



On the development of a refined finite element free from discretization-dispersion for elastic wave propagation modeling
by Venkatachalam Lakshminarayanan

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY in Civil Engineering
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Abstract:

The problems associated with the application of the finite element method in two-dimensional elastic wave propagation in solids are investigated in detail. Two serious defects associated with the conventional displacement compatible finite elements are discretization-dispersion and artificial farfield coupling.

Discretization-dispersion causes the wavefronts of different frequencies to propagate at different velocities. Lumped mass models propagate waves at slower, while, models with consistent mass representation propagate waves at faster than continuum velocities. Lumped mass representation of the displacement compatible finite element models is identical to finite difference models, and, hence the same amount of discretization-dispersion is present in the finite difference models as well. The magnitude of discretization-dispersion is also a function of the geometry of the model. A test procedure is developed to evaluate the magnitude of the discretization-dispersion in a two-dimensional continuum, modelled by finite elements. For the displacement compatible rectangular finite elements, below the cut off frequency, the maximum discretization-dispersion is of the order of 92% for dilatational waves and 20% for shear waves. When used in modelling a longitudinal bar, less discretization-dispersion is noted for dilatational waves (6.5% at 10.000 cps), while more for shear waves (52.5% at 2000 cps).

The artificial farfield coupling causes a given node to oscillate before the arrival of the wave front at the node. It arises due to the changes in the inertial properties of the continuum associated with the discretization. Its magnitude is found to increase with increasing frequency of the wave front and increasing distance of the node from the point of application of the external force.

A refined rectangular finite element is developed. The refined element is free from the above two defects. The refined element has compatibility of strains across the interelement boundaries, in addition to the compatibility of displacements. First order Hermitian interpolation polynomials are used to represent the displacements in the interior of the element.

Responses of a longitudinal bar, modelled by assemblies of both refined and conventional finite elements of square and nonsquare geometries, are determined under the action of several kinds of external force excitations. The kinds of force excitations include single and multifrequency harmonic excitations and a step-rise force input. Propagation of dilatational and shear waves are studied. Models built with conventional finite elements propagate wavefronts faster due to the discretization-dispersion while the refined element models propagate waves at continuum velocities. Low pass characteristics of the finite element models are verified. All the finite element models attenuate the wave fronts above the cut off frequency. A two-dimensional problem of the propagation of dilatational and shear waves past a step change elevation is studied. While eliminating the discretization-dispersion, the model does not suppress the natural dispersion which would be seen in the corresponding continuum.

ON THE DEVELOPMENT OF A REFINED FINITE ELEMENT FREE FROM DISCRETIZATION-
DISPERSION FOR ELASTIC WAVE PROPAGATION MODELLING

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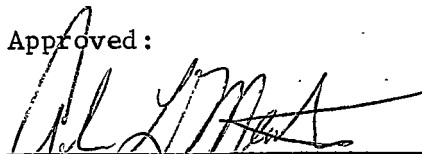
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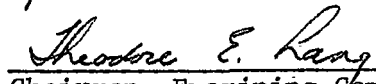
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कर्मण्येवाधिकारस्ते मा फलेषु कदाचन ।

मा कर्मफलहेतुर्भूर्मा सगोस्त्वकर्मणि ॥



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ABSTRACT

The problems associated with the application of the finite element method in two-dimensional elastic wave propagation in solids are investigated in detail. Two serious defects associated with the conventional displacement compatible finite elements are discretization-dispersion and artificial farfield coupling.

Discretization-dispersion causes the wavefronts of different frequencies to propagate at different velocities. Lumped mass models propagate waves at slower, while, models with consistent mass representation propagate waves at faster than continuum velocities. Lumped mass representation of the displacement compatible finite element models is identical to finite difference models, and, hence the same amount of discretization-dispersion is present in the finite difference models as well. The magnitude of discretization-dispersion is also a function of the geometry of the model. A test procedure is developed to evaluate the magnitude of the discretization-dispersion in a two-dimensional continuum, modelled by finite elements. For the displacement compatible rectangular finite elements, below the cut off frequency, the maximum discretization-dispersion is of the order of 92% for dilatational waves and 20% for shear waves. When used in modelling a longitudinal bar, less discretization-dispersion is noted for dilatational waves (6.5% at 10,000 cps), while more for shear waves (52.5% at 2000 cps).

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A refined rectangular finite element is developed. The refined element is free from the above two defects. The refined element has compatibility of strains across the interelement boundaries, in addition to the compatibility of displacements. First order Hermitian interpolation polynomials are used to represent the displacements in the interior of the element.

Responses of a longitudinal bar, modelled by assemblies of both refined and conventional finite elements of square and nonsquare geometries, are determined under the action of several kinds of external force excitations. The kinds of force excitations include single and multifrequency harmonic excitations and a step-rise force input. Propagation of dilatational and shear waves are studied. Models built

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I. INTRODUCTION

1.1 General

Rigid body dynamics is founded on the assumption that when a force is applied at any point in a body, the resultant stresses set every other point in motion instantaneously. The force is considered as producing a linear acceleration of the whole body together with an angular acceleration about its center of gravity. In the theory of elasticity, on the other hand, the body is considered to be in equilibrium under the action of applied forces and the elastic deformations are assumed to have reached their static values. These assumptions lead to sufficiently accurate answers for problems in which the time between the application of force and the setting up of effective equilibrium is short in comparison with the times in which the observations are made. When, however, we are considering forces which are applied only for very short periods of time, or are changing rapidly, the effects must be considered in terms of propagation of stress waves. A stress wave is a mechanical disturbance travelling across a body. By disturbance we mean a deviation from the state of static equilibrium which is characterized by a minimum of the potential energy. Since a disturbance involves energy, wave propagation implies the transfer of energy from one region to another in the body.

1.2 Definition and Scope of the Study

Consider a homogeneous isotropic elastic simply connected body occupying a domain C surrounded by a surface S in a three-dimensional Euclidean space as shown in Fig. 1.2-1. Let the displacements and surface tractions be specified as functions of time on the parts S_I and S_{II} respectively, of the surface, S . There may be a part of the surface, S_{III} , which is neither subjected to any surface tractions nor restrained against any possible deformations. This is a special case of S_{II} with identically zero surface tractions at all times. Further, an external force vector field of definite magnitude may be acting at any or all points of the body inside the domain C .

