mathematics, science instruction can easily degenerate into description, demonstration, and memorization. Without the intrinsic authority of inference, the authority found in science becomes extrinsic, hence heretical: students believe what teachers tell them, not what they have logically demonstrated from evidence.

The theoretical basis which underlies our study stems from Steen's thoughts, more than from any other thinker's.

Teachers of mechanics use calculus pervasively. It is time that teachers of calculus use physics and chemistry as pedagogical tools in the classroom, drawing illuminating examples from both sciences to impress upon the students the proper idea of the close links between mathematics and the natural sciences. There seems to be much to gain and nothing to lose.

In consonance with the aforementioned concepts, NCTM (1991, p. 70) recommends "connecting mathematics to other subjects and to the world outside the classroom," while AMATYC (1995, p. 16) stresses the fact that " students must have the opportunity to observe the inter-relatedness of scientific and mathematical investigations."

Review of Literature

The Precursors of Integrated Calculus

Kline (1967) was among the first in modern times to warn of the dangers of excessive rigor in the presentation of the calculus. He advocated an intuitive approach. Besides, he saw the need to use physics extensively in the calculus classes:

> The second essential respect in which this book differs from current ones is that the relationship of mathematics to science is taken seriously. The present trend to separate mathematics from science is tragic. There are chapters of mathematics that have value in and for themselves. However, the calculus divorced from applications is meaningless. We should also keep in mind that most of the students taking calculus will be scientists and engineers and these students must learn how to use mathematics. But the step from mathematics to its applications is not simple and straightforward and it creates difficulties for the student from the time he is called upon to solve verbal problems in algebra. The mathematics courses fail to teach students how to formulate physical problems mathematically. The science and engineering courses, on the other hand, assume that students know how to translate physical problems into mathematical language and how to make satisfactory idealizations. The gap between mathematics and science instruction must be filled, and we can do so to our own advantage because thereby we give meaning and motivation to the calculus (Kline 1967, p. vii).

The same author (Kline 1970) quite forcefully advocates an intuitive approach to calculus teaching, with physical arguments playing an important role. This proposal is a continuation of the criticism voiced three years earlier.

Boas (1971) called attention to the advisability of using an approach to calculus teaching similar to the one used by scientists when they teach science. Whether in physics, chemistry, or biology, teachers of science base many of their lectures on well-established experiments carried out in the past. Often these experiments are difficult to replicate or time-consuming, so scientists take them for granted in the classroom setting and continue ahead. Boas maintains that proofs are to mathematics what experiments are to the natural sciences, an analogy that can be used with great profit by teachers of mathematics:

> ... I claim that the teacher of calculus would do well to follow the lead of the experimental scientist. Let him give proofs when they are easy and justify unexpected things; let him omit tedious or difficult proofs, especially those of plausible things. Let him give easy proofs under simplified assumptions rather than complicated proofs under general hypotheses. Let him by all means give correct statements, but not necessarily the most general ones that he knows (Boas 1971, p. 666).

Despite the warnings of noted teachers as Kline and Boas, the same inclination toward rigor at the expense of understanding was followed in the seventies with regard to calculus teaching. In fact, two trends coexisted during this decade: excessive rigor and "watered-down" versions with no rigor at all. The latter trend had dominated calculus teaching for most part of the first half of this century, in concordance with the belief that the theory behind the calculus could only be understood in advanced courses on real analysis. The pendulum started to swing to the other extreme after the launching of the Sputnik when great changes in education were started. Both trends lacked links to physics or to any other natural science.

In the sixties and seventies, students were failing in calculus at an alarming rate, so several solutions to the problem were proposed. Self-pacing, discovery, collaborative learning, computers in calculus instruction, and programmed learning were among them (Teles 1992). However, none of them offered conclusive evidence as to their merits. In some cases there was a small, but not significant difference when compared to traditional calculus courses taught through the lecture format. In others it was difficult to replicate the experiments, leaving the impression that a superb teacher could do marvels with whatever approach he/she adopted.

The Calculus Reform Movement

The situation came to a crisis around 1984 when Ralston (1984) presented the idea to put calculus on a coequal role with discrete mathematics in the first two years of college. This would amount to downsizing calculus, a subject that for a long time had been the center of first and second year college mathematics. The idea was hotly disputed by many mathematicians, Daniel Kleitman and Peter D. Lax among them. Kleitman and Lax contended that calculus should continue to be the core of first year mathematics, since its methods and ideas are at the origin of some of the most impressive developments in mathematics. Lax wrote: "As to calculus, mathematicians need not less, but more of it. The real crisis is that it is badly taught, the syllabus has remained stationary, and modern points of view, especially those having to do with the role of applications and computing are poorly represented" (Ralston 1984, p. 380).

At the 1986 Tulane University conference there was a consensus with regard to the need of using modern technology in the classroom, and of striking a balance between the conceptual and computational aspects of calculus, together with significant applications. The Mathematical Association of America published the proceedings of the conference (Douglass 1986) and a continuation of the proceedings (Steen 1987), both of which have become very influential.

Several reform projects were developed after the Tulane conference. The best known, and the one that has had the greatest impact on calculus teaching, is the project started by a consortium based at Harvard University (Hughes-Hallet et al. 1994). Two basic principles guided their efforts:

1. Every topic should be presented geometrically, numerically, and algebraically.
2. Formal definitions and procedures evolve from the investigation of practical problems.

Research on the Impact of Reform Calculus

Does research support the claims of the Harvard Consortium when compared to traditional approaches? Ratay (1993) used a preliminary edition of the Harvard Consortium textbook with the class of '95 at the United States Merchant Marine Academy, while the class of '94 had used a traditional textbook. The grades of the classes of '94 and '95 were compared for each of the three quarters of their freshman year. Students were grouped according to their mathematics scholastic aptitude test (SAT-math) scores into four categories (500 - 540, 550 - 590, 600 - 640, 650 - 800 ranges), and their average grades were plotted both for the '94 and '95 classes. Results show that the class of '95 consistently outperforms the class of '94, especially for the first quarter among those in the 500 - 540 range. The mean difference is almost a full letter grade for those in the lowest aptitude group. A similar result was obtained when, instead of the SAT-math scores, the researcher used the CPT scores (CPT is an algebra examination administered to all entering freshman at the Academy). The students were divided into four groups according to their CPT scores (20 - 49, 50 - 69, 70 - 89, 90- 120 ranges) and their average grade was calculated. In summary, Ratay found that students earned higher calculus grades using the Harvard consortium book and the benefit was larger among those students with less preparation and aptitude in mathematics. It is to be noted that the work under consideration is of a preliminary nature. The fact that the groups that were compared took different examinations at different times, does diminish the validity of its conclusions. Besides, no significance tests were conducted; the results could be due to chance. Certain trends can be noticed from the graphs of Ratay's paper, but no conclusive statement can be done since no statistical tests were performed.

At Brigham Young University, three calculus programs are taught simultaneously as part of the regular curriculum (Armstrong, Garner and Wynn 1994). Two of them are reform calculus (Harvard Calculus and CUM, calculus using Mathematica), and the third is a traditional approach. The reform calculus programs are characterized as programs where essential use of technology is made, with students learning pertinent applications in an environment where teaching is innovative (group work, interactive teaching methods, et cetera). Two early evaluations compared grades of students in several courses with calculus as a prerequisite. Grades of former calculus students in courses in nine areas, namely biology, chemical engineering, chemistry, civil engineering, electrical engineering, electrical engineering technology, mathematics, physics, and statistics were surveyed.

The first study concluded that Harvard Calculus students did better in six of the nine areas, CUM students did better in mathematics, while traditional calculus students obtained the highest grades in electrical engineering and in statistics courses. However, only the better achievement of the CUM students in subsequent mathematics courses was statistically significant (0.05 level). The second study comprised more students (the authors do not specify the number of students involved in each study). Again, Harvard Calculus students did best in the same six areas, CUM students did best in mathematics and electrical engineering courses, while traditional calculus students did best in statistics. None of the differences were statistically significant.

A third study was also conducted at Brigham Young University, considering calculus grades and ACT (American College Testing) scores. Neither student selection strategies nor instructor differences were taken into consideration. Besides, the CUM group was very small compared to those in the other two programs. No statistical differences in grade point averages in subsequent courses (linear algebra, multidimensional calculus, engineering mathematics, mathematical statistics, and two principles of physics courses) were found.

> While these statistics were not definitive, we were happy to find that reformed calculus students did not do worse statistically than traditional students in any of the courses surveyed. This is so despite the fact that subsequent courses depend upon traditional calculus information. Also, such surveys do not account for other factors making reformed calculus more advantageous, such as more positive student and instructor attitudes, better mastery of concepts and applications, benefits obtained through group study, and advantages to students due to their increased technological expertise (Armstrong, Garner and Wynn 1994, p. 309).

A secondary finding was reported: Students who took second semester traditional calculus after having taken Harvard Calculus the first semester, suffered a significant drop in grades (0.05 alpha level).

Kerry Johnson conducted a four-semester study at Oklahoma State University, comparing Harvard calculus with traditional Calculus (Johnson 1995). Answers were sought to the following questions:

1. Do Harvard students get better grades in calculus than traditional calculus students?
2. Are Harvard students more likely to enroll in subsequent mathematics courses?
3. Do Harvard students perform better in subsequent mathematics courses than other students?
4. How do students that go from Harvard Calculus 1 into Traditional Calculus 2 perform?

With regard to the first question, the answer found is that a higher percentage of the Harvard Calculus students pass the course and make a C or better in the course than traditional calculus students (67% vs. 62% in Calculus 1, 80% vs. 71% in Calculus 2.) There was a varied response to the second question, depending on the course. For example, among students who got a D or better in Calculus 1, 63% of Harvard Calculus students took Calculus 2 compared to 56% of traditional calculus students that took

Calculus 2. Enrollments in Differential Equations were 36% Harvard, 33% traditional, while in Linear Algebra it was Harvard 20%, traditional 27%. With regard to the third question, Johnson found that the answer is no. For example, in Differential Equations, 50% of the D or better Calculus 2 students maintained or improved their grades, while 58% of the traditional calculus students maintained or improved their grades. In Linear Algebra, the difference was Harvard 60% vs. Traditional 69%. Both Differential Equations and Linear Algebra are traditional courses. Finally, as one might expect, it is not advisable to go from Calculus 1 - Harvard Calculus into Calculus 2 - Traditional Calculus. Only 55.3% of these students made a C or better in Calculus 2 compared to more than 80% in the other three possible combinations (Calculus 1 - traditional into calculus 2 - traditional, Calculus 1 - traditional into Calculus 2 - Harvard, and Calculus 1 - Harvard into Calculus 2 - Harvard.) The greater emphasis on algebraic skills in traditional calculus may explain these percentages. No statistical analysis of any kind (besides simple percentages) is reported in the paper under consideration.

The three papers mentioned above have shortcomings which are diverse. For instance, there are no common measures of achievement or carefully set conditions with an experimental and a control group. Some of these difficulties are recognized by the authors, when they write about the "preliminary nature" of their research. There has been little published research concerning reform calculus initiatives (Becker and Pence 1994, p.6), so the research done at the USA Merchant Marine Academy, Brigham Young University, and Oklahoma State University, are a first step forward.

Physical Applications within the Calculus Courses

Joan Ferrini-Mundy and Karen Geuther Graham (1991) put in the forefront of future research the idea tò determine whether or not examples drawn from physics can help in the learning of mathematical concepts related to calculus:

A number of mathematics education research questions arise in conjuction with the calculus effort. Several questions relate to the scope and sequence of the mathematics content. Examples include: How does one decide which parts of the traditional curriculum can best be omitted? Is it more helpful to students to introduce the idea of limit before the idea of derivative? What are the effects of introducing substantial physical applications within the calculus course? Many questions arise that relate directly to student learning and background matters. Examples of relatively broad questions include: Does lack of algebraic facility truly hinder calculus learning? How does the experience of secondary school calculus relate to the experience of college calculus? How does student "intuition" develop? Do physical examples help in the learning of concepts?" (Ferrini-Mundy and Geuther Graham 1991, p. 633).

These research questions, and others mentioned by both authors in the same paper, could have very important consequences since, annually, 600,000 students enroll in some type of calculus course in four-year colleges and universities in the United States of America. Almost half of these students are in mainstream "engineering" calculus, and only 46% finish the year with a grade of D or higher (Ferrini-Mundy and Geuther Graham 1991, p. 627.)

A pilot project was conducted at Dutchess Community College (Poughkeepsie, New York) by Wesley Ostertag, a mathematician, and Tony Zito, a physicist (Ostertag and Zito 1995), fully integrating first year Harvard Calculus with first year physics. Both of them teach this rather unique course, which blends all the topics in first year calculus and physics. Students at Dutchess Community College can enroll in the integrated course or enroll separately in a two semester Harvard Calculus and in a two semester "traditional" physics course. They found that 65% of those in the integrated one-year course obtained a grade of C or better in both semesters, while only 50% of those enrolled in the non-integrated sequence accomplished this goal. The authors do not mention whether or not the objectives and tests were the same. Moreover, a standardized test designed to measure student's understanding of basic kinematics was given as a pre- and post-test to students in both integrated and non-integrated sections. The mean post-test score for the integrated section was 69% (an improvement of 22% over the pre-test mean score), whereas the mean score for the non-integrated section was 61% (an improvement of 15% over the pre-test mean score). The authors do not report whether these results are significant or not.

This type of fully integrated first year calculus-physics course might be of crucial importance for community colleges, since they offer two-year degrees, and thus cannot afford to have first-semester calculus as a prerequisite for first-semester physics (as often happens in four-year colleges and universities).

Integrating College Mathematics with the Natural Sciences

There has been a limited number of efforts toward integrating mathematics teaching and the natural sciences at the college level. Among these we can mention Helfgott (1990), Jean and Iglesias (1990), and Helfgott (1995). The first one describes an integrated approach to differential equations, used in the classroom setting by the author, blending mathematics and several aspects of chemistry and physics:

> Student proficiency and surveys conducted among former pupils show that an integrated approach to several aspects of the natural sciences together with differential equations is highly recommended. Students who followed the integrated approach, instead of the classical differential equations course with few examples of applications outside mathematics, found it less difficult to do the work in later courses in control theory, heat transfer, transport phenomena and chemical kinetics (Helfgott 1990, p. 1014).

The paper by Jean and Iglesias is based on a course developed by the authors, wherein biology and mathematics are blended in a coherent whole. This is a radical departure from traditional courses in mathematics for biology majors. The third paper describes a first-year calculus course where history and the natural sciences (physics and chemistry) are used extensively:

> In teaching the calculus, it is useful to supply many examples of applications from the natural sciences. We go further than usual, developing classical examples from mechanics and chemical kinetics. The latter are particularly helpful because they are simple, require few prerequisites, and use a significant amount of readily available data. We stress the meaning of the scientific method in its different stages of building models, obtaining consequences, and contrasting them with data. Applications appear everywhere, not necessarily at the end of a section (Helfgott 1995, p. 136).

K-12 Integration of Mathematics and Science

The problem of integrating mathematics and science in the K-12 curriculum in the United States has a long history that goes back to E.H. Moore at the turn of the century. Moore, Professor of Mathematics at the University of Chicago, advocated teaching mathematics in close relationship to problems in physics, chemistry, and engineering. His idea found a great deal of support among teachers of high school mathematics and also by teachers of the High School sciences, and eventually led to the formation of the Central Association of Science and Mathematics Teachers, and its influential publication "School Science and Mathematics." Breslich (1936, p. 58) wrote about the Association in the following terms:

> One of the major purposes of the association was to find and establish legitimate contacts between the mathematical subjects and the sciences. It was hoped that the constant training which the pupil derives from applying mathematics to problems in science would increase his mathematical power and that his interest in mathematics would grow with the opportunities of using it in other school subjects. Indeed, some of the leaders of the movement were advocating that algebra, geometry, and physics be organized into a coherent course. If possible, this course was to be taught by the same teacher or at least by two teachers who were in sympathy with the ideas of correlation.

What happened? The efforts toward integration have not been successful, especially at the high school level. The trend toward specialization and the lack of proper training among teachers have conspired against integration. All too often mathematics teachers had little knowledge about science, and teachers of science had little knowledge about mathematics.

Few steps were taken with regard to how mathematics teaching could be improved by using examples drawn from the natural sciences. Fortunately, in the last decade there has been a renewed interest in the subject. Systemic initiatives have started to foster an interdisciplinary approach to mathematics, among these the Systemic Initiative in Montana Mathematics and Science (SIMMS) project in Montana. Its first objective (SIMMS 1993) is to redesign the 9-12 mathematics curriculum using an integrated approach for all students. The project is in its fifth year, and has had a marked impact on high school education in Montana, through its workshops, the publication of high-quality, fully integrated modules, technological support to schools, and the like. Besides, high school teachers all over the nation are developing ways to integrate mathematics and the natural sciences, trying to close the gap between them (Abad 1994, Longhart and Hughes 1995).

At the elementary and middle school level some research has been done about the effect of science and mathematics integration (Friend 1985, Kren and Huntsberger 1977). Friend's main purpose was to determine how integrating science and mathematics in a seventh grade physics unit affected achievement in science. Divided into 4 classes, 108 seventh graders were involved in the study. Two classes consisted of students with standardized reading and mathematics scores at least two years above grade level (AGL), while two classes consisted of students with standardized reading and mathematics scores on grade level (GL). One AGL class and one GL class followed the science and mathematics integration approach.

The investigation lasted 10 weeks, and students were assessed on a common test of physics. An analysis of variance showed that only AGL students taught by the integrated approach, demonstrated significantly greater achievement (0.01 level) than AGL students taught by the non-integrated approach. For GL students there was no significant difference between those in the integrated and non-integrated classes. Moreover, analysis of variance showed that AGL students that followed the non-integrated approach scored significantly higher (0.01 level) than GL students that followed the integrated approach. Friend recommends, on the basis of his findings, that science and mathematics should be integrated for AGL students.

Kren and Huntsberger investigated the effect of integrating science and mathematics instruction on fourth and fifth-grade student achievement in two mathematical skills (measuring and constructing angles, and interpreting and constructing graphs). A total of 161 children from eight classrooms participated in the study. The authors report (Kren and Huntsberger 1977, p. 558) that the treatments of the study were:

1. To present the concept in mathematics first so that the child may be able to apply it in science at a later date.
2. To present the concept in science and mathematics concurrently so that the two disciplines will enhance each other.
3. To present the mathematical concept in science, and follow the presentation with a similar one in mathematics.

An analysis of variance showed that there was no significant difference among the groups. Thus, the aforementioned mathematical skills could be taught with equal effectiveness under any of the three approaches. The authors recommend further investigations of ways science can be used to enhance the teaching of mathematics. The issue of integration is actively discussed nowadays at the middle school level, both by its advocates and by those that have some concerns about its implementation (George 1996, Beane 1996).

The Critics

Reform calculus has provoked a backlash in the academic community. Some critics adopt an emotional stance without justifying their claims that the use of applications in calculus courses may harm the integrity of the subject (Kleinfeld 1996, p. 230) while others raise important questions with regard to calculus teaching, which have to be addressed and discussed objectively. Hu (1996, p. 1538) quite appropriately singularizes four main problems that seem to affect several reform calculus books:

1. Confusion between heuristics and mathematical proof.
2. Less emphasis on symbolic manipulation.
3. Use of computers as a replacement of mathematical thinking.
4. Lack of mathematical closure in the discussion of applications.

There is an ongoing discussion, a healthy development in the mathematical community, which paid little attention to pedagogical issues in the past (Cipra 1996, Wilson 1997). Proponents and detractors of reform calculus, and, in particular, of the utilization of applications as a learning device in the classroom, have laid out their arguments. Everyone hopes that the ongoing debate will remain civil and fruitful.

Final Remarks

Should mathematics and science lose their identity in the quest of integration? Lynn A. Steen, former President of the Mathematical Association of America, gives a negative answer (Steen, 1992). He would rather recommend teachers to employ mathematical methods thoroughly in science, and scientific methods thoroughly in mathematics. That is to say, rather than blending content, Steen advocates blending methods. He sees mathematics and the sciences as different enterprises, one revealing order and pattern, the other seeking to understand nature. Nonetheless, they can contribute a lot to each other through cross-fertilization.

CHAPTER 2

METHODOLOGY

Sample and Population Description

The sample comprised all the students that completed second semester reform calculus (Math 182) at Montana State University-Bozeman, during the fall semester of 1996 (Math 182 is a four credit course that meets five days per week.) Five sections with five different instructors were scheduled at different hours. Three sections were set aside for the experimental treatment, while two sections acted as the control group. Students chose the sections according to their timetables and none of them knew who his or her instructor was going to be until the first day of classes. The students did not know which sections were in the experimental group and which sections were in the control group. In other words, the experimental group did not know that they were being "experimented upon." The population under consideration is intended to simulate the students that enroll and complete second-semester calculus in land-grant institutions of the United States.

Statistical Hypotheses

The questions to be answered by this study, stated in hypothesis form, are the following:

1. There is no difference between the adjusted Calculus II achievement means, of the integrated and the non-integrated groups, when statistically equated with respect to previous knowledge of calculus and scholastic aptitude in mathematics.

2. Method of instruction (integrated or non-integrated) and previous knowledge of calculus do not interact on Calculus II achievement.

3. Method of instruction (integrated or non-integrated) and scholastic aptitude in mathematics do not interact on Calculus II achievement.

4. When statistically equated with respect to previous knowledge of calculus and scholastic aptitude in mathematics, there is no difference in the adjusted Calculus II achievement means between those students that have not yet taken Physics I, those students that are concurrently taking Physics I and Calculus II, and those students that have taken Physics I before Calculus II.

5. Method of instruction (integrated or non-integrated) and when students take Physics I (not yet, concurrently with Calculus II, or before Calculus II) do not interact on Calculus II achievement.

6. There is no difference between the adjusted Physics I achievement means, of the integrated and the non-integrated groups, when statistically equated with respect to previous knowledge of calculus and scholastic aptitude in mathematics.

7. Method of instruction (integrated or non-integrated) and previous knowledge of calculus do not interact on Physics I achievement.

8. Method of instruction (integrated or non-integrated) and scholastic aptitude in mathematics do not interact on Physics I achievement.

9. Method of instruction (integrated or non-integrated) and achievement in Calculus II do not interact on Physics I achievement.

10. There is no difference in the adjusted Calculus II achievement means between female and male students, when statistically equated with respect to previous knowledge of calculus and scholastic aptitude in mathematics.

11. There is no difference in the adjusted Physics I achievement means between female and male students, when statistically equated with respect to previous knowledge of calculus and scholastic aptitude in mathematics.

12. Method of instruction (integrated or non-integrated) and gender do not interact on Calculus II achievement.

13. Method of instruction (integrated or non-integrated) and gender do not interact on Physics I achievement.

Explanation of Experimental Treatments

The main objective of this study was to determine whether or not the integrated method of instruction, which interrelates calculus and mechanics, affected performance in calculus and physics. The three experimental sections covered the same basic materials as the two control sections, namely chapters 6 through 10 of the Harvard Calculus textbook. However, the experimental sections discussed in class many examples from physics. These examples go beyond those presented in the textbook, have strong mathematical components, and are related to topics covered in Physics I. In addition, students in the experimental sections received enrichment notes that supplemented the textbook, a sample of which can be found in appendix E.

The original sample of students that completed Calculus II was statistically analyzed in order to answer hypothesis 5. This sample was smaller whenever Calculus I achievement or SAT scores intervened as factors or covariables, because not every student had taken Calculus I the previous semester and not all the mathematics scholastic aptitude scores were available. Similar considerations had to be applied when Physics I scores were analyzed, corresponding to students that took this course concurrently with

Calculus II. The need to obtain an unbiased score, limited the sample to those students that took Calculus I the previous semester. In other words, the need of common examinations to judge achievement in first semester calculus determined a smaller sample. The latter was analyzed in order to answer several questions of this research, namely questions 1, 2, 6, 7, and 10.

It should be emphasized that students that dropped out from Calculus II were not considered in the study. They had to take all the exams, including the two-hour final, to remain in the sample . The same applies to Physics I students wherever their scores were analyzed. Quizzes, homework, or group work were not analyzed because of the impossibility of obtaining a common measure acceptable to all instructors. These three activities together determined only one sixth of the total score in Calculus I or Calculus II. Neither are physics labs considered.

The Maxmincon Principle

The difference between the integrated and non-integrated groups was made as large as possible, so as to maximize the systematic variance. There were constraints, due to the fact that all sections of Calculus II had to cover the same core materials. Nonetheless, the experimental group dealt with enrichment notes, mainly related to several mathematical aspects of physics that are not normally covered in calculus classes. Besides, the physics problems in the textbook were heavily stressed in the experimental group.

