

An analytical approach to finite slope stability analysis
by William Arthur Vischer

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of
MASTER OF SCIENCE in Civil Engineering
Montana State University
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Abstract:
The development of an analytic approach to the finite slope stability problem is initialized in this paper. This method is similar to the "method of slices," in terms of the static analysis; however, exact integration is used for determining the actuating and resisting forces.

An equation was derived expressing the safety factor of a homogeneous, finite slope in terms of the slope geometry, geometry of a circular failure arc, and soil parameters. Safety factors obtained from the derived equation, were compared with those obtained by methods currently in use. Differences of up to five percent were noted in the comparison. The equation for the safety factor was then differentiated, with respect to the radius of the failure arc, in a futile attempt to derive an analytical expression for the radius that yields the minimum factor of safety for any given center. Results of the differentiated expression and the basic expression were compared. This comparison showed that when the differentiated expression was nearly satisfied, the center yielding the minimum safety factors was normally defined.

Further extensive studies are required before any definite conclusion can be made concerning the characteristics of the differentiated expression.

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Date December 5, 1969

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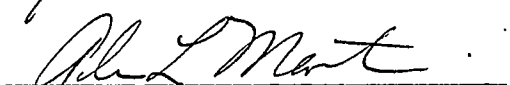
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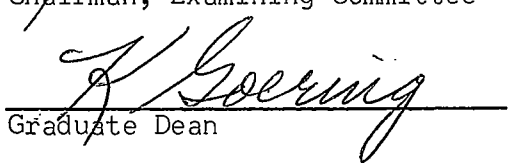
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Bozeman, Montana

December, 1969

ACKNOWLEDGEMENT

The writer wishes to express his appreciation to Dr. Glen L. Martin for his patient guidance and assistance in the preparation of this thesis. His unique qualities of understanding and genuine interest in the student are truly attributes to be commended.

Appreciation is extended to the typist, Patricia Rinaldi.

Special thanks are extended to the writer's wife, Mary Ellen, and his children, Robbie and Terri, for their understanding and encouragement during the course of this study.

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ABSTRACT

The development of an analytic approach to the finite slope stability problem is initialized in this paper. This method is similar to the "method of slices," in terms of the static analysis; however, exact integration is used for determining the actuating and resisting forces.

An equation was derived expressing the safety factor of a homogeneous, finite slope in terms of the slope geometry, geometry of a circular failure arc, and soil parameters. Safety factors obtained from the derived equation, were compared with those obtained by methods currently in use. Differences of up to five percent were noted in the comparison. The equation for the safety factor was then differentiated, with respect to the radius of the failure arc, in a futile attempt to derive an analytical expression for the radius that yields the minimum factor of safety for any given center. Results of the differentiated expression and the basic expression were compared. This comparison showed that when the differentiated expression was nearly satisfied, the center yielding the minimum safety factors was normally defined.

Further extensive studies are required before any definite conclusion can be made concerning the characteristics of the differentiated expression.

CHAPTER I

INTRODUCTION

Probably the most widely used method of analysis for finite slopes is the "method of slices" based on circular failure surfaces. According to Taylor (1948), K. E. Petterson is thought to be the first to use such a method in the study of a quay wall in 1915. Further investigations and studies revealed that actual failure surfaces do not deviate greatly from this assumed circular failure surface.

The method of slices basically consists of dividing the soil mass into vertical slices and performing a static analysis on the soil above the assumed failure surface. As there are many possible circular arcs for a given cross-section, a trial and error procedure must be used to locate both the location of the center of the critical arc and the radius of the critical arc. The time and labor involved in the graphical trial and error solutions is excessive. With the advent of the digital computer, the problems associated with this trial and error analysis were alleviated.

The above remarks indicate the benefit of a direct analytical analysis to the engineering profession. An analytical analysis would not only eliminate the repetitive calculations, but would also allow more efficient use of the computer in slope stability problems.

CHAPTER II

ANALYTICAL DEVELOPMENT

The initial portion of the derivation of the analytic expression is similar to the analysis used in the "method of slices;" the only difference being that exact integration is used to obtain the actuating and resisting forces. After performing the required exact integrations, the actuating and resisting forces can be expressed in terms of the radius, R, and the known geometric and soil parameters. The factor of safety can similarly be expressed as it is a function of these actuating and resisting forces. Differentiating the factor of safety with respect to the radius, R, and setting the resultant equal to zero, maximizes or minimizes the factor of safety with respect to the radius. Solving the expression for R should then yield the radius, for a given slope geometry and arc center, that produces the minimum factor of safety.

For a simple finite slope, three geometric failure possibilities, as shown in Figures 1, 2, and 3, must be investigated. The failure geometry shown in Figure 1 is probably the most common failure noted in soft, cohesive materials, whereas the failure geometry shown in Figure 2 is most common in "mixed" soils. The failure geometry of Figure 1 is the general case and degenerates to the cases shown in Figures 2 and 3 when the appropriate geometric substitutions are made.

Definition of Geometry

The initial step of the derivation consists of defining the intercepts of the slopes and failure arc.

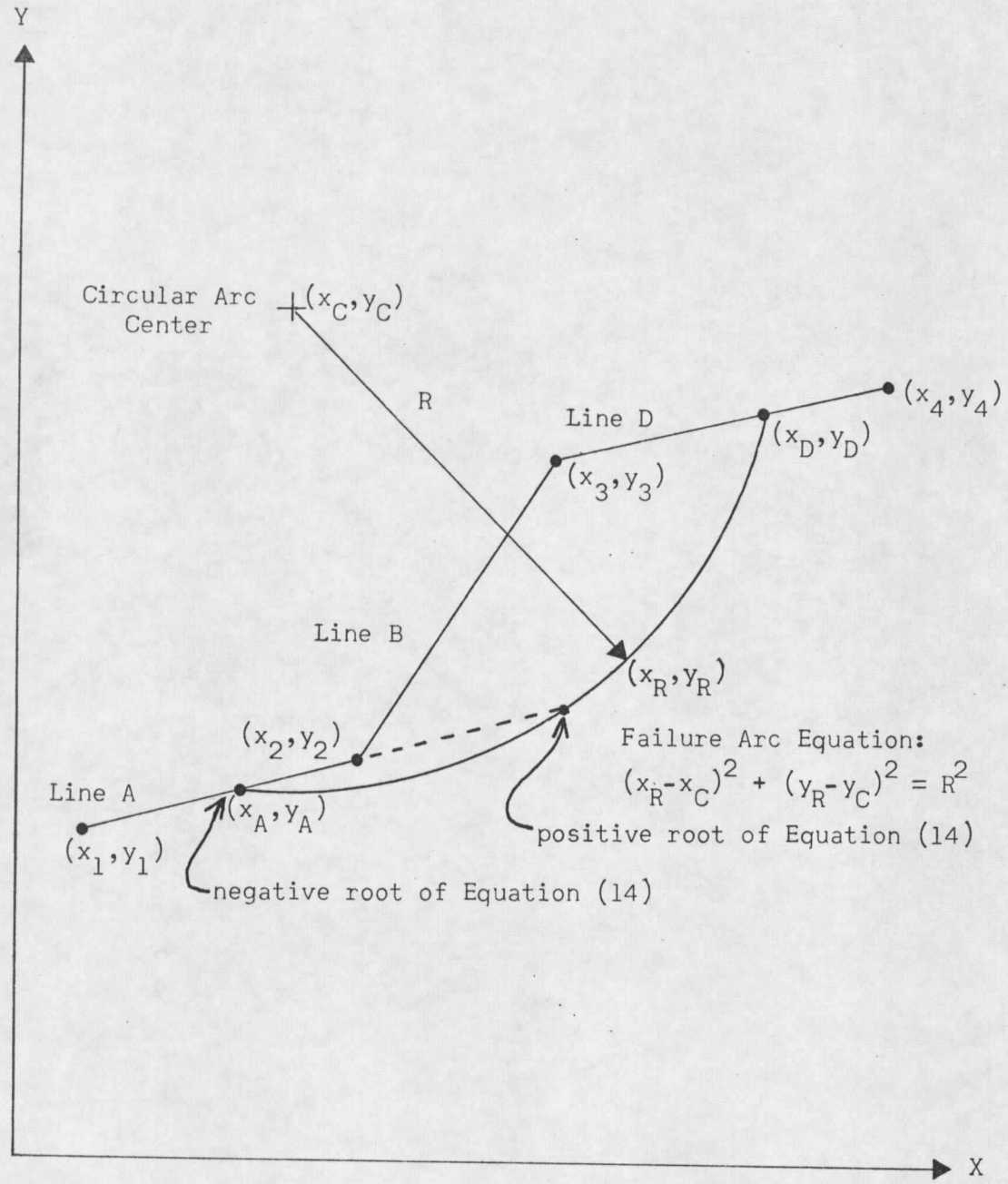


Figure 1. Possible failure geometry of a simple, finite slope, Case 1.

