MONTANA STATE UNIVERSITY

Optimal control of linear distributed-parameter systems
by Robert Clifford Kolb

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Electrical Engineering
Montana State University
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Abstract:
The thesis research is centered on the development of optimal design concepts and techniques for linear distributed-parameter systems with outputs which are functions of both spatial variables and the time variable these systems are denoted as class-II-type distributed-parameter systems. Such systems are characterized by partial differential equations in one or more spatial variables and the time variable. In this thesis, only linear, time-invariant, distributed-parameter systems which are defined on a one-dimensional spatial domain are given detailed attention.

The primary purpose of the thesis project is to develop a general body of theory for effecting the optimal control of class-II-type distributed-parameter systems, and in so doing, to make significant advances in certain individual areas of distributed-parameter control theory.

The content of the thesis is summarized as follows: First, a review of the literature concerning the analysis, synthesis and optimization of distributed-parameter control systems is given which enfolds over 200 references. Second, performance measures are developed and interpreted for both deterministic and stochastic class-II-type, distributed-parameter control systems. The concepts of e-reachability and e-maintainability are introduced in this regard. Third, several forms of block-diagram models are developed for class-II-type distributed-parameter systems by using finite integral transforms in conjunction with the infinite (one-sided) Laplace transform. These models form a basis for the application of optimal design techniques for a wide class of linear distributed-parameter systems. Fourth, a theoretical basis is laid for the statistical analysis of class-II-type distributed-parameter systems which are subject to stationary random inputs. Distributed correlation functions and distributed spectral-density functions are obtained for a general second-order partial differential equation; three basic types of boundary conditions are considered. Fifth, a procedure is developed for minimizing the spatial-averaged-integral-square error of class-II-type distributed-parameter systems which are subject to integral constraints. An optimal open-loop control law is derived by use of the Wiener-Hopf spectrum-factorization technique. A procedure of rational fraction approximation is formulated in order to realize the optimal control law. Sixth, a procedure is developed for minimizing the integral-square error of class-II-type, distributed-parameter, feedback control systems. The Wiener-Hopf spectrum-factorization technique is used to obtain the optimal compensator for the feedback system.

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A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree

of

DOCTOR OF PHILOSOPHY

in

Electrical Engineering

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December, 1965

ACKNOWLEDGMENT

The author wishes to express his appreciation to Professor D. A. Pierre for his untiring guidance and encouragement during the course of his graduate work and thesis research. The financial support of the thesis research afforded by the National Science Foundation Grants GP-1440 and GK-38 and the Engineering Experiment Station at Montana State University is gratefully acknowledged.

The author expresses his appreciation to his wife, Nancy, for her willing assistance with the computer programming and with the typing of the manuscript and for her cooperation and encouragement given during the course of the author's graduate study.

The author is grateful to the Montana State University Computing Center for the use of the computing facilities which allowed the author to carry out the numerical examples given in this thesis.

Finally, the author would express his appreciation of family and friends who encouraged the author to undertake work toward the Ph. D. degree.

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ABSTRACT

The thesis research is centered on the development of optimal design concepts and techniques for linear distributed-parameter systems with outputs which are functions of both spatial variables and the time variable --- these systems are denoted as class-II-type distributed-parameter systems. Such systems are characterized by partial differential equations in one or more spatial variables and the time variable. In this thesis, only linear, time-invariant, distributed-parameter systems which are defined on a one-dimensional spatial domain are given detailed attention.

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The content of the thesis is summarized as follows: First, a review of the literature concerning the analysis, synthesis and optimization of distributed-parameter control systems is given which enfolds over 200 references. Second, performance measures are developed and interpreted for both deterministic and stochastic class-II-type, distributed-parameter control systems. The concepts of ε-reachability and ε-maintainability are introduced in this regard. Third, several forms of block-diagram models are developed for class-II-type distributed-parameter systems by using finite integral transforms in conjunction with the infinite (one-sided) Laplace transform. These models form a basis for the application of optimal design techniques for a wide class of linear distributed-parameter systems. Fourth, a theoretical basis is laid for the statistical analysis of class-II-type distributed-parameter systems which are subject to stationary random inputs. Distributed correlation functions and distributed spectral-density functions are obtained for a general second-order partial differential equation; three basic types of boundary conditions are considered. Fifth, a procedure is developed for minimizing the spatial-averaged-integral-square error of class-II-type distributed-parameter systems which are subject to integral constraints. An optimal open-loop control law is derived by use of the Wiener-Hopf spectrum-factorization technique. A procedure of rational fraction approximation is formulated in order to realize the optimal control law. Sixth, a procedure is developed for minimizing the integral-square error of class-II-type, distributed-parameter, feedback control systems. The Wiener-Hopf spectrum-factorization technique is used to obtain the optimal compensator for the feedback system.

CHAPTER 1

INTRODUCTION

1.1 PURPOSE OF THE THESIS PROJECT

The purpose of the thesis project is fivefold:

1. To review and evaluate in an organized manner the published work on the analysis, synthesis, and optimization of control systems enfolding distributed-parameter elements.

2. To develop and interpret performance measures for use as design criteria for the optimal design of distributed-parameter control systems which have outputs and/or inputs that are distributed in space as well as in time --- these systems are denoted herein as class-II-type distributed-parameter systems.

3. To develop block-diagram models for linear class-II-type distributed-parameter systems to form a basis for the application of optimal design techniques to these systems.

4. To develop a general body of theory, based on distributed correlation functions, for the analysis of stochastic class-II-type, distributed-parameter control systems.

5. To develop optimal design techniques, based on minimization of integral-square error, for class-II-type, distributed-parameter control systems.

1.2 VALUE OF THE THESIS RESEARCH

In reality, all physical systems are distributed-parameter systems; in some cases, however, the energy of a system can be considered to be concentrated at various "points" throughout the system. In this case, the dynamics of the system are adequately characterized, at least over a limited range of operation, by a set of ordinary differential equations. Hence, a lumped-parameter system, which is characterized by a set of ordinary differential equations, is merely a special case of a distributed-parameter system in which the spatial variables are discretized to reduce the partial differential equations to ordinary differential equations.

If the energy of a system is distributed throughout the system, however, a lumped-parameter mathematical model seldom gives a satisfactory characterization of dynamic performance. When this is the case, it is generally necessary to describe the system dynamics by partial differential equations, i.e., to use a distributed-parameter model of the system.

As the control of more physical processes is automated and as more accurate control is required, the availability of a body of control theory which is applicable to distributed-parameter systems is of ever-increasing importance. The following account gives several typical areas wherein distributed-parameter elements appear in control loops.

When a control loop contains transmission lines (electrical, hydraulic, or pneumatic), the control system designer must necessarily concern himself with the distributed-parameter characteristics of these

lines (59, 75, 188, 207, 212, 254, 275). In aircraft and missile control and in many industrial process control systems, transmission lines are often an integral part of control loops.

In industrial process control, many diverse forms of distributed-parameter elements appear in control loops; for example, heat exchangers (95, 219, 250, 255), chemical distillation columns (168, 177), nuclear reactors (201, 286), chemical reactors (225, 283), and continuous strip processes (130, 170).

In electronics, the development of distributed-RC networks has posed some formidable distributed-parameter analysis and design problems (70, 112, 122, 123, 159, 196, 220, 247, 258, 278, 279).

A relatively new area is the application of control theory to the study and simulation of biological systems (134, 173, 176, 177, 239); most biological system models involve distributed parameters or at least the special case of time delay. Also, when a human being is an integral part of a control system, his distributed-parameter characteristics must necessarily be considered in determining the over-all performance of the system (133, 172, 178).

When the control of a distributed-parameter system is automated, it is desirable to control the process in a suitably defined optimal fashion. Thus, the purpose of an optimal design technique is to generate optimal control laws from which the variables of the system are determined with the result that a performance measure is minimized (or maximized in some cases) subject to constraints on these variables.

As evidenced in the literature review presented in Section 2.4, certain theoretical aspects of the optimal control problem for class-II-type distributed-parameter systems have been investigated by a number of authors. The following comments indicate, however, that these authors feel that additional research is necessary. A. G. Butkovskii and A. Ya. Lerner (111) state: that the known works on the theory of optimal control do not give any more or less general method of solution for systems which contain distributed parameters.

Thus far, no easily implemented methods exist for effecting the optimal design of general distributed-parameter systems. P. K. C. Wang and F. Tung (167, 209, 256) recommend: that future investigations concerning distributed-parameter systems should be directed toward establishing control theories for particular classes of systems, and at the same time, developing practical computational procedures for solving the functional equations derived by use of optimization theory.

Also, Pierre (135) states: that a topic worthy of additional research is the development of a general body of theory for effecting the design of optimal distributed-parameter control systems, as based on minimization of integral-square or mean-square error, subject to stated constraints. Therefore, the primary contribution of this thesis research consists of the development of a body of theory which satisfies this need to a large extent.

In order to form a basis for the application of optimal design techniques, several forms of block-diagram models are developed for class-II-type distributed-parameter systems; and because the choice of the per-

formance measure is an important aspect of the optimal design problem, the development and interpretation of performance measures for class-II-type distributed-parameter systems is an integral part of the theory developed in this thesis. By using high-order rational fraction approximations to realize the optimal design, the bulk of the design problem can be performed on a general-purpose digital computer, with the result that design time and effort are reduced considerably.

1.3 SCOPE OF THE THESIS

Chapter 2 comprises a concise review of literature relative to the analysis, synthesis, and optimization of distributed-parameter control systems. This review enfolds over 200 papers, reports and books which are ordered topically to enable easy location of particular aspects of interest.

Chapter 3 embodies a development and interpretation of performance measures for class-II-type, distributed-parameter control systems. Performance measures are formulated for both deterministic and stochastic class-II-type systems.

In Chapter 4, a general body of theory is advanced for obtaining block-diagram models of linear class-II-type distributed-parameter systems. Several forms of models are considered, and in each case emphasis is placed upon simplicity and upon the ease with which the model lends itself to the application of optimal design techniques. Chapter 4 is complemented by Appendix A. Appendix A contains a brief review of the theory of the finite Laplace and finite Fourier transforms. In

Chapter 4, these finite integral transforms are used in conjunction with the infinite (one-sided) Laplace transform to derive Laplace-transform transfer functions for the block-diagram models.

Chapter 5 comprises a theory for the statistical analysis of linear class-II-type, distributed-parameter control systems. The theory of the distributed correlation function is developed for the analysis of distributed-parameter systems which are excited by stationary random inputs and/or are subject to stationary random disturbances.

Chapter 6 is devoted to developing the use of the Wiener-Hopf spectrum-factorization technique for the design of linear class-II-type, distributed-parameter control systems. The spatial-averaged-integral-square-error metric (introduced in Chapter 3) is used to obtain open-loop control laws for class-II-type distributed-parameter systems.

In Chapter 7, a technique is advanced for the minimum-integral-square-error design of class-II-type, distributed-parameter feedback control systems. A feedback signal is formed by using values of the output measured at equally spaced points along the spatial domain of the system. The Wiener-Hopf spectrum-factorization technique is used to obtain the optimal lumped-parameter compensator for the feedback system.

Chapters 6 and 7 are complemented by Appendices B, C, and D. In Appendix B a necessary condition for optimality is derived. Appendix C contains a brief account of the rational fraction approximations used to realize the optimal designs developed in Chapters 6 and 7. And Appendix D contains a digital computer program for performing multiplications, additions, and scalar multiplications of polynomials.

Finally, the concluding Chapter 8 provides a summary of the essential values of the thesis research and a statement of some aspects of distributed-parameter control theory which merit additional research.

CHAPTER 2

REVIEW AND EVALUATION OF OPTIMAL DESIGN TECHNIQUES FOR DISTRIBUTED-PARAMETER CONTROL SYSTEMS

2.1 DEFINITIONS AND PROPERTIES OF DISTRIBUTED-PARAMETER SYSTEMS

The purpose of this chapter is to present a systematic review and evaluation of the literature pertaining to distributed-parameter control systems. Before a meaningful literature review can be presented, however, some pertinent definitions and properties of distributed-parameter systems must necessarily be formulated. With this purpose in mind, a distributed-parameter element is defined herein as an element which can be characterized by one or more partial differential equations in two or more independent variables (usually time and one or more space variables). Furthermore, a distributed-parameter system is defined as one having one or more distributed-parameter elements in addition, possibly, to lumped-parameter elements. Because distributed-parameter systems contain distributed-parameter elements, the dynamics of such systems are usually characterized by partial differential equations. In some cases, it is convenient to reduce the partial differential equations to integral equations before proceeding with the analysis; and in special cases, differential-difference equations and other equation forms may be useful. In general, distributed-parameter systems can be nonlinear, time varying, and/or stochastic in nature.

In the control of distributed-parameter systems, interest is usually centered on certain distinct "input" or "output" quantities; it is convenient, therefore, both conceptually and analytically, to classify systems according to the nature of these quantities. Such a classification is formulated in reference 222, wherein distributed-parameter systems are divided into two major classes: Class I, those distributed-

parameter systems which ostensibly contain no space-distributed inputs or outputs; and Class II, those distributed-parameter systems which contain space-distributed inputs or space-distributed outputs or both. A simple example will serve to illustrate this method of classification. Consider a control system containing a uniform transmission line; if the time relationship between the input voltage and the output voltage at the corresponding terminals of the line is of interest, the system is a class-I-type distributed-parameter system: e.g., pure time delay for a distortionless transmission line. In this case, the actual distributed nature of the problem is somewhat disguised in that time is the only independent variable present; the space variable appears only as a constant, the length of the line. However, if the relationship between the input voltage applied at the input terminals of the line and the potential distribution along the line as a function of both distance and time is of interest, the system is a class-II-type distributed-parameter system.

As indicated above, a system in which the output response is a scaled replica of the input --- delayed in time --- is a special form of a class-I-type distributed-parameter system. This type of behavior is herein designated by the term time-delay behavior. Many other terms describing this behavior are found in the literature --- "dead-time" (33, 34, 36, 77, 98, 146, 169, 189, 237), "time-lag" (3, 4, 10, 90, 92, 94, 151, 182), "transport-delay or -lag" (69, 102, 105, 147, 152), "retarded control or system" (8, 16, 68, 93, 119), "distance-velocity lag" (12), "lagging argument" (141), and "delayed argument" (235).

Quite often, the dynamics of a linear distributed-parameter element can be represented by a single linear partial differential equation of the form

$$\frac{\partial^m v}{\partial t^m} = H(t, x, f, v_x, v_t, \ldots) \tag{2.1}$$

where $v \equiv v(x,t)$ represents a state variable of the system, $f \equiv f(x,t)$ represents a distributed input function, v_x equals $\frac{\partial v(x,t)}{\partial x}$, v_t equals $\frac{\partial v(x,t)}{\partial t}$, and $H(t, x, f, v_x, v_t, \ldots)$ represents a given function of its arguments (the dots denote higher-order partial derivatives with respect to t and x). The left-hand member of equation (2.1) is assumed to be the highest-order derivative with respect to t that appears in (2.1). In order to complete the description of the element, the initial and boundary conditions for the particular element considered must necessarily be specified.

While equation (2.1) gives a description of a general distributed-parameter element, systems which enfold time delay are characterized more directly by ordinary or partial differential-difference equations. Thus, the dynamics of many time-delay systems can be characterized by one or more mixed ordinary differential-difference equations of the form

$$\sum_{n=0}^{k} a_n \frac{d^n v(t-\tau_n)}{dt^n} = f(t) \tag{2.2}$$

where τ_n is a constant, f(t) is an input function, and v(t) is a state variable of the system.

In the remaining sections of this chapter, the literature on distributed-parameter control systems is reviewed and evaluated. In view of the varied nature of this literature, it is convenient to divide this review into three major sections, as follows: in Section 2.2, the works pertaining to the analysis and synthesis of distributed-parameter control systems, including time-delay systems; in Section 2.3, the works pertaining to the optimal design of class-I-type, distributed-parameter control systems; and in Section 2.4, the works pertaining to the optimal design of class-II-type, distributed-parameter control systems.

2.2 METHODS OF ANALYSIS AND SYNTHESIS OF DISTRIBUTED-PARAMETER SYSTEMS

The material in this section is presented in two parts: first, the literature devoted to the analysis and synthesis of control systems enfolding time delay, hereafter designated as CSETD; and second, the literature devoted to the analysis and synthesis of more general types of distributed-parameter control systems. In this way, the methods which have been proposed for the two types of systems are easily distinguished from one another.

The literature on distributed-parameter control systems published prior to 1962 is well epitomized by Pierre (135). The all-encompassing review presented by Pierre includes over four hundred references related to the analysis, synthesis and optimization of distributed-parameter systems. Proir to the review presented by Pierre, Weiss (73) in 1959

and Choksy (92) in 1960 gave extensive bibliographies on time-delay systems. Because these extensive bibliographies on CSETD are already available, only a representative sample of the literature concerning CSETD is considered here. Major emphasis is placed on the work which has appeared since 1962.

2.2.1 Control Systems Enfolding Time Delay

As indicated in Section 2.1, the dynamics of many time-delay systems can be described by ordinary differential-difference equations of the form given by equation (2.2). The study of equations of this type was begun in the eighteenth century and since that time, hundreds of papers have appeared on this subject (see, for example, the extensive bibliography in reference 44). The classical methods of solution of ordinary differential-difference equations are presented in the recent books by Bellman and Cooke (153) and by Pinney (44).

In addition to the classical methods of solution, the operational calculus has been applied to obtain the solution of differential-difference equations. In 1934, Neufeld (2) developed the use of Heaviside's operational calculus for mixed differential-difference equations; and in 1936, Hartree, et al. (3, 4) developed an analysis technique for CSETD using Heaviside's operational calculus. Heins (5) in 1940 formalized the use of the Laplace transform for the solution of differential-difference equations. The solution of linear differential-difference equations by Laplace transform methods or by Heaviside's operational calculus leads to equations involving nonrational terms of

the form e^{-sT} and e^{-pT}, respectively. Because of this, many of the conventional methods of finding the inverse Laplace transform are not applicable or are at least more difficult to apply.

Because the nonrational term e^{-sT} appears when Laplace transform methods are applied to CSETD, many novel techniques have been developed for obtaining the transient response of CSETD. Pipes (8) and later Elgerd (113) obtained the time response c(t) for a single-loop CSETD by use of the following form of inversion integral:

$$c(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{A(s)\, e^{st}}{1 + B(s)\, e^{-sT}}\, ds \qquad (2.3)$$

where A(s) and B(s) are proper rational fraction functions of s. The procedure is to expand the denominator of the integrand in a power series in e^{-sT} and then to invert the series term-by-term using only a finite number of terms. Although this method is straightforward for simple systems, it is quite difficult to apply to more complex systems.

Many of the standard methods of frequency domain analysis, which are available for lumped-parameter systems, can be applied to time-delay systems. Thus, analytic methods for obtaining transient response from frequency response, as typified by the work of Solodovnik (46), Levadi (74), and Levy (72), are directly applicable. In addition, Blokh (85) developed a method for estimating the overshoot and the control time of CSETD from frequency-response data.

Because CSETD are characterized, in general, by characteristic

equations which have an infinite number of roots, effecting the proper partial fraction expansion may require the use of Mittag-Leffler's theorem. Pierre (135) and Pierre and Higgins (164) applied Mittag-Leffler's theorem in finding the transient response of CSETD.

The root-locus method of analysis was first applied to single-loop CSETD by Chu (13) and later by Powell (36) and Tyner (237). But because the characteristic equation of CSETD has an infinite number of roots, the root-locus method is much more difficult to apply than in the lumped-parameter case. In order to avoid difficulties, Chu (13) used a dominant root analysis to obtain the approximate response of a system; however, McAvoy (213) showed that dominant root analysis is not valid in many instances.

Smith (61) considered some general properties of systems which have both polynomials and transcendental terms in their transfer functions and presents various transform techniques which are useful in the analysis of such systems. The z-transform was used by Boxer and Thaler (27) to obtain the approximate response of CSETD. When the z-transform method is used, however, the sampling period is restricted to an integral multiple of the time delay. This limitation is overcome by using the modified z-transform, as formulated by Schroeder (26), Siljak (105), and Pai (149). For sampled-data CSETD, Siljak (105) combined the use of the modified z-transform and Mitrovic's algebraic-graphic method (49).

The stability of CSETD is a very important aspect of control system design; and as is indicated in the next few paragraphs, it is a subject which has received considerable attention. Just as for lumped-parameter

systems, the location of the roots of the characteristic equation is important in the determination of the stability of linear CSETD. In order to insure stability, all of the roots of the characteristic equation must lie in the left-half of the s plane, Re $s<0$. Although the characteristic equations of CSETD contain the transcendental term e^{-sT}, various stability criteria which are useful in determining the stability of lumped-parameter systems have been modified to render them applicable to CSETD. Satche's modification of the Nyquist criterion (11) and Chu's work on root-locus techniques (13) are typical examples of the modifications which have been accomplished.

In recent years, many papers have appeared, especially in Soviet literature, in which the stability problem for CSETD has been investigated. The Mikhailov stability criterion, which is similar to the Nyquist criterion, was used by Popov (129) and Sakawa and Sugata (190) to investigate the stability of CSETD. Pyatnikskii (142) determined necessary conditions for the structural stability of linear, single-loop CSETD --- structural stability is that property of a system which insures that the system can be made stable by choosing its parameters, without changing its structure. In another paper, Pyatnikskii (182) determined the upper bound on the degree of stability for a single-loop CSETD; that is, he determined the limit on the degree of stability when some of the parameters of the system are fixed. The stability of a time-varying CSETD has been investigated by Krasovskii and Osipov (192) using Lyapunov's stability theory and the theory of controllability. In addition to the works mentioned above, many other papers have been

published on various aspects of the stability problem for CSETD, among which are references 15, 21, 22, 24, 48, 51, 67, 78, 87, 93, 94, 98, 141, 191, 230, 234, and 235.

Numerous papers are devoted to the presentation of a variety of techniques for approximating and/or simulating time delay. In many cases, a simplified model of the system is used in order to avoid treating the actual time-delay behavior of the system. This procedure is, in general, unsatisfactory unless the time delay is quite small; and even then, it may lead to erroneous results.

An often-used approach is the approximation of the nonrational term e^{-sT} by some form of rational function in s. If a suitable approximation --- one which is valid over the frequency range of interest --- is obtained, the usual methods of Laplace transform analysis for lumped-parameter systems can be applied to the approximate model. One of the first approximations to be used was the truncated Maclaurin series for e^{-sT}; however, this method may lead to erroneous results because the approximation does not converge to the actual function along the Laplace-transform inversion contour.

As compared to the truncated Maclaurin series, the Padé approximations for e^{-sT} (7, 54, 89, 91, 97) yield a more accurate characterization; consequently, they are frequently used in practice (20, 30, 102, 118, 225). Many papers are devoted to finding methods of simulating time delay for analog computer studies of CSETD, as typified by references 19, 35, 62, 102, 117, 121, 152, and 188. And, in yet others, various electrical or electromechanical methods for simulating time delay are

presented (34, 69, 71, 137 138, 146, 156, 183, 187).

Some of the methods of nonlinear analysis have been modified for use in the analysis of both linear and nonlinear CSETD. For example, Cunningham (42), Kuwahara, et al. (189), and Strakhov (234) have applied phase-plane methods of analysis to CSETD. A sampled-data CSETD was analyzed by Strakhov (234) using a multisheet phase plane, the first sheet of which is valid for $0 \leq t \leq T$, the second for $T \leq t \leq 2T$, etc. Uhlrich (216) developed several finite difference techniques for the analysis of nonlinear sampled-data CSETD. The analysis of linear and nonlinear CSETD by describing function techniques (157) was effected by Kislyakov (88) and Sakawa (190), and the steady-state behavior of CSETD was also investigated by Kislyakov (83, 140) using the Krylov-Bogolybov method (157). Some additional references pertaining to the nonlinear analysis of CSETD are 86, 119, 120, and 136.

The synthesis of controllers for CSETD and/or the introduction of a time-delay element into the controller of a system has been investigated by a number of authors (16, 18, 23, 33, 37, 38, 44, 45, 52, 77, 82, 106). A novel method of block diagram manipulation was developed by Smith (33, 38, 77) for synthesizing a controller for a CSETD: minor feedback loops around the controller are used which appropriately include an analog of the time-delay and a model of the plant. In a somewhat different approach, Cohen and Coon (16) studied the effects of various parameters of proportional, proportional-plus-rate, proportional-plus-integral, and proportional-plus-integral-plus-rate controllers for CSETD; and Pinney (44) presented a systematic procedure for setting system parameters to

obtain maximum damping in a particular system.

The use of "state-variable" techniques for CSETD was discussed by Kalman and Bertram (50); they state that their technique is applicable to systems with time delay if the state of the system can be predicted T seconds in advance, where T is the time delay constant. In a more recent paper, Koepcke (285) developed a synthesis procedure, using state variable techniques, by reducing the differential-difference equations to an equivalent infinite dimensional difference equation which can be solved using a general-purpose digital computer.

2.2.2 Distributed-Parameter Systems --- Other Than Time-Delay Systems

In the remaining portion of this section, a review of the literature pertaining to the analysis and synthesis of distributed-parameter systems of a more general nature is presented. As indicated in Section 2.1, the dynamics of these systems are, in general, described by partial differential equations of the form given by equation (2.1). The classical approach to partial differential equations is given in several recent books (56, 126, 202). However, methods of solving partial differential equations are not as well developed as those which exist for ordinary differential equations; this is one of the major difficulties encountered in the analysis and synthesis of distributed-parameter systems.

In a somewhat more applied approach, Fisher (227) derived the equations which describe the vibrations occuring in flexible airframes and discussed the classical mathematical techniques which are applicable to the solution of such problems. In a similar manner, Murray (217)

investigated two methods for determining the eigenvalues of linear distributed-parameter systems. For electrical engineers, the wave and diffusion analogies given by Moore (81, 207) are helpful, both conceptually and analytically, because they relate several types of diffusive processes to the commonly-known electrical transmission line.

In addition to the classical solution of partial differential equations, the numerical solution of these equations has received considerable attention since the advent of high-speed digital computers which make such solutions feasible (see the extensive bibliography in reference 43). Several recent books (29, 43, 79, 202) contain numerical techniques which are applicable to the solution of partial differential equations. These computer-oriented techniques are quite useful in determining the time response of open-loop distributed-parameter systems.

In addition, several integral transforms can be used to obtain the solution of partial differential equations, as demonstrated in references 31, 40, 58, and 154. Although Laplace-transform methods yield transcendental equations in s (the Laplace variable) when applied to typical partial differential equations, they have been applied with varying degrees of success by several authors (154, 249, 259, 262). The solution of boundary-value problems by Laplace transform methods was demonstrated by Speigel (262) and Smyth (249). However, Smyth's total transform method --- which entails taking the Laplace transform with respect to both t and x, among other things --- is restricted to the solution of homogeneous partial-differential equations with boundary conditions specified for specific values of only one of the independent variables.

In a recent paper, Higgins and Oesterlei (221) presented the general theory of the finite Laplace transform (see also Appendix A). This transform technique can be applied to either ordinary or partial linear differential equations which are defined on finite domains. In Chapter 4, the utility of this method is demonstrated for linear distributed-parameter systems defined on a finite spatial domain.

Because the Laplace-transform transfer functions of distributed-parameter systems contain transcendental functions of s, the determination of the stability of such systems in the s-domain requires the use of some novel techniques. For example, Pierre (135), Pierre and Higgins (260), and Ball (132, 223) used a Riemann surface to represent the two-valued, distributed-lag transfer function $\exp[-(sD)^{1/2}]$. Using this Riemann surface representation, Pierre (253) developed an electric potential analog for obtaining frequency-response data. On the other hand, Ball used the transformation $w=s^2$ to map the s-domain Riemann surface for $\exp[-(sD)^{1/2}]$ into a w plane, wherein the root-locus technique can be applied to determine the stability of the system. In addition to the above work, the root-locus method has been applied to distributed-lag systems by Chu (13) and Rekoff (145) and to other forms of distributed-parameter systems by Stone (59) and Radant (25). However, Chu and Rekoff did not use a Riemann surface to represent the two-valued, distributed-lag transfer function.

The Nyquist criterion was applied to some simple distributed-parameter systems by Radant (25) and Leggett (17). Although Radant and Leggett gave no justification for its use, Desoer (284) recently

examined the Nyquist criterion in a more rigorous manner and justified its use for linear, time-invariant distributed-parameter systems.

Using the Nyquist and related criteria, Kadymov (64) considered the stability of a system characterized by equations of the form $(1 - e^{-kv})/W(s) = 0$ where $v = (cs^2 + bs + a)^{1/2}$. In yet another paper, Papoulis (104) formulated several stability theorems for some single-loop distributed-parameter control systems. And, the stability of some nonlinear control systems with distributed parameters was investigated by Geleg (274).

Brin (143) developed a graphical and an analytical criterion --- similar to the Mikhailov (Nyquist) and Hurwitz criterion, respectively --- for transfer functions of the form $F(\sqrt{s}) = A(\sqrt{s})/B(\sqrt{s})$, where $A(\sqrt{s})$ and $B(\sqrt{s})$ are polynomials in $\sqrt{s}$. In a recent paper, Kolomeitseva and Netushil (273) have investigated the stability and transient response of systems of a similar form.

In several recent papers, Wang (195, 197, 209, 246) and Friedly (163) developed the use of Lyapunov's stability theory for certain classes of distributed-parameter systems. Using Lyapunov's direct method, Wang (209) formulated several stability theorems for linear distributed-parameter systems; and in references 195, 197, and 246, Wang considered the stability of a time-delayed-diffusive system which is described by partial differential-difference equations. For systems described by linear partial differential equations with linear boundary conditions, Friedly (163) developed a set of transformations that reduces the equations to the canonic form of Lurė (157); hence, the simplified criterion of Lurė can

be applied.

To date, most of the analysis and synthesis of distributed-parameter systems has been accomplished by using some form of approximate model of the system. The approximations which have been used are varied in nature, but one common approach is the discretizing or "lumping" of the spatial variables which reduces the partial differential equations to ordinary differential equations (see, for example, 109, 161, 198, 215, or 290). A similar approach requires lumping both the space and time variables to reduce the partial differential equations to difference equations (130, 170). The difficulty with each of the above approaches is that as yet there is no general method for determining the error introduced by the lumping process; however, the errors introduced in specific examples or in problems of a specific class have been investigated. For example, Landon (200) derived bounds on the steady-state error due to lumping the space variable in the special class of problems characterized by the heat-conduction equation defined on an infinite or semi-infinite, one-dimensional spatial domain.

An alternate approach for linear systems is to approximate in the s-domain; that is, to approximate the nonrational functions of the Laplace variable s by a rational fraction function of s. If rational fraction approximations are available for the particular distributed-parameter characteristic considered, this type of approximation is especially helpful because it allows the use of many of the standard Laplace-transform methods of analysis that have been developed for lumped-parameter systems.

Rational fraction approximations have been developed for some common forms of nonrational distributed-parameter characteristics: $\exp[-(sD)^{1/2}]$ in (135, 257); $(s)^{1/2}$ in (60, 103, 210, 248, 277); and [cosh Ts + B sinh Ts] in (254). However, the infinite product approximation for [cosh Ts + B sinh Ts], when B is a function of s, is difficult to obtain because it requires finding the roots of [cosh Ts + B(s) sinh Ts] = 0.

These rational fraction approximations are especially useful for the analog computer simulation of nonrational Laplace-transform transfer functions (see, for example, 103 and 248). In Chapters 6 and 7, the rational fraction approximations for $\exp[-(sD)^{1/2}]$ and $(s)^{1/2}$ --- developed by Pierre (135, 257) and Kopal (60), respectively --- are used to demonstrate the optimal design techniques developed herein for class-II-type distributed-parameter systems.

In an attempt to determine the error introduced when lumped-parameter models are used to describe the distributed characteristics of the actual system, Ball (250) plotted dimensionless-parameter Bode diagrams of the ratio of the approximate distribution to the "exact" distribution as a function of normalized frequency (the "exact" distribution corresponds to the solution to the partial differential equation). For the two process-control systems considered, both the magnitude and the phase of the error introduced by several forms of lumped-parameter models were determined as a function of the normalized frequency.

Yet other authors (55, 174, 265, 270) have developed methods whereby simulation can be accomplished using an analog computer. Hong and Larson (174) developed a method for simulating a distributed-

parameter system which can be characterized by a first-order partial differential equation defined on a finite spatial domain. And for distributed-parameter systems defined on a finite, one-dimensional, spatial domain, Meredith and Freeman (55) presented an analog computer simulation technique which consists of finding a solution that is valid for $0 \leq x \leq L$, in the time increment $0 \leq t \leq T$. The procedure is then iterated, with the now-known new initial conditions, to find the solution for the time increments $T \leq t \leq 2T$, $2T \leq t \leq 3T$, ..., until the desired solution is generated.

Interest in the availability of analysis techniques for distributed-parameter systems increased considerably after the development of distributed-RC networks. In 1959, Hager (70) published one of the first papers dealing with the analysis of distributed-RC networks; and since then, many papers have appeared in which this subject is treated with varying degrees of rigor, as typified by the following: 112, 122, 123 159, 196, 214, 220, 247, 258, 263, 278, and 279. Because the four-terminal parameters of distributed-RC networks are usually transcendental functions in the Laplace variable s, the exact model is difficult to treat analytically. Therefore, the majority of the papers which have appeared thus far contain low-order lumped-parameter approximations for the actual distributed network; and the more or less standard frequency response techniques, such as Bode diagrams, Nyquist diagrams, etc., are employed (70, 112, 122, 123, 196, 247, 263, 278).

Similar approximate methods of analysis have been developed for process control systems: Lamb and Simpkins (171) investigated the

frequency response of some practical process-control systems using dimensionless-parameter Bode diagrams; and Thal-Larsen and Loscutoff (255) and Schuder and Binder (75) developed some approximate methods for finding the transient response of particular distributed-parameter systems.

In a somewhat different vein, Murray (218) used an eigenfunction expansion corresponding to the distributed-parameter plant to obtain an approximate model; subsequently, this approximate model was implemented in a closed-loop configuration, thereby allowing experimental verification of his mathematical analysis.

As an aid in the design and analysis of solenoid and transformer cores, Cooper (283) obtained a formal solution for the transient response of a rectangular region subject to Joulean heating.

For class-I-type distributed-parameter systems of a general nature, Pierre (135) and Pierre and Higgins (252) developed a sampled-data representative system in which the distributed-parameter elements of the control system are represented by an approximately equivalent sampled-data model. After the sampled-data representative system of the distributed-parameter element is formed, the analysis and/or synthesis of the resulting sampled-data system can be effected by use of the available techniques in sampled-data theory (57, 158).

For stochastic distributed-parameter systems, Wang and Tung (199) and Senin (233, 238, 271) developed methods of analysis. Wang and Tung (199) analyzed a distributed-parameter system that can be characterized by a first-order, linear partial differential equation which has either a random coefficient or a random input. In reference 238, Senin

considered the passage of stationary random signals through linear distributed-parameter systems. And in two subsequent papers, Senin outlined the use of the statistical techniques, developed in his first paper (238), for the extrapolation of a random, distributed signal (233) and for the statistical design of class-I-type distributed-parameter systems (271). Although Senin's papers are extremely brief, the techniques appear to have considerable merit for the analysis and/or design of stochastic, distributed-parameter systems; therefore, in Chapter 5 of this thesis, his work is extended and clarified to render it more readily applicable to actual systems design and/or analysis.

The synthesis of a compensator for linear class-I-type, distributed-parameter systems was considered by Pierre (135) and Pierre and Higgins (282); a novel method of time-domain synthesis was developed which incorporates the use of a sampled-data representative system for the distributed-parameter elements (135, 252). Yet other methods of time domain synthesis (150, 194, 240) have been developed which might possibly be extended to render them applicable to synthesis of distributed-parameter systems.

Using frequency domain transformations, Wyndrum (220) and O'Shea (280) developed synthesis techniques for certain types of distributed-RC networks. Wyndrum (220) used the hyperbolic transform $w = \tanh[a(s)^{1/2}/2]$ in order to develop a synthesis procedure in the w plane; the procedure may be used to synthesize a prescribed immitance magnitude function with distributed-RC networks. In a similar manner, O'Shea (280) developed a transfer-function synthesis procedure in the $p = \cosh[(sRC)^{1/2}]$ plane.

2.3 OPTIMAL DESIGN TECHNIQUES APPLICABLE TO CLASS-I-TYPE, DISTRIBUTED-PARAMETER CONTROL SYSTEMS

In this section, the optimal design techniques which have been applied to class-I-type distributed-parameter systems are summarized; herein, in general, the phrase "optimal design technique" indicates that the application of the design technique results in a minimum --- or, in some cases, a maximum --- of a given performance measure, i.e., a mathematical measure of system performance. Many novel techniques have been applied to effect the optimal design of class-I-type distributed-parameter systems; however, the majority of these techniques are applicable only to CSETD, and the literature devoted to the optimization of more general forms of class-I-type distributed-parameter systems is quite limited.

In one of the first papers on the optimization of CSETD, Oldenbourg (12) minimized the integral of the nonoscillatory error of a system controlled by a "step-by-step" controller by determining the optimal location of the roots of the characteristic equation. However, in this method, the length of the control interval is limited to being greater than, or equal to, the time delay of the system if the error is to remain nonoscillatory.

The use of nonlinear switching controllers for CSETD has been investigated by several authors (65, 68, 236). In reference 68, Mitsumskii indicated that self-oscillations will occur in certain systems if the time-delay value shifts from a specified design value. Mitsumakii suggested the use of a linear controller in the region of small error for such systems.

Several authors have developed systematic methods for adjusting system parameters to obtain the optimal performance of CSETD, as typified by references 96, 224, 237, 266, and 276. In their study of high-order CSETD. Freeman and Owen (96) effected the minimization of a time-weighted, integral-square error through automatic adjustment of four variable loop-parameters by a steepest descent searching process. Further, Law and Weaver (224) developed a numerical technique for obtaining the optimal parameter settings for CSETD; however, only systems with one adjustable parameter were considered. The performance measure used by Law and Weaver is the weighted sum of the integral-square error, the integral of the error, the maximum absolute value of the error, and the time duration of the process. Each term in the performance measure has an associated cost factor, or weighting factor, which can be chosen to fit the particular problem considered. In a somewhat different manner, Tyner (237) presented a graphical technique for finding the gain required to achieve any desired degree of damping; and Haalman (276) used an analog computer to find the controller-parameter settings which minimize the mean-square error of CSETD.

The use of complementary feedback, that is, canceling the original plant transfer function and inserting the desired characteristic, was developed by Galoi (169) to obtain an optimal controller for CSETD. Galoi indicated several methods which can be used to realize and to adjust the resulting feedback network; and for systems with a variable time delay, he discussed the use of an adaptive controller.

Newton, et al. (30) demonstrated the use of the Wiener-Hopf spectrum-factorization technique to effect the minimum-integral-square or mean-square-error design of CSETD. In order to effect the factorization, Padé approximations for e^{-sT} (7, 20) are used which then allows the application of the Wiener-Hopf technique in the usual manner. In connection with the design of stochastic CSETD, Itskovich (66) made an interesting comment to the effect that if the correlation function of a disturbing stimulus is such that the quantity entering the delay element is negligibly correlated with the measured and controlled output quantity, no type of control other than elimination of the delay will work effectively.

Recently, Pierre (226, 268) formalized the use of the Wiener-Hopf technique for class-I-type distributed-parameter systems. The methods developed by Pierre were demonstrated for a distributed-parameter system with a transfer function of the form $\exp[-(sD)^{1/2}]$; and the explicit form of the compensator which resulted in the minimum-mean-square error was obtained. In Chapters 6 and 7, the use of the Wiener-Hopf technique is developed for class-II-type distributed-parameter systems.

Using correlation functions, Itskovich (180) investigated the problem of instrumenting a random, distributed field; he considered the automatic determination of the value of a chosen quantity at any point in a distributed field with a prescribed accuracy and with a minimum number of discrete "pickups". In addition, when the number of pickups was limited by structural or technical considerations, Itskovich evaluated the accuracy with which the characteristics of the entire field can be

determined when the distributed field is controlled by the discrete pickups.

A technique for the synthesis of distributed-RC networks was proposed by Rohrer, et al. (281); wherein, they posed the problem of minimizing an integral of the following form:

$$\int_0^\infty \frac{[K\,\hat{v}(t) - v(L,t)]^2}{2}\,dt \tag{2.4}$$

where $\hat{v}(t)$ is the desired output voltage, K is a constant, and $v(L,t)$ is the output of the distributed-RC network at x=L. Using the calculus of variations, functional equations were derived for the distributed resistance R(x) and the distributed capacitance C(x); however, no solutions were given for the partial differential equations obtained.

The application of Bellman's principle of optimality (28, 125) for the optimal design of CSETD has been demonstrated by several authors: Salukvadze (144) and Krasovskii (151, 184) combined the principle of optimality and Lyapunov's method to synthesize an optimal controller for CSETD; Kramer (90) considered a terminal-control problem subject to energy constraints; Chang Jên Wei (139) presented an approximate method for synthesizing an optimal controller; and Ash (201) derived a minimum-time control policy for a nuclear reactor subject to random reactivity fluctuations. In reference 151, Krasovskii showed that the optimal control law depends on the state of the controlled object during the

preceding time interval T (where T is the time-delay constant).

The optimal control of systems having time delay in the phase coordinates but having no delay in the control action has been considered by Haralisvili (110) and by Popov and Khalanay (179); although they used different methods of solution, their results are the same.

Using Pontryagin's maximum principle (128), Takahashi and Thal-Larsen (219) developed a time-optimal control policy for a dual-heat-exchanger system; they applied the maximum principle only to various forms of lumped-parameter models of the system.

Zhivoglyadov (229) formulated an algorithm for the synthesis of both discrete and continuous controllers for CSETD.

In a recent paper, Goodson and Khatri (269) proposed two methods for the minimum-integral-square error design of class-I-type distributed-parameter systems. With the first method, the so-called exact method, the partial differential equations are reduced to integral equations; then the calculus of variations is used to derive a necessary condition for optimality. Although this method is called an "exact" method, it requires the truncation of an infinite series in order to obtain the explicit form of the optimal feedback control system. In the second method, the transcendental Laplace-transform transfer function of the distributed-parameter plant is approximated using the infinite product approximations developed by Oldenburger and Goodson (254); then Pontryagin's maximum principle (128) is applied to obtain the optimal feedback control law for the approximate model. Goodson and Khatri (269) applied each of these methods to a one-dimensional and a two-dimensional

heat transfer problem, producing results found to be in favorable agreement.

2.4 OPTIMAL DESIGN TECHNIQUES APPLICABLE TO CLASS-II-TYPE, DISTRIBUTED-PARAMETER CONTROL SYSTEMS

Early in 1960, A. G. Butkovskii and A. Ya. Lerner (84) in the U.S.S.R. published the first paper which treated the optimal design of class-II-type, distributed-parameter control systems in a rigorous manner. From the time of this pioneering work to the present, A. G. Butkovskii and A. Ya. Lerner (84, 111, 124) and A. G. Butkovskii (107, 108, 109, 131, 175, 181, 185) have published at least nine papers on this subject. All of the papers mentioned above and yet others (165, 231, 232, 272) stress the application of Pontryagin's maximum principle (128) in various forms.

The techniques used by Butkovskii can be grouped into one of two major categories: category 1, techniques which are dependent upon one's ability to reduce the partial differential equations involved to ordinary differential equations; and category 2, techniques which are dependent upon one's ability to reduce the partial differential equations to integral equations. If one of the above-mentioned procedures can be effected, Pontryagin's maximum principle will, in some cases, yield relationships which the control variables must satisfy to obtain an optimal response.

In reference 109, Butkovskii gave two examples in which the distributed nature of the problem is by-passed by discretizing (lumping) the

space variable. As a result of this lumping process, the equations which govern the approximate system are functions of time only and, for the examples treated by Butkovskii, are of a variety to which Pontryagin's maximum principle can be applied.

It must be emphasized, however, that lumping is not a cure-all process; it fails to lead to beneficial results for even some of the most elementary distributed-parameter optimal design problems. See, for example, page 19 of reference 222.

For distributed-parameter systems which can be characterized by integral equations, Butkovskii (107, 108) developed two theorems, based on the maximum principle, for optimal terminal-control problems, that is, for finding a control law which takes a system from an arbitrary initial state to a specified final state in an optimal fashion.

Further, Pontryagin's maximum principle has been applied to other forms of class-II-type distributed-parameter systems by Sirazetdinov (231) and Egorov (232). In an attempt to extend the work of Butkovskii, Sirazetdinov (231) proved several theorems which give the necessary conditions for optimality of a class of distributed-parameter systems characterized by a set of quasi-linear partial differential equations in n variables. And for systems characterized by linear partial differential equations, a locally sufficient condition for the optimum was given by both Sirazetdinov (231) and Egorov (232).

Even when the above formulations are applicable, the maximum principle often leads to equations which are very difficult to solve; and this still presents a formidable barrier to the complete solution of the

optimal control problem.