There were two contaminating variables that had to be controlled: previous knowledge of calculus and scholastic aptitude in the area of mathematics. Both were controlled by means of an analysis of covariance. The best way to control all extraneous variables would be through randomization (Kerlinger and Pedhazur 1973, p. 82).

However, in the setting of our study, random assignment to treatments could not be accomplished since students chose the section to be in, according to their class schedules in many different courses. It is to be noted that they did not know who their instructor was until the first day of classes, when groups were already immutable. Thus, because of these conditions, we can assume that there was not any systematic selection bias.

The teacher variable, always present in this type of research, was controlled by having five different instructors; three in the experimental group, one of them the researcher, and two in the control group. The instructors were all graduate teaching assistants with similar backgrounds.

Minimization of error variance was accomplished through reduction of errors of measurement. Students in all five sections of Calculus II took common examinations simultaneously, at a certain common time in the evening especially set aside for this purpose (outside the regular class hours). Moreover, the reliability of the graders of the essay type questions in Calculus II was controlled carefully by establishing a common rubric for partial credit. With regard to the physics examinations, there is confidence that there were not any errors of measurement in grading beyond the normal standard errors since all the exam questions were of the multiple-choice type.

The Hawthorne effect

The Hawthorne effect holds that the researcher's impact on his or her subjects may actually affect the research results. This effect did not affect the results since students in Calculus II did not know that a research project was under way. If students in the experimental and control groups compared their class notes or hand-outs, they may have noticed that there was a difference on how some topics are developed, with applications being stressed, but they could not have noticed anything else.

Enrichment Notes

A sample of the type of additional materials -- to be called "enrichment notes"-- that students in the integrated group used, can be found in appendix E. These notes constituted a distinctive difference between the experimental and control groups. The topics covered there went beyond the common textbook and the common syllabus used by the five sections (experimental and control), and stressed the close links between calculus and physics. In these notes, several models of physics problems were built from first principles; not as accepted differential equations whose origin was unknown to the student. Also, some basic aspects of chemical kinetics were covered in class due to their strong mathematical content. Mathematical closure was sought in the experimental group, in the sense that the enrichment notes served to highlight important aspects of calculus. These notes did not discuss isolated topics, but were fully integrated with the textbook .

Besides, the enrichment notes dealt with some proofs and techniques that cannot be found in the Harvard Calculus book. For instance, the enrichment notes dealt with the fundamental theorem of calculus, the criteria for comparison of improper integrals, and the usual tests to determine the convergence of series. Even though students were not tested on the theoretical aspects of calculus, the instructors in the experimental group considered that it was advisable to provide proofs of some very important propositions.

The Instructors in the Experimental Group

Two instructors were invited by the researcher to participate in the experimental group. They willingly accepted, even though they knew that the integrated method required greater effort on the part of the teacher. A long session took place before classes started, wherein the researcher explained the purpose and methodology of the study.

Thereafter, each week the three instructors met on a regular basis to discuss the enrichment notes, practice exams, quizzes and the like. These meetings went beyond the regular meetings of the five instructors with the course supervisor -- the latter a regular member of the faculty in the department of mathematical sciences. A careful record of the experimental group meetings was kept, and is shown in Appendix C.

Methods of Data Collection

Collection of Quantitative Data

Achievement in Calculus II was measured by three one-hour exams and a comprehensive final, comprising a total of 500 points, 100 for each term exam and 200 for the final. All were common examinations, taken outside the regular hours. A first draft of each of them was made by the course supervisor, an experienced faculty member that was not in charge of teaching a section. This draft was carefully analyzed by each instructor, whose collective responsibility was to check for its content validity. All the instructors got then together and discussed the final version.

With regard to grading, each instructor was assigned the job of grading one question of all the tests. Since the questions were open-ended, of the essay type, the grader had to adopt a consistent policy, especially concerning partial credit. This was achieved by adopting a common rubric, which was strictly followed.

The conditions under which the study was carried out, did not allow a reliability analysis of the test itself (test-retest technique, for example), but we may assume that these tests are reliable due to the extensive body of knowledge accumulated on calculus testing since reform calculus was adopted at MSU-Bozeman for the 181-182 series.

Physics I was taught by one member of the physics department faculty, in a large lecture setting. Achievement in Physics I was measured by the midterm exam and a comprehensive final, both common examinations of the multiple-choice type.

This study deals only with raw scores. Moreover, as was mentioned before, 100 points allotted in Calculus II to group work and quizzes was not considered. These were given at the discretion of each instructor, who could adopt the policy that he/she found best suited for the group. The very nature of the group work and quizzes did not allow a common standard of measurement, thus determining an insurmountable barrier for statistical analysis. For a similar reason, laboratory work in physics or chemistry was not taken into account. The Calculus II exams taken by the students during the fall of 96 are included in appendix A.

The SAT math scores were provided by the Office of Admissions at MSU-Bozeman. Since some students had taken the ACT math but not the SAT math test, a concordance table between SAT and ACT -- supplied by the ACT company and shown in appendix D-- was used. The other covariate scores (Calculus I) were provided by the instructors of first semester calculus, on the basis of three one-hour exams and a comprehensive two-hour final (a total of 500 points).

Instructor Feedback

The researcher interviewed the two instructors that accompanied him in the experimental group. These interviews were conducted at the end of the semester, with a pre-established questionnaire geared toward their experiences in teaching reform calculus with a strong applied component. The almost verbatim transcriptions of the interviews are to be found in the next chapter.

Analytical Techniques and Research Design

Definition of Variables

The scores obtained in the one-hour exams and the final by all students in Math 182 (Fall 96) were added, constituting a number called CalcII. Similarly, their scores obtained in their previous semester calculus (Math 181) -- both in the one-hour exams and the final -- were added; this value was called CalcI. Under the symbol Phy we have scores for those students enrolled in Math 182 that were taking Physics I (Physics 211) concurrently. These scores were obtained in the same way as in Math 181 and Math 182, by adding the scores corresponding to the midterm exam and the final. Each student's scholastic aptitude test in mathematics can be found under a column called SAT. Furthermore, those students that belonged to the experimental group (three sections of Math 182) were coded 1 with regard to Method, while those that belonged to the control group (two sections of Math 182) were coded 2 with regard to Method. The variable WhenPh had three levels (1= not yet taken Physics 211, 2= taking Physics 211 concurrently with Math 182, 3= taken Physics 211 before), while female students were coded 0 and male students were coded 1.

Categorization of Some Variables and Display of Data

In order to study possible interactions, some of the variables were categorized. With this purpose in mind, the lower and upper quartiles (Q1 and Q3, respectively) of CalcI, CalcII, and SAT data were calculated to divide the students in three groups (low, medium, and high) for each of the aforementioned variables. Thus, three new columns were added: CalcIG, CalcIIG, and SATG. The following ten entries were the basis for all subsequent analyses:

Method CalcI CalcII Phy SAT CalcIG CalcIIG SATG WhenPh Gender

Methods of Analysis

Two-way analysis of variance (ANOVA) was the statistical technique used in order to answer questions 2, 3, 5, 7, 8, 9, 12, and 13, while analysis of covariance (ANCOVA) with two covariates, CalcI and SAT, was the chosen statistical tool to answer questions 1, 4, 6, 10, and 11.

Two-way ANOVAS , on eight different instances, were calculated:

- ANOVA for CalcII, with Method and CalcIG as factors
- ANOVA for CalcII, with Method and SATG as factors
- ANOVA for CalcII, with Method and WhenPh as factors
- ANOVA for Phy, with Method and CalcIG as factors
- ANOVA for Phy, with Method and SAT as factors
- ANOVA for Phy, with Method and CalcIIG as factors
- ANOVA for CalcII, with Method and Gender as factors
- ANOVA for Phy, with Method and Gender as factors

Five different ANCOVAS (each of them with two covariates: SAT and CalcI) were calculated:

- ANCOVA for CalcII, with Method as factor
- ANCOVA for CalcII, with WhenPhy as factor
- ANCOVA for Phy, with Method as factor
- ANCOVA for CalcII, with Gender as factor
- ANCOVA for Phy, with Gender as factor

Alpha Level

An alpha level of 0.05 was set before the collection of data took place. This level was adopted instead of a more conservative 0.01 because it is hard to imagine any harm that could be done to the students by advocating an integrated approach to calculus even if no measurable advantages exist. That is to say, a type I error (rejecting the null even though it is true) could not possibly have serious negative effects. This researcher was quite concerned about the possibility of making a type II error (failing to reject the null even though it is false).

Limitations and Delimitations

Limitations

1. One limitation to the study is that students of Calculus II were not randomly assigned to the experimental and control groups. They chose the section to be in well before the beginning of the semester and according to their timetables. Calculus II was taught in five sections, at five different times and by five different instructors. Students did not know that there were going to be experimental and control groups. Thus, despite lack of random assignment, one might expect that chance was not absent from the process of selection of sections. In other words, there is no apparent selection bias.

2. Another limitation of the study was the fact that some students enrolled in Calculus II did not concurrently take Physics I. Only those students that concurrently took Calculus II and Physics I were included as part of the sample used in order to answer questions 6, 7, 8, 9, 11, and 13. A smaller sample does create problems in the realm of statistical analysis. However, the data collected from all students enrolled in Calculus II was used to answer questions 1, 2, 3, 4,

and 10 (provided the student had taken, at MSU-Bozeman, Calculus I the previous semester, and the student's scholastic aptitude test in mathematics was available). Questions 5 and 12 had no restrictions, in the sense that the scores of all students that completed Calculus II could be analyzed.

3. For one reason or another, some students dropped from courses during the semester, reducing the size of the sample. Furthermore, some students had not taken Calculus I at MSU-Bozeman, or their scholastic aptitude test was not kept at the Admissions Office. These two factors also reduced the size of the sample.

4. The background and experience of the teachers in charge of Calculus II were not exactly the same. This fact introduces the well-known "teacher effect" phenomenon, maybe unavoidable in educational research of this type, but nonetheless a limitation. In order to minimize the teacher effect, three sections followed the integrated approach: one taught by the researcher, two others by instructors willing to participate in the experiment. The five sections used the same textbook and followed the same syllabus, a fact that lessened the influence of teaching styles.

5. Once classes start, the groups are immutable. Nonetheless, it can happen that a student may want to move from an integrated group to a non-integrated group or vice versa, for some compelling reason.

6. The course load taken by each student is an uncontrolled variable that may affect student's achievement in Calculus II or Physics I. However, this researcher trusts that the variability due to course load is reflected in Calculus I achievement. A similar consideration can be applied to uncontrolled variables such as number of hours spent on homework, or on part-time jobs. The grades in Calculus I reflect not only mathematical aptitude -- the correlation with SAT scores is rather low as

can be seen in the next chapter -- but also the life style of the student in the sense of how much time he/she devotes to study.

7. Some sections, following the integrated or non-integrated approaches, did group work once per week. In one of the integrated sections, students were encouraged to meet on weekends in order to complete the assigned group problems. Thus, academic activities outside the classroom were an uncontrolled variable.

Delimitations

1. The study was conducted during the 1996 fall semester, at MSU-Bozeman. It can be assumed that this study could be replicated with a sample of students of similar background, in land-grant institutions of the United States.

2. Another delimitation of the study is related to the fact that the study was carried out with students enrolled in a reform calculus course. Reform calculus welcomes applications, thus creating an atmosphere where the integrated approach can be tried. Traditional calculus courses, offered at many institutions (Math 170, 175, and 176 at MSU-Bozeman are an example), may not be as receptive as reform calculus courses with regard to an integrated approach.

CHAPTER 3

RESULTS

The Pilot Study

A pilot study was conducted by the researcher during the spring of 96. All the students that completed Math 182 (CalcII), second semester reform calculus, participated in the study. There were six sections, in charge of six different instructors. The experimental group, taught by the researcher, met daily (Monday through Friday) for 50 minutes, while the other five sections (constituting the control group) also met five times per week for 50 minutes.

There were 32 students in the experimental group (Method 1) and 178 students in the control group (Method 2). The next table shows the number of participants in the study, as a function of the four main variables. MINITAB was the statistical software used.

Table 1. Number of students by method and the four main variables

Row: Variable	Column: Method 1	2	total
CalcII	32	178	210
Phy	18	52	70
CalcI	20	136	156
SAT	28	150	178

In other words, among the 210 students, 156 took Math 181 (CalcI) the previous semester, while 70 students took Physics 211 (first semester physics) during the spring of 96. Through the MSU-Bozeman Office of Admissions, it was possible to obtain the scholastic aptitude scores in mathematics with regard to 178 of the total number of 210 students enrolled in Calculus II.

Some students had taken the ACT, others -- a majority -- the SAT; an official concordance table allowed us to assign a single SAT score to each one of the 178 students.

It is to be noted that all the scores of Calculus I - Fall 95 (three one-hour exams and the final), Calculus II - Spring 96 (three one-hour exams and the final), and Physics 211 (one midterm exam and the final), were obtained directly from the instructors. All the tests were common examinations.

The quartiles (Q1 and Q3) corresponding to Calculus I, Calculus II, and SAT were obtained in order to categorize students in three groups (low, medium, and high). The maximum score for both Calculus I and Calculus II was 500 points, while students could achieve a maximum of 150 points in physics.

Table 2. Means, Medians, and Quartiles Corresponding to the Main Variables

Variable	N	Mean	Median	Q1	Q3
CalcI	156	406.1	409	383.25	441.75
CalcII	210	386.66	391.5	347.00	440.50
SAT	178	613.71	620.00	570.00	662.50
Phy	70	113.74	115.50	106.75	127.00

Table 3. Ranks According to Quartiles

	CalcI	CalcII	SAT
Low	1 - 383	1 - 347	1 - 570
Medium	384 - 441	348 - 441	571 - 662
High	442 - 500	442 - 500	663 - 800

How strong is the linear association between the four main variables? The following table gives an answer to this question.

Table 4. Pearson Correlations Among the Four Main Variables

	CalcI	CalcII	SAT
CalcII	0.504		
SAT	0.366	0.337	
Phy	0.414	0.805	0.289

From this table it can be concluded that there is a strong linear association (0.805) between CalcII and Phy, a moderate linear association (0.504) between CalcI and CalcII, and a weak linear association (0.289) between SAT and Phy. Next, it is important to look at a table of descriptive statistics comparing methods.

Table 5. Descriptive Statistics Comparing Methods

Variable	Method	N	Mean	Median	StDev
CalcII	1	32	386.8	378.0	62.9
	2	178	386.63	393.50	67.16
Phy	1	18	114.17	114.00	20.00
	2	52	113.60	118.00	22.95

There is a striking similarity between the means in CalcII, when method 1 (integrated approach) and method 2 (non-integrated approach) are compared: 386.8 and 386.63, respectively. The means in physics are also very close between both methods (114.17 and 113.60). Their medians do not differ much either. In addition, students in the experimental and control groups were compared with regard to their SAT and CalcI scores.

Table 6. SAT and CalcI Scores in the Experimental and Control Groups

Variable	Method	N	Mean	Median	StDev
SAT	1	28	620.4	650	71.8
	2	150	612.47	620	74.23
CalcI	1	20	391.1	391	46.8
	2	136	408.19	409.50	47.68

Five different two-way ANOVAS, with their respective interaction plots, were calculated so as to reject or retain the null hypothesis related to questions 2, 3, 7, 8, and 9. During the pilot project the variables Gender and WhenPhy were not considered.

Figure 1. Interaction Plot for CalcII Grades with Method and CalcIG as Factors

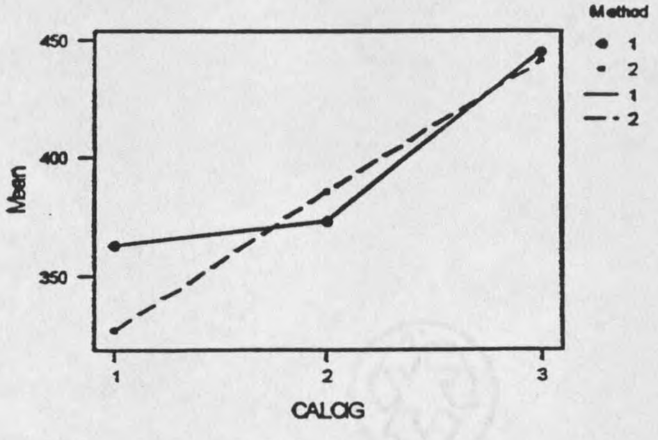


Table 7. ANOVA for CalcII with Method and CalcIG as factors

Source	DF	Seq SS	Adj SS	Adj MS	F	P
Method	1	74	1458	1458	0.46	0.498
CalcIG	2	231448	97292	48646	15.40	0.000
Method*CalcIG	2	7544	7544	3772	1.19	**0.306**
Error	150	473797	473797	3159		
Total	155	712863				

Figure 2. Interaction Plot for CalcII with Method and SATG as Factors

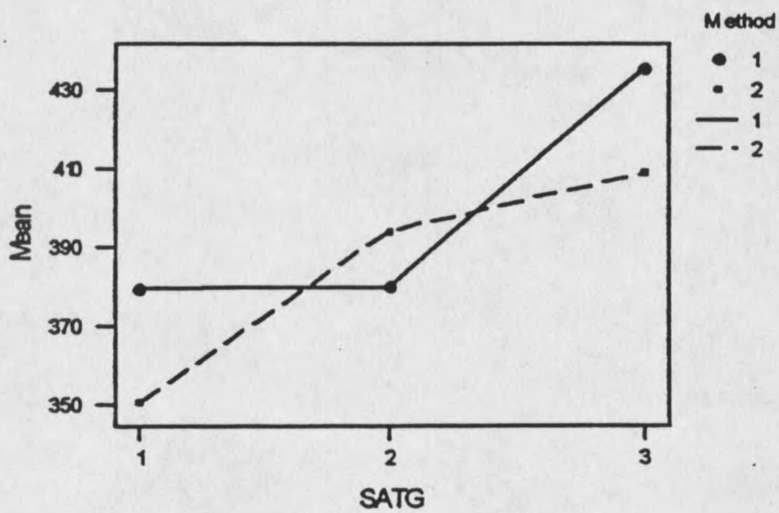


Table 8. ANOVA for CalcII with Method and SATG as Factors

Source	DF	Seq SS	Adj SS	Adj MS	F	P
Method	1	1380	4130	4130	1.00	0.318
SATG	2	80986	40041	20020	4.86	0.009
Method*SATG	2	10608	10608	5304	1.29	**0.279**
Error	172	708592	708592	4120		
Total	177	801565				

Figure 3. Interaction Plot for Physics with Method and CalcIG as Factors

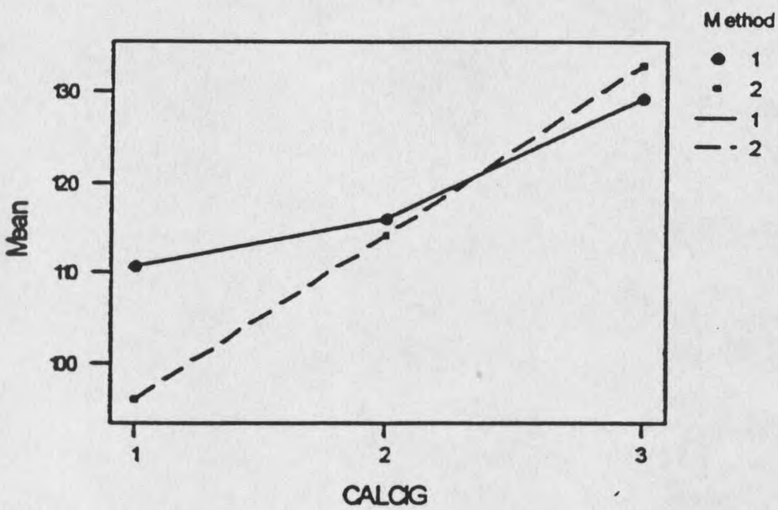


Table 9. ANOVA for Physics with Method and CalcIG as Factors

Source	DF	Seq SS	Adj SS	Adj MS	F	P
Method	1	235.9	153.7	153.7	0.41	0.527
CalcIG	2	6814.2	3902.5	1951.3	5.15	0.010
Method*CalcIG	2	482.2	482.2	241.1	0.64	0.534
Error	46	17414	17414	378.6		
Total	51	24946.3				

Figure 4. Interaction Plot for Physics with Method and SATG as Factors

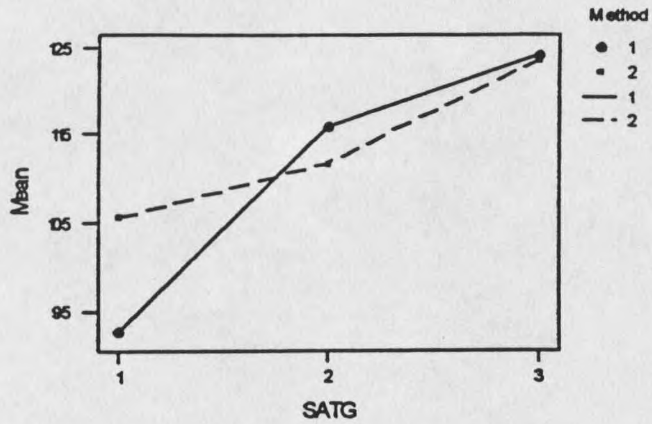


Table 10. ANOVA for Physics with Method and SATG as Factors

Source	DF	Seq SS	Adj SS	Adj MS	F	P
Method	1	17.1	78.5	78.5	0.16	0.693
SATG	2	3485.1	3532.9	1766.4	3.54	0.035
Method*SATG	2	520.0	520.0	260.0	0.52	**0.597**
Error	58	28930.7	28930.7	498.8		
Total	63	32953.0				

Figure 5. Interaction Plot for Physics with Method and CalcIIG as Factors

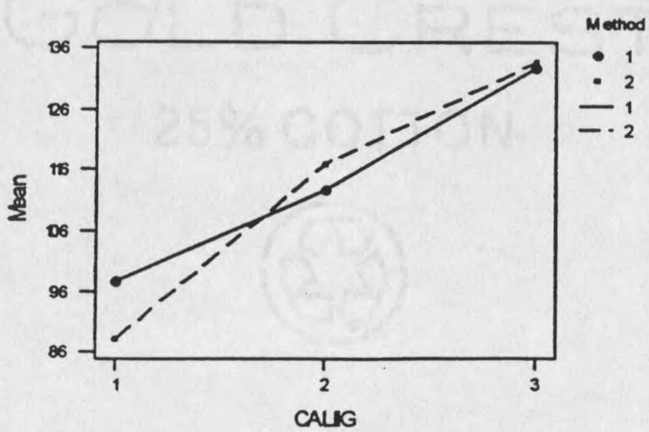


Table 11. ANOVA for Physics with Method and CalcIIG as Factors

Source	DF	Seq SS	Adj SS	Adj MS	F	P
Method	1	4.4	26.7	26.7	0.09	0.760
CalcIIG	2	15028.2	11379.1	5689.6	20.02	0.000
Method*CalcIIG	2	445.9	445.9	222.9	0.78	**0.461**
Error	64	18187.0	18187.0	284.2		
Total	69	33665.4				

None of the analyses led to significant interactions (p-values 0.306, 0.279, 0.534, 0.597, 0.461 respectively). Thus, we retain the null hypotheses corresponding to questions 2, 3, 7, 8, 9, and proceed to analyze the principal effects.

From the preceding five tables it is possible to ascertain that the CalcI categories (low, medium, high) are significantly different with regard to CalcII and Phy, while the SAT categories (low, medium, high) are significantly different with regard to CalcII and Phy. Moreover, the CalcII categories (low, medium, high) are significantly different with respect to Phy. None of the tables allow to conclude that there is a significant difference between methods; the p-values are just too large.

Next, two different ANCOVAS, with two covariates (CalcI and SAT), were performed so as to answer questions 1 and 6. Instead of the variable WhenPh, which was not used in the pilot project, it was possible to define the variable TakePh (0 = not taking Physics I concurrently with CalcII, 1 = taking Physics I concurrently with CalcII). The variable TakePh is less encompassing than WhenPh. The former was used in the pilot study because no reliable way of obtaining the information on whether a student had taken Physics 211 before the fall of 96 could be found (self-reporting was avoided). This shortcoming was solved in the fall of 96 thanks to the collaboration of the administration personnel at MSU-Bozeman. The TakePh variable led to a third ANCOVA table.