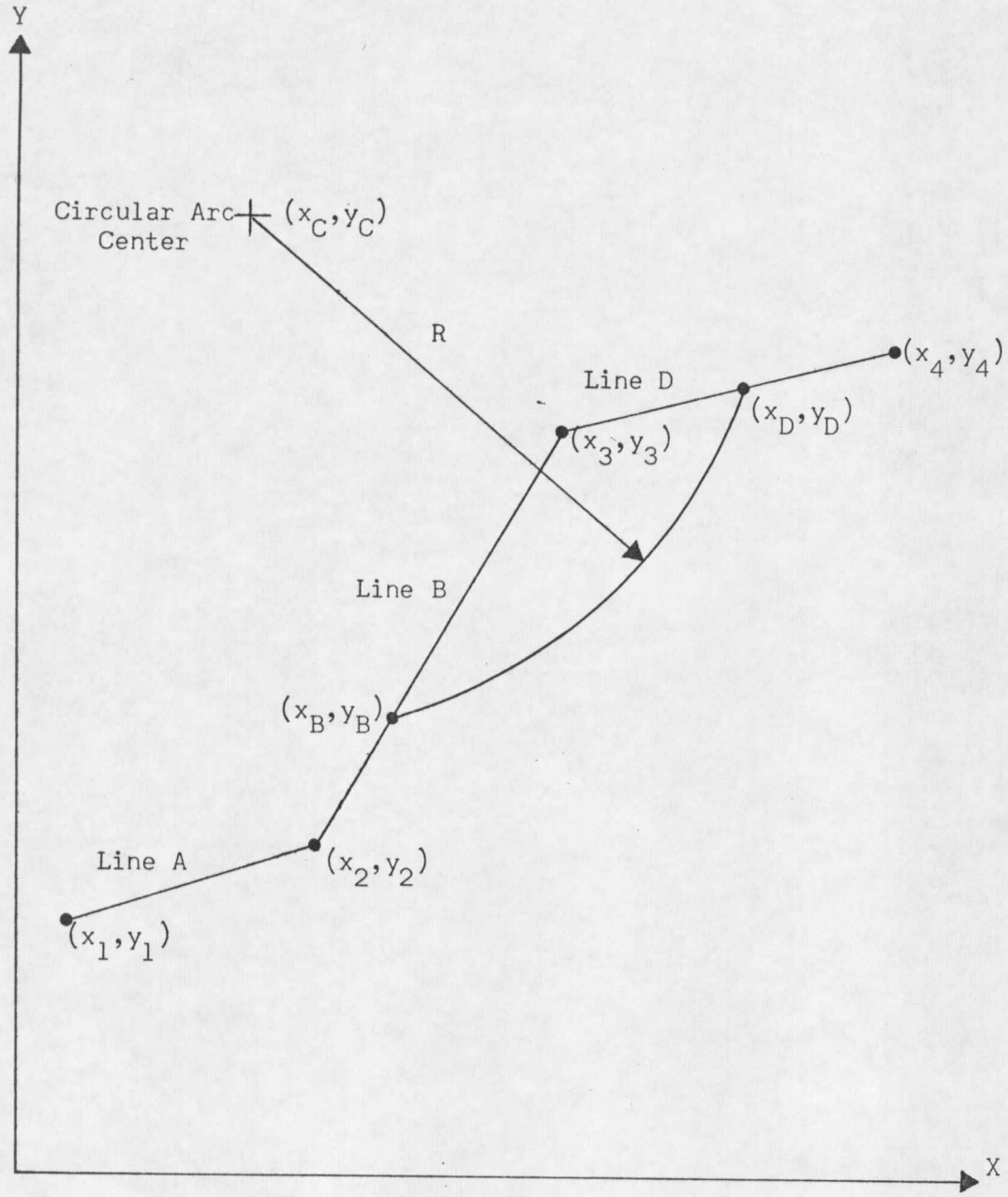


Figure 2. Possible failure geometry of a simple, finite slope, Case 2.

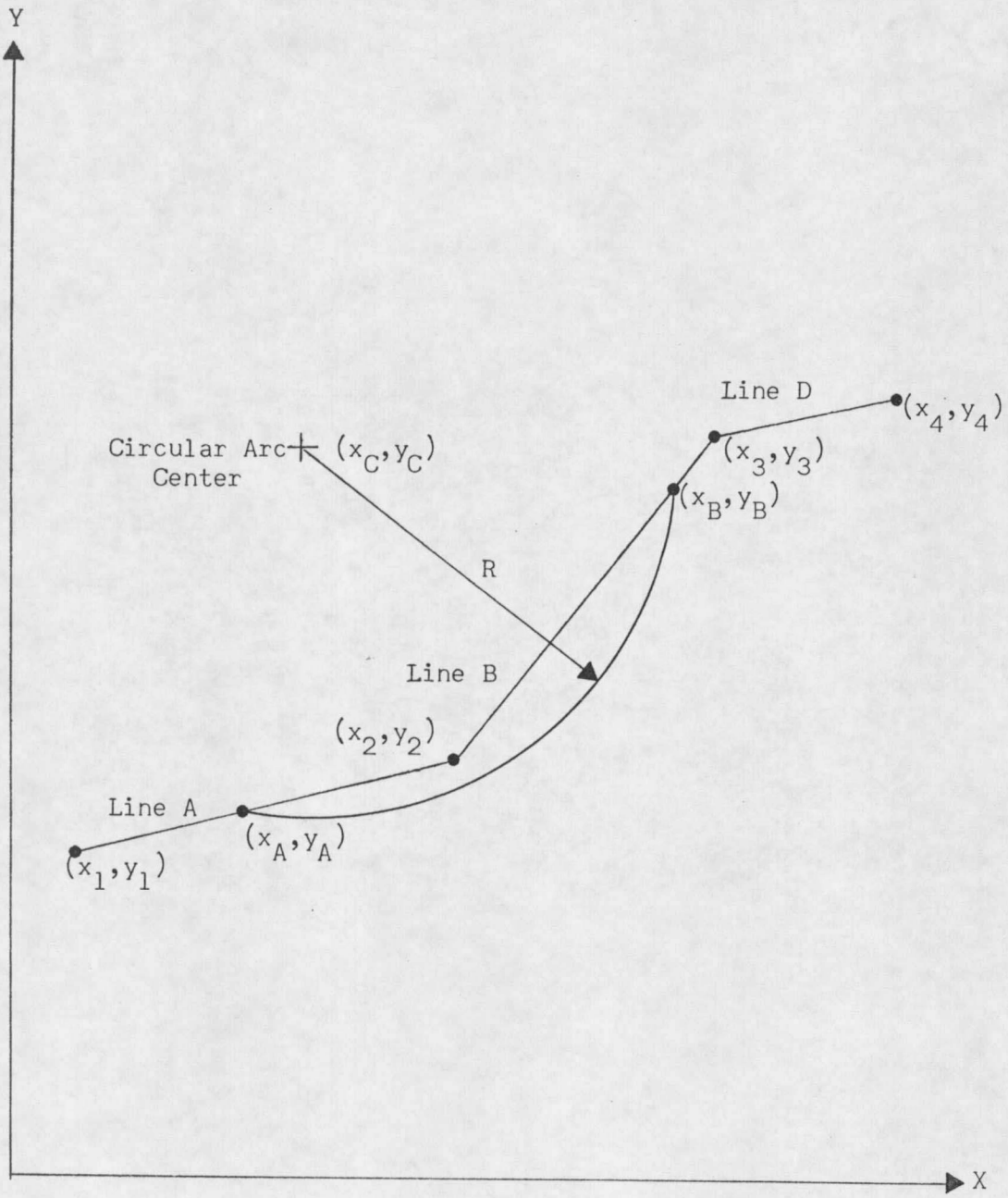


Figure 3. Possible failure geometry of a simple, finite slope, Case 3.