In the United States, the first major contributors to the theory of optimal control of class-II-type distributed-parameter systems were P. K. C. Wang and F. Tung (167, 256). In these initial works, Wang and Tung presented the basic mathematical framework needed to solve a large class of distributed-parameter control problems. Definitions and general concepts were given in a rigorous mathematical manner, so much so that the aforementioned papers are very abstract: the theory presented is not easily related to practical problems.

In brief, the papers by Wang and Tung (167, 256) contain a general description of the following: the domain of definition, the state variables, the input functions, the mathematical models, controllability, observability, performance measures, and the optimal control problem for distributed-parameter systems.

Using Bellman's principle of optimality (28, 125), Wang and Tung derived a maximum principle which is applicable to a general class of distributed-parameter systems. This maximum principle results, however, in a functional partial differential equation for which there is no convenient method of solution at this time.

On the other hand, Marcus (166) successfully applied dynamic programming techniques in the design of an optimal controller for a particular distributed-parameter system. Before the optimization procedure of Marcus can be applied, however, the partial differential equations characterizing the system must be approximated by a set of difference equations.

In addition, Wang (242) investigated the optimal control problem for class-II-type distributed-parameter systems which contain time delay, that is, systems which are described by partial differential-difference equations. Here again, Wang applied the principle of optimality to derive the functional equations associated with the optimal control problem considered.

In his latest work (209), Wang expanded the material contained in his previous works and, in addition, he included the more important aspects of Butkovskii's results. Because lumped-parameter systems are a special case of distributed-parameter systems, Wang has attempted to extend the basic concepts of lumped-parameter control system theory to include the more general case, distributed-parameter control systems.

Most of the works of Butkovskii (107, 108, 109, 175), Butkovskii and Lerner (84, 111, 124), and Wang and Tung (167, 256) are reviewed in more detail in reference 222.

Recently, McCausland (161, 198, 290) developed three approximate methods for synthesizing bang-bang controllers to obtain minimum-time performance for a linear diffusion process whose control inputs, applied at the spatial boundaries, are subject to absolute-value constraints. The form of the optimal control signal for this problem had previously been determined by Butkovskii (108, 109, 111) to be bang-bang, but the functional equations given by Butkovskii could not easily be solved for the switching times. To circumvent this problem, McCausland (161, 198, 290) developed the following: a subdivision, or finite difference, method; a harmonic, or Fourier series, method; and a parabolic method to

solve for the approximate controller switching times. The relative merit of each method is discussed in considerable detail in reference 290.

In addition, McCausland (161, 290) and others (147, 167, 209, 256, 287) investigated the controllability of linear class-II-type, distributed-parameter systems.

In a somewhat different vein, Sakawa (245) formulated an optimal terminal-control problem for a class-II-type distributed-parameter system; he then investigated the solution of this problem using each of the following techniques: a variational technique, a nonlinear programming technique, and a linear programming technique. For the first case, an integral-square-error performance measure was used, and the necessary condition was given as a Fredholm integral equation of the first kind. An approximate solution to this problem was then formulated using nonlinear programming techniques (203, 292); the explicit solution of the problem using the first two techniques was not attempted however. In Sakawa's third approach, the performance measure was taken to be the integral of the absolute value of the error; and for this problem, an approximate solution was obtained by using the simplex method of linear programming (205, 292) --- as formulated for optimal control problems by Zadeh and Whalen (148, 162).

In a recent paper, Axelband (267) formulated the use of function space methods (288,289, 291) to effect the optimal control of a class of distributed-parameter systems. The solution to the optimal control problem posed by Axelband was obtained <u>implicitly</u> using function space methods and <u>explicitly</u> using a steepest descent searching process (203).

CHAPTER 3

DEVELOPMENT AND INTERPRETATION OF PERFORMANCE MEASURES FOR DISTRIBUTED-PARAMETER CONTROL SYSTEMS

3.1 INTRODUCTION

In control theory, the phrase performance measure is used to denote some form of mathematical measure of control system performance[1]. Such mathematical measures of system performance are used in conjunction with various analytical control-system-design techniques; these design techniques are usually classified under the general heading of optimal control theory. The current connotation of the term optimal system, as it appears in control theory, is that the system has been designed by using some analytical technique which minimizes, or in some cases maximizes, some form of performance measure. In recent years, the development of optimal control theory has advanced at a rapid pace, and in addition to the hundreds of papers which have appeared on the subject, several recent books are devoted to the presentation of optimal design techniques (28, 100, 116, 125, 128, 204, 208, 292).

Although there are many forms of optimal control problems, most of them can be grouped into at least one of the following basic forms: the minimum-time control problem, the terminal-control problem, or the minimum-integral control problem. The minimum-time control problem may be described as the problem of determining admissible control signals --- control signals which satisfy specified constraints placed on system performance --- so that the process is taken from a specified initial

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[1]The term "performance measure" is used here, but other terms are also found in the literature: criterion function (206), error criterion (53), performance index (30, 116, 208), payoff function (127), and performance criterion (32, 76, 193, 204, 243).

state to a desired final state in minimum time. The terminal-control problem is the problem of determining admissible control signals such that, in a given time interval T, the system is taken from an initial state to as close as possible to a desired final state; and in the process, a specified function of the state variables, evaluated at the end of the trajectory, is minimized (or maximized). The minimum-integral control problem is the problem of determining admissible control signals which minimize the time integral of a specified function of the state variables, the control variables, and the time.

In optimal control system design, the system which is obtained is only "optimal" with respect to the performance measure used in the design procedure. Therefore, the performance measure must be chosen in such a way that it provides an adequate mathematical measure of the performance desired in the actual system, and at the same time, it should be one that can be treated with the analytical design techniques which have been developed. In the mathematical sense, performance measures are usually metrics because they are a measure of the "distance" between the desired performance and the actual performance of the system. If a metric, a performance measure satisfies the following properties: (1) It is always positive or zero, (2) it is zero if and only if the distance it represents is zero, and (3) it is numeric. Thus, performance measures yield a number or a set of numbers which may be used to evaluate the performance of the system.

As indicated in Section 3.2, a number of authors have considered the development of performance measures for lumped-parameter systems. For

distributed-parameter systems, however, this subject has received little attention. Wang (167, 209, 256) discusses some forms of performance measures for distributed-parameter systems, but his discussion is quite abstract; and in most cases, the explicit form of the performance measure discussed is not given. Thus, in order to fill the above-noted gap in the theory, this chapter is devoted to the development and interpretation of performance measures for class-II-type distributed-parameter systems. Before presenting this development, however, a brief review of some of the performance measures developed for lumped-parameter systems is given in Section 3.2. These performance measures can be applied, in general, to class-I-type distributed-parameter systems; and in some cases, they can be shown to be limiting cases of the more general forms of performance measures developed in Sections 3.3 and 3.4 for class-II-type distributed-parameter systems.

3.2 A REVIEW OF THE PERFORMANCE MEASURES DEVELOPED FOR LUMPED-PARAMETER, CONTROL SYSTEM DESIGN

Performance measures for optimal control systems design have been studied in detail by Schultz and Rideout (32, 39, 47, 114) and yet other authors have formulated various forms of performance measures --- as typified by the following references: 53, 63, 76, 115, 116, 186, 193, 211, 241, 243, and 264. In 1961, Schultz and Rideout (114) presented an excellent discussion of performance measures; wherein, the authors trace the development of control-system performance measures from their origin, in about 1942, to 1961. Because this excellent review of performance

measures for lumped-parameter systems already exists, the material presented in this section is limited in scope; it is given to serve as background material for later sections and because it applies to class-I-type, distributed-parameter control systems.

For lumped-parameter deterministic systems, a general performance measure,

$$J_L = \int_0^T F[e(t),t]\, dt \tag{3.1}$$

encompasses many of the often-used forms of performance measures. In equation (3.1), e(t) is the system error, t is time, F is a specified function of its arguments, and [0,T] is the time interval of interest. The system error e(t) can be defined in a number of ways, but one common method is illustrated in Figure 3-1: in which, r(t) is the input; G(s) is the system transfer function; $G_i(s)$ is the ideal, or desired, system transfer function; c(t) is the system output; $c_d(t)$ is the desired system output; and $e(t) = c_d(t) - c(t)$ is the system error. By specifying the form of F in equation (3.1), many different performance measures can be obtained. For example, F can be chosen to be $e^2(t)$, $|e(t)|$, $t|e(t)|$, $te^2(t)$, etc. Thus, the general performance measure given by equation (3.1) allows the control system designer to pick the form of the integrand F and, in addition, to choose the form of the desired system transfer function $G_i(s)$ for the particular problem considered. The relative advantages and disadvantages of the above-noted and other forms of F are

considered in detail by Schultz and Rideout.

When T of equation (3.1) equals infinity and when any one of the forms of F given in the preceding paragraph is used, the error must satisfy

$$\lim_{t\to\infty} e(t) = 0 \tag{3.2}$$

in order for J_L of equation (3.1) to exist. This means that only systems which have zero steady-state error to the specified input can be considered when $T=\infty$. If the system can be shown to be "reachable" and "maintainable" (160, 244), the requirement given by equation (3.2) is satisfied, thus assuring a finite value of the performance measure (3.1).

To illustrate the use of a particular form of performance measure, the often-used integral-square error is considered below; i.e., let $F = e^2(t)$ and $T=\infty$ in equation (3.1) to obtain

$$J_L = \int_0^\infty e^2(t)\, dt \tag{3.3}$$

A graphical interpretation of this particular performance measure is given in Figure 3-3: where $G_i(s) = 1$, $r(t) = \delta_{-1}(t)$ (the unit step function), and $e(t) = c_d(t) - c(t)$. Illustrations similar to that shown in Figure 3-3 are presented by Schultz and Rideout (114) for several forms of the function F.

An integral-square-error performance measure, similar but not identical to equation (3.3), is used in Chapter 7 as a design criterion for approximate models of class-II-type distributed-parameter systems. A similar design criterion is used by Butkovskii (109, 111). In addition, the integral-square error has been used as a design criterion for class-I-type, distributed-parameter control systems by a number of authors; see references 12, 139, 151, 179, 224, and 269, for example.

For stochastic systems, the mean-square error,

$$\overline{e^2(t)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^2(t)\, dt \tag{3.4}$$

is used extensively as a measure of system performance. Again, a common origin of the error e(t) is illustrated in the block diagram of Figure 3-1. If the process is stationary and ergodic (30), it is convenient to express this performance measure in terms of correlation functions, that is,

$$R_{ee}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e(t)e(t+\tau)\, dt \tag{3.5}$$

where $R_{ee}(\tau)$ is the autocorrelation function of the error e(t). From equation (3.5), the mean-square error is obtained by letting $\tau\to 0$. The use of the minimum-mean-square error criterion for the design of class-I-type distributed-parameter systems is demonstrated in references 30,

226, and 268.

3.3 PERFORMANCE MEASURES FOR DETERMINISTIC, CLASS-II-TYPE, DISTRIBUTED-PARAMETER SYSTEMS

In this section, a number of performance measures are developed for deterministic, class-II-type, distributed-parameter systems. Using an approach similar to that of Schultz and Rideout (114), a general performance measure is proposed; then specific performance measures, which can be obtained from the general form, are discussed.

A class-II-type distributed-parameter system is shown in Figure 3-2. The performance measure to be developed here applies to yet more general systems than that shown, but a concrete example will better serve to illustrate the principles involved. In Figure 3-2, $G_f^t(x,s)$ represents the Laplace-transform transfer function of the distributed-parameter system[2]; and $G_s(s)$ is a known transfer function which relates a constrained signal $v_s(t)$ to the control signal $u(t)$. The distributed-parameter system is assumed to be defined on either a finite spatial domain, $0 \leq x \leq L$, or a semi-infinite spatial domain, $0 \leq x \leq \infty$; however, if a system with a semi-infinite spatial domain is considered, it is assumed that the control of only a finite interval, $0 \leq x \leq L$, is of interest. If a system of finite length is considered, the input signal $u(t)$ is assumed to be applied at both spatial boundaries, $x=0$ and $x=L$.

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[2]The superscript t denotes that $G_f^t(x,s)$ is the Laplace transform with respect to time t of the impulse response $g_f(x,t)$. See Section 4.2.1 for additional clarification of this notation.

On the other hand, for a semi-infinite spatial domain, the input u(t) is assumed to be applied only at x=0.

As shown in Figure 3-2, the system error e(x,t) is defined as the difference between the desired output $v_d(x)$ and the actual output v(x,t). Figure 3-4 gives a graphical representation of the error e(x,t) as it might appear for a distributed-parameter system defined for $0 \leq x \leq L$ and for $0 \leq t \leq t_f$.

The general performance measure proposed here for class-II-type distributed-parameter systems is given by equation (3.6):

$$J_D = \int_0^L F[t_f,x,e(x,t_f)]\, dx + \int_0^L \int_{t_i}^{t_f} H[t,x,e(x,t)]\, dt\, dx \qquad (3.6)$$

where F and H are specified functions of their arguments. In equation (3.6), the first integral represents a terminal-time performance measure, and the second integral represents a performance measure over the entire time interval $(t_f - t_i)$. Although this general performance measure is formulated for systems defined on a one-dimensional spatial domain, it can easily be extended to include systems with multidimensional spatial domains by adding integrals appropriately with respect to the spatial variables.

If H[t,x,e(x,t)] is a positive definite function of e(x,t) and if an infinite-time process is considered, $t_i=0$ and $t_f=\infty$, the error e(x,t) must satisfy the following:

$$\lim_{t\to\infty} e(x,t) = 0 , \qquad 0 \leq x \leq L \tag{3.7}$$

If the condition given in equation (3.7) is not satisfied, the performance measure formulated in equation (3.6) is meaningless for an infinite-time process. In regard to considerations such as this for lumped-parameter systems, LeMay (160, 244) defined the terms reachability and maintainability. In essence, a system is said to have the "reachable" property if it can be driven from its initial state to the desired final state in finite time, and a system is said to have the "maintainable" property if controlling actions are capable of maintaining the system at this desired state throughout the remainder of the process. Because of the nature of distributed-parameter systems, however, more general definitions are necessary; since, in most cases, it is only possible to take the system to within an ε-neighborhood of the desired state in a finite time. Therefore, the terms "ε-reachability" and "ε-maintainability" are defined as follows[3].

A system is said to be ε-reachable for a set of states if admissible control signals exist which will take the system from a given initial state to within an ε-neighborhood of any given desired state from the set of desired states in a finite time $T(\varepsilon)$, where ε is an arbitrary

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[3]The terms "ε-reachability" and "ε-maintainability" evolved from an informal discussion of performance measures --- held at Montana State University in August 1965 --- by Dr. D. A. Pierre, Vichit Lorch-orachoonkul, and the author.

positive constant.

A system is said to be <u>ε-maintainable</u> for a set of states if it is ε-reachable for the set of states and if, in addition, admissible control signals exist which will maintain any of the states in the set within a specified ε-neighborhood for all t greater than $T(\varepsilon)$.

Figures 3-5 and 3-6 illustrate the form of the ε-neighborhood for a lumped-parameter system and a distributed-parameter system, respectively. To further illustrate these definitions, consider a system, characterized by the heat-flow equation, with identical boundary inputs at x=0 and x=L. Obviously, this system is not ε-reachable if the desired state is from the set of states which are not symmetric about x=L/2. On the other hand, if the desired state $v_d(x)$ is from the set of states that are symmetric about x=L/2 and that are piece-wise continuous with a finite number of discontinuities, the system would be ε-reachable but not necessarily ε-maintainable. However, if the desired state $v_d(x)$ is from the set of states that are uniform, $v_d(x) = K$ (K a constant), and if the controlling action u(t) is unrestricted, the system is both ε-reachable and ε-maintainable.

Referring to equation (3.6), several metrics are developed for distributed-parameter systems which require close control during the entire duration of the process, in other words, for the minimum-integral control problem. In the text to follow, the performance measures are formulated for finite-time processes; but if the particular system considered is ε-reachable and ε-maintainable, the performance measures are also applicable to infinite-time processes.

If $F = 0$ and $H = [e(x,t)]^n/L$, the general performance measure of equation (3.6) reduces to

$$J_D = \frac{1}{L} \int_0^L \int_{t_i}^{t_f} [e(x,t)]^n \, dt \, dx \tag{3.8}$$

where n is an even, positive integer. Note that if n were allowed to be odd, the performance measure given by equation (3.8) would not be a metric because it could have a negative value and because it could be zero for nonzero error. For n=2, equation (3.8) is analogous to the familiar integral-square-error performance measure, equation (3.3); therefore, when n=2, the performance measure given by equation (3.8) is herein designated as the spatial-averaged-integral-square error. In Chapter 6, the Wiener-Hopf spectrum-factorization technique is used to effect the minimum spatial-averaged-integral-square error design of class-II-type distributed-parameter systems.

A similar form of performance measure is obtained by letting F=0 and $H = |e(x,t)|$,

$$J_D = \int_0^L \int_{t_i}^{t_f} |e(x,t)| \, dt \, dx \tag{3.9}$$

but this performance measure is difficult to handle analytically.

If it is desirable to weight the errors as a function of their position and/or their time of occurence, performance measures of the

form shown below can be used.

$$J_D = \int_0^L \int_{t_i}^{t_f} a(x)\, b(t)\, [e(x,t)]^n \, dt\, dx \tag{3.10}$$

or

$$J_D = \int_0^L \int_{t_i}^{t_f} a(x)\, b(t)\, |e(x,t)| \, dt\, dx \tag{3.11}$$

where a(x) and b(t) are specified functions of their arguments. For example, if large initial errors can be tolerated but considerable accuracy is required near the end of the process, a(x)=1 and b(t)=c_1t, c_1 equals a constant, can be used to decrease the contribution of the initial errors and to emphasize the "late" errors.

On the other hand, if the state of the system near the boundary x=L is of primary interest, a(x)=c_1+ c_2x and b(t)=1 can be used to emphasize the errors which occur near x=L and, in turn, to decrease the contribution near x=0. A performance measure of this type is applicable to systems which have only one boundary input, applied at x=0: because the output near x=L is at a distance from the input, it may be desirable to weight the errors near x=L more heavily than those near x=0. Yet other forms of weighting functions, a(x) and b(t), could be specified for particular problems considered.

Although the utility of the spatial-averaged-integral-square error is demonstrated in Chapter 6, the performance measures given by equations

(3.9), (3.10), and (3.11), have not, as yet, been used in practice. This may be due in part to the increased computational difficulties introduced by these performance measures when the optimization techniques now available for class-II-type distributed-parameter systems are used.

The minimum-time control problem for class-II-type distributed-parameter systems has been considered in a number of papers (107, 109, 161, 185, 198, 290). Expressed in the form of an integral, the performance measure for the problem is obtained from the general form (3.6) by letting F=0 and $\int_0^L H(\ldots)\, dx = 1$ to obtain

$$J_D = \int_{t_i}^{t_f} dt \tag{3.12}$$

Certain theoretical aspects of the minimum-time problem for systems with bounded input constraints are developed by Butkovskii (107, 109, 185) using Pontryagin's maximum principle; and subsequently, McCausland (161, 198, 290) developed some approximate methods for implementing the optimal controller. In order to obtain a finite value of J_D in equation (3.12), the system must be ε-reachable for the desired state considered. Although the concept of ε-reachability is not used, both McCausland (161, 290) and Egorov (147) investigate the existence of a control signal which will drive the system from an initial state to the desired state in finite time.

Turning now from the minimum-integral control problem to the terminal-control problem, performance measures for this problem are

formulated by specifying the form of F and setting H equal to zero in equation (3.6). Figure 3-4 illustrates that the terminal-control problem requires a metric of the error $e(x,t_f)$. Consider the following metric:

$$J_D = \int_0^L [e(x,t_f)]^2 \, dx \tag{3.13}$$

which is herein called the spatial-integral-square error. Butkovskii (107, 108, 109) and Sakawa (245) used the spatial-integral-square-error criterion in their study of the terminal-control problem for class-II-type distributed-parameter systems. Note that Figure 3-3 gives a graphical illustration of the spatial-integral-square-error performance measure if the coordinate t is replaced by x.

Also, Butkovskii (185) and Sakawa (245) formulated an optimal control problem which required the minimization of a performance measure of the following form:

$$J_D = \int_0^L |e(x,t_f)| \, dx \tag{3.14}$$

which can be obtained from the general form (3.6) by letting $F=|e(x,t_f)|$ and H=0. Using the performance measures given by equations (3.13) and (3.14), Sakawa (245) investigated the terminal-control problem using nonlinear and linear programming techniques. In this study, Sakawa used various numerical techniques to evaluate the integral performance

measures given by equations (3.13) and (3.14) and to render the performance measures into a form that could be treated by the programming techniques.

If the error at particular regions of the spatial domain is of special importance, a spatial-weighted performance measure of the form given in equation (3.15) or (3.16) could be used.

$$J_D = \int_0^L a(x) \, [e(x,t_f)]^2 \, dx \tag{3.15}$$

or

$$J_D = \int_0^L a(x) \, |e(x,t_f)| \, dx \tag{3.16}$$

where a(x) is a specified function of x. One simple form of weighting factor is obtained by setting a(x)=x; thus, the errors which occur are weighted in proportion to their distance from the boundary x=0. When only one input is applied, at x=0, this type of performance measure appears to have considerable merit; however, this criterion adds to the analytical difficulties which are already plentiful in the optimal control problem for distributed-parameter systems.

At this point, it is appropriate to discuss some of the constraints which are applicable to distributed-parameter systems. In order to obtain a physically realizable control system, constraints must necessarily be imposed upon it. These constraints describe, mathematically, the

physical limitations imposed upon control signals and/or the state variables of the system. A control signal which satisfies the constraints imposed upon it is an admissible control signal.

One common form of constraint imposed upon a control signal u(t) is the absolute-value constraint, $|u(t)| \leq M$, where M is a specified constant. The use of constraints of this form is demonstrated in a number of papers (107, 108, 109, 111, 161, 185, 198, 245, 290).

For some optimization techniques, it is necessary, or at least desirable, to express the constraints in integral form --- sometimes called "soft" constraints in the literature (211) in contrast to the "hard" constraint given in the preceding paragraph. When a constraint on a control signal is expressed in integral form, a convenient representation is obtained by specifying the transfer function $G_s(s)$ --- see Figure 3-2 --- which relates the constrained signal to the control signal. For example, if the integral-square value of the derivative of the control signal is to be kept below a specified value, the constraint can be expressed as

$$\int_0^\infty \left[\frac{d\,u(t)}{dt}\right]^2 dt = \int_0^\infty [v_s(t)]^2\, dt \leq M \qquad (3.17)$$

where $V_s(s) = G_s(s)\, U(s)$, and for this particular case, $G_s(s) = s$. Constraints of this form can be added to the performance measures using a Lagrange multiplier or a "cost" factor to form an augmented functional; then, the optimal control signal which minimizes the augmented functional

is determined (See Chapters 6 and 7). Many different forms of integral constraints can be imposed upon the system by specifying the form of the transfer function $G_s(s)$, and in addition, several constraints can be considered by adding more transfer functions $G_{si}(s)$, i=1,2,....

Yet other forms of analytical constraints can be used to characterize physical limitations imposed upon control signals and/or state variables of systems. Just as for performance measures, the selection of the constraints which adequately describe the physical restrictions imposed upon a given system and which can be treated by analytical design methods requires considerable insight on the part of the control-system designer.

3.4 PERFORMANCE MEASURES FOR STOCHASTIC, CLASS-II-TYPE, DISTRIBUTED-PARAMETER SYSTEMS

In this section, a performance measure is formulated for class-II-type distributed-parameter systems which are excited by stationary random inputs and/or are subject to stationary random disturbances. It is assumed that the stochastic inputs and/or disturbances satisfy the ergodic property; as demonstrated in Chapter 5, therefore, the systems can be analyzed using distributed correlation functions. The distributed correlation function concept was introduced in a recent paper by Senin (238). Because Senin's paper suffers from a lack of detail and from a poor translation from the Russian, the ideas presented in his paper are clarified and expanded in Chapter 5.

Referring to Figure 3-7, a typical optimal control problem is formulated with the following data given: (1) the correlation function of the stochastic input or disturbance m(t), (2) the Laplace-transform transfer function of the distributed-parameter plant $G_f^t(x,s)$, (3) a specified transfer function $G_s(s)$ which may be used to place a constraint on the control signal u(t), and (4) a desired stochastic distribution $v_d(x,t)$. The problem is to find a compensator $W_c(s)$ which minimizes a specified measure of the system error $e(x,t) = v_d(x,t) - v(x,t)$ and which satisfies the constraint on the control signal u(t).

The purpose of this section is to develop a suitable performance measure for a problem of the type posed above. This performance measure is formulated in terms of the <u>distributed autocorrelation function</u> of the error, which is defined by

$$R_{ee}(x,y,\tau) \equiv \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e(x,t)\, e(y,t+\tau)\, dt \tag{3.18}$$

where x is the spatial coordinate of the error used in conjunction with the time coordinate t, and y is the spatial coordinate of the error used in conjunction with a shifted time coordinate t+τ --- see Figure 5-1. If x=y and τ=0, equation (3.18) reduces to the <u>distributed-mean-square-error</u>,

$$R_{ee}(x,0) \equiv R_{ee}(x,y,0)\Big|_{y=x} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [e(x,t)]^2\, dt \tag{3.19}$$

Equation (3.19) gives the mean-square error as a function of the spatial coordinate x, and therefore, it is numeric only at specific values of x. Consequently, it is not a suitable performance measure for class-II-type distributed-parameter systems.

To circumvent this difficulty, the distributed-mean-square error given by equation (3.19) is averaged with respect to x to give

$$J_S \equiv \frac{1}{L}\int_0^L R_{ee}(x,0)\,dx = \frac{1}{L}\int_0^L \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[e(x,t)]^2\,dt\,dx \qquad (3.20)$$

which is herein called the spatial-averaged-mean-square error. The spatial-averaged-mean-square error is a metric, and it is analogous to the mean-square error for lumped-parameter systems. Thus, it is an appropriate design criterion for many stochastic, distributed-parameter systems.

For class-I-type distributed-parameter systems, the ordinary mean-square error is an appropriate metric; this metric can be obtained from the distributed-mean-square error by specifying the particular value of x that is of interest.

3.5 REMARKS

In optimal control theory, the choice of the performance measure is an important aspect of the design. Therefore, the development and interpretation of performance measures of class-II-type distributed-parameter systems which is given in this chapter is a definite contri-

bution to the optimal design of these systems.

The review of the performance measures developed for lumped-parameter systems (Section 3.2) serves as background material for the work of Sections 3.3 and 3.4. Also, lumped-parameter performance measures are, in general, applicable to class-I-type distributed-parameter systems.

In Section 3.3, performance measures for deterministic, class-II-type, distributed-parameter systems are discussed; a general performance measure is formulated, and from it, several specific performance measures are developed.

A performance measure for stochastic, class-II-type, distributed-parameter systems is formulated in Section 3.4. This performance measure is posed in terms of the statistical analysis techniques developed in Chapter 5.

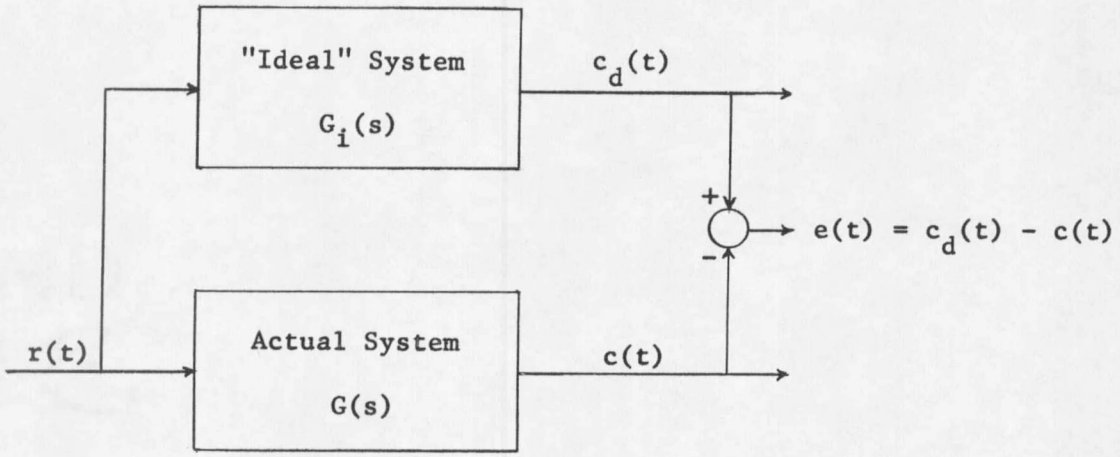


Figure 3-1. Error representation for lumped-parameter systems (Section 3.2).

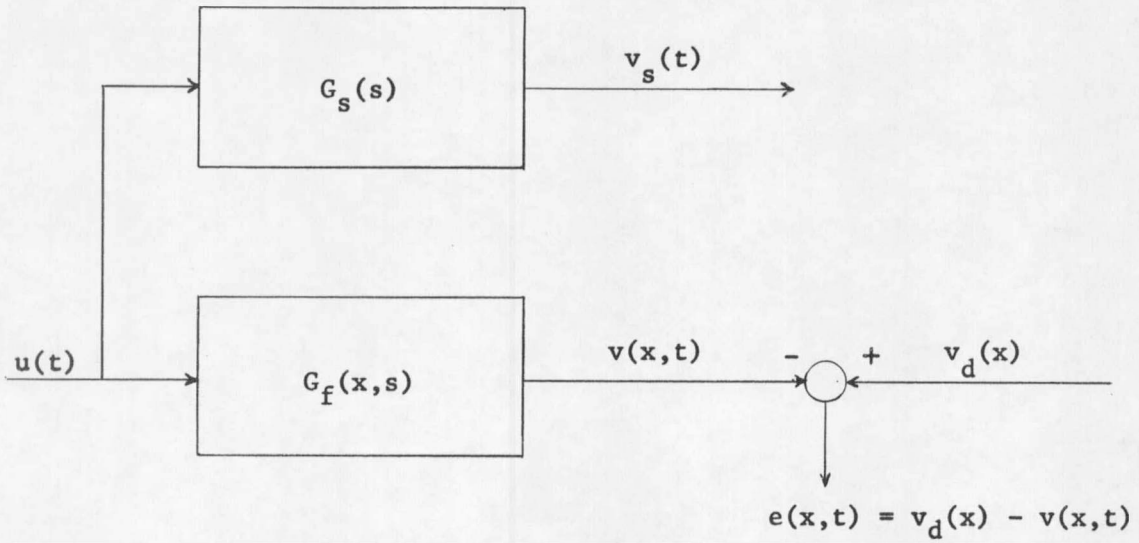


Figure 3-2. Error representation for deterministic class-II-type distributed-parameter systems (Section 3.3).

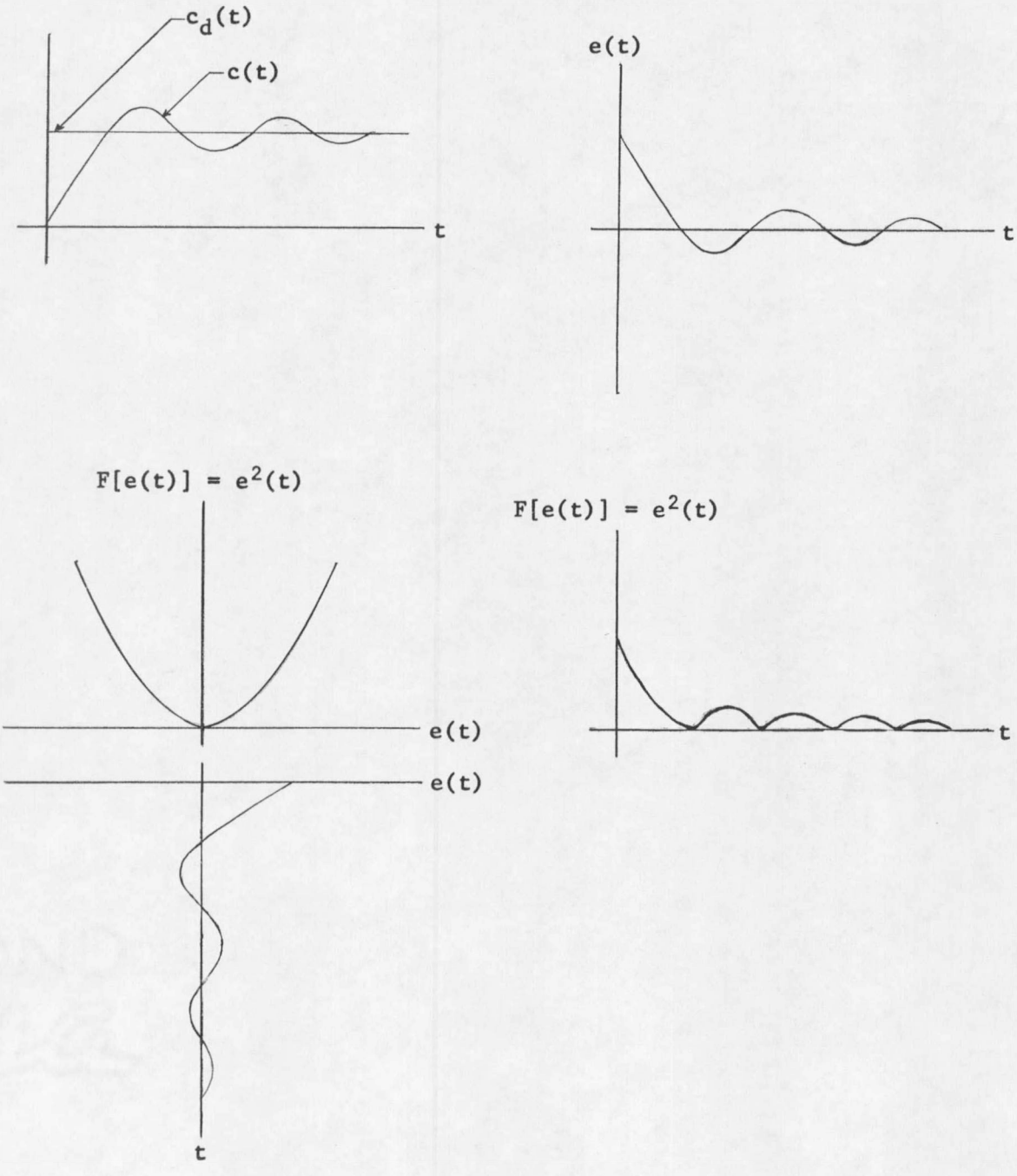


Figure 3-3. The integral-square-error performance measure for $c_d(t) = \delta_{-1}(t)$.

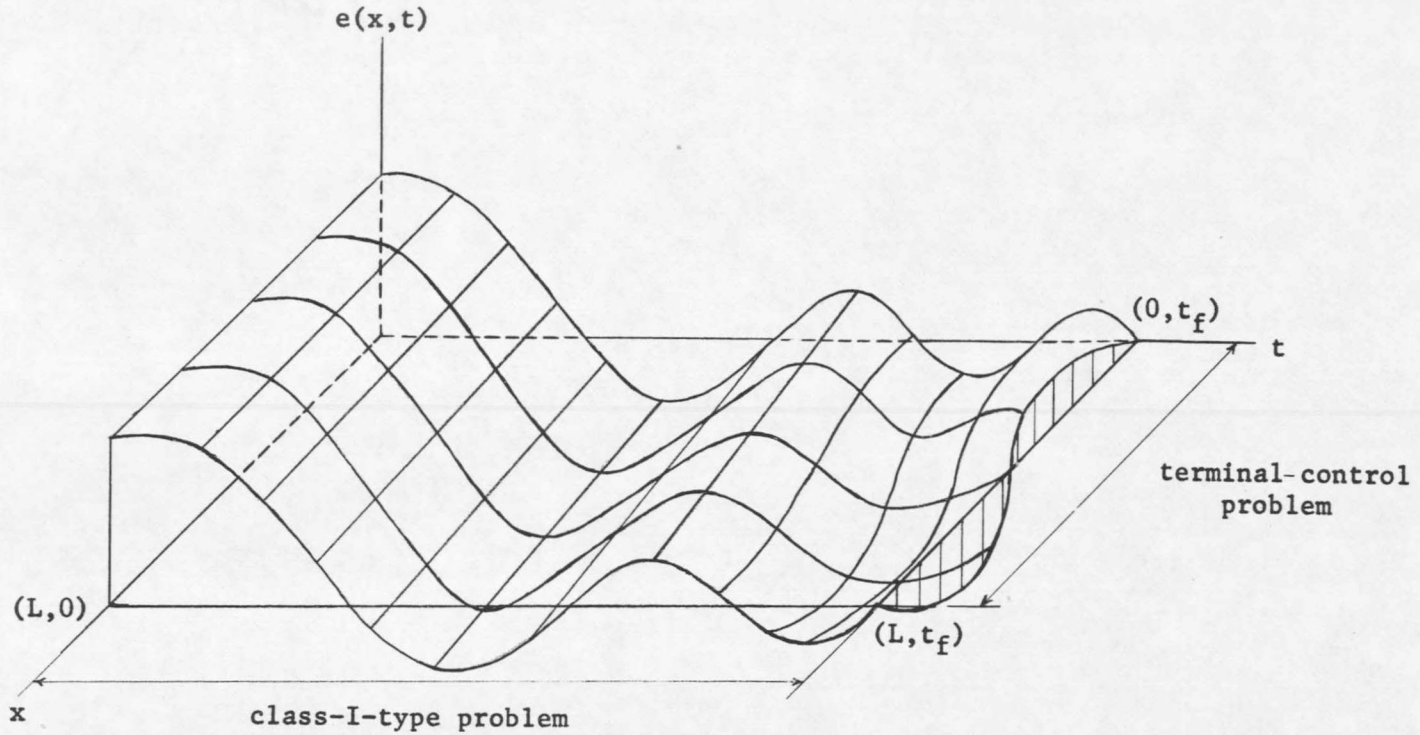


Figure 3-4. Graphical illustration of the error in a typical distributed-parameter system.

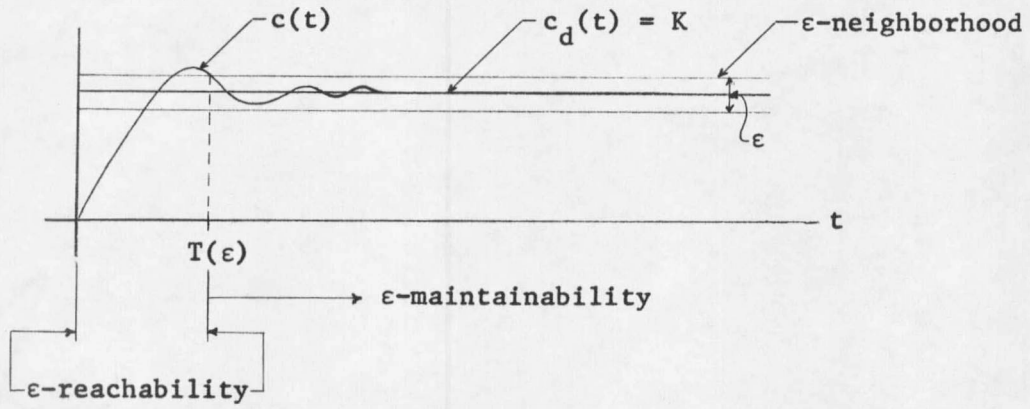


Figure 3-5. Example of ε-reachability and ε-maintainability for lumped-parameter systems.

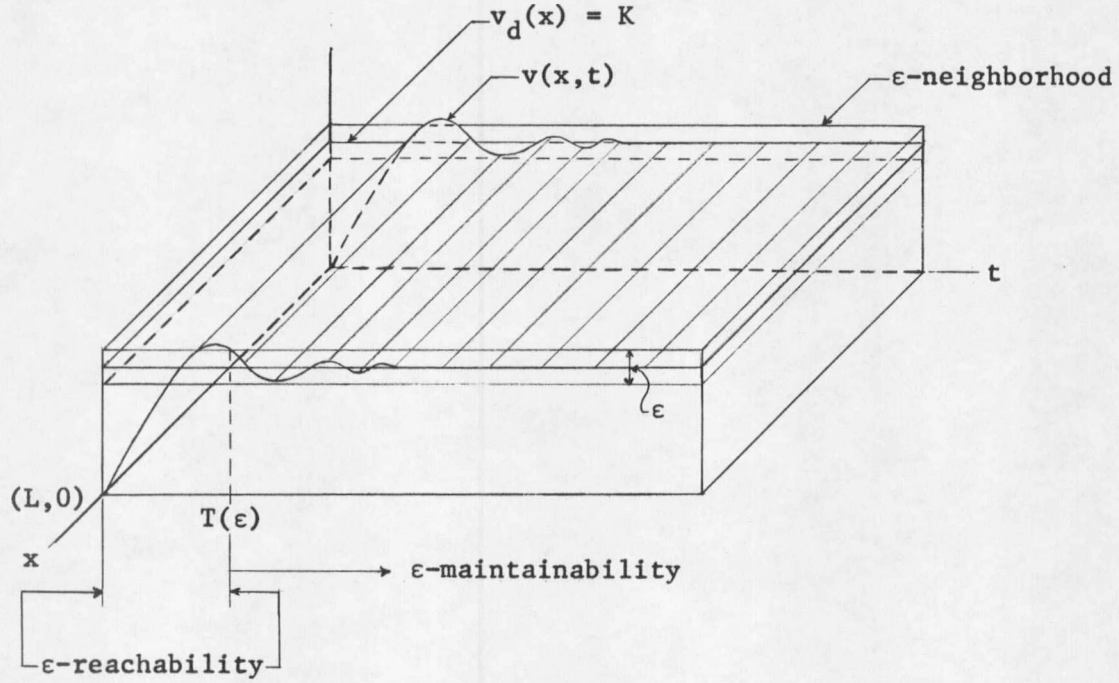


Figure 3-6. Example of ε-reachability and ε-maintainability for distributed-parameter systems with boundary inputs.

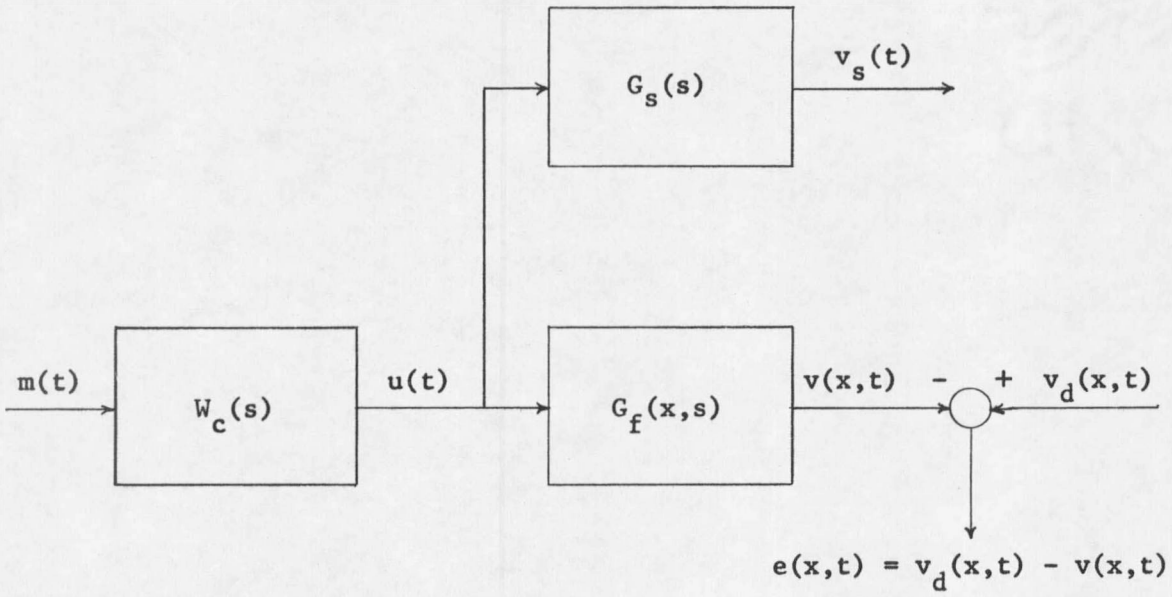


Figure 3-7. Error representation for stochastic class-II-type distributed-parameter systems (Section 3.4).

CHAPTER 4

BLOCK-DIAGRAM MODELS

FOR

CLASS-II-TYPE DISTRIBUTED-PARAMETER SYSTEMS

4.1 INTRODUCTION

Block-diagram models, such as exist for lumped-parameter systems, are not generally available for class-II-type distributed-parameter systems. Because of this, several general forms of block-diagram models are developed in the following sections for class-II-type systems. Some of these block-diagram models were developed by the author in a previous work (226). These models form a basis for the application of optimal design techniques to linear distributed-parameter systems. Several forms of block-diagram models are considered, and in each case emphasis is placed on simplicity and on the ease with which the model lends itself to the application of optimization techniques.