The motion of any point C under the action of external forces and surface tractions is represented by Navier's equation of motion:

$$\rho \ddot{u}_i - (\lambda + G) (u_{k,k})_{,i} - G u_{i,jj} = \phi_i (x_1, x_2, x_3, t)$$

$$x_1, x_2, x_3 \in C, t > t_0 \text{ and } i = 1, 2, 3. \quad (1.2-1)$$

and the boundary conditions:

$$u_i (x_1, x_2, x_3, t) = U_i (x_1, x_2, x_3) \text{ on } S_I, t > t_0 \quad (1.2-2)$$

$$(\lambda u_{i,i} \delta_{ij} + 2G u_{i,j}) n_j = \phi_i (x_1, x_2, x_3, t) \text{ on } S_{II}, t > t_0 \quad (1.2-3)$$

$$(\lambda u_{i,i} \delta_{ij} + 2G u_{i,j}) n_j = 0 \text{ on } S_{III}, t > t_0 \quad (1.2-4)$$

and the initial conditions:

$$\dot{u}_i (x_1, x_2, x_3, t) = V_i (x_1, x_2, x_3) \text{ when } t = t_0 \quad (1.2-5)$$

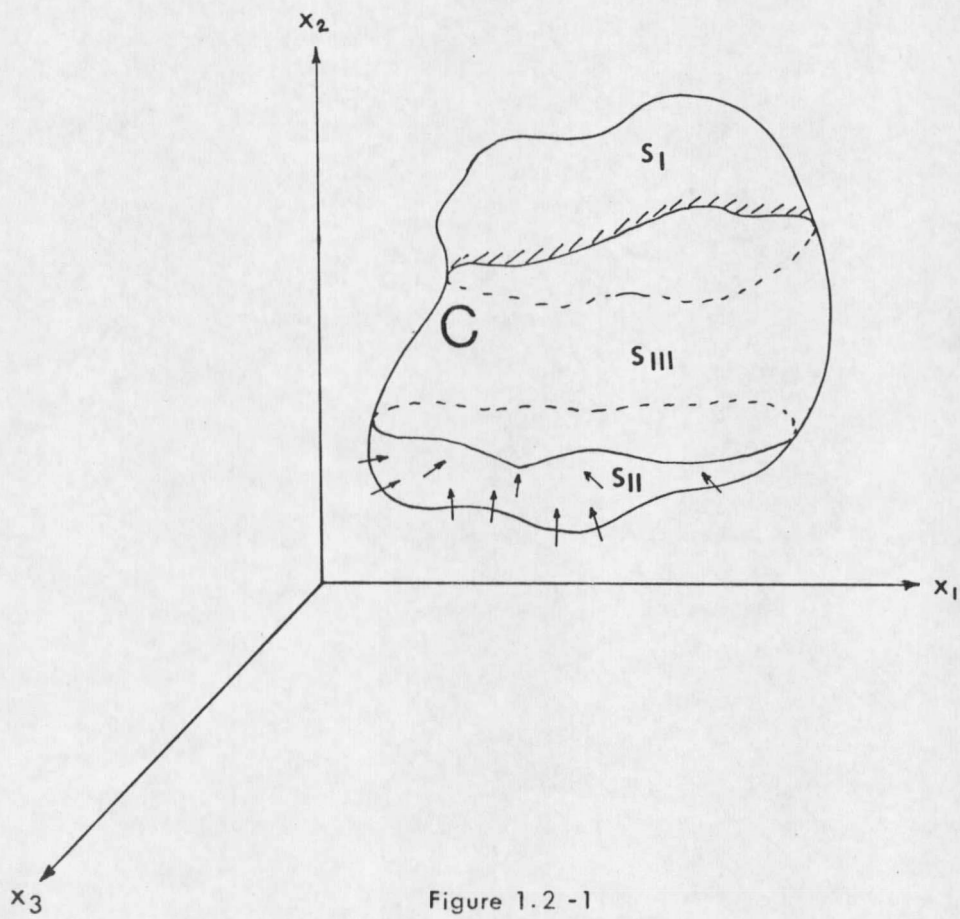


Figure 1.2 -1

and

$$u_i(x_1, x_2, x_3, t) = W_i(x_1, x_2, x_3) \text{ when } t = t_0 \quad (1.2-6)$$

$$x_1, x_2, x_3 \in C.$$

where: ρ is the mass density

λ and G are Lamé's constants

V_i is the initial velocity vector field

W_i is the initial displacement vector field and

ϕ_i is the external force vector field.

For the existence of unique solutions to the above differential equations and boundary conditions, certain conditions are to be satisfied by the constants and functions [10]. The constants ρ , λ and G are to be real and positive to insure the propagative nature of the medium. In any physical problem, the mass density is always greater than zero. Positive-definiteness of the strain energy demands that the modulus of elasticity E and the rigidity modulus G and hence the constant λ be greater than zero. So, a real elastic solid propagates waves at definite velocities depending on its physical properties and the nature of the disturbance. The functions $U_i(x_1, x_2, x_3, t)$ should be such that all its nine spatial derivatives are at least sectionally continuous in order that infinite strains may not develop.

Some of the methods of deriving exact solutions to the above differential equations and boundary conditions are: (i) D'Alembert solution which is also known as the method of characteristics, (ii) Integral transform methods such as Fourier and Laplace Transformations and (iii) Reduction to a Sturm-Liouville problem and solving for eigenvalues and eigenfunctions which is the same as the modal superposition method. The latter method is a discretization of a continuous system in a generalized sense and the procedure leads to solutions converging to the exact solution in the limit. It is possible to use the above methods only in problems where the domains are defined by simple geometries. When the geometry becomes complex, they are not of much use and one must use a numerical procedure, settling for approximate solutions.

It is the purpose of this dissertation to describe the problems associated with the treatment of stress waves in solids by the use of the finite element method, and to develop a refined finite element for an accurate representation of linear elastic wave propagation. The conventional finite elements used in elastostatic analysis yield incorrect solutions when used in wave propagation because of the presence of the defect of discretization-dispersion [52]. The refined element, free from this defect, is developed by imposing stronger compatibility conditions in the development of the element. To give sufficient background to the investigation, detailed

discussion of the related aspects of the classical theory of wave propagation and a comprehensive review of the literature on the application of finite elements to wave propagation are presented in the next chapter.

II. ON CERTAIN ASPECTS OF STRESS WAVE PROPAGATION IN SOLIDS -

EXACT AND APPROXIMATE SOLUTIONS PROCEDURES

2.1 Introduction

In this chapter we attempt to present a brief coverage of the history, development and the current state of the art of certain aspects of classical methods, discretized approaches and the recent developments of computer oriented solution techniques of finite difference and finite elements applied to wave propagation problems. Wide diversity of interest in the subject makes it clear that no one general survey could justifiably cover all the aspects. The review on classical theory is confined to the propagation of longitudinal and flexural waves and some half-space problems. Discussion on wave propagation in rods centers around the idea of dispersion - the dependence of phase velocity on the wavelength. This is done to orient the reader to distinguish and appreciate the new concept of discretization-dispersion, which this investigation encompasses. Discussions are presented on finite difference solution procedures to wave propagation problems. Literature on the application of finite elements to wave propagation is limited to several papers published in the last seven years. A detailed and critical review of these investigations is presented.