Table 12. ANCOVA for CalcII with Method as Factor

Source	DF	ADJ SS	MS	F	P
Covariates	2	198252	99126	29.97	0.000
Method	1	5000	5000	1.51	**0.221**
Error	134	443239	3308		
Total	137	642661			
Covariate	Coeff	StDev	t-value	P	
CalcI	0.7363	0.1125	6.545	0.000	
SAT	0.1107	0.0794	1.395	0.165	

Table 13. ANCOVA for Physics with Method as Factor

Source	DF	ADJ SS	MS	F	P
Covariates	2	5184.2	2592.1	6.24	0.004
Method	1	766.1	766.1	1.84	**0.181**
Error	47	19526.0	415.4		
Total	50	24945.9			
Covariate	Coeff	StDev	t-value	P	
CalcI	0.23045	0.0805	2.8632	0.006	
SAT	0.02553	0.0449	0.5691	0.572	

Table 14. ANCOVA for CalcII with TakePh as Factor

Source	DF	ADJ SS	MS	F	P
Covariates	2	194313	97157	29.05	0.000
TakePh	1	66	66	0.02	**0.888**
Error	134	448174	3345		
Total	137	642661			
Covariate	Coeff	StDev	t-value	P	
CalcI	0.7190	0.1122	6.407	0.000	
SAT	0.12	0.0795	1.509	0.134	

Table 12 leads to the conclusion that there is no significant difference (p-value = 0.221) in the adjusted Calculus II means between the integrated and non-integrated groups, when statistically adjusted with respect to previous knowlwdge of calculus and scholastic aptitude in mathematics. Furthermore, the covariate CalcI plays an important role in this ANCOVA model (p = 0.000), while the covariate SAT does not (p = 0.165). Thus, it would be possible to eliminate the latter without affecting the model.

Table 13 asserts that there is no significant difference (p-value = 0.181) in the adjusted Physics I means between the integrated and non-integrated groups, when statistically equated with respect to previous knowledge of calculus and scholastic aptitude in mathematics. Again, the covariate CalcI plays an important role in the ANCOVA model under consideration (p = 0.006), while the covariate SAT does not (p = 0.572).

Finally, Table 14 implies that there is no significant difference (p = 0.888) in the adjusted Calculus II means between those students that were taking Physics I concurrently with Calculus II, and those students that were not taking Physics I concurrently with

Calculus II, when statistically adjusted with respect to previous knowledge of calculus and scholastic aptitude in mathematics. The covariate CalcI exerts an important role in this ANCOVA model too (p = 0.000), but the covariate SAT does not (p = 0.134).

The pilot study provided valuable information for the research design and the acquisition of data procedure, as well as allowed the researcher the opportunity to write the enrichment notes and test them in class.

Certain trends can be observed, for instance with regard to Calculus II and Physics grades, when CalcI and SAT are categorized in three groups. Nevertheless, none of them is significant (comparing methods) despite the fact that for low achievers the interaction plots indicate an apparently wide difference. One explanation could be that the number of students in the experimental group was not big enough, leading to a lack of statistical evidence. Besides, the design was highly unbalanced, with many more students in the control group than in the experimental group.

Physics Questionnaire

A questionnaire, based in part on the "Force Concept Inventory" test (Hestenes, Wells, and Swackhamer 1992) was developed. A copy is to be found in appendix B. It was taken by 132 students enrolled in Math 182 during the spring of 96, as part of the materials developed during the pilot project carried out in the above-mentioned period.

The purpose of the questionnaire was to find out how much physics students learned in high school and how able they were to deal with simple physical problems involving only high school algebra and physics. The dependent variable was achievement in basic physics.

The integrated approach, used by the experimental group, relied heavily on physics. Thus, teachers in this group needed to have a clear picture about their student's background. The questionnaire provided valuable information for course planning. This information was taken into consideration in the management of the three experimental sections during the fall of 96.

The Fall 96 Experimental Study

The main study was conducted during the fall of 96. A total of 151 students were enrolled and completed Calculus II, in the sense that they took the three one-hour exams and the final. Among those students, 72 had taken Calculus I at MSU-Bozeman the previous semester, 55 took all the tests in Physics I, and it was possible to obtain the scholastic aptitude scores in mathematics (SAT and/or ACT) for 135 of them. The latter were incorporated into the statistical analysis as SAT scores using a concordance table between SAT and ACT scores. MINITAB was the statistical software of choice.

The following three tables provide the basic descriptive statistics of the study. The quartiles (Q1 and Q3) are needed to categorize the Calculus I, Calculus II, and SAT scores.

Table 15. Students Involved in the Research by Available Information and Method

Course or Test	Method		Total
	Experimental	Control	
Calculus II	97	54	151
Calculus I	55	17	72
Physics	33	22	55
SAT	88	47	135

Table 16. Descriptive Statistics of Scores

Variable	N	Mean	Median	StDev	Min	Max	Q1	Q3
SAT	135	599.70	600.00	61.51	460	740	560	640
CalcII	151	333.77	341.00	86.21	77	491	282	397
CalcI	72	389.24	393.00	45.93	206	470	367	415

Table 17. Categorization of Scores

Group	Calculus I	Calculus II	SAT
Low	1-367	1-282	1-560
Medium	368-415	283-397	561-640
High	416-500	398-500	641-800

Table 18. Number of Students by Category and Method

Category	Method	Calculus I	Calculus II	SAT
Low	1	13	15	29
	2	5	23	9
Medium	1	27	53	40
	2	10	23	25
High	1	15	29	19
	2	2	8	13

Table 19. Descriptive Statistics of Scores by Method

Course or Test	Method	N	Mean	Median	StDev	Q1	Q3
Calculus I	1	55	394.93	393	40.07	369	428
	2	17	370.8	392	58.9	341	406
Calculus II	1	97	356.99	353	71.75	320	412
	2	54	292.1	295	94.6	236	372
SAT	1	88	596.02	600	64.18	560	640
	2	47	606.6	600	56.19	580	660
Physics I	1	33	144.39	141	19.96	129	160
	2	22	132.45	132	20.83	116	148

The Calculus I and SAT medians of the students in the experimental and control groups are remarkably similar (actually equal for the SAT), as may be inferred from the preceding table. The corresponding Calculus I and SAT means do not differ much either. Through an analysis of covariance it will be possible to determine whether or not the differences among the means between the two groups are statistically significant, when Calculus II and Physics I scores are analyzed.

The next table is related to questions 4 and 5 of this study. It provides information about the number of students in the experimetal and control groups, according to their status in Physics I (not yet taken Physics I, concurrently taking Physics I with Calculus II, taken Physics I before Calculus II). It is to be noted that the sum of all the students in this list, taking into account their status in Physics I and the two groups, has to be the total number of students that completed Calculus II; namely 151.

Table 20. Number of Students by Method and When They Took Physics

Status	Method	Number
Not Yet	1	31
	2	20
Concurrently	1	35
	2	23
Before	1	31
	2	11

Pearson correlations can be found in Table 21. It can be seen that there is a rather strong linear association (0.585) between Calculus I and Calculus II, and between Calculus II and Physics I (0.583), while there is a less strong linear association between Calculus I and Physics I (0.459). The linear association between SAT and Calculus II (0.293), and between SAT and Physics I (0.264), is low.

Table 21. Pearson Correlations Among The Four main variables

	SAT	CalcII	CalcI
CalcII	0.293		
CalcI	0.146	0.585	
Phy	0.264	0.583	0.459

Further descriptive statistics, related to students' scores in Calculus I and SAT by instructor, are to be found in Appendix F.

The next two figures are highly suggestive, since they show graphically why the linear association between Calculus I and Calculus II is much higher than the linear association between SAT and Calculus II.

Figure 6. Calculus II Scores and Calculus I Scores

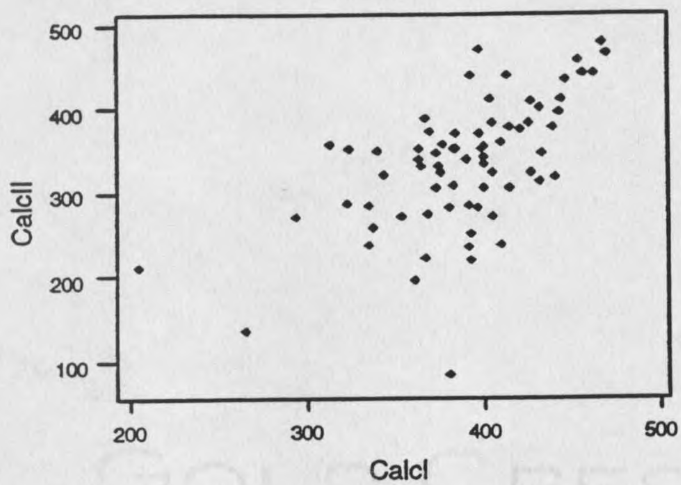
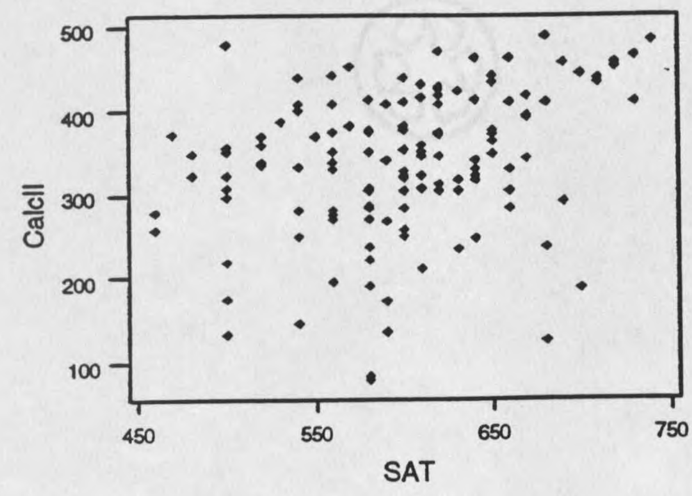


Figure 7. Calculus II Scores and SAT Scores



Calculus II and Physics I Achievement

The results of the ANCOVAS related to the null hypothesis 1, 4, and 6 can be found in the next three tables. They are followed by statements of the hypotheses and the decision taken in each case. It is to be remembered that the alpha level, chosen before the collection of data, is 0.05.

Table 22. ANCOVA for Calculus II with Method as Factor

Source	DF	Adj SS	MS	F	P
Covariates	2	115327	57663	15.31	0.000
Method	1	23533	23533	6.25	**0.015**
Error	62	233570	3767		
Total	65	409082			

Covariate	Coeff	StDev	T	P
CalcI	0.9021	0.172	5.2557	0.000
SAT	0.1301	0.143	0.9071	0.368

Table 23. ANCOVA for Calculus II with WhenPhy as Factor

Source	DF	Adj SS	MS	F	P
Covariates	2	135689	67845	16.15	0.000
WhenPh	2	820	410	0.10	**0.907**
Error	61	256283	4201		
Total	65	409082			

Covariate	Coeff	StDev	T	P
CalcI	0.9961	0.181	5.4948	0.000
SAT	0.1084	0.153	0.7109	0.48

Table 24. ANCOVA for Physics I with Method as Factor

Source	DF	Adj SS	MS	F	P
Covariates	2	3435	1717.5	7.09	0.004
Method	1	507.1	507.1	2.09	**0.161**
Error	24	5815.7	242.3		
Total	27	9729.2			

Covariate	Coeff	StDev	T	P
SAT	0.1104	0.0533	2.072	0.049
CalcI	0.2343	0.0850	2.756	0.011

Hypothesis 1: There is no difference between the adjusted Calculus II achievement means, of the integrated and the non-integrated groups, when statistically equated with respect to previous knowledege of calculus and scholastic aptitude in mathematics.

Decision: Reject the null hypothesis. There was a statistically significant difference between the adjusted Calculus II achievement means of both groups (p = 0.015). The adjusted means were 343.82 (N = 50) for the experimental group, and 298.19 (N=16) for the control group.

Hypothesis 4: When statistically equated with respect to previous knowledge of calculus and scholastic aptitude in mathematics, there is no difference between the adjusted Calculus II achievement means of those students that have not yet taken Physics I, those students that are concurrently taking Physics I and Calculus II, and those students that have taken Physics I before Calculus II.

Decision: Fail to reject the null hypothesis. No statistically significant difference between the adjusted Calculus II achievement means of the three groups was detected (p = 0.907). The adjusted means were 327.42 (N=17) for the group of students that had not taken Physics in the past and were not taking it concurrently with Calculus II, 336.31 (N=30) for those students that were taking Physics I concurrently with Calculus II, and 331.92 (N=19) for those students that had taken Physics I before.

Hypothesis 6: There is no difference between the adjusted Physics I means, of the integrated and the non-integrated groups, when statistically equated with respect to previous knowledge of calculus and scholastic aptitude in mathematics

Decision: Fail to reject the null hypothesis. No statistically significant difference between the adjusted Physics I achievement means of both groups was detected (p = 0.161). The adjusted means were 146.98 (N=22) for the experimental group and 136.57 (N=6) for the control group.

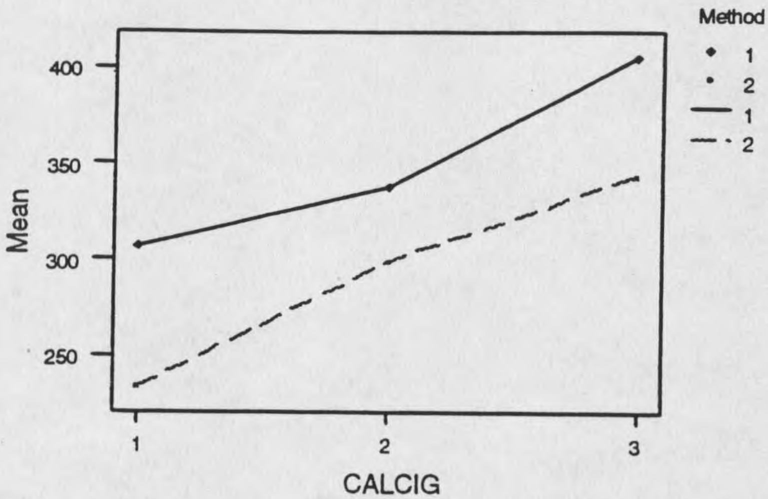
Interaction Analyses

Six interaction analyses were performed in order to make a decision with regard to the null hypotheses 2, 3, 5, 7, 8, and 9. Each analysis of variance is followed by the interaction plot, the corresponding hypothesis, and the decision made.

Table 25. Analysis of Variance for Calculus II with Method and CalcIG as Factors

Source	DF	Seq SS	AdjSS	AdjMS	F	P
Method	1	53856	31493	31493	7.84	0.007
CalcIG	2	94815	54782	27391	6.82	0.002
Method*CalcIG	2	3429	3429	1714	0.43	**0.665**
Error	66	265249	265249	4019		
Total	71	417349				

Figure 8. Interaction Plot for Calculus II Scores by Method and Calculus I Groups



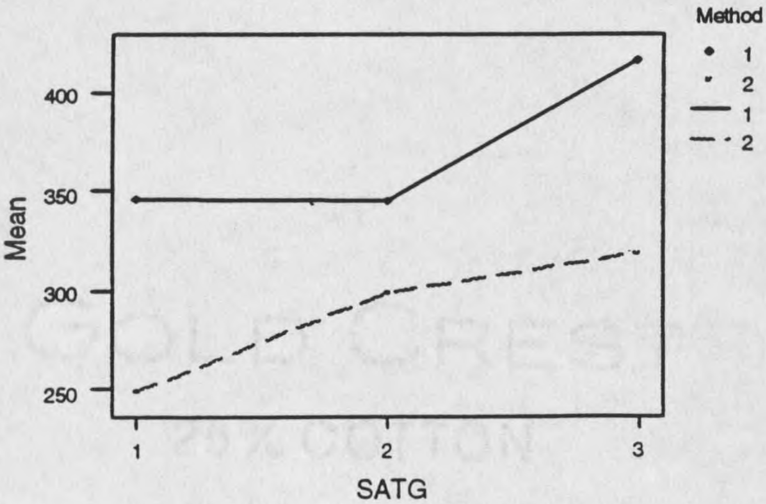
Hypothesis 2: Method of instruction (integrated or non-integrated) and previous knowledge of calculus do not interact on Calculus II achievement.

Decision: Fail to reject the null hypothesis. No statistically significant interaction was detected (p = 0.655).

Table 26. Analysis of Variance for Calculus II with Method and SATG as Factors

Source	DF	Seq SS	Adj SS	Adj MS	F	P
Method	1	136640	174382	174382	29.19	0.000
SATG	2	83466	77176	38588	6.46	0.002
Method*SATG	2	19747	19747	9873	1.65	**0.196**
Error	129	770531	770531	5973		
Total	134	1010383				

Figure 9. Interaction Plot for Calculus II Scores by Method and SAT Groups



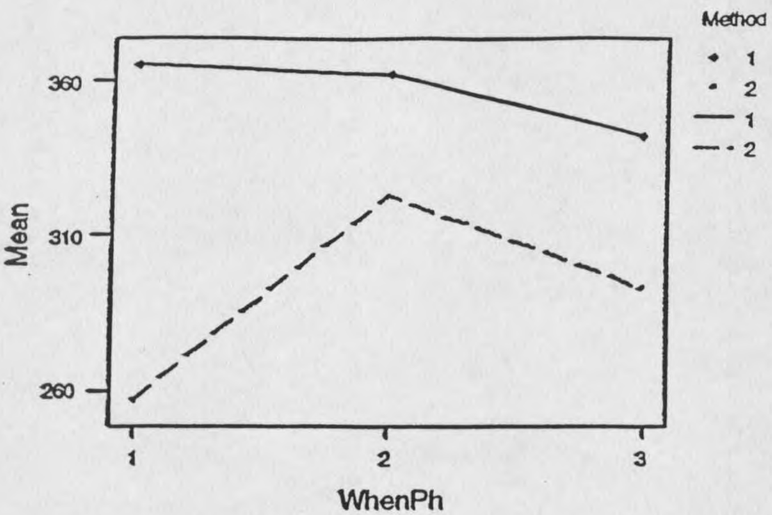
Hypothesis 3: Method of instruction (integrated or non-integrated) and scholastic aptitude in mathematics do not interact on Calculus II achievement.

Decision: Fail to reject the null hypothesis. No statistically significant interaction was found (p = 0.196).

Table 27. Analysis of Variance for Calculus II with Method and WhenPh as Factor

Source	DF	Seq SS	Adj SS	Adj MS	F	P
Method	1	146180	141863	141863	22.58	0.000
WhenPh	2	22651	28648	14324	2.28	0.106
Method*WhenPh	2	35019	35019	17510	2.79	**0.065**
Error	145	910955	910955	6282		
Total	150	1114804				

Figure 10. Interaction Plot for Calculus II Scores by Method and WhenPh



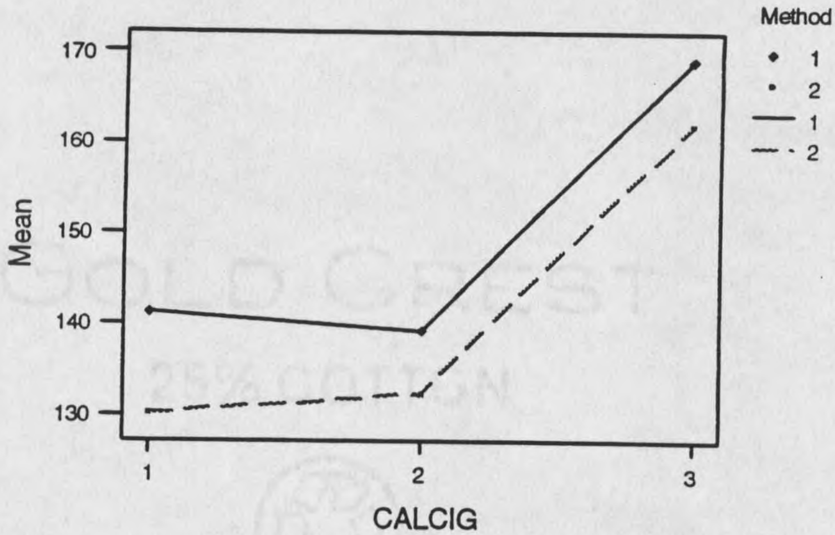
Hypothesis 5: Method of instruction (integrated or non-integrated) and when students take Physics I (not yet, concurrently with Calculus II, or before Calculus II) do not interact on Calculus II achievement.

Decision: Fail to reject the null hypothesis. No statistically significant interaction was detected (p = 0.065).

Table 28. Analysis of Variance for Physics I with Method and CalcIG as Factors

Source	DF	Seq SS	Adj SS	Adj MS	F	P
Method	1	504.3	238.8	238.8	1.26	0.272
CalcIG	2	4728.2	2531.7	1265.8	6.70	0.005
Method*CalcIG	2	12.7	12.7	6.3	0.03	**0.967**
Error	24	4535.8	4535.8	189.0		
Total	29	9781.0				

Figure 11. Interaction Plot for Physics Scores by Method and Calculus I Groups



Hypothesis 7: Method of instruction (integrated or non-integrated) and previous knowledge of calculus do not interact on Physics I achievement.

Decision: Fail to reject the null hypothesis. No statistically significant interaction was detected ($p = 0.967$).

Table 29. Analysis of Variance for Physics I with Method and SATG as Factors

Source	DF	Seq SS	Adj SS	Adj MS	F	P
Method	1	866.8	1213.3	1213.3	3.85	0.056
SATG	2	1810.4	1133.3	566.6	1.80	0.178
Method*SATG	2	3301.6	3301.6	1650.8	5.24	**0.009**
Error	42	13222.4	13222.4	314.8		
Total	47	19201.3				

Figure 12. Interaction Plot for Physics Scores by Method and SAT groups

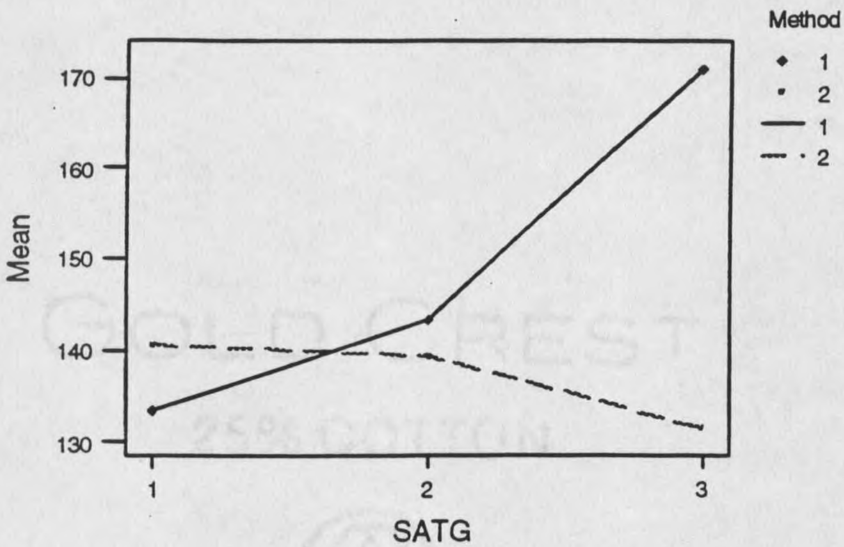


Table 30. Comparison Test for Physics I Scores Considering Only SATG 3 Students

Method	N	Mean	T	P
1	5	171.2	3.935	0.000
2	8	131.4		

The program MSUSTAT was used (COMPARE subroutine, with DF = 42 and MSE = 314.8; both values come from table 29)

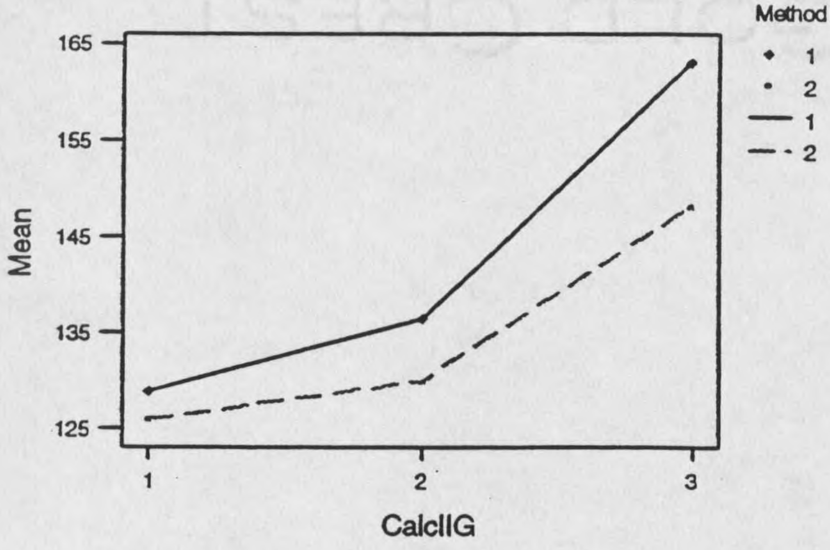
Hypothesis 8: Method of instruction (integrated or non-integrated) and scholastic aptitude in mathematics do not interact on Physics I achievement.