In Section 4.2, four different models are formulated for linear class-II-type distributed-parameter systems with boundary inputs and with a distributed output. For each of these block-diagram models, various methods are given to obtain Laplace-transform transfer functions for a general class of distributed-parameter systems. In Section 4.3, a block-diagram model for a linear class-II-type distributed-parameter system which has both a distributed output and input is developed.

Although the models developed here are restricted to systems which have one space variable and the time variable; the methods can be generalized to systems of higher dimension, unless otherwise specified.

4.2 BLOCK-DIAGRAM MODELS: BOUNDARY INPUTS

In this section, block-diagram models are developed for a general class of linear independent-variable-invariant distributed-parameter

systems which are defined on a one-dimensional spatial domain X, with its boundary denoted by $\overline{X}$, its closure denoted by $X^+ (X^+ = X + \overline{X})$, and with the spatial coordinate variable x. Since only one spatial dimension is considered here, X^+ is taken as the closed interval [0,L], i.e., $0 \leq x \leq L$, and the time variable t is allowed to vary over the interval $[0,\infty]$, i.e., $0 \leq t \leq \infty$.

Let $v \equiv v(x,t)$ represent the state of the system at any time t, and let $u_1(t)$ and $u_2(t)$ represent the boundary input functions at x=0 and x=L, respectively. In general, the dynamics of the linear independent-variable-invariant system can be represented by a linear partial differential equation of the form

$$\frac{\partial^m v}{\partial t^m} = H(v, v_t, v_x, \ldots) \quad \text{for } 0 \leq x \leq L \tag{4.1}$$

where $H(v, v_t, v_x, \ldots)$ represents a given linear function of its arguments and where the dots denote higher-order partial derivatives with respect to t and x; the left-hand member of (4.1) is assumed to be the highest-order derivative with respect to t that appears in (4.1). The boundary conditions are

$$v(0,t) = u_1(t)$$

and (4.2)

$$v(L,t) = u_2(t)$$

The initial condition v(x,0) is assumed to be zero.

4.2.1 The Distributed Model

The distributed model represents the distributed-parameter system considered here by a single block. This model has a distributed transfer function, one which depends explicitly upon the spatial coordinate variable x as well as the Laplace variable s (the transformed time variable). For this case, the transfer function is obtained directly from the partial differential equation (4.1) and its associated boundary conditions (4.2).

Several different approaches may be used to obtain distributed transfer functions, each of which has its own merits; but possibly the most direct approach, herein called method 1, is presented below.

First, the system equation (4.1) is transformed with respect to t by use of the infinite (one-sided) Laplace transform. Although the resulting equation is still a partial differential equation, it contains only derivatives with respect to the space variable x; hence, the equation may be solved using ordinary differential equation theory (126). In general, if the $u_1(t) = u_2(t) = u(t)$ in equation (4.2), the distributed transfer function[1] $G^t(x,s)$ corresponding to equation (4.1) is of the form $G^t(x,s) = V^t(x,s)/U(s)$ where U(s) is the Laplace transform of u(t) and

[1]The notation used throughout is as follows: the right-sided Laplace transform with respect to t of g(x,t) is denoted by $G^t(x,s)$; the finite Laplace transform of g(x,t) with respect to x is denoted by $G^x(p,t)$; and the finite Laplace transform of $G^t(x,s)$ with respect to x is denoted by G(p,s).

$V^t(x,s)$ is the one-sided Laplace transform of $v(x,t)$ with respect to t. Note that this approach can be used even if equation (4.1) is not spatial-invariant.

An alternate approach, called method 2, incorporates use of both the finite Laplace transform and the infinite (one-sided) Laplace transform. When applying this approach, both a finite transform with respect to the space variable and an infinite Laplace transform with respect to the time variable are applied to equation (4.1). Because transforms are applied with respect to both independent variables t and x, only equations of the form (4.1) which are independent-variable-invariant can be subjected to this approach, in general. (Both methods can be applied, however, if a Laplace transformable function of x and t is simply added to the right-hand member of 4.1.) Although the finite Laplace transform introduces some unknown boundary conditions, these are determined in terms of the known boundary conditions by using the analytic property of the finite Laplace transform (see Appendix A). The transfer function obtained is a function of the finite Laplace variable p and the infinite Laplace variable s. The inverse finite Laplace transform of the transfer function may then be effected to yield a function of the frequency variable s and the space variable x.

Of the two methods mentioned, method 1 is easier to apply in simple cases, but not so in complicated ones. For example, method 1 is not easily applied to systems which have a distributed input or to systems which have a space dimension greater than one, while the application of method 2 is straightforward in both cases.

The transfer function for the distributed model is, in general, a transcendental function of the frequency variable s and the space variable x, as is evidenced in the example of Section 4.2.3.

4.2.2 Class-I Representations of Class-II Systems

Use of values of the state variable obtained at discrete points along the space dimension of the system is quite practical and desirable. In the first place, it is usually easier to apply optimization techniques to distributed-parameter systems when values of distributed state variables at specific points, rather than the entire distributed state variable, are used. In the second place, it is easier to implement a distributed-parameter system in a feedback configuration using the data obtained at specific points (see Chapter 7).

The state variable is assumed to be measurable at equally spaced points along the spatial domain of the system. In this way, a class-I model is obtained for the class-II system (for convenience, called the class-I representation) in either a series or a parallel configuration. If the class-I representation, in either form, were to be used as a feedback system, it would result in a multi-output and hence, a multi-loop feedback system. Because a boundary input as a function of time only is considered in this section, the discrete values (in space) of the state variable which are fed back have to be weighted or averaged in some fashion with respect to the space variable in order to control the input function. If open loop control is considered, the control problem is still quite complex because of the generally large number of values of

the state variable to monitor.

Although the distributed model is easier to represent than its class-I counterpart, class-I representation has practical merit for purposes of feedback control; and for this reason, the method of obtaining this model is presented below and is illustrated in the example given in Section 4.2.3.

The transfer functions for the parallel-form of the class-I representation are obtained by dividing the spatial domain [0,L] of the system into n+1 (n odd) intervals of width Δ, and then writing a transfer function from the boundary x=0 to x=iΔ for i=1,2,...,(n+1)/2. Suppose that the same input is applied at each of the boundaries x=0 and x=L. Then the transfer functions and the values of the state variable are symmetric about x=L/2=(n+1)Δ/2; thus, only the first (n+1)/2 transfer functions are needed to represent the system. These transfer functions, denoted by $G^t_{ip}(\Delta,s) = V^t(i\Delta,s)/U(s)$, i=1,2,...,(n+1)/2, are easily obtained from the distributed transfer functions of the system; that is, by substituting iΔ for x in the transfer function obtained in Section 4.2.3.

In a similar manner, the transfer functions for the series-form class-I representation are derived by again dividing the interval [0,L] into n+1 (n odd) equal intervals of width Δ; however, in this case, a transfer function is written from (i-1)Δ to iΔ, i=1,2,...,(n+1)/2. Symbolically, the series-form transfer function is denoted by $G^t_{is}(\Delta,s) = V^t(i\Delta,s)/V^t(\overline{i-1}\Delta,s)$, i=1,2,...,(n+1)/2 where the subscript s denotes that it is for the series form and the argument s represents the infinite

Laplace variable (the transformed time variable). For equal boundary inputs, the state variables are symmetric about $x=L/2=(n+1)\Delta/2$; as before therefore, only the first $(n+1)/2$ values are needed to represent the system.

The series-form and parallel-form transfer functions are quite closely related, and in fact this relationship can be expressed by the following equations.

$$\frac{V^t(\Delta,s)}{U(s)} = G^t_{1s}(\Delta,s) = G^t_{1p}(\Delta,s) \tag{4.3}$$

$$\frac{V^t(i\Delta,s)}{V^t(\overline{i-1}\Delta,s)} = G^t_{is}(\Delta,s) = \frac{G^t_{ip}(\Delta,s)}{G^t_{(i-1)p}(\Delta,s)}, \quad i=2,3,\ldots,(n+1)/2 \tag{4.4}$$

Once the transfer function for the distributed model is obtained, therefore, the transfer functions for the parallel-form and series-form class-I representations follow directly.

Although the class-I representations are obtained in the following example for a system with a one-dimensional spatial domain and with the same input applied at the spatial boundaries, this procedure can be extended to include both unequal boundary inputs and certain higher dimension systems, but the complexity of the model increases quite rapidly in both cases.

4.2.3 Illustrative Example I

Consider as an example, a distributed-parameter plant which is characterized adequately by the partial differential equation

$$\frac{\partial^2 v(x,t)}{\partial x^2} = K \frac{\partial v(x,t)}{\partial t}, \quad 0 \leq x \leq L, \; 0 \leq t \leq \infty \tag{4.5}$$

with the initial condition at t=0 of

$$v(x,0) = 0, \quad 0 \leq x \leq L \tag{4.6}$$

and with the boundary conditions at x=0 and x=L of

$$v(0,t) = v(L,t) = u(t), \quad 0 \leq t \leq \infty \tag{4.7}$$

where K is a constant and u(t) is the input to the system applied at the boundaries x=0 and x=L.

The transfer function of the distributed model is derived for this problem using each of the two methods described in Section 4.2.1. Then, from the distributed transfer function, the parallel-form and series-form class-I representations are obtained.

To illustrate method 1, first take the infinite Laplace transform, with respect to the time variable t, of the system equation (4.5); the transformed equation is:

$$\frac{\partial^2 V^t(x,s)}{\partial x^2} = K[s\ V^t(x,s) - v(x,0)] \tag{4.8}$$

where it is given that $v(x,0) = 0$.

Although equation (4.8) is still a partial differential equation, since $V^t(x,s)$, like $v(x,t)$, is a function of two variables, it has the advantage of involving differentiation with respect to x only and thus may be solved as an ordinary differential equation.

Proceeding formally, equation (4.8) has a general solution of the form

$$V^t(x,s) = a \exp[(Ks)^{1/2}\ x] + b \exp[-(Ks)^{1/2}\ x], \quad 0 \leq x \leq L \tag{4.9}$$

where a and b are independent of x. Assuming the input function u(t) is transformable, the boundary conditions on $V^t(x,s)$ are

$$V^t(0,s) = V^t(L,s) = U(s) \tag{4.10}$$

The coefficients a and b, which are functions of s, are obtained by applying the boundary conditions (4.10) to equation (4.9). After effecting the indicated operation, the transfer function for this system is expressed as:

$$\frac{V^t(x,s)}{U(s)} = G^t(x,s) = \frac{A(x,s) + B(x,s)}{C(L,s)}, \quad 0 \leq x \leq L \tag{4.11}$$

where

$$A(x,s) = \exp[-(Ks)^{1/2}(L-x)] - \exp[-(Ks)^{1/2}(L+x)]$$

$$B(x,s) = \exp[-(Ks)^{1/2}x] - \exp[-(Ks)^{1/2}(2L-x)]$$

$$C(L,s) = 1 - \exp[-(Ks)^{1/2}2L]$$

For comparison, the transfer function of the distributed-parameter plant considered in this example is again derived, only this time using method 2; that is, using the finite Laplace transform in conjunction with the infinite Laplace transform. Pertinent properties of the finite Laplace transform are given in Appendix A.

Using equation (A.2), the finite Laplace transform of equation (4.5), with respect to x for $0 \leq x \leq L$, is

$$p^2 V^x(p,t) - [p\, v(0,t) + v_x(0,t)] + [p\, v(L,t) + v_x(L,t)]e^{-Lp} = K \frac{\partial V^x(p,t)}{\partial t}, \quad 0 \leq x \leq L \tag{4.12}$$

where

$$v_x(0,t) = \left.\frac{\partial v(x,t)}{\partial t}\right|_{x=0}, \qquad v_x(L,t) = \left.\frac{\partial v(x,t)}{\partial x}\right|_{x=L},$$

and $V^x(p,t)$ is the finite Laplace transform of $v(x,t)$.

The next step is to take the infinite Laplace transform of equation (4.12), which gives

$$p^2\, V(p,s) - [p\, V^t(0,s) + V_x^t(0,s)] + [p\, V^t(L,s) + V_x^t(L,s)]e^{-Lp} = K[s\, V(p,s) - V^x(p,0)] \tag{4.13}$$

where the finite Laplace transform of the initial condition, equation (4.6), is

$$V^x(p,0) = 0 \tag{4.14}$$

and the infinite Laplace transform of the boundary conditions, equation (4.7), is (assuming the input u(t) is Laplace transformable)

$$V^t(0,s) = V^t(L,s) = U(s) \tag{4.15}$$

Substituting the identities of equations (4.14) and (4.15) into (4.13) and then solving for V(p,s) gives an equation for the transformed state

variable, as follows:

$$V(p,s) = \frac{[p\ U(s) + V_x^t(0,s)] - [p\ U(s) + V_x^t(L,s)]e^{-Lp}}{p^2 - Ks} \qquad (4.16)$$

In order to solve for the boundary conditions $V_x^t(0,s)$ and $V_x^t(L,s)$, the condition employed is that $V(p,s)$ must be analytic in finite regions of the p plane. Holding s fixed, the denominator of the finite transform $V(p,s)$ has zeros at $p = \pm(Ks)^{1/2}$; hence, as explained in Appendix A, the numerator of equation (4.16), evaluated at $p = \pm(Ks)^{1/2}$, must equal zero.

This requirement yields two equations which may be solved for $V_x^t(0,s)$ and $V_x^t(L,s)$ in terms of U(s).

$$p\ U(s) + V_x^t(0,s) - [p\ U(s) + V_x^t(L,s)]e^{-Lp}\Big|_{p=(Ks)^{1/2}} = 0 \qquad (4.17)$$

and

$$p\ U(s) + V_x^t(0,s) - [p\ U(s) + V_x^t(L,s)]e^{-Lp}\Big|_{p=-(Ks)^{1/2}} = 0 \qquad (4.18)$$

Simultaneous solution of equations (4.17) and (4.18) gives

$$V_x^t(0,s) = (Ks)^{1/2}U(s)\ \{\exp[-(Ks)^{1/2}L](\tanh[\frac{(Ks)^{1/2}L}{2}] + 1) - 1\} \qquad (4.19)$$

and

$$V_x^t(L,s) = (Ks)^{1/2}U(s)\ \tanh[\frac{(Ks)^{1/2}L}{2}] \tag{4.20}$$

In order to obtain the transfer function in the form of equation (4.11), the inverse finite Laplace transform of equation (4.16) must be effected. As explained in Appendix A, this may be accomplished by using a table of finite Laplace transforms or by using equation (A.5) and a table of infinite Laplace transforms.

Since an extensive table of finite Laplace transforms is not available, equation (4.16) is inverted using the second of the two methods mentioned above. Appropriate substitution of equations (4.19) and (4.20) into (4.16) yields the following equation for which the inverse infinite Laplace transform must be obtained.

$$V(p,s) = \{p + (Ks)^{1/2}(\tanh[(Ks)^{1/2}L/2] + 1)\ \exp[-(Ks)^{1/2}L] - (Ks)^{1/2}\}\ \frac{U(s)}{p^2 - Ks} \tag{4.21}$$

Holding s fixed, equation (4.21) is inverted

$$V^t(x,s) = \frac{U(s)}{2}\ \{D(L,s)\ \exp[-(Ks)^{1/2}x] + E(L,s)\ \exp[(Ks)^{1/2}x]\} \tag{4.22}$$

where

$$D(L,s) = 2 - \exp[-(Ks)^{1/2}L](\tanh[(Ks)^{1/2}L/2] + 1)$$

and

$$E(L,s) = \exp[-(Ks)^{1/2}L](\tanh[(Ks)^{1/2}L/2] + 1)$$

Equation (4.22) is simplified to obtain the transfer function $G^t(x,s)$ which agrees with equation (4.11) obtained by method 1; that is,

$$\frac{V^t(x,s)}{U(s)} = G^t(x,s) = \frac{A(x,s) + B(x,s)}{C(L,s)}, \quad 0 \leq x \leq L \tag{4.23}$$

where A(x,s), B(x,s), and C(L,s) are defined below equation (4.11).

To obtain a more convenient form, equations (4.11) and (4.23) are written in terms of hyperbolic functions and are simplified using hyperbolic identities, as follows:

$$G^t(x,s) = \frac{2 \sinh[(Ks)^{1/2}L/2] \cosh[(Ks)^{1/2}(2x-L)/2]}{\sinh[(Ks)^{1/2}L]}, \quad 0 \leq x \leq L \tag{4.24}$$

Using equation (4.24), the distributed model of this distributed-parameter plant is illustrated in Figure 4-1; note that it contains both a distributed transfer function and a distributed output.

To illustrate the class-I representations of a class-II-type system with equal boundary inputs, both the series-form and parallel-form representations are given for this example. As stated previously, the transfer functions for these representations may be obtained directly from the distributed transfer function which has already been derived.

A preliminary step in obtaining the class-I representations is to divide the spatial domain [0,L] of the system into n+1 equal intervals of width Δ; that is, $\Delta=L/n+1$ (n odd).

The i^{th} transfer function for the parallel form is obtained by substituting $(n+1)\Delta$ for L and $i\Delta$, $i=1,2,\ldots,(n+1)/2$, for x in equation (4.24). For the system considered in this example, the same input, u(t), is applied at x=0 and x=L, and thus, the distributed state variable is symmetric about the center $x=L/2=(n+1)\Delta/2$. Using this fact, the first (n+1)/2 transfer functions give sufficient information to completely represent the system. Hence,

$$\frac{V^t(i\Delta,s)}{U(s)} = G^t_{ip}(\Delta,s)$$

$$= \frac{2\sinh[(Ks)^{1/2}(n+1)\Delta/2]}{\sinh[(Ks)^{1/2}(n+1)\Delta]} \cosh[(Ks)^{1/2}[2i-(n+1)]\Delta/2] \qquad (4.25)$$

where $G^t_{ip}(\Delta,s)$ represents the transfer functions from x=0 to $x=i\Delta$, $i=1,2,\ldots,(n+1)/2$.

Because each transfer function has a common term, namely,

$2\ \sinh[(Ks)^{1/2}(n+1)\Delta/2]\ /\ \sinh[(Ks)^{1/2}(n+1)\Delta]$,

the block diagram may be represented as shown in Figure 4-2.

For the series-form class-I representation, the transfer functions are written from the input to the output of each interval Δ; that is, from $(i-1)\Delta$ to $i\Delta$, $i=1,2,\ldots,(n+1)/2$.

Using equations (4.3) and (4.4), the series-form transfer functions are obtained directly from those derived for the parallel form. For this example, the series transfer functions are given by

$$G^t_{1s}(\Delta,s) = G^t_{1p}(\Delta,s) \tag{4.26}$$

$$= \frac{2\ \sinh[(Ks)^{1/2}(n+1)\Delta/2]}{\sinh[(Ks)^{1/2}(n+1)\Delta]}\ \cosh[(Ks)^{1/2}(n-1)\Delta/2]$$

and

$$G^t_{is}(\Delta,s) = \frac{G^t_{ip}(\Delta,s)}{G^t_{(i-1)p}(\Delta,s)} \tag{4.27}$$

$$= \frac{\cosh\{(Ks)^{1/2}[2i - (n+1)]\Delta/2\}}{\cosh\{(Ks)^{1/2}[2(i-1) - (n+1)]\Delta/2\}}$$

for $i=2,3,\ldots,(n+1)/2$.

Just as for the parallel case, only the first (n+1)/2 values of the state variable, and hence, of the transfer functions are needed to represent the system; the values of the state variable are symmetric about $x=L/2=(n+1)\Delta/2$. The series-form block diagram for this example is shown in Figure 4-3.

4.2.4 The Harmonic Model

The spatial-harmonic representation --- representing a distributed-parameter system by a harmonic series in the spatial variable x --- is used by McCausland (161, 198, 290) and Murray (217, 218) for the analysis of class-II-type distributed-parameter systems. McCausland (161, 198, 290) did not give a method for obtaining this representation; and, although Murray (217, 218) presented two methods for obtaining this type of model, he did not demonstrate the utility of the finite sine and finite cosine transforms for its derivation. Several authors have applied the finite sine and finite cosine transforms to the solution of boundary-value problems (for example, references 6, 31, 40, and 58) but these transform techniques have not been used, in general, for the analysis of class-II-type, distributed-parameter control systems. In this subsection, therefore, a method which uses these transform techniques is developed for determining transfer functions for the harmonic-block-diagram models of class-II-type distributed-parameter systems.

The method of obtaining the transfer functions for the harmonic model, as presented here, is a straightforward one which makes use of the infinite (one-sided) Laplace transform together with the finite sine

transform or the finite cosine transform, whichever is applicable. The choice of the transform, finite sine or finite cosine, depends upon the form of the boundary conditions: it is well known that if the boundary conditions are specified in terms of the state variable $v(x,t)$, the finite sine transform is applicable; but, if the boundary conditions are specified in terms of the first derivative $\partial v(x,t)/\partial x$, the finite cosine transform is applicable (see Appendix A for more detail). The procedure used herein is as follows: First, the partial differential equation characterizing the system is transformed with respect to x by the finite sine (cosine) transform; second, the resulting equation is transformed with respect to t by the infinite (one-sided) Laplace transform; and third, the inverse finite sine (cosine) transform of the transformed equation which results from steps one and two is effected.

In general, if $u_1(t) = u_2(t) = u(t)$, the system can be represented by one infinite set of blocks which contains only odd spatial harmonics. However, if $u_1(t) \neq u_2(t)$, two infinite sets of blocks which contain both even and odd harmonics are required, and the outputs of the two sets, for which inputs are $u_1(t)$ and $u_2(t)$, respectively, are summed to yield the distributed output $V^t(x,s)$.

Although an infinite number of blocks is required to represent the system, an approximate analysis of the system can be accomplished by using only a finite number of harmonics --- see references 161, 198, 217, 218, and 290. In addition, Vichit Lorchirochoonkul, a graduate student at Montana State University, is incorporating the use of this model in his development of various approximations for optimal design of

distributed-parameter systems.

The following example illustrates the steps incident to the formation of this type of model.

4.2.5 Illustrative Example II

Again, consider as an example, a distributed-parameter plant which is characterized adequately by the partial differential equation

$$\frac{\partial^2 v(x,t)}{\partial x^2} = K \frac{\partial v(x,t)}{\partial t} , \quad 0 \leq x \leq L , \quad 0 \leq t \leq \infty \tag{4.28}$$

with the initial condition at t=0 of

$$v(x,0) = 0 , \quad 0 \leq x \leq L \tag{4.29}$$

and with the boundary conditions at x=0 and x=L of

$$v(0,t) = v(L,t) = u(t) , \quad 0 \leq t \leq \infty \tag{4.30}$$

where K is a constant and u(t) is the input to the system applied at the boundaries x=0 and x=L.

Because the boundary conditions are specified in terms of the state variable v(x,t), rather than in terms of the first derivative $\partial v(x,t)/\partial x$, the finite sine transform is used.

Using equation (A.14), the finite sine transform with respect to x of equation (4.28) is

$$-\left(\frac{n\pi}{L}\right)^2 V_s(n,t) + \left(\frac{n\pi}{L}\right)[(-1)^{n+1}v(L,t) + v(0,t)] = K\frac{dV_s(n,t)}{dt} \qquad (4.31)$$

where $V_s(n,t)$ denotes the finite sine transform with respect to x of $v(x,t)$. Substituting the boundary conditions given by equation (4.30) and rearranging, equation (4.31) reduces to

$$K\frac{dV_s(n,t)}{dt} + \left(\frac{n\pi}{L}\right)^2 V_s(n,t) = \left(\frac{n\pi}{L}\right)[(-1)^{n+1} + 1]\, u(t) \qquad (4.32)$$

Now, assuming that the input function $u(t)$ is Laplace transformable, the Laplace transform of equation (4.32) is

$$K[s\, V_s^t(n,s) - V_s(n,0)] + \left(\frac{n\pi}{L}\right)^2 V_s^t(n,s) = \left(\frac{n\pi}{L}\right)[(-1)^{n+1} + 1]\, U(s) \qquad (4.33)$$

where $V_s^t(n,s)$ represents the Laplace transform with respect to t of $V_s(n,t)$. Taking the finite sine transform of the initial conditions given by equation (4.29), it is found that $V_s(n,0) = 0$; therefore, equation (4.33) can be written as follows:

$$V_s^t(n,s) = \frac{(n\pi/L)[(-1)^{n+1} + 1]\, U(s)}{K[s + (n\pi/L)^2/K]} \qquad (4.34)$$

Next, using the inversion formula given by equation (A.8), the inverse finite sine transform of equation (4.34) is found to be

$$V^t(x,s) = \frac{2\pi}{KL^2} \sum_{n=1}^{\infty} n \frac{[(-1)^{n+1} + 1] \sin(n\pi x/L)}{[s + (n\pi/L)^2/K]} U(s) \qquad (4.35)$$

Note that when n is even, the term which appears within the summation in equation (4.35) is zero; therefore, only odd spatial harmonics are present. Thus, if n is replaced by 2k-1, equation (4.35) reduces to:

$$V^t(x,s) = \frac{4\pi}{KL^2} \sum_{k=1}^{\infty} \frac{(2k-1) \sin[(2k-1)\pi x/L]}{s + [(2k-1)\pi/L]^2/K} U(s) \qquad (4.36)$$

Finally, from equation (4.36), the transfer function for the harmonic model is obtained, as follows:

$$\frac{V^t(x,s)}{U(s)} = \frac{4\pi}{KL^2} \sum_{k=1}^{\infty} \frac{(2k-1) \sin[(2k-1)\pi x/L]}{s + [(2k-1)\pi/L]^2/K}, \quad 0 \leq x \leq L \qquad (4.37)$$

As shown in Figure 4-4, the harmonic model of the system considered in this example is represented by one infinite set of simple blocks.

Although only the use of the finite sine transform is illustrated here, a similar procedure is followed when the finite cosine transform is applicable --- that is, when the boundary conditions are specified in

terms of the derivatives, $\left.\frac{\partial v(x,t)}{\partial x}\right|_{x=0} = u_3(t)$ and $\left.\frac{\partial v(x,t)}{\partial x}\right|_{x=L} = u_4(t)$.

4.3 BLOCK-DIAGRAM MODELS: DISTRIBUTED INPUTS

Thus far in this chapter, all of the block diagrams have been formulated for systems which have boundary inputs; not all distributed-parameter systems fit into this class, however. In this section, block-diagram models for linear distributed-parameter systems which have distributed input functions, as well as distributed output functions, are treated. For this case, method 1, described in Section 4.2.1, is not readily applicable; but method 2 can be applied to this case and is so done in Section 4.3.1.

Consider a general distributed-parameter system with the following properties: The system is defined on a one-dimensional spatial domain X, with its boundary denoted by $\overline{X}$, its closure by $X^+ (X^+ = X + \overline{X})$, and with a spatial coordinate variable x. As in Section 4.2, the spatial domain X^+ is taken as the interval [0,L], i.e., $0 \leq x \leq L$, and the time variable t is taken over the semi-infinite range, $0 \leq t \leq \infty$.

The following symbols are used in this section: $v \equiv v(x,t)$ represents the state of the system at any time t, $u_1(t)$ and $u_2(t)$ represent, respectively, the inputs at the boundaries x=0 and x=L, and $f \equiv f(x,t)$ represents the distributed input function which is assumed to be Laplace transformable. It is assumed that the dynamics of the linear system can be represented by a linear partial differential equation of the form

$$\frac{\partial^m v(x,t)}{\partial x^m} = H(v, v_t, v_x, \ldots) + f(x,t) \tag{4.38}$$

where $H(v, v_t, v_x, \ldots)$ represents a given linear function of its arguments and where the dots denote higher-order partial derivatives with respect to t and x; the left-hand member of (4.38) is assumed to be the highest-order derivative with respect to t that appears in (4.38). The boundary conditions are

$$v(0,t) = u_1(t)$$

and (4.39)

$$v(L,t) = u_2(t)$$

The initial condition v(x,0) is assumed to be zero.

4.3.1 An Implicit Transfer Function

For class-II-type distributed-parameter systems having both a distributed input and output, an explicit transfer function (output/input) cannot be obtained in general; it is possible, however, to obtain an implicit transfer function. An implicit transfer function is defined as one in which the ratio of the transformed output to the transformed input cannot be obtained explicitly, but one which gives, nevertheless, the Laplace-transform relationship between the output and the input

(or inputs). To obtain the implicit transfer function, a procedure similar to method 2, given in Section 4.2.1, is used.

As in the case of boundary-input class-II-type distributed-parameter systems, the transforms effected are a finite Laplace transform with respect to the space variable x and an infinite Laplace transform with respect to the time variable t. As before, the finite Laplace transform introduces some unknown boundary conditions which must be resolved by using the fact that the finite Laplace transform must be analytic in finite regions of the p plane. In this case, however, the above-noted operation introduces some added complexities because the distributed input, which is a function of both Laplace variables p and s, appears in the equations for the unknown boundary conditions. Thus, as will be demonstrated in Section 4.3.2, the distributed input function must be evaluated at the zeros with respect to p of the denominator of the finite transform V(p,s); usually, these zeros are themselves functions of s.

As is to be expected, the block diagrams for distributed input systems are more complex than those obtained for boundary input systems. For boundary-input distributed-parameter systems, the output is obtained as a function of the space variable x and the frequency variable s; but in general, the output from a system which has distributed inputs cannot be obtained in this form because the finite Laplace transform cannot be inverted until after the explicit form of the distributed input is given. Thus, an implicit transfer function is obtained as a function of the Laplace variables p and s; this is illustrated by the following example.

4.3.2 Illustrative Example III

The problem considered is similar to that of Sections 4.2.3 and 4.2.5; a distributed-parameter plant is described by the partial differential equation

$$\frac{\partial^2 v(x,t)}{\partial x^2} - K \frac{\partial v(x,t)}{\partial t} = f(x,t) , \quad 0 \leq x \leq L, \quad 0 \leq t \leq \infty \tag{4.40}$$

with the initial condition at t=0 of

$$v(x,0) = 0 , \quad 0 \leq x \leq L \tag{4.41}$$

and with the boundary conditions at x=0 and x=L of

$$v(0,t) = v(L,t) = u(t) , \quad 0 \leq t \leq \infty \tag{4.42}$$

where K is a constant, u(t) is a boundary input, and f(x,t) is the distributed input to the system.

Using equation (A.2), the finite Laplace transform, $V^x(p,t)$, with respect to x ($0 \leq x \leq L$) of v(x,t) in equation (4.40) is an element of the following equation:

$$p^2 V^x(p,t) - [p\, v(0,t) + v_x(0,t)] + [p\, v(L,t) + v_x(L,t)]e^{-Lp} - K \frac{\partial V^x(p,t)}{\partial t} = F^x(p,t) \tag{4.43}$$

where

$$v_x(0,t) \equiv \left.\frac{\partial v(x,t)}{\partial x}\right|_{x=0}, \qquad v_x(L,t) \equiv \left.\frac{\partial v(x,t)}{\partial x}\right|_{x=L}$$

and $F^x(p,t)$ is the finite Laplace transform of $f(x,t)$.

Equation (4.43) is transformed by the infinite Laplace transform with respect to time to give

$$p^2\, V(p,s) - [p\, V^t(0,s) + V_x^t(0,s)] + [p\, V^t(L,s) + V_x^t(L,s)]e^{-Lp} - K[s\, V(p,s) - V^x(p,0)] = F(p,s) \tag{4.44}$$

The initial condition is transformed by taking the finite Laplace transform of equation (4.41),

$$V^x(p,0) = 0 \tag{4.45}$$

Assuming u(t) is Laplace transformable, the boundary conditions are transformed by taking the infinite Laplace transform of equation (4.42),

$$V^t(0,s) = V^t(L,s) = U(s) \tag{4.46}$$

and finally, the identities given by equations (4.45) and (4.46) are substituted into (4.44), which is then solved to obtain

$$V(p,s) = \frac{[p\ U(s) + V_x^t(0,s) + F(p,s)] - [p\ U(s) + V_x^t(L,s)]e^{-Lp}}{(p^2 - Ks)} \qquad (4.47)$$

In the above expression, the boundary conditions $V_x^t(0,s)$ and $V_x^t(L,s)$ are determined by the requirement that $V(p,s)$ must satisfy the following basic property of finite Laplace transform theory: namely, a finite Laplace transform must be analytic in finite regions of the p plane. In the case of would-be first-order poles, the numerator, as evaluated at the zeros of the denominator of $V(p,s)$, must equal zero. This requirement yields

$$[p\ U(s) + V_x^t(0,s) + F(p,s)] - [p\ U(s) + V_x^t(L,s)]e^{-Lp}\Big|_{p=-(Ks)^{1/2}} = 0 \qquad (4.48)$$

and

$$[p\ U(s) + V_x^t(0,s) + F(p,s)] - [p\ U(s) + V_x^t(L,s)]e^{-Lp}\Big|_{p=(Ks)^{1/2}} = 0 \qquad (4.49)$$

which are solved simultaneously to obtain

$$V_x^t(L,s) = \frac{F[-(Ks)^{1/2},s] - F[(Ks)^{1/2},s]}{2\ \sinh[(Ks)^{1/2}L]} + U(s)\ P(L,s) \qquad (4.50)$$

and

$$V_x^t(0,s) = F[-(Ks)^{1/2},s]M(L,s) - F[(Ks)^{1/2},s]N(L,s) + U(s)P(L,s) \qquad (4.51)$$

where

$$M(L,s) = \exp[-(Ks)^{1/2}L] \;/\; 2\sinh[(Ks)^{1/2}L]$$

$$N(L,s) = \exp[(Ks)^{1/2}L] \;/\; 2\sinh[(Ks)^{1/2}L]$$

$$P(L,s) = (Ks)^{1/2}\tanh[(Ks)^{1/2}L/2]$$

$$F[-(Ks)^{1/2},s] = F(p,s)\Big|_{p=\,-(Ks)^{1/2}}$$

$$F[(Ks)^{1/2},s] = F(p,s)\Big|_{p=(Ks)^{1/2}}$$

By substituting the identities given by equations (4.50) and (4.51) appropriately into equation (4.47) and then simplifying, the implicit transfer function for this example is obtained as shown below.

$$V(p,s) = \frac{1}{p^2 - Ks}\{U(s)[(1-e^{-Lp})p + Q(L,s)] + F(p,s) + F[-(Ks)^{1/2},s]R(L,s) + F[(Ks)^{1/2},s]S(L,s)\} \qquad (4.52)$$

where

$$Q(L,s) = (Ks)^{1/2}(1+e^{-Lp})(1 - \exp[-(Ks)^{1/2}L]) \;/\; (1 + \exp[-(Ks)^{1/2}L])$$

$$R(L,s) = (\exp[-(Ks)^{1/2}L] - e^{-Lp}) \;/\; 2\sinh[(Ks)^{1/2}L]$$

$$S(L,s) = (e^{-Lp} - \exp[(Ks)^{1/2}L]) \;/\; 2\sinh[(Ks)^{1/2}L]$$

The terms in the right-hand member of equation (4.52) represent, respectively, the effect on the output of the system due to:

1. the boundary conditions;
2. the distributed input;
3. the distributed input with p replaced by $-(Ks)^{1/2}$; and
4. the distributed input with p replaced by $(Ks)^{1/2}$.

The block diagram of this system is shown in Figure 4-5, where the diamond-shaped block represents symbolically the operation

$$F(p,s)\Big|_{p=(Ks)^{1/2}}$$

and

$$F(p,s)\Big|_{p=-(Ks)^{1/2}}$$

4.4 REMARKS

For the distributed model (Section 4.2.1), the Laplace-transform transfer function is obtained for linear distributed systems defined on a one-dimensional spatial domain. Quite often, the same techniques can be applied to similar systems with spatial domains of more than one dimension.

In Section 4.2.2, class-I representations are developed for a general class-II-type distributed-parameter system with equal boundary inputs and with a one-dimensional spatial domain. Here too, the general approach is applicable to many systems of more than one dimension; for higher dimensional systems, however, the model is a grid of blocks instead of the simple series of blocks obtained for the one-dimensional spatial domain.

In Section 4.2.4, the utility of the finite sine and finite cosine transforms is demonstrated for the derivation of a spatial-harmonic-model representation of a class-II-type distributed-parameter system with boundary inputs. Although the harmonic model requires an infinite number of simple blocks to represent a distributed-parameter system, it is a convenient representation because an approximate analysis of the system can be accomplished by using only a finite number of harmonics.

Block-diagram models for class-II-type distributed-parameter systems with distributed inputs are obtained by use of a new block-diagram operation, one which is not encountered in lumped-parameter systems. Even for one spatial dimension, an explicit transfer function cannot be obtained; but the implicit transfer function developed gives sufficient

information to represent the system in block-diagram form. Apart from this, however, the diagram involves only conventional operations.

As is evidenced, finite integral transforms play an important role in the derivation of transfer functions for the block diagrams of distributed-parameter systems. By using a finite integral transform for variables defined on a finite domain and the infinite (one-sided) Laplace transform for variables defined on a semi-infinite range, the transfer functions for a wide class of higher dimension systems can be derived.

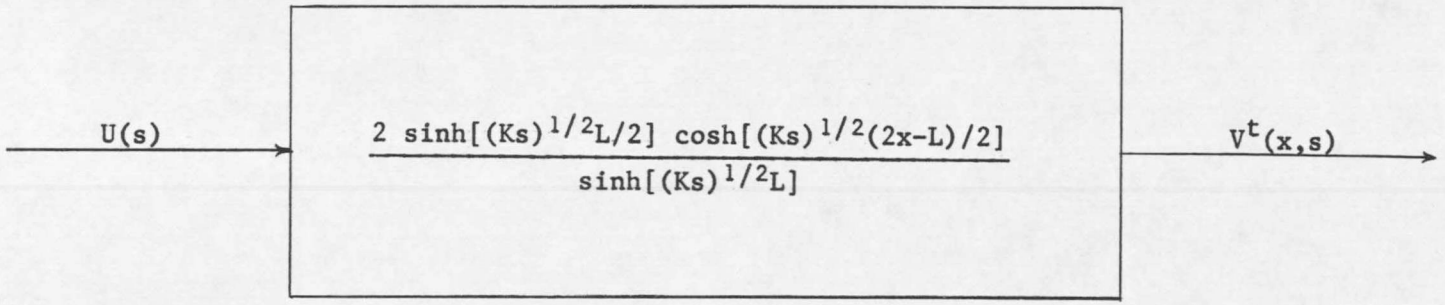


Figure 4-1. The distributed model (Example 4.2.3).

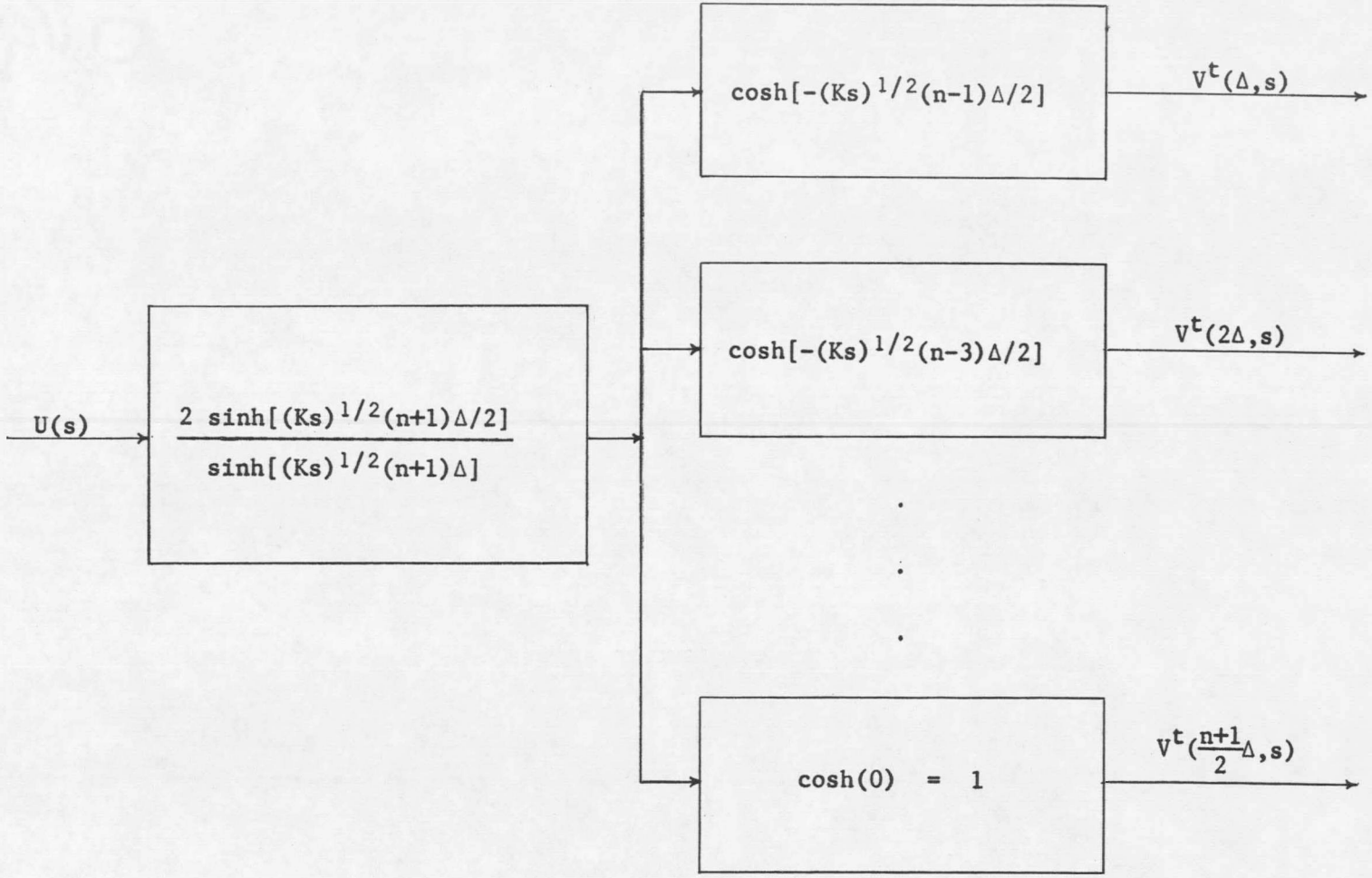


Figure 4-2. The parallel-form class-I representation (Example 4.2.3).

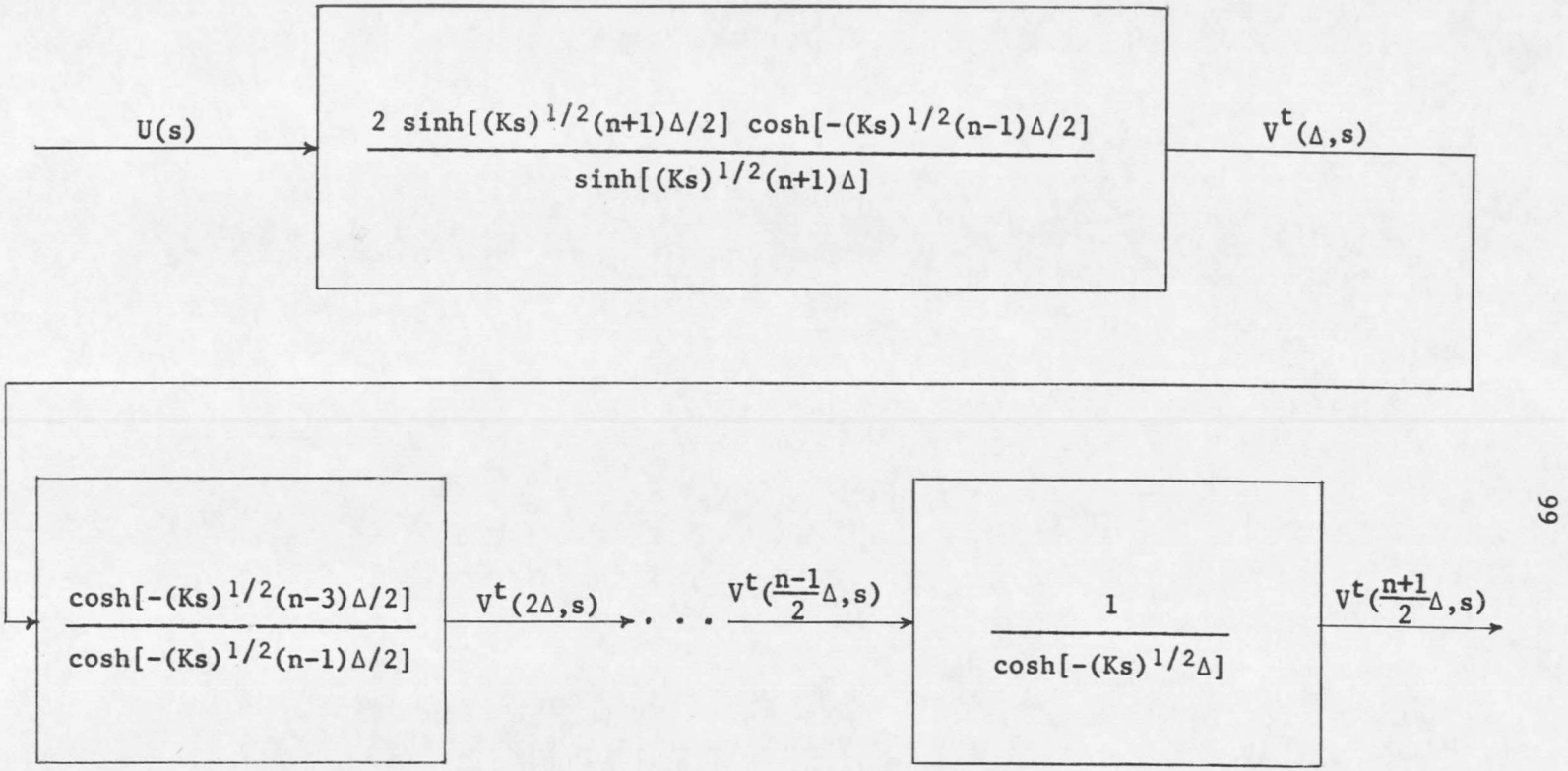


Figure 4-3. The series-form class-I representation (Example 4.2.3).