2.2 A Brief History of the Development of the Theory of Stress Wave Propagation

Brilliant work of French mathematicians at École Polytechnique in France during the first half of the nineteenth century led to a systematic development of the Theory of Elasticity and consequently stress wave propagation [56]. Poisson [43] succeeded in integrating the equations of motion in elasticity and showed the possibility of propagation of two types of waves, namely dilatational and equivoluminal. He also developed the equations of longitudinal, torsional and lateral vibrations of a bar and calculated the frequencies and modes of vibration. Navier [35] developed the corresponding equation for the lateral vibration of plates. Fifty years later Saint Venant [49] investigated the lateral and longitudinal impacts on bars and forced vibrations of bars in detail.

During the later half of the nineteenth century, mathematicians at Cambridge University in England made significant contributions in wave propagation. Stokes [55] studied the vibrations of a single-degree-of-freedom system and gave solutions for arbitrary initial displacements and velocities and forcing functions. Rayleigh [46] made a significant contribution by recognizing the existence of surface waves and their importance in seismological studies because of their ability to travel longer distances than the dilatational and equivoluminal waves without significant attenuation. His other

principal contributions in wave propagations and vibrations are documented in his treatise, "Theory of Sound" [47]. Love [27] explained the presence of horizontally polarized surface waves due to the difference in the elasticity and density of the outer layer of the earth from that of the interior. In the beginning of the twentieth century, Lamb [23] studied the propagation of a pulse in a semi-infinite half-space. He treated several cases of surface normal line load, point load and internal dilatational line and point sources.

Even with all these investigations spanning about one hundred years, techniques of analyzing wave propagation in practical problems of bounded geometries were slow to come. Wave propagation in bounded geometries are associated with reflection and refraction at the boundaries depending on the nature of the problem. In general an incident wave produces both reflected and refracted waves. Boundary conditions determine their relative amplitudes. These often couple with the divergence and curl of the displacement potentials so that a dilatational wave (P-wave), for example, produces both dilatational and equivoluminal waves (also known as shear or S-waves). The propagation of surface or interfacial waves also must be considered. When all these wave types are considered together, the problem becomes quite involved. Because of this, exact theory of elasticity solutions are possible only for very simple and idealized problems

and even these solutions are too involved to be used in engineering applications. During the first half of the twentieth century, there was no significant development in the wave propagation theory, except for the appearances of solutions to some half space problems. Interest was revived after World War II and substantial progress has been made in the last twenty-five years, especially due to the development of many approximate numerical solution procedures which depend upon the modern high speed computer.

2.3 Stress Wave Propagation in Rods and Beams

It is of interest to review briefly the analytical work done so far on elastic stress wave propagation in rods and beams for an appreciation of the concept of dispersion and group velocity. These include compressional, torsional and flexural waves. In recent years, several surveys and monographs have appeared both primarily and partly centering on the topic [1, 22, 33]. The analyses include exact theories and theories based on restrictive assumptions and simplifications.

2.3-1 Compressional Waves

Pochhammer and Chree [42, 9] independently investigated the simple case of vibrations of a solid circular cylinder of infinite length in terms of general elastic equations. Since they used the exact governing equations of elasticity, it is also known as the

exact theory. For convenience the equations of motion are written in cylindrical coordinates, given as:

$$\left. \begin{aligned} (\lambda+2G) \frac{\partial \Delta}{\partial r} - \frac{2G}{r} \cdot \frac{\partial \omega_x}{\partial \theta} + 2G \frac{\partial \omega_\theta}{\partial x} &= \rho \frac{\partial^2 u_r}{\partial t^2} \\ (\lambda+2G) \cdot \frac{1}{r} \cdot \frac{\partial \Delta}{\partial \theta} - 2G \frac{\partial \omega_r}{\partial x} + 2G \frac{\partial \omega_x}{\partial r} &= \rho \frac{\partial^2 u_\theta}{\partial t^2} \\ (\lambda+2G) \frac{\partial \Delta}{\partial x} - \frac{2G}{r} \cdot \frac{\partial}{\partial r} (r \omega_\theta) + \frac{2G}{r} \cdot \frac{\partial \omega_r}{\partial \theta} &= \rho \frac{\partial^2 u_x}{\partial t^2} \end{aligned} \right\} \quad (2.3-1)$$

where

$$\left. \begin{aligned} \Delta &= \frac{1}{r} \cdot \frac{\partial}{\partial r} [r u_r] + \frac{1}{r} \cdot \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_x}{\partial x}, \\ 2\omega_r &= \frac{1}{r} \cdot \frac{\partial u_x}{\partial \theta} - \frac{\partial u_\theta}{\partial x}, \\ 2\omega_\theta &= \frac{\partial u_r}{\partial x} - \frac{\partial u_x}{\partial r} \\ \text{and } 2\omega_x &= \frac{1}{r} \left[\frac{\partial}{\partial r} (r u_\theta) - \frac{\partial u_r}{\partial \theta} \right] \end{aligned} \right\} \quad (2.3-2)$$

where u_r , u_θ and u_x are the displacements in the radial, tangential and axial directions, λ and G are Lamé's constants and the three rotation components satisfy the identity

$$\frac{1}{r} \cdot \frac{\partial}{\partial r} [r \omega_r] + \frac{1}{r} \cdot \frac{\partial \omega_\theta}{\partial \theta} + \frac{\partial \omega_x}{\partial x} = 0 \quad (2.3-3)$$

An axisymmetric displacement solution is assumed:

$$\left. \begin{aligned} u_r &= U(r) \exp [2\pi i(vt-ax)] \\ u_\theta &= 0 \\ u_x &= W(r) \exp [2\pi i(vt-ax)] \end{aligned} \right\} \quad (2.3-4)$$

In this relation, v is the frequency, and the wavelength $\Lambda = 1/a$, where a is the wavenumber. When these assumed displacements are substituted into the partial differential equations (2.3-1), two ordinary differential equations are obtained for the functions $U(r)$ and $W(r)$. Solution of these equations yields:

$$\left. \begin{aligned} U(r) &= A \frac{\partial}{\partial r} J_0(h'r) + C a J_1(\kappa'r) \\ \text{and} \\ W(r) &= Aia J_0(h'r) + \frac{iC}{r} \frac{\partial}{\partial r} [r J_1(\kappa'r)] \end{aligned} \right\} \quad (2.3-5)$$

A and C are arbitrary constants, J_0 and J_1 are Bessel functions of first kind and the parameters h' and κ' are given by:

$$\left. \begin{aligned} h'^2 &= \frac{4\pi^2 v_p^2}{\lambda + 2G} - 4\pi^2 a^2 \\ \text{and} \\ \kappa'^2 &= \frac{4\pi^2 v_p^2}{G} - 4\pi^2 a^2 \end{aligned} \right\} \quad (2.3-6)$$

Substituting these equations into the boundary conditions of zero normal and shear stresses along the boundary results in two equations of zero surface stress:

$$A\{2G[\frac{\partial^2 J_0(h'r)}{\partial r^2}]_{r=r_0} - \frac{4\pi^2 v_p^2 \lambda}{\lambda + 2G} J_0(h'r_0)\} + 2GC[\frac{\partial J_1(\kappa'r)}{\partial r}]_{r=r_0} = 0$$

and

$$4\pi Aa[\frac{\partial J_0(h'r)}{\partial r}]_{r=r_0} + C[8\pi^2 a^2 - \frac{4\pi^2 v_p^2}{G}]J_1(\kappa'r) = 0 \quad (2.3-7)$$

where r_0 is the radius of the cylinder.