The equation of lines A, B, and D can be expressed in the general form:

$$y = mx + b \tag{1}$$

where: m is the slope of the line, and
b is the y intercept.

Letting:

$$e = \frac{y_2 - y_1}{x_2 - x_1} \tag{2}$$

$$g = \frac{y_3 - y_2}{x_3 - x_2} \tag{3}$$

$$i = \frac{y_4 - y_3}{x_4 - x_3} \tag{4}$$

$$f = \frac{y_1x_2 - x_1y_2}{x_2 - x_1} \tag{5}$$

$$h = \frac{y_2x_3 - y_3x_2}{x_3 - x_2} \text{ , and} \tag{6}$$

$$j = \frac{y_3x_4 - y_4x_3}{x_4 - x_3} \tag{7}$$

and substituting yields:

$$y_A = ex_A + f \tag{8}$$

$$y_B = gx_B + h \tag{9}$$

$$y_D = ix_D + j \tag{10}$$

The general equation for the assumed circular failure arc is:

$$r^2 = (x - h)^2 + (y - k)^2 \tag{11}$$

where: h and k are the respective x and y values denoting the center of the circle, and r is the radius.

Substituting the appropriate variables, as shown in Figure 1, into Equation (11) yields:

$$R^2 = (x_R - x_C)^2 + (y_R - y_C)^2 \tag{12}$$

Solving for y_R in Equation (12) gives:

$$y_R = y_C \pm \sqrt{R^2 - (x_R - x_C)^2} \tag{13}$$

Respectively setting Equations (8), (9), and (10) equal to Equation (13) the expressions for x_A, x_B, and x_D are:

$$x_A = \frac{ey_C - ef + x_C - \sqrt{-2efx_C + 2ey_Cx_C + e^2R^2 - e^2x_C^2 - f^2 + 2fy_C - y_C^2 + R^2}}{e^2 + 1} \tag{14}$$

$$x_B = \frac{gy_C - gh + x_C \pm \sqrt{-2ghx_C + 2gy_Cx_C + g^2R^2 - g^2x_C^2 - h^2 + 2hy_C - y_C^2 + R^2}}{g^2 + 1} \quad (15)$$

$$x_D = \frac{iy_C - ij + x_C + \sqrt{-2ijx_C + 2iy_Cx_C + iR^2 - i^2x_C^2 - j^2 + 2jy_C - y_C^2 + R^2}}{i^2 + 1} \quad (16)$$

The solution of Equation (14) involves a quadratic with the negative root being rational (see Figure 1). The solution of Equation (15) can involve either the negative or positive root depending on the failure geometry, while for Equation (16), the positive root is rational.

Computation of Forces

The stability of the slope is dependent on the resisting moment (made up of the resisting forces acting about the assumed center of rotation) being greater than the actuating moment.

From elementary soil mechanics, the actuating force, F_{act}, can be expressed in terms of that component of the weight force acting tangentially to the assumed failure arc, as shown in Figure 4:

$$F_{act} = \sum \text{Tangential Components of Weight Force} = \sum T \quad (17)$$

Figure 4 is a profile of a lineal slope having an infinite length normal to the cross-section shown. Assuming a unit length of slope, the volume of the differential parallelepiped shown in Figure 4 is:

$$dV = (y - y_R)dx \quad (18)$$

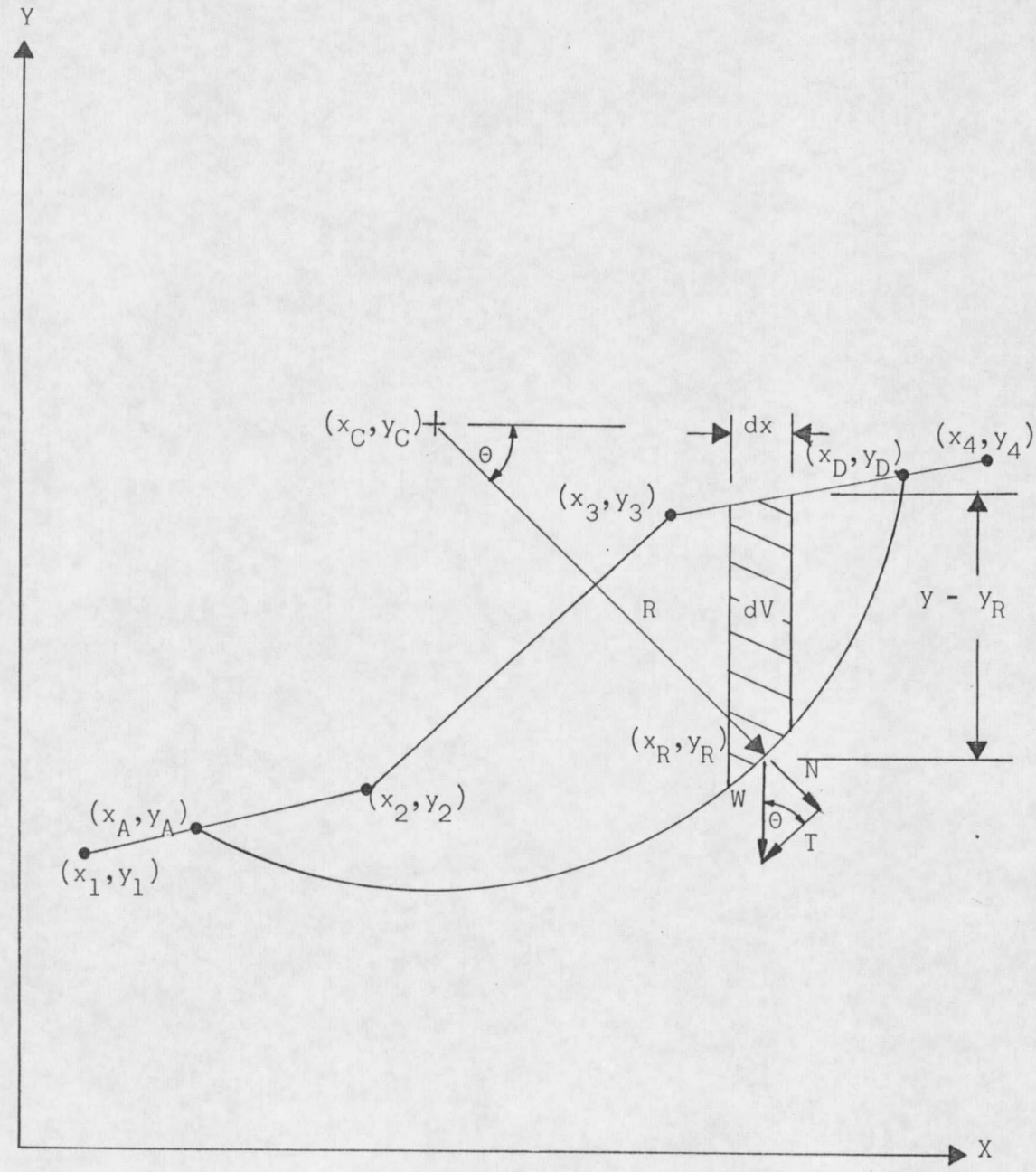


Figure 4. Diagram of forces and geometry.