Decision: Reject the null hypothesis. There is a statistically significant interaction ($p = 0.009$). Furthermore, there is a statistically significant difference between the experimental and control groups when the physics scores of the high scholastic aptitude mathematics students (SATG 3) are analyzed ($p < 0.001$).

Table 31. Analysis of Variance for Physics I Scores with Method and CalcIIG as Factors

Source	DF	Seq SS	Adj SS	Adj MS	F	P
Method	1	1881.6	735.9	735.9	2.55	0.116
CalcIIG	2	7498.9	6157.2	3078.6	10.69	0.000
Method*CalcIIG	2	247.3	247.3	123.6	0.43	**0.653**
Error	49	14115.2	14115.2	288.1		
Total	54	23743.0				

Figure 13. Interaction Plot for Physics Scores by Method and CalcII Groups



Hypothesis 9: Method of Instruction (integrated or non-integrated) and achievement in Calculus II do not interact on Physics I achievement.

Decision: Fail to reject the null hypothesis. No statistically significant interaction was detected (p = 0.653).

Gender

The number of female students who completed Calculus II was 25, out of a total of 151 students who completed the course. The next four tables, followed by interaction plots when pertinent, will help to determine whether or not to reject the last four null hypotheses of this study (namely, hypotheses 10 - 13).

Table 32. Analysis of Covariance for Calculus II with Gender as Factor

Source	DF	Adj SS	MS	F	P
Covariates	2	148289	74144	18.73	0.000
Gender	1	11624	116.24	2.94	**0.092**
Error	62	245480	3959		
Total	65	409082			
Covariate	Coeff	StDev	T	P	
SAT	0.1224	0.147	0.8330	0.408	
CalcI	0.9992	0.170	5.8732	0.000	

Hypothesis 10: There is no difference in the adjusted Calculus II achievement means between female and male students, when statistically equated with respect to previous knowledge of calculus and scholastic aptitude in mathematics.

Decision: Fail to reject the null hypothesis. No statistically significant interaction was detected (p = 0.092). The adjusted means were 362.48 (N = 11) for female students, and 326.81 (N = 55) for male students.

Table 33. Analysis of Covariance for Physics I with Gender as Factor

Source	DF	Adj SS	MS	F	P
Covariates	2	2747.6	1373.8	5.88	0.008
Gender	1	715.7	715.7	3.06	**0.093**
Error	24	5607.1	233.6		
Total	27	9729.2			

Covariate	Coeff	StDev	T	P
SAT	0.08624	0.0531	1.623	0.118
CalcI	0.23006	0.0836	2.753	0.011

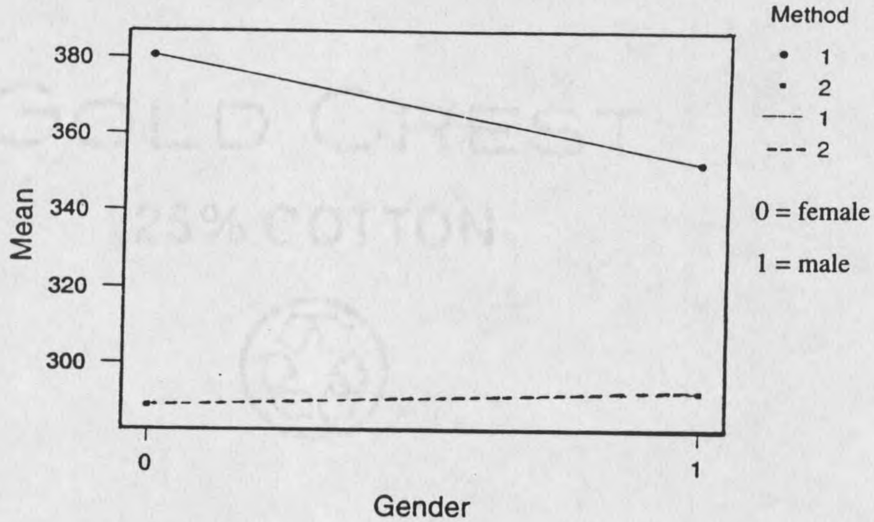
Hypothesis 11: There is no difference in the adjusted Physics I achievement means between female and male students, when statistically equated with respect to previous knowledge of calculus and scholastic aptitude in mathematics.

Decision: Fail to reject the null hypothesis. No statistically significant difference was detected between the adjusted Physics I achievement means of female and male students (p = 0.093). The adjusted means were 134.83 (N = 6) for female students, and 147.46 (N = 22) for male students.

Table 34. Analysis of Variance for Calculus II with Method and Gender as Factors

Source	DF	Seq SS	Adj SS	Adj MS	F	P
Method	1	146180	110085	110085	16.89	0.000
Gender	1	5789	2830	2830	0.43	0.511
Method*Gender	1	4976	4976	4976	0.76	**0.384**
Error	147	957860	957860	6516		
Total	150	1114804				

Figure 14. Interaction Plot for Calculus II Scores with Method and Gender as Factors



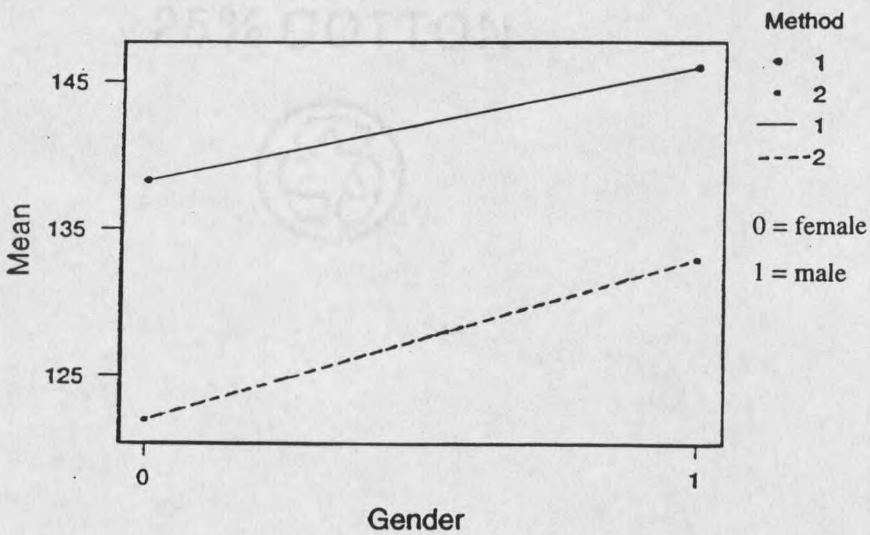
Hypothesis 12: Method of instruction (integrated or non-integrated) and gender do not interact on Calculus II achievement.

Decision: Fail to reject the null hypothesis. No statistically significant interaction was detected (p = 0.384).

Table 35. Analysis of Variance for Physics I with Method and Gender as Factors

Source	DF	Seq SS	Adj SS	Adj MS	F	P
Method	1	1881.6	702.0	702.0	1.67	0.202
Gender	1	437.7	284.7	284.7	0.68	0.414
Method*Gender	1	8.3	8.3	8.3	0.02	**0.889**
Error	51	21415.3	2145.3	419.9		
Total	54	23743.0				

Figure 15. Interaction Plot for Physics I Scores with Method and Gender as Factors



Hypothesis 13: Method of instruction (integrated or non-integrated) and gender do not interact on Physics I achievement.

Decision: Fail to reject the null hypothesis. No statistically significant interaction was detected (p = 0.889).

Additional Analyses

Calculus II Scores by Instructor

Although it is not related to any hypotheses of this study, Calculus II scores by instructor were analyzed. The following boxplot (where horizontal segments denote medians), and tables 36 and 37, are a first attempt in this direction. It is to be noted that instructors 1, 2, 3 were in the experimental group, while the two other instructors were in the control group.

Figure 16. Calculus II Scores by Instructor

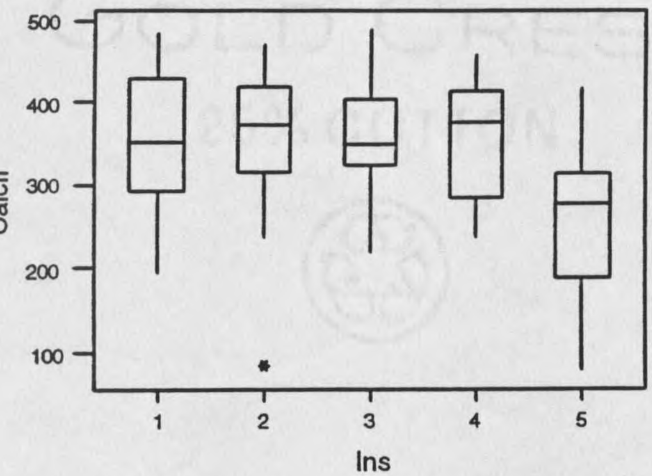


Table 36. Calculus II Mean Scores by Instructor

Instructor	Number of Students	Mean	StDev
1	31	353.06	75.36
2	30	361.80	82.47
3	36	356.36	59.87
4	19	355.58	72.00
5	35	257.60	87.92

Table 37. Analysis of Variance for Calculus II by Instructor

Source	DF	SS	MS	F	P
Instructor	4	265586	66397	11.42	**0.000**
Error	146	849218	5817		
Total	150	1114804			

The preceding table shows that there were statistically significant differences between the means. A multiple comparison based on Tukey's post-test implied that the differences were between the fifth instructor and the other four instructors.

For the topic under consideration, the next step is to discuss an analysis of covariance with Instructor as factor, and Calculus I and SAT as covariables. Through an ANCOVA it will be possible to control the latter.

Table 38. Analysis of Covariance for Calculus II with Instructor as Factor

Source	DF	Adj SS	MS	F	P
Covariates	2	108389	54195	14.32	0.000
Instructor	4	33842	8461	2.24	**0.076**
Error	59	223261	3784		
Total	65	409082			
Covariate	Coeff	StDev	T	P	
CalcI	0.8764	0.174	5.05	0.000	
SAT	0.1480	0.145	1.019	0.312	

It follows that there were no significant differences ($p = 0.076$) between the mean scores, when the five sections were compared. The adjusted means were 326 (N= 13) for instructor 1; 353.94 (N=17) for instructor 2; 347.24 (N= 20) for instructor 3; 322.06 (N=5) for instructor 4; and 286.55 (N=11) for instructor 5.

The next goal is to make a plot of the residuals of the regression model to predict CalcII scores on the basis of CalcI and SAT scores, by instructor.

Table 39. Regression Analysis on Calculus II with CalcI and SAT as Predictors

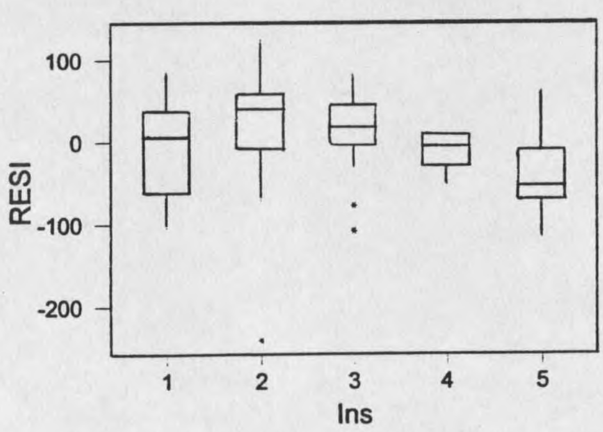
Predictor	Coef	StDev	T	P
Constant	-129.8	102.5	-1.27	0.210
CalcI	1.0137	0.1725	5.88	0.000
SAT	0.1146	0.1491	0.77	0.445
S = 63.88	R-Sq = 37.2%	R-Sq (Adj) = 35.2%		

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	2	151979	75990	18.62	**0.000**
Error	63	257103	4081		
Total	65	409082			

This regression model explains about 37% of the variability of Calculus II scores. The residuals of the regression model, by instructor, will explain how each instructor -- or some unknown factor or factors -- brought the scores of Calculus II students above or below the predicted scores on the basis of CalcI and SAT scores. The next figure shows this phenomenon.

Figure 17. Residuals of Regression Model to Predict CalcII Scores



From the preceding figure it can be concluded that most students in the first three sections (taught by the experimental group instructors) did better than expected on the basis of their CalcI and SAT scores, especially students in section 2. On the other hand, most students in sections 4 and 5 (especially the latter) did worse than expected on the basis of their CalcI and SAT scores.

It is to be noted that the design of this research -- with several instructors in both groups and the use of a common syllabus, a common textbook, and common exams -- lessens the "teacher effect" but cannot eliminate it completely.

A Homogenizing Effect of the Integrated Method

From figure 9 it can be seen that the integrated method "homogenizes" the Calculus II scores of groups 1 and 2 according to SAT, a trend that is not observed with regard to the non-integrated method. This homogenizing effect deserves closer attention.

Table 40. Calculus II Scores by Method and SAT Group

		Method 1				Method 2	
SATG	N	Mean	StDev	SATG	N	Mean	StDev
1	29	345.9	63	1	9	248.2	82.8
2	40	345.2	73.4	2	25	298.2	95.9
3	19	417.1	54.2	3	13	318.2	100.2

There is practically no difference between the means of SAT groups 1 and 2 in the integrated group, while there is a marked difference between the means of SAT groups 1 and 2 among students that followed the non-integrated approach. Moreover, the

Calculus II scores of students in the integrated group are considerably less dispersed than the scores of their peers in the non-integrated group. A similar homogenizing effect of the integrated method can be seen in figure 10.

Teacher Interviews

The two instructors who accompanied the researcher in the experimental group were interviewed by the latter at the end of the semester. Ten basic questions about calculus teaching were asked.

Question 1: Do you think that applications, mainly to physics, enhance calculus learning?

> **Teacher A**: My answer is yes. Physicists should not try to do our job, nor should we try to do their job. But there should be enough overlap so that students can see the interrelationship between mathematics and physics, or mathematics and chemistry. Applications are good, they are fun too! It is neat to see where a new concept of calculus is used, you get to understand better the calculus and its applications.
>
> The calculus course I took in my freshman year didn't have many applications. There were lots of practice problems in integration by parts and substitution, hundreds of them. It was only when I got to physics that I understood its relationship to what we had learned in calculus. I do like to have very many applications in calculus, it gives a better understanding of what the tool is all about.

> **Teacher B**: What do we think we should be teaching at the level of a second semester calculus course? I think that students should walk out of there feeling that they can apply what they know. It makes them feel better. If nothing else, it helps their motivation. On the other hand, our students were not explicitly aware that they were being taught more physics examples than the sections in the control group. I am not sure what their reaction would be if it is known to them. They thought that there was the right mix of application questions. I believe that it is important for them to realize that physics is the subject that has the most readily available examples for us.

Question 2: Last semester, the experimental group spent more time than usual dealing with applications. What was student's reaction in your section about it?

Teacher A: Students' reaction to applications? In the experimental group we covered more applications than usual. This was only their second semester in calculus, so they didn't have much to compare to. To them maybe that was how a course should be taught. I received no complaints. There was an isolated case of a student wining, in the sense that he had not had any physics yet. But he had to understand that the applications we discussed in class did not presuppose any knowledge of physics or chemistry. The course was arranged in such a way that students in physics would not have an advantage.

The enrichment notes helped the best students in the class. The reality is that there are many students that don't read the book, so it is unlikely that they will read additional material. On the quizzes, sometimes I put questions related to the enrichment notes. Thus, they had to read them . However, there was a constraint. We had to prepare the students for the common exams, covering a considerable amount of material from the textbook . There wasn't much time left to discuss in detail very interesting problems like the catenary or falling bodies when the resistance of air is proportional to the square of the velocity. If there were not to be common exams, (i.e., if I could independently prepare my syllabus and teach a calculus course) I would pay more attention to the enrichment notes, absolutely. I will save those handouts for the future.

Teacher B: I didn't receive any complaints, neither personal nor written. There were some students that maintained that this was a math course and that there should be more math and less requirement on them to understand the physics. They thought that we were trying to have them understand some applied problems that should be taught in the physics class. Anyhow, some course, somewhere in the students' program , should put together all the ideas. Some capstone course at the end of their career. At the 182 level though, the way we teach math in the United States, we don't teach out of Apostol's books and begin proving rigorously all the results of calculus. We are trying to give them a geometric feeling and a sort of usefulness feeling for the calculus that they learn. So, any extreme is probably the wrong direction. Has reform calculus gone the wrong direction because there is less time for mathematics? I am not sure.

I think that you need examples for the students, which shows them where the mathematics is applied. Real world examples. Whether we have to spend our time trying to teach them calculus through those examples or teach them the mathematics of the calculus and then present some real world examples, I am not sure. But the examples should be there one way or the other.

Let me make some comments about the enrichment notes. Maybe the noblest endeavor that we have is to teach them to model the world with math, so I think it is important that we at least present to them the information. If they would like to know where these equations came from, here is the explanation! In most of the enrichment notes that we gave to them, or presented to them, the idea was that the book picked up at step 8, and if you actually are trying to model something you need to start back a little sooner. We were not requiring them to learn the techniques of modeling until the end of the course when we presented differential equations. But that is maybe the whole usefulness of teaching the math to start with. So, not only shouldn't it be part of the enrichment notes, we should teach it. But then again, we do not have time for all that.

Maybe that is what reform calculus has done more than anything else, is have the courage to say we only have time to teach so many things, and these are the topics we think should be there. People may disagree on the topics, but the approach in the past has been as follows: give me a calculus book that has every single topic in it , so that none is missed, so that my book will be sold to everyone. I think that reform calculus had the courage to knock some of those out. We might not agree about what topics should have been eliminated from first-year calculus, but then we can present our students with extra-notes . Maybe that is the right approach rather than to put all in a book and ask the student to pull out of there what you think is important. I prefer to give them a bare bones book like the ones in reform calculus.

I liked the balance that we had. Our book concentrated less on modeling in the beginning, although it tried to at the end. If I were to teach a calculus course all by myself, I would choose a book like the Harvard Calculus book, and then I would insert a few of the proofs that are easier to present, simply for the sake of presenting the idea of proof, not because any particular proof is so indispensable to calculus, and then I would try to fit in the applications.

I found it very difficult trying to integrate. I think this is my lack of experience. I had a book that was given to me to teach out of, I had your advice, and the enrichment notes, and I just picked through all these and decided what I was going to give the student. All by myself I don't know if I am qualified to decide what I am going to teach in a calculus course. I will probably have to turn to some standard, some book, someone that is telling me what should be taught, and then , maybe, I can make decisions in little chunks at a time; that I don't like this section, that this other section needs more applications. I feel woefully inadequate to do it...

Question 3: What should be the role of proofs in second-semester calculus?

Teacher A: There should be more proofs than the book provides. I know that you put in some proofs, I did as well. I like having proofs in a calculus textbook; if not in the main body of the book, at least in the appendices so that you could always refer to them . Especially the best students could look at the proofs. Calculus students should get a glimpse about what mathematics is.

I am not advocating ε - δ proofs. When I was an undergraduate, my calculus book had these type of proofs. They seemed a mystery to me when we ran into them for the first time. My initial reaction is no to the introduction of ε - δ proofs in first year calculus. But certain proofs can be understood by students, if not completely at least some steps of it. Memorizing and regurgitating proofs in an exam is out of the question. Rather test them in such a way that to solve a problem, some idea or ideas from a proof have to be used. If you don't make them responsible for the proofs, they probably won't even read them!

Teacher B: At some point, by the time a student has finished second semester calculus, he/she should understand the basis of mathematics. That we begin with some things that we hope to be true, and we prove results from there. I am not acquainted with all the reform calculus books, but I do not think that the Harvard Calculus book does a particularly good job presenting that idea. It does not. You have to stop and teach a student about proof, regardless what you are trying to prove. It is hard when there are not many examples in the book about proofs that the students should try to read and digest and then ask questions. Maybe there should be more proofs. But again, I think reform calculus has the right idea, something is got to go, we got to sacrifice something in order to teach the rest well.

I would like to see a treatment of the idea of proof, not necessarily many proofs. It is fine with me if we are going to tell a student that you have to believe me, if you are interested in a proof just stop by my office. Even though I would like to see more of the idea of proof, I am not adamant about which proofs should be there. It is not the omission of a particular proof I am worried about, it is the omission of the idea of proof.

Question 4: How was your experience with group work during the course?

Teacher A: We didn't do group work in my section last semester. The biggest positive result that can come out of group work is that it gets students working together outside of class. Realistically speaking, students can not finish group work in a 50 minute class period. I am skeptical about taking class-time in second semester calculus for group work. In Calculus I or precalculus I would do group work; they are starting their college studies and need support from their peers.

The situation is different in Calculus II. I believe that it is more beneficial to them to see collective work. I like to have one day per week to relax, I don't lecture on wednesdays but stand in front of the class answering questions. I do pose questions too, and everyone has the opportunity to think about them before the smart guy in the back gets the answer!

Teacher B: In Math 182 I didn't do much group work at all. When I taught Math 181 I did a great deal of it. You have so many days a week to teach, are you going to sacrifice those days to do group work? In a beginning calculus course you should, but only from a social aspect. Entering students could be immature people that will seat in class and not know the name of the person next to them; they are a little scared, they just need an excuse to talk to each other.

I think entering students need to realize the power of working together, certainly industry wants to have them work together so that they are better prepared to work as a team once they graduate from college. But I really think the group work we do in class is simply a replacement from what they should be doing outside of class. And maybe the reality is that students don't work together outside of class. We are just demanding that they do in the classroom what we wish they would do outside of the classroom. If all students were to work together outside of the classroom, there would not be a need to have them in groups in the classroom. That is a little too strong, there are some problems that are better handled with the collaboration of more than one head.

We, the instructors, are the experts, we are supposed to give the students our knowledge. However, it is true what some people say: I can't just give you what I know . You have to do it yourself, discover it on your own. But that is really the way our university system is set up . A student comes in, listens to an expert, then the expert summarizes, highlights certain topics and points out the pitfalls of the subject.

Some teachers are better on a one-to-one basis, and group work allows them to circulate around the room and see what students are doing, while some instructors are very good, excellent, lecturers. So, it should be up to each instructor. In other words, if a teacher can convey his/her knowledge and get to the student through the setting of group work, with well-crafted problems, then lecturing may not be as important.

Question 5: What is your opinion about the textbook used in Calculus II? Do students read it?

Teacher A: There are some things that I do not like about the book and I will explain why. I think there are some fantastic problems in there. Also some very good examples, but I think that in an attempt to make it readable, they have thrown away a good deal of mathematics. I have a theory: The better a book is as a learning tool, the worst it is as a reference and vice versa. If it has too many words, it is hard to make it a good reference; if it is to be a good reference it has to be concise. When students ask me whether they should keep the textbook after the semester has come to an end, I tell them not to do so. Sell it immediately and go buy a real calculus book. By real calculus book I mean one that could serve you as a reference book. This book is a horrible reference. You can't go back in there and look for theorems.

In the Harvard Calculus book often the hypotheses are missing. I would like to see more theorems listed. The authors went to the other extreme when compared to books of the type theorem, proposition, lemma, corollary. In the Harvard Calculus book you find a lot of words without any good theorems you can refer to. In the past one was looking for a book that would be good for learning and reference. Harvard Calculus lacks in the latter sense. There is not a balance between readability and mathematical consistency.

I really do like the problems in the Harvard Calculus book. There you can find some hard problems. If you , as an instructor, don't think about them before going to class, you might get stuck. However, the textbook treats the whole idea of series very lightly, many important things are missing. New editions could improve some of the shortcomings.

Before calculus reform, everyone had his/her pet interests, so books started to be bigger and bigger , up to one thousand pages long. Some teachers thought that you had to cover every single topic, an evident wrong decision.

If you are a mathematics or physics major that is planning to go into graduate school, where do you start to see ε - δ proofs? Maybe you should start to see a little of them in first-year calculus. Understanding these techniques requires a long maturation process, so the sooner the better. The hardest thing in calculus is the idea of limit. In the Harvard Calculus book this idea is missed, it is gone. Then students get into real trouble when they have to deal with improper integrals, for example. You can't find in this book any list of properties of limits. In every calculus course one has to spend some time dealing with limits!

Teacher B: The Harvard Calculus text is a great book. Again, I think they had the courage to leave out some things that no one would touch before. They had the courage to leave out the ε - δ presentation of limits, something with which I agree. I think the authors did a good job, a book that could be read by students. Although some students I have talked to seemed to complain about the book and how hard it is to read.

The Harvard Calculus book is not the type of book where you have example after example and then all the problems follow the examples. The authors are trying to strive for a higher level of understanding on the part of the students, not just to apply a pattern. I really like the book. There has been support and non-support in our department by different instructors, but I think you can take the book and effectively teach with it.

They've done a great job putting a book together. Maybe the authors could seat down and agree on the philosophy that we should present the idea of proof. A preliminary section at the beginning could be a possibility. Besides, I don't think that leaving out partial fractions was a good idea; there is very little of it in the book. Trigonometric substitutions were left out, that is fine. It is a fact that the algebraic skills for those that go through Harvard Calculus are less than those that go through a traditional course.