This is a set of two homogeneous equations in A and C. In order to have a nontrivial solution the determinant of the coefficient matrix must vanish. The resulting equation expresses the relation between the frequency v and the wavelength λ . The product of v and λ is the phase velocity, c , which provides the relation between the phase velocity and the wavelength. Such a phase velocity-wavelength relationship is known as a dispersion relation. Physically, this means that waves of different wavelengths travel at different phase velocities. A pulse having a wide band of frequencies in its Fourier spectrum will undergo change in shape as it travels along the rod, which is a distortion or natural dispersion of the pulse. Dispersion curves for the first two modes are shown in Fig. 2.3-1. It may be noted that the waves of long wavelengths travel at velocity

$$c_0 = \sqrt{\frac{E}{\rho}} \text{ known as the bar velocity.}$$

An analysis of this phenomenon of dispersion was discussed by Davies [16] by employing Kelvin's method of stationary phase. In this method of analysis a pulse of infinite amplitude and zero duration is imagined to be applied to the infinite bar at some point

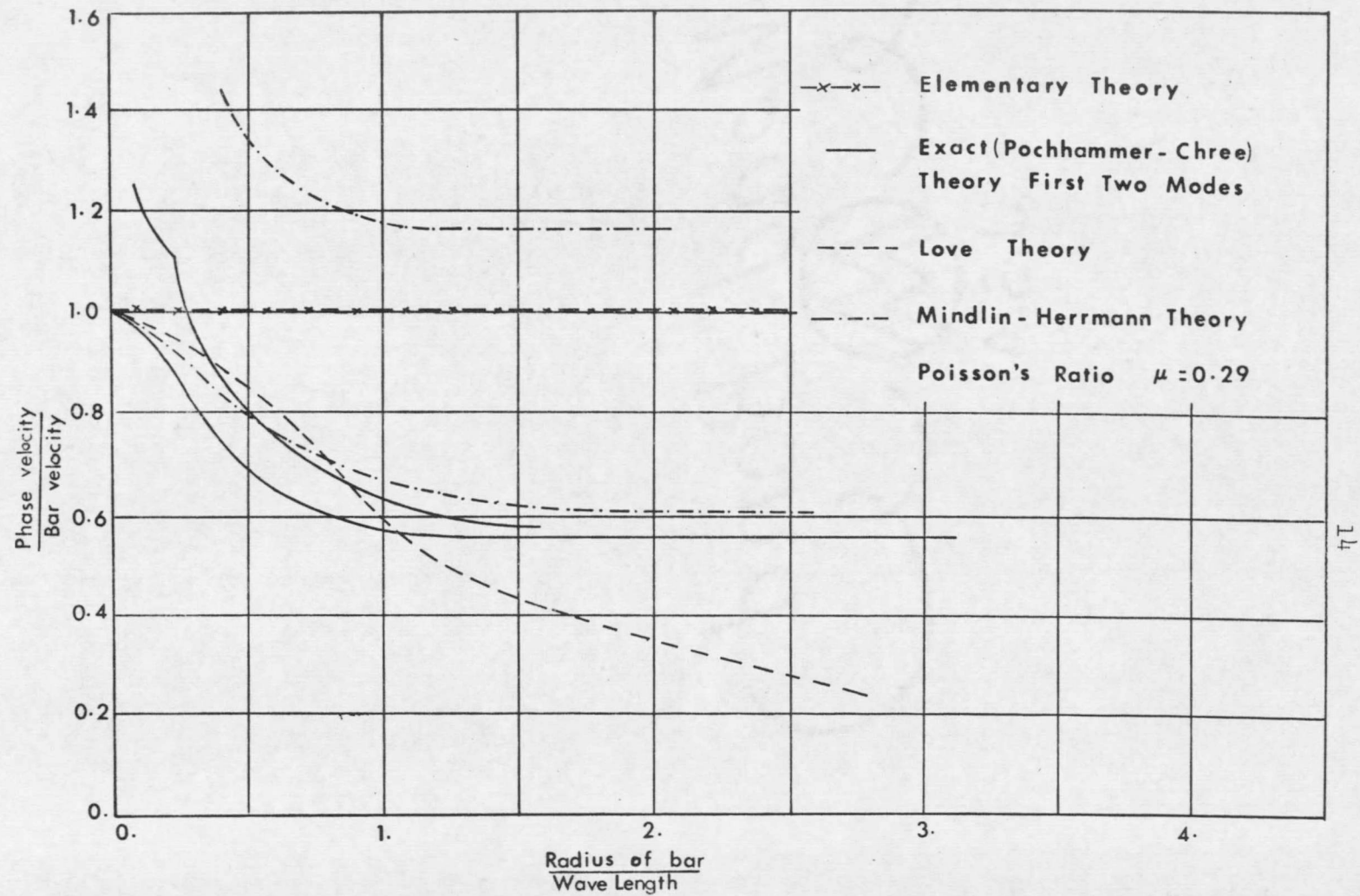


Figure 2.3-1 Dispersion Curves For Longitudinal Waves In Cylindrical Bars

$x = 0$ at time $t = 0$. The Fourier spectrum of such a pulse, sometimes called the Dirac-delta function, contains all frequencies with equal amplitudes. At the time of application, $t = 0$, the entire set of sinusoidal components are phased such that they reinforce one another at the point $x = 0$ to produce infinite amplitude. At all other locations they interfere with one another to produce zero amplitude. Because all the Fourier components propagate at different phase velocities, it should be expected that at some station x' along the bar, wave groups of different wavelengths arrive at different times. At certain times there would be a grouping of considerable number of waves of nearly the same wavelength which reinforce one another producing a local disturbance large in comparison with the nearly completely cancelled components surrounding it. This larger disturbance travels at group velocity c_g which is given by:

$$c_g = c - \lambda \frac{dc}{d\lambda} \quad (2.3-8)$$

The group velocity represents the velocity of propagation of energy contained in a packet of waves whose wavelengths do not differ greatly from that of the dominant group.