The differential weight, dW, of the differential volume is equal to the volume of soil multiplied by the unit weight of soil, γ , or:

$$dW = \gamma(y - y_R)dx \tag{19}$$

Multiplying dW by cos θ (see Figure 4) yields the tangential component of the weight, or:

$$dT = \gamma(y - y_R)\cos\theta dx \tag{20}$$

and from Figure 4:

$$\cos\theta = \frac{x_R - x_C}{R} \tag{21}$$

Converting Equation (17) to its integral form and substituting Equation (20) yields:

$$F_{act} = \int_{x_A}^{x_2} \gamma(y_A - y_R)\cos\theta dx + \int_{x_2}^{x_3} \gamma(y_B - y_R)\cos\theta dx$$

$$+ \int_{x_3}^{x_D} \gamma(y_D - y_R)\cos\theta dx \tag{22}$$

Substituting Equations (8), (9), (10), (13), and (21) into Equation (22), and simplifying yields:

$$F_{act} = A'\left[\int_{x_A}^{x_2} (B' + C' + D' + E')dx + \int_{x_2}^{x_3} (F' + G' + H' - I')dx + \int_{x_3}^{x_D} (J' + K' + L' - M')dx\right] \quad (23)$$

where: $A' = \frac{\gamma}{R}$;

$B' = ex^2 + fx - y_C x$;

$C' = G' = K' = x(-x^2 + 2x_C x + R^2 - x_C{}^2)^{\frac{1}{2}}$;

$D' = -ex_C x - fx_C + y_C x_C$;

$E' = I' = M' = x_C(-x^2 + 2x_C x + R^2 - x_C{}^2)^{\frac{1}{2}}$;

$F' = gx^2 + hx - y_C x$;

$H' = -gx_C x - hx_C + y_C x_C$;

$J' = ix^2 + jx - y_C x$; and

$L' = ix_C x - jx_C + y_C x_C$

Note that the smallest root of Equation (13) was used as it is applicable

as shown in Figure 1. Integrating Equation (23) produces:

$$F_{act} = A'(B'' + C'' + D'' + E'' + F'' + G'' + H'' - I'' + J'' + K'' + L'' - M'') \quad (24)$$

where: $$B'' = \int_{x_A}^{x_2} B'dx = (\tfrac{1}{3})e(x_2^3 - x_A^3)$$

$$+ (\tfrac{1}{2})(f - y_C)(x_2^2 - x_A^2) \quad ; \quad (25)$$

$$C'' = \int_{x_A}^{x_2} C'dx = (-\tfrac{1}{3})(-x_2^2 + 2x_Cx_2 + R^2 - x_C^2)^{\frac{3}{2}}$$

$$+(\tfrac{1}{3})(-x_A^2 + 2x_Cx_A + R^2 - x_C^2)^{\frac{3}{2}}$$

$$- (\tfrac{1}{2})(x_C)(x_C - x_2)(-x_2^2 + 2x_Cx_2 + R^2 - x_C^2)^{\frac{1}{2}}$$

$$+ (\tfrac{1}{2})(x_C)(x_C - x_A)(-x_A^2 + 2x_Cx_A + R^2 - x_C^2)^{\frac{1}{2}}$$

$$- (\tfrac{1}{2})(x_CR^2)(\sin^{-1}\frac{x_C - x_2}{R}) + (\tfrac{1}{2})(x_CR^2)(\sin^{-1}\frac{x_C - x_A}{R}) \; ; \quad (26)$$

$$D'' = \int_{x_A}^{x_2} D'dx = (-\tfrac{1}{2})(ex_C)(x_2^2 - x_A^2) + (y_Cx_C - fx_C)(x_2 - x_A); \quad (27)$$

$$E'' = \int_{x_A}^{x_2} E'dx = (\tfrac{1}{2})(x_C)(x_2-x_C)(-x_2^2+2x_Cx_2+R^2-x_C^2)^{\frac{1}{2}}$$

$$-(\tfrac{1}{2})(x_C)(x_A-x_C)(-x_A^2+2x_Cx_A+R^2-x_C^2)^{\frac{1}{2}}$$

$$-(\tfrac{1}{2})(x_CR^2)(\sin^{-1}\frac{x_C-x_2}{R}) + (\tfrac{1}{2})(x_CR^2)(\sin^{-1}\frac{x_C-x_A}{R}) \quad ; \qquad (28)$$

$$F'' = \int_{x_2}^{x_3} F'dx = \text{(same form as Equation (25) with g, h, } x_3 \text{ and } x_2 \text{ replacing e, f, } x_2\text{, and } x_A \text{ respectively);} \qquad (29)$$

$$G'' = \int_{x_2}^{x_3} G'dx = \text{(same form as Equation (26) with } x_2 \text{ and } x_3 \text{ replacing } x_A \text{ and } x_2 \text{ respectively);} \qquad (30)$$

$$H'' = \int_{x_2}^{x_3} H'dx = \text{(same form as Equation (27) with g, h, } x_2 \text{ and } x_3 \text{ replacing e, f, } x_A\text{, and } x_2 \text{ respectively);} \qquad (31)$$

$$I'' = \int_{x_2}^{x_3} I'dx = \text{(same form as Equation (28) with } x_2 \text{ and } x_3 \text{ replacing } x_A \text{ and } x_2 \text{ respectively);} \qquad (32)$$

$$J'' = \int_{x_3}^{x_D} J'dx = \text{(same form as Equation (25) with i, j, } x_3 \text{ and } x_D \text{ replacing e, f, } x_A \text{ and } x_2 \text{ respectively)}; \quad (33)$$

$$K'' = \int_{x_3}^{x_D} K'dx = \text{(same form as Equation (26) with } x_3 \text{ and } x_D \text{ replacing } x_A \text{ and } x_2 \text{ respectively)}; \quad (34)$$

$$L'' = \int_{x_3}^{x_D} L'dx = \text{(same form as Equation (27) with i, j, } x_3 \text{ and } x_D \text{ replacing e, f, } x_A \text{ and } x_2 \text{ respectively)}; \quad (35)$$

$$M'' = \int_{x_3}^{x_D} M'dx = \text{(same form as Equation (28) with } x_3 \text{ and } x_D \text{ replacing } x_A \text{ and } x_2 \text{ respectively)}. \quad (36)$$

Substituting the above values into Equation (24) and reducing, yields the following expression for the activating force:

$$F_{act} = \frac{\gamma}{R}\left[\left(\frac{-e}{3}\right)x_A^3 + \left(\frac{-f+y_C+ex_C}{2}\right)x_A^2 + \right.$$

$$(f-y_C)x_Cx_A + \left(\frac{1}{3}\right)(-x_A^2+2x_Cx_A+R^2-x_C^2)^{\frac{3}{2}} +$$

$$\left(\frac{e-g}{3}\right)x_2^3 + \left(\frac{f-ex_C-h+gx_C}{2}\right)x_2^2 + (h-f)x_Cx_2$$

$$+\left(\frac{g-i}{3}\right)x_3^3 + \left(\frac{h-gx_C-j+ix_C}{2}\right)x_3^2 + (j-h)x_Cx_3 + \left(\frac{i}{3}\right)x_D^3$$

$$+ \left(\frac{j-y_C-ix_C}{2}\right)x_D^2 + (y_C-j)x_Cx_D - \left(\frac{1}{3}\right)\left(-x_D^2+2x_Cx_D+R^2-x_C^2\right)^{\frac{3}{2}}\Big] \quad (37)$$

The strength of the soil material, which resists any impending rotation or failure, can be expressed by Coulomb's Law:

$$S = \sigma \tan\phi + c \quad (38)$$

where: S = shear strength,
σ = normal effective stress,
ϕ = angle of internal friction, and
c = cohesion.