I would use a reform calculus book because I could not come up with all the great problems one finds in Harvard Calculus. For the typical manipulation problems, that are also important, I could come up with problems of my own. For instance, problems that involve partial fractions. But it is more difficult to come up with the type of problems that Harvard Calculus has in abundance.

I am not really sure which one is more readable from a student's standpoint, a traditional or a reform calculus book. But I would hate for a student to get out of calculus and not have the ideas that the Harvard Calculus elaborates on. So, I think I prefer it as my resource and as a textbook for the students.

Question 6: What policies should be adopted with regard to homework and quizzes?

Teacher A: I assign lots of problems, but do not collect them. The quizzes are based on them. Feedback from the students shows that they have a favorable opinion about this arrangement. I have had a few students that would have liked to have the homework collected and graded. However, many students are happy that I don't collect homework every day, because some days they are very busy. Let us not forget that engineering students have lots of work to do in several courses. In first-semester calculus I did grade homework, for the same reason that I had them work sometimes in groups. They were not supposed to be responsible enough at that stage of their college experience, they were just starting out.

Teacher B: We are all busy and have obligations and if I am not demanding some of their time, someone else is demanding a lot of their time. Homework is absolutely the most important aspect of the course, and quizzes function for me as a replacement for grading homework on a daily basis. It is also a good preparation for them . You can fool yourself into thinking that you have mastered a topic when you have a book by your side and you can refer to it. Quizzes are generally a good idea but if I had to choose I would pick up homework daily and mark it. The reality is that I just do not always have time to do that. I did assign homework daily, which was due the next day but I didn't collect it . However, in each semester that I've taught, there always comes a time in the semester when they have to turn in homework for a week or two just to get them back on track.

If I am not grading the homework, on the quizzes I put problems from the assigned list. So, if you have done your homework, there is your reward. Again, somehow they should be motivated to do this on their own but you have a lot of responsibilities. So, if I can make it a reward for you on the quiz, so much the better. I am not trying to use it so much as another method of assigning you a grade, I am trying to use it as a teaching tool, I like to put things on the quizzes that will reward you for doing homework. It is just a motivational tool, and I also like to use it to ask the students questions that they might not have thought about.

Thus, the quizzes were mostly off the homework but maybe 20% were problems that required an extra step, to put some ideas together.

Question 7: Calculus II meets five hours per-week despite the fact that it is a four credit course. Do you consider that this is a good policy?

Teacher A: No student has complained. In fact, occasionally I ran an extra help-session. As I mentioned before, I put aside one day per-week for an informal meeting with the students, a sort of question and answer period that proceeded quite informally. Thus, I lectured three days per week and at the end of the week I always gave them a fifty minute long quiz so as to prepare them for the exams. Taking into consideration that the amount of material to be covered in Calculus II is quite impressive, an extra hour is a positive decision taken by the department of mathematical sciences at MSU-Bozeman.

Teacher B: From the students standpoint, they are only getting four credits and have to come to class five days per week. Part of the reasoning behind why we are having students attend more days each week is the idea of a lab. For example, in a physics lab you would go for three hours and it would typically contribute one extra credit to your physics class. So, it is not always hour for hour, the credit hours are not necessarily exactly the hours that you spend in class. On the other hand, students are getting to pay for four credits and they are getting five credits of instruction. From their point of view it is a sort of trade-off, they are not getting as many credits, but they do not have to pay for them.

How many days a week you should teach calculus? I can do it in less than five. I think I could actually do it if I were not to give them quizzes. One day was completely sacrificed to the quiz, no instruction was given that day; maybe a couple of questions briefly at the beginning of the hour. If I were to grade homework every night I would not need that fifth day. It is just important to give them some feedback.

None of my students complained about the extra hour, but I am sure they talked about it. Some students are quick to complain about an "injustice". However, they didn't have to pay for the five credits, they got a deal!

Question 8: Should Calculus I be a pre-requisite for first-semester physics?

Teacher A: I don't know how they teach physics here. Where I went to school it was a pre-requisite. Students could not take it even concurrently, unless there were special circumstances. I have the impression that the physics course there, was significantly more challenging than the one they teach here. In my college, even if you had one year of high school physics and took a non-calculus based physics course the first semester, the equivalent of Physics I was tough. So, it all depends on what level first-semester physics is taught.

Teacher B: We would have to talk to the physics department. They seem to have two sequences of physics courses. One is non-calculus based, the other is calculus-based. The reality is that they cannot demand much calculus knowledge from their students because some of them are not taking calculus and those that are taking calculus have not had much integration theory until well into second semester calculus.

Calculus and physics, one helps the other. If you had physics before, it helps your calculus, and vice versa. I would really have to talk to the physics instructors and see how handicapped they feel when presenting ideas that are simple applications of the derivative or the integral. I am sure that they would like their students to have more calculus tools at their side. I think that taking physics and calculus concurrently is fine.

Question 9 : At MSU-Bozeman three different types of calculus courses have been under way; namely, reform calculus (Math 181-Math 182), traditional calculus (Math 175-Math 176), and honors calculus (using Spivak's book). Should it continue to be so? Can these three approaches coexist peacefully?

Teacher A: There should be separate calculus courses for engineering majors and, say, business majors. If there were just one course, it would be slowed -down or watered-down. For engineers this is a tool they will be using all the time, so they need to know calculus really well and move at a good pace or get out from engineering school. If you cannot do well in a Math 181-2 setting (the type of courses we've been teaching), you should not be in engineering.

There is no agreement among the business departments about the need to have their students learn calculus. I believe they should take it, although at a different level than the engineering or the natural sciences students. Among the engineering people there are no doubts about the need of having their students take a strong calculus course. After all, we not only prepare their students in mathematics, but weed out the weakest students, those that are not suited to become engineers. The truth of the matter is that we weed out more engineering students than they do.

Honors calculus? Logistically it is difficult to implement . Nonetheless, it is good for students with a strong background. But there is a drawback, we might need these students in the regular Math 181-2 classes in order to drive the whole class forward. They are an important asset for the class. So, there is no easy solution. The best students need to be challenged, and an honors section might be a good way. But not having them in the regular classes is a real pity. What should be done?

Teacher B: The different series of calculus courses was originally probably a good idea, but is has gotten a little out of hand. We have a one-semester Math 170 for business students, a kind of survey of calculus. We also have the sequence 175-176. Its content is basically what we cover in Math 181-182, but from a traditional book. Maybe students in Math 181-182 are more sophisticated , but the material is not more sophisticated. Not at all. At our university, each department has a certain policy with regard to calculus. For instance, students majoring in electrical, chemical, mechanical, or civil engineering have to take Math 181-182. Students in engineering technology take math 175-176. I think that the series 175-176 and 181-182 should be put together, conglomerated. Anyhow, we teach only one or two sections of math 175-176; few students take it. Math 181-182 can stay just like it is if we want a reform or a non-reform book.

In our honors section, I think we should do whatever we do in Math 181-182 and demand that they have the ability to prove many things by the time they finish first year calculus. If 181-182 follows the reform calculus approach, then honors calculus would be reform + proofs. Similarly with traditional calculus. Engineering departments who are perfectly happy to have their students omit the proof part of calculus, and just let them have the manipulative techniques and the understanding, and the ability to apply it to real world problems, would be happy with the 181-182 series. The very best engineering students or any students could still take the honors section, and we would demand that our math majors take the honors section.

Question 10: Should precalculus at MSU-Bozeman be a prerequisite for calculus? Should more students take it?

Teacher A: The existing system --having all students take a placement exam before starting their first semester-- is good. It is not perfect, but it is good enough. Those students that show a lack of algebraic expertise have to take precalculus. I like a lot the text used in this course, by Warren Esty. Can they learn in one semester what they should have learned in High School along several years of hard study? It is difficult to give an answer .

Most of my worst students in Calculus II had not taken precalculus or had gotten a C in it. So, maybe we should raise the standards of the placement exam too. It is a pathetic situation, they don't even understand the idea of an equality sign (i.e., if you do something to one side of the equation, you have to do it to the other side). They just don't seem to get that. A book like Esty's attempts to clear up those misunderstandings.

A lack of algebraic skills hurts a lot a sizable number of students in Calculus II. For example, some of them believe that partial fractions decomposition is an advanced topic. I tell them, this is a topic from math 103, a course for which you don't even get college credit! Definitely, weak algebraic skills are the downfall of many students and I think that Esty's book is good at fixing those problems. It is a tough book, some students find it hard to read even though Warren tries to make it really readable. It is geared toward students that were not successful in learning algebra, that is why they are in precalculus.

Teacher B: Precalculus at MSU-Bozeman should not be a prerequisite for calculus. Math 160 (precalculus) is not designed for students that have had a successful mathematical career, it is designed for those students who have not been good students all along, students that didn't quite learn the Algebra II they took in high school, but yet they still want to be engineers. So this is their one last-ditch effort to understand what mathematics is about, and perhaps be able to tackle calculus, which they would need to proceed to their chosen career.

It is true that no matter how good you were in high school algebra, math 160 will improve your algebraic skills, your outlook on what algebra is about. This course teaches students to use graphics calculators and many other things we ask in Math 181-182. It is a good preparation. However, if you had the ideal student who wants to be an electrical or chemical engineer, this student already has pretty good skills.

Math 160 gives you another chance if you are not prepared. I don't think we need to require it for every student. It is a great course though, I really like it. Why are algebraic skills so low among students in calculus? It is really a problem in the high schools, so it becomes a problem here. Perhaps we should raise the standards of the placement exam. On the other hand, many people flunk the calculus courses because they do not work.

Symbolic manipulation is important. Many students seem not to understand the idea of a dummy variable. That is at the root of any trouble they have with symbolic manipulations. If students arrive in my calculus course and understand what a dummy variable is, and understand the functional notation, but don't know how to complete the square, I can teach them that. But if they do not have those understandings, then it becomes just another sequence of steps that they have to memorize, and it is daunting to them. The misunderstanding is deeper, it is not whether they did or didn't learn completion of squares.

I wish all students had time and take the course based on Esty's "Language of Mathematics". The contents of this book should be taught at all high schools, there is where it belongs. High school students are perfectly capable of understanding the material in the aforementioned book. I took Esty's book when I first arrived, and despite the fact that I had taken algebra at high school, I suddenly understood what algebra was all about. He has definitely put his finger on what students have trouble in understanding mathematics.

Question 11: What should be the role played by graphics calculators in a calculus class?

Teacher A: I think that they are a great tool, but they are abused. Instead of using them as a tool, many students use it not even as a crutch but as a wheelchair. People will not think by themselves first and then use the tool to verify what they have come up with. They just turn on their calculator before starting to think. Former engineering students of mine are grateful to me because I insisted that they should not just grab their calculator without a previous thought process. They have to be able to picture what some graphs look like, to visualize the generic functions, to know how to translate them. It is good that the calculators are there, but I think that they are not used the way that they should be.

I ask my students, for example, give me an upper bound of the sine function. Some of them try to use the calculator immediately, are incapable to have a mental image of the sine function! So, the use of the calculators is often abused. By not making the effort of visualizing functions, students are missing a huge facet of calculus. Calculators should be used to explore, to check a conjecture, to perform a long calculation, and the like. However, they should not replace mathematical reasoning.

If you give the student a TI-92 they may think that they do not have to learn anything. Just push a button and get an answer for an integration process. The machine will integrate it for you. Let us not forget that there are some topics students have to learn: integration by parts, substitution. These basic techniques have to be learned by hand, without calculators. It would be pretty sad if we were to ask them to find $\int x^2\,dx$ and they turn on their calculators. I am in favor of keeping that kind of technology out of the classroom!

Teacher B: If you always use a graphics calculator to generate a graph, then you haven't learned what we all learned in calculus and that is how to graph functions. The whole reason we learned how to find a maximum or a minimum, or how to analyze the concavity of a function, was to draw pictures. On the other hand, any technology used correctly simply allows us to see immediately whether a conjecture is true or not. Students need to explore, experiment, and then go back and determine the mathematical basis of their conjectures. So, I think that if it is used to explore it becomes a great tool.

So long as we test correctly and we ask questions that demand understanding, I don't see how a calculator can hurt. It ends up hurting because sometimes we don't ask the right questions, we don't emphasize the right things.

What should we done with calculators that do symbolic manipulations like the TI-92? A student can claim that he/she could bring a PC to class and "fire up" Maple and ask it to do their quizzes problems for them. So, just because technological innovations lead to smaller powerful calculators doesn't make their argument, that they should use it, any more valid. We should dispel that to start with. Just because it has gotten to calculator size does not have anything to do with whether we allow it in class. Let us consider statistics. You have an introductory course in statistics at every college. What you really want students to take out of this type of course is to have them read and understand articles that claim something about statistics. You can't really get a hold on understanding statistics unless you do it yourself. If you calculate a few of the typical standard deviations, means, confidence intervals and the like, you have a feeling for what they are telling you. The same with integrals . Could you understand everything that can be understood about integrals without having done some simple integrations by pencil and paper? I doubt it. It is the same situation as with arithmetic.

Do you need to understand integral calculus? Well, if you are going to understand differential equations you certainly do. You need to know what integration is all about. Besides, you need to understand those things if nothing more because when you are in higher level classes of your major and you are told that to go from here to there one has to integrate. Whether or not you could actually do that in your head quickly is not important, but to understand what that means is important.

There is absolutely nothing wrong with any technology, but we should realize that technology cannot replace mathematical reasoning.

CHAPTER 4

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

Summary of the Study

The question addressed in this study was whether there is a difference in achievement between students that followed the Harvard Calculus textbook, and those students who, besides using the Harvard Calculus textbook, received supplemental materials and studied several mathematical aspects of physics that went beyond the textbook.

The subjects in the study were the 151 students registered in Calculus II at MSU-Bozeman during the Fall semester of 1996. In parts of the analysis the number of subjects was smaller, namely when information for some of the variables was not available for all. There were a total of five sections of Calculus II with different instructors. Three sections were assigned to the experimental group (Harvard Calculus + additional materials), also called Method 1, and two sections were assigned to the control group (Harvard Calculus alone), also called Method 2.

The main variables under study were the total scores in Calculus II (three one-hour exams and the final), and the total scores in Physics I (one midterm and the final) for those students that took this course concurrently with Calculus II. The other variables considered in the analysis were the total scores in Calculus I (for the students that took it at MSU-Bozeman the previous semester), and the SAT-math scores (to be denoted SAT for short). Calculus I and SAT acted as covariables. The information about the time in which the students took Physics I (not yet, concurrently with Calculus II, or before, to be represented by the variable WhenPh) was also considered, as well as gender.

Calculus I scores were obtained directly from the Spring 1996 instructors (corresponding to three one-hour common exams and the common final), while the SAT scores were delivered by the university's office of admissions. The latter also provided the information with regard to the students' Physics I status. The scores in Physics I were sent by the physics instructor to the researcher.

The statistical tools used were Analysis of Covariance, Analysis of Variance, and Regression, accompanied by scatter-plots, box-plots, and interaction plots. The alpha level was set at 0.05.

Despite the fact that Calculus II is a four credit course, classes met five days per-week for fifty minutes. Four of the instructors (two in the experimental group and two in the control group) were of similar background and experience, while the researcher -- who taught one section in the experimental group -- had comparatively more teaching experience. All of them were graduate teaching assistants at the Department of Mathematical Sciences (MSU-Bozeman) during the time of the study.

The three instructors in the experimental group met on a weekly basis and closely coordinated their lectures and other activities. Besides, they and the instructors in the control group attended the regular meetings of the course. All the one-hour exams and the final for Calculus II were common to the five sections. They were taken simultaneously by all students, at a common time set aside in the evenings. Each instructor graded one or more questions for every student, setting a common rubric for partial credit. A regular faculty member, who did not teach a Calculus II class during the fall of 96, was assigned by the Department of Mathematical Sciences as course supervisor. His main duties were to draw a syllabus and make a first draft of every exam, as well as to verify that every section was delivering the material on schedule.

The experimental group covered the main topics of chapters 6-10 of the Harvard Consortium Calculus textbook, stressing the problems about applications as well as the conceptual and computational aspects of calculus. Besides, as was mentioned before, students in this group received enrichment notes, mainly on several mathematical aspects of basic mechanics -- a sample of which can be found in appendix E. These materials were related to several problems in the textbook, wherein the differential equations are presented but not deduced from first principles. Moreover, problems of elementary chemical kinetics comprised part of the supplemental materials. They show the underlying common mathematical structure of some aspects of physics and chemistry. The control group followed the textbook closely -- covering the same chapters as the experimental group -- without departing from it.

The two instructors in the experimental group, who willingly and voluntarily participated in the experiment, were interviewed by the researcher at the end of the semester. Their responses are to be found in the pertinent section.

Conclusions

Analysis of Calculus II Scores

For the students that took Calculus I at MSU-Bozeman the previous semester, an analysis of covariance was performed to compare the two methods in terms of the Calculus II scores, using SAT and Calculus I scores as covariates. The latter variables could represent the scholastic mathematical aptitude of the student and his/her past performance in college-level mathematics courses. A significant difference was found between the experimental and control groups ($p = 0.015$).

In order to look for possible interactions between method of teaching, previous performance in calculus, and scholastic math aptitude, the Calculus I and the SAT scores were categorized in three groups using the respective quartiles to define the groups (below Q1, between Q1 and Q3, above Q3). Two separate analyses were performed using two-way ANOVAS. Working with method and Calculus I groups, no interaction was found ($p = 0.665$) and the difference between methods was significant ($p = 0.007$). The difference between Calculus I groups was also significant ($p = 0.002$). The performance of the students in the experimental group was better in the three groups defined by Calculus I.

It can be concluded that the experimental method produces better results regardless of the students' performance in Calculus I. Working with the SAT groups, the interaction was not significant either ($p = 0.196$), but significant differences were found among the SAT groups ($p = 0.002$) and between the methods ($p < 0.001$).

There is no interaction between method and when they take Physics I ($p = 0.065$). Neither was there a significant difference with regard to Physics I status (not yet, concurrently with Calculus II, before). There is a significant difference between methods ($p < 0.001$). An analysis of covariance with WhenPh as factor, using Calculus I and SAT scores as covariables, showed no significant differences ($p = 0.907$).

Analysis of Physics I Scores

Interaction was found ($p = 0.009$) between method and SAT groups when Physics I scores were used as the response variable. For the lower and medium groups of SAT there was no significant difference in Physics I scores between the two methods. However, for the students in the upper group defined by the SAT scores, the difference in Physics I scores was significant ($p < 0.001$). In other words, high scholastic mathematics aptitude students that followed the integrated method achieved better scores in Physics I.

No interaction was found when Physics I scores were analyzed by method and Calculus I groups ($p = 0.967$). There was no significant difference between methods ($p = 0.272$), while there was a significant difference between Calculus I groups ($p = 0.005$). On the other hand, no interaction was detected when Physics I scores were analyzed by method and Calculus II groups ($p = 0.653$); it is to be noted that Calculus II scores were categorized in three groups too (below Q1, between Q1 and Q3, above Q3). No significant differences between methods were found ($p = 0.116$), while there was a significant difference among Calculus II groups ($p < 0.001$).

An analysis of covariance for Physics I as the response variable, using Calculus I and SAT scores as covariates, did not find a significant difference between methods ($p = 0.161$).

Gender

An analysis of covariance showed no significant differences between female and male students ($p = 0.092$), when Calculus II scores were analyzed. Similarly, an analysis of covariance for Physics I, with gender as factor, showed no significant differences ($p = 0.093$). In both ANCOVAS, Calculus I and SAT acted as covariables.

Method of instruction and gender did not interact on Calculus II achievement ($p = 0.384$). There was not a significant difference between female and male students ($p = 0.511$), but there was a significant difference between methods ($p < 0.001$). Moreover, gender and method of instruction did not interact on Physics I achievement ($p = 0.889$). Neither method nor gender led to significant differences ($p = 0.202$, $p = 0.414$ respectively).

Additional Analyses

Calculus II scores were analyzed by instructor through an ANOVA; a significant difference was found ($p < 0.001$). Sections 1, 2, and 3 were in the experimental group, while sections 4 and 5 were in the control group. The students in section 5, as a group, got significantly lower scores than their peers in the other four sections. However, when Calculus II scores were analyzed by instructor through an ANCOVA, with Calculus I and SAT acting as covariables, no significant differences were found ($p = 0.076$).

It should come as no surprise that no significant differences were found in the preceding ANCOVA. From comparing two groups of 50 and 16 subjects (table 22), we ended up comparing five sections with 13, 17, 20, 5, and 11 subjects respectively.

A regression analysis on Calculus II, with Calculus I and SAT as predictors, allowed an analysis of residuals of the regression model that led to the conclusion that most students in sections 4 and 5 -- especially the latter -- did worse than expected on the basis of their Calculus I and SAT scores (figure 17).

The integrated method exerts a "homogenizing effect" on Calculus II scores of groups 1 and 2 according to SAT, a trend that is not observed with regard to the non-integrated method. A similar effect of the integrated method is observed when Calculus II scores are analyzed, with the variables WhenPh and Method as factors.

Teacher Feedback

Based on the teacher interviews, to be found in the preceding chapter, several conclusions can be drawn. Instructors liked the applied emphasis of the course, and in particular the enrichment notes used in the experimental group. They thought that a balance had been reached between conceptual ideas, computational problems and topics of physics and chemistry that have a high mathematical content. These topics helped to convey mathematical ideas relevant to calculus.

There was the perception that the aforementioned notes helped the best students the most. They would use them in the future, although there was always a time constraint that limited the number of topics that could be covered in class.

There should be some proofs in the textbook, especially the good students could profit from them . The instructors meant the treatment of the idea of proof, not necessarily the presentation of many proofs. It was not the omission of a particular proof what worried them, it was the omission of the idea of proof in a mathematics course.

They didn't use the group work setting. One of them chose one day per week for a question-answer period with no lecturing. Students were encouraged to ask questions therein. The other instructor was of the opinion that it should be up to each instructor whether to do group work or not. Some teachers, he asserted, are excellent lecturers. So they should lecture. Others are very good working with groups, then they should choose the latter alternative.

Both instructors found some weaknesses in the Harvard Calculus book, for instance the lack of the idea of proof mentioned before. But both had a positive opinion about the examples and problems to be found in the book. One instructor praised the authors for having had the courage to throw out some materials from traditional books, while the other instructor thought that the book was not a good reference. Its readability was pointed out, but the disappearance of some aspects of the theory of limits was regretted.

Homework was assigned but not collected in both sections. Weekly quizzes were mainly based on the list of assigned problems. Although the importance of grading homework was recognized, one instructor was of the opinion that graduate teaching assistants usually did not have the time to grade it on a daily basis.

There was a consensus in the sense that meeting 5 days per-week was a good idea, even though Calculus II is a 4 credit course. The extra day gave instructors the opportunity to have the students take a 50 minutes quiz each week. No student complained about this arrangement, adopted by the department of mathematical sciences at MSU-Bozeman.

The instructors in the experimental group had no definite opinions on whether Calculus I should be a prerequisite for first-semester physics. It all depended on the level at what Physics I was taught. One instructor noted that calculus helps physics and vice versa. He thought that it is fine to take physics and calculus concurrently.

No strong feelings were expressed about the fact that at MSU-Bozeman three types of calculus courses are offered: traditional, reform, and honors. They are intended for different audiences. One instructor thought that although honors calculus is a good challenge for the best students, the regular classes suffer because of the absence of them.

Students are coming into calculus with a preoccupying lack of algebraic skills, which is one of the main reasons for students' failure. What is even more serious, basic ideas like dummy variables and functional notation are absent from the minds of many students. Maybe the standards of the placement examination should be raised. On the other hand, probably more of them should take precalculus, a highly praised college course intended for those that want to be engineers but do not have a good background in mathematics. These ideas were shared, one way or another, by both instructors.

Finally, the issue of technology in the calculus class drew sharp responses. The instructors felt that quite often the use of calculators was being abused, in the sense that they were not used for exploration but as a replacement of mathematical reasoning. One instructor was in favor of keeping the TI-92 technology outside the calculus class altogether, while the other instructor pointed out that what is important is to test correctly, asking questions that demand understanding.

Recommendations

Recommendations for Curriculum and Instruction

On the basis of the conclusions and the experiences accumulated in the present study, an integrated approach to second-semester calculus is a possibility that should be considered by those instructors that are scheduled to teach the course using the Harvard Calculus textbook. Such an approach does not cause any harm to the students. On the contrary, it might lead to better achievement in calculus and physics. Furthermore, it is recommended that the Harvard Calculus textbook should not be used verbatim. Each instructor can stress what he/she considers are the strong aspects of the book while supplementing it with enrichment notes, a sample of which can be found in Appendix E.