The Pochhammer-Chree theory is applicable only to an infinitely long bar in which sinusoidal wave trains of infinite wavelengths are propagated in either direction. It cannot be used in a finite or even semi-infinite bar having arbitrarily prescribed displacement

or stress distribution on the end cross-section, since the infinite set of modes in the theory do not form a complete set with respect to section coordinates. In addition, the complexity of the frequency equation introduces great difficulties in the analysis of the propagation of an arbitrary pulse. As a result approximate theories have been constructed which contain the essential features of the exact problem in a simplified form, or some restrictive assumptions introduced, to simplify the problem. Such are the elementary theory, Love's approximate theory and the Mindlin-Herrmann theory. They are discussed below briefly.

The elementary theory is the simplest with the assumptions that plane cross-sections remain plane and that only axial stresses are present, being uniformly distributed over the cross-section. When the equation of motion is substituted into the equation of continuity and constitutive equations, the well known wave equation results:

$$\frac{\partial^2 \sigma_{xx}}{\partial t^2} = c_o^2 \frac{\partial^2 \sigma_{xx}}{\partial x^2} \quad (2.3-9)$$

in which $c_o = \sqrt{E/\rho}$, the bar velocity in a uniform bar. The general solution to this differential equation is of the form:

$$\sigma_x = f(x - c_o t) + F(x + c_o t) \quad (2.3-10)$$

The solution represents the sum of two waves travelling in opposite directions, the functions $f(x)$ and $F(x)$ representing the travelling

pulse. It is seen that each travelling wave has its shape preserved as it proceeds along the bar with velocity c_0 .

A particular solution of eqn. (2.3-9) is that which describes the steady state vibration of an infinitely long uniform rod. It is assumed that the axial stress σ_{xx} varies sinusoidally with time and axial coordinates as follows:

$$\sigma_{xx} = (\sigma_{xx})_0 \sin \frac{2\pi}{\Lambda} (x - ct) \quad (2.3-11)$$

where Λ is the wavelength of the sinusoidal wave propagating with a velocity c . On substitution of eqn. (2.3-11) into (2.3-9) it is found that the velocity c is equal to c_0 regardless of the value of the wavelength Λ , i.e., there is no dispersion. Since any pulse can be represented in terms of its sinusoidal components by means of the Fourier expansion, and each sinusoidal component is propagated with the same velocity c_0 , no phase differences develop at remote stations along the bar and hence the pulse travels with no change in shape.

In the approximate theory given by Love [26], the assumptions are made that the plane cross-sections remain plane, and that the axial stresses are uniformly distributed over the cross-section as in the elementary theory. However, terms are included to account for the inertia associated with the lateral expansion or contraction due to the Poissonic effect connected with the axial deformations.

The assumption is made that the displacement in the radial direction is proportional to the radial coordinate r and to the axial strain $\partial u_x / \partial x$, that is, $u_r = -\mu r \partial u_x / \partial x$. Love developed his wave equations by applying Hamilton's principle to the energy expression which includes transverse inertia.

Adequacy of Love's theory can be estimated by a comparison with the exact theory using the dispersion curves. The curve labelled 'Love theory' in Fig. 2.3-1 is in agreement with the lowest branch of the exact theory as long as the disturbance in the rod does not contain wavelengths less than about ten times the radius of the bar.

In the neighborhood of the wave front, there would rather be a large rate of change of lateral dimension especially when short wavelengths are present. Shearing strains of non-negligible magnitude are associated with such deformations. An approximate theory considering these shearing strains and the corresponding shearing stresses was proposed by Mindlin and Herrmann [34]. As in Love's theory it is assumed that the radial displacements are proportional to the radial coordinate, initially plane cross-sections remain plane, and that the problem is axisymmetric. While Love postulated a direct proportionality between radial and axial strain, Mindlin and Herrmann do not impose this relation in advance. In order to allow more freedom in the correction for shear and inertia, they introduced two new parameters. The resulting frequency

equations or dispersion relations are presented in Fig. 2.3-1.

The new parameters are adjusted so as to make the lower mode curve agree with the exact theory for short wavelengths.

It may be said that the exact theory of longitudinal wave propagation is not suited to engineering applications. Since its analysis is extremely difficult mathematically, progress in this direction is slow, though understanding of the basic mechanism of propagation is not lacking. Experimental results have helped to indicate the nature of mechanism while the highly developed approximate theories have made it possible to predict analytically the essential propagational characteristics. For practical engineering applications the available approximate theories are enough to yield reasonably good answers for design purposes.

2.3-2 Flexural Waves

In speaking of flexural waves we mean that type of vibrations of a beam in which some portions are in flexure and the elements of the neutral axis move laterally during the motion. The problem of flexural vibrations is somewhat more difficult than longitudinal vibrations because the elastic deformations are more complex. Further, it will be seen that even the elementary theory leads to dispersion. In the following presentation a brief discussion of the elementary theory precedes a presentation of the exact theory of

Pochhammer and Chree and this is followed by an approximate theory developed by Timoshenko.

The vibration equation of a beam based on the Euler-Bernoulli theory of bending is:

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = 0 \quad (2.3-12)$$

where A is the area of cross-section, I is the moment of inertia of cross-section. The elementary theory is primarily based on the assumption that the cross-sectional dimensions are small compared with the length of the beam and that the plane cross-sections before bending remain plane during bending. A wave type solution may be taken of the form:

$$w = C \cos (2\pi vt - ax) \quad (2.3-13)$$

where v is the frequency and a is the wavenumber. Substituting this solution into the differential equation, one obtains the expression for phase velocity:

$$c = \frac{2\pi R c_0}{\Lambda} \quad (2.3-14)$$

where $R = \sqrt{I/A}$, the radius of gyration and c_0 is the bar velocity.

The corresponding dispersion relation is illustrated in Fig. 2.3-2.