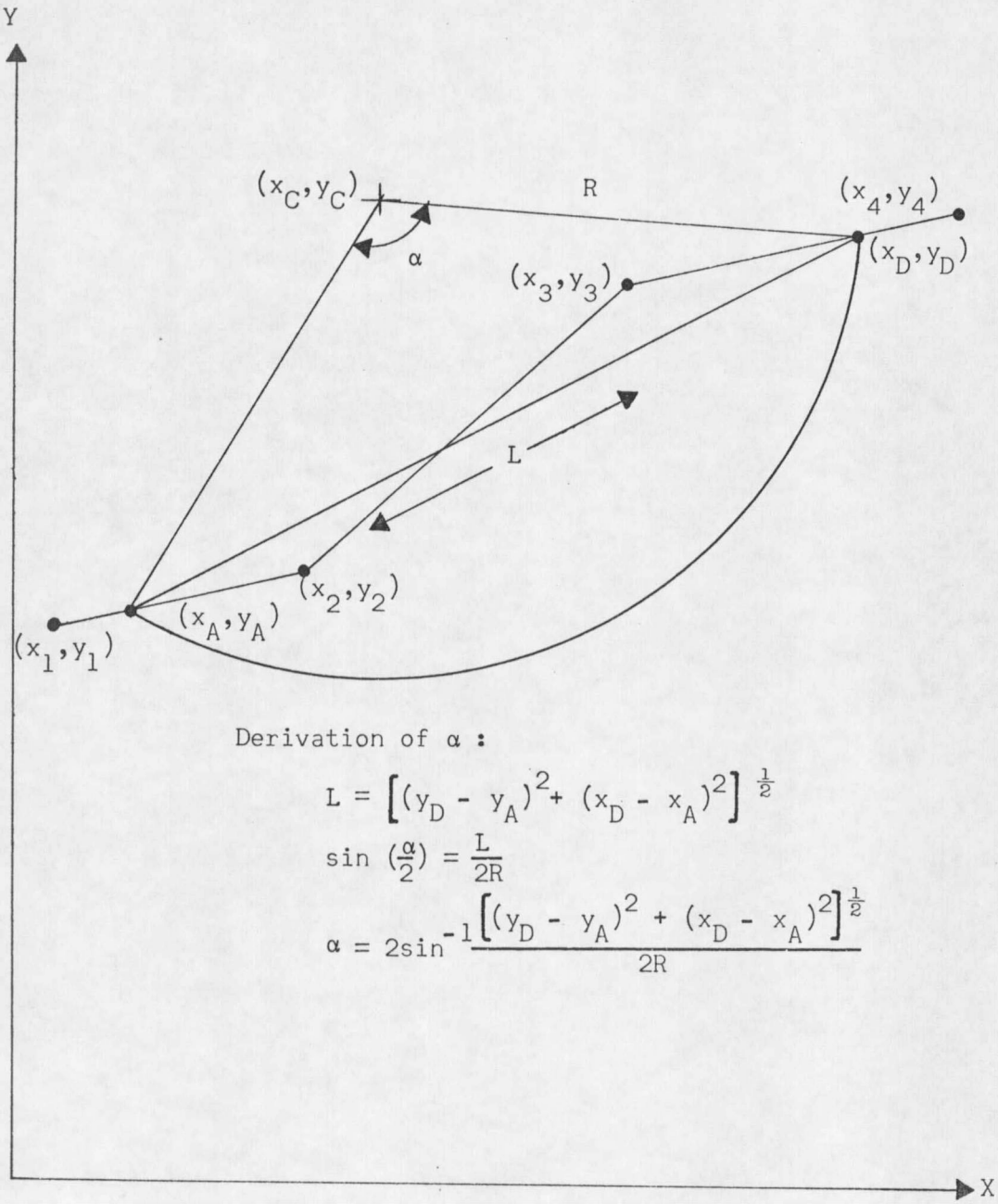
The resisting force, F_{res}, per unit length of the lineal slope (Taylor, 1948, p. 437), is expressed as:

$$F_{res} = \sum N \tan\phi + R\alpha c \quad (39)$$

where: N = normal component of differential weight (see Figure 4),
R = arc radius, and
α = central angle (see Figure 5).

Once again using the differential expression for the slice weight:

$$dW = \gamma (y - y_R)dx \quad (40)$$



Derivation of α :

$$L = \left[(y_D - y_A)^2 + (x_D - x_A)^2\right]^{\frac{1}{2}}$$

$$\sin\left(\frac{\alpha}{2}\right) = \frac{L}{2R}$$

$$\alpha = 2\sin^{-1}\frac{\left[(y_D - y_A)^2 + (x_D - x_A)^2\right]^{\frac{1}{2}}}{2R}$$

Figure 5. Derivation of alpha angle.

and multiplying by $\sin\theta$, yields the normal component of the differential weight:

$$dN = \gamma(y - y_R)\sin\theta dx \tag{41}$$

From Figure 4:

$$\sin\theta = \frac{y_C - y_R}{R} \tag{42}$$

Expressing Equation (39) in its integral form and substituting the value of dN from Equation (41) yields:

$$F_{res} = \tan\phi\left[\int_{x_A}^{x_2} \gamma(y_A - y_R)\sin\theta\, dx + \int_{x_2}^{x_3} \gamma(y_B - y_R)\sin\theta\, dx\right.$$

$$\left. + \int_{x_3}^{x_D} \gamma(y_D - y_R)\sin\theta\, dx\right] + R\alpha\, c \tag{43}$$

Substituting Equations (8), (9), (10), (13) and (42) into Equation (43), integrating, and simplifying yields:

$$F_{res} = \frac{\gamma\tan\phi}{R}\left[\frac{x_A^3}{3} - x_C x_A^2 + \frac{e}{3}x_A^{\frac{3}{2}} + (x_C^2 - R^2)x_A\right.$$

$$+ (\tfrac{1}{2})(ex_C^2 - ex_Cx_A - fx_A + fx_C + y_Cx_A - y_Cx_C)X_A^{\frac{1}{2}}$$

$$+ (\tfrac{1}{2})(R^2)(ex_C + f - y_C)S_A + (\tfrac{1}{3})(g - e)X_2^{\frac{3}{2}}$$

$$+ (\tfrac{1}{2})(-ex_C + ex_Cx_2 + fx_2 - fx_C + gx_C^2 - gx_Cx_2$$

$$- hx_2 + hx_C)X_2^{\frac{1}{2}} + (\tfrac{1}{2})(R^2)(-ex_2 - f + gx_C + h)S_2$$

$$+ (\tfrac{1}{3})(i - g)X_3^{\frac{3}{2}} + (\tfrac{1}{2})(-gx_C^2 + gx_Cx_3 + hx_3 - hx_C + ix_C^2$$

$$- ix_Cx_3 - jx_3 + jx_c)X_3^{\frac{1}{2}} + (\tfrac{1}{2})(R^2)(-gx_C - h + ix_C + j)S_3$$

$$- \frac{x_D^3}{3} + x_Cx_D^2 - \frac{i}{3}X_D^{\frac{3}{2}} + (R^2 - x_C^2)x_D + (\tfrac{1}{2})(-ix_C^2$$

$$+ ix_Cx_D + jx_D - jx_C - y_Cx_D + y_Cx_C)X_D^{\frac{1}{2}}$$

$$+ (\tfrac{1}{2})(R^2)(-ix_C - j + y_C)S_D \Big] + R\alpha c \qquad (44)$$

where: $X_{subscript} = (-x^2_{subscript} + 2x_Cx_{subscript} + R^2 - x_C^2)$,

$S_{subscript} = \sin^{-1}(\dfrac{x_C - x_{subscript}}{R})$, and

α is derived from Figure 5.

The factor of safety, FS, for the stability of a slope can then be expressed as:

$$FS = \text{Factor of safety} = \frac{\text{Resisting Force}}{\text{Actuating Force}} \tag{45}$$

where the actuating and resisting forces are expressed by Equations (37) and (44), respectively.

As mentioned previously the expressions developed thus far are applicable to the geometry shown in Figure 1. To analyze either of the remaining two cases certain substitutions are necessary. To analyze the geometry shown in Figure 2, the following substitutions are required:

g for e, h for f, i for g, j for h, and x_3 for x_2 in Equations (37) and (44).

To analyze the geometry shown in Figure 3, the following substitutions are required:

g for i, and h for j in Equations (37) and (44).