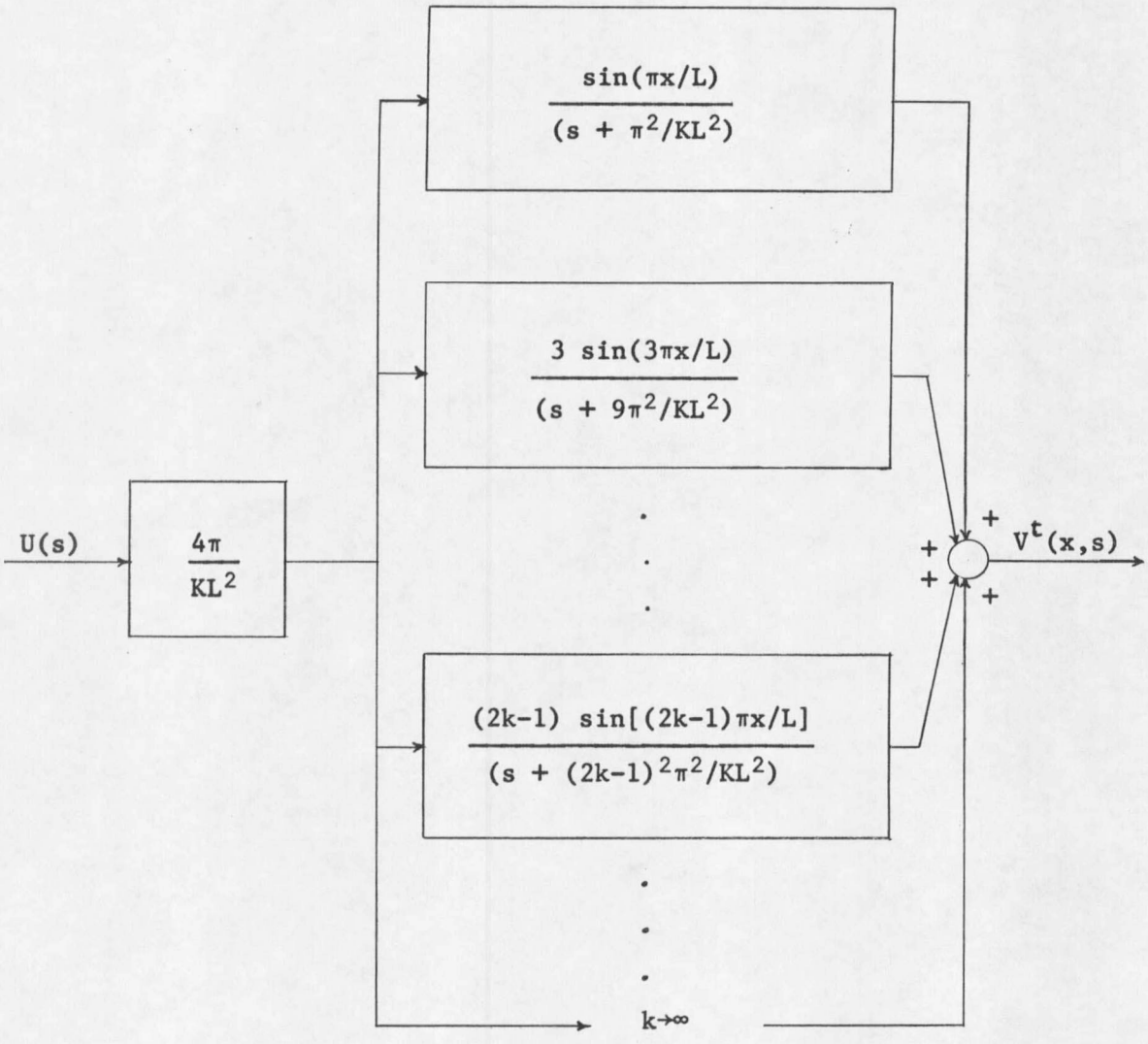


Figure 4-4. The harmonic model (Example 4.2.5).

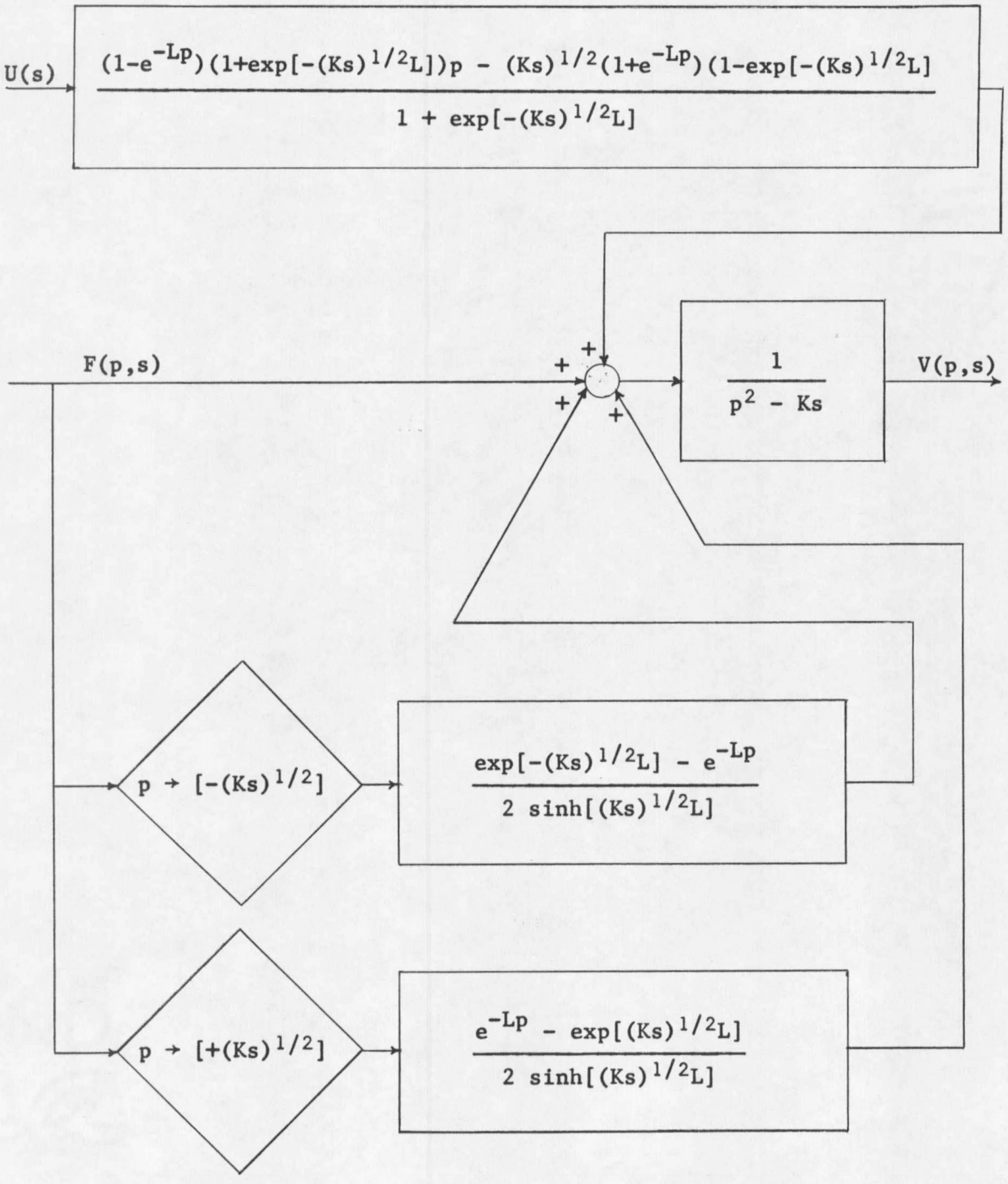


Figure 4-5. Block diagram for distributed-input system (Example 4.3.2).

CHAPTER 5

STATISTICAL ANALYSIS OF DISTRIBUTED-PARAMETER CONTROL SYSTEMS

5.1 INTRODUCTION

The statistical analysis and design of lumped-parameter control systems has been investigated by numerous authors (20, 30, 80, 101, 261, and others), and it is now an integral part of most graduate-level courses in control theory. In the analysis of stationary random processes, the autocorrelation and cross-correlation functions together with their respective spectral density functions play an important role when the processes analyzed satisfy the ergodic hypothesis. The ergodic hypothesis is satisfied by a given system if the following two procedures lead to identical results in regard to the statistical properties of the signals in the system: (1) the signals of the system are recorded over n randomly spaced intervals of width τ, and these records are used to compute statistical properties; or (2) the signals of a set of n systems, each of which is identical in character to the given system, are recorded over a single time interval of width τ, and these records are used to compute statistical properties (for a more rigorous treatment, see reference number 9).

In 1963, Itskovich (180) used a discrete form of correlation function to determine the optimum location of pickups in a distributed field. In a more recent paper, Senin effected statistical analysis of class-II-type distributed-parameter systems which are subject to stationary, ergodic, random disturbances or inputs. Subsequently, Senin used these techniques for the extrapolation of signals (233) and for the design of stochastic distributed-parameter systems (271). Because Senin's first paper on this subject (238) is very brief and because the translation from the Russian

is very poor, this chapter is devoted to clarifying and expanding the theory advanced therein. In Section 5.2, the basis of the method of analysis is presented by means of an example; in Section 5.3, a convenient procedure is given for obtaining the distributed spectral-density function of a general, second-order, linear partial differential equation; and in Sections 5.4 and 5.5, various boundary conditions are incorporated into the analysis to obtain explicit distributed spectral-density functions.

5.2 A METHOD OF ANALYSIS FOR STOCHASTIC DISTRIBUTED-PARAMETER SYSTEMS

In the development of the techniques which follow, the time-invariant, stochastic, distributed-parameter systems considered are assumed to be in the steady-state mode of operation; that is, deterministic transients are not present, and the initial conditions do not influence the statistical properties of the process. Also, the process and the disturbances which act on the process are assumed to satisfy the ergodic hypothesis.

It is convenient to develop the method of analysis by means of an example. For this purpose, consider a distributed-parameter system defined on a finite spatial domain, $0 \leq x \leq L$. The system has inputs applied at the spatial boundaries and is characterized by

$$\frac{\partial^2 v(x,t)}{\partial x^2} - K \frac{\partial v(x,t)}{\partial t} = 0 \qquad (5.1a)$$

In order to determine the "correlation" between the values of the state variable measured at various points along the spatial domain of the system, it is convenient to consider two models of the single system, herein introduced as the <u>two-model representation</u>: the first model has the spatial coordinate x, $0 \leq x \leq L$, and the state variable $v(x,t)$; and the second model has the spatial coordinate y and the state variable $v(y,t+\tau)$, where τ is a time-shift parameter. This method of representation is illustrated in Figure 5-1. The equation governing the second system is

$$\frac{\partial^2 v(y,t+\tau)}{\partial y^2} - K \frac{\partial v(y,t+\tau)}{\partial t} = 0 \tag{5.1b}$$

and by definition,

$$v(x,t) = v(y,t+\tau)\Big|_{\substack{y=x \\ \tau=0}} \tag{5.2}$$

For the systems considered in this section, the correlation between the value of the state variable at x and at time t and the value of the state variable at y and at time $t+\tau$ is expressed by the <u>distributed autocorrelation function</u> defined by

$$R_{vv}(x,y,\tau) \equiv \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} v(x,t)\, v(y,t+\tau)\, dt \tag{5.3}$$

To find the correlation function for the example considered here, corresponding members of equations (5.1a) and (5.1b) are multiplied, and the product is averaged with respect to time to obtain

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \{\frac{\partial^2 v(x,t)}{\partial x^2} \frac{\partial^2 v(y,t+\tau)}{\partial y^2} - K \frac{\partial^2 v(y,t+\tau)}{\partial y^2} \frac{\partial v(x,t)}{\partial t} - K \frac{\partial^2 v(x,t)}{\partial x^2} \frac{\partial v(y,t+\tau)}{\partial t} + K^2 \frac{\partial v(x,t)}{\partial t} \frac{\partial v(y,t+\tau)}{\partial t}\} \, dt = 0 \tag{5.4}$$

In this equation, the partial derivatives with respect to t are replaced by equivalent convolution integrals as follows:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \{\frac{\partial^2 v(x,t)}{\partial x^2} \frac{\partial^2 v(y,t+\tau)}{\partial y^2} - K \frac{\partial^2 v(y,t+\tau)}{\partial y^2} \int_{-\infty}^{\infty} v(x,t-t_1)\dot{\delta}(t_1) \, dt_1 - K \frac{\partial^2 v(x,t)}{\partial x^2} \int_{-\infty}^{\infty} v(y,t+\tau-t_2)\dot{\delta}(t_2) \, dt_2 + K^2 \int_{-\infty}^{\infty} v(x,t-t_3)\dot{\delta}(t_3) \, dt_3 \int_{-\infty}^{\infty} v(y,t+\tau-t_4)\dot{\delta}(t_4) \, dt_4\} \, dt = 0 \tag{5.5}$$

where $\delta(t)$ (the Dirac delta function) and $\dot{\delta}(t) = d\delta(t)/dt$ are generalized functions. Because equation (5.5) satisfies Fubini's theorem and because the partial derivatives which appear in the integrands are continuous, both the order of integration and the order of partial

differentiation and integration can be interchanged. Thus, equation (5.5) is rewritten as follows:

$$\frac{\partial^4}{\partial x^2\,\partial y^2}\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} v(x,t)\,v(y,t+\tau)\,dt\right] -$$

$$K\frac{\partial^2}{\partial y^2}\left\{\int_{-\infty}^{\infty}\dot{\delta}(t_1)\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} v(x,t-t_1)\,v(y,t+\tau)\,dt\right]dt_1\right\} - \tag{5.6}$$

$$K\frac{\partial^2}{\partial x^2}\left\{\int_{-\infty}^{\infty}\dot{\delta}(t_2)\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} v(x,t)\,v(y,t+\tau-t_2)\,dt\right]dt_2\right\} +$$

$$K^2\int_{-\infty}^{\infty}\dot{\delta}(t_3)\int_{-\infty}^{\infty}\dot{\delta}(t_4)\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} v(x,t-t_3)\,v(y,t+\tau-t_4)\,dt\right]dt_4\,dt_3 = 0$$

and by use of the definition of the distributed autocorrelation function given by equation (5.3), equation (5.6) is rewritten as

$$\frac{\partial^4}{\partial x^2\,\partial y^2}R_{vv}(x,y,\tau) - K\frac{\partial^2}{\partial y^2}\int_{-\infty}^{\infty}\dot{\delta}(t_1)\,R_{vv}(x,y,\tau+t_1)\,dt_1 -$$

$$K\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\dot{\delta}(t_2)\,R_{vv}(x,y,\tau-t_2)\,dt_2 + \tag{5.7}$$

$$K^2\int_{-\infty}^{\infty}\dot{\delta}(t_3)\int_{-\infty}^{\infty}\dot{\delta}(t_4)\,R_{vv}(x,y,\tau+t_3-t_4)\,dt_4\,dt_3 = 0$$

After carrying out the integrations indicated in equation (5.7), a partial differential equation in terms of the distributed autocorrelation function is obtained; that is

$$\frac{\partial^4 R_{vv}(x,y,\tau)}{\partial x^2\,\partial y^2} + K\frac{\partial^3 R_{vv}(x,y,\tau)}{\partial y^2\,\partial \tau} - K\frac{\partial^3 R_{vv}(x,y,\tau)}{\partial x^2\,\partial \tau} - K^2\frac{\partial^2 R_{vv}(x,y,\tau)}{\partial \tau^2} = 0 \tag{5.8}$$

The distributed autocorrelation function can now be determined by solving the partial differential equation (5.8); the solution of this partial differential equation is facilitated by the use of the two-sided Laplace transform. Herein, the two-sided Laplace transform with respect to τ of $R_{vv}(x,y,\tau)$ is denoted by $S^{\tau}_{vv}(x,y,s)$:

$$S^{\tau}_{vv}(x,y,s) \equiv \int_{-\infty}^{\infty} R_{vv}(x,y,\tau)\, e^{-s\tau}\, d\tau \tag{5.9}$$

and

$$R_{vv}(x,y,\tau) = \frac{1}{2\pi j}\int_{-jw}^{jw} S^{\tau}_{vv}(x,y,s)\, e^{s\tau}\, ds \tag{5.10}$$

From $S^{\tau}_{vv}(x,y,s)$, the <u>distributed spectral-density</u> $S^{\tau}_{vv}(x,y,jw)$ is obtained by letting $s=jw$.

Using the notation given above, the two-sided Laplace transform with respect to τ of equation (5.8) is

$$\frac{\partial^4 S^{\tau}_{vv}(x,y,s)}{\partial x^2\, \partial y^2} + Ks \frac{\partial^2 S^{\tau}_{vv}(x,y,s)}{\partial y^2} - Ks \frac{\partial^2 S^{\tau}_{vv}(x,y,s)}{\partial x^2} - K^2 s^2 S^{\tau}_{vv}(x,y,s) = 0 \tag{5.11}$$

Equation (5.11) is now in a form which can be solved by the method of separation of variables (126); therefore, $S^{\tau}_{vv}(x,y,s)$ is assumed to be of the form

$$S^{\tau}_{vv}(x,y,s) = X^{\tau}(x,s)\, Y^{\tau}(y,s) \tag{5.12}$$

where $X^{\tau}(x,s)$ is a function of x and the independent parameter s, and $Y^{\tau}(y,s)$ is a function of y and the independent parameter s. Letting s=jw and substituting the right-hand side of equation (5.12) into equation (5.11), it follows that

$$\ddot{X}^{\tau}(x,jw)\, \ddot{Y}^{\tau}(y,jw) + jwK\, X^{\tau}(x,jw)\, \ddot{Y}^{\tau}(y,jw) - jwK\, \ddot{X}^{\tau}(x,jw)\, Y^{\tau}(y,jw) + w^2K^2X^{\tau}(x,jw)\, Y^{\tau}(y,jw) = 0 \tag{5.13}$$

and, in factored form,

$$[-jwK\, X^{\tau}(x,jw) - \ddot{X}^{\tau}(x,jw)][jwK\, Y^{\tau}(y,jw) - \ddot{Y}^{\tau}(y,jw)] = 0 \qquad (5.14)$$

where $\ddot{X}^{\tau}(x,jw) \equiv \frac{d^2X^{\tau}(x,jw)}{dx^2}$ and $\ddot{Y}^{\tau}(y,jw) \equiv \frac{d^2Y^{\tau}(y,jw)}{dy^2}$.

Because the distributed autocorrelation function evaluated at x=y $R_{vv}(x,y,\tau)\Big|_{x=y}$ is an even function of τ, the distributed spectral-density function evaluated at x=y $S^{\tau}_{vv}(x,y,jw)\Big|_{x=y}$ is an even function of w. Using this property of the distributed spectral-density function, it follows that

$$Y^{\tau}(y,jw) = \overline{X}^{\tau}(x,jw)\Big|_{x=y} \qquad (5.15)$$

where $\overline{X}^{\tau}(x,jw)$ is the complex conjugate of $X^{\tau}(x,jw)$. Thus, it follows from equations (5.14) and (5.15) that

$$jwK\, X^{\tau}(x,jw) + \ddot{X}^{\tau}(x,jw) = 0 \qquad (5.16)$$

and

$$jwK\, Y^{\tau}(y,jw) - \ddot{Y}^{\tau}(y,jw) = 0 \qquad (5.17)$$

Consequently, by using the method of separation of variables, the partial differential equation (5.11) is reduced to two ordinary differential

equations which may be easily solved for $X^{\tau}(x,jw)$ and $Y^{\tau}(y,jw)$ to obtain

$$X^{\tau}(x,jw) = A(jw)\exp[-(-jwK)^{1/2}x] + B(jw)\exp[+(-jwK)^{1/2}x] \qquad (5.18)$$

and

$$Y^{\tau}(y,jw) = C(jw)\exp[-(jwK)^{1/2}y] + D(jw)\exp[+(jwK)^{1/2}y] \qquad (5.19)$$

where the coefficients A(jw), B(jw), C(jw) and D(jw) are determined from the boundary conditions which are necessarily specified for the particular problem considered. From equations (5.18) and (5.19), the distributed spectral-density function for the example considered here is

$$S^{\tau}_{vv}(x,y,jw) = \{A(jw)\exp[-(-jwK)^{1/2}x] + B(jw)\exp[(-jwK)^{1/2}x]\} \cdot \{C(jw)\exp[-(jwK)^{1/2}y] + D(jw)\exp[(jwK)^{1/2}y]\} \qquad (5.20)$$

5.3 OBTAINING THE DISTRIBUTED SPECTRAL-DENSITY FUNCTION

Although the process just completed is useful for illustrating the theory, a more straightforward method for obtaining the distributed spectral-density function is illustrated below. First, the two-sided Laplace transform with respect to t of the partial differential equation (5.1a) characterizing the system is obtained; that is, letting s=jw

$$[\frac{\partial^2}{\partial x^2} - jwK]\ V^t(x,jw) = 0 \qquad (5.21)$$

Replacing $\frac{\partial^2}{\partial x^2}$ by r^2 the characteristic equation of the system is obtained

$$(r^2 - jwK) = 0 \qquad (5.22)$$

and solving for the roots of this equation, denoted by r_1 and r_2, $Y^T(y,jw)$ can be written in terms of these roots

$$Y^T(y,jw) = C(jw)\ \exp[r_1 y] + D(jw)\ \exp[r_2 y] \qquad (5.23)$$

where $r_1 = -(jwK)^{1/2}$ and $r_2 = (jwK)^{1/2}$. Because $Y^T(y,jw)$ and $X^T(x,jw)$ at x=y are complex conjugates, $X^T(x,jw)$ can be determined from the conjugate characteristic equation,

$$(\bar{r}^2 + jwK) = 0 \qquad (5.24)$$

Having solved for the roots of the conjugate characteristic equation (5.24), denoted by $\bar{r}_1$ and $\bar{r}_2$, $X^T(x,jw)$ can be written as

$$X^T(x,jw) = A(jw)\ \exp[\bar{r}_1 x] + B(jw)\ \exp[\bar{r}_2 x] \qquad (5.25)$$

where $\bar{r}_1 = -(-jwK)^{1/2}$ and $\bar{r}_2 = (-jwK)^{1/2}$.

For a general second-order steady-state system described by

$$a \frac{\partial^2 v(x,t)}{\partial t^2} + b \frac{\partial v(x,t)}{\partial t} + c \frac{\partial^2 v(x,t)}{\partial t\, \partial x} + d \frac{\partial v(x,t)}{\partial x} + e \frac{\partial^2 v(x,t)}{\partial x^2} + f\, v(x,t) = 0 \tag{5.26}$$

where a, b, c, d, e, and f are constants, the distributed spectral-density function can be written as

$$S_{vv}^{\tau}(x,y,jw) = \{A(jw) \exp[\overline{r}_1 x] + B(jw) \exp[\overline{r}_2 x]\} \cdot \{C(jw) \exp[r_1 y] + D(jw) \exp[r_2 y]\} \tag{5.27}$$

where r_1 and r_2 are the roots of the characteristic equation,

$$er^2 + (d + jwc)r + (jwb - aw^2 + f) = 0 \tag{5.28}$$

and $\overline{r}_1$ and $\overline{r}_2$ are the roots of the conjugate characteristic equation,

$$e\overline{r}^2 + (d - jwc)\overline{r} + (-jwb - aw^2 + f) = 0 \tag{5.29}$$

5.4 INCORPORATION OF BOUNDARY CONDITIONS

To complete the formulation of the distributed spectral-density function of equation (5.27), the coefficients A(jw), B(jw), C(jw), and D(jw) must necessarily be determined. These coefficients are determined by expressing the boundary conditions in terms of spectral density functions. To illustrate the procedure, three basic types of boundary conditions are considered: Boundary conditions of the first type,

$$v(x,t)\Big|_{x=0} = u(t) \tag{5.30}$$

of the second type,

$$\frac{\partial v(x,t)}{\partial x}\Big|_{x=0} = q(t) \tag{5.31}$$

and of the third type,

$$\frac{\partial v(x,t)}{\partial x}\Big|_{x=0} = k[\theta(t) - v(x,t)\Big|_{x=0}] \tag{5.32}$$

where k is a constant; and u(t), q(t), and $\theta(t)$ are stationary random functions for which correlation functions are assumed known or may be determined by known methods (41). The corresponding spectral density functions for these boundary conditions are derived by first finding the distributed autocorrelation functions and then taking the Laplace

transform of the results and letting s=jw.

To illustrate this procedure, a boundary condition of the first type is now considered. Again employing the two-model representation shown in Figure 5-1, a boundary condition of the first type is expressed as follows:

$$v(x,t)\Big|_{x=0} = u(t) \tag{5.33}$$

and

$$v(y,t+\tau)\Big|_{y=0} = u(t+\tau) \tag{5.34}$$

Corresponding members of equations (5.33) and (5.34) are multiplied and averaged with respect to time with the result that

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} v(x,t)\, v(y,t+\tau)\, dt\Big|_{x=y=0} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} u(t)\, u(t+\tau)\, dt \tag{5.35}$$

which reduces to

$$R_{vv}(x,y,\tau)\Big|_{x=y=0} \equiv R_{vv}(x,\tau)\Big|_{x=0} = R_{uu}(\tau) \tag{5.36}$$

where $R_{uu}(\tau)$ is the autocorrelation function of the time function u(t), and $R_{vv}(x,\tau)\Big|_{x=0}$ is defined to be the distributed autocorrelation function $R_{vv}(x,y,\tau)$ evaluated at x=y=0. Taking the Laplace transform of equation

(5.36) and letting s=jw, the boundary condition of the first type is obtained in terms of spectral density functions.

$$S^{\tau}_{vv}(x,y,jw)\Big|_{x=y=0} \equiv S^{\tau}_{vv}(x,jw)\Big|_{x=0} = S_{uu}(jw) \tag{5.37}$$

Following a similar procedure, boundary conditions of the second and third type are obtained in terms of spectral density functions: For boundary conditions of the second type,

$$\frac{\partial^2 S^{\tau}_{vv}(x,jw)}{\partial x^2}\Big|_{x=0} = S_{qq}(jw) \tag{5.38}$$

and for boundary conditions of the third type,

$$\left[\frac{\partial^2 S^{\tau}_{vv}(x,jw)}{\partial x^2} + 2k\,\frac{\partial S^{\tau}_{vv}(x,jw)}{\partial x} + k^2 S^{\tau}_{vv}(x,jw)\right]\Big|_{x=0} = k^2 S_{\theta\theta}(jw) \tag{5.39}$$

where $S_{qq}(jw)$ and $S_{\theta\theta}(jw)$ are the spectral density functions corresponding to the time functions $q(t)$ and $\theta(t)$, respectively.

In addition, the disturbances applied to the opposite boundaries of the system may be correlated; the distributed spectral-density function $S^{\tau}_{vv}(x,y,jw)$ must necessarily enfold this correlation. By determining the cross-correlation function and the corresponding spectral density of the disturbance applied at x=0 and the disturbance applied at x=L, the

cross-correlation effect can be included in the distributed spectral-density function $S^{\tau}_{vv}(x,y,jw)$ when the coefficients A(jw), B(jw), C(jw), and D(jw) are determined. From the three types of boundary conditions considered here, six distinct combinations are possible, as follows:

1. Boundary conditions of the first type at both ends,

$$v(x,t)\Big|_{x=0} = u_1(t) \tag{5.40}$$

$$v(x,t)\Big|_{x=L} = u_2(t) \tag{5.41}$$

In this case, the distributed spectral-density function $S^{\tau}_{vv}(x,y,jw)$ must satisfy the following conditions

$$S^{\tau}_{vv}(x,y,jw)\Big|_{\substack{x=0\\y=L}} = S_{u_1u_2}(jw) \tag{5.42}$$

and

$$S^{\tau}_{vv}(x,y,jw)\Big|_{\substack{x=L\\y=0}} = S_{u_2u_1}(jw) = S_{u_1u_2}(-jw) \tag{5.43}$$

where $S_{u_1u_2}(jw)$ and $S_{u_2u_1}(jw) = S_{u_1u_2}(-jw)$ are the spectral densities corresponding to the cross-correlation functions $R_{u_1u_2}(\tau)$ and $R_{u_2u_1}(\tau)$, respectively.

2. Boundary conditions of the second type at both ends,

$$\left.\frac{\partial v(x,t)}{\partial x}\right|_{x=0} = q_1(t) \tag{5.44}$$

$$\left.\frac{\partial v(x,t)}{\partial x}\right|_{x=L} = q_2(t) \tag{5.45}$$

To illustrate the procedure for finding the corresponding spectral density functions, the steps required for this case are given below. Using the two-model representation (Figure 5-1), the product of the boundary conditions at x=0 and at y=L is obtained.

$$\left.\frac{\partial v(x,t)}{\partial x}\right|_{x=0} \left.\frac{\partial v(y,t+\tau)}{\partial y}\right|_{y=L} = q_1(t)\, q_2(t+\tau) \tag{5.46}$$

Taking the time average of both sides of equation (5.46) and interchanging the order of differentiation and integration in the resulting equation, the boundary conditions are found in terms of correlation functions.

$$\left.\frac{\partial^2 R_{vv}(x,y,\tau)}{\partial x\, \partial y}\right|_{\substack{x=0\\y=L}} = R_{q_1 q_2}(\tau) \tag{5.47}$$

Effecting the two-sided Laplace transform with respect to τ of equation (5.47) and letting s=jw, the corresponding spectral density functions

are obtained.

$$\left.\frac{\partial^2 S^{\tau}_{vv}(x,y,jw)}{\partial x\ \partial y}\right|_{\substack{x=0\\y=L}} = S_{q_1 q_2}(jw) \tag{5.48}$$

Similarly, the spectral density function corresponding to the cross-correlation function between the inputs at x=L and y=0 is obtained.

$$\left.\frac{\partial^2 S^{\tau}_{vv}(x,y,jw)}{\partial x\ \partial y}\right|_{\substack{x=L\\y=0}} = S_{q_2 q_1}(jw) = S_{q_1 q_2}(-jw) \tag{5.49}$$

where $S_{q_1 q_2}(jw)$ and $S_{q_2 q_1}(jw) = S_{q_1 q_2}(-jw)$ are the spectral density functions which correspond to respective cross-correlation functions $R_{q_1 q_2}(\tau)$ and $R_{q_2 q_1}(\tau)$.

3. Boundary conditions of the third type at both ends,

$$\left.\frac{\partial v(x,t)}{\partial x}\right|_{x=0} = k\left[\theta_1(t) - v(x,t)\Big|_{x=0}\right] \tag{5.50}$$

$$\left.\frac{\partial v(x,t)}{\partial x}\right|_{x=L} = k\left[\theta_2(t) - v(x,t)\Big|_{x=L}\right] \tag{5.51}$$

The same procedure as used before yields

$$\left[\frac{\partial^2 S^{\tau}_{vv}(x,y,jw)}{\partial x\,\partial y} + k\,\frac{\partial S^{\tau}_{vv}(x,y,jw)}{\partial y} + k\,\frac{\partial S^{\tau}_{vv}(x,y,jw)}{\partial x} + k^2 S^{\tau}_{vv}(x,y,jw)\right]\Bigg|_{\substack{x=0\\y=L}} = k^2 S_{\theta_1\theta_2}(jw) \tag{5.52}$$

and

$$\left[\frac{\partial^2 S^{\tau}_{vv}(x,y,jw)}{\partial x\,\partial y} + k\,\frac{\partial S^{\tau}_{vv}(x,y,jw)}{\partial y} + k\,\frac{\partial S^{\tau}_{vv}(x,y,jw)}{\partial x} + k^2 S^{\tau}_{vv}(x,y,jw)\right]\Bigg|_{\substack{x=L\\y=0}} = S_{\theta_2\theta_1}(jw) = S_{\theta_1\theta_2}(-jw) \tag{5.53}$$

where $S_{\theta_1\theta_2}(jw)$ and $S_{\theta_2\theta_1}(jw) = S_{\theta_1\theta_2}(-jw)$ are the spectral density functions corresponding to respective cross-correlation functions $R_{\theta_1\theta_2}(\tau)$ and $R_{\theta_2\theta_1}(\tau)$.

4. Mixed boundary conditions of the first and second type,

$$v(x,t)\Big|_{x=0} = u(t) \tag{5.54}$$

$$\left.\frac{\partial v(x,t)}{\partial x}\right|_{x=L} = q(t) \tag{5.55}$$

from which result

$$\left.\frac{\partial S^{\tau}_{vv}(x,y,jw)}{\partial y}\right|_{\substack{x=0\\y=L}} = S_{uq}(jw) \tag{5.56}$$

and

$$\left.\frac{\partial S^{\tau}_{vv}(x,y,jw)}{\partial x}\right|_{\substack{x=L\\y=0}} = S_{uq}(-jw) \tag{5.57}$$

where $S_{uq}(jw)$ and $S_{qu}(jw) = S_{uq}(-jw)$ are the spectral density functions resulting from the cross-correlation functions $R_{uq}(\tau)$ and $R_{qu}(\tau)$, respectively.

5. Mixed boundary conditions of the first and third type,

$$v(x,t)\Big|_{x=0} = u(t) \tag{5.58}$$

$$\left.\frac{\partial v(x,t)}{\partial x}\right|_{x=L} = k[\theta(t) - v(x,t)\Big|_{x=L}] \tag{5.59}$$

which result in

$$\left[\frac{\partial S^{\tau}_{vv}(x,y,jw)}{\partial y} + k\, S^{\tau}_{vv}(x,y,jw)\right]\Bigg|_{\substack{x=0\\y=L}} = k\, S_{u\theta}(jw) \tag{5.60}$$

and

$$\left[\frac{\partial S_{vv}(x,y,jw)}{\partial x} + k\, S^{\tau}_{vv}(x,y,jw)\right]\Bigg|_{\substack{x=L\\y=0}} = k\, S_{u\theta}(-jw) \tag{5.61}$$

where $S_{u\theta}(jw)$ and $S_{\theta u}(jw) = S_{u\theta}(-jw)$ are the spectral densities which correspond to the respective cross-correlation functions $R_{u\theta}(\tau)$ and $R_{\theta u}(\tau)$.

6. Mixed boundary conditions of the second and third type,

$$\frac{\partial v(x,t)}{\partial x}\Bigg|_{x=0} = q(t) \tag{5.62}$$

$$\frac{\partial v(x,t)}{\partial x}\Bigg|_{x=L} = k[\theta(t) - v(x,t)\Big|_{x=L}] \tag{5.63}$$

which result in

$$\left[\frac{\partial^2 S^{\tau}_{vv}(x,y,jw)}{\partial x\, \partial y} + k\, \frac{\partial S^{\tau}_{vv}(x,y,jw)}{\partial x}\right]\Bigg|_{\substack{x=0\\y=L}} = k\, S_{q\theta}(jw) \tag{5.64}$$

and

$$\left[\frac{\partial^2 S^{\tau}_{vv}(x,y,jw)}{\partial x\,\partial y} + k\,\frac{\partial S^{\tau}_{vv}(x,y,jw)}{\partial y}\right]\Bigg|_{\substack{x=L\\y=0}} = k\,S_{q\theta}(-jw) \qquad (5.65)$$

where $S_{q\theta}(jw)$ and $S_{\theta q}(jw) = S_{q\theta}(-jw)$ are the spectral density functions which correspond to $R_{q\theta}(\tau)$ and $R_{\theta q}(\tau)$, respectively.

Note that when this method of analysis is applied to systems defined on a semi-infinite spatial domain, $0 \leq x \leq \infty$, the cross-correlation effects of the disturbances applied at the boundaries can be neglected.

By making use of the appropriate boundary conditions given above, the coefficients A(jw), B(jw), C(jw), and D(jw) which appear in the distributed spectral-density function, given by equation (5.27), can be determined. In Section 5.5, the explicit form of the distributed spectral-density function is determined for a distributed-parameter system with boundary conditions of the first type.

5.5 THE DISTRIBUTED SPECTRAL-DENSITY FUNCTION FOR SYSTEMS WITH BOUNDARY CONDITIONS OF THE FIRST TYPE

In this section, the explicit form of the distributed spectral-density function is derermined for a general, second-order, distributed-parameter system (characterized by equation 5.26) which is defined on a finite spatial domain, $0 \leq x \leq L$, and which is subject to boundary conditions of the first type at the spatial boundaries. The general

form of the distributed spectral-density function for a problem of this type is given by equation (5.27); after carrying out the indicated multiplications, it is written in the form

$$S_{vv}^{\tau}(x,y,jw) = \{AC \exp[\overline{r}_1 x + r_1 y] + BC \exp[\overline{r}_2 x + r_1 y] + AD \exp[\overline{r}_1 x + r_2 y] + BD \exp[\overline{r}_2 x + r_2 y]\} \tag{5.66}$$

where $A \equiv A(jw)$, $B \equiv B(jw)$, $C \equiv C(jw)$, and $D \equiv D(jw)$.

For a system with boundary conditions of the first type at x=0 and at x=L, the distributed spectral-density function (5.66) must satisfy a condition of the form given by equation (5.37) at x=y=0 and at x=y=L; and in addition, it must satisfy the conditions given by equations (5.42) and (5.43). Therefore, for this case, the distributed spectral-density function must satisfy the four equations shown below.

$$S_{vv}^{\tau}(x,y,jw)\Big|_{x=y=0} = AC + BC + AD + BD = S_{11}(jw) \tag{5.67}$$

$$S_{vv}^{\tau}(x,y,jw)\Big|_{x=y=L} = \{AC \exp[(\overline{r}_1 + r_1)L] + BC \exp[(\overline{r}_2 + r_1)L] + AD \exp[(\overline{r}_1 + r_2)L] + BD \exp[(\overline{r}_2 + r_2)L]\} = S_{22}(jw) \tag{5.68}$$

$$S^{\tau}_{vv}(x,y,jw)\Big|_{\substack{x=0\\y=L}} = \{AC\ \exp[r_1 L] + BC\ \exp[r_1 L] + AD\ \exp[r_2 L] + BD\ \exp[r_2 L]\} = S_{12}(jw) \tag{5.69}$$

$$S^{\tau}_{vv}(x,y,jw)\Big|_{\substack{x=L\\y=0}} = \{AC\ \exp[\bar{r}_1 L] + BC\ \exp[\bar{r}_2 L] + AD\ \exp[\bar{r}_1 L] + BD\ \exp[\bar{r}_2 L]\} = S_{12}(-jw) \tag{5.70}$$

where $v(0,t) = u_1(t)$, $v(L,t) = u_2(t)$, $S_{11}(jw) \equiv S_{u_1u_1}(jw)$, $S_{22}(jw) \equiv S_{u_2u_2}(jw)$, $S_{12}(jw) \equiv S_{u_1u_2}(jw)$, and $S_{12}(-jw) \equiv S_{u_1u_2}(-jw) = S_{u_2u_1}(jw)$. Note that the equations given above are nonlinear in the variables A, B, C, and D; however, the explicit solution for these unknowns is not necessary because only the products AC, BC, AD, and BD appear in $S^{\tau}_{vv}(x,y,jw)$. Therefore, equations (5.67), (5.68), (5.69), and (5.70) are solved simultaneously for the products AC, BC, AD, and BD to obtain

$$AC = \{S_{11}(jw)\ \exp[(\bar{r}_2 + r_2)L] - S_{12}(-jw)\ \exp[r_2 L] - S_{12}(jw)\ \exp[\bar{r}_2 L] + S_{22}(jw)\} / N(jw) \tag{5.71}$$

$$BC = \{-S_{11}(jw)\ \exp[(\bar{r}_1 + r_2)L] + S_{12}(-jw)\ \exp[r_2 L] + S_{12}(jw)\ \exp[\bar{r}_2 L] - S_{22}(jw)\} / N(jw) \tag{5.72}$$

$$AD = \{-S_{11}(jw)\ \exp[(\bar{r}_2 + r_1)L] + S_{12}(-jw)\ \exp[r_1 L] + S_{12}(jw)\ \exp[\bar{r}_2 L] - S_{22}(jw)\} / N(jw) \tag{5.73}$$

$$BD = \{S_{11}(jw)\ \exp[(\bar{r}_1 + r_1)L] - S_{12}(-jw)\ \exp[r_1 L] - S_{12}(jw)\ \exp[\bar{r}_1 L] + S_{22}(jw)\} / N(jw) \tag{5.74}$$

where

$$N(jw) = \{\exp[\bar{r}_2 L] - \exp[\bar{r}_1 L]\} \cdot \{\exp[r_2 L] - \exp[r_1 L]\} \tag{5.75}$$

If the disturbances applied at x=0 and at x=L are uncorrelated, i.e., if $S_{12}(jw) = S_{12}(-jw) = 0$, then the coefficients are given by

$$AB' = \{S_{11}(jw)\ \exp[(\bar{r}_2 + r_2)L] + S_{22}(jw)\} / N(jw) \tag{5.76}$$

$$BC' = \{-S_{11}(jw)\ \exp[(\overline{r}_1 + r_2)L] - S_{22}(jw)\} / N(jw) \tag{5.77}$$

$$AD' = \{-S_{11}(jw)\ \exp[(\overline{r}_2 + r_1)L] - S_{22}(jw)\} / N(jw) \tag{5.78}$$

$$BD' = \{S_{11}(jw)\ \exp[(\overline{r}_1 + r_1)L] + S_{22}(jw)\} / N(jw) \tag{5.79}$$

where the prime is used to denote the coefficients for the uncorrelated case and where N(jw) is given be equation (5.75).

To obtain the distributed spectral-density function for the example considered in Section 5.2, the identities $r_1 = -(jwK)^{1/2}$, $r_2 = (jwK)^{1/2}$, $\overline{r}_1 = -(-jwK)^{1/2}$, and $\overline{r}_2 = (-jwK)^{1/2}$ are simply substituted into equation (5.65) and into the coefficient equations (5.71), (5.72), (5.73), (5.74), and (5.75).

In a similar manner, expressions for AC, BC, AD, and BD can be determined for boundary conditions of the second and third type and for combinations of the three types. Although the procedure is straightforward, the steps required to obtain the solution are somewhat tedious.

5.6 REMARKS

In this chapter, a theoretical basis is laid for the statistical analysis of class-II-type distributed-parameter systems which are time-invariant and which are excited by stationary random inputs. In essence, the concepts of correlation function analysis are extended to include distributed-parameter systems.

In Sections 5.2 and 5.3, the theory of the distributed autocorrelation function and its corresponding spectral-density function is developed, and it is shown that the distributed spectral-density function can be obtained from the characteristic equation and conjugate characteristic equation of the distributed-parameter plant.

In Section 5.4, three basic types of boundary conditions are considered; and in Section 5.5, the distributed spectral-density function is derived for a general, second-order, class-II-type distributed-parameter system with boundary conditions of the first type applied at the spatial boundaries. Hence, the distributed spectral-density function for a particular problem of this type can be obtained by substituting appropriately into the general equation.

If the distributed transfer function of the class-II-type system $G^t(x,s)$ is known, another procedure, different from that presented in this chapter, can be used to determine the spectral-density function. The two-model representation developed in this chapter also provides a convenient base from which to apply this method. Using the two-model representation, the distributed transfer function for each of the models is determined, i.e., the transfer functions $G^t(x,s)$ and $G^t(y,s)$. Then, following a procedure similar to that given in reference 30 for lumped-parameter systems, the distributed spectral-density function can be determined for class-II-type systems. For control systems which contain both lumped-parameter and distributed-parameter elements, the use of the transfer function method appears to be the more convenient procedure to

use. On the other hand, for general second-order systems, the method presented in this chapter is easier to apply. This is especially true if the system is subject to boundary conditions of the first type because the distributed spectral-density function can be obtained directly by using the results presented in Section 5.5. Also, for more general class-II-type systems, considerable effort may be required to determine the distributed transfer function.

The techniques developed here are of use in the statistical synthesis and optimization of class-II-type distributed-parameter systems. The minimization of averaged mean-square error can be effected by a procedure which is directly analogous to that which is used to minimize averaged integral-square error in the following chapter.