The phase velocity, c , is inversely proportional to the wavelength implying that the waves of infinitely short wavelengths propagate at

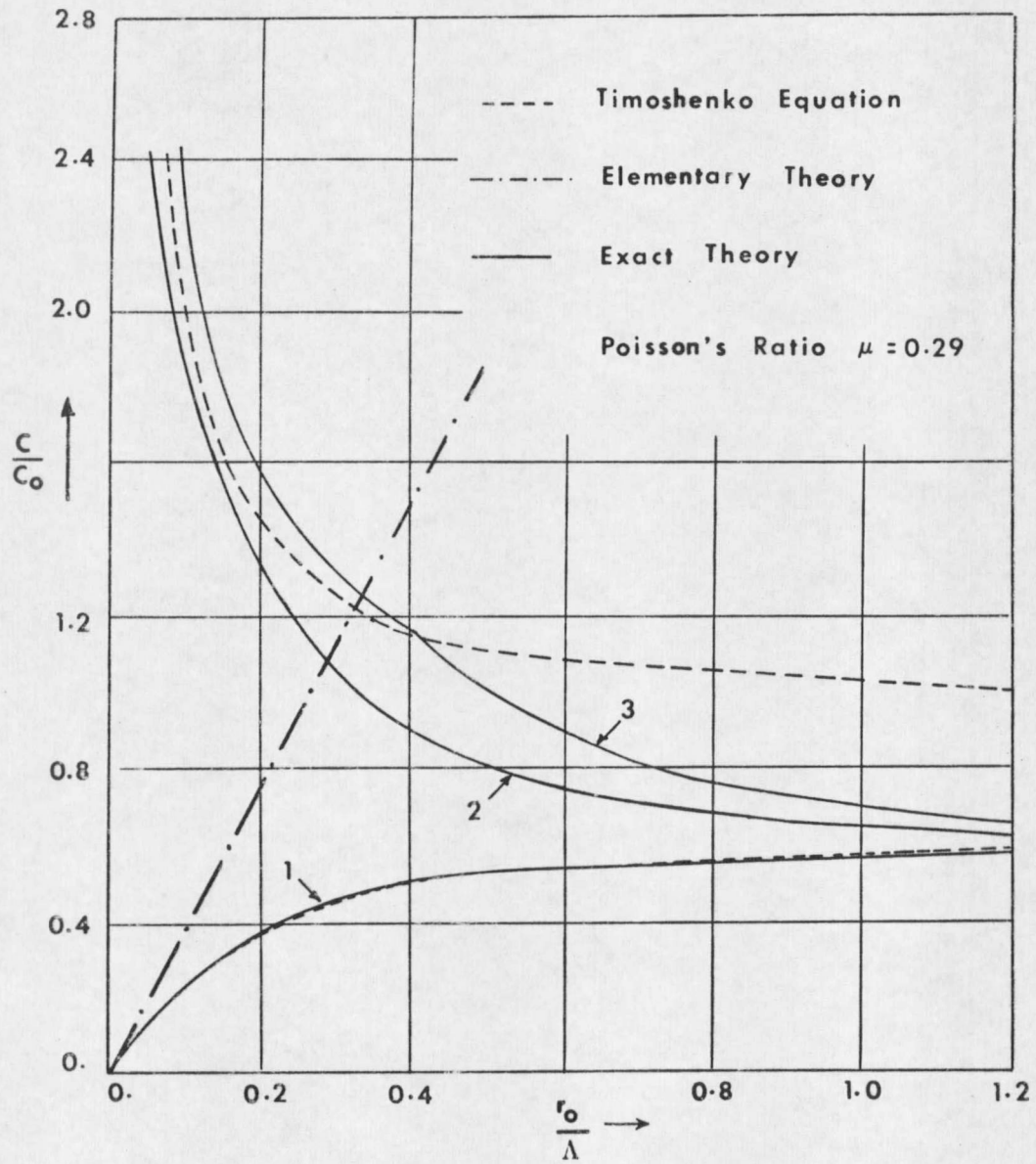


Figure 2.3-2

Phase Velocity Curves For Flexural Waves in
a Solid Circular Cylinder

infinite velocities, which is not physically possible.

The case of an infinitely long beam of uniform circular cross-section was investigated independently by Pochhammer [42] and Chree [9]. Taking the axis of the circular cylinder to be oriented in the x-direction, the equations of motion written in cylindrical coordinates (2.3-1) are valid here also. Solutions considered are of the form:

$$\left. \begin{aligned} u_r &= U \cos \theta \exp[2\pi i(vt - ax)] \\ u_\theta &= V \sin \theta \exp[2\pi i(vt - ax)] \\ u_x &= W \cos \theta \exp[2\pi i(vt - ax)] \end{aligned} \right\} \quad (2.3-15)$$

where U, V and W are functions of r only according to:

$$\left. \begin{aligned} U &= A \frac{\partial}{\partial r} [J_1(h'r)] + Bf \frac{\partial}{\partial r} [J_1(\kappa'r)] + \frac{C}{r} J_1(\kappa'r) \\ V &= -\frac{A}{r} J_1(h'r) - \frac{Bf}{r} J_1(\kappa'r) - C \frac{\partial}{\partial r} [J_1(\kappa'r)] \\ W &= ifA J_1(h'r) - iB\kappa'^2 J_1(\kappa'r) \end{aligned} \right\} \quad (2.3-16)$$

where A, B and C are constants, J_1 is the Bessel function of the first kind and

$$\left. \begin{aligned} \kappa'^2 &= \frac{4\pi^2 v^2 \rho}{G} - 4\pi^2 a^2 \\ \text{and} \\ h'^2 &= \frac{4\pi^2 v^2 \rho}{\lambda + 2G} - 4\pi^2 a^2 \\ \text{and } f &= 2\pi a \end{aligned} \right\} \quad (2.3-17)$$

The equations for displacements may now be expressed in terms of the arbitrary constants A, B and C and then introduced in the radial stress expression. Specification of zero stresses along the boundary leads to three homogeneous equations in A, B and C. The determinant of the coefficients must vanish and the expansion of the determinant yields the dispersion relation based on the elastic constants. By expressing it in a nondimensional form, an equation of two variables c/c_0 and r_0/Λ with Poisson's ratio as a parameter results. Abramson [1a] calculated the lowest three modes which are shown in Fig. 2.3-2.

The Pochhammer-Chree theory just described is too complicated to be useful in practical engineering. On the other hand, the elementary theory is too limited to describe adequately the transient behavior resulting from sharp bending impacts. The gap between the over simplified and the too highly complex theories is well bridged by an intermediate theory by Timoshenko [57]. This theory adds terms which correct for shearing strains and rotatory inertia of beam elements. Whereas the elementary theory is built on the assumption that plane sections remain plane during bending and remain perpendicular to the deformed middle curve of the beam, the Timoshenko theory is less restrictive in that a section remains plane but may be inclined at different angles to the deformed middle curves. Also, whereas the conventional equilibrium relations are assumed in the elementary theory, the Timoshenko theory includes the inertia term

arising from the rotation of the beam elements in the plane of vibration of the beam.