The enrichment notes should cover not only relevant applications, beyond those given in the book, but also acceptable proofs of several important results such as the fundamental theorem of calculus or the criteria for comparison of improper integrals. Important tools, like the main properties of limits and the usual tests in the theory of series, should also be covered in the notes.

Recommendations for Future Research

1. It is advisable to implement new studies, using large samples, comparing the integrated and non-integrated approaches to second-semester calculus teaching. Only a meta-analysis, based on multiple studies, could lead to a definitive answer.

2. There is a trend to be observed in figure 10, when Calculus II scores are analyzed with the variables Method and WhenPh as factors. New research needs to be done which looks for interaction.

3. A replication of this study can be done, adding more covariates. Besides CalcI and SAT scores, one should contemplate adding, for instance, college GPA, having in mind lessening the number of uncontrolled variables.

4. Second semester calculus-based chemistry scores should be analyzed in future studies, provided the chosen sample includes a sufficient number of students that are taking chemistry concurrently with Calculus II.

5. Students' future academic performance, beyond their first year, can give further valuable information with regard to possible advantages or disadvantages of the integrated method. Longitudinal (one or two years in length) studies, collecting data on how well students do on courses such as control theory, physical chemistry, electrical circuits and the like, could further substantiate an integrated approach to calculus teaching.

6. Further studies have to be conducted with the participation of many instructors, so as to minimize the "teacher effect" phenomenon that usually haunts educational research of this type. Besides, random assignment of subjects to treatments is a goal to have in mind, no matter how difficult and unattainable it is in actual practice.

7. At MSU-Bozeman, male students far outnumber female students in Calculus II. Does this imbalance occur at other land-grant universities in the USA? Research should be done to find out what keeps women from enrolling in calculus courses.

8. Replicating the study with a traditional calculus textbook also needs to be done.

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APPENDICES

APPENDIX A

CALCULUS II SYLLABUS AND EXAMS

MATH 182 Fall 1996

Required Text: Calculus, Hughes-Hallett, Gleason, et al
Also required: A Graphic Calculator

WEEK #	STARTS ON	DAYS	SECTIONS
1	Sept. 3	⌘TWRF	6.1, 6.2
2	Sept. 9	MTWRF	6.3, 6.4, 7.1
3	Sept. 16	MTWRF	7.1, 7.2, 7.3
4	Sept. 23	MTWRF	7.3, 7.4, 7.5
5	Sept. 30	MTWRF	7.5, REVIEW, TEST 1, 7.6
6	Oct. 7	MTWRF	7.7, 7.8, 7.9
7	Oct. 14	MTWRF	8.1, 8.2, 8.3
8	Oct. 21	MTWRF	8.4, 8.5, 8.6
9	Oct. 28	MTWRF	8.6, REVIEW, TEST 2, 9.1
10	Nov. 4	M⌘WRF Election Day	9.2, 9.3
11	Nov. 11	⌘TWRF Veterans Day	9.4, 9.5, 9.6
12	Nov. 18	MTWRF	9.6, REVIEW, TEST 3, 10.1
13	Nov. 25	MTW⌘⌘ Thanksgiving	10.1, 10.2
14	Dec. 2	MTWRF	10.2, 10.3
15	Dec. 9	MTWRF	10.3, REVIEW
16	Dec. 13	Finals Week	

Final Monday, December 16th, 10:00-11:50 AM

Important Dates to Remember:
Last day to ADD: September 16
Last day to DROP: September 23
Last day to DROP with a W: October 28

Two of the most important components in order for a student to be successful in study of calculus: ADEQUATE MATHEMATICAL SKILLS IN THE PREREQUISITE TOPICS AND MASTERING THE NEW TOPICS IN A TIMELY FASHION. Each topic covered in this class depends on the previous topics that have been covered. Mathematics is not a spectator sport.

Calculators will be used for the homework and tests. See the back side of this sheet for recommendations.

The course grade will be based on three examinations (100 points each), quiz/homework results (100 points), and a final examination (200 points).

Students are required to take all hour exams at the scheduled hours as they appear on the syllabus. Any exceptions must be approved by the course supervisor, and in no case will exceptions be made for two exams. In the event a hour exam is unavoidably missed for a reason approved by the course supervisor, the final comprehensive exam will count 300 points instead of 200 points. No makeup exam will be given.

Math 182

October 8, 1996

Exam 1

Show all work

Name____________________

Instructor____________________

Problem	1	2	3	4	Total
Possible	8	52	20	20	100
Score					

1. a. Find the derivative of $(\ln(x^3))^2$.

b. Find the indefinite integral $\int 6\frac{\ln(x^3)}{x}\,dx$

2. Evaluate the following integrals exactly. [Do not estimate.]

(6 pts) a) $\int\left(\frac{1}{x^2}+7\cos x\right)dx$

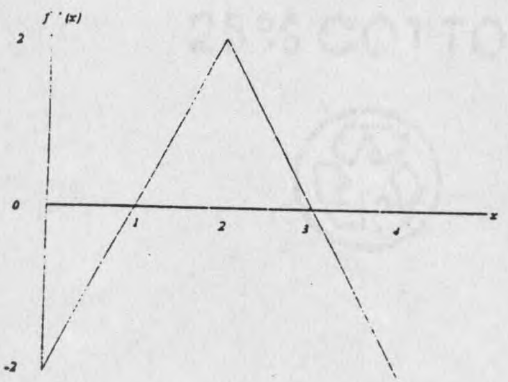
(10 pts) b) $\int_0^1 x\sqrt{1-x^2}\,dx$

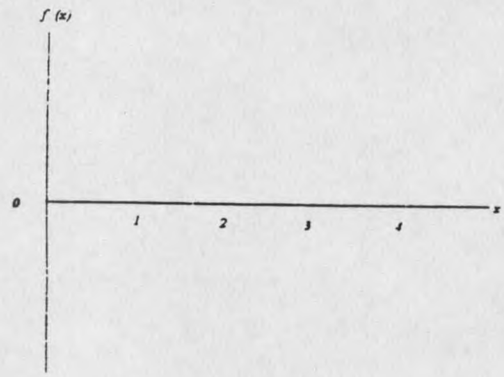
(10 pts) c) $\int\frac{z+1}{z^2+1}\,dz$ *Recall:* $\frac{d}{d\xi}\arctan\xi = \frac{1}{1+\xi^2}$

(12 pts) d) $\int_1^e x\ln x\,dx$

(14 pts) e) $\int x^5\left(x^3-\frac{1}{2}\right)^4 dx$

3. Below is the graph of the derivative $f'(x)$ of a function $f(x)$, with $f(0) = 1$. Sketch the graph of $f(x)$. Show all critical points and inflection points giving their coordinates. [Be clear in showing how you calculate the coordinates.]





4. A skier on a particular section of the downhill skiing speed trial at Pete's Hill is observed entering the section with velocity 40 m/s. 100 meters later the skier's velocity was measured to be 54 m/s.
<u>Assuming the acceleration is constant</u>:

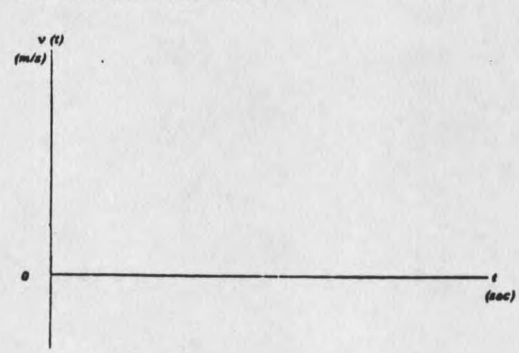
a) Sketch the graph of velocity, $v(t)$, versus t for this section of the course.

b) Label the following quantities on the graph from part a).

  i) distance = 100 meters

  ii) the acceleration at any point.

c) Find the acceleration.



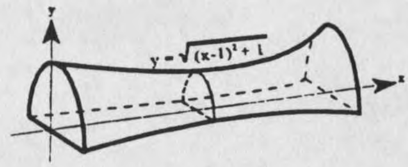
Math 182 Exam 2 Name____________________

October 29, 1996 ***Show all work*** Instructor____________________

Problem	1	2	3	4	5	6	Total
Possible	15	16	15	12	22	20	100
Score							

Note: Problem 2b is the only problem in which an approximation is acceptable.

1. At any point x, the cross section of a solid is a semicircle of radius $\sqrt{(x-1)^2+1}$ where $0 \le x \le 2$. Find the volume of the solid.



2. The integral $\int_0^1 e^{-0.1r} 2\pi r\, dr$ gives the total mass M of a circular plate of radius 1 meter and density

$\rho(r) = e^{-0.1r}$ kg/m².

a) Explain why this integral gives us the value of M.

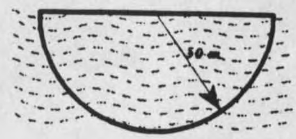
b) Using the appropriate calculator program, evaluate Trap(50) and Mid(50). Why is Trap(50) < Mid(50)? Are we going to have Trap(n) < Mid(n) for all n ? Justify your answer.

3. Determine whether or not $\int_5^\infty e^{-x^2} \sin^2 x\, dx$ converges. If it converges, without using the calculator find numbers a and b such that such that $a \le \int_5^\infty e^{-x^2} \sin^2 x\, dx \le b$. Explain carefully how you found a and b.

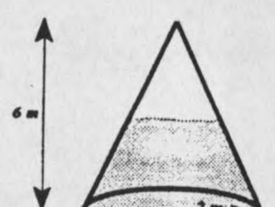
4. Does $\int_1^4 \frac{dx}{x\left(\ln(x^2)\right)^{1/4}}$ converge or diverge? If it converges, find its value. (An approximation is not acceptable.)

For problems 5 and 6, either explain or indicate on the figures any variables you use, especially where you have placed the axis. Also, recall that the weight density of water is 9800 Newtons per cubic meter.

5. Find the total force exerted by water on the dam drawn to the right, which has the shape of a semicircle with radius 50 m. The water level reaches to the very top of the dam.



6. Calculate the work to be done to empty the cone, half full (by height) of water, raising the liquid to a height 3 meters above the vertex.



Math 182 Exam 3 Name________________

November 26, 1996 ***Show all work*** Instructor________________

Problem	1	2	3	4	5	6	7	Total
Possible	9	7	16	12	12	18	26	100
Score								

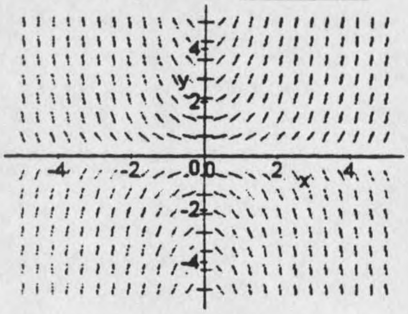
1. Is $y(x) = xe^{-x}$ a solution to the differential equation $\frac{dy}{dx} + y = \frac{y}{x}$?

2. Given $y(\theta) = A\cos(3\theta) + B\sin(3\theta)$ is a solution of $\frac{d^2y}{d\theta^2} + 9y = 0$, find constants A and B such that $y(\theta)$ is a solution to the Initial Value Problem: $\frac{d^2y}{d\theta^2} + 9y = 0$, $y(0) = 2$, $y'(0) = 0$.

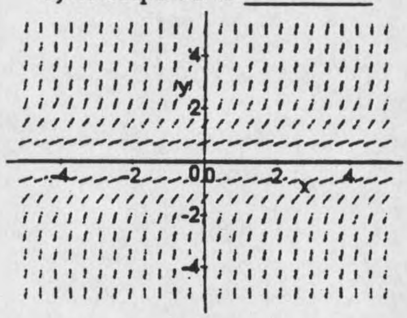
3. (a) Slope fields for 2 of the 3 DE's have been drawn below. For each slope field, identify the corresponding DE and briefly state a valid reason for each choice.

A: $\frac{dy}{dx} = xy$ B: $\frac{dy}{dx} = y^2$ C: $\frac{dy}{dx} = x^2 + x$

i) corresponds to ________



ii) corresponds to ________



(b) Sketch the solution curve passing through the point (0,1) on the slope field i). [above to the left]

4. Use Euler's method with $\Delta x = 0.25$ to approximate the solution curve to the differential equation $\frac{dy}{dx} = \frac{-x}{y}$ passing through the point (0, 1) and ending at $x = 1$.
(Keep the approximate function values to 3 decimal places.) Summarize your results in the table provided, but clearly show work for each step. **No work - No credit.**

	x_i	y_i
(x_0, y_0)		
(x_1, y_1)		
(x_2, y_2)		
(x_3, y_3)		
(x_4, y_4)		

5. A five liter tank initially contains fresh water. Beginning at $t = 0$ minutes, water containing 3 grams of salt per liter is pumped into the tank at a rate of 0.5 liter per minute. The salt water instantly mixes with the water in the tank. The excess mixture is drained from the tank at the same rate (0.5 liter per minute). Let s(t) represent the mass of salt in the tank at time t.
Write a differential equation for the amount of salt in the tank (ie, $\frac{ds}{dt}$ = ...).

Math 182 | Exam 3 *__Show all work__* | Name________________________ Instructor________________________

6. Suppose that at midnight the heater in your house breaks down. At the time of the heater malfunction the temperature in your house is 65° F and the temperature outside is - 35° F (minus 35).

(6 pts) (a) Assuming that the temperature T in your home obeys Newton's Law of cooling, **write the differential equation satisfied by T.**

{Newton's Law of cooling: Rate of change of temperature is proportional to the temperature difference.}

(12 pts) (b) If you also observe that at 2 AM the temperature in your house is 50° F, solve the differential equation above, and determine at what time the temperature in the house will be 32° F.

7. Solve the following DE's by separation of variables. Note: Your solution must be solved for the dependent variable; that is, of the form y(t) = ... Also, your answers must be simplified.

(8 pts) (a) $\frac{dy}{dt} = t^2 e^y$; $y(0) = 0$

(10 pts) (b) $\frac{dy}{dt} = \frac{y}{1-t}$; $y(0) = 2$ *for* $t < 1$

(8 pts) (c) $\frac{dy}{dt} = y(y+5)$; $y(0) = 5$ **Hint, using partial fraction decomposition:** $\frac{1}{y(y+5)} = \frac{1}{5y} - \frac{1}{5(y+5)}$

December 16, 1996

Final Exam

Instructor____________________

Problem	1	2	3	4	5	6	7	8	Total
Possible	60	18	22	20	20	20	20	20	200
Score									

1. Find integrals below. Approximations are not acceptable, show all steps.

(a) $\int_{-1}^{3} \frac{x \ln(2+x^2)}{2+x^2}\,dx$

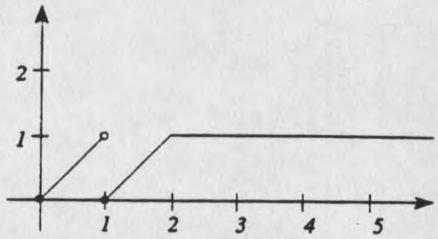
(b) $\int \frac{x-4}{x^2+x-6}\,dx$

(c) $\int_{-2}^{2} (1-x)\sqrt{4-x^2}\,dx$

(d) $\int x e^{-2x}\,dx$

2. Does $\int_0^{\infty} \frac{dy}{1+e^y}$ converge or diverge? [An answer without proper justification will receive no credit.]

3. Solve the initial value problem $\frac{dy}{dx} = f(x)$, $y(0) = 0$, where the function f is defined by its graph to the right. You may proceed graphically or analytically. However, your answer must be of the form y(x) = ...



4. (a) Find the Taylor series for $f(x) = \frac{1}{1-x}$ expanded about x = 0. Show your work.

(b) State the interval (open) of convergence for the Taylor series above. Explain how you determined the interval.

Math 182 **Final Exam** **Name** ____________________

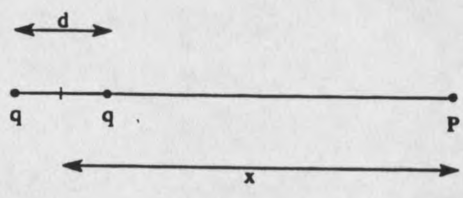
Show all work **Instructor** ____________________

5. Use Taylor series to give an estimate for $\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{\frac{-x^2}{2}}\,dx$. (Work with the first 3 terms only.)

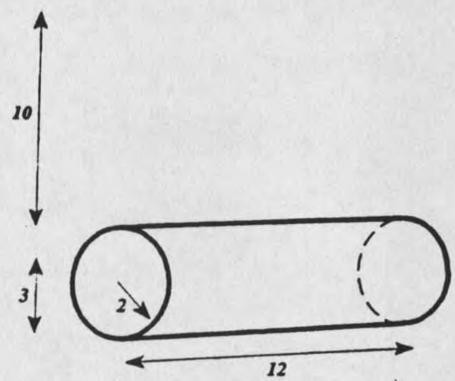
6. Two positive electric charges of magnitude q are separated by a distance d. The magnitude of the electric field E at a point P (see figure) is given by,

$$E=\frac{1}{4\pi E_0}\frac{q}{\left(x-\frac{d}{2}\right)^2}+\frac{1}{4\pi E_0}\frac{q}{\left(x+\frac{d}{2}\right)^2}$$ where E_0 is a specified constant.

Assume $x \gg d$ ("x is much greater than d", thus d/x is a very small number). Using Taylor series expansions (work with the first two terms only) find an approximation of E that involves q, a power of $\frac{1}{x}$ and a constant.



7. Recall that the weight density of water is 62.4 pounds per cubic meter. A cylinder with radius 2 ft and height 12 ft lies on its side, filled with water to a depth of 3 ft [see picture]. Give an integral expression that represents the work to raise all of the water 10 feet above the top of the tank. **Do not integrate.** Explain clearly the meaning of any variables used and indicate where you place the axes.



8. Find the solution of the initial value problem $\frac{dy}{dx} = \sqrt{3-y^2}$, $y(0) = 0$. Indicate the largest open interval around zero where your solution is defined. (Recall that $\frac{d}{dz}\arcsin\left(\frac{z}{a}\right) = \frac{1}{\sqrt{a^2-z^2}}$).

APPENDIX B

PHYSICS QUESTIONNAIRE

I. On a scale from 0 to 3 (0 = I don't remember, 1 = we never covered it, 2 = we touched on it, 3 = we covered it in depth), which of the following topics and to what extent did you study them in High School? Please, circle the number indicating your response*.

1. Galileo's experiments with inclined planes	0 1 2 3
2. Newton's three laws of mechanics	0 1 2 3
3. Newton's law of universal gravitation	0 1 2 3
4. The phenomenon of reflection of light on plane mirrors	0 1 2 3
5. The phenomenon of reflection of light on parabolic mirrors	0 1 2 3
6. The phenomenon of refraction of light	0 1 2 3
7. Newton's theory of propagation of light	0 1 2 3
8. Fermat's principle of least time	0 1 2 3
9. The general gas law	0 1 2 3
10. Avogadro's number	0 1 2 3

*** If you did not take High School Physics, do not complete part I of the questionnaire.**

II. DEMOGRAPHIC DATA

1. WHAT WAS THE SIZE OF YOUR GRADUATING HIGH SCHOOL CLASS?

Less than 40 students__ Between 40 and 100 students___
More than 100 students____

2. DID YOU TAKE CALCULUS IN HIGH SCHOOL?
No____ ....Last math course taken in High School_____________
Yes___.......................... Number of semesters____
Did you take the AP Calculus exam?
No____
Yes____.................Grade obtained____

3. SIMMS IS THE SYSTEMIC INITIATIVE FOR MONTANA MATHEMATICS AND SCIENCE. Were SIMMS curricula materials used in your mathematics classes in High School?
No____
Yes___ ....Location of your High School_______________

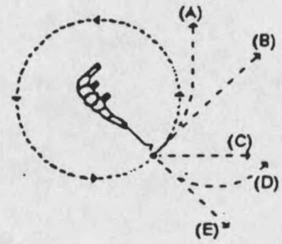
4. DID YOU TAKE PHYSICS COURSES IN HIGH SCHOOL?
No___
Yes__ ....How many years? ____
Did you take the AP Physics exam?
No___
Yes___ .... Grade obtained____

5. HAVE YOU TAKEN PHYSICS COURSES AT THE COLLEGE LEVEL PREVIOUS TO THIS SEMESTER?
No____
Yes___ .... Title of the course______________________________
6. ARE YOU TAKING A PHYSICS COURSE THIS SEMESTER ?
No__
Yes___............................. Title of the course_______________________

If your answer was NO to questions 4, 5, and 6, please complete only the first four questions of the next section. Otherwise (that is to say, if you answered YES to either 4,5, or 6) complete the eight questions of the next section (section III).

III. The following questions* do not require a knowledge of Calculus. Please, circle the option that you consider is the correct one.

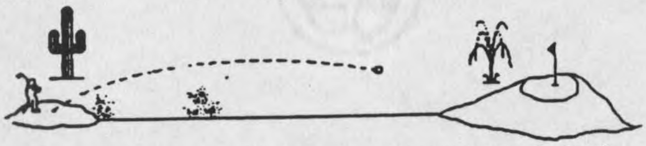
1. A heavy ball is attached to a string and swung in a circular path in a horizontal plane as illustrated in the diagram to the right. At the point indicated in the diagram, the string suddenly breaks at the ball. If these events were observed from directly above, indicate the path of the ball after the string breaks.



2. A stone falling from the roof of a single story building to the surface of the earth;

(A) reaches its maximum speed quite soon after release and then falls at a constant speed thereafter.
(B) speeds up as it falls, primarily because the closer the stone gets to the earth, the stronger the gravitational attraction.
(C) speeds up because of the constant gravitational force acting on it.
(D) falls because of the intrinsic tendency of all objects to fall toward the earth.
(E) falls because of a combination of the force of gravity and the air pressure pushing it downward.

3. A golf ball driven down a fairway is observed to travel through the air with a trajectory (flight path) similar to that in the depiction below.

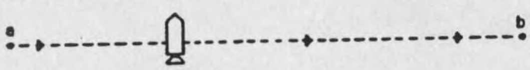


Which following force(s) is(are) acting on the golf ball during its entire flight?

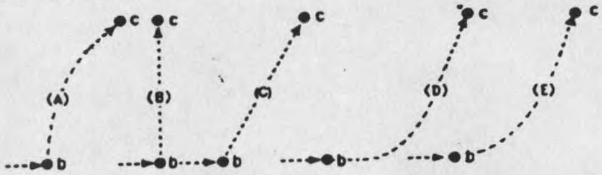
1. the force of gravity
2. the force of the "hit"
3. the force of air resistance

(A) 1 only
(B) 1 and 2
(C) 1, 2, and 3
(D) 1 and 3
(E) 2 and 3

4. A rocket, drifting sideways in outer space from position "a" to position "b", is subject to no outside forces. At "b", the rocket's engine starts to produce a constant thrust at right angles to line "ab". The engine turns off again as the rocket reaches some point "c".



Which path below best represents the path of the rocket between "b" and "c"?



* The first four questions come from the "Force Concept Inventory" test by Hestenes, Wells, and Swackhamer.

5. A ball, starting from rest at C, rolls down an inclined plane. Find its velocity at the bottom (point A) in terms of the height h. Friction is not to be considered. Recall that $g = 32$ ft/sec ^{2}, **where g is the acceleration on the surface of the earth due to gravity.**

(A) 32 h ft/sec (B) 32 ft/sec (C) 8 $\sqrt{h}$ ft/sec

(D) 16 $\sqrt{h}$ ft/sec (E) 16 ft/sec

6. Let G be the constant of universal gravitation, M the mass of the earth, and r the radius of the earth. The relationship between G, M, r, and g is given by:

(A) $g = (rG) / M$ (B) $M = (r^2 G) / g$ (C) $G = (r g) / M$

(D) $G = (r^2 g) / M$ (E) $r = (g^2 G) / M$

7. One of the first tests of Newton's law of universal gravitation had to do with the period T of the moon (T is the time the moon takes to make one complete revolution around the earth). Assuming that the moon's orbit is circular, and recalling that the centripetal force acting on the moon is given by the expression $(m v^2) / R$, where v is the moon's velocity, and R is the distance between the center of the earth and the center of the moon, find T in terms of r, R, and g.

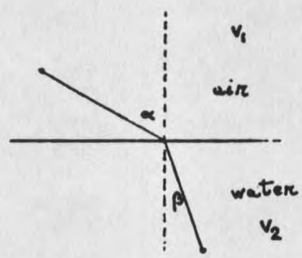
(A) $T = [(2 \pi R) / r] \sqrt{R / g}$ (B) $T = (1 / \pi R) \sqrt{r / g}$ (C) $T = (\pi R / r) \sqrt{R / g}$

(D) $T = (2 \pi R^2) / r g$ (E) $T = (2 \pi R) / r g$

8. Using his theory of light, Newton found the formula $\sin \beta = (v_1 / v_2) \sin \alpha$ for the phenomenon of refraction. Fermat had reached the formula $\sin \beta = (v_2 / v_1) \sin \alpha$ on the basis of his principle of least time. It Looks like Newton's formula, except that v_1 and v_2 are interchanged. Who was right, Newton or Fermat? Recall that the speed of light in air is greater than the speed of light in water.