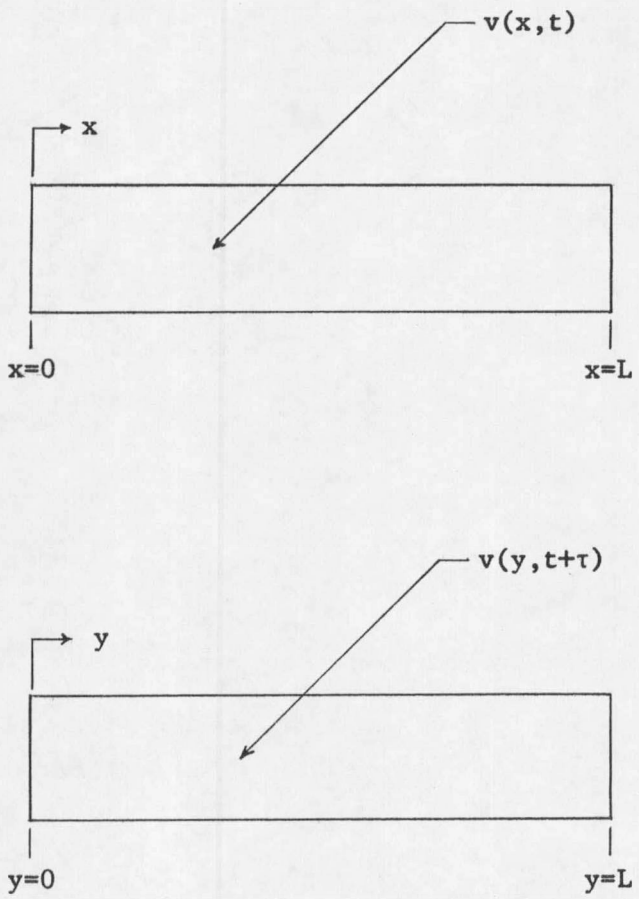


Figure 5-1. Two-model representation of a class-II-type distributed-parameter system.

CHAPTER 6

AVERAGED INTEGRAL-SQUARE ERROR AND OPTIMAL DESIGN

6.1 INTRODUCTION

The original work of Wiener (9,1) has led to minimum-integral-square-error design techniques and minimum-mean-square-error design techniques for several types of linear control systems: lumped-parameter control systems (30, 292); time-variant control systems (14, 80); and control systems enfolding time delay (30). The recent work of Pierre (226, 268) formalized the use of the Wiener-Hopf spectrum factorization technique to effect the minimum-mean-square-error design of class-I-type distributed-parameter systems, but this technique has not been developed previously for the design of class-II-type distributed-parameter systems. This chapter and Chapter 7 are therefore devoted to developing the use of the Wiener-Hopf technique for the design of class-II-type distributed-parameter systems.

In this chapter, the spatial-averaged-integral-square-error metric (introduced in Chapter 3) is used as a performance measure to obtain open-loop control laws for class-II-type distributed-parameter systems. Section 6.2 contains a detailed statement of the control problem and a derivation of a necessary condition for optimality. In Section 6.3, an approximate method for the solution of the control problem is formulated; and in Section 6.4, an example is presented to demonstrate the theory developed in the preceding sections.

6.2 THE OPTIMAL CONTROL PROBLEM

A class-II-type distributed-parmeter system is shown in Figure 6-1. The theory developed here applies to yet more general systems than that

shown, but an example will serve to illustrate the principles involved. In Figure 6-1, the fixed elements of the distributed-parameter plant are represented by the Laplace-transform transfer function $G_f^t(x,s)$; and the known transfer function $G_s(s)$ relates the constrained signal $v_s(t)$ to the control signal $u(t)$. The system error $e(x,t)$ is defined to be the difference between the desired distribution $v_d(x)$ and the actual distribution $v(x,t)$. The distributed-parameter plant is assumed to be defined on either a finite spatial domain $0 \leq x \leq L$ or a semi-infinite spatial domain $0 \leq x \leq \infty$. If the system is defined on a semi-infinite domain, however, it is assumed that control of the system is required on only a finite portion of the spatial domain, say, on the interval $0 \leq x \leq L$. In addition, for a system with a semi-infinite spatial domain, the control signal $u(t)$ is assumed to be applied at $x=0$; and for a system with a finite spatial domain, the control signal $u(t)$ is assumed to be applied at $x=0$ and at $x=L$.

The problem is to find the control signal $u(t)$ which minimizes the augmented performance measure J_D,

$$J_D = \frac{1}{L} \int_0^L \int_0^\infty [e(x,t)]^2 \, dt \, dx + \lambda \int_0^\infty [v_s(t)]^2 \, dt \qquad (6.1)$$

which is an augmented version of equation (3.8) where λ is a "cost factor" associated with the constrained signal $v_s(t)$. Equation (6.1) is equivalent to

$$J_D = \frac{1}{L} \int_0^L \int_0^\infty [v_d(x) - v(x,t)]^2 \, dt \, dx + \lambda \int_0^\infty [v_s(t)]^2 \, dt \qquad (6.2)$$

where $e(x,t) = v_d(x) - v(x,t)$. Note that the limits on the time integral in equation (6.2) are from 0 to ∞; therefore, in order to obtain a finite value of the performance measure, the system must be ε-maintainable for the class of desired distributions considered.

In this chapter and in Chapter 7, the two-sided Laplace transform is used. Herein, $G_s(s)$ and $G_f^t(x,s)$ represent two-sided Laplace-transform transfer functions for which the (two-sided) inverse Laplace transforms are assumed to be zero for $t<0$ and for which the inverse transforms are assumed to be stable real functions of time. The phrase stable real functions of time is defined by Pierre (226, 268) to mean that the Laplace transforms $G_s(s)$ and $G_f^t(x,s)$ exist, are real for real s, and are analytic at all points to the right of the imaginary axis (of the principal sheet) of the s plane, as well as on the imaginary axis.

It is important to bear in mind that $G_f^t(x,s)$ may be a multivalued function of s; the domain of which can be viewed as a Riemann surface. The principal sheet of such a surface is that sheet which contains the strip of convergence for the inversion process (where the strip of convergence of $G_f^t(x,s)$ may be a function of x). Again using the terminology formulated by Pierre (226, 268), the abbreviated phrase "analytic in the $\overline{\text{RHP}}$" is used throughout this work in place of the more cumbersome phrase "analytic in the right-half plane of the principal sheet of the

Riemann surface and on the imaginary axis of said sheet".

With the above-noted conditions in mind, Parseval's theorem is applied to equation (6.2) to obtain

$$J_D = \lim_{\varepsilon \to 0+} \frac{1}{2\pi j L} \int_0^L \int_{-j\infty}^{j\infty} \left[\frac{v_d(x)}{(s+\varepsilon)} - V^t(x,s)\right]\left[\frac{v_d(x)}{(-s+\varepsilon)} - V^t(x,-s)\right] ds\, dx + \frac{\lambda}{2\pi j} \int_{-j\infty}^{j\infty} V_s(s)\, V_s(-s)\, ds \tag{6.3}$$

where $V^t(x,s) = G_f^t(x,s)\, U(s)$ and $V_s(s) = G_s(s)\, U(s)$, as indicated in Figure 6-1. Substituting these identities for $V^t(x,s)$ and $V_s(s)$ into equation (6.3), the performance measure may be expressed as follows:

$$J_D = \lim_{\varepsilon \to 0+} \frac{1}{2\pi j L} \int_0^L \int_{-j\infty}^{j\infty} \left\{\left[\frac{v_d(x)}{(s+\varepsilon)} - G_f^t(x,s)\, U(s)\right] \cdot \left[\frac{v_d(x)}{(-s+\varepsilon)} - G_f^t(x,-s)\, U(-s)\right]\right\} ds\, dx + \frac{\lambda}{2\pi j} \int_{-j\infty}^{j\infty} G_s(s)\, U(s)\, G_s(-s)\, U(-s)\, ds \tag{6.4}$$

After performing the indicated multiplications and interchanging the order of integration in the first term on the right-hand side of equation (6.4), the following equation is obtained

$$J_D = \lim_{\varepsilon \to 0+} \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \{ [\frac{1}{L(s+\varepsilon)(-s+\varepsilon)} \int_0^L v_d^2(x)\, dx] +$$

$$U(-s)[\frac{-1}{L(s+\varepsilon)} \int_0^L v_d(x)\, G_f^t(x,-s)\, dx] +$$

$$U(s)[\frac{-1}{L(-s+\varepsilon)} \int_0^L v_d(x)\, G_f^t(x,s)\, dx] +$$

$$U(s)U(-s)[\frac{1}{L} \int_0^L G_f^t(x,s)\, G_f^t(x,-s)\, dx + \lambda\, G_s(s)\, G_s(-s)]\}\, ds \qquad (6.5)$$

Equation (6.5) is of a general form which is considered by Pierre (292); namely,

$$J_D = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} [A_4(s) + A_3(s)U(-s) + A_2(s)U(s) + A_1(s)U(s)U(-s)]\, ds \qquad (6.6)$$

where in this case,

$$A_1(s) = \frac{1}{L} \int_0^L G_f^t(x,s)\, G_f^t(x,-s)\, dx + \lambda\, G_s(s)\, G_s(-s)$$

$$A_2(s) = \lim_{\varepsilon \to 0+} \frac{-1}{L(-s+\varepsilon)} \int_0^L v_d(x)\, G_f^t(x,s)\, dx$$

$$A_3(s) = \lim_{\varepsilon \to 0+} \frac{-1}{L(s+\varepsilon)} \int_0^L v_d(x)\, G_f^t(x,-s)\, dx$$

and

$$A_4(s) = \lim_{\varepsilon \to 0+} \frac{1}{L(s+\varepsilon)(-s+\varepsilon)} \int_0^L v_d^2(x)\, dx$$

Note that for a specific problem, the transfer functions $G_f^t(x,s)$ and $G_s(s)$ and the desired distribution $v_d(x)$ will be known. Therefore, if the integrations with respect to x can be effected in the above equations, $A_1(s)$, $A_2(s)$, $A_3(s)$, and $A_4(s)$ will be explicit Laplace transforms which do not depend upon the spatial variable x.

The derivation of the necessary condition for the function $U^o(s)$ which minimizes the general equation (6.6) is given in Appendix B. In this derivation, it is assumed that at least one of the $A_2(s)$ and $A_3(s)$ is nonzero and that $L^{-1}\{U(s)\} = u(t) = 0$ for $t<0$. If these conditions are satisfied, it is shown that the optimal control signal $U^o(s)$ is given by

$$U^o(s) = \frac{-\left\{\dfrac{[A_2(-s) + A_3(s)]}{[A_1(s) + A_1(-s)]}\right\}_+}{[A_1(s) + A_1(-s)]^+} \qquad (6.7)$$

where

$$[A_1(s) + A_1(-s)] = 2A_1(s) = \frac{2}{L} \int_0^L G_f^t(x,s)\, G_f^t(x,-s)\, dx + 2\lambda G_s(s)\, G_s(-s)$$

and

$$[A_2(-s) + A_3(s)] = 2A_3(s) = \lim_{\varepsilon \to 0+} \frac{-2}{L(s+\varepsilon)} \int_0^L v_d(x)\, G_f^t(x,-s)\, dx$$

The + and - superscripts and subscripts appearing in equation (6.7) are interpreted as follows:

$$[A_1(s) + A_1(-s)] = [A_1(s) + A_1(-s)]^+ [A_1(s) + A_1(-s)]^- \tag{6.8}$$

and

$$\left\{\frac{[A_2(-s)+A_3(s)]}{[A_1(s)+A_1(-s)]^-}\right\} = \left\{\frac{[A_2(-s)+A_3(s)]}{[A_1(s)+A_1(-s)]^-}\right\}_+ + \left\{\frac{[A_2(-s)+A_3(s)]}{[A_1(s)+A_1(-s)]^-}\right\}_- \tag{6.9}$$

For $t<0$,

$$L^{-1}\{[A_1(s) + A_1(-s)]^+\} = 0$$

$$L^{-1}\{\, 1 \,/\, [A_1(s) + A_1(-s)]^+\} = 0$$

and

$$L^{-1}\{\left(\frac{[A_2(-s) + A_3(s)]}{[A_1(s) + A_1(-s)]^-}\right)_+\} = 0$$

For $t>0$,

$$L^{-1}\{[A_1(s) + A_1(-s)]^-\} = 0$$

$$L^{-1}\{ 1 / [A_1(s) + A_1(-s)]^-\} = 0$$

and

$$L^{-1}\{\left(\frac{[A_2(-s) + A_3(s)]}{[A_1(s) + A_1(-s)]^-}\right)_-\} = 0$$

The problem of finding $[A_1(s) + A_1(-s)]^+$ and $[A_1(s) + A_1(-s)]^-$ is called the spectrum-factorization problem, and the method used to obtain equation (6.7) is known as the Wiener-Hopf spectrum-factorization technique. When $[A_1(s) + A_1(-s)]$ is a rational fraction function, the process is straightforward (30, 292); for this case, $[A_1(s) + A_1(-s)]^+$ contains all of the poles and zeros of $[A_1(s) + A_1(-s)]$ which lie in the left-half of the s plane and $[A_1(s) + A_1(-s)]^-$ contains all of the poles and zeros of $[A_1(s) + A_1(-s)]$ which lie in the right-half of the s plane. When $[A_1(s) + A_1(-s)]$ contains nonrational terms, the Wiener-Hopf

spectrum-factorization technique is still valid (226, 268), but the spectrum-factorization problem is no longer straightforward. Because the transfer function $G_f^t(x,s)$ is, in general, a nonrational function of s, it is difficult to obtain the optimal control signal $U^o(s)$ from equation (6.7) directly. For this reason, a method for finding an approximation for $U^o(s)$ is formulated in the next section.

6.3 AN APPROXIMATE SOLUTION

In this section, an approximate solution of the optimal control problem is formulated which incorporates the use of rational fraction approximations of the nonrational terms appearing in $[A_1(s) + A_1(-s)]$ and $[A_2(-s) + A_3(s)]$. As indicated in Chapter 2, rational fraction approximations have been developed for several common forms of distributed-parameter characteristics: for $\exp[-(sD)^{1/2}]$ in (135, 257), for e^{-sT} in (7, 89, 91, 97, 117), for $(s)^{1/2}$ in (60, 103, 210, 248), and for [cosh Ts + B sinh Ts] in (254). For a specific design problem, the plant transfer function $G_f^t(x,s)$ and the desired distribution $v_d(x)$ are generally known, and these are used to determine $[A_1(s) + A_1(-s)]$ and $[A_2(-s) + A_3(s)]$, assuming that the integrations indicated below equation (6.7) can be effected. The procedure formulated here requires that the nonrational terms in $[A_1(s) + A_1(-s)]$ and $[A_2(-s) + A_3(s)]$ be approximated by rational fraction approximations. If the nonrational terms which appear are of a form for which approximations have already been developed or which appear as combinations of these forms, the rational fraction approximation can be obtained, in general, by direct substitution

of these known approximations for the nonrational terms which appear[1]. If the nonrational functions are not of this form, rational fraction approximations must be developed.

In order to obtain a good approximation for $U^o(s)$, the rational fraction approximation for the nonrational terms appearing in $[A_1(s) + A_1(-s)]$ and $[A_2(-s) + A_3(s)]$ must converge to the actual function along the Laplace transform inversion contour. Assuming that a suitable approximation can be obtained, the approximation for $U^o(s)$, denoted by $U_a^o(s)$, can be obtained as follows:

$$U_a^o(s) = \frac{-\left\{\dfrac{[A_2(-s) + A_3(s)]_a}{[A_1(s) + A_1(-s)]_a^-}\right\}+}{[A_1(s) + A_1(-s)]_a^+} \tag{6.10}$$

where $[A_1(s) + A_1(-s)]_a$ and $[A_2(-s) + A_3(s)]_a$ represent the rational fraction approximations of $[A_1(s) + A_1(-s)]$ and $[A_2(-s) + A_3(s)]$, respectively. Because $[A_1(s) + A_1(-s)]_a$ and $[A_2(-s) + A_3(s)]_a$ are ratios of polynomials in s, the spectrum factorization required to obtain $U_a^o(s)$ is straightforward, as explained previously, but the following details merit attention.

After all nonrational terms have been approximated, the functions $[A_1(s) + A_1(-s)]_a$ and $[A_2(-s) + A_3(s)]_a$ must be arranged as ratios of

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[1]That this is not as simply done as one might at first suspect is illustrated in the example of Section 6.4.

polynomials is s. Even this seemingly simple operation is quite tedious when performed by hand because of high-order rational fraction approximations which are generally used for accuracy. For this reason, a computer program was devloped to perform polynomial multiplications, additions, and scalar multiplications on a digital computer (the Fortran program is given in Appendix D).

Also, because $[A_1(s) + A_1(-s)]_a$ and $[A_2(-s) + A_3(s)]_a$ are usually of relatively high order, the bulk of spectrum factorization is effected on a digital computer; that is, the roots of $[A_1(s) + A_1(-s)]_a$ and $[A_2(-s) + A_3(s)]_a$ are determined using a digital computer routine. For the example given in the next section, program number 7.0.032 in the Montana State University Computing Center Library was used. This program uses a Newton process to find the roots, both real and complex, of a real polynomial of nineteenth order or less. By using a program for multiplying polynomials and a program for finding the roots of real polynomials, almost all of the steps required to find $U_a^o(s)$ are performed on a digital computer. The required design time and effort are thereby reduced considerably, as is demonstrated in the next section.

6.4 ILLUSTRATIVE EXAMPLE

Referring to Figure 6-1, consider as an example a system for which the following data are given: (1) $G_f^t(x,s) = \exp[-(s)^{1/2}x]$, (2) $G_s(s) = s$, and (3) $v_d(x) = B$, where B is a specified constant. The problem is to find the control signal u(t) which minimizes the performance measure given by equation (6.1).

A common origin of the transfer function $G_f^t(x,s) = \exp[-(s)^{1/2}x]$ is the one dimensional heat-flow equation

$$\frac{\partial^2 v(x,t)}{\partial x^2} = K \frac{\partial v(x,t)}{\partial t}, \quad 0 \leq x \leq \infty, \quad 0 \leq t \leq \infty \tag{6.11}$$

with the initial condition

$$v(x,0) = 0, \quad 0 \leq x \leq \infty \tag{6.12}$$

and with the boundary conditions

$$v(0,t) = u(t) \quad \text{and} \quad v(\infty,t) = 0, \quad 0 \leq t \leq \infty \tag{6.13}$$

The transfer function for this example can be obtained by applying method 1 given in Section 4.2.1; that is, Laplace transforming equation (6.11) with respect to t to obtain

$$\frac{d^2 V^t(x,s)}{dx^2} = K[s\, V^t(x,s) - v(x,0)] \tag{6.14}$$

By making use of the initial condition (6.12) and the boundary conditions (6.13), the ordinary differential equation (6.14) is solved to obtain the desired result,

$$\frac{V^t(x,s)}{U(s)} = G_f^t(x,s) = \exp[-(Ks)^{1/2}x], \qquad 0 \leq x \leq \infty \tag{6.15}$$

where for this example K=1. Also, it is assumed that close control of the plant is required only on the finite interval $0 \leq x \leq 1$, i.e., L=1.

In Section 6.2, it is shown that the performance measure (6.1) can be written in the general form (6.6). Using this result, the given quantities for this example are substituted appropriately into the equations for $A_i(s)$, i=1,2,3,4, which are given below equation (6.6). It follows that

$$\begin{aligned} A_1(s) &= \int_0^1 \exp[-(s)^{1/2}x]\, \exp[-(-s)^{1/2}x]\, dx + \lambda\,(s)(-s) \\ &= \frac{1 - \exp[-(s)^{1/2} - (-s)^{1/2}]}{[(s)^{1/2} + (-s)^{1/2}]} + \lambda(-s^2) \end{aligned} \tag{6.16}$$

$$\begin{aligned} A_2(s) &= \lim_{\varepsilon \to 0+} \frac{-B}{(-s+\varepsilon)} \int_0^1 \exp[-(s)^{1/2}x]\, dx \\ &= \lim_{\varepsilon \to 0+} \frac{B}{(-s+\varepsilon)} \frac{(\exp[-(s)^{1/2}] - 1)}{(s)^{1/2}} \end{aligned} \tag{6.17}$$

$$A_3(s) = \lim_{\varepsilon \to 0+} \frac{-B}{(s+\varepsilon)} \int_0^1 \exp[-(-s)^{1/2}x]\, dx$$

$$= \lim_{\varepsilon \to 0+} \frac{B}{(s+\varepsilon)} \frac{(\exp[-(-s)^{1/2}] - 1)}{(-s)^{1/2}} \qquad (6.18)$$

and

$$A_4(s) = \lim_{\varepsilon \to 0+} \frac{1}{(s+\varepsilon)(-s+\varepsilon)} \int_0^1 B^2\, dx$$

$$= \lim_{\varepsilon \to 0+} \frac{B^2}{(s+\varepsilon)(-s+\varepsilon)} \qquad (6.19)$$

Note that $A_1(-s) = A_1(s)$ and $A_2(-s) = A_3(s)$; therefore, $[A_1(s) + A_1(-s)]$ and $[A_2(-s) + A_3(s)]$ are given by the following equations.

$$[A_1(s)+A_1(-s)] = 2A_1(s) = 2\{\frac{1 - \exp[-(s)^{1/2} - (-s)^{1/2}]}{(s)^{1/2} + (-s)^{1/2}} + \lambda(-s^2)\} \qquad (6.20)$$

and

$$[A_2(-s)+A_3(s)] = 2A_3(s) = \lim_{\varepsilon \to 0+} \frac{2B}{(s+\varepsilon)} \frac{(\exp[-(-s)^{1/2}] - 1)}{(-s)^{1/2}} \qquad (6.21)$$

Next, the nonrational terms appearing in equations (6.20) and (6.21) are approximated by rational fraction approximations. For this example,

Pierre's approximations for $\exp[-(sD)^{1/2}]$ given in (135, 257) and Kopal's approximations for $(s)^{1/2}$ given in (60) are used --- see Appendix C. Thus, for this example, the following approximations are used:

$$\exp[-(s)^{1/2}] \cong \frac{P_2(s)}{P_3(s)} = \frac{3s^2+ 7s + 1}{3s^3+ 13s^2+ 12s + 1} \tag{6.22}$$

$$\frac{1}{(s)^{1/2}} = \frac{(s)^{1/2}}{s} \cong \frac{N_2(s)}{s\, M_1(s)} = \frac{s^2+ 6s + 1}{s(4 + 4s)} \tag{6.23}$$

and

$$\frac{1}{(s)^{1/2}+ (-s)^{1/2}} \cong \frac{N_2(s)N_2(-s)}{sN_2(-s)M_1(s) - sN_2(s)M_1(-s)} = \frac{s^4- 34s^2+ 1}{8s^4- 40s^2} \tag{6.24}$$

Using the approximations given by equations (6.22) and (6.24), the rational fraction approximation for the nonrational terms in $[A_1(s) + A_1(-s)]$ is of the following form:

$$\frac{1 - \exp[-(s)^{1/2}-(-s)^{1/2}]}{(s)^{1/2} + (-s)^{1/2}} \cong \frac{[P_3(s)P_3(-s) - P_2(s)P_2(-s)]N_2(s)N_2(-s)}{P_3(s)P_3(-s)[sN_2(-s)M_1(s) - sN_2(s)M_1(-s)]} \tag{6.25}$$

where $P_3(s)$, $P_2(s)$, $N_2(s)$, and $M_1(s)$ are the polynomials given in equations (6.22) and (6.23). The computer program given in Appendix D is used to obtain the approximation in the form of a ratio of polynomials.

After factoring $(-s^2)$ out of the numerator and the denominator, the following rational fraction approximation is obtained.

$$\frac{1 - \exp[-(s)^{1/2} - (-s)^{1/2}]}{(s)^{1/2} + (-s)^{1/2}} \cong \frac{9s^8 - 394s^6 + 3076s^4 - 2638s^2 + 75}{72s^8 - 1136s^6 + 4824s^4 - 4728s^2 + 40} \tag{6.26}$$

In Figure 6-2, the exact function and its approximation are plotted along the inversion contour, s=jw. Note that they are real functions of w.

Similarly, the approximation for the nonrational terms in $[A_2(-s) + A_3(s)]$ is found to be

$$\frac{\exp[-(s)^{1/2}] - 1}{(-s)^{1/2}} \cong \frac{[P_2(-s) - P_3(-s)]N_2(-s)}{(-s)M_1(-s)P_3(-s)} \tag{6.27}$$

$$\cong \frac{-3s^4 + 28s^3 - 68s^2 + 40s - 5}{12s^4 - 64s^3 + 100s^2 - 52s + 4}$$

The magnitude and phase of the exact function and its approximation for s=jw are plotted in Figures 6-3 and 6-4, respectively.

Notice that in both cases, the approximations are quite poor for low values of w. Consequently, if the approximations are used --- without modification --- the approximation to the control signal $U^o(t)$ which results will not give the desired steady-state value at the output of the system. A logical assertion would be that higher order

approximations should be used. However, it was found that higher order approximations did indeed give a slightly better fit for higher values of w, but in turn, they increased the error for small values of w. Thus, in order to correct this discrepancy, a correction term is added to the approximations given by equations (6.26) and (6.27). A correction term is chosen which makes the exact function and its approximation equal at w=0 and which decreases the error for low values of w but does not affect the approximation for high values of w.

Because the nonrational terms in $[A_1(s) + A_1(-s)]_{s=jw}$ and their approximations are real functions of w, the correction term is also chosen to be a real function of w. The simplest correction term which satisfies the above-noted conditions for the approximation (6.26) is

$$\frac{-1}{C_1(s)} = \frac{-1}{(-as^2 + 1.143)} \tag{6.28}$$

where a is a positive constant which is yet to be determined. The constant term 1.143 is determined on the basis that the approximation is to equal the exact value at w=0. In order to determine the value of a, the modified approximation,

$$\frac{1 - \exp[-(s)^{1/2} - (-s)^{1/2}]}{(s)^{1/2} + (-s)^{1/2}} \cong \left\{\frac{9s^8 - 394s^6 + 3076s^4 - 2638s^2 + 75}{72s^8 - 1136s^6 + 4824s^4 - 4728s^2 + 40} - \frac{1}{-as^2 + 1.143}\right\} \tag{6.29}$$

with s=jw is plotted in Figure 6-2 for a = 100, 200, 300, and 400. A comparison of the results shown in Figure 6-2 indicates that a=200 yields the most improvement of the approximation; therefore, the modified approximation, equation (6.29), with a=200 is used to approximate the nonrational terms in $[A_1(s) + A_1(-s)]$.

Following a similar procedure, the correction term for the approximation of the nonrational terms in $[A_2(-s) + A_3(s)]$ is selected to be

$$\frac{1}{C_2(s)} = \frac{1}{-bs + 4} \tag{6.30}$$

where b is a positive constant. Note that the pole of this correction term is in the right-half of the s plane because the poles and zeros of the approximation given in equation (6.27) are all located in the right-half of the s plane. In order to determine the value of b, the magnitude and phase of the modified approximation,

$$\frac{\exp[-(-s)^{1/2}] - 1}{(-s)^{1/2}} = \frac{-3s^4 + 28s^3 - 68s^2 + 40s - 5}{12s^4 - 64s^3 + 100s^2 - 52s + 4} + \frac{1}{-bs + 4} \tag{6.31}$$

for s=jw and for b = 100, 150, 200, and 300 were calculated and then plotted in Figures 6-3 and 6-4, respectively. After comparing the results, b=100 was selected.

By substituting appropriately the modified approximation (6.29) with a=200 into equation (6.20), the approximation for $[A_1(s) + A_1(-s)]$,

denoted by $[A_1(s) + A_1(-s)]_a$, is of the following form:

$$[A_1(s) + A_1(-s)]_a = \frac{2 \sum_{i=0}^{6} (-1)^i C_{2i} s^{2i}}{\sum_{i=0}^{5} (-1)^i D_{2i} s^{2i}} \tag{6.32}$$

where the C_{2i}'s and D_{2i}'s are given in Table 6-4. Because the numerator of $[A_1(s) + A_1(-s)]_a$ depends upon the value of the cost factor λ, the coefficients C_{2i}, i=1,2,...,6, are given in Table 6-4 for λ = 0, 0.5, 1, 2, and 3.

The rational fraction approximation of $[A_2(-s) + A_3(s)]$, denoted by $[A_2(-s) + A_3(s)]_a$, is obtained by substituting appropriately the modified approximation (6.31) with b=100 into equation (6.21) which yields

$$[A_2(-s) + A_3(s)]_a = \lim_{\varepsilon \to 0+} \frac{2B}{(s+\varepsilon)} \frac{Q_1(s)}{Q_2(s)} \tag{6.33}$$

where

$$Q_1(s) = 300s^5 - 2800s^4 + 6848s^3 - 4172s^2 + 608s - 16$$

and

$$Q_2(s) = -1200s^5 + 6448s^4 - 10256s^3 + 5600s^2 - 608s + 16$$

With the approximations $[A_1(s) + A_1(-s)]_a$ and $[A_2(-s) + A_3(s)]_a$ determined, the approximation for $U^o(s)$, denoted by $U_a^o(s)$, can be obtained using equation (6.10). In order to carry out the spectrum factorization, the roots of the numerator and denominator of $[A_1(s) + A_1(-s)]_a$ and $[A_2(-s) + A_3(s)]_a$ must be found. This is accomplished directly on a digital computer[2]. Table 6-5 contains the roots of $[A_2(-s) + A_3(s)]_a$, and the roots of $[A_1(s) + A_1(-s)]_a$ are given in Table 6-6 for $\lambda = 0, 0.5, 1, 2,$ and 3.

To illustrate the procedure, $U_a^o(s)$ for $\lambda=1$ is determined in the text which follows. Referring to Table 6-6, it is seen that

$$[A_1(s) + A_1(-s)]_a^+ = \frac{Q_3(s)}{Q_4(s)} \tag{6.34}$$

where

$$Q_3(s) = (s+0.06410)(s+0.1494)(s+0.6677)(s+1.437)(s+2.008)(s+3.056)$$

and

$$Q_4(s) = (s+0.07630)(s+0.09180)(s+1.178)(s+2.236)(s+3.063)$$

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[2]For this example, program number 7.0.032 in the Montana State University Computing Center Library was used.

and

$$[A_1(s) + A_1(-s)]_a^- = \frac{Q_5(s)}{Q_6(s)} \tag{6.35}$$

where

$$Q_5(s) = 2(14400)(s-0.06410)(s-0.1494)(s-0.6677)(s-1.437)(s-2.008)(s-3.056)$$

and

$$Q_6(s) = (14400)(s-0.07630)(s-0.09180)(s-1.178)(s-2.236)(s-3.063)$$

where the constant multiplier 14400 which appears in the numerator and the denominator of equation (6.35) is the coefficient of the highest power of s appearing in both the numerator and the denominator of the right-hand member of equation (6.32). Although these constant terms are equal for $\lambda=1$, this is not the case for the other values of λ. Note also that the constant terms could be carried equally well with $[A_1(s) + A_1(-s)]_a^+$, just so long as they are included in one or the other.

From Table 6-5, $[A_2(-s) + A_3(s)]_a$ is obtained in factored form:

$$[A_2(-s) + A_3(s)]_a = \lim_{\varepsilon \to 0+} \frac{2B}{(s+\varepsilon)} \frac{Q_7(s)}{Q_8(s)} \tag{6.36}$$

where

$$Q_7(s) = (300)(s-0.03367)(s-0.1629)(s-0.6165)(s-2.719)(s-5.802)$$

and

$$Q_8(s) = (-1200)(s-0.0400)(s-0.09238)(s-1.000)(s-1.178)(s-3.063)$$

and where the constant multipliers (300) and (-1200) are the coefficients of the highest power of s appearing in $Q_1(s)$ and $Q_2(s)$, respectively, of equation (6.33).

Note that $([A_2(-s) + A_3(s)]_a / [A_1(s) + A_1(-s)]_a^-)$ has only one pole in the left-half of the s plane, i.e., at $s = -\varepsilon$. Therefore, the residue of the pole at $s = -\varepsilon$ as $\varepsilon \to 0+$ is evaluated to obtain

$$\left\{\frac{[A_2(-s) + A_3(s)]_a}{[A_1(s) + A_1(-s)]_a^-}\right\}_+ = -\frac{B}{s} \tag{6.37}$$

After substituting the right-hand sides of equations (6.34) and (6.37) into equation (6.10), the approximation for $U^o(s)$ when $\lambda=1$ is found to be

$$U_a^o(s) = \frac{B}{s}\frac{Q_4(s)}{Q_3(s)} \tag{6.38}$$

where $Q_3(s)$ and $Q_4(s)$ are defined below equation (6.34). In Table 6-7,

the $U_a^o(s)$ are given for $\lambda = 0$, 0.5, 1, 2, and 3.

To obtain $u_a^o(t)$, the partial fraction expansion of equation (6.38) is effected, and the inverse Laplace transform of the result is found to be

$$u_a^o(t) = B[\delta_{-1}(t) - 0.0928 \exp(-0.06410t) + 0.574 \exp(-0.1494t) - 1.27 \exp(-0.6677t) - 0.373 \exp(-1.437t) + 0.127 \exp(-2.008t) + 0.0008 \exp(-3.056t)] \quad (6.39)$$

where $\delta_{-1}(t)$ represents the unit step function. In Table 6-8, the poles and the associated residues of $U_a^o(s)$ are given for $\lambda = 0$, 0.5, 1, 2, and 3. And in Figure 6-5, the same values of λ are used in plotting the control signals $u_a^o(t)$ as a function of time.

6.5 REMARKS

In this chapter, complemented by Appendices B, C, and D, a procedure is advanced for minimizing the spatial-averaged-integral-square error of class-II-type distributed-parameter systems. An optimal open-loop control signal is derived by use of the Wiener-Hopf spectrum-factorization technique. In Section 6.2, a detailed statement of the problem and the derivation of necessary conditions for optimality are given; in Section 6.3, a procedure of rational fraction approximation is advanced in order to realize the optimal control law; and in Section 6.4, the technique is applied to an illustrative example.

In working the example of Section 6.4, the author was mildly surprised to find that certain composite rational fraction approximations (equations 6.26 and 6.27) were quite poor near s=0, this occurring in spite of the fact that the component approximations used for the nonrational terms $(s)^{1/2}$ and $\exp[-(s)^{1/2}]$ were not so afflicted. Upon reflecting on this, however, it was observed that the nonrational members of equations (6.26) and (6.27) are evaluated by use of L'Hospital's rule at s=0, because numerators and denominators of both approach zero as $s \to 0$. Hence, slight deviations in the component approximations were magnified in the composite approximations at s=0, and corrections terms had to be added to the initial approximations in order to obtain accurate results.

The theory developed in this chapter has a logical counterpart in the minimization of averaged mean-square error. The distributed spectral-density functions of Chapter 5 are of use in this regard.

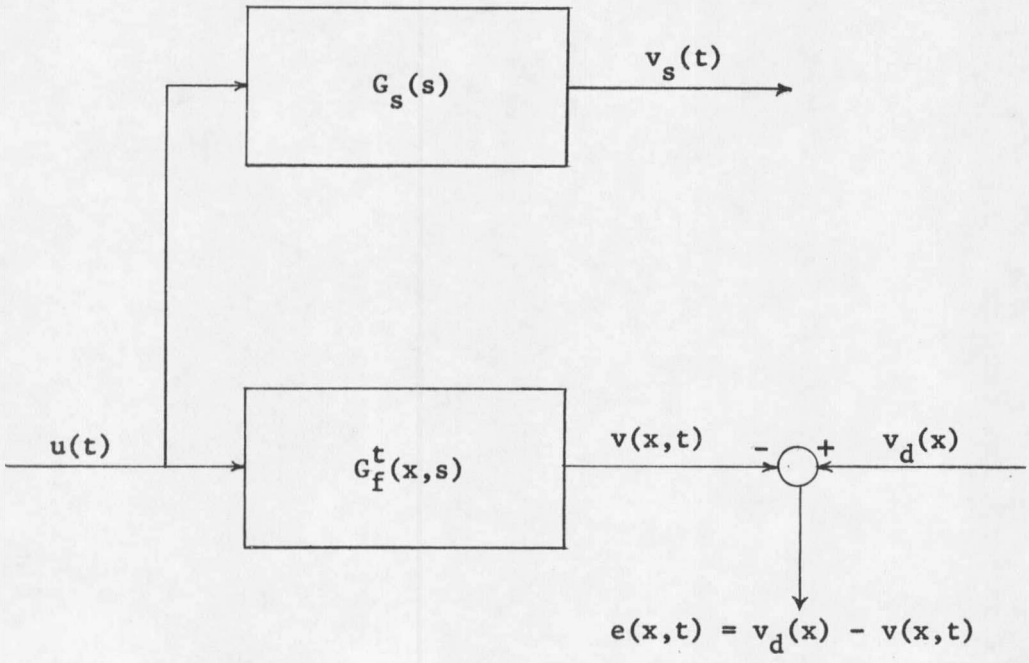


Figure 6-1. Block diagram representation of a class-II-type distributed-parameter system.

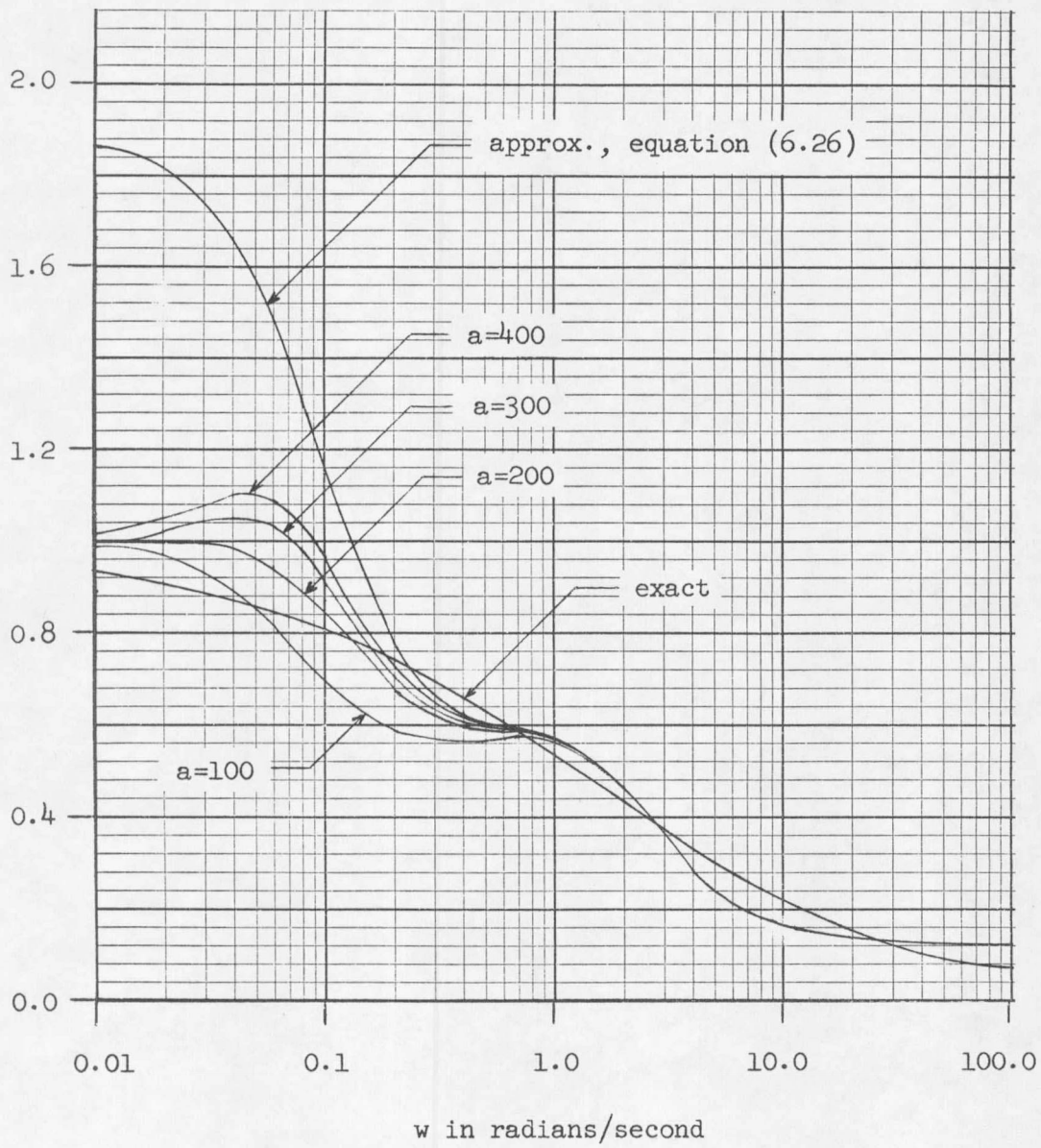


Figure 6-2. The frequency response of the nonrational terms and the rational fraction approximations, equations (6.26) and (6.29).

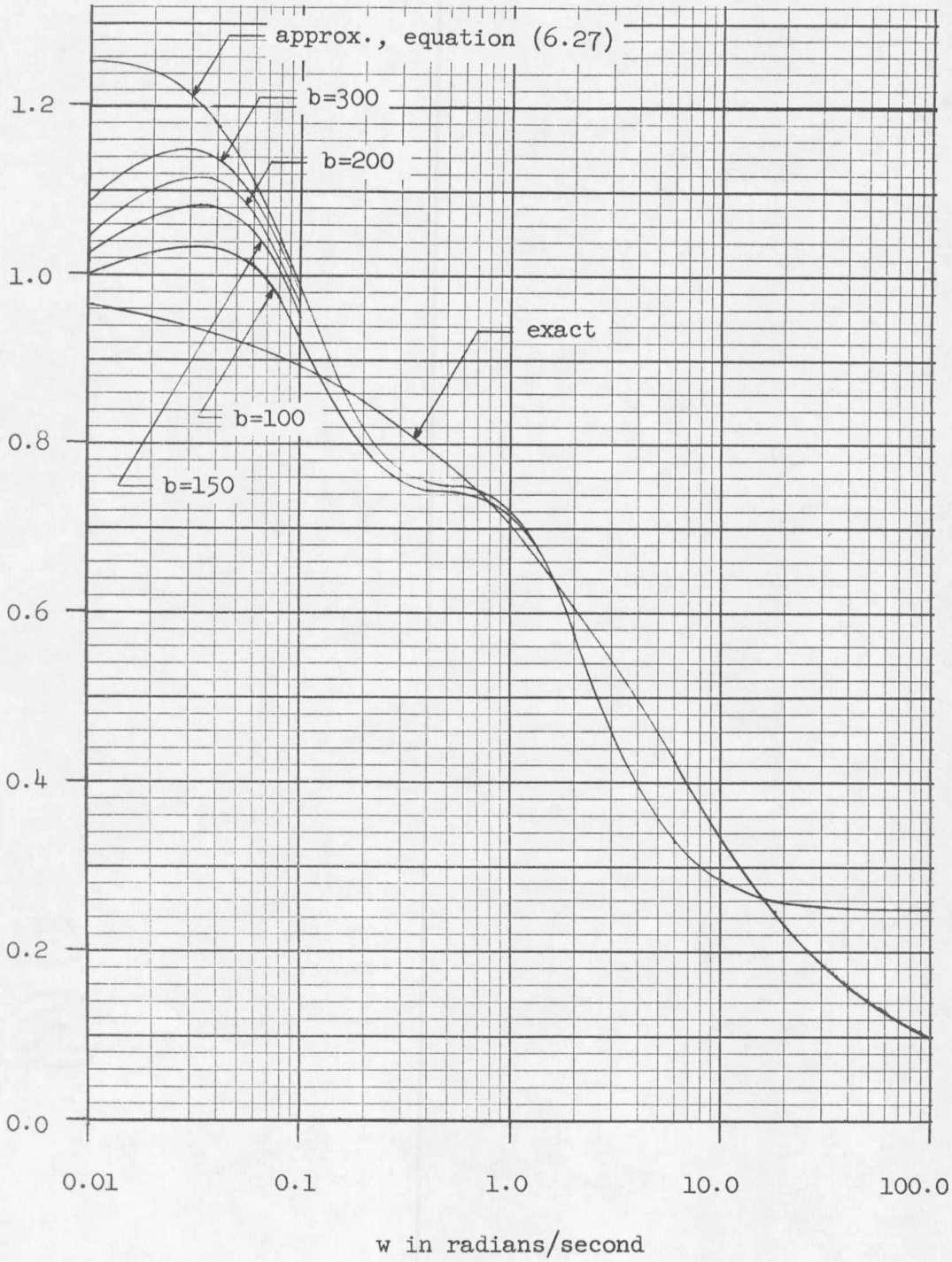


Figure 6-3. The magnitude of the nonrational terms and the rational fraction approximations, equations (6.27) and (6.31), with s=jw.