Minimizing the Safety Factor for a Fixed Center

In order to minimize the safety factor, differentiation of the safety factor with respect to the radius is necessary. Differentiating Equation (45) with respect to the radius, R:

$$\frac{d(FS)}{dR} = \frac{\left[F_{act}\right]\left[\frac{d(F_{res})}{dR}\right] - \left[F_{res}\right]\left[\frac{d(F_{act})}{dR}\right]}{\left[F_{act}\right]^2} \tag{46}$$

The derivative of the actuating force, F_{act}, as given in Equation (37), is:

$$\frac{d(F_{act})}{dR} = -\frac{\gamma}{R^2}\Big[-(\tfrac{e}{3})x_A^3 + (\frac{-f + y_C + ex_C}{2})x_A^2 + (f-y_C)x_Cx_A$$

$$+ (\tfrac{1}{3})(-x_A^2 + 2x_Cx_A + R^2 - x_C^2)^{\frac{3}{2}} + (\tfrac{e-g}{3})x_2^3$$

$$+(\frac{f - ex_C - h + gx_C}{2})x_2^2 + (h - f)x_Cx_2$$

$$+ (\tfrac{g-i}{3})x_3^3 + (\frac{h - gx_C - j + x_C}{2})x_3^2 + (j-h)x_Cx_3$$

$$+ (\tfrac{i}{3})x_D^3 + (\frac{j - y_C - ix_C}{2})x_D^2 + (y_C-j)x_Cx_D$$

$$- (\tfrac{1}{3})(-x_D^2 + 2x_Cx_D + R^2 - x_C^2)^{\frac{3}{2}}\Big] + \frac{\gamma}{R}\Big[- ex_A^2x_A'$$

$$+ (- f + y_C + ex_C)x_Ax_A' + (f - y_C)x_Cx_A' + (-x_A^2$$

$$+ 2x_Cx_A + R^2 - x_C^2)^{\frac{1}{2}}(-x_Ax_A' + x_Cx_A' + R) + ix_D^2x_D'$$

$$+ (j - y_C - ix_C)x_D x_D' + (y_C - j)x_C x_D' - (-x_D^2 +$$

$$2x_C x_D + R^2 - x_C^2)^{\frac{1}{2}}(-x_D x_D' + x_C x_D' + R)\Big] \qquad (47)$$

The derivative of the resisting force, F_{res}, as given in Equation (44), is:

$$\frac{d(F_{res})}{dR} = -\frac{\gamma \tan\phi}{R^2}\Big[\frac{x_A^3}{3} - x_C x_A^2 + (\frac{e}{3})X^{\frac{3}{2}} + (x_C - R^2)x_A$$

$$+ (\tfrac{1}{2})(ex_C^2 - ex_C x_A - fx_A + fx_C + y_C x_A - y_C x_C)X_A^{\frac{1}{2}}$$

$$+ (\tfrac{1}{2})(R^2)(ex_C + f - y_C)\, S_A + (\tfrac{1}{3})(g-e)X_2^{\frac{3}{2}}$$

$$+ (\tfrac{1}{2})(-ex_C^2 + ex_C x_2 + fx_2 - fx_C + gx_C^2 - gx_C x_2$$

$$- hx_2 + hx_C)X_2^{\frac{1}{2}} + (\tfrac{1}{2})(R^2)(-ex_C - f + gx_C + h)S_2$$

$$+ (\tfrac{1}{3})(i-g)X_3^{\frac{3}{2}} + (\tfrac{1}{2})(-gx_C^2 + gx_C x_3 + hx_3 - hx_C$$

$$+ ix_C^2 - ix_C x_3 - jx_3 + jx_C)X_3^{\frac{1}{2}} + (\tfrac{1}{2})(R^2)(-gx_C - h$$

$$+ ix_C + j)S_3 - \frac{x_D^3}{3} + x_C x_D^2 - (\tfrac{i}{3})X_D^{\frac{3}{2}} + (R^2 - x_C^2)x_D$$

$$+ (\tfrac{1}{2})(-ix_C^2 + ix_Cx_D + jx_D - jx_C - y_Cx_D + y_Cx_C)X_D^{\frac{1}{2}}$$

$$+ (\tfrac{1}{2})(R^2)(-ix_C - j + y_C)S_D\Big] + \frac{\gamma\tan\phi}{R}\Big[x_A^2x_A'$$

$$- 2x_Cx_Ax_A' + (\tfrac{1}{2})(e)X_A^{\frac{1}{2}}X_A' + (x_C^2 - R^2)x_A' - 2Rx_A$$

$$+ (\tfrac{1}{4})(ex_C^2 - ex_Cx_A - fx_A + fx_C + y_Cx_A - y_Cx_A)X_A^{-\frac{1}{2}}X_A'$$

$$+ (\tfrac{1}{2})(-ex_Cx_A' - fx_A' + y_Cx_A')X_A^{\frac{1}{2}} + (\tfrac{1}{2})(R^2)(ex_C$$

$$+ f - y_C)S_A' + R(ex_C + f - y_C)S_A + (\tfrac{1}{2})(g-e)X_2^{\frac{1}{2}}X_2'$$

$$+ (\tfrac{1}{4})(-ex_C^2 + ex_Cx_2 + fx_2 - fx_C + gx_C^2 - gx_Cx_2$$

$$- hx_2 + hx_C)X_2^{-\frac{1}{2}}X_2' + (\tfrac{1}{2})(R^2)(-ex_C - f + gx_C + h)S_2'$$

$$+ R(-ex_C - f + gx_C + h)S_2 + (\tfrac{1}{2})(i-g)X_3^{\frac{1}{2}}X_3'$$

$$+ (\tfrac{1}{2})(-gx_C^2 + gx_Cx_3 + hx_3 - hx_C + ix_C^2 - ix_Cx_3$$

$$- jx_3 + jx_C)X_3^{-\frac{1}{2}}X_3' + (\tfrac{1}{2})(R^2)(-gx_C - h + ix_C+j)S_3'$$

$$+ R(-gx_C - h + ix_C + j)S_3 - x_D{}^2x_D' + 2x_Cx_Dx_D'$$

$$- (\tfrac{1}{2})X_D{}^{\frac{1}{2}}X_D' + (R^2 - x_C{}^2)x_D' + 2Rx_D + (\tfrac{1}{4})(-ix_C{}^2$$

$$+ ix_Cx_D + jx_D - jx_C - y_Cx_D + y_Cx_C)X_D{}^{-\frac{1}{2}}X_D' + (\tfrac{1}{2})(ix_Cx_D'$$

$$+ jx_D' - y_Cx_D')X_D{}^{\frac{1}{2}} + (\tfrac{1}{2})(R^2)(-ix_C - j + y_C)S_D'$$

$$+ R(-ix_C - j + y_C)S_D] + \alpha c + R\,\alpha' c \qquad (48)$$

where: x_A, x_D, X_A, X_D, X_2, X_3, S_A, S_D, S_2, S_3, α, y_A, and y_D are as defined earlier and whose derivatives are:

$$\frac{dx_A}{dR} = x_A' = - R(-2efx_C + 2ey_Cx_C + e^2R^2 - e^2x_C{}^2$$

$$- f^2 + 2fy_C - y_C{}^2 + R^2)^{-\frac{1}{2}} \qquad (49)$$

$$\frac{dx_D}{dR} = x_D' = R(-2ijx_C + 2\,iy_Cx_C + i^2R^2 - i^2x_C{}^2 - j^2$$

$$+ 2jy_C - y_C{}^2 + R^2)^{-\frac{1}{2}} \qquad (50)$$

$$\frac{dX_A}{dR} = X_A' = - 2x_Ax_A' + 2\,x_Cx_A' + 2R \qquad (51)$$

$$\frac{dX_D}{dR} = X_D' = -2x_D x_D' + 2x_C x_D' + 2R \tag{52}$$

$$\frac{dX_2}{dR} = \frac{dX_3}{dR} = X_2' + X_3' = 2R \tag{53}$$

$$\frac{dS_A}{dR} = S_A' = \frac{-Rx_A' + x_A - x_C}{R^2\sqrt{1 - \left(\frac{x_C - x_A}{R}\right)^2}} \tag{54}$$

$$\frac{dS_D}{dR} = S_D' = \frac{-Rx_D' + x_D - x_C}{R^2\sqrt{1 - \left(\frac{x_C - x_D}{R}\right)^2}} \tag{55}$$

$$\frac{dS_2}{dR} = S_2' = \frac{x_2 - x_C}{R^2\sqrt{1 - \left(\frac{x_C - x_2}{R}\right)^2}} \tag{56}$$