(A) Newton (B) Fermat (C) There is not enough information to give an answer

(D) Neither Newton nor Fermat were right (E) It is an open question until today



INSTRUMENT ANALYSIS

1. Did you find that all the questions were clearly stated?

2. Would you suggest any improvements in sections I, II, or III?

3. Do you think that the allotted time (30 minutes) was enough to complete the questionnaire in a leisurely fashion?

4. Please, make any comments you find worthwhile.

APPENDIX C

RECORD OF MEETINGS OF THE EXPERIMENTAL GROUP

The instructors in charge of the experimental group met for an hour or so on a weekly basis and planned their activities. A log of the meetings was kept by the researcher. These weekly meetings were preceded by a long encounter between the researcher and the other two instructors, wherein the former explained the purpose and methodology of the research to be conducted during the fall, 1996.

First Week

The Fundamental Theorem of Calculus (FTC) plays an important role in the course, so we may as well prove it. The presentation given by the textbook in section 3.4 is a heuristic argument that needs to be refined. The students are to receive an enrichment note wherein the FTC is proven along the ideas set in problem 15 p. 132. The proof is given first for increasing or decreasing continuous functions. Thereafter it is to be generalized to continuous functions that are not necessarily increasing or decreasing.

Special attention is to be paid to problem 10, p. 319. More examples are to be analyzed in class, dealing with initial value problems of the type $y'(t) = f(t)$, $y(0) = \alpha$, where α is a given number and f is sectionally continuous, i.e. continuous with a finite number of discontinuities; problem 17 on page 324 is an example of this sort. Students should realize that the solution will be continuous everywhere, but not differentiable at the points of discontinuity of the function f.

Second Week

Lots of problems about motion. Problems 48-54 (section 6.4), 1-4 (section 6.5) and 34-40 (Review of Chapter 6) are particularly interesting. Let students understand that one is dealing with differential equations of the type $s''(t) = -k$, where k could be g (when a body is subjected to the force of gravity) or some fixed number if one were to deal with accelerating or decelerating cars or airplanes. The importance of fixing coordinate axes. Have students solve the differential equations each time a problem is given. One of the keys to solve the application problems is to realize what the physical conditions are, and what is a mathematics derivation.

Handout on basic chemical kinetics is given to the students. This is a nice opportunity to talk about the essence of the scientific method and the role that mathematics plays in science.

Third Week

Let students understand that the underlying foundation of the method of substitution is the chain rule and the FTC. Present a proof of the method along the lines of thought set by the book on page 350, but do it for definite integrals. Advantages and disadvantages of the algorithmic approach to substitution. Strategies to be used in substitution problems. Beware! A common mistake made by students takes place when they have to calculate a definite integral through substitution and, despite all the warnings, they forget about the proper handling of the limits of integration.

Have students deduce the equation given in problem 45 (page 354) from the differential equation $m\,v'(t) = mg - k\,v(t)$. Many practice problems! Let students know that the equation presented in problem 32 (p. 359) will be deduced from first principles once we learn about the method of partial fractions.

Fourth Week

Old saying: "When you do not know what to do, apply integration by parts". Students need to realize that there is not a unique way to solve a problem using this technique. For instance, one can "solve" $\int e^x \cos x\,dx$ in two ways. There is even a third possibility that leads to the equality $0 = 0$, an unpleasant situation that shouldn't demoralize anyone. It is just a matter of trying another path!

Problem 50 on page 365 and similar ones are good for practice purposes. Completion of squares and partial fractions should be given a higher status, the book down plays both of them; this is a pity since these methodologies -- especially the latter-- are widely used in mathematics, physics, and chemistry. As an example, discuss in class the enrichment notes about falling objects that are subjected to air resistance proportional to the square of the velocity and chemical reactions whose solution leads to a problem in partial fractions. Have students justify items 24-31 of the short table of indefinite integrals (p. 367).

Fifth Week

Example 3 (p. 381) is particularly nice. Discuss in class similar problems, for instance $\int 1/(1+x^3)\,dx$ between the limits 0 and 1. Let the students work on most if not all the problems in section 7.6. Stress the fact that "approximating by parabolas" leads to Simpson's method (along the lines of problems 20, 21 of section 7.7).

Sixth Week

Handout on the proof of the criteria of comparison for improper integrals. Section 7.8 has a good collection of problems, including problems 29, 30. Example 5 (p. 404) is particularly well done, let students read it before coming to class. All the problems for section 7.9 look nice, especially problem 25 related to Planck's radiation law. Cover in class section 7.10, even though it is not in the syllabus; students will realize that the developments in p. 410 were covered in the first enrichment note. Many good problems in the review section for chapter seven (pp. 414-417).

Seventh Week

Lots of applications! Students usually have trouble with work and water pressure problems. With particular care discuss in class problems about cylinders full of water or gasoline, and the work to be done to empty them when the cylinder is standing on its base or lying on its side (problem 6, p. 444). The hydrostatic force on an object varies according to the shape of the submerged object and the depth of the column of water. Let students analyze triangular and circular plates.

Enrichment note on an alternative calculation for the escape velocity of an object on the earth's surface. This alternative approach is a good opportunity for the students; they can become acquainted with simple qualitative methods in the elementary theory of differential equations.

Show students how mathematicians used to calculate volumes of cones and spheres, before Newton and Leibniz-- so as to impress upon them the power of differential and integral calculus.

Eighth Week

Students will understand sections 8.5 and 8.6 only through the problems. Set aside enough time to discuss in class as many problems as possible, chosen from the nice collection laid out in the book. Assign as homework several problems from the review section for chapter eight.

Ninth Week

Section 9.1 should not be a novelty, since from the very beginning of the course we have been discussing the idea of solution of a differential equation. This section can be dealt with rather rapidly so as to concentrate on 9.2. Slope fields are something new for the students. Handout on more slope fields.

Tenth Week

Handout on the theoretical justification of Euler's method. Students need to understand the underlying geometrical idea lying behind the method, before doing actual computations. A balance between paper and pencil computations and the use of calculators has to be reached. For the former, choose some simple problems that do not require extensive computations.

Eleventh Week

Enrichment note on the catenary, starting from basic ideas in physics. This note explains where the equation in problem 8 (p. 482) comes from. The catenary changes into a parabola when one analyzes a suspension bridge under suitable simplifications. Enrichment note on Newton's law of cooling and radioactive decay (validation of the model, solution of the corresponding differential equation, and solution of problems that arise in actual practice).

Twelfth Week

General solution of linear first order differential equations. Do cover in class the most salient features of sections 9.10 and 9.12. These are not contemplated in the syllabus; however, students that are taking Physics I concurrently with Calculus II, have to know how to solve second order linear differential equations.

The book does a particularly nice job with section 9.6. Follow it closely. Enrichment note on enzyme kinetics. Simple ideas in chemistry lead to basic techniques in mathematics.

Thirteenth Week

Have students plot on their calculators different Taylor polynomials. Enrichment note on the approximation $\ln(1+x) \approx x$ and its appearance in the natural sciences.

Fourteenth Week

Geometric series. Handout on series of numbers. Students need to know the main tests (ratio, root, and integral tests). Parallelism between the theory of improper integrals and the theory of series of numbers. Taylor series as a particularly important case of series of functions. Start dealing with Taylor series, then develop the basic facts about power series.

Fifteenth Week

Follow closely section 10.3 of the book. The material is well-presented and the problems at the end of the section are very good. Wrap up everything and separate two days for review. No quiz this week!

APPENDIX D

CONCORDANCE TABLE

Concordance Between ACT Mathematics Score and SAT I Recentered Mathematical Score

ACT Mathematics score	Concordant SAT I Mathematical score
36	800
35	790
34	780
33	760
32	730
31	700
30	680
29	660
28	640
27	620
26	600
25	580
24	560
23	540
22	520
21	500
20	480
19	460
18	440
17	410
16	390
15	360
14	330
13	300
12	280
11	260

Source: American College Testing.

These estimates are based on the test scores of 53,326 applicants to 22 postsecondary institutions who took both the ACT and the SAT I between March 1994 and February 1995. Questions about the concordance study may be directed to ACT's Research Division (319/337-1471).

APPENDIX E

A SAMPLE OF ENRICHMENT NOTES

Elementary Chemical Kinetics

A single reactant is put in contact with water, and its concentration c(t) is measured:

time (hours)	concentration (moles/lt.)
0	0.1039
3.15	0.0896
4.10	0.0859
6.2	0.0776

According to basic principles of chemistry, a possible model for the reaction is the differential equation

$$c'(t) = -kc(t)$$

where k is a positive parameter. If this were the case, we would have

$$\frac{c'(t)}{c(t)} = -k$$

So

$$\int_0^t \frac{c'(s)}{c(s)} ds = \int_0^t -k ds$$

But

$$(\ln c(t))' = \frac{c'(t)}{c(t)}$$

for any t. Therefore:

$$[\ln c(s)]_0^t = [-ks]_0^t$$

Thus $$\ln c(t) - \ln c(0) = -kt$$

In other words:

$$\frac{1}{t}\ln\frac{c(0)}{c(t)} = k \qquad (1)$$

A prediction of the model is that

$$\frac{1}{t}\ln\frac{c(0)}{c(t)}$$

will remain constant for different values of t. That is to say, our model predicts that this mathematical expression is independent of t. The model

$$c'(t) = -kc(t)$$

will attain validity insofar as the prediction is in agreement with the data given at the beginning. Using a calculator we get the following values:

t	c(0)/c(t)	ln(c(0)/c(t))	(1/t)ln(c(0)/c(t))
3.15	1.15959	0.14807	0.04700
4.1	1.20954	0.19024	0.04640
6.2	1.33891	0.29186	0.04707

The last column indicates that

$$\frac{1}{t}\ln\frac{c(0)}{c(t)}$$

remains practically constant. Then we accept the model and proceed to approximate the value of k as the simple mean of the three numbers 0.04700, 0.04640, and 0.04707.

Notice that the validity of the model has to be checked before any attempt to approximate k. An equivalent approach, often favored by scientists, can be followed: Let us rewrite (1) as $\ln c(t) = -k t + \ln c(0)$. This equation implies a straight line, with t on the x-axis and $\ln c(t)$ on the y-axis. Using the experimental values, given in the first table, the following table is built:

t	ln c(t)
0	-2.26432
3.15	-2.41239
4.1	-2.45457
6.2	-2.55618

We expect that these points will be distributed in such a way that an imaginary line can be drawn passing through them. Thereafter we draw the line that best fits the four points. Its slope provides the value of k (there is a standard statistical procedure to obtain the line of "best fit" through any number of points). If the four points are arranged in a haphazard way, not suggesting the presence of a line, we reject the model because a mathematical consequence of it does not agree with the experimental values! The latter is the "supreme judge" that will determine the acceptance or rejection of the model

$$c'(t) = -kc(t)$$

If this model is not acceptable, the next step would be to consider the model

$$c'(t) = -k(c(t))^2$$

Let us try to find a consequence of it that may be compared with a table of experimental values involving t and the corresponding c(t). We notice that

$$\frac{c'(t)}{(c(t))^2} = -k$$

Thus

$$\int_0^t \frac{c'(s)\,ds}{(c(s))^2} = \int_0^t k\,ds$$

Using the chain rule we get

$$\frac{d}{ds}\left(\frac{1}{c(s)}\right) = -\frac{c'(s)}{(c(s))^2}$$

Therefore

$$[1/c(s)]_0^t = kt \qquad i.e. \qquad \frac{1}{c(t)} = kt + \frac{1}{c(0)} \qquad (2)$$

If we were to have t (time) on the x-axis and 1/c(t) on the y-axis, our model predicts a straight line with positive slope and y intercept 1/c(0) . Let us consider the following data about the gas-phase decomposition of nitrogen dioxide at 300 degrees centigrade (Brown, Lemay &Bursten, “Chemistry”, p. 500):

t (secs.)	c(t)
0	0.0100
50	0.0079
100	0.0065
200	0.0048
300	0.0038

We can calculate 1/c(t) for different values of t: 100, 126.58, 153.85, 208.33, 263.16(t=0, 50, 100, 200, 300 respectively). We plot these five points, and notice that they gather around a straight line. So, one important prediction of the model holds! The points are spread out in such a way that it is possible to draw a line passing through them. The model $d/dt(c(t)) = -k(c(t))^2$ is then accepted and we proceed to calculate k as the slope of the aforementioned line.There is another procedure to validate simple models of chemical reactions. It has to do with the concept of half-life. By definition, the half life $t_{1/2}$ is the time it takes for the conversion of half of the reactant, i.e. $t_{1/2}$ is such that $c(t_{1/2}) = c(0)/2$. For a first order reaction, by (1) it follows that $(1/t_{1/2}) \ln 2 = k$. So $t_{1/2} = \ln 2/k$. Thus, $t_{1/2}$ does not depend on the initial amount c(0). On the other hand, if the reaction is of second order, from (2) we get $1/(c(0)/2) = k\, t_{1/2} + 1/c(0)$. Hence $t_{1/2} = 1/kc(0)$. We notice that, in this case, $t_{1/2}$ depends on c(0). Thus, a strong indication that a reaction cannot be of first order is to start with different amounts of reactant, measure the half-life, and realize that it varies. If the reaction were to be of first order, this could not happen.

Wind Resistance Proportional to the Square of Velocity

Let us make some comments about problem 32 on page 359. The book provides an expression for v(t). It looks complicated. Where does it come from? The model for a falling object of unit mass is $(d/dt)\, v(t) = g - k\,(v(t))^2$. Then

$$\frac{v'(t)}{g-k(v(t))^2} = 1$$

Thus

$$\frac{v'(t)}{(\sqrt{g}+\sqrt{k}v(t))(\sqrt{g}-\sqrt{k}v(t))} = 1$$

By the method of partial fractions we know that there exist numbers p and q such that

$$\frac{1}{(\sqrt{g}+\sqrt{k}v(t))(\sqrt{g}-\sqrt{k}v(t))} = \frac{p}{\sqrt{g}+\sqrt{k}v(t)} + \frac{q}{\sqrt{g}-\sqrt{k}v(t)}$$

The usual procedure leads to the values of p and q:

$$p(g^{1/2} - k^{1/2}\, v(t)) + q(g^{1/2} + k^{1/2}\, v(t)) = 1$$

So $p\, g^{1/2} + q\, g^{1/2} = 1$ and $-k^{1/2}\, p + k^{1/2}\, q = 0$. Therefore $p = q = 1/(2g^{1/2})$. Using these values and integrating, we arrive to:

$$\frac{1}{2\sqrt{kg}}\int_0^t \frac{\sqrt{k}v'(s)\,ds}{\sqrt{g}+\sqrt{k}v(s)} - \frac{1}{2\sqrt{kg}}\int_0^t \frac{-\sqrt{k}v'(s)ds}{\sqrt{g}-\sqrt{k}v(s)} = t$$

Thus

$$\frac{1}{2\sqrt{kg}}[\ln(\sqrt{g}+\sqrt{k}v(s))]_0^t - \frac{1}{2\sqrt{kg}}[\ln(\sqrt{g}-\sqrt{k}v(s))]_0^t = t$$

Then

$$\ln\frac{\sqrt{g}+\sqrt{k}v(t)}{\sqrt{g}-\sqrt{k}v(t)} = 2\sqrt{kg}\,t$$

After a little bit of elementary algebra we finally arrive to

$$v(t) = \sqrt{\frac{g}{k}\,\frac{\exp(\sqrt{kg}\,t) - \exp(-\sqrt{kg}\,t)}{\exp(\sqrt{kg}\,t) + \exp(-\sqrt{kg}\,t)}}$$

This is the expression we were looking for!

<u>A Problem in Chemical Kinetics where Partial Fractions Are Used</u>

Let us consider the chemical reaction $Na_2\,S_2\,O_3$ + CH_3 I ----> products, with k as the parameter of the reaction. From a series of experiments we get the following data:

t (hours)	$Na_2S_2O_3$	CH_3I (moles/lt.)
0	35.35	18.25
4.75	30.5	13.4
10	27	9.9
20	23.2	6.1
35	20.3	3.2
55	18.6	1.5

Let A(t), B(t) denote the concentration of $Na_2S_2O_3$ and CH_3I respectively. Moreover, let A(0) = a, B(0) = b. According to the law of mass action, a tentative model is given by the pair of differential equations:

$$A'(t) = -kA(t)B(t) \quad B'(t) = -kA(t)B(t)$$

Since both derivatives are equal we can assert that A(t) = B(t) + L for some constant L. In particular A(0) = B(0) + L. So L= a - b. Consequently B(t) = A(t) + (b - a). Therefore

$$A'(t) = -kA(t)(A(t) + b - a) \qquad (1)$$

This is a separable variables differential equation, an important type of differential equations that will be used extensively in chapter 9 of the textbook. We notice that

$$\frac{A'(t)}{A(t)(A(t) + b - a)} = -k \qquad (2)$$

The method of partial fractions implies the existence of constants p,q such that

$$\frac{1}{A(t)+b-a} = \frac{p}{A(t)} + \frac{q}{A(t)+(b-a)}$$

Hence p(A(t)+b-a) + qA(t) = 1. Thus p + q = 0, pb - pa = 1. Therefore p = 1/(b-a), q = -1/(b-a). Going back to (2) we get:

$$\frac{1}{b-a}\int_0^t \frac{A'(s)\,ds}{A(s)} - \frac{1}{b-a}\int_0^t \frac{A'(s)\,ds}{A(s)+b-a} = \int_0^t -k\,ds$$

Thus

$$\frac{1}{b-a}[\ln A(s)]_0^t - \frac{1}{b-a}[\ln(A(s)+b-a)]_0^t = -kt$$

Consequently

$$\frac{1}{b-a}\ln\frac{\frac{A(t)}{a}}{\frac{A(t)+b-a}{b}} = -kt$$

i.e.

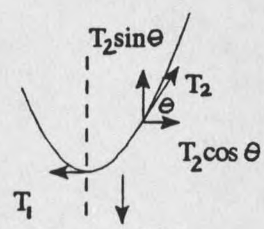
$$\frac{1}{(a-b)t}\ln\frac{bA(t)}{aB(t)} = k \qquad (3)$$

Thus, a prediction of our model is that the expression to the left of (3) will remain nearly constant through time. Using the data given above, this expression adopts the values 1.9863 x 10^{-3}, 2 x 10^{-3}, 1.9729 x 10^{-3}, 1.9821 x 10^{-3}, 1.974 x 10^{-3} (for t = 4.75, 10, 20, 35, and 55 respectively). So, we accept the validity of the model. Thereafter, the parameter k can be calculated in the usual way through a plot of t versus (1/(b-a))ln((bA(t)/aB(t)) on the y-axis. We will be dealing with a straight line that goes through the origin.

The Catenary

In problem 21 (p. 436) we are given the function $y = (e^x + e^{-x})/2$, and we are told that it is called the catenary and that it represents the shape in which a cable hangs. Is that so?

Let us consider a flexible and uniform chain, suspended between its endpoints and hanging under its own weight. We draw the y-axis in such a way that it passes through the lowest point of the chain. For the time being we do not specify where the origin of the coordinate system is.



The chain, between 0 and an arbitrary point x, is in equilibrium. The only forces acting on this piece of the chain are T_1 and T_2 (the "tension" forces), and its weight. Thus $T_1 = T_2 \cos\theta$, $\rho L(x) = T_2 \sin\theta$ where L(x) is the length of the chain between 0 and x, while ρ is the uniform density. Recalling that

$$L(x) = \int_0^x \sqrt{1 + (f'(x))^2}$$

we have

$$\tan\theta = \frac{\rho}{T_1} L(x), \textit{ i.e.} \qquad f'(x) = \frac{\rho}{T_1}\int_0^x \sqrt{1+(f'(x))^2}$$

This is the integro-differential equation satisfied by the chain! Let us try to solve it. We notice that

$$f''(x) = \alpha\sqrt{1+(f'(x))^2}$$

where, by definition, $\alpha = \rho / T_1$. Can we solve this differential equation? Let g(x) = f′(x). Then

$$g'(x) = \alpha\sqrt{1+(g(x))^2}$$

This equation looks much simpler, it is a differential equation that can be solved by "separating variables". We get

$$\frac{g'(x)}{\sqrt{1+(g(x))^2}} = \alpha$$

Consequently

$$\int_0^x \frac{g'(t)\,dt}{\sqrt{1+(g(t))^2}} = \alpha \int_0^x dt$$

Formula 29 (p. 367 of our textbook) leads to the following equality:

$$[\ln(g(t) + \sqrt{1+(g(t))^2}\,]_0^x = \alpha x$$

Therefore

$$\ln \frac{g(x) + \sqrt{1+(g(x))^2}}{g(0) + \sqrt{1+(g(0))^2}} = \alpha x$$

However, g(0) = f ′(0). By the way we chose the y-axis, the derivative of f at the origin has the value zero. Hence ln(g(x)+√(1+(g(x))^2) = α x. Thus

$$g(x) + \sqrt{1 + (g(x)^2} = \exp(\alpha x)$$

Hence

$$\frac{1}{g(x) + \sqrt{1+(g(x))^2}} = \exp(-\alpha x)$$

Using the last two equalities and some elementary algebra we get:

$$f'(x) = g(x) = \frac{\exp(\alpha x) - \exp(-\alpha x)}{2}$$

A simple integration leads to the equality

$$f(x) - f(0) = \frac{\exp(\alpha x) - 1 + \exp(-\alpha x) - 1}{2\alpha}$$

If we choose the origin of the coordinate system in such a way that $f(0) = 1/\alpha$, then

$$f(x) = \frac{\exp(\alpha x) + \exp(-\alpha x)}{2\alpha}$$

This is precisely the equation given in the book, when $\alpha = 1$.

Three Stages in Modeling with Differential Equations

To illustrate what we have in mind, let us consider Newton's law of cooling. Based on the observation of how the temperature of a body immersed in running water goes down, we put forward, as a working hypothesis, the differential equation

$$T'(t) = -k(T(t) - T_M)$$

where T_M is the temperature of flowing water (a constant) and T(t) denotes the temperature of the body that has been put in contact with water at time t = 0. In the first place, we have to validate the model. Once this is done, our next job is to find the solution of the differential equation. Finally, we can start solving practical problems.

a) Validation of the model

An experiment has to be done, in the sense of acquiring a table for T(t) at various values of t. From the differential equation it follows that

$$\frac{T'(t)}{T(t) - T_M} = -k$$

Therefore

$$\int_0^t \frac{T'(s)\,ds}{T(s)-T_M} = -\int_0^t k\,ds$$

Hence

$$[\ln(T(s)-T_M)]_0^t = -kt$$

Thus

$$\ln\frac{T(0)-T_M}{T(t)-T_M} = kt \qquad (1)$$

A prediction of the model is that when t is on the x-axis and ln $\{(T(0)-T_M)/(T(t)-T_M)\}$ is on the y-axis, the experimental values will cluster around a straight line that goes through the origin. If this happens, we accept the model and proceed to calculate k as the slope of the "line of best fit". Thus, k is calculated once the process of data contrasting has come to an end, and the validity of the model has been accepted.

b) Finding the solution of the differential equation

From (1) we get

$$\frac{T(0)-T_M}{T(t)-T_M} = \exp(kt)$$

Hence

$$T(t) = T_M + (T(0)-T_M)\exp(-kt) \qquad (2)$$

Obtaining the solution of the differential equation was pretty easy. However, in some cases it is not possible to obtain an explicit solution. Nonetheless, the process of validation, which usually precedes the search of a solution of the differential equation, can be done as was shown in the first part.

c) Solving practical problems

Suppose we wish to find the temperature at which a certain chemical reaction takes place. For some reason, it is not possible to disturb the reaction while it is under way, i.e. we can not place a thermometer and measure the temperature of the reaction. We then bring the reaction to a stop (say, by some physical device) and the container is put in contact with a cooling system where water is circulating. Two measurements are then made: $(t_1, T(t_1))$ and $(t_2, T(t_2))$. The unknown is T(0). From (2) it follows that

$$T(0) - T_M = (T(t_1) - T_M)\exp(kt_1) \qquad (3)$$

and

$$T(0) - T_M = (T(t_2) - T_M)\exp(kt_2) \qquad (4)$$

Therefore

$$1 = \frac{T(t_1) - T_M}{T(t_2) - T_M}\exp(k(t_1 - t_2))$$

So

$$k = \frac{1}{t_1 - t_2}\ln\frac{T(t_2) - T_M}{T(t_1) - T_M}$$

Once k is calculated, the value of T(0) can be found by using (3) or (4):

$$T(0) = T_M + (T(t_1) - T_M)\exp(kt_1)$$

A proof of the Fundamental Theorem of Calculus

Let us recall the statement of the theorem: Let f: $I \rightarrow \Re$ be a continuous function, where $\Re$ denotes the set of real numbers. Suppose G is any antiderivative of f. Then

$$\int_a^b f = G(b) - G(a)$$

for any points a, b lying in the interval I.