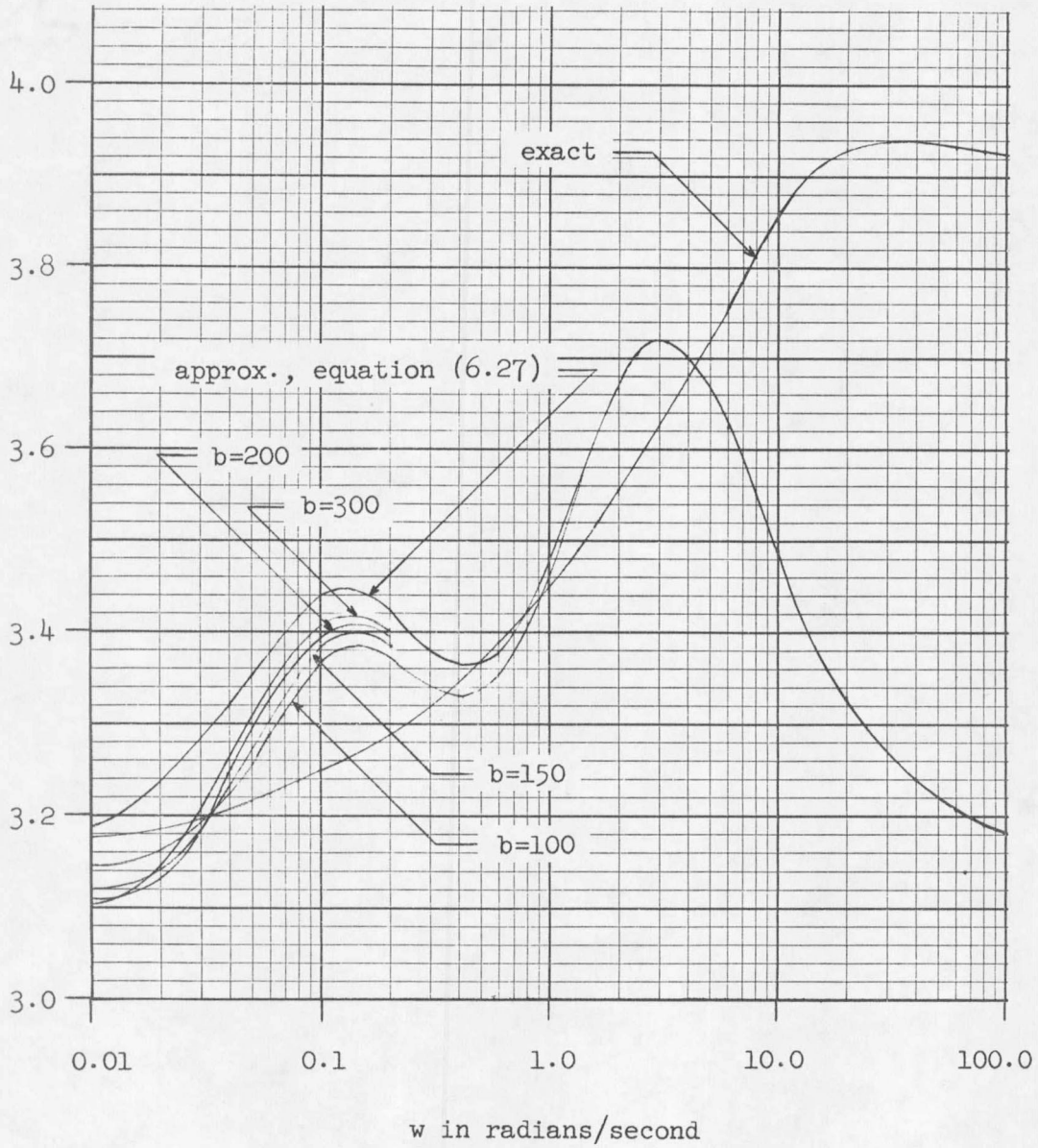


Figure 6-4. The phase (in radians) of the nonrational terms and the rational fraction approximations, equations (6.27) and (6.31), with s=jw.

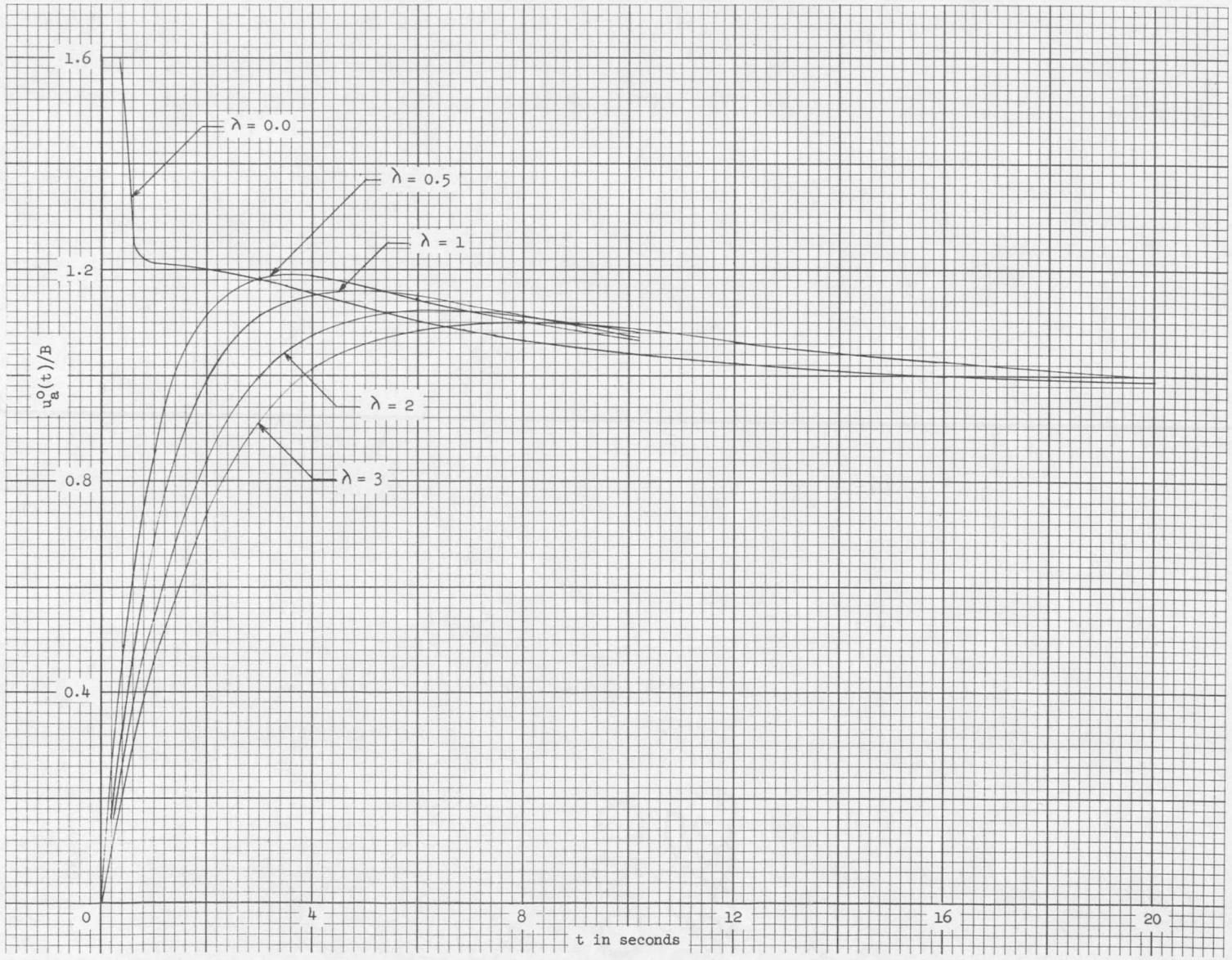


Figure 6-5. Time response of the control signal $u_a^o(t)$ for $\lambda = 0$, 0.5, 1, 2 and 3.

Table 6-1. The frequency response of the nonrational terms and the rational fraction approximations, equations (6.26) and (6.29).

w	equation (6.26)		equation (6.29)			
	exact	approx.	a=100	a=200	a=300	a=400
.01	.932508	1.859612	.992310	.999767	1.007097	1.014304
.02	.906346	1.815519	.970211	.997858	1.023753	1.048059
.04	.871020	1.665267	.897807	.981740	1.049124	1.104414
.06	.845176	1.480933	.815597	.944165	1.031091	1.093786
.08	.824200	1.305891	.745038	.893179	.979413	1.035839
.10	.806309	1.158658	.692023	.840491	.917287	.964219
.20	.741102	.782853	.588414	.673480	.706767	.724520
.40	.660935	.624484	.566152	.594312	.604136	.609134
.60	.607615	.595446	.568523	.581775	.586284	.588557
.80	.567419	.584494	.569143	.576751	.579317	.580605
1.00	.535197	.573333	.563446	.568361	.570012	.570840
2.00	.432332	.466503	.464010	.465255	.465670	.465878
4.00	.332656	.283964	.283339	.283651	.283756	.283808
6.00	.279639	.208369	.208091	.208230	.208276	.208299
8.00	.245421	.175019	.174863	.174941	.174967	.174980
10.00	.221052	.158028	.157928	.157978	.157995	.158003
20.00	.157830	.133621	.133596	.133608	.133613	.133615
40.00	.111788	.127179	.127173	.127176	.127177	.127177
60.00	.091285	.125970	.125967	.125969	.125969	.125969
80.00	.079056	.125546	.125544	.125545	.125545	.125545
100.00	.070710	.125349	.125348	.125349	.125349	.125349

Table 6-2. The magnitude of the nonrational terms and the rational fraction approximations, equations (6.27) and (6.31), with s=jw.

	equation (6.27)		equation (6.31)			
w	exact	approx.	b=100	b=150	b=200	b=300
.01	.965263	1.244911	1.008088	1.024403	1.044077	1.084941
.02	.951230	1.230216	1.024589	1.064404	1.099418	1.147488
.04	.931731	1.178791	1.038343	1.087064	1.115309	1.142888
.06	.917042	1.112541	1.014738	1.053496	1.072518	1.089614
.08	.904839	1.045898	.973857	1.003390	1.016963	1.028940
.10	.894223	.986572	.931066	.953966	.964205	.973228
.20	.853764	.819650	.798140	.806852	.810670	.814091
.40	.799673	.752701	.745187	.748123	.749432	.750633
.60	.760532	.747577	.742791	.744581	.745404	.746178
.80	.729055	.739638	.735647	.737086	.737765	.738417
1.00	.702432	.721110	.717470	.718752	.719367	.719965
2.00	.607372	.574584	.572006	.572881	.573312	.573740
4.00	.495802	.399669	.398339	.398786	.399008	.399229
6.00	.425854	.331725	.330948	.331209	.331338	.331468
8.00	.375991	.300453	.299954	.300121	.300204	.300287
10.00	.338137	.283913	.283569	.283684	.283741	.283799
20.00	.233069	.259124	.259028	.259060	.259076	.259092
40.00	.158553	.252327	.252302	.252311	.252315	.252319
60.00	.128726	.251038	.251027	.251031	.251032	.251034
80.00	.111603	.250584	.250578	.250580	.250581	.250582
100.00	.099940	.250374	.250370	.250371	.250372	.250373

Table 6-3. The phase (in radians) of the nonrational terms and the rational fraction approximations, equations (6.27) and (6.31), with s=jw.

w	equation (6.27)		equation (6.31)			
	exact	approx.	b=100	b=150	b=200	b=300
.01	3.176531	3.191239	3.144526	3.121666	3.104978	3.087905
.02	3.190759	3.238833	3.160566	3.141065	3.136534	3.145125
.04	3.210637	3.321372	3.224301	3.229463	3.241085	3.260680
.06	3.225695	3.381612	3.289052	3.305296	3.319189	3.336970
.08	3.238259	3.419748	3.335007	3.354670	3.368077	3.383657
.10	3.249229	3.440139	3.362461	3.382867	3.395401	3.409290
.20	3.291373	3.425435	3.370955	3.387974	3.396974	3.406247
.40	3.348533	3.364552	3.332887	3.343247	3.348509	3.353819
.60	3.390455	3.377621	3.356249	3.363295	3.366851	3.370425
.80	3.424489	3.424979	3.408942	3.414242	3.416910	3.419590
1.00	3.453484	3.480885	3.467947	3.472227	3.474379	3.476540
2.00	3.558303	3.667087	3.659616	3.662094	3.663337	3.664584
4.00	3.682383	3.695086	3.689780	3.691545	3.692429	3.693313
6.00	3.758792	3.622126	3.617676	3.619158	3.619900	3.620641
8.00	3.811014	3.549162	3.545344	3.546616	3.547252	3.547889
10.00	3.848271	3.489894	3.486584	3.487688	3.488239	3.488791
20.00	3.927830	3.334189	3.332296	3.332927	3.333242	3.333558
40.00	3.938055	3.240631	3.239645	3.239974	3.240138	3.240302
60.00	3.930016	3.207972	3.207310	3.207530	3.207641	3.207751
80.00	3.926916	3.191471	3.190973	3.191139	3.191222	3.191305
100.00	3.926388	3.181530	3.181131	3.181264	3.181330	3.181397

Table 6-4. The coefficients of the numerator and the denominator of $[A_1(s) + A_1(-s)]_a$, equation (6.32).

denominator	numerator $\lambda=0$	numerator $\lambda=0.5$
	C_{12} = 0.00	C_{12} = 7,200.00
D_{10} = 14,400.00	C_{10} = 1,800.00	C_{10} = 115,441.00
D_8 = 227,282.29	C_8 = 78,738.28	C_8 = 561,787.00
D_6 = 966,098.44	C_6 = 614,514.34	C_6 = 1,090,070.00
D_4 = 951,113.83	C_4 = 526,291.86	C_4 = 532,993.00
D_2 = 13,404.10	C_2 = 13,287.23	C_2 = 13,310.00
D_0 = 45.72	C_0 = 45.72	C_0 = 45.72

numerator $\lambda=1$	numerator $\lambda=2$	numerator $\lambda=3$
C_{12} = 14,400.00	C_{12} = 28,800.00	C_{12} = 43,200.00
C_{10} = 229,082.29	C_{10} = 456,364.58	C_{10} = 683,646.87
C_8 = 1,044,836.70	C_8 = 2,010,935.00	C_8 = 2,977,033.50
C_6 = 1,565,628.10	C_6 = 2,516,741.90	C_6 = 3,467,855.70
C_4 = 539,695.96	C_4 = 533,100.06	C_4 = 566,504.17
C_2 = 13,332.95	C_2 = 13,378.67	C_2 = 13,424.39
C_0 = 45.72	C_0 = 45.72	C_0 = 45.72

Table 6-5. The roots of the numerator and the denominator of $[A_2(-s) + A_3(s)]_a$, equation (6.33).

roots of numerator	roots of denominator
s = 0.03367	s = $-\varepsilon$, $\varepsilon \to 0+$
s = 0.1629	s = 0.04000
s = 0.6165	s = 0.09238
s = 2.719	s = 1.000
s = 5.802	s = 1.178
	s = 3.063

Table 6-6. The roots of the numerator and the denominator of $[A_1(s) + A_1(-s)]_a$, equation (6.32).

roots of denominator	roots of numerator, λ=0	roots of numerator, λ=0.5
s = ±0.07630	s = ±0.06399	s = ±0.06407
s = ±0.09180	s = ±0.1477	s = ±0.1485
s = ±1.178	s = ±0.9725	s = ±0.8209
s = ±2.236	s = ±2.973	s = +1.783 ± j0.4059
s = ±3.063	s = ±5.816	s = -1.783 ± j0.4059
		s = ±3.050

roots of numerator, λ=1	roots of numerator, λ=2	roots of numerator, λ=3
s = ±0.06410	s = ±0.06400	s = ±0.06375
s = ±0.1494	s = ±0.1514	s = ±0.1538
s = ±0.6677	s = ±0.4926	s = ±0.4018
s = ±1.437	s = ±1.271	s = ±1.234
s = ±2.008	s = ±2.145	s = ±2.178
s = ±3.056	s = ±3.059	s = ±3.061

Table 6-7. The control signal $U_a^o(s)$ for $\lambda = 0$, 0.5, 1, 2, and 3.

$\lambda = 0$

$$U_a^o(s) = \frac{2.86B(s+0.07630)(s+0.09180)(s+1.178)(s+2.236)(s+3.063)}{s(s+0.06399)(s+0.1477)(s+0.9725)(s+2.973)(s+5.816)}$$

$\lambda = 0.5$

$$U_a^o(s) = \frac{1.414B(s+0.07630)(s+0.09180)(s+1.178)(s+2.236)(s+3.063)}{s(s+0.06407)(s+0.1485)(s+0.8209)(s^2+3.566s+3.344)(s+3.050)}$$

$\lambda = 1$

$$U_a^o(s) = \frac{B(s+0.07630)(s+0.09180)(s+1.178)(s+2.236)(s+3.063)}{s(s+0.06410)(s+0.1494)(s+0.6677)(s+1.437)(s+2.008)(s+3.056)}$$

$\lambda = 2$

$$U_a^o(s) = \frac{0.707B(s+0.07630)(s+0.09180)(s+1.178)(s+2.236)(s+3.063)}{s(s+0.06400)(s+0.1514)(s+0.4926)(s+1.271)(s+2.145)(s+3.059)}$$

$\lambda = 3$

$$U_a^o(s) = \frac{0.577B(s+0.07630)(s+0.09180)(s+1.178)(s+2.236)(s+3.063)}{s(s+0.06375)(s+0.1538)(s+0.4018)(s+1.234)(s+2.178)(s+3.061)}$$

Table 6-8. The residues of the poles of $U_a^o(s)$ for λ = 0, 0.5, 1, 2, and 3.

λ=0		λ=0.5	
pole	residue	pole	residue
s	B	s	B
(s+0.06399)	-0.109B	(s+0.06407)	-0.0912B
(s+0.1477)	+0.433B	(s+0.1485)	+0.523B
(s+0.9725)	-0.172B	(s+0.8209)	-0.856B
(s+2.973)	+0.0201B	(s+1.783+j0.4059)	(-0.290-j0.301)B
(s+5.816)	+1.63B	(s+1.783-j0.4059)	(-0.290+j0.301)B
		(s+3.050)	+0.003B

λ=1		λ=2		λ=3	
pole	residue	pole	residue	pole	residue
s	B	s	B	s	B
(s+0.06410)	-0.0928B	(s+0.06400)	-0.099B	(s+0.06375)	-0.105B
(s+0.1494)	+0.574B	(s+0.1514)	+0.672B	(s+0.1538)	+0.787B
(s+0.6677)	-1.27B	(s+0.4926)	-1.52B	(s+0.4018)	-1.66B
(s+1.437)	-0.373B	(s+1.271)	-0.071B	(s+1.234)	-0.035B
(s+2.008)	+0.127B	(s+2.145)	+0.021B	(s+2.178)	+0.01B
(s+3.056)	+0.0008B	(s+3.059)	0.000B	(s+3.061)	0.000B

Table 6-9. Time response of the control signal $u_a^o(t)$ for $\lambda = 0$, 0.5, 1, 2 and 3.

t	$u_a^o(t)/B$ $\lambda = 0$	$u_a^o(t)/B$ $\lambda = 0.5$	$u_a^o(t)/B$ $\lambda = 1$	$u_a^o(t)/B$ $\lambda = 2$	$u_a^o(t)/B$ $\lambda = 3$
.20	1.69121	.25364	.16009	.13560	.10679
.40	1.35035	.45959	.32552	.25435	.20698
.60	1.24837	.62435	.46592	.36041	.29821
.80	1.21947	.75513	.58472	.45495	.38117
1.00	1.21205	.85828	.68500	.53908	.45654
2.00	1.20165	1.11816	.99189	.83736	.74016
3.00	1.17851	1.18240	1.11466	.99760	.91124
4.00	1.15165	1.18490	1.15553	1.07870	1.01094
5.00	1.12611	1.16783	1.15983	1.11462	1.06553
6.00	1.10342	1.14554	1.14842	1.12519	1.09191
7.00	1.08382	1.12328	1.13113	1.12205	1.10102
8.00	1.06712	1.10292	1.11258	1.11201	1.09990
9.00	1.05298	1.08500	1.09487	1.09901	1.09307
10.00	1.04108	1.06954	1.07883	1.08526	1.08338
12.00	1.02274	1.04513	1.05255	1.05972	1.06176
14.00	1.01003	1.02771	1.03328	1.03920	1.04211
16.00	1.00141	1.01545	1.01955	1.02386	1.02640
18.00	.99572	1.00696	1.00994	1.01285	1.01466
20.00	.99213	1.00120	1.00335	1.00518	1.00626
30.00	.98912	.99261	.99296	.99273	.99221
40.00	.99273	.99429	.99430	.99394	.99344
50.00	.99582	.99658	.99655	.99631	.99601
60.00	.99771	.99810	.99808	.99795	.99777
70.00	.99877	.99898	.99896	.99889	.99880
80.00	.99935	.99945	.99944	.99941	.99936
90.00	.99965	.99971	.99970	.99968	.99966
100.00	.99981	.99984	.99984	.99983	.99982

CHAPTER 7

FEEDBACK CONTROL OF CLASS-II-TYPE DISTRIBUTED-PARAMETER SYSTEMS

7.1 INTRODUCTION

In this chapter, a technique is developed for the minimum-integral-square-error design of class-II-type, distributed-parameter, feedback control systems. This technique incorporates the use of the class-I representations (Chapter 4) of class-II-type distributed-parameter systems. In using class-I representations for feedback control, the distributed state variable is measured at specific points of the spatial domain, and the signals so obtained are used as feedback signals. After the system has been implemented in a feedback configuration, the Wiener-Hopf spectrum-factorization technique is used to obtain a compensator which results in minimal-integral-square error.

In Section 7.2, block-diagram models for class-II-type, distributed-parameter, feedback control systems are developed; and in Section 7.3, an optimal control problem and an approximate method of solution are presented for the feedback system. The example in Section 7.4 illustrates a typical application of the theory. In this example, the distributed-parameter plant considered in Section 6.4 is again used; hence, the results obtained using the methods developed in this chapter can be compared with the results obtained in Chapter 6.

7.2 BLOCK-DIAGRAM MODELS OF CLASS-II-TYPE, DISTRIBUTED-PARAMETER, FEEDBACK CONTROL SYSTEMS

When the value of the state variable is measurable at equally spaced points along the one-dimensional spatial domain of the system, the class-I representations developed in Chapter 4 can be used to obtain a

feedback model of a class-II-type, distributed-parameter control system. The feedback model which results is shown in Figure 7-1 for the parallel-form class-I representation and in Figure 7-2 for the series-form class-I representation. For the models shown in these figures, the output of the system $v(x,t)$ is measured at three points, $x=\Delta$, 2Δ, and 3Δ, where Δ is a positive constant. If necessary, more values of the state variable may be used. The constants W_1, W_2, and W_3 are used to weight the measured values, and $G_c(s)$ is the transfer function of the lumped-parameter compensator.

Note that only one pick-off point is all that may be required --- e.g., the optimal system obtained in the example of the preceding chapter could be placed in feedback form by designing a compensator which has an error input signal (the desired value minus the signal obtained at a specific point within the range $0 < x < L$) and which has an optimal output signal (the control law which was obtained in the example). Even in this case, however, it should be evident that the use of several feedback signals would lead to better overall results, especially in regard to sensitivity of the response as a function of parameter changes. Also, the incorporation of the W_i weighting factors allows the designer to penalize those errors which he considers to be most critical. Finally, if enough equally-spaced pick-off points are used, certain integrations of the preceding chapter --- those of equations (6.16), (6.17) and (6.18), for example --- are effectively replaced by summations.

The feedback models shown in Figures 7-1 and 7-2 can be used to represent control systems which contain class-II-type distributed-

parameter plants of the following two forms: one, a plant defined on a semi-infinite spatial domain, $0 \leq x \leq \infty$, which has an input r(t) applied at x=0; or two, a plant defined on a finite spatial domain, $0 \leq x \leq L$, which has an input r(t) applied both at x=0 and at x=L. For systems of the first form, the state variable is measured at equally spaced points along a finite portion of the spatial domain, say, on the interval $0 \leq x \leq L$. The class-I representation of systems of the second form is discussed in detail in Section 4.2.2.

These feedback models can be reduced to the equivalent, but more convenient form shown in Figure 7-3. For the system in Figure 7-3, the equivalent transfer function $G_f^t(\Delta,s)$ for the parallel-form model (Figure 7-1) is given by

$$G_f^t(\Delta,s) = \frac{C(s)}{U(s)} = \frac{\sum_{i=1}^{3} W_i G_{ip}^t(\Delta,s)}{\sum_{i=1}^{3} W_i} \tag{7.1}$$

and, the equivalent transfer function for the series-form model (Figure 7-2) is given by

$$G_f^t(\Delta,s) = \frac{C(s)}{U(s)} = \frac{\sum_{i=1}^{3} W_i \prod_{k=1}^{i} G_{ks}^t(\Delta,s)}{\sum_{i=1}^{3} W_i} \tag{7.2}$$

where $\prod_{k=1}^{i} G_{ks}^{t}(\Delta,s) = G_{ip}^{t}(\Delta,s)$. If the state variable is measured at n equally-spaced points instead of 3, equations (7.1) and (7.2) are yet valid if the upper indices on the summations are changed to n.

This model may be further reduced to yield the equivalent open-loop model shown in Figure 7-4, wherein

$$W_c(s) = \frac{G_c(s)}{1 + G_c(s)\ G_f^t(\Delta,s)} \tag{7.3}$$

7.3 THE OPTIMAL CONTROL PROBLEM AND ITS SOLUTION

7.3.1 Problem Formulation

Consider a system of the form shown in Figure 7-5. To obtain this representation, the class-II-type, distributed-parameter, feedback control system of the form shown in Figure 7-1 or 7-2 is reduced to the equivalent open-loop model developed in the preceding section. For the control problem considered here, the things assumed given are the input r(t), the desired value of the feedback signal $c_d(t)$, and the transfer function $G_s(s)$. The problem is to find the compensator $W_{cm}(s)$ --- from which the compensator $G_c(s)$ for the feedback system can be obtained --- which minimizes the augmented performance measure shown below

$$J = \int_0^\infty [e(t)]^2\ dt + \lambda \int_0^\infty [c_s(t)]^2\ dt \tag{7.4}$$

where λ is a cost factor associated with the constraint on $c_s(t) = L^{-1}\{G_s(s)\ U(s)\}$. For the system considered here, the error e(t) is equal to $[c_d(t) - c(t)]$; therefore, equation (7.4) can be written as follows:

$$J = \int_0^\infty [c_d(t) - c(t)]^2\ dt + \lambda \int_0^\infty [c_s(t)]^2\ dt \qquad (7.5)$$

7.3.2 The Necessary Condition for Optimality

As in Chapter 6, the two-sided Laplace transform is used. The functions $G_s(s)$ and $G_f^t(\Delta,s)$ represent Laplace-transform transfer functions for which the (two-sided) inverse Laplace transforms are assumed to be zero for $t<0$ and for which the inverse transforms are assumed to be stable real functions of time[1]. In addition, the time functions $[c_d(t) - c(t)]$ and $c_s(t)$ are assumed to be two-sided Laplace transformable functions whose strip of convergence includes the jw axis in the s plane.

With the above-noted conditions in mind, Parseval's theorem is applied to equation (7.5) to obtain

$$J = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} [C_d(s) - C(s)][C_d(-s) - C(-s)]\ ds + \frac{\lambda}{2\pi j} \int_{-j\infty}^{j\infty} C_s(s)\ C_s(-s)\ ds \qquad (7.6)$$

[1]See Section 6.2 for more detail.

where $C(s) = W_c(s)\, G_f^t(\Delta,s)\, R(s)$ and $C_s(s) = W_c(s)\, G_s(s)\, R(s)$. After appropriately substituting these identities for $C(s)$ and $C_s(s)$ into equation (7.6), it follows that

$$J = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} [C_d(s) - W_c(s)G_f^t(\Delta,s)R(s)][C_d(-s) - W_c(-s)G_f^t(\Delta,-s)R(-s)]\, ds + \frac{\lambda}{2\pi j} \int_{-j\infty}^{j\infty} W_c(s)G_s(s)R(s)W_c(-s)G_s(-s)R(-s)\, ds \tag{7.7}$$

which can be written as

$$J = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \{C_d(s)C_d(-s) + W_c(-s)[-C_d(s)G_f^t(\Delta,-s)R(-s)] + W_c(s)[-C_d(-s)G_f^t(\Delta,s)R(s)] + W_c(s)W_c(-s)[\{G_f^t(\Delta,s)G_f^t(\Delta,-s) + \lambda G_s(s)G_s(-s)\}R(s)R(-s)]\}\, ds \tag{7.8}$$

Equation (7.8) is of a general form which is considered by Pierre (292); namely,

$$J = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} [A_4(s) + A_3(s)W_c(-s) + A_2(s)W_c(s) + A_1(s)W_c(s)W_c(-s)]\, ds \tag{7.9}$$

where in this case,

$$A_1(s) = R(s)R(-s)[G_f^t(\Delta,s)G_f^t(\Delta,-s) + \lambda G_s(s)G_s(-s)]$$

$$A_2(s) = -C_d(-s)G_f^t(\Delta,s)R(s)$$

$$A_3(s) = -C_d(s)G_f^t(\Delta,-s)R(-s)$$

$$A_4(s) = C_d(s)C_d(-s)$$

If at least one of the $A_2(s)$ and $A_3(s)$ is nonzero and if $L^{-1}\{W_c(s)\} = w_c(t) = 0$ for $t<0$, it is shown in Appendix B that the compensator $W_{cm}(s)$ which minimizes equation (7.9) can be obtained from the following relationship:

$$W_{cm}(s) = \frac{-\left\{\dfrac{[A_2(-s) + A_3(s)]}{[A_1(s) + A_1(-s)]^-}\right\}^+}{[A_1(s) + A_1(-s)]^+} \qquad (7.10)$$

where

$$[A_1(s) + A_1(-s)] = 2A_1(s)$$

$$= 2R(s)R(-s)[G_f^t(\Delta,s)G_f^t(\Delta,-s) + \lambda G_s(s)G_s(-s)]$$

$$[A_2(-s) + A_3(s)] = 2A_3(s) = -2C_d(s)G_f^t(\Delta,-s)R(-s)$$

and where the notation used in equation (7.10) is discussed in Section 6.2.

7.3.3 Solution by Rational Fraction Approximations

Because the transfer function $G_f^t(\Delta,s)$ is a nonrational function of s, the spectrum factorization required to obtain $W_{cm}(s)$ can be performed more easily by analytical and numerical techniques when accurate rational fraction approximations of the nonrational functions are used. When used, these high-order rational fraction approximations must converge to the actual function along the Laplace transform inversion contour.

Let $G_a(s)$ denote a high-order rational fraction approximation of $G_f^t(\Delta,s)$; an approximation to $W_{cm}(s)$, which is denoted by $W_{cm}^a(s)$, can be obtained from the following equation

$$W_{cm}^a(s) = \frac{-\left\{\dfrac{[A_2(-s) + A_3(s)]_a}{[A_1(s) + A_1(-s)]_a^-}\right\}^+}{[A_1(s) + A_1(-s)]_a^+} \qquad (7.11)$$

where

$$[A_1(s) + A_1(-s)]_a = 2R(s)R(-s)[G_a(s)G_a(-s) + \lambda G_s(s)G_s(-s)]$$

$$[A_2(-s) + A_3(s)]_a = -2C_d(s)G_a(-s)R(-s)$$

both of which are rational fraction functions. Hence, the spectrum factorization required to obtain $W_{cm}^a(s)$ is straightforward; that is, $[A_1(s) + A_1(-s)]_a^+$ contains all of the poles and zeros of $[A_1(s) + A_1(-s)]_a$ which lie in the left-half of the s plane whereas $[A_1(s) + A_1(-s)]_a^-$ contains all of the poles and zeros of $[A_1(s) + A_1(-s)]_a$ which lie in the right-half of the s plane. With the aid of a digital computer, the compensator $W_{cm}^a(s)$ --- even when high-order rational fraction approximations are used --- can be determined with relative ease. A computer program, such as the one given in Appendix D, can be used to perform the polynomial manipulations required to obtain $[A_1(s) + A_1(-s)]_a$ and $[A_2(-s) + A_3(s)]_a$ as ratios of polynomials, and the roots of the resulting rational fraction functions can be determined using a standard program for finding the roots of real polynomials.

Once the compensator $W_{cm}^a(s)$ is determined, the compensator for the feedback system $G_c^a(s)$ can be obtained by the use of the following relationship

$$G_c^a(s) = \frac{W_{cm}^a(s)}{1 - G_a(s)W_{cm}^a(s)} \tag{7.12}$$

where $G_a(s)$ designates the rational fraction approximation of the nonrational transfer function $G_f^t(\Delta,s)$.

7.4 ILLUSTRATIVE EXAMPLE

To illustrate the design procedure developed in this chapter, again consider the distributed-parameter plant characterized by the transfer function $G_f^t(\Delta,s) = \exp[-(s)^{1/2}x]$, $0 \leq x \leq \infty$ [2]. As in Section 6.4, the system is to be controlled on the finite interval $0 \leq x \leq 1$ by means of a control signal u(t) applied at the spatial boundary x=0. For this example, the control system is to operate in a feedback configuration.

In order to implement this system in a feedback configuration, the value of the state variable v(x,t) is measured at three equally-spaced points along the spatial domain of the system: at x=1/3, 2/3, and 1, i.e., Δ=1/3. Then, these measured values of the state variable are weighted, summed and averaged to yield the feedback signal c(t) (see Figures 7-1 and 7-2). Using the method developed in Chapter 4, the transfer functions for the parallel-form class-I representations (Figure 7-1) are obtained by substituting $i\Delta=i/3$ for x in $G_f^t(x,s)$:

$$G_{ip}^t(\Delta,s) = \exp[-(s)^{1/2}i/3] , \quad i = 1, 2, 3 \tag{7.13}$$

And by using equations (4.3) and (4.4), the transfer functions for the series-form class-I representations (Figure 7-2) can be obtained from the parallel-form transfer functions. For this case, they are found to be

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[2] A common origin of this transfer function is discussed in Section 6.4.

$$G_{is}^{t}(\Delta,s) = \exp[-(s)^{1/2}/3] , \quad i = 1, 2, 3 \tag{7.14}$$

Next, these feedback models are reduced to an equivalent open-loop model of the form shown in Figure 7-4. By substituting the parallel-form transfer functions given in equation (7.13) into equation (7.1), the transfer function for the equivalent open-loop model is

$$G_{f}^{t}(\Delta,s) = \frac{1}{\sum_{i=1}^{3} W_i} \{W_1 \exp[-(s)^{1/2}/3] + W_2 \exp[-(s)^{1/2}\ 2/3] + W_3 \exp[-(s)^{1/2}]\} \tag{7.15}$$

And, using equation (7.2) and the series-form transfer functions given by equation (7.14), the equivalent transfer function is obtained in the following form

$$G_{f}^{t}(\Delta,s) = \frac{1}{\sum_{i=1}^{3} W_i} \{W_1 \exp[-(s)^{1/2}/3] + W_2(\exp[-(s)^{1/2}/3])^2 + W_3(\exp[-(s)^{1/2}/3])^3\} \tag{7.16}$$

The two forms of the equivalent transfer function, given by equations (7.15) and (7.16), are used in the text which follows to illustrate two methods for obtaining a rational fraction approximation for $G_{f}^{t}(\Delta,s)$.

Now that the equivalent transfer function $G_f^t(\Delta,s)$ has been derived, consider the system shown in Figure 7-5, for which the following data are given: (1) $G_s(s) = s$; (2) $r(t) = B\ \delta_{-1}(t)$, where B is a positive constant; and (3) $c_d(t) = r(t) = B\ \delta_{-1}(t)$.

The problem is to find the $W_{cm}(s)$ which minimizes the augmented performance measure given by equation (7.4). Once this optimal compensator $W_{cm}(s)$ is determined, the optimal compensator $G_c(s)$ for the feedback system can be formed by the use of the relationship given by equation (7.12).

In Section 7.3.2, it was shown that the performance measure (7.4) can be written in the general form (7.9). Using this result, the given quantities for this example are substituted appropriately into the equations for $A_i(s)$, i=1,2,3,4, given below equation (7.9) to obtain

$$A_1(s) = \lim_{\varepsilon\to 0+} \frac{B^2}{(s+\varepsilon)(-s+\varepsilon)} [G_f^t(\Delta,s)G_f^t(\Delta,-s) + \lambda(-s^2)] \tag{7.17}$$

$$A_2(s) = \lim_{\varepsilon\to 0+} \frac{-B^2}{(s+\varepsilon)(-s+\varepsilon)} G_f^t(\Delta,s) \tag{7.18}$$

$$A_3(s) = \lim_{\varepsilon\to 0+} \frac{-B^2}{(s+\varepsilon)(-s+\varepsilon)} G_f^t(\Delta,-s) \tag{7.19}$$

$$A_4(s) = \lim_{\varepsilon\to 0+} \frac{B^2}{(s+\varepsilon)(-s+\varepsilon)} \tag{7.20}$$

from which it follows that

$$[A_1(s) + A_1(-s)] = \lim_{\varepsilon \to 0+} \frac{2B^2}{(s+\varepsilon)(-s+\varepsilon)} [G_f^t(\Delta,s)G_f^t(\Delta,-s) + \lambda(-s^2)] \qquad (7.21)$$

and

$$[A_2(-s) + A_3(s)] = \lim_{\varepsilon \to 0+} \frac{-2B^2}{(s+\varepsilon)(-s+\varepsilon)} G_f^t(\Delta,-s) \qquad (7.22)$$

where $G_f^t(\Delta,s)$ is given by equations (7.15) and (7.16).

Because $[A_1(s) + A_1(-s)]$ and $[A_2(-s) + A_3(s)]$ contain the nonrational transfer function $G_f^t(\Delta,s)$, an approximation for the compensator $W_{cm}(s)$ is determined by using the methods presented in Section 7.3.3. In order to determine $W_{cm}^a(s)$, the transcendental transfer function $G_f^t(\Delta,s)$ is replaced by a rational fraction approximation. Two distinct ways of doing this are presented below, one of which proves to be considerably better than the other. Both methods make use of the following rational fraction approximations developed by Pierre (135, 257):

$$\exp[-(s/9)^{1/2}] \cong \frac{P_2(s/9)}{P_3(s/9)} = \frac{27s^2 + 567s + 729}{3s^3 + 117s^2 + 972s + 729} \qquad (7.23)$$

$$\exp[-(4s/9)^{1/2}] \cong \frac{P_2(4s/9)}{P_3(4s/9)} = \frac{432s^2 + 2268s + 729}{192s^3 + 1872s^2 + 3888s + 729} \qquad (7.24)$$

and

$$\exp[-(s)^{1/2}] \cong \frac{P_2(s)}{P_3(s)} = \frac{3s^2+ 7s + 1}{3s^3+ 13s^2+ 12s + 1} \qquad (7.25)$$

A brief discussion of these approximations is given in Appendix C. The first approximation to $G_f^t(\Delta,s)$, designated $G_{ap}(s)$, is obtained by substituting the approximations given by equations (7.23), (7.24), and (7.25) into equation (7.15). Letting $W_1= W_2= W_3= 1$ (in order to compare results with those of the example in Chapter 6) and using the digital computer program given in Appendix D, the following approximation is obtained

$$G_{ap}(s) = \frac{Q_1(s)}{Q_2(s)} \qquad (7.26)$$

where

$$Q_1(s) = 0.211680s^8+ 8.22780s^7+ 103.261s^6+ 573.070s^5+ 1545.09s^4+ 2020.80s^3+ 1189.50s^2+ 256.272s + 15.9432$$

and

$$Q_2(s) = 0.0518400s^9+ 2.75184s^8+ 48.7166s^7+ 390.374s^6+ 1564.18s^5+ 3240.72s^4+ 3384.55s^3+ 1639.59s^2+ 297.606s + 15.9432$$

The second approximation, denoted by $G_{as}(s)$, can be obtained by substituting the approximation given by equation (7.23) into equation (7.16). The rational fraction approximations $G_{as}(s)$ and $G_{ap}(s)$ are of the same form, but the coefficients of the numerator and denominator polynomials are different.

For comparison, the rational fraction approximations $G_{ap}(s)G_{ap}(-s)$ and $G_{as}(s)G_{as}(-s)$ are determined and are found to be of the following form

$$G_a(s)G_a(-s) = \frac{\sum_{i=1}^{8} (-1)^i M_{2i} s^{2i}}{\sum_{i=1}^{9} (-1)^i N_{2i} s^{2i}} \qquad (7.27)$$

where the coefficients M_{2i} and N_{2i} are given in Table 7-1 for both approximations. In order to check the approximations, the values of $G_f^t(\Delta,s)G_f^t(\Delta,-s)$, $G_{ap}(s)G_{ap}(-s)$ and $G_{as}(s)G_{as}(-s)$ are calculated along the inversion contour s=jw (see Table 7-2), and the results are plotted in Figure 7-6.

As can be seen in Figure 7-6, $G_{ap}(s)G_{ap}(-s)$ is the better approximation and is therefore used in the following; in fact, it is a good approximation over a relatively wide frequency range. Also, the approximation and the exact function are each equal to unity as s approaches zero, and consequently the approximation need not be modified as was necessary in the example of Chapter 6.

Replacing $G_f^t(\Delta,s)$ by $G_{ap}(s)$ in equation (7.22), it follows that

$$[A_2(-s) + A_3(s)]_a = \lim_{\varepsilon \to 0+} \frac{-2B^2}{(s+\varepsilon)(-s+\varepsilon)} G_{ap}(-s) \tag{7.28}$$

where $G_{ap}(-s)$ is obtained from equation (7.26) by replacing s by -s. And, the rational fraction approximation of $[A_1(s) + A_1(-s)]$ is of the following form:

$$[A_1(s) + A_1(-s)]_a = \lim_{\varepsilon \to 0+} \frac{2B^2}{(s+\varepsilon)(-s+\varepsilon)} \frac{\sum_{i=0}^{10} (-1)^i C_{2i} s^{2i}}{\sum_{i=0}^{9} (-1)^i D_{2i} s^{2i}} \tag{7.29}$$

where the coefficients C_{2i} and D_{2i} are given in Table 7-3. Because the coefficients C_{2i} depend upon the cost factor λ, they are given in Table 7-3 for λ = 0, 0.5, 1, 2, and 3.

The roots of these approximations are determined on a digital computer[3]. Because the computer routine used to find the roots was written for polynomials of nineteenth order or less, the roots of the numerator of $[A_1(s) + A_1(-s)]_a$, which is a twentieth order polynomial in this example, are determined by letting $y=s^2$ and then solving for the roots of the tenth order polynomial in y. The roots of the polynomial

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[3]Program number 7.0.032 in the Montana State University Computing Center was used.

in s are then determined by taking the square roots of the roots obtained for the polynomial in y. The roots of the numerator and the denominator of $[A_2(-s) + A_3(s)]_a$ and $[A_1(s) + A_1(-s)]_a$ are given in Tables 7-4 and 7-5, respectively.

To illustrate the procedure, the details of the solution for the compensator $W^a_{cm}(s)$ and for the corresponding compensator $G^a_c(s)$ are now given for $\lambda=2$. First, the spectrum factorization of $[A_1(s) + A_1(-s)]_a$ is carried out. Referring to Table 7-5, note that the pole-zero pairs near $s = \pm 3, \pm 7, \pm 10$, and ± 27 are very close together. Because they are close together and because they lie relatively far from the origin, they will have little effect upon the response of the compensated system. Therefore, these pole-zero pairs are canceled with the result that

$$[A_1(s) + A_1(-s)]^+_a \cong \lim_{\varepsilon \to 0+} \frac{1}{(s+\varepsilon)} \frac{Q_3(s)}{Q_4(s)} \tag{7.30}$$

where

$$Q_3(s) = (s+0.1052)(s+0.2579)(s+0.6325)(s+0.6814)(s+1.144)(s+2.640)$$

and

$$Q_4(s) = (s+0.09238)(s+0.2079)(s+0.8314)(s+1.178)(s+2.651)$$

Likewise

$$[A_1(s) + A_1(-s)]_a^- \cong \lim_{\varepsilon \to 0+} \frac{2B^2(5.37477 \times 10^{-5})}{(-s+\varepsilon)(2.68738 \times 10^{-5})} \frac{Q_5(s)}{Q_6(s)} \tag{7.31}$$

where

$$Q_5(s) = (s-0.1052)(s-0.2579)(s-0.6325)(s-0.6814)(s-1.144)(s-2.640)$$

and

$$Q_6(s) = (s-0.09238)(s-0.2079)(s-0.8314)(s-1.178)(s-2.651)$$

The constants, (5.37477×10^{-5}) and (2.68738×10^{-5}), are the coefficients of the highest powers of s appearing in the numerator and the denominator of $[A_1(s) + A_1(-s)]_a$, respectively.