A discussion of the essential details of the Timoshenko theory follows. The forces, moments and directions of motion of a beam element are shown in Fig. 2.3-3. The equations of the behavior of the beam element are as follows:

$$\left. \begin{aligned} \frac{\partial M}{\partial x} - Q &= \rho I \frac{\partial \omega}{\partial t} \text{ (rotational)} \\ \frac{\partial Q}{\partial x} &= \rho A \frac{\partial \dot{w}}{\partial t} \text{ (transverse)} \end{aligned} \right\} \text{ equations of motion} \quad (2.3-18)$$

$$\left. \begin{aligned} \frac{\partial K}{\partial t} &= \frac{\partial \omega}{\partial t} \text{ (bending)} \\ \frac{\partial \gamma}{\partial t} &= \frac{\partial}{\partial x} [\dot{w}] + \omega \text{ (shear)} \end{aligned} \right\} \text{ equations of continuity} \quad (2.3-19)$$

and

$$\left. \begin{aligned} M &= EIK \text{ (bending)} \\ Q &= A_s \gamma \text{ (shear)} \end{aligned} \right\} \text{ constitutive equations} \quad (2.3-20)$$

where M is the bending moment

Q is the shearing force

K is the axial rate of change of section angle, $= \frac{-\partial \psi}{\partial x}$

γ is the shearing strain, $= \frac{\partial w}{\partial x} - \psi$

$\dot{w}$ is the transverse velocity, $\frac{\partial w}{\partial t}$

A is the section area

A_s is the area parameter defined by $\iint \gamma(Z) dA = \gamma A_s$ where

ω is the angular velocity

$\gamma(Z)$ is the true variation of γ over the area. When all but one

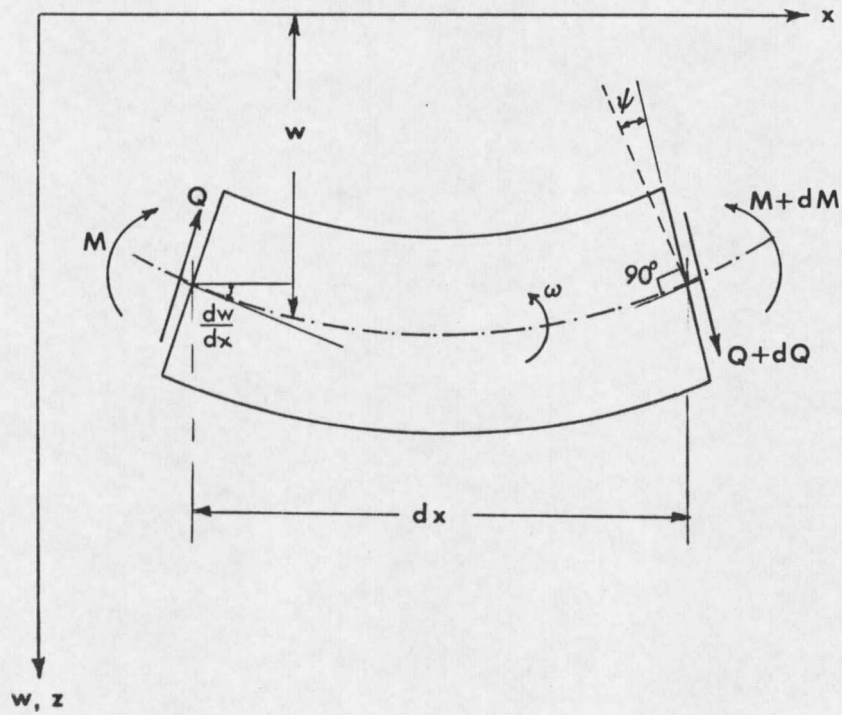


Fig. 2.3-3 Element of Timoshenko beam in bending

variable is eliminated from the six equations above, a single fourth order partial differential equation results. The equation is:

$$\frac{\partial^4 M}{\partial x^4} - \left(\frac{1}{C_o^2} + \frac{1}{C_Q^2} \right) \frac{\partial^4 M}{\partial x^2 \partial t^2} + \frac{1}{C_o^2} \cdot \frac{1}{C_Q^2} \cdot \frac{\partial^4 M}{\partial t^4} + \frac{1}{C_o^2 k^2} \frac{\partial^2 M}{\partial t^2} = 0 \quad (2.3-21)$$

$$\text{where } C_o^2 = \frac{E}{\rho}; \quad C_Q^2 = \frac{GA_s}{\rho A}; \quad k^2 = \frac{I}{A} \quad (2.3-22)$$

This is the form usually designated as the Timoshenko equation.

The quantity $\frac{GA_s}{\rho A}$ is a modified form of the square of the shear wave velocity which is $\frac{G}{\rho}$. This modification results from the assumption of plane cross-sections.

The steady state equation to the Timoshenko equation is of the form:

$$M = M_o \exp [2\pi i(vt - ax)] \quad (2.3-23)$$

For a bar of radius r_o , the resulting equation is:

$$16\pi^4 a^4 - \left(\frac{C^2}{C_o^2} + \frac{C^2}{C_Q^2} \right) 16\pi^4 a^4 + \frac{C^4}{C_o^2 C_Q^2} 16\pi^4 a^4 - \frac{C^2}{C_o^2 k^2} 4\pi^2 a^2 = 0$$

or

$$1 - \left(\frac{C^2}{C_o^2} + \frac{C^2}{C_Q^2} \right) + \frac{C^4}{C_o^2 C_Q^2} - \frac{C^2}{C_o^2 k^2} \cdot \frac{1}{4\pi^2 a^2} = 0 \quad (2.3-24)$$

The solution of this equation as a function of $\frac{C}{C_o}$ and $\frac{r_o}{\lambda}$ yields two separate branches as shown in Fig. 2.3-2 and labeled 'Timoshenko Theory.' One of the branches agrees well with the Pochhammer-Chree theory discussed previously. Recent experimental and theoretical

investigations [48,54] have established the adequacy of the Timoshenko bending mechanism to describe the response of beams to sharp impacts.

2.4 Waves in Semi-infinite Half-space

Wave propagation in infinite and semi-infinite elastic media are subjects of long standing interest in seismology. Recently they have become of increasing interest in engineering. The underlying practical reason is the need for nuclear blast-resistant and earthquake resistant structures. Complexities of these half-space problems stem from the mode conversion of dilatational and shear waves at the free surface and the generation of Rayleigh surface waves. Rayleigh's discovery demonstrated the fundamental difference in the characteristics of the half space problem from the infinite medium problem. Lamb [23] was the first to study the propagation of a pulse in the half space. He treated the cases of normal surface line load and point loads, and internal dilational line and point sources. For the surface source problems Lamb evaluated the surface displacements bringing forth the important fact that the largest disturbance in the farfield was due to the Rayleigh wave. Several other investigators have studied the surface source problems. Of note are the solution derived by Sauter [51], Lang [24] and Pekeris [40,41].

Of particular interest in nuclear blast half-space wave studies are the problems involving travelling line and point surface loads. Sneddon [53] treated the problem of a line load moving with constant velocity deriving a steady state solution. Cole and Huth [13] extended Sneddon's work to transonic and supersonic cases. They presented a numerical study of stress and acceleration fields as a function of the ratio of load velocity to shear wave velocity. Their results show that the maximum stresses and accelerations increase with load velocity but the influence of Rayleigh wave singularity on these results was not assessed. Interesting are Cole and Huth's comparison of the stresses with their static counterparts (Boussinesque-Flamant solution). For a travelling load with the velocity near the Rayleigh wave velocity, stresses show multifold increase over the corresponding static values.

2.5 A Brief History of Discretization Approaches to Wave Propagation Problems

A detailed account of the historical development of discretized models for wave propagation studies is given by Brillouin [8]. Development of such models date back to the time of Newton [36] when he derived a formula for the velocity of sound in air by the use of a one-dimensional lattice made up of springs and lumped masses connected in series. The formula was good only for wavelengths

great in comparison with the distance between the lumped masses in the lattices. The reason why Newton considered a discretized rather than a continuous structure was that nothing was known about partial differential equations at that time. In 1830, Cauchy used Newton's model in an attempt to account for the dispersion of light waves. Cauchy assumed that light waves were just elastic waves but with very high frequencies. He obtained the result that the velocity of propagation was independent of the wavelength for waves with wavelengths long compared with the distance between the lumped masses. However, for shorter waves, he showed that the velocity of propagation was a function of the wavelength. The result is correct for elastic waves.

In 1841, Baden-Powell [8] computed the velocity of a wave propagating along one axis of a cubic lattice structure as a function of the wavelength. The model consisted of point masses of mass m spaced along a straight line at a distance d from one another. These masses are elastically bound to each of its neighbors with the same restoring force. His equation for the velocity of propagation V of the waves as a function of wavelength is:

$$V = V_{\infty} \frac{\left| \sin \frac{\pi d}{\Lambda} \right|}{\frac{\pi d}{\Lambda}} \quad (2.5-1)$$

where Λ is the wavelength and V_{∞} is the velocity for waves of

infinitely long wavelength. This is the discretization-dispersion relation. It should be mentioned here that the present day one-dimensional finite-difference models with lumped mass representation are the same as that developed by Baden-Powell and hence have the same discretization-dispersion relation. The above discretization studies were done by theoretical physicists and were not used in engineering applications until recently along with the development of finite difference and finite element methods applied to elasticity.

2.6 Finite Difference Method and Its Application to Wave Propagation Studies

The finite difference method is an approximate numerical procedure for solving partial differential equations. It replaces the governing differential equations and boundary conditions by finite difference approximations in terms of a finite number of unknown values of the dependent variables at a number of points (nodes) within, or just outside, the domain of integration [2]. The finite difference equations form a set of algebraic equations which are solved for the unknowns. The set of unknowns are comprised of the discrete values of the dependent variables at nodes. To solve for the unknowns one requires the availability of an equal number of algebraic equations. These equations are provided by: (i) corresponding to each unknown at an internal node, the finite difference approximation to the governing differential equations at the node

is formed, and (ii) corresponding to each unknown at an external node the finite difference approximation is made of the boundary condition.

The finite difference method has been applied to wave propagation problems in axisymmetric solids, both elastic and elastoplastic, and elastoplastic layered media by Ang et al. [3,4,50]. The continuum is represented by a lattice of lumped masses and springs evenly spaced based on the choice of the coordinate system. Nodal strain components, which are the spatial derivatives of displacement functions at nodes, are written in terms of adjacent nodes by a central differencing scheme. They are related to the stress components through the constitutive equations of the material. Equilibrium equations of elasticity relate the spatial derivatives of the stress components to the second order time derivatives of the displacements. Accelerations at nodes are related to the displacements at surrounding node points and the problem is reduced to a set of difference equations. These difference equations are integrated by one of several available numerical procedures.

2.7 Finite Element Method

The finite element method may be defined as a systematic procedure for solving partial differential equations, commonly occurring in Mathematical Physics by an approximate variational formulation

analogous to the Ritz method. It involves modifying the problem of solving the differential equation to the problem of minimizing a functional subject to specified boundary conditions. A solution is assumed as a function of several unknown parameters and these parameters are determined by minimizing the functional. In elastostatic finite element analysis the unknown parameters are the nodal displacements and the functional represents the total potential energy, whose minimum is a necessary and sufficient condition for equilibrium. The minimization equations yield the so called stiffness matrix, a relation between the generalized (nodal) displacements and the generalized (nodal) forces.

The method was developed in the 1950's for structural analysis problems [60] in aircraft design. Since that time it has seen significant development and application to a wide variety of problems. Though the method has been used extensively for the last two decades, it is only recently that its mathematical foundations are becoming better understood [17,38]. It has been sufficiently generalized so that a finite element model can be generated for any type of continuous abstract function whose values, for example, can be scalar, vectors or tensors of any order in a given Hilbert Space [39].

The finite element method is not used in wave propagation problems extensively because of the discretization-dispersion present in

the currently available finite elements. This dispersion causes wave fronts of different wavelengths to travel at different velocities, resulting in an inadequate representation of the continuum. In the following section is presented a detailed and critical review of several investigations on the subject during the past decade.

2.8 Current State-of-the-Art in the Application of Finite Elements to Stress Wave Propagation in Solids

Shipley et al.[52] use displacement compatible elements to solve several wave propagation problems and compare solutions with existing solutions. Using a lumped mass representation, they report that high-frequency waves travel at reduced velocities while low frequency waves travel at approximately continuum group velocities. This results in a predominance of high frequency oscillations in the trailing portions of the disturbances. This is the characteristic of a lumped mass representation, as seen in the model developed by Baden-Powell in 1842 and discussed in Sec. 2.5. Problems solved by Shipley et al. include a step force applied at one end of a one-dimensional rod, a spherically symmetric wave motion, a dilatational cylindrical wave problem, and a plane strain problem of an elastic half-space subjected to a travelling surface load. In the computed results velocity responses show wild oscillations. The authors suggest an interpretative scheme of averaging the envelopes of peaks and valleys. The displacement response is smoother as it

is the integral of the velocity function. Strains are computed from the central differencing of the nodal displacements. The corresponding stress response curves are smooth, similar to the displacement response curves. However, in the above study, computations were carried out for a short period after the application of external forces, and hence, the accumulated effects of discretization-dispersion are not brought out clearly.

Costantino [14] investigated the application of finite elements to stress wave propagation. He also uses displacement compatible rectangular and triangular elements and a lumped mass representation, but apparently, does not recognize the presence of discretization dispersion. He solves a spherical wave problem and the problem of a step-force applied to a one-dimensional bar and computes the displacement response for a short period after the application of the external force. As may be expected the displacement response shows close agreement with the exact solution. He also solved the problem of a half-space subjected to a dynamic surface load. The computed stress response at several node points inside the half space are shown to be in good agreement with the solutions obtained elsewhere by finite difference method [3]. In fact, the finite element formulation by