We will first prove a lemma, following the idea given in problem 15(p. 312) of the textbook.

Lemma

Let f be the function given before. Define F: $I \rightarrow \Re$ by

$$F(x) = \int_a^x f(t)\, dt$$

where the lower limit of integration is any fixed point in I. Then

$$\frac{d}{dx} F(x) = f(x)$$

for every x in I.

Proof

To start with, we will present a proof with the added assumption that f is increasing or decreasing. Suppose f is increasing. We notice that for any h>0

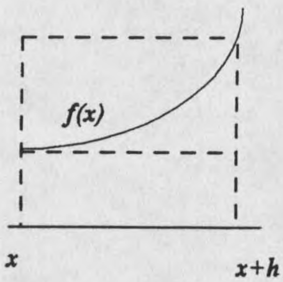
$$F(x+h) - F(x) = \int_a^{x+h} f - \int_a^x f = \int_x^{x+h} f$$

Since f is increasing we can assert that $f(x) \leq f(t) \leq f(x+h)$ for every t in the interval [x, x+h]. Therefore:

$$f(x)\,h = \int_x^{x+h} f(x)\,dt \le \int_x^{x+h} f(t)\,dt \le \int_x^{x+h} f(x+h)\,dt = f(x+h)\,h$$

By looking to the graph to the right, you will notice the geometrical significance of the preceding formula.
Thus:

$$f(x) \le \frac{F(x+h) - F(x)}{h} \le f(x+h)$$



But f is continuous at x. Consequently

$$\lim_{h\to 0^+} f(x+h) = f(x)$$

Thus, by the "sandwich" property of limits we get

$$\lim_{h\to 0^+} \frac{F(x+h) - F(x)}{h} = f(x)$$

In a completely analogous fashion one can show that

$$\lim_{h\to 0^-} \frac{F(x+h) - F(x)}{h} = f(x)$$

The lemma has been proven for continuous increasing functions. The proof for decreasing functions is similar. If you are satisfied with this proof, and don't have the time to consider the more general case when only the continuity of f is assumed, go straight to the proof of the fundamental theorem of calculus below. Otherwise, let us accept the following fact about any function p(x) defined on an interval [c,d]: If p(x) is continuous, there exists $c \le \theta \le d$ such that

$$\int_c^d p(x)\,dx = p(\theta)(d-c)$$

In other words, it is possible to build a rectangle whose area is equal to the area of the region that lies between the curve defined by p(x) and the x-axis. It is "evident", isn't it? Going back to the central theme of the proof, we can assert that (for h>0)

$$\int_x^{x+h} f(t)\,dt = f(\theta(h))\,h$$

for some $\theta(h)$, where $x \le \theta(h) \le x+h$. Obviously

$$\lim_{h\to 0+} \theta(h) = x$$

(Again, we have used the "sandwich" property of limits). Moreover, the continuity of f at x assures that:

$$\lim_{h\to 0+} f(\theta(h)) = f(\lim_{h\to 0+} \theta(h))$$

Then

$$\lim_{h\to 0+} \frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h\to 0+} f(\theta(h)) = f(\lim_{h\to 0+} \theta(h)) = f(x)$$

Thus

$$\lim_{h\to 0+} \frac{F(x+h) - F(x)}{h} = f(x)$$

The limit from the left is proved in a similar way.

QED

<u>A Proof of the FTC</u>

Since G′ = f and F′ = f, where F is the function defined in the lemma, it follows that G′ = F′. So, for some k, G(x) = F(x) + k for every x. In particular G(a) = F(a) + k. But F(a) = 0. So k = G(a). Thus G(x) = F(x) + G(a) for every x in I. In particular:

G(b) = F(b) + G(a), i.e.

$$G(b) - G(a) = \int_a^b f$$

QED

Remark

The lemma is discussed on pp. 409-410 of our textbook. Notice how important it is: Given any continuous function we can assert that it has an antiderivative, precisely

$$F(x) = \int_a^x f(t)dt$$

This function might not be "simple", but it exists!

The Comparison Test for Improper Integrals

Let f,g: $[a,\infty) \to \Re$ be continuous functions such that $0 < f(x) \le g(x)$ for all $x \ge a$. Then:

$$(i)\ \int_a^\infty g(x)\,dx \text{ converges} \Rightarrow \int_a^\infty f(x)\,dx \text{ converges}$$

$$(ii)\ \int_a^\infty f(x)\,dx \text{ diverges} \Rightarrow \int_a^\infty g(x)\,dx \text{ diverges}$$

Proof

Define the following two functions:

$$F(x) = \int_a^x f(t)\,dt, \qquad G(x) = \int_a^x g(t)\,dt$$

Since $f(x) \le g(x)$, we can conclude that $F(x) \le G(x)$, $x \ge a$. Moreover, since f and g are positive functions it follows that F and G are increasing functions. Let us try to prove (i): By hypothesis, the limit of G(x) when $x \to \infty$ exists. So G has to be bounded, say $G(x) \le M$

for some number M and every $x \geq a$. Then $F(x) \leq M$, $x \geq a$. The function F is increasing and bounded above, so it has no alternative but to converge. In other words, lim F(x) when $x \to \infty$ exists, i.e. the integral of f(x) between a and ∞ converges. Furthermore, since $F(x) \leq G(x)$, $x \geq a$, and the limit of both functions exist when $x \to \infty$, we can assert that

$$\int_a^\infty f \leq \int_a^\infty g$$

The proof of (ii) is now almost immediate: it is nothing but its logical equivalent (recall that for any two statements p and q, $p \Rightarrow q$ is equivalent to $\sim q \Rightarrow \sim p$, where the symbol ~ denotes the operation of negation).

QED

Usual Tests in the Theory of Series

Let $a_n = (1/2)^n$ $(n = 0,1,2,...)$ and define $s_n = 1+(1/2)+...+(1/2)^n$
Question: Does the limit of s_n when $n \to \infty$ exist? We notice that

$$(1/2)s_n = (1/2) + (1/2)^2 + ... + (1/2)^{n+1}$$

So $s_n - (1/2)s_n = 1 - (1/2)^{n+1}$. Hence $s_n = 2 - 2(1/2)^{n+1}$. But $(1/2)^{n+1} \to 0$. Therefore $s_n \to 2$. Notice that

$$s_n = \sum_{i=0}^{n} (1/2)^i$$

We have just shown that

$$\lim_{n \to \infty} \sum_{i=0}^{n} (1/2)^i = 2$$

Another way of writing this fact is as follows:

$$\sum_{i=0}^{\infty} (1/2)^i = 2$$

Since "i" is a "dummy" variable, we may rewrite the preceding equality as follows:

$$\sum_{n=0}^{\infty} (1/2)^n = 2$$

or, as a matter of notation, we agree that

$$1 + 1/2 + (1/2)^2 + (1/2)^3 + \dots\dots\dots = 2$$

The sequence (s_n) is usually called the "series" determined by the sequence (a_n). Quite often the series (s_n) itself is denoted by the symbol

$$\sum_{n=0}^{\infty} a_n$$

Strictly speaking, this symbol denotes a number. Nonetheless, it is used both to denote (s_n) and its limit — when the latter exists. Beware about this double meaning! From the context of a particular problem you will be able to decide which meaning is intended.

Let us discuss another example: Determine whether the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

converges or not. If it converges, where does it converge to? Let

$$s_n = 1/1(1+1) + 1/2(2+1) + \dots + 1/n(n+1)$$

Since $1/n(n+1) = 1/n - 1/(n+1)$, we arrive to $s_n = (1-1/2) + \dots + (1/n - 1/(n+1))$, i.e.

$S_n = 1 - 1/(n+1)$. Thus $s_n \to 1$. Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Does $\sum(1/n^2)$ converge? In order to answer this question we will state a proposition, which has a striking similarity with the comparison test for improper integrals.

Proposition 1

Let (a_n) and (b_n) be sequences of non-negative terms, such that $0 < a_n \le K b_n$ for some fixed constant K and all n. The following two implications are true:

$$(i)\ \sum_{n=1}^{\infty} b_n \quad converges \Rightarrow \sum_{n=1}^{\infty} a_n \quad converges$$

$$Moreover \quad \sum_{n=1}^{\infty} a_n \le K \sum_{n=1}^{\infty} b_n$$

$$(ii)\ \sum_{n=1}^{\infty} a_n \quad diverges \Rightarrow \sum_{n=1}^{b_n} \quad diverges$$

(We could weaken the hypothesis in the sense that the inequality only has to hold for all n except a finite number).

Let us use this proposition in order to determine whether or not $\sum(1/n^2)$ converges. We have the inequality $n(n+1) \le 2n^2$ for n = 1, 2, 3,....... Thus $0 < 1/n^2 \le 2/n(n+1)$ N = 1, 2, ... Since $\sum 1/n(n+1)$ converges, we can conclude that $\sum 1/n^2$ converges. Where does it converge to? The preceding proposition does not give an answer to this question. All we can assert is that

$$0<\sum_{n=1}^{\infty} \frac{1}{n^2} \le 2 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 2$$

For a long time, the "exact" value of $\sum 1/n^2$ was an open question, until Leonhard Euler -- probably the greatest 18th century mathematician-- used a very clever technique and showed that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Unexpected, isn't it? There is another result that we have to keep in mind.

Proposition 2 (the ratio test)

Let (a_n) be a sequence of non-zero numbers. Let

$$L = \lim_{n\to\infty} |a_n / a_{n+1}|$$

The following two implications are true:

$$(i)\ 0 \leq L \leq 1 \Rightarrow \sum_{n=1}^{\infty} \quad converges$$

$$(ii)\ L > 1 \ \vee \ L = \infty \Rightarrow \sum_{n=1}^{\infty} a_n \quad diverges$$

Using this proposition let us determine whether $\sum$ 1/n! converges. We notice that

1/(n+1)! divided by 1/n! gives n!/(n+1)!, which in turn is the quotient 1/(n+1). The latter sequence obviously converges to zero. Thus $\sum$ 1/n! converges. Amazingly enough, it can be shown that:

$$\sum_{n=1}^{\infty} \frac{1}{n!} = e - 1$$

where e is the basis of natural logarithms.

There is a deep connection between improper integrals of the type

$$\int_1^{\infty} f(x)\, dx$$

and series of numbers, as the following proposition shows.

Proposition 3 (Integral Test)

Let f: $[1, \infty) \rightarrow \Re$ be continuous and decreasing. Then

$$\int_1^\infty f(x)\,dx \quad converges \leftrightarrow \sum_{n=1}^\infty f(n) \quad converges$$

Let us consider an example. In chapter 7 of our textbook, we proved that

$$\int_1^\infty \frac{1}{x^p}$$

converges for $p > 1$. So $\sum 1/n^p$ converges for $p > 1$. On the other hand, since

$$\int_1^\infty \frac{dx}{x}$$

diverges we can assert that $\sum 1/n$ diverges. The divergence of this series implies the divergence of other series, for instance $\sum 1/\log n$ (starting with $n = 2$). This is so because $\log n < n$ $(n > 1)$, implying $0 < 1/n < 1/\log n$.

Remark

Given any series $\sum a_n$, the following equivalence holds:

$$\sum_{n=p}^\infty a_n \quad converges \leftrightarrow \sum_{n=q}^\infty a_n \quad converges$$

where p and q are any natural numbers. In other words, with regard to convergence it doesn't make any difference where one starts. Beware of the following fact: Suppose

$$\sum_{n=p}^\infty a_n$$

converges and $q > p$. Then:

$$\sum_{n=p}^{\infty} a_n = a_p + a_{p+1} + \ldots + a_{q-1} + \sum_{n=q}^{\infty} a_n$$

For instance:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \sum_{n=4}^{\infty} \frac{1}{n^2}$$

Enzyme Kinetics

The basic model of enzyme kinetics is given by

$$S + E \rightleftharpoons C \rightarrow E + P$$

with parameters k_1, k_{-1} for the reversible part of the reaction and k_2 for the irreversible part. The substrate S combines with the enzyme E giving birth to an intermediate compound C through a reversible reaction. C decomposes into the product P and regenerates the enzyme E. Notice that the values of the parameters k_1, k_{-1}, k_2 are unknown, and will have to be determined by analyzing the data, once the model has been validated.

According to a basic law of chemistry (the law of mass action), the acceptance of a first order kinetics leads to the following differential equations:

$$d/dt\, C(t) = k_1\, E(t)\, S(t) - k_{-1}\, C(t) - k_2\, C(t) \qquad (1)$$

$$d/dt\, S(t) = -k_1\, E(t)\, S(t) + k_{-1}\, C(t) \qquad (2)$$

$$d/dt\, E(t) = -k_1\, E(t)\, S(t) + k_{-1}\, C(t) + k_2\, C(t) \qquad (3)$$

$$d/dt\, P(t) = k_2\, C(t) \qquad (4)$$

Furthermore,

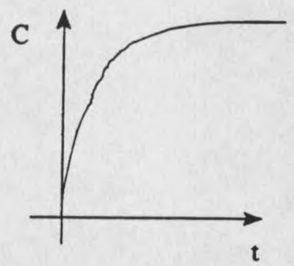
$$E_T = E(t) + C(t) \qquad (5)$$

where E_T denotes the total amount of enzyme in the process. A fraction of the enzyme is free (E(t)) while the rest is bounded to the intermediate compound.

The aforementioned system of differential equations looks quite complicated, and it is complicated indeed! Around 1925 two British scientists, Briggs and Haldane, put forward a daring hypothesis: At the beginning of the experiment, substrate and enzyme combine quite rapidly giving birth to C and thereafter a "stationary state" ensues, during which the amount of C remains practically constant; the latter situation takes place because each time a molecule of P is formed by a rearrangement of C, a molecule of the enzyme is regenerated and combines rapidly with a molecule of the substrate (there is a high affinity between both of them). This mechanism lasts while there is substrate left. Thus, during considerable time one should expect that d/dt C(t) = 0.

Let us see what happens to the system of differential equations if we were to accept the stationary state hypothesis:

From (1) and (5) we have



$d/dt\ C(t) = k_1 (E_T - C(t))\ S(t) - k_{-1} C(t) - k_2 C(t)$

i.e. $d/dt\ C(t) = k_1 E_T S(t) - (k_1 S(t) + k_{-1} + k_2)\ C(t)$

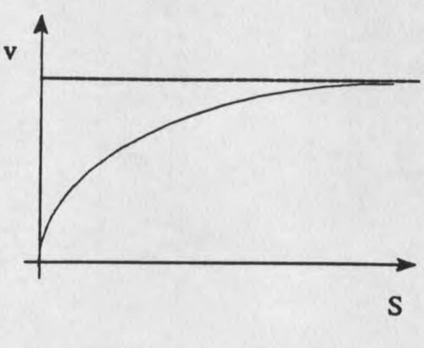
So, $0 = k_1 E_T S(t) - (k_1 S(t) + k_{-1} + k_2)\ C(t)$. Therefore

$$C(t) = \frac{E_T S(t)}{K_m + S(t)} \qquad (6)$$

where we define $K_m = (k_{-1} + k_2)/k_1$. Furthermore, using (4) and (6) we arrive to

$$P'(t) = \frac{k_2 E_T\, S(t)}{K_m + S(t)} \qquad (7)$$

But d/dt P(t) is precisely the reaction velocity v(t). The graph of v(t) corresponds to a well-known mathematical function, shown to the right. By adding more and more substrate we can increase the speed of the reaction. However, a point is reached when S becomes much bigger than K_m and v cannot surpass the quantity $k_2 E_T$.



So, it is advisable to write $v_{max} = k_2 E_T$.

A little bit of elementary algebra leads to the equality - d/dt S(t) = d/dt P(t) provided d/dt C(t) = 0. That is to say, under a stationary state scenario we arrive to the differential equation

$$S'(t) = \frac{-v_{max} S(t)}{K_m + S(t)} \qquad (8)$$

called the Micaelis-Menten equation. Let us pay a closer look at this equation. We notice that it is a separable variables differential equation! Thus we can separate variables and then integrate:

$$K_m \frac{S'(t)}{S(t)} + S'(t) = -v_{max}$$

$$K_m \int_0^t \frac{S'(x)\,dx}{S(x)} + \int_0^t S'(x)\,dx = \int_0^t -v_{max}\,dx$$

Hence

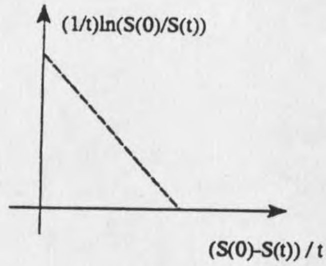
$$K_m [\ln S(x)]_0^t + S(t) - S(0) = -v_{max} t$$

Therefore

$$\frac{1}{t} \ln \frac{S(0)}{S(t)} = -\frac{1}{K_m} \frac{S(0) - S(t)}{t} + \frac{v_{max}}{K_m} \qquad (9)$$

This equality predicts that if (S(0)-S(t))/t is placed on the x-axis and (1/t)ln(S(0)/S(t)) on the y-axis, a straight line with negative slope will appear when we start taking into consideration the experimental values S(t). Very many experiments have been carried out, and always the points

$$\left(\frac{S(0)-S(t)}{t}, \frac{1}{t}\ln\frac{S(0)}{S(t)}\right)$$



were found to be distributed as predicted. This is a crucial finding, which helps to validate the original model and the stationary state hypothesis.

Once the model has been validated,our job is to calculate the parameters K_m and v_{max} (they vary according to the type of enzymatic reaction being analyzed).From (9) we notice that v_{max} / K_m is the y-intercept, while the x-intercept is found from the equality

$$0 = -\frac{1}{K_m}\frac{S(0)-S(t)}{t} + \frac{v_{\max}}{K_m}$$

So

$$\frac{S(0)-S(t)}{t} = v_{\max}$$

Then we draw the line of best fit that goes through the experimental points. From this graph we can find v_{max} and v_{max}/K_m. So, v_{max} and K_m can be calculated. Since $v_{max} = k_2 E_T$, we are able to find k_2. We cannot calculate k_1 and k_{-1} yet, despite the fact that K_m is known already. We will do so later, by employing the elementary theory of second order differential equations.

One further question remains to be solved: Can we find a graph of S(t)? In order to answer this question, an approach that stresses an approximation technique is needed. We have:

$$\ln\frac{S(t)}{S(0)} = \ln\left(1+\frac{S(t)}{S(0)}-1\right) = \ln\left(1+\frac{S(t)-S(0)}{S(0)}\right)$$

At the beginning of the experiment, the quantity (S(t) - S(0)) / S(0) is very small, so

$$\ln\left(1+\frac{S(t)-S(0)}{S(0)}\right) \approx \frac{S(t)-S(0)}{S(0)}$$

wherein we have used the well-known fact that ln(1+x) ≈ x provided x is very small in absolute value. The aforementioned approximation, when applied to (9), leads to:

$$S(t) - S(0) + K_m \frac{S(t)-S(0)}{S(0)} = -v_{max} t$$

Therefore

$$S(t) = S(0) - \frac{v_{max} t}{1+\frac{K_m}{S(0)}} \qquad (10)$$

Thus, at the beginning of the experiment we should have a straight line with negative slope. After a while, a lot of substrate will have been transformed into the product. Hence S(t) will be small compared to S(0). Then ln(S(t) / S(0)) will be much bigger than | S(t)-S(0) |. The equality (9) can be written

$$S(t) - S(0) + K_m \ln\frac{S(t)}{S(0)} = -v_{max} t$$

Therefore

$$K_m \ln\frac{S(t)}{S(0)} = -v_{max} t$$

Consequently

$$S(t) = S(0)\, e^{-\frac{v_{max} t}{K_m}} \qquad (11)$$

Thus, the curve described by S(t) behaves like a straight line with negative slope at the beginning of the experiment, and later on it becomes a negative exponential. This is another prediction of the model being discussed, which was duly corroborated by doing experiments.

All that remains to be done is find a way to calculate k_1 and k_{-1}. Evidently, it is enough to calculate k_1 because $k_{-1} = k_1K_m - k_2$. With this purpose in mind we will analyze the basic model of enzyme kinetics before the steady state, a very short period of time, at the beginning of the experiment, during which it is a good approximation to assume that S(t) is practically S_T (the total amount of substrate with which we started). It is to be noticed that, in general, $S_T(t)$ varies because the substrate is being used.

We differentiate the differential equation (4). Taking into account (1), we can assert that

$$P''(t) + a_1 P'(t) = a_2$$

where $a_1 = k_1 S_T + k_{-1} + k_2$, $a_2 = k_2 k_1 E_T S_T$. It should be mentioned that we have used the equality $E(t) = E_T - C(t)$ and the approximation $S(t) = S_T$. In front of us there is a second order linear homogeneous differential equation. Its characteristic polynomial is $r^2+a_1 r$ with roots $r_1 = 0$ and $r_2 = -a_1$. By inspection of the differential equation it is evident that the function $(a_2 / a_1)\, t$ is a particular solution. Then, its general solution is given by

$$P(t) = c_1 + c_2 e^{-a_1 t} + \frac{a_2}{a_1} t$$

The initial conditions are $P(0) = P_o$, $d/dt\, P(0) = k_2\, C(0) = 0$. These lead to the algebraic system

$$c_1 + c_2 = P_o$$

$$-a_1 c_2 + a_2 / a_1 = 0$$

in the unknowns c_1 and c_2. Then

$$P(t) = P_o - \frac{a_2}{a_1^2} + \frac{a_2}{a_1^2} e^{-a_1 t} + \frac{a_2}{a_1} t$$

However, when t is very small we have the following approximation:

$$e^{wt} = \sum_{n=0}^{\infty} \frac{(wt)^n}{n!} \approx 1 + wt + \frac{w^2t^2}{2}$$

Therefore:

$$P(t) = P_o - \frac{a_2}{a_1^2} + \frac{a_2}{a_1^2}(1 - a_1 t + \frac{a_1^2 t^2}{2}) + \frac{a_2}{a_1} t$$

Simplifying this expression we arrive to $P(t) = P_o + (a_2/2)t^2$. Consequently $P(t) - P_o = (k_1 k_2 E_T S_T t^2)/2$. Thus

$$k_1 = \frac{2(P(t) - P_o)}{t^2 v_{max} S_T} \qquad (12)$$

In summary, the basic model of enzyme kinetics predicts that if it is possible to measure P(t) before the steady state, then the expression to the right of (12) will be constant. When around 1950 scientists developed experimental techniques to deal with very short spans of time, they could corroborate that the quantity $2(P(t)-P_o) / t^2 v_{max} S_T$ remained practically constant. The parameter k_1 was obtained as the mean of these numbers at different values of t before the steady state.

The prediction that stems from (12) was a driving force in developing advanced techniques in what is now called ultrarrapid enzyme kinetics. After all, scientists wanted to know whether or not the prediction could withstand experimental verification!

Through the discussion of several aspects of enzyme kinetics we have had the opportunity to deal with some qualitative methods of approximation in the theory of elementary differential equations, as well as to realize the important role played by mathematics in the development of a scientific theory.

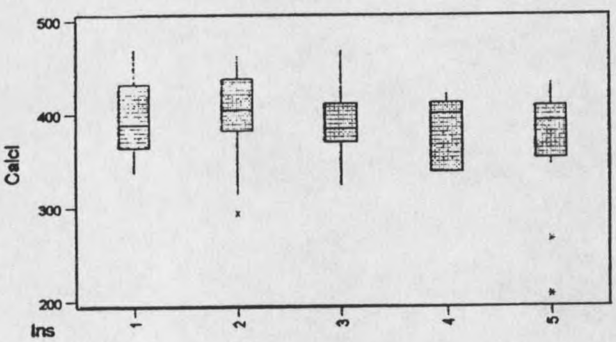
APPENDIX F

FURTHER DESCRIPTIVE STATISTICS

The following data is related to the analysis of Calculus II scores by instructor (p. 64):

Variable	Ins	N	Mean	Median	StDev
Calc I	1	14	396	388.5	39.7
	2	20	399	404.5	46.9
	3	21	389.62	384	34.15
	4	5	379.2	400	39.4
	5	12	367.3	392	66.6
SAT	1	29	611	610	75.2
	2	24	591.7	600	51.2
	3	35	586.6	600	61.7
	4	17	614.1	610	65.1
	5	30	602.33	600	51.17

Boxplots of CalcI by Ins



Boxplots of SAT by Ins