Referring to equation (7.28) and Table 7-4, it can be seen that

$$[A_2(-s) + A_3(s)]_a = \lim_{\varepsilon \to 0+} \frac{-2B^2(0.211680)}{(s+\varepsilon)(-s+\varepsilon)(0.0518400)} \frac{Q_7(s)}{Q_8(s)} \tag{7.32}$$

where $Q_7(s)$ and $Q_8(s)$ are eighth and ninth order polynomials[4], respectively, which have only right-half plane roots. The constants (0.211680) and (0.0518400) are the coefficients of the highest powers of s appearing

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[4] $Q_7(s)$ and $Q_8(s)$ can be obtained in factored form from Table 7-4.

in the numerator and the denominator of $[A_2(-s) + A_3(s)]_a$, respectively.

Notice that $\{[A_2(-s) + A_3(s)]_a / [A_1(s) + A_1(-s)]_a^-\}$ has only one pole in the left-half of the s plane: at $s= -\varepsilon$ as $\varepsilon \to 0+$. The residue of the pole at $s= -\varepsilon$ as $\varepsilon \to 0+$ is evaluated to obtain

$$\left\{\frac{[A_2(-s) + A_3(s)]_a}{[A_1(s) + A_1(-s)]_a^-}\right\}_+ = \frac{-0.707}{s} \tag{7.33}$$

After substituting the right-hand sides of equations (7.30) and (7.33) into equation (7.11), the compensator $W_{cm}^a(s)$ for $\lambda=2$ is found to be

$$W_{cm}^a(s) \cong \frac{0.707(s+0.09238)(s+0.2079)(s+0.8314)(s+1.178)(s+2.651)}{(s+0.1052)(s+0.2579)(s+0.6325)(s+0.6814)(s+1.144)(s+2.640)} \tag{7.34}$$

The compensator $W_{cm}^a(s)$ is given in Table 7-6 for several additional values of λ. Note that the compensator for the unconstrained case, $\lambda=0$, has a Dirac delta function in its step response.

In order to compare these results with those obtained in Section 6.4, the control signal $u_a^o(t) = L^{-1}\{R(s)W_{cm}^a(s)\}$ is determined. For $\lambda=2$, the control signal $u_a^o(t)$ is

$$\begin{aligned} u_a^o(t) = B[&\delta_{-1}(t) + 0.138\ \exp(-0.1052t) + 0.561\ \exp(-0.2579t) - \\ &5.67\ \exp(-0.6325t) + 3.93\ \exp(-0.6814t) + \\ &0.030\ \exp(-1.144t) + 0.001\ \exp(-2.640t)] \end{aligned} \tag{7.35}$$

The poles and the associated residues of the control signal $U_a^o(s)$ are given in Table 7-7 for λ = 0, 0.5, 1, 2, and 3; in addition, the control signals $u_a^o(t)$ are plotted in Figure 7-7. Observe that the general shapes of the curves are the same as those shown in Figure 6-5. However, the control signals shown in Figure 7-7 do not rise as rapidly as those obtained in Section 6.4. This is due in part to the pole-zero cancellations performed in $[A_1(s) + A_1(-s)]_a$ and in part to the fact that only three pick-off points were used.

To complete the solution, the compensator $G_c^a(s)$ for the feedback system is determined by substituting $W_{cm}^a(s)$, equation (7.34), and $G_{ap}(s)$, equation (7.26), into equation (7.12). Again, the pole-zero pairs near s = 3, 7, 10, and 27 are canceled with the result that for λ=2,

$$G_c^a(s) = \frac{0.707(s+0.09238)(s+0.2079)(s+0.8314)(s+1.178)(s+2.651)}{(s+0.0007761)(s+0.1037)(s+0.2372)(s+0.8866)(s+1.266)(s+2.743)} \quad (7.36)$$

The compensators $G_c^a(s)$ obtained for λ = 0.5, 1, 2, and 3 are given in Table 7-9.

7.5 REMARKS

When the value of the distributed state variable of a class-II-type distributed-parameter system is open to measurement at equally spaced points along the spatial domain of the system, the class-I representations are shown to constitute a convenient method for obtaining the block-diagram model of the feedback system (Section 7.2). Using block-

diagram models as a basis, a minimum-integral-square-error design technique is developed for the feedback system (Section 7.3).

In Section 7.2, the utility of class-I representations is demonstrated for implementing a class-II-type distributed-parameter system in a feedback configuration. By using standard block-diagram manipulations, the feedback models are reduced to an equivalent open-loop model which is used as a basis for applying the optimal design technique formulated in Section 7.3.

In Section 7.3.1, a detailed statement of the optimal control problem is given; and in Section 7.3.2, complemented by Appendix B, the necessary condition for optimality is derived. As shown in Section 7.3.3, the solution of the optimal control problem is expedited when accurate rational fraction approximations are available. This method of solution has the advantage that most of the calculations can be performed on a digital computer which reduces considerably the time and effort required to effect the design.

The techniques developed in this chapter can also be used for the minimum-mean-square-error design of class-II-type, distributed-parameter, feedback control systems which are excited by stationary random inputs.

In practice, the optimal controllers given by Table 7-9 would be further approximated --- additional poles and zeros would be canceled and various constants would be rounded off --- because of the usual inability to physically set pole-zero locations to the degree of accuracy obtained analytically. Also, for zero steady-state error to a step input, the pole at s=-0.0007761 in equation (7.36) should be at s=0.

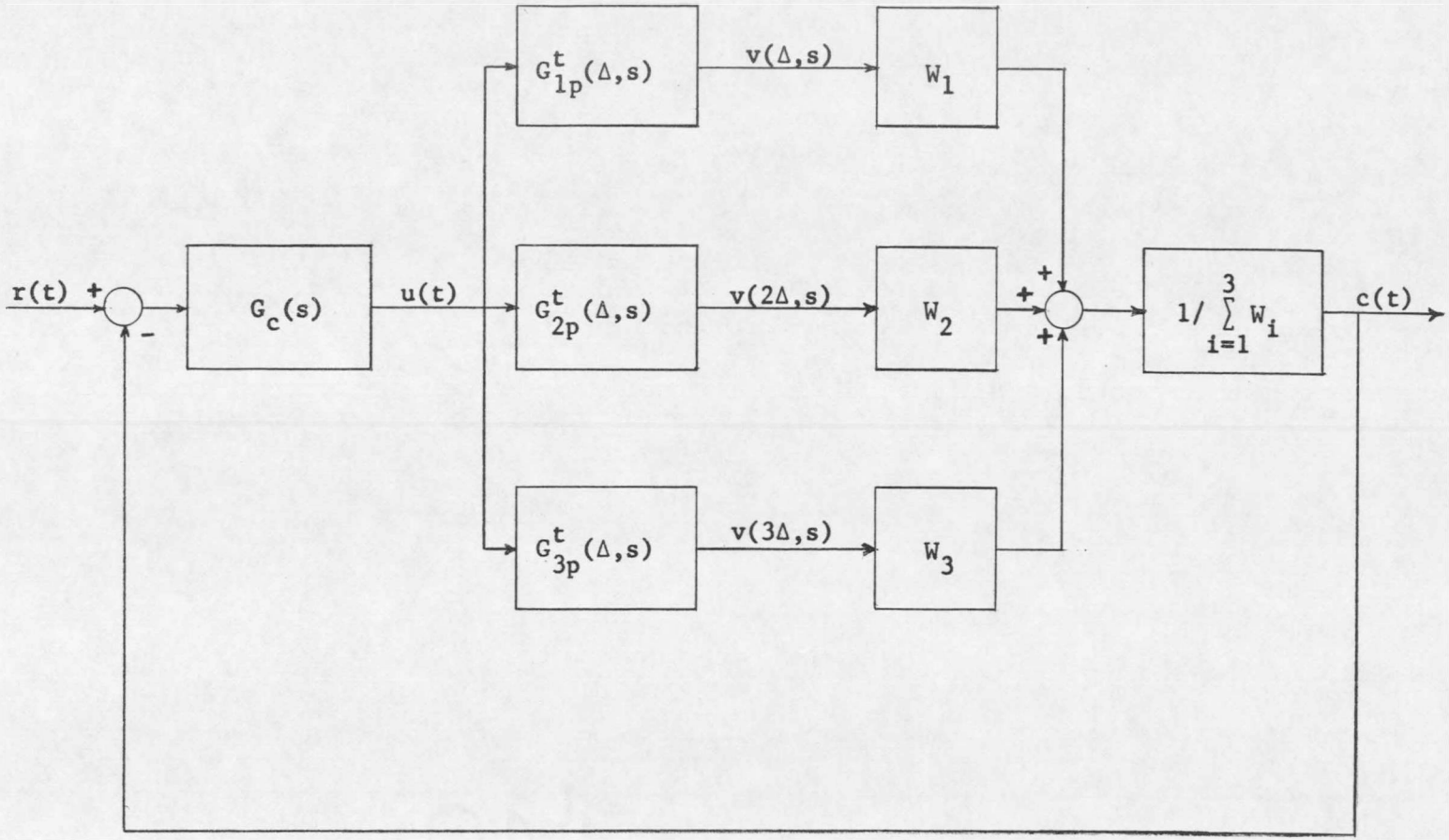


Figure 7-1. A feedback model of a class-II-type, distributed-parameter control system obtained by using a parallel-form class-I representation.

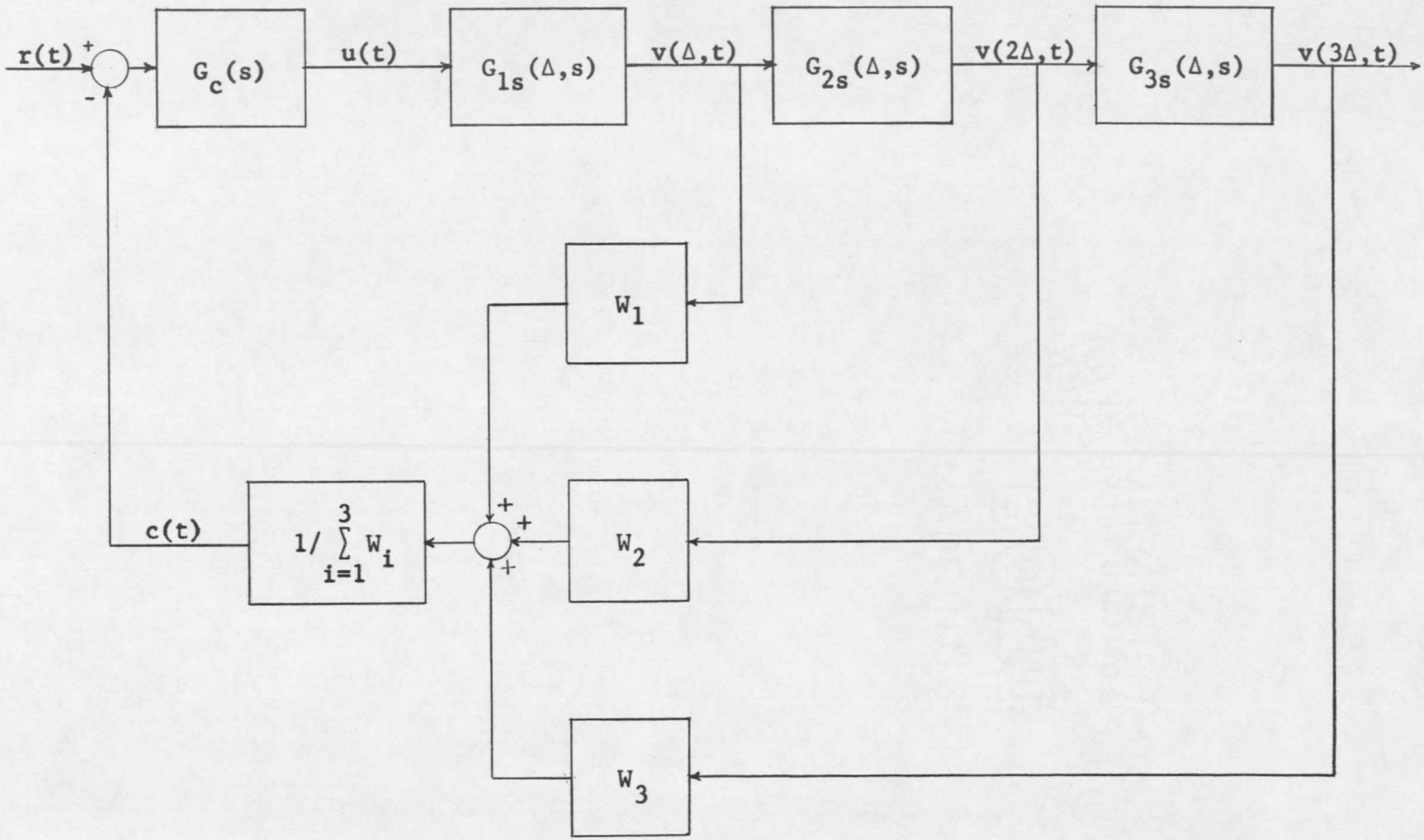


Figure 7-2. A feedback model of a class-II-type, distributed-parameter control system obtained by using a series-form class-I representation.

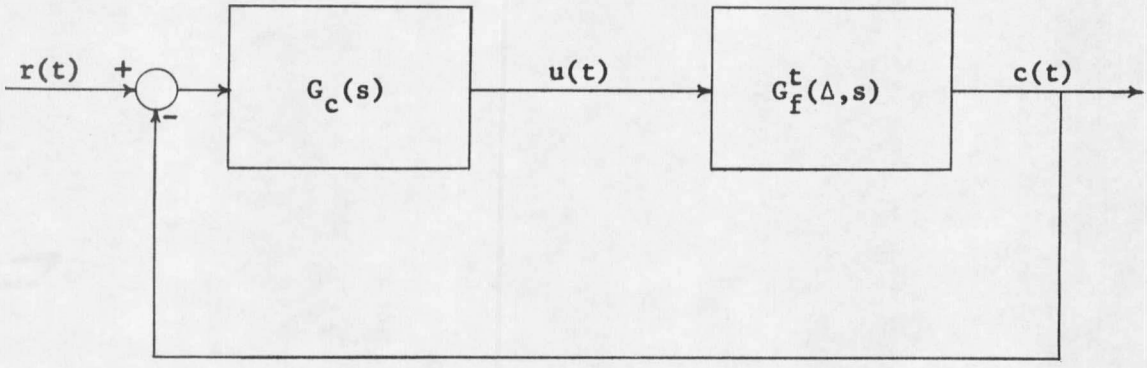


Figure 7-3. An equivalent representation of the feedback models shown in Figures 7-1 and 7-2.

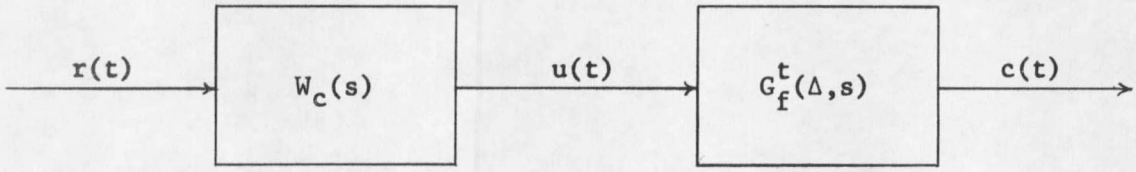


Figure 7-4. An equivalent open-loop model of the feedback representation shown in Figure 7-3.

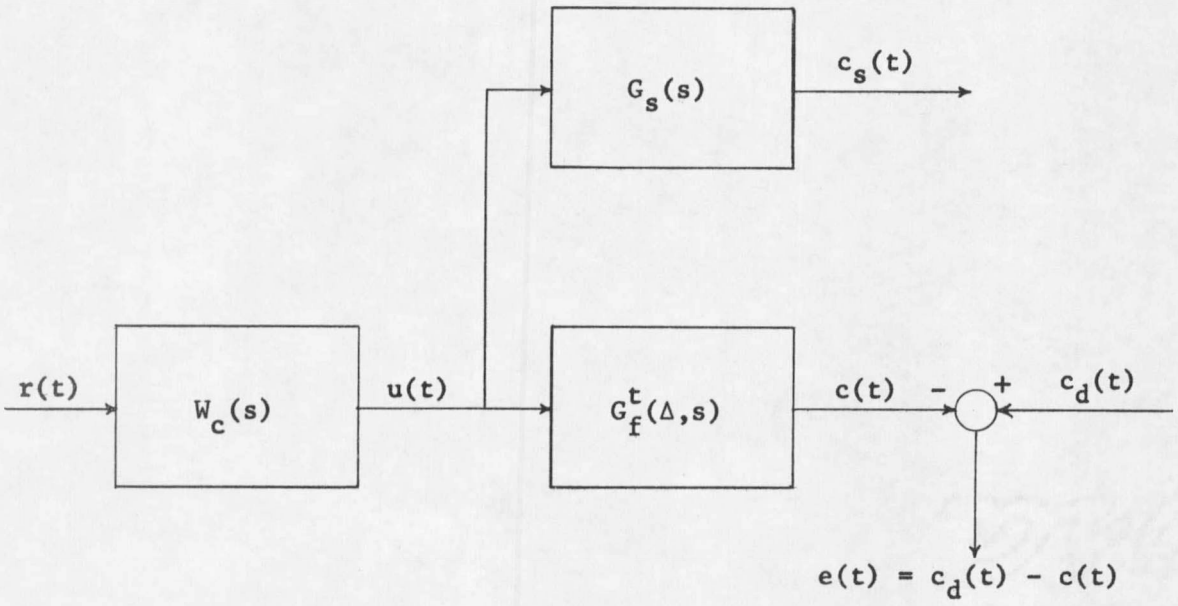


Figure 7-5. A block-diagram representation of the system considered in Sections 7.3 and 7.4.

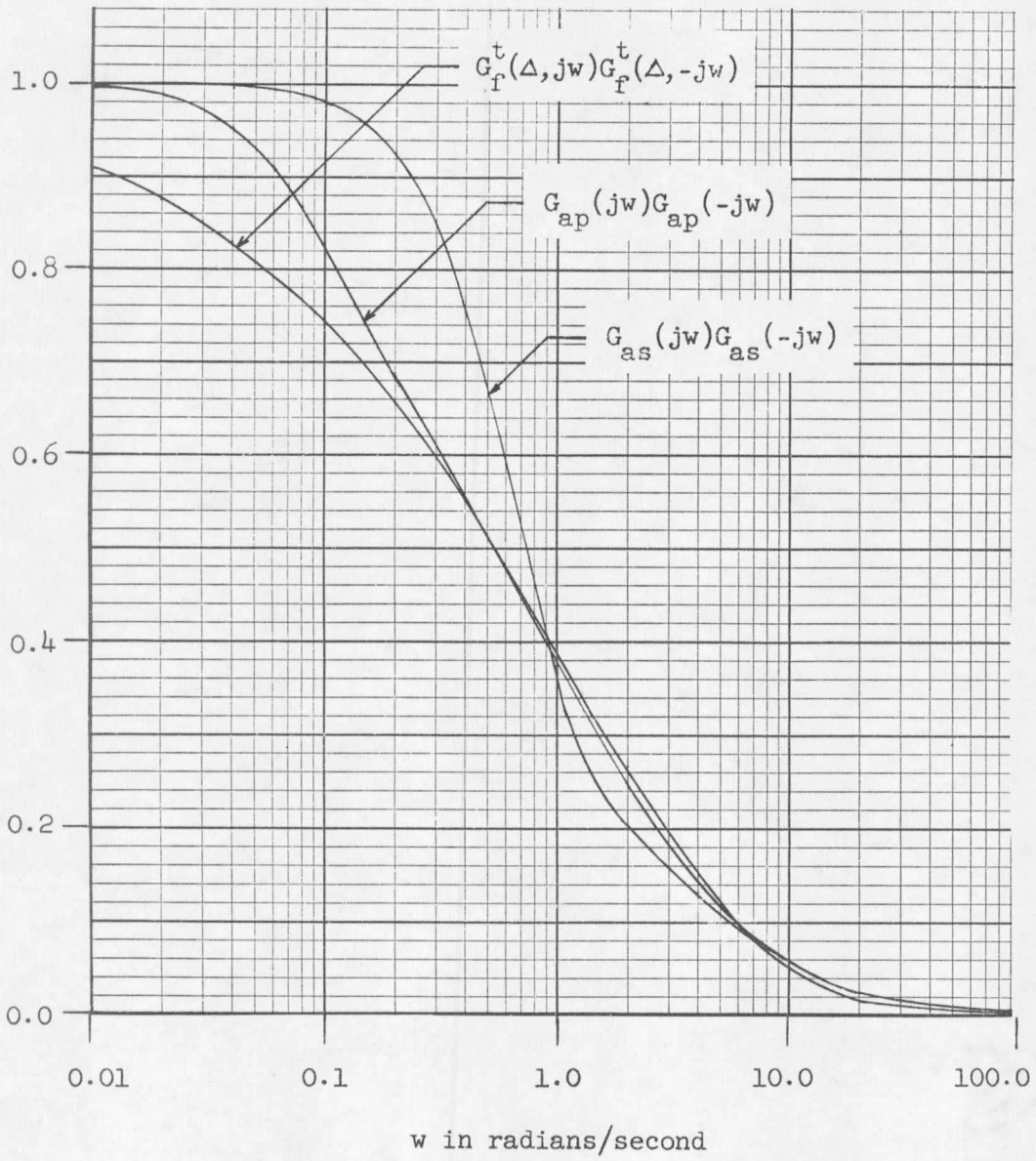


Figure 7-6. Frequency response of the transcendental function and the rational fraction approximations (Example 7.4).

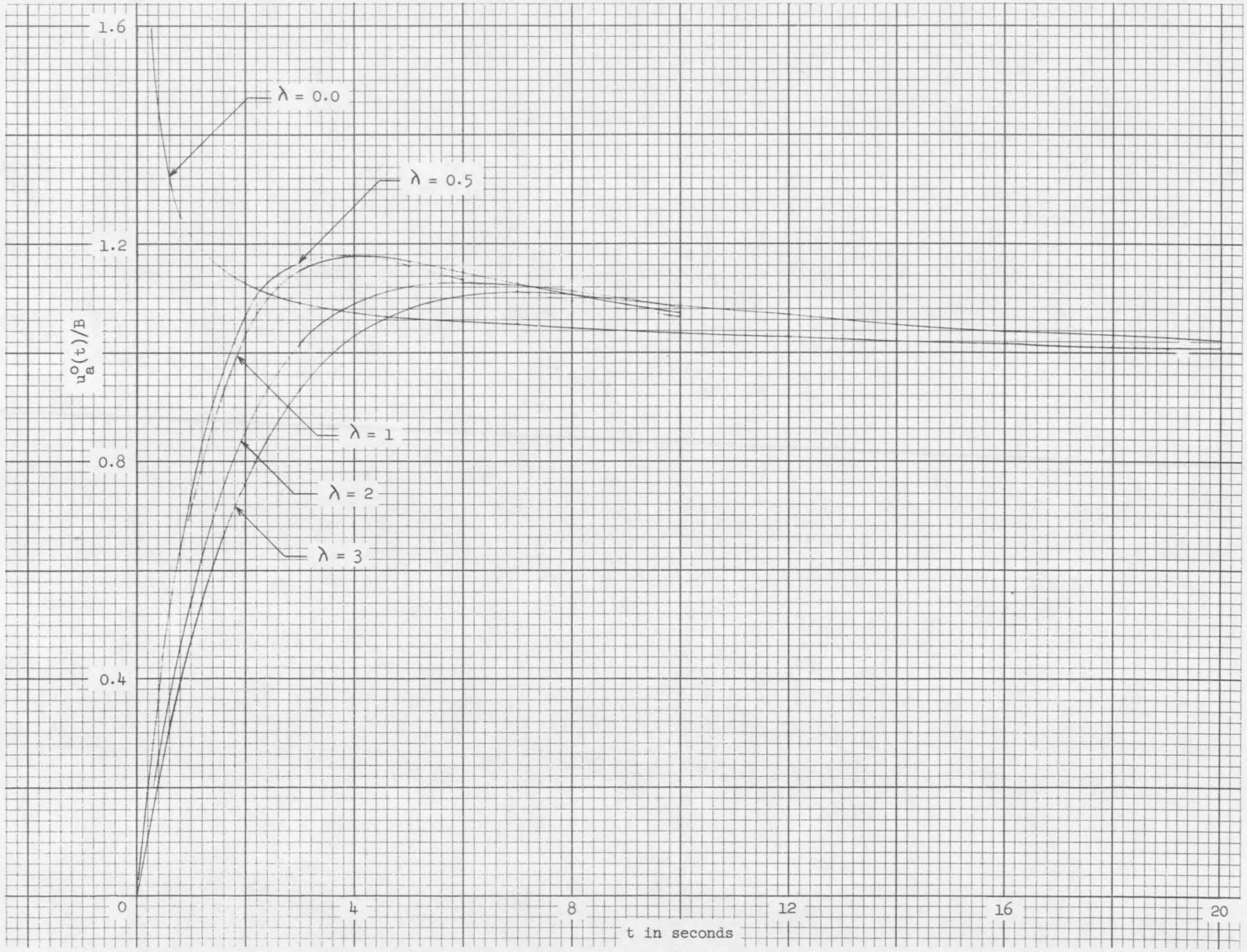


Figure 7-7. Time response of the control signal $u_a^o(t)$ for $\lambda = 0$, 0.5, 1, 2 and 3.

Table 7-1. The coefficients of the numerator and the denominator of $G_{ap}(s)G_{ap}(-s)$ and $G_{as}(s)G_{as}(-s)$, equation (7.27).

$G_{ap}(s)G_{ap}(-s)$	
numerator	denominator
	$N_{18} = 0.268738 \times 10^{-4}$
$M_{16} = 0.448084 \times 10^{-3}$	$N_{16} = 0.252168 \times 10^{-1}$
$M_{14} = 0.239800 \times 10^{0}$	$N_{14} = 0.386987 \times 10^{1}$
$M_{12} = 0.188675 \times 10^{2}$	$N_{12} = 0.174741 \times 10^{3}$
$M_{10} = 0.420643 \times 10^{3}$	$N_{10} = 0.237245 \times 10^{4}$
$M_{8} = 0.312633 \times 10^{4}$	$N_{8} = 0.116538 \times 10^{5}$
$M_{6} = 0.698272 \times 10^{4}$	$N_{6} = 0.174684 \times 10^{5}$
$M_{4} = 0.428447 \times 10^{4}$	$N_{4} = 0.777067 \times 10^{4}$
$M_{2} = 0.277464 \times 10^{3}$	$N_{2} = 0.362890 \times 10^{3}$
$M_{0} = 0.254186 \times 10^{1}$	$N_{0} = 0.254186 \times 10^{1}$

Table 7-1. (continued).

$G_{as}(s)G_{as}(-s)$	
numerator	denominator
	$N_{18} = 0.656100 \times 10^{-8}$
$M_{16} = 0.590490 \times 10^{-7}$	$N_{16} = 0.171832 \times 10^{-4}$
$M_{14} = 0.140300 \times 10^{-3}$	$N_{14} = 0.166941 \times 10^{-1}$
$M_{12} = 0.125323 \times 10^{0}$	$N_{12} = 0.732272 \times 10^{1}$
$M_{10} = 0.497549 \times 10^{2}$	$N_{10} = 0.143809 \times 10^{4}$
$M_{8} = 0.894452 \times 10^{4}$	$N_{8} = 0.128237 \times 10^{6}$
$M_{6} = 0.643947 \times 10^{6}$	$N_{6} = 0.435100 \times 10^{7}$
$M_{4} = 0.246069 \times 10^{7}$	$N_{4} = 0.866039 \times 10^{7}$
$M_{2} = 0.312419 \times 10^{7}$	$N_{2} = 0.590372 \times 10^{7}$
$M_{0} = 0.135085 \times 10^{7}$	$N_{0} = 0.135085 \times 10^{7}$

Table 7-2. Frequency response of the transcendental function and the rational fraction approximations (Example 7.4).

w	$G_f^t(\Delta,jw)G_f^t(\Delta,-jw)$	$G_{ap}(jw)G_{ap}(-jw)$	$G_{as}(jw)G_{as}(-jw)$
.01	.91002710	.99667310	.99979429
.02	.87517352	.98708127	.99917766
.04	.82815017	.95365101	.99671907
.06	.79378891	.91057613	.99264926
.08	.76593167	.86654097	.98700933
.10	.74220071	.82578904	.97985557
.20	.65599006	.68379500	.92421728
.40	.55091480	.55258970	.75594256
.60	.48188493	.47954954	.58801251
.80	.43048827	.42435683	.46027465
1.00	.38979923	.37964098	.37207318
2.00	.26432041	.24590508	.20316383
4.00	.15340147	.14293283	.13121805
6.00	.10177675	.10000871	.09673209
8.00	.07254160	.07580003	.07184594
10.00	.05421383	.05990041	.05392386
20.00	.01886682	.02506275	.01762636
40.00	.00558496	.00856311	.00506128
60.00	.00258937	.00419468	.00236359
80.00	.00143381	.00245687	.00135921
100.00	.00087077	.00160442	.00087988

Table 7-3. The coefficients C_{ij} and D_{ij} for $[A_1(s) + A_1(-s)]_a$, equation (7.29).

denominator	numerator, λ=0	numerator, λ=0.5
	$C_{20} = 0.000000 \times 10^{0}$	$C_{20} = 0.134369 \times 10^{-4}$
$D_{18} = 0.268738 \times 10^{-4}$	$C_{18} = 0.000000 \times 10^{0}$	$C_{18} = 0.126084 \times 10^{-1}$
$D_{16} = 0.252168 \times 10^{-1}$	$C_{16} = 0.448084 \times 10^{-3}$	$C_{16} = 0.193538 \times 10^{1}$
$D_{14} = 0.386987 \times 10^{1}$	$C_{14} = 0.239800 \times 10^{0}$	$C_{14} = 0.876104 \times 10^{2}$
$D_{12} = 0.174741 \times 10^{3}$	$C_{12} = 0.188675 \times 10^{2}$	$C_{12} = 0.120509 \times 10^{4}$
$D_{10} = 0.237245 \times 10^{4}$	$C_{10} = 0.420643 \times 10^{3}$	$C_{10} = 0.624758 \times 10^{4}$
$D_{8} = 0.116538 \times 10^{5}$	$C_{8} = 0.312633 \times 10^{4}$	$C_{8} = 0.118605 \times 10^{5}$
$D_{6} = 0.174684 \times 10^{5}$	$C_{6} = 0.698272 \times 10^{4}$	$C_{6} = 0.108680 \times 10^{5}$
$D_{4} = 0.777067 \times 10^{4}$	$C_{4} = 0.428447 \times 10^{4}$	$C_{4} = 0.446591 \times 10^{4}$
$D_{2} = 0.362890 \times 10^{3}$	$C_{2} = 0.277464 \times 10^{3}$	$C_{2} = 0.278735 \times 10^{3}$
$D_{0} = 0.254186 \times 10^{1}$	$C_{0} = 0.254186 \times 10^{1}$	$C_{0} = 0.254186 \times 10^{1}$

Table 7-3. (continued).

numerator, λ=1	numerator, λ=2	numerator, λ=3
$C_{20} = 0.268738 \times 10^{-4}$	$C_{20} = 0.537477 \times 10^{-4}$	$C_{20} = 0.806215 \times 10^{-4}$
$C_{18} = 0.252168 \times 10^{-1}$	$C_{18} = 0.504336 \times 10^{-1}$	$C_{18} = 0.756504 \times 10^{-1}$
$C_{16} = 0.387031 \times 10^{1}$	$C_{16} = 0.774018 \times 10^{1}$	$C_{16} = 0.116100 \times 10^{2}$
$C_{14} = 0.174981 \times 10^{3}$	$C_{14} = 0.349722 \times 10^{3}$	$C_{14} = 0.524463 \times 10^{3}$
$C_{12} = 0.239132 \times 10^{4}$	$C_{12} = 0.476378 \times 10^{4}$	$C_{12} = 0.713624 \times 10^{4}$
$C_{10} = 0.120745 \times 10^{5}$	$C_{10} = 0.237283 \times 10^{5}$	$C_{10} = 0.353822 \times 10^{5}$
$C_{8} = 0.205948 \times 10^{5}$	$C_{8} = 0.380632 \times 10^{5}$	$C_{8} = 0.555317 \times 10^{5}$
$C_{6} = 0.147534 \times 10^{5}$	$C_{6} = 0.225240 \times 10^{5}$	$C_{6} = 0.302947 \times 10^{5}$
$C_{4} = 0.464736 \times 10^{4}$	$C_{4} = 0.501025 \times 10^{4}$	$C_{4} = 0.537314 \times 10^{4}$
$C_{2} = 0.280006 \times 10^{3}$	$C_{2} = 0.282548 \times 10^{3}$	$C_{2} = 0.285089 \times 10^{3}$
$C_{0} = 0.254186 \times 10^{1}$	$C_{0} = 0.254186 \times 10^{1}$	$C_{0} = 0.254186 \times 10^{1}$

Table 7-4. The roots of the numerator and the denominator of $[A_2(-s) + A_3(s)]_a$, equation (7.28).

roots of numerator	roots of denominator
s = 0.1049	s = $-\varepsilon$, $\varepsilon \to 0+$
s = 0.2468	s = ε, $\varepsilon \to 0+$
s = 0.9311	s = 0.09238
s = 1.509	s = 0.2079
s = 2.932	s = 0.8314
s = 4.353	s = 1.178
s = 7.691	s = 2.650
s = 21.10	s = 3.063
	s = 6.892
	s = 10.60
	s = 27.57

Table 7-5. The roots of the numerator and the denominator of $[A_1(s) + A_1(-s)]_a$, equation (7.29).

roots of denominator	roots of numerator, λ=0	roots of numerator, λ=0.5
$s = \pm 0.09238$	$s = \pm 0.1048$	$s = \pm 0.1049$
$s = \pm 0.2079$	$s = \pm 0.2468$	$s = \pm 0.2489$
$s = \pm 0.8314$	$s = \pm 0.9312$	$s = -0.941 \pm j0.366$
$s = \pm 1.178$	$s = \pm 1.509$	$s = +0.941 \pm j0.366$
$s = \pm 2.651$	$s = \pm 2.932$	$s = \pm 1.021$
$s = \pm 3.063$	$s = \pm 4.353$	$s = \pm 2.605$
$s = \pm 6.892$	$s = \pm 7.691$	$s = \pm 3.051$
$s = \pm 10.60$	$s = \pm 21.10$	$s = \pm 6.887$
$s = \pm 27.57$		$s = \pm 10.60$
$s = \pm \varepsilon$, $\varepsilon \to 0+$		$s = \pm 27.56$

Table 7-5. (continued).

roots of numerator, λ=1	roots of numerator, λ=2	roots of numerator, λ=3
$s = \pm 0.1050$	$s = \pm 0.1052$	$s = \pm 0.1053$
$s = \pm 0.2514$	$s = \pm 0.2579$	$s = \pm 0.2686$
$s = +0.774 \pm j0.238$	$s = \pm 0.6325$	$s = \pm 0.4274$
$s = -0.774 \pm j0.238$	$s = \pm 0.6814$	$s = \pm 0.7798$
$s = \pm 1.102$	$s = \pm 1.144$	$s = \pm 1.559$
$s = \pm 2.629$	$s = \pm 2.640$	$s = \pm 2.643$
$s = \pm 3.057$	$s = \pm 3.060$	$s = \pm 3.061$
$s = \pm 6.889$	$s = \pm 6.891$	$s = \pm 6.891$
$s = \pm 10.60$	$s = \pm 10.60$	$s = \pm 10.60$
$s = \pm 27.57$	$s = \pm 27.57$	$s = \pm 27.57$

Table 7-6. The open-loop compensator $W_{cm}^{a}(s)$ for $\lambda = 0,\ 0.5,\ 1,\ 2,$ and 3.

$\lambda = 0$

$$W_{cm}^{a}(s) = (0.245)[(s+0.09238)(s+0.2079)(s+0.8314)(s+1.178)(s+2.651)(s+3.063)(s+6.892)(s+10.60)(s+27.57)] \,/\, [(s+0.1048)(s+0.2468)(s+0.9312)(s+1.509)(s+2.932)(s+4.353)(s+7.691)(s+21.10)]$$

$\lambda = 0.5$

$$W_{cm}^{a}(s) \cong \frac{1.41(s+0.09238)(s+0.2079)(s+0.8314)(s+1.178)(s+2.651)}{(s+0.1049)(s+0.2489)(s^2+1.882s+1.019)(s+1.021)(s+2.605)}$$

$\lambda = 1$

$$W_{cm}^{a}(s) \cong \frac{1.0(s+0.09238)(s+0.2079)(s+0.8314)(s+1.178)(s+2.651)}{(s+0.1050)(s+0.2514)(s^2+1.548s+0.6556)(s+1.102)(s+2.629)}$$

$\lambda = 2$

$$W_{cm}^{a}(s) \cong \frac{0.707(s+0.09238)(s+0.2079)(s+0.8314)(s+1.178)(s+2.651)}{(s+0.1052)(s+0.2579)(s+0.6325)(s+0.6814)(s+1.144)(s+2.640)}$$

$\lambda = 3$

$$W_{cm}^{a}(s) \cong \frac{0.577(s+0.09238)(s+0.2079)(s+0.8314)(s+1.178)(s+2.651)}{(s+0.1053)(s+0.2686)(s+0.4274)(s+0.7798)(s+1.559)(s+2.643)}$$

Table 7-7. Poles and associated residues of the control signal $U_a^o(s)$ for λ = 0, 0.5, 1, 2, and 3.

λ=0		λ=0.5	
pole	residue	pole	residue
	0.245B		
s	B	s	B
(s+0.1048)	+0.106B	(s+0.1049)	+0.125B
(s+0.2468)	+0.0068B	(s+0.2489)	+0.295B
(s+0.9312)	+0.187B	(s+0.941+j0.366)	(-0.875+j0.301)B
(s+1.509)	+0.056B	(s+0.941-j0.366)	(-0.875-j0.301)B
(s+2.932)	+0.0244B	(s+1.021)	+0.322B
(s+4.353)	+0.662B	(s+2.605)	+0.014B
(s+7.691)	+0.176B		
(s+21.10)	+0.989B		

λ=1		λ=2		λ=3	
pole	residue	pole	residue	pole	residue
s	B	s	B	s	B
(s+0.1050)	+0.132B	(s+0.1052)	+0.138B	(s+0.1053)	+0.153B
(s+0.2514)	+0.363B	(s+0.2579)	+0.561B	(s+0.2686)	+1.003B
(s+0.774+j0.238)	(-0.812+j0.00005)B	(s+0.6325)	-5.67B	(s+0.4274)	-2.30B
(s+0.774-j0.238)	(-0.812-j0.00005)B	(s+0.6814)	+3.93B	(s+0.7798)	+0.132B
(s+1.102)	+0.123B	(s+1.144)	+0.030B	(s+1.559)	+0.014B
(s+2.629)	+0.004B	(s+2.640)	+0.001B	(s+2.643)	+0.0007B

Table 7-8. Time response of the control signal $u_a^o(t)$ for $\lambda = 0$, 0.5, 1, 2 and 3.

t	$u_a^o(t)/B$ $\lambda = 0$	$u_a^o(t)/B$ $\lambda = 0.5$	$u_a^o(t)/B$ $\lambda = 1$	$u_a^o(t)/B$ $\lambda = 2$	$u_a^o(t)/B$ $\lambda = 3$
.20	1.77242	.19152	.18952	.13136	.11208
.40	1.46742	.35734	.35236	.25250	.21333
.60	1.32754	.50311	.49294	.36139	.30600
.80	1.25281	.62948	.61366	.45899	.39065
1.00	1.20855	.73766	.71675	.54623	.46781
2.00	1.12272	1.06268	1.03458	.85645	.75924
3.00	1.09272	1.16646	1.15226	1.01942	.93321
4.00	1.07684	1.17789	1.17927	1.09666	1.03146
5.00	1.06649	1.15921	1.17003	1.12638	1.08246
6.00	1.05871	1.13499	1.14924	1.13116	1.10488
7.00	1.05231	1.11302	1.12711	1.12389	1.11069
8.00	1.04681	1.09483	1.10721	1.11162	1.10733
9.00	1.04197	1.08009	1.09039	1.09804	1.09937
10.00	1.03768	1.06811	1.07650	1.08493	1.08950
12.00	1.03042	1.05015	1.05565	1.06274	1.06947
14.00	1.02458	1.03764	1.04134	1.04634	1.05256
16.00	1.01988	1.02869	1.03125	1.03460	1.03958
18.00	1.01609	1.02216	1.02397	1.02619	1.02997
20.00	1.01302	1.01729	1.01861	1.02010	1.02290
30.00	1.00454	1.00551	1.00585	1.00615	1.00686
40.00	1.00158	1.00188	1.00199	1.00208	1.00231
50.00	1.00055	1.00065	1.00069	1.00072	1.00080
60.00	1.00019	1.00022	1.00024	1.00025	1.00028
70.00	1.00006	1.00008	1.00008	1.00008	1.00009
80.00	1.00002	1.00002	1.00002	1.00003	1.00003
90.00	1.00000	1.00000	1.00001	1.00001	1.00001
100.00	1.00000	1.00000	1.00000	1.00000	1.00000

Table 7-9. The compensator $G_c^a(s)$ for the feedback system for $\lambda = 0.5$, 1, 2, and 3.

$\lambda = 0.5$

$$G_c^a(s) = \frac{1.41(s+0.09238)(s+0.2079)(s+0.8314)(s+1.178)(s+2.651)}{(s+0.005993)(s+0.1045)(s+0.2417)(s+0.9043)(s+1.324)(s+2.798)}$$

$\lambda = 1$

$$G_c^a(s) = \frac{1.0(s+0.09238)(s+0.2079)(s+0.8314)(s+1.178)(s+2.651)}{(s+0.003599)(s+0.1041)(s+0.2397)(s+0.8958)(s+1.294)(s+2.767)}$$

$\lambda = 2$

$$G_c^a(s) = \frac{0.707(s+0.09238)(s+0.2079)(s+0.8314)(s+1.178)(s+2.651)}{(s+0.0007761)(s+0.1037)(s+0.2372)(s+0.8866)(s+1.266)(s+2.743)}$$

$\lambda = 3$

$$G_c^a(s) = \frac{0.577(s+0.09238)(s+0.2079)(s+0.8314)(s+1.178)(s+2.651)}{(s+0.0006370)(s+0.1035)(s+0.2355)(s+0.8808)(s+1.253)(s+2.724)}$$

CHAPTER 8

SUMMARY AND SUGGESTED FUTURE RESEARCH

8.1 SUMMARY OF ESSENTIAL VALUES OF THE THESIS RESEARCH

The major accomplishment of the thesis research is fourfold:

1. Performance measures for class-II-type, distributed-parameter control systems are developed in Chapter 3. For deterministic systems, a general performance measure is formulated; and from this general form, specific metrics are developed and discussed for both the minimum-integral-control problem and the terminal-control problem. For stochastic systems, a performance measure is developed for class-II-type systems which are subject to stationary random inputs. Because the choice of the performance measure is an important aspect of optimal design, the development and interpretation of performance measures for class-II-type, distributed-parameter control systems is a definite contribution to the optimal design of these systems.

2. Block-diagram models of class-II-type distributed-parameter systems with boundary inputs and with a distributed input are developed in Chapter 4, complemented by Appendix A. These models form a basis for the application of optimal design techniques to a wide class of linear distributed-parameter systems. Several different forms of block-diagram models are considered, and in each case, emphasis is placed upon simplicity and upon the ease with which the model lends itself to the application of optimization techniques. In essence, the material presented in Chapter 4 demonstrates the utility of finite integral transforms for the derivation of block-diagram models of linear class-II-type distributed-parameter systems defined on a finite spatial domain.

3. A procedure is advanced in Chapter 6, complemented by Appendices B, C, and D, for minimizing the spatial-averaged-integral-square error of class-II-type distributed-parameter systems subject to stated constraints. An optimal open-loop control signal is derived by use of the Wiener-Hopf spectrum-factorization technique. A procedure of rational fraction approximation is formulated in order to realize the optimal control law. With the aid of a digital computer the high-order rational fraction approximations required to obtain the desired degree of accuracy can be handled with relative ease, as the example in Section 6.4 clearly demonstrates.

4. A procedure is developed in Chapter 7, complemented by Appendices B, C, and D, for minimizing the integral-square error of class-II-type, distributed-parameter, feedback control systems subject to integral constraints. The Wiener-Hopf spectrum-factorization technique is used to obtain the optimal compensator for the feedback system. To realize this lumper-parameter compensator, the nonrational Laplace-transform transfer functions of the distributed-parameter plant are replaced by rational fraction approximations. After the nonrational terms have been approximated, a digital computer can be used to perform nearly all of the steps required in the design of the compensator.

The following three points, while of lesser significance, are worthy of note:

1. The review of the literature on the analysis, synthesis and

optimization of distributed-parameter systems, both class-I-type and class-II-type, presented in Chapter 2 enfolds over 200 of the references listed in the bibliography.