$$\frac{dS_3}{dR} = S_3' = \frac{x_3 - x_C}{R^2\sqrt{1 - \left(\frac{x_C - x_3}{R}\right)^2}} \tag{57}$$

$$\frac{d\alpha}{dR} = \alpha' = \left\{R\left[(y_D - y_A)(y_D' - y_A') + (x_D - x_A)(x_D' - x_A')\right] - \left[(y_D - y_A)^2 + (x_D - x_A)^2\right]^{\frac{1}{2}}\right\}\left\{R^2\left[(y_D - y_A)^2\right.\right.$$

$$+ (x_D - x_A)^2\Big]^{\frac{1}{2}} \left[1 - \left(\frac{(y_D-y_A)^2 + (x_D-x_A)^2}{4R^2}\right)\right]^{\frac{1}{2}}\Bigg\}^{-1} \qquad (58)$$

$$\frac{dy_A}{dR} = y_A' = ex_A' \qquad (59)$$

$$\frac{dy_D}{dR} = y_D' = ix_D' \qquad (60)$$

Setting Equation (46) equal to zero yields:

$$\left[F_{act}\right]\left[\frac{d(F_{res})}{dR}\right] - \left[F_{res}\right]\left[\frac{d(F_{act})}{dR}\right] = 0 \qquad (61)$$

After intensive investigation it was deemed beyond the scope of this study to explicitly solve Equation (61) for the radius in terms of the other parameters. A digital computer program was therefore written in which the approximate solution of Equation (61) was sought while systematically varying the radius, R, keeping all other parameters constant.

CHAPTER III

RESULTS AND DISCUSSION

Three different conditions were used to study the results of the derived expressions. These conditions are shown in Table I.

TABLE I

GEOMETRY AND SOIL CONDITIONS
USED IN EVALUATING DERIVED EQUATIONS

CASE	SLOPE OF LINES* A	B	D	HEIGHT OF SLOPE	UNIT WT. OF SOIL	ANGLE OF INTERNAL FRICTION	COHESION
1	0	1:1	0	100	120pcf	8	400
2	0	1:1	0	100	120pcf	30	0
3	0	1:1	0	100	100pcf	20	900

*See Figure 1 for definitions of lines A, B, and D.

The majority of the investigative work was performed on Case 1, Table I, thus, the results below are primarily applicable to the conditions of Case 1.

The safety factor, determined from Equation (45), was compared with the safety factor as determined from two existing digital computer slope stability programs. The comparison showed that there is only a slight difference between the results from the exact integration method and the numerical integration methods. For the comparisons made, the numerical integration results exhibited a zero to five percent difference from those obtained by exact integration. The majority of the comparisons, however, were much closer, being in the zero to one percent range

(see Table II).

TABLE II

COMPARISON OF THE MINIMUM SAFETY FACTORS AS OBTAINED BY THE EXACT AND NUMERICAL INTEGRATION METHODS

Case No.	Exact Integration Method	Numerical Method #1	Numerical Method #2
1	.438	.445	.434
2	.606	.602	.620*
3	1.159	1.175	1.046

*This factor of safety is not necessarily the minimum.

Results of the differentiated expression, Equation (61), and the basic expression, Equation (45), were compared. This comparison showed that when the differentiated expression was nearly satisfied, the center yielding the minimum safety factor was also defined (Figure 6). Ideally, if a radius other than the radius defining the maximum or minimum safety factor for the fixed center was used, a "residual" should result in the solution of Equation (61). By varying the radius, in increments of 10 ft. to 0.05 ft., an attempt was made to force the residual to zero. The attempt proved unsuccessful even when changes in the radius, R, were as small as 0.05 ft. In all cases the residuals, as well as the differences in residuals, ranged in value from plus to minus several million to only as low as plus to minus several thousand (Figure 7). These results seem to indicate that Equation (61) is extremely sensitive, even to slight

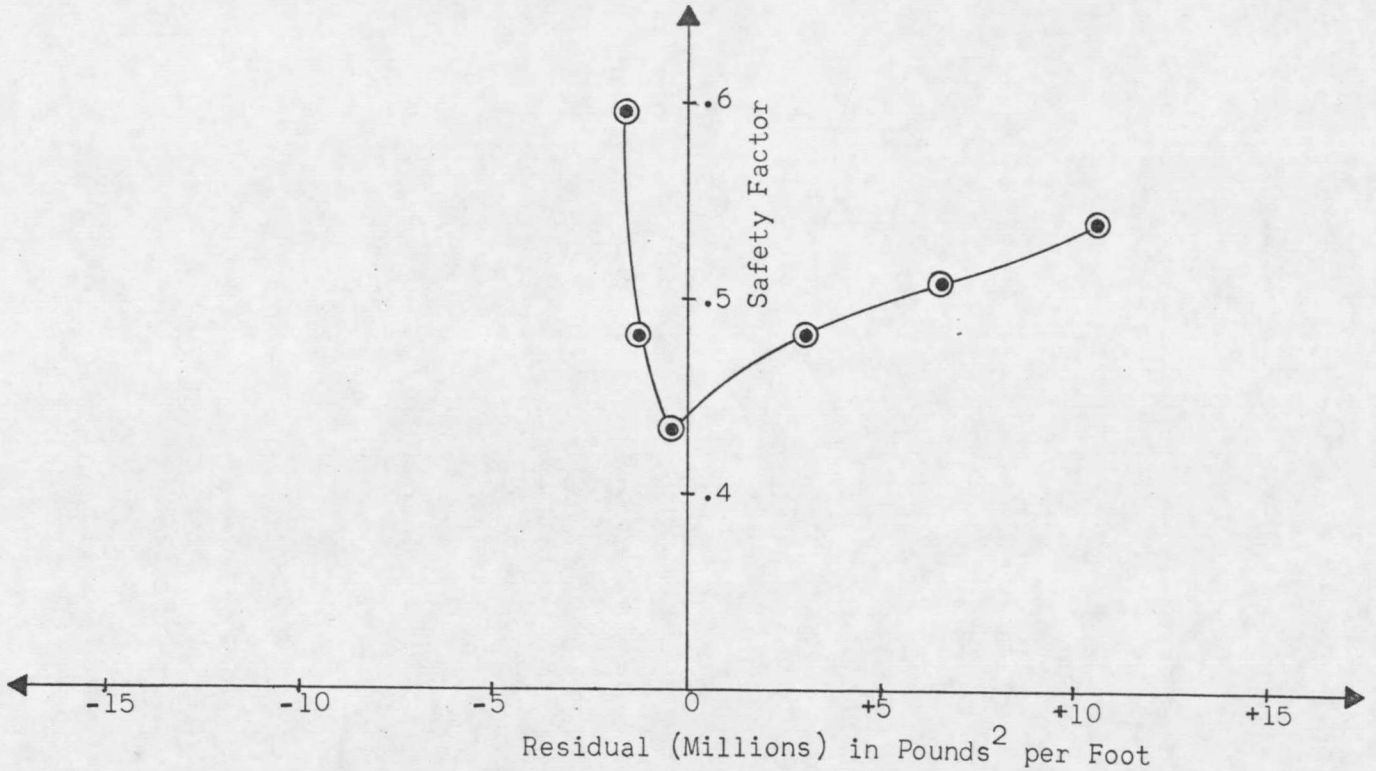


Figure 6. Typical graph of safety factor versus residual of Equation (61). (Note: Relationship shown is for the trial center of Case 1, which produces the minimum safety factor of 0.438.)