2. A theoretical basis is laid in Chapter 5 for the analysis of class-II-type distributed-parameter systems which are time-invariant and which are excited by stationary random inputs. A two-model representation is advanced as a convenient base from which to obtain distributed correlation functions.

3. The concepts of ε-reachability and ε-maintainability, formulated in Chapter 3, are useful for determining the existance of optimal control signals for distributed-parameter systems.

8.2 ASPECTS MERITING ADDITIONAL STUDY

The following four aspects of distributed-parameter control theory, the author believes, warrant additional development and study:

1. As evidenced in Chapters 6 and 7, high-order rational fraction approximations can be used to advantage for the optimal design of distributed-parameter systems. Although rational fraction approximations exist for some nonrational distributed-parameter characteristics, the derivation of additional rational fraction approximations, or preferrably, the derivation of a general method for obtaining rational fraction approximations merits further research. An integral part of such a study would be an error analysis of the approximations. In conjunction with

this study, analog simulation techniques might be included.

2. As indicated in Section 2.4, certain theoretical aspects of the optimal design of class-II-type distributed-parameter systems using Pontryagin's maximum principle, Bellman's principle of optimality, and other modern control system design techniques have been investigated by a number of authors. However, the computational aspects of the design of practical control systems, particularly feedback systems, by the use of these methods are not well developed and require additional research.

3. Another important aspect of the optimal design problem is that of establishing the existance of optimal control signals for distributed-parameter systems; as is evidenced in Chapter 3, very little work has been published concerning this problem. Thus, the development of a general body of existence theory, perhaps based upon the concepts of ε-reachability and ε-maintainability as formulated in Chapter 3, for distributed-parameter systems would be a significant contribution to the theory on the optimal design of these systems.

4. The development of a general body of theory for the analysis and synthesis of sampled-data, class-II-type, distributed-parameter systems, similar to that developed by Pierre for class-I-type systems (135), would provide a worthy subject of investigation. The logical starting point for this investigation would be the development of sampled-data models for class-II-type distributed-parameter systems. Then, the techniques for

the analysis and synthesis of lumped-parameter sampled-data systems could be extended to include class-II-type distributed-parameter systems.

APPENDICES

APPENDIX A

DEFINITIONS AND PERTINENT PROPERTIES OF THE FINITE LAPLACE AND FINITE FOURIER TRANSFORMS

A.1 THE FINITE LAPLACE TRANSFORM

In this section the basic properties of the finite Laplace transform are presented. These properties are given without proof, but the interested reader may refer to reference 221 for further information.

The finite Laplace transform of $f(x)$ over the range $0 \leq x \leq L$ is defined by

$$L^x\{f(x)\} = F^x(p) \equiv \int_0^L f(x)e^{-px}\,dx \tag{A.1}$$

when the integral exists. If the integral exists for a particular value of p, it can be shown that it exists for any complex value of p in finite regions of the p plane. The function $F^x(p)$ is therefore an entire function, i.e., $F^x(p)$ is analytic in all finite regions of the p plane.

The finite Laplace transform of the n^{th} derivative of $f(x)$ is given by

$$L^x\{d^nf(x)/dx^n\} = p^nF^x(p) - [p^{n-1}f(0) + p^{n-2}\dot{f}(0) + \ldots + f^{(n-1)}(0)] + [p^{n-1}f(L) + p^{n-2}\dot{f}(L) + \ldots + f^{(n-1)}(L)]e^{-Lp} \tag{A.2}$$

and this is readily verified by use of the definition (A.1) in conjunction with integration by parts.

For an n^{th} order linear differential equation, it is well-known that n <u>independent</u> boundary conditions give rise to a unique solution, a

solution designated by g(x) for example. When solving such an equation for g(x) by use of the finite Laplace transform, however, 2n boundary conditions appear in the process of solution, as is indicated by (A.2). Fortunately, the 2n boundary conditions are related implicitly because of the fact that the finite Laplace transform $G^x(p)$ of the solution g(x) must be analytic in finite regions of the p plane. Thus, if the denominator of $G^x(p)$ contains roots at various values of p, the numerator of $G^x(p)$ must contain roots at corresponding values of p of the same order as those in the denominator. In general, n of the 2n boundary conditions are specified in the process of satisfying the above requirement.

Conceivably, the inverse of a given finite Laplace transform could be found in a table, but an extensive table of them is not now in print (a short table is given in reference 221, however).

In order to effect the inverse transformation, use is made of a relationship between the finite Laplace transform and the infinite Laplace transform, and this is given below. By definition,

$$F^x(p) = \int_0^L f(x)e^{-px}\,dx \tag{A.3}$$

which can be written as

$$F^x(p) = \int_0^\infty f(x)e^{-px}\,dx - \int_L^\infty f(x)e^{-px}\,dx \tag{A.4}$$

where f(x) on the interval $[L,\infty]$ is assumed to be the analytic contin-

uation of f(x) on the interval [0,L] and where the integrals are assumed to exist. By letting x=v+L in the last integration of equation (A.4), it follows that

$$F^x(p) = F(p) - e^{-Lp} \int_0^\infty f(v+L)e^{-pv}\, dv \qquad (A.5)$$

If $F^x(p)$ is expressed in the form of equation (A.5), the inverse finite Laplace transform is obtained using a table of infinite Laplace transforms; the desired relationship is

$$F(x) = L^{-1}\{F(p)\}\,, \quad 0 \leq x \leq L \qquad (A.6)$$

A.2 THE FINITE SINE AND FINITE COSINE TRANSFORMS

The definitions and fundamental properties of the finite Fourier transforms, the finite sine and finite cosine transforms, are presented in this section. For more detail, the interested reader may refer to references 31, 40, and 58.

If f(x) satisfies Dirichlet's conditions in the interval $0 \leq x \leq L$[1], its finite sine transform is defined to be

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[1]The function f(x) is said to satisfy Dirichlet's conditions in the interval $0 \leq x \leq L$ if

1. f(x) has only a finite number of maxima and minima in $0 \leq x \leq L$.
2. f(x) has only a finite number of discontinuities in $0 \leq x \leq L$ --- and no infinite discontinuities.

$$F_s(n) \equiv \int_0^L f(x) \sin\frac{n\pi x}{L}\, dx \qquad (A.7)$$

where n is a positive integer. At each point in the interval $0 \leq x \leq L$ at which f(x) is continuous, the inverse finite sine transform is given by

$$f(x) = \frac{2}{L} \sum_{n=1}^{\infty} F_s(n) \sin\frac{n\pi x}{L} \qquad (A.8)$$

If f(x) satisfies Dirichlet's conditions in the interval $0 \leq x \leq L$, its finite cosine transform is defined to be

$$F_c(n) \equiv \int_0^L f(x) \cos\frac{n\pi x}{L}\, dx \qquad (A.9)$$

where n is a nonnegative integer. At each point in the interval $0 \leq x \leq L$ at which f(x) is continuous, the inverse finite cosine transform is given by

$$f(x) = \frac{1}{L} F_c(0) + \frac{2}{L} \sum_{n=1}^{\infty} F_c(n) \cos\frac{n\pi x}{L} \qquad (A.10)$$

where $F_c(0)$ denotes $\int_0^L f(x)\, dx$.

Using the definitions of the finite sine and finite cosine transforms, given by equations (A.7) and (A.8), in conjunction with integration by parts, the following fundamental results are obtained:

$$\int_0^L \frac{d\,f(x)}{dx} \sin\frac{n\pi x}{L}\,dx = f(x)\,\sin\frac{n\pi x}{L}\Bigg|_0^L - \frac{n\pi}{L}\int_0^L f(x)\,\cos\frac{n\pi x}{L}\,dx$$

$$= -\frac{n\pi}{L}\,F_c(n) \qquad \text{(A.11)}$$

and

$$\int_0^L \frac{d\,f(x)}{dx} \cos\frac{n\pi x}{L}\,dx = f(x)\,\cos\frac{n\pi x}{L}\Bigg|_0^L + \frac{n\pi}{L}\int_0^L f(x)\,\sin\frac{n\pi x}{L}\,dx$$

$$= (-1)^n\,f(L) - f(0) + \frac{n\pi}{L}\,F_s(n) \qquad \text{(A.12)}$$

where $F_s(n)$ and $F_c(n)$ denote, respectively, the finite sine transform and finite cosine transform of $f(x)$.

The finite Fourier transforms of higher derivatives can be obtained by repeated application of equations (A.11) and (A.12). For example, the finite sine transform of the second derivative of $f(x)$ is determined as follows: First, equation (A.11) is used to obtain

$$\int_0^L \frac{d^2f(x)}{dx^2} \sin\frac{n\pi x}{L}\,dx = -\frac{n\pi}{L}\int_0^L \frac{d\,f(x)}{dx}\cos\frac{n\pi x}{L}\,dx \qquad \text{(A.13)}$$

and second, equation (A.12) is applied to the right-hand side of equation (A.13) to obtain the desired result

$$\int_0^L \frac{d^2f(x)}{dx^2} \sin\frac{n\pi x}{L}\, dx = \frac{n\pi}{L}[(-1)^{n+1}f(L) + f(0)] - (\frac{n\pi}{L})^2 F_s(n) \qquad (A.14)$$

Following a similar procedure, the finite cosine transform of the second derivative of f(x) is

$$\int_0^L \frac{d^2f(x)}{dx^2} \cos\frac{n\pi x}{L}\, dx = (-1)^n \dot{f}(L) - \dot{f}(0) - (\frac{n\pi}{L})^2 F_c(n) \qquad (A.15)$$

where $\dot{f}(L) \equiv \left.\frac{d\ f(x)}{dx}\right|_{x=L}$ and $\dot{f}(0) \equiv \left.\frac{d\ f(x)}{dx}\right|_{x=0}$.

As evidenced in equations (A.14) and (A.15), the choice of the transform, finite sine or finite cosine, depends upon the form of the boundary conditions: For second-order differential equations, if the boundary conditions are specified in terms of f(x), the finite sine transform is applicable; if the boundary conditions are specified in terms of d f(x)/dx, however, the finite cosine transform is applicable.

APPENDIX B

NECESSARY CONDITION

FOR OPTIMALITY

B.1 PRELIMINARY REMARKS

The performance measures considered in Chapters 6 and 7 are reduced to a common functional form, equations (6.6) and (7.9). Thus, the optimal design problem considered in each of these chapters reduces to the minimization of a general functional equation in the Laplace variable s (the transformed time variable). This general functional equation is considered by Pierre (292), and is solved by use of the calculus of variations and the Wiener-Hopf spectrum-factorization technique. The derivation of the necessary condition for optimality given here is essentially that given by Pierre (292).

B.2 THE DERIVATION OF THE NECESSARY CONDITION FOR OPTIMALITY

B.2.1 Problem Statement

In Chapters 6 and 7, the optimal design problem reduced to that of minimizing a general equation of the form

$$J = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} [A_1(s)F(s)F(-s) + A_2(s)F(s) + A_3(s)F(-s) + A_4(s)]\, ds \qquad (B.1)$$

where $A_i(s)$, i=1,2,3,4, is a known two-sided Laplace-transform function of s and where the Laplace-transform function F(s)[1] is to be selected so

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[1] In Chapter 6, F(s) corresponds to the control signal U(s), and in Chapter 7, F(s) corresponds to the compensator $W_c(s)$.

as to minimize J. In the proof which follows, it is assumed that at least one of the functions $A_2(s)$ and $A_3(s)$ is nonzero and that the F(s) which is selected must satisfy the following property: $L^{-1}\{F(s)\} \equiv f(t) = 0$ for $t<0$.

B 2.2 Initial Steps in the Derivation

For the initial steps in the derivation, the calculus of variations is used. Thus, first assume that $F_m(s)$, the optimal value of F(s), exists and that

$$F(s) = F_m(s) + \varepsilon\, \Delta(s) \tag{B.2}$$

where ε is a real variable and $\Delta(s)$ is any allowable variation of F(s) from $F_m(s)$. An allowable variation is one that satisfies the condition that $L^{-1}\{\Delta(s)\} \equiv \Delta(t) = 0$ for $t<0$ because f(t) and $f_m(t)$ are so restricted.

Appropriate substitution of the right-hand side of equation (B.2) into equation (B.1) results in

$$J(\varepsilon) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \{A_1(s)[F_m(s)F_m(-s) + \varepsilon\, \Delta(s)F_m(-s) + \varepsilon\, \Delta(-s)F_m(s) + \varepsilon^2\Delta(s)\Delta(-s)] + A_2(s)[F_m(s) + \varepsilon\, \Delta(s)] + A_3(s)[F_m(-s) + \varepsilon\, \Delta(-s)] + A_4(s)\}\, ds \tag{B.3}$$

It is well-known from the theory of the calculus of variations that the extremum of $J(\varepsilon)$ is obtained at $\varepsilon=0$; and at $\varepsilon=0$, the value of $\partial J/\partial\varepsilon$ is equal to zero. Thus, a necessary condition that $F_m(s)$ must satisfy is obtained by differentiating both sides of equation (B.3) with respect to ε and by setting the result equal to zero, that is

$$\left.\frac{\partial J(\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} = 0$$

$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \{A_1(s)[\Delta(s)F_m(-s) + \Delta(-s)F_m(s)] + A_2(s)\Delta(s) + A_3(s)\Delta(-s)\}\, ds$$

$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Delta(-s)[F_m(s)A_1(s) + A_3(s)]\, ds + \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Delta(s)[F_m(-s)A_1(s) + A_2(s)]\, ds \qquad (B.4)$$

By applying Parseval's theorem, the second integral on the right-hand side of equation (B.4) is shown to be equivalent to

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Delta(-s)[A_1(-s)F_m(s) + A_2(-s)]\, ds \qquad (B.5)$$

which is substituted appropriately into equation (B.4) to obtain

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\Delta(-s)\{[A_1(s) + A_1(-s)]F_m(s) + A_2(-s) + A_3(s)\}\,ds = 0 \qquad (B.6)$$

Thus, the problem is to determine the Laplace-transform function $F_m(s)$ which satisfies equation (B.6).

In order to simplify the notation, the following identities are introduced

$$A_{11}(s) \equiv A_1(s) + A_1(-s) \qquad (B.7)$$

$$A_{23}(s) \equiv A_2(-s) + A_3(s) \qquad (B.8)$$

and

$$A_5(s) \equiv A_{11}(s)F_m(s) + A_{23}(s) \qquad (B.9)$$

The identity given by equation (B.9) is substituted into equation (B.6) to obtain

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\Delta(-s)A_5(s)\,ds = 0 \qquad (B.10)$$

And, the application of Parseval's theorem to equation (B.10) yields

$$\int_{-\infty}^{\infty} \Delta(t)\, a_5(t)\, dt = 0 \qquad \text{(B.11)}$$

where $a_5(t) = L^{-1}\{A_5(s)\}$ and $\Delta(t) = L^{-1}\{\Delta(s)\}$.

By definition, $\Delta(t)$ is an arbitrary Laplace transformable function for $t>0$ and zero for $t<0$. In order for equation (B.11) to be satisfied, therefore, $a_5(t)$ must equal zero for $t>0$, but $a_5(t)$ is not necessarily zero for $t<0$. Because of the latter condition, $A_5(s)$ cannot simply be set equal to zero to determine $F_m(s)$.

The above-noted problem is effectively resolved, however, by use of the Wiener-Hopf spectrum-factorization technique. Even though $A_i(s)$, $i=1,2,3,4$ contains nonrational terms, the Wiener-Hopf spectrum factorization technique is applicable as shown by Pierre (226, 268). Before the Wiener-Hopf technique is applied, however, the following pertinent notation is presented.

B.2.3 Notation for Spectrum Factorization

Let $A_{11}(s)$ be factored in the following manner:

$$A_{11}(s) = A_{11}^{+}(s)A_{11}^{-}(s) \qquad \text{(B.12)}$$

where $L^{-1}\{A_{11}^{+}(s)\} = 0$ for $t<0$ and $L^{-1}\{A_{11}^{-}(s)\} = 0$ for $t>0$. Also, $L^{-1}\{1/A_{11}^{+}(s)\} = 0$ for $t<0$ and $L^{-1}\{1/A_{11}^{-}(s)\} = 0$ for $t>0$. Stated in an equivalent, but somewhat different manner, $A_{11}(s)$ must be factored in

the form shown in equation (B.12) such that, (1) both $A_{11}^{+}(s)$ and $1/A_{11}^{+}(s)$ are analytic in the $\overline{\text{RHP}}$, and (2) both $A_{11}^{-}(s)$ and $1/A_{11}^{-}(s)$ are analytic in the $\overline{\text{LHP}}$ [2].

In addition, $A_5(s)$ and $[A_{23}(s)/A_{11}^{-}(s)]$ are factored as follows

$$A_5(s) = A_5(s)_+ + A_5(s)_- \tag{B.13}$$

and

$$\frac{A_{23}(s)}{A_{11}^{-}(s)} = \{\frac{A_{23}(s)}{A_{11}^{-}(s)}\}_+ + \{\frac{A_{23}(s)}{A_{11}^{-}(s)}\}_- \tag{B.14}$$

where, for $t<0$,

$$L^{-1}\{A_5(s)_+\} = 0$$

$$L^{-1}\{[A_{23}(s)/A_{11}^{-}(s)]_+\} = 0$$

and, for $t>0$,

$$L^{-1}\{A_5(s)_-\} = 0$$

$$L^{-1}\{[A_{23}(s)/A_{11}^{-}(s)]_-\} = 0$$

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[2] As noted in Section 6.2, the term "analytic in the $\overline{\text{RHP}}$ ($\overline{\text{LHP}}$)" was employed by Pierre (226, 268) to replace the more cumbersome phrase "analytic in the right-half (left-half) plane of the principal sheet of the Riemann surface and on the imaginary axis of said sheet."

B.2.4 Application of the Wiener-Hopf Spectrum Factorization

Consider the function $A_5(s)_-$:

$$A_5(s)_- = A_5(s) - A_5(s)_+ \tag{B.15}$$

but, because $a_5(t)$ is zero for $t>0$, as determined in Section B.2.2, it follows that

$$A_5(s)_- = A_5(s)$$

$$= A_{11}(s)F_m(s) + A_{23}(s) \tag{B.16}$$

where $L^{-1}\{A_5(s)_-\} = 0$ for $t>0$.

Equation (B.16) is now used to derive an equation which contains $F_m(s)$ and which has the property that the inverse transform of its left-hand side is zero for $t>0$ while the inverse transform of its right-hand side is zero for $t<0$. After this is accomplished, it follows that each side of the equation must be zero, independent of the other side, and $F_m(s)$ is obtained explicitly.

The left-hand side of equation (B.16) exhibits the desired property, but the right-hand member does not. By use of equation (B.12), equation (B.16) is written as follows:

$$\frac{A_5(s)_-}{A_{11}^-(s)} = A_{11}^+(s)F_m(s) + \frac{A_{23}(s)}{A_{11}^-(s)} \tag{B.17}$$

And by use of equation (B.14), equation (B.17) is rearranged into the form

$$\frac{A_5(s)_-}{A_{11}^-(s)} - \{\frac{A_{23}(s)}{A_{11}^-(s)}\}_- = A_{11}^+(s)F_m(s) + \{\frac{A_{23}(s)}{A_{11}^-(s)}\}_+ \qquad (B.18)$$

Note that the inverse transform of the left-hand side of equation (B.18) is zero for t>0, and that the inverse transform of the right-hand side is zero for t<0; whence, equation (B.18) is in the desired form.

The right-hand side of equation (B.18) is equated to zero and solved for $F_m(s)$ to obtain

$$F_m(s) = \frac{-1}{A_{11}^+(s)} \{\frac{A_{23}(s)}{A_{11}^-(s)}\}_+ \qquad (B.19)$$

$$= \frac{-\{\dfrac{[A_2(-s) + A_3(s)]}{[A_1(s) + A_1(-s)]^-}\}_+}{[A_1(s) + A_1(-s)]^+}$$

In order to determine the extremizing nature of $F_m(s)$, the second partial derivative of $J(\varepsilon)$, equation (B.3), with respect to ε is examined.

$$\frac{\partial^2 J}{\partial \varepsilon^2} = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} 2\, A_1(s)\, \Delta(s)\, \Delta(-s)\, ds \qquad (B.20)$$

If $\partial^2 J(\varepsilon)/\partial\varepsilon^2$ is greater (less) than zero for all allowable values of $\Delta(s)$, then the minimum (maximum) of J is obtained when F(s) equals $F_m(s)$ given by equation (B.19). Because only one function $F_m(s)$ satisfies the necessary condition $\partial J/\partial\varepsilon\Big|_{\varepsilon=0} = 0$, this $F_m(s)$ yields the absolute minimum (maximum) of J.

APPENDIX C

RATIONAL FRACTION APPROXIMATIONS

C.1 PRELIMINARY REMARKS

In Chapters 6 and 7, high-order rational fraction approximations are used to realize the optimal designs developed therein. The nonrational terms $\exp[-(sD)^{1/2}]$ and $(s)^{1/2}$ are approximated by the rational fraction approximations developed by Pierre (135, 257) and Kopal (60), respectively. A brief account of Pierre's rational fraction approximations for $\exp[-(sD)^{1/2}]$ is given in Section C.2; and in Section C.3, Kopal's approximations for $(s)^{1/2}$ are presented.

C.2 RATIONAL FRACTION APPROXIMATIONS FOR $\exp[-(sD)^{1/2}]$

Pierre (135, 257) developed high-order rational fraction approximations for $\exp[-(sD)^{1/2}]$ by embedding the problem in a class of related problems. An iterative technique was then developed for obtaining approximations of any order.

The derivation of these approximations is based on a common origin of the transfer function $\exp[-(sD)^{1/2}]$; namely, the diffusion equation with boundary conditions $v(\infty,t) = 0$ and $v(0,t) = u(t)$ and with the initial condition $v(x,0) = 0$. After Laplace transforming this equation with respect to t, the resulting equation is approximated by a set of difference equations which are solved in an iterative manner to obtain increasingly higher order approximations, as follows:

$$\left.\frac{P_{n-m}(q)}{P_n(q)}\right|_{q=sD/m^2} \longrightarrow \exp[-(sD)^{1/2}] \qquad (C.1)$$

as $m \to \infty$ with $n \propto m^2$. In equation (C.1), $P_{n-m}(q)$ and $P_n(q)$ are polynomials in q; m and n are positive integers where $m<n$. The polynomials P_1 through P_7 are given in Table C-1. Higher-order polynomials can be generated by use of the following equation:

$$P_n(q) = \sum_{k=0}^{n-1} \left[\binom{n-1+k}{n-1-k} + \binom{n+k}{n-1-k} q_e\right] q^k \qquad (C.2)$$

where $q_e = hq$ and h is a real positive constant. The parameter q_e is used to improve the fit for low-order approximations[1].

Using the polynomials shown in Table C-1, the rational fraction approximations for $\exp[-(sD)^{1/2}]$ used in Chapters 6 and 7 are formed as follows: let n=3, m=1, and $q_e=3q$ to obtain:

for D = 1,

$$\exp[-(s)^{1/2}] \cong \left.\frac{P_2(q)}{P_3(q)}\right|_{q=s} = \frac{P_2(s)}{P_3(s)} = \frac{3s^2 + 7s + 1}{3s^3 + 13s^2 + 12s + 1} \qquad (C.3)$$

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[1]As shown by Pierre (135), q_e has a direct analog in RC-transmission line theory: q_e represents the terminating impedance of the line and should be selected to match the characteristic impedance of the line as much as possible.

for D = 1/9,

$$\exp[-(s/9)^{1/2}] \cong \left.\frac{P_2(q)}{P_3(q)}\right|_{q=s/9} = \frac{P_2(s/9)}{P_3(s/9)}$$

$$\cong \frac{27s^2 + 567s + 729}{3s^3 + 117s^2 + 972s + 729} \qquad (C.4)$$

and for D = 4/9,

$$\exp[-(4s/9)^{1/2}] \cong \left.\frac{P_2(q)}{P_3(q)}\right|_{q=4s/9} = \frac{P_2(4s/9)}{P_3(4s/9)}$$

$$\cong \frac{432s^2 + 2268s + 729}{192s^3 + 1872s^2 + 3888s + 729} \qquad (C.5)$$

As shown by Pierre (135, 257), these rational fraction approximations yield a relatively good approximation to the transcendental terms over a wide frequency range.

C.3 RATIONAL FRACTION APPROXIMATIONS FOR $(s)^{1/2}$

Rational fraction approximations of $(s)^{1/2}$ and $(1/s)^{1/2}$ have been investigated by several authors (60, 103, 210, 248, 277). For the example in Section 6.4, the rational fraction approximations developed by Kopal (60) are used. A brief discussion of these approximations is given in the text which follows; for a detailed derivation, the interested

reader may consult reference 60.

Kopal's high-order rational fraction approximations for $(s)^{1-a}$, $0 \leq a \leq 1$, are based upon generalized Laguerre polynomials. For $a = 1/2$, Kopal obtains the following rational fraction approximations:

$$(s)^{1/2} \cong \frac{N_1(s)}{M_0(s)} = \frac{s + 1}{2} \tag{C.6}$$

$$(s)^{1/2} \cong \frac{N_2(s)}{M_1(s)} = \frac{s^2 + 6s + 1}{4(s+1)} \tag{C.7}$$

$$(s)^{1/2} \cong \frac{N_4(s)}{M_3(s)} = \frac{s^4 + 28s^3 + 70s^2 + 28s + 1}{8(s + 1)(s^2 + 6s + 1)} \tag{C.8}$$

In Section 6.4, it was necessary to approximate $(1/s)^{1/2}$ and $1/[(s)^{1/2} + (-s)^{1/2}]$. The nonrational term $(1/s)^{1/2}$ is approximated by using equation (C.7) in the following way:

$$\frac{1}{(s)^{1/2}} = \frac{(s)^{1/2}}{s} \cong \frac{N_2(s)}{sM_1(s)} = \frac{s^2 + 6s + 1}{4s(1 + s)} \tag{C.9}$$

It can be shown that a very poor approximation is obtained if $(1/s)^{1/2}$ is approximated by the reciprocal of the right-hand member of equation (C.7)

Again using the approximation given by equation (C.7), the rational fraction approximation of $1/[(s)^{1/2} + (-s)^{1/2}]$ is obtained as follows:

$$\frac{1}{(s)^{1/2} + (-s)^{1/2}} = \frac{1}{(1/s)^{1/2} + (1/-s)^{1/2}} \cdot \frac{1}{(s)^{1/2}(-s)^{1/2}}$$

$$\cong \frac{1}{\frac{N_2(s)}{sM_1(s)} + \frac{N_2(-s)}{-sM_1(-s)}} \cdot \frac{N_2(s)N_2(-s)}{s(-s)M_1(s)M_1(-s)}$$

$$\cong \frac{N_2(s)N_2(-s)}{sN_2(-s)M_1(s) - sN_2(s)M_1(-s)}$$

$$\cong \frac{s^4 - 34s^2 + 1}{8s^4 - 40s^2} \qquad \text{(C.10)}$$

Table C-1. Polynomials used in the approximation of $\exp[-(sD)^{1/2}]$.

n	P_n
1	$(1 + q_e)$
2	$(1 + q) + q_e(2 + q)$
3	$(1 + 3q + q^2) + q_e(3 + 4q + q^2)$
4	$(1 + 6q + 5q^2 + q^3) + q_e(4 + 10q + 6q^2 + q^3)$
5	$(1 + 10q + 15q^2 + 7q^3 + q^4) + q_e(5 + 20q + 21q^2 + 8q^3 + q^4)$
6	$(1 + 15q + 35q^2 + 28q^3 + 9q^4 + q^5) + q_e(6 + 35q + 56q^2 + 36q^3 + 10q^4 + q^5)$
7	$(1 + 21q + 70q^2 + 84q^3 + 45q^4 + 11q^5 + q^6) + q_e(7 + 56q + 126q^2 + 120q^3 + 55q^4 + 12q^5 + q^6)$

APPENDIX D

FORTRAN PROGRAM FOR PERFORMING POLYNOMIAL MANIPULATIONS

D.1 DESCRIPTION OF THE FORTRAN PROGRAM

The Fortran program described herein was developed to perform the polynomial manipulations required to obtain the rational fraction approximations $[A_1(s) + A_1(-s)]_a$ and $[A_2(-s) + A_3(s)]_a$ as ratios of polynomials in s for the examples given in Sections 6.4 and 7.4. Using this program, the multiplication, addition and scalar multiplication of real polynomials can be performed on a digital computer. The flow-diagram for this program is given in Figure D-1.

The component polynomials required to form $[A_1(s) + A_1(-s)]_a$ and $[A_2(-s) + A_3(s)]_a$ and the sequence of operations required to obtain the numerator and denominator polynomials are read into the computer. The component polynomials must each be of sixth order or less. As the component polynomials are read into the computer, they are numbered consecutively to identify each polynomial; the fixed point variable NP is used for this purpose (see Figure D-1). After each manipulation, the resulting polynomial is numbered in the same manner.

The "sequence cards" specify the polynomials which are to be operated upon and the operation which the computer is to perform (see Section D.2 for more detail). After all of the manipulations are performed, both the numerator and the denominator polynomials of the rational fraction approximation are punched. The resulting numerator polynomial must be of twentieth order or less, and the denominator polynomial must be of nineteenth order or less.

Note that although this program was written specifically for obtaining $[A_1(s) + A_1(-s)]_a$ and $[A_2(-s) + A_3(s)]_a$, as ratios of polynomials

in s, it can be used to perform the multiplication, addition and scalar multiplication of any polynomials that are of sixth order or less.

For the example in Section 6.4, the program, as shown in Table D-1, was used. Note that the coefficients of the output polynominals were punched in a fixed-point format. For the example in Section 7.4, however, the resulting coefficients were too large for the fixed-point format; and for this reason, the output format was changed to an exponential format.

The following is a description of the input cards required for this program. The first input card is the control card which contains the following data:

1. In columns 1-2, IN (format I2). IN determines the heading to be punched on the output data: If IN=1, the heading $[A_1(s) + A_1(-s)]$ is punched; and if IN=0, the heading $[A_2(-s) + A_3(s)]$ is punched.

2. In columns 3-4, NA (format I2). NA is chosen to be an integer greater than or equal to the largest number of coefficients occurring in the output polynomials. NA must be 12, 16, or 20.

3. In columns 5-6, N (format I2). N is the number of component polynomials to be read into the computer.

4. In columns 7-8, NN (format I2). NN is the identifying number of the resulting numerator polynomial.

5. In columns 9-10, ND (format I2). ND is the identifying number of the resulting denominator polynomial.

6. In columns 11-12, NB (format I2). NB is the identifying number of the last polynomial plus one. In general, this will be either NN+1 or ND+1, whichever is the greater.

7. In columns 13-18, XL (format F6.2). XL is the length of the spatial domain L. If IN=0, it is not necessary to specify XL[1].

8. In columns 19-24, XLAM (format F6.2). XLAM is equal to the cost factor λ. If IN=0, it is not necessary to specify XLAM[1].

The control card is followed by N cards which contain the coefficients of the input polynomials. Each of these cards is punched as follows:

1. M(NP) is punched in columns 1-2 and again in columns 3-4 in format I2. M(NP) is equal to the number of coefficients to follow on the card.

2. In the remaining columns, the coefficients of the polynomial are punched in adjacent fields, each in format F9.3, starting in column 5 with the constant term and continuing with succeeding powers of s.

The coefficient cards are followed by the sequence cards. These cards are punched as follows:

---

[1]The variables XL and XLAM are used only to identify the output data when $[A_1(s) + A_1(-s)]_a$ is determined.

1. In columns 1-2, L1 (format I2). L1 is the identifying number of the polynomial involved in the polynomial multiplication, addition, or scalar multiplication.

2. In columns 3-4, L2 (format I2). L2 is the identifying number of the other polynomial involved in the polynomial multiplication or polynomial addition. This field is left blank if scalar multiplication is to be performed.

3. In columns 5-6, L3 (format I2). L3 may be +1, 0, or -1. If L3=+1, polynomial number L1 and polynomial number L2 are multiplied; if L3=0, polynomial number L1 and polynomial number L2 are added; and if L3=-1, polynomial number L1 is multiplied by a scalar x. The value of x must be punched on the next card in format F6.2.

D.2 APPLICATION OF THE PROGRAM

Tables D-1 and D-2 give the program and the input data used to calculate $[A_1(s) + A_1(-s)]_a$, equation (6.32). In order to obtain equation (6.32), the polynomial manipulations indicated in the following expression must be performed

$$[A_1(s) + A_1(-s)]_a = 2\frac{E(s)}{F(s)} \qquad (D.1)$$

where

$$E(s) = \{[P_3(s)P_3(-s) - P_2(s)P_2(-s)]N_2(s)N_2(-s)C_1(s) - P_3(s)P_3(-s)[sN_2(-s)M_1(s) - sN_2(s)M_1(-s)]\}W(s)W(-s) + \lambda Q(s)Q(-s)C_1(s)P_3(s)P_3(-s)[sN_2(-s)M_1(s) - sN_2(s)M_1(-s)]$$

and

$$F(s) = W(s)W(-s)C_1(s)P_3(-s)P_3(s)[sN_2(-s)M_1(s) - sN_2(s)M_1(-s)]$$

Each of the terms which appears in E(s) and F(s) is a polynomial in s:

$$M_1(s) = 4 + 4s \qquad (D.2)$$

$$N_2(s) = 1 + 6s + s^2 \qquad (D.3)$$

$$P_3(s) = 1 + 12s + 13s^2 + 3s^3 \qquad (D.4)$$

$$P_2(s) = 1 + 7s + 3s^2 \qquad (D.5)$$

$$Q(s) = 0 + s \qquad (D.6)$$

$$W(s) = 1 \qquad (D.7)$$

$$C_1(s) = 1.143 + 0s - 200s^2 \qquad (D.8)$$

Thus, to calculate $[A_1(s) + A_1(-s)]_a$, the following polynomials are read into the computer in the order shown: $M_1(s)$, $M_1(-s)$, $N_2(s)$, $N_2(-s)$, $P_3(s)$, $P_3(-s)$, $P_2(s)$, $P_2(-s)$, $Q(s)$, $Q(-s)$, s, $-s$, $W(s)$, $W(-s)$, and $C_1(s)$ (see Table D-2). Note that the last polynomial in Table D-2, 4 - 100s, is not required for the determination of $[A_1(s) + A_1(-s)]_a$. It was included because the same set of input polynomials was used for the determination of $[A_2(-s) + A_3(s)]_a$, and it was required for the latter calculation.

Shown in Table D-2, following the control card and the sixteen coefficient cards are the sequence cards used to determine $[A_1(s) + A_1(-s)]_a$. The first sequence card contains the numbers 7, 8, and 1. Thus, the computer multiplies polynomial number 7, $P_2(s)$, by polynomial number 8, $P_2(-s)$, and obtains the polynomial, $P_2(s)P_2(-s)$, which is then assigned the number 17. Proceeding in this manner, the remainder of the sequence cards are used. In this example, the numerator polynomial is polynomial number 38, and the denominator polynomial is polynomial 39. These polynomials are given in Table D-3. Note that in order to obtain the coefficients in the form listed in Table 6-4, it is necessary that $(-s^2)$ be factored out of the polynomials shown in Table D-3.

The input data shown in Table D-2 can be used to calculate $[A_1(s) + A_1(-s)]_a$ for any value of λ simply by changing the thirty-ninth card to correspond to the value of λ considered and by appropriately modifying the value of XLAM on the control card.

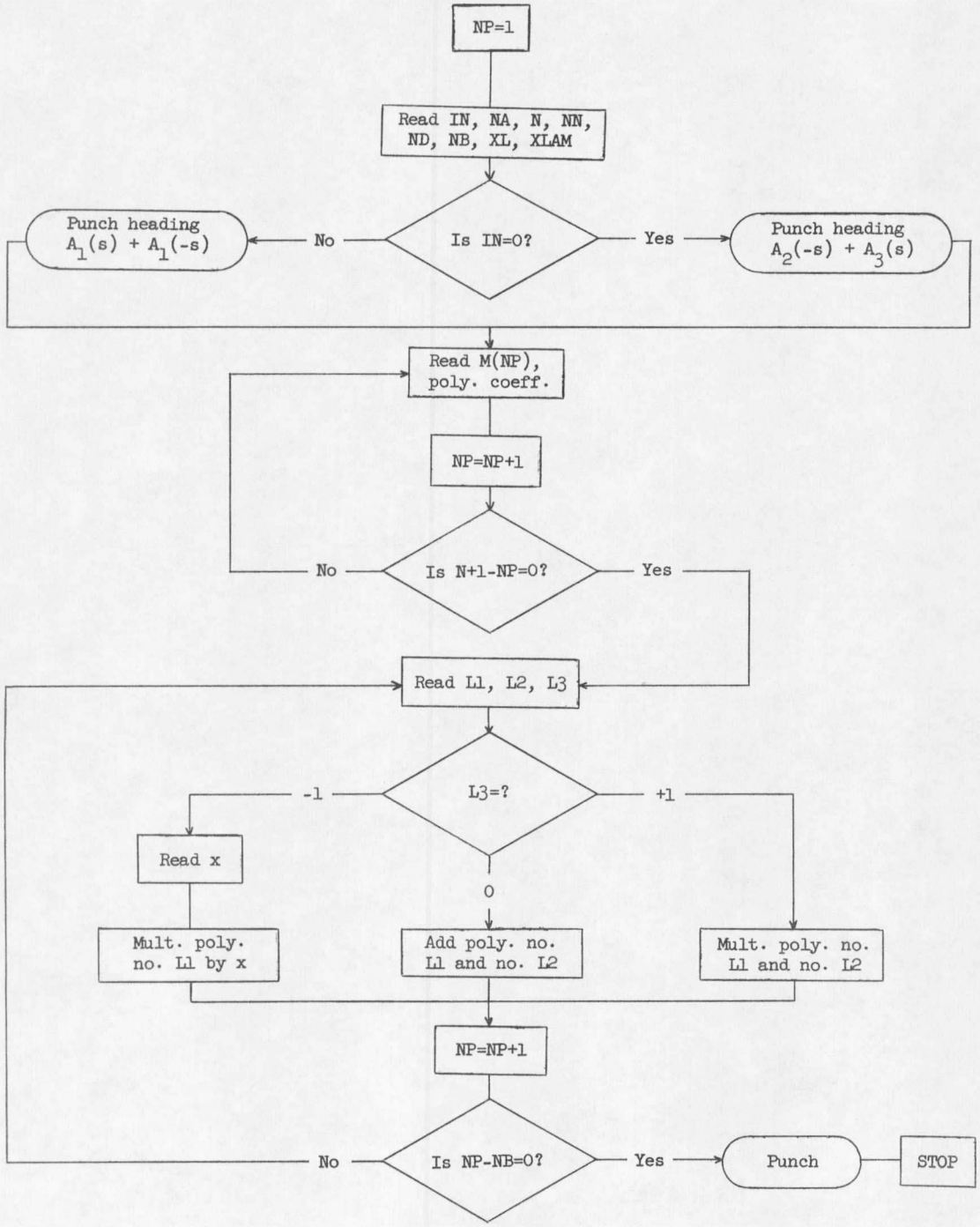


Figure D-1. Flow diagram of digital computer program used to perform polynomial multiplication, addition and scalar multiplication.

Table D-1. Fortran program used to perform polynomial multiplication, addition and scalar multiplication.

```
   DIMENSION C(50,22),M(50)
 1 FORMAT (2I2,7F9.3)
 2 FORMAT (3I2)
 3 FORMAT (F6.2)
 4 FORMAT(//2X, 12HNUMERATOR = ,F12.2, 3X, F12.2, 5HS     ,
  1F12.2, 5HS**2 , F12.2, 5HS**3 )
 5 FORMAT (/12X,F12.2,5HS**4 ,F12.2,5HS**5 ,
  1F12.2,5HS**6 ,F12.2,5HS**7 )
 6 FORMAT (/12X,F12.2,5HS**8 ,F12.2,5HS**9 ,
  1F12.2,5HS**10,F12.2,5HS**11)
 7 FORMAT(//14HDENOMINATOR =    ,F12.2, 3X, F12.2, 5HS     ,
  1F12.2, 5HS**2 , F12.2, 5HS**3 )
 8 FORMAT (/12X,F12.2,5HS**12,F12.2,5HS**13,
  1F12.2,5HS**14,F12.2,5HS**15)
 9 FORMAT (/12X,F12.2,5HS**16,F12.2,5HS**17,
  1F12.2,5HS**18,F12.2,5HS**19)
11 FORMAT (//4X, 12HA2(-S)+A3(S))
13 FORMAT (6I2,2F6.2)
17 FORMAT (//4X,30HL/2(A1(S)+A1(-S)) FOR LAMBDA =,
  1F6.2,8H AND L =,F6.2)
18 FORMAT(/12X,F12.2,5HS**20)
   NP=1
   READ 13,IN,NA,N,NN,ND,NB,XL,XLAM
   IF(IN) 14,14,16
14 PUNCH 11
   GO TO 10
16 PUNCH 17,XLAM,XL
10 READ 1, M(NP),L, (C(NP,I),I=1,L)
   IA=M(NP)+1
   DO 12 I=IA,NA
12 C(NP,I)=0.
   NP=NP+1
   IF (N-NP+1) 15,15,10
15 READ 2, L1,L2,L3
   IF (L3) 20,45,80
20 READ 3,X
   DO 30 K=1,NA
30 C(NP,K)=X*C(L1,K)
   M(NP)=M(L1)
   GO TO 110
```

Table D-1. (continued).

```
 45 IF (M(L1)-M(L2)) 50,50,60
 50 M(NP)=M(L2)
    GO TO 65
 60 M(NP)=M(L1)
 65 DO 70 J=1,NA
 70 C(NP,J)= C(L1,J)+C(L2,J)
    GO TO 110
 80 M(NP)=M(L1)+M(L2)-1
    L=M(NP)
    K1=M(L1)
    K2=M(L2)
    DO 100 K=1,L
    C(NP,K)=0.
    DO 100 I=1,K1
    DO 100 J=1,K2
    IF (I+J-K-1) 100,90,100
 90 C(NP,K)=C(NP,K) + C(L1,I)*C(L2,J)
100 CONTINUE
101 IC=M(NP)+1
    DO 105 K=IC,NA
105 C(NP,K)=0.
110 NP=NP+1
111 IF(NP-NB) 15,120,120
120 PUNCH 4, (C(NN,I),I=1,4)
    PUNCH 5, (C(NN,I), I=5,8)
    PUNCH 6, (C(NN,I),I=9,12)
    IF (NA-12) 123,123,121
121 PUNCH 8, (C(NN,I),I=13,16)
    IF(NA-16) 123,123,122
122 PUNCH 9, (C(NN,I),I=17,20)
    IF(NA-20) 123,123,130
130 PUNCH 18, C(NN,21)
123 PUNCH 7, (C(ND,I),I=1,4)
    PUNCH 5, (C(ND,I),I=5,8)
    PUNCH 6, (C(ND,I),I=9,12)
    IF (NA-12) 126,126,124
124 PUNCH 8, (C(ND,I),I=13,16)
    IF (NA-16) 126,126,125
125 PUNCH 9, (C(ND,I),I=17,20)
126 CALL EXIT
    END
```

Table D-2. Input data for the calculation of $[A_1(s) + A_1(-s)]_a$ for $\lambda = 1$, equation (D.1).

```
 11616383940   1.      1.
 2 2      4.          4.
 2 2      4.         -4.
 3 3      1.          6.          1.
 3 3      1.         -6.          1.
 4 4      1.         12.         13.          3.
 4 4      1.        -12.         13.         -3.
 3 3      1.          7.          3.
 3 3      1.         -7.          3.
 2 2      0.          1.
 2 2      0.         -1.
 2 2      0.          1.
 2 2      0.         -1.
 1 1      1.
 1 1      1.
 3 3      1.143       0.       -200.
 2 2      4.       -100.
 7 8 1
 5 6 1
 3 4 1
1314 1
 910 1
 4 1 1
 3 2 1
2211 1
2312 1
2425 0
1826 1
27  -1
 -1.0
17  -1
 -1.0
1829 0
3019 1
3115 1
3228 0
3320 1
21  -1
  1.
2715 1
3536 1
3437 0
2036 1
```

Table D-3. The numerator and the denominator of $[A_1(s) + A_1(-s)]_a$ for $\lambda = 1$, equation (D.1).

L/2(A1(S)+A1(-S)) FOR LAMBDA = 1.00 AND L = 1.00

NUMERATOR =	0.00	0.00S	-45.72S**2	0.00S**3
	13332.95S**4	0.00S**5	-539695.96S**6	0.00S**7
	1565628.10S**8	0.00S**9	-1044836.70S**10	0.00S**11
	229082.29S**12	0.00S**13	-14400.00S**14	0.00S**15
DENOMINATOR =	0.00	0.00S	-45.72S**2	0.00S**3
	13404.10S**4	0.00S**5	-951113.83S**6	0.00S**7
	966098.44S**8	0.00S**9	-227282.29S**10	0.00S**11
	14400.00S**12	0.00S**13	0.00S**14	0.00S**15

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