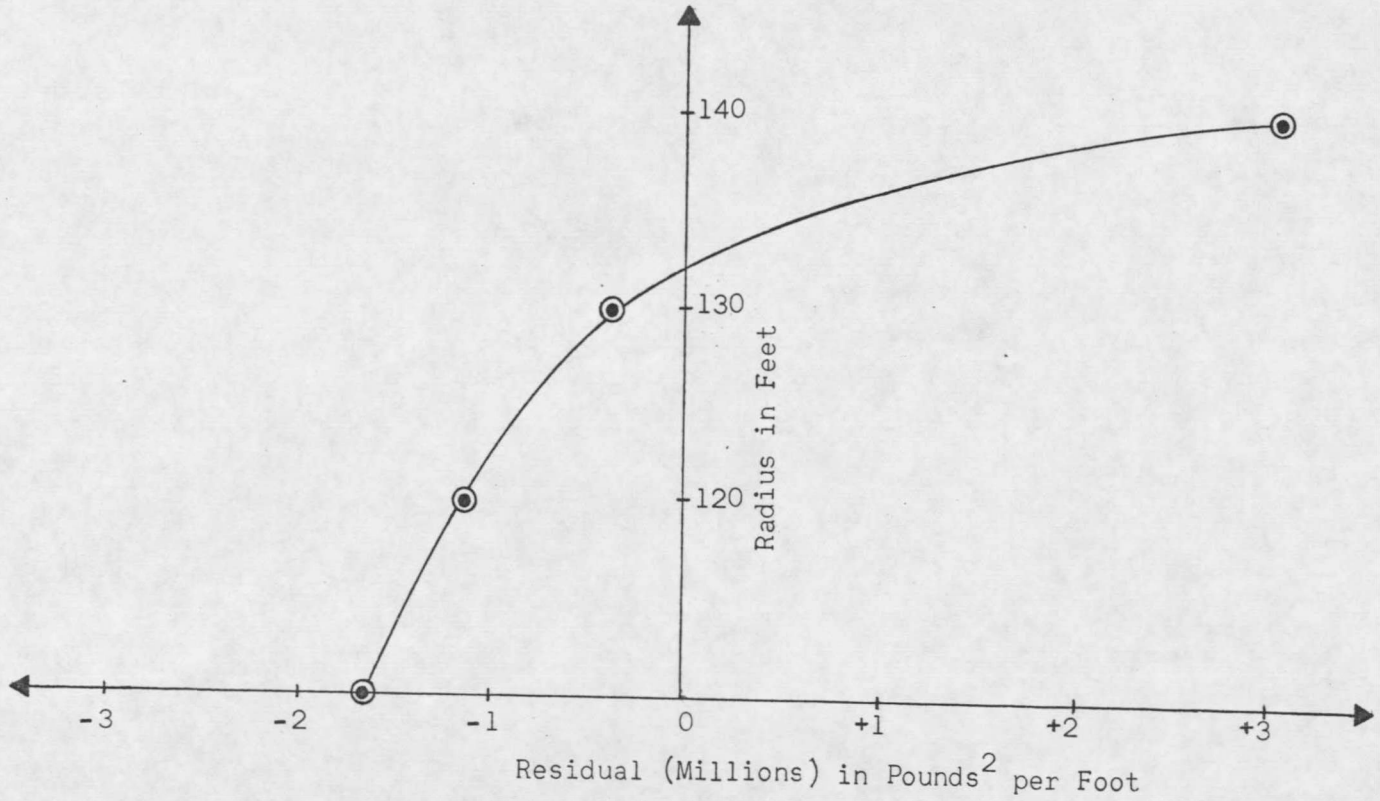


Figure 7. Typical graph of radius versus residual of Equation (61). (Note: Relationship shown is for the trial center of Case 1, which produces the minimum safety factor of 0.438.)

changes in the value of the radius.

It was also noted that the small incremental changes of the radius, although having a very notable effect on the residual, had only a slight effect on the safety factor (the change occurring in the third or fourth digit only), as shown in Figures 7 and 8. It was further noted that the radius that made the residual change from positive to negative, as shown in Figure 6, did not always define the radius associated with the minimum safety factor (and it never defined the radius corresponding to the maximum safety factor). In all cases investigated, this radius defined a safety factor very close to that defined by the existing slope stability computer programs.

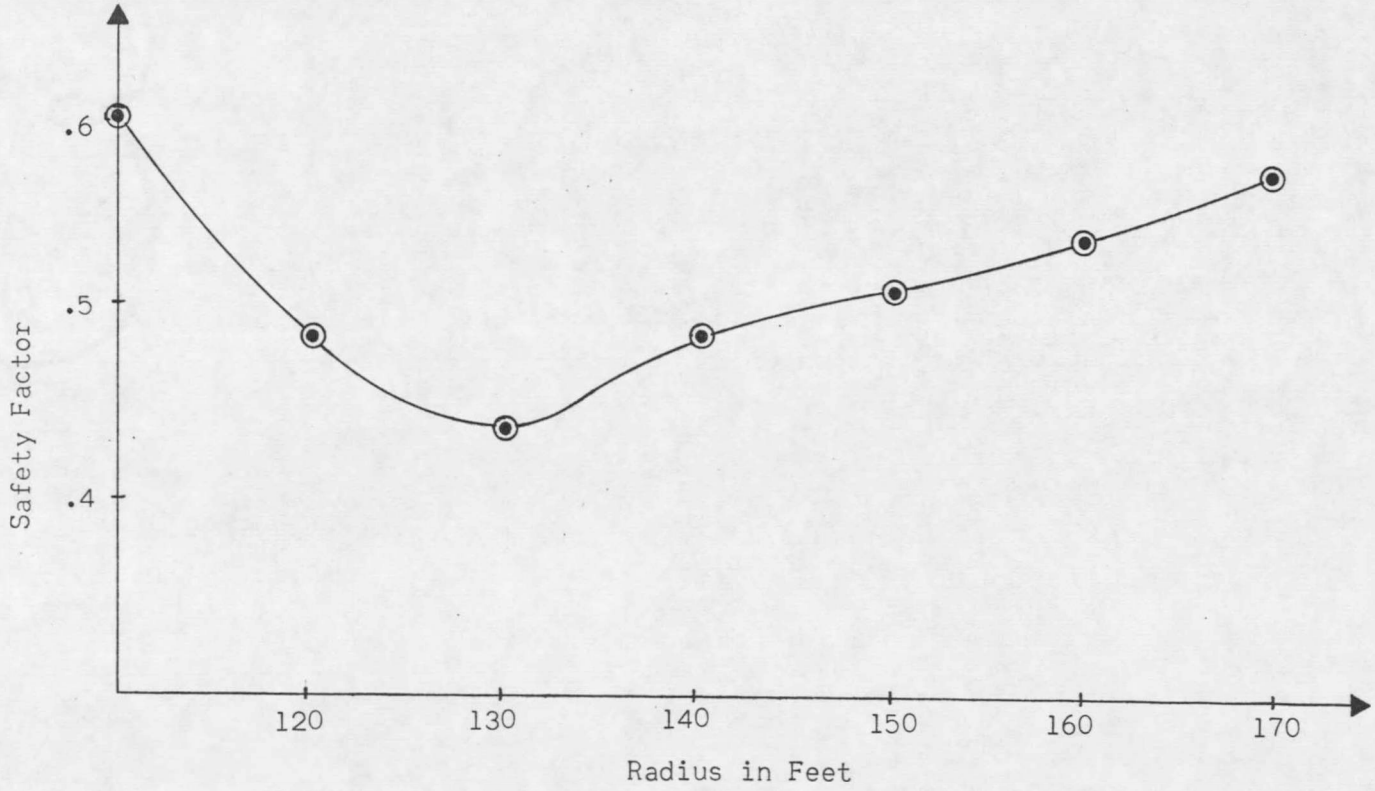


Figure 8. Typical graph of safety factor versus radius. (Note: Relationship shown is for the trial center of Case 1, which produces the minimum safety factor of 0.438.)

CHAPTER IV

CONCLUSIONS AND RECOMMENDATIONS

The attempt to develop a completely analytical solution solving explicitly for either the minimum factor of safety, or the radius associated with the minimum factor of safety, was unsuccessful. The study was, however, successful in establishing correct analytical expressions for the actuating and resisting forces. The results from these expressions compare favorably with the numerical integration methods presently in use.

Recommendations for further research are: 1) to solve Equation (61) explicitly for the radius, R, in terms of the other variables; and 2) to substitute this explicit value of the radius into Equation (45). The resulting expression would then give the minimum factor of safety in terms of the known geometric and soil parameters, and be independent of the radius. It may be that no exact method is available, but probably an approximate trigonometric series substitution would suffice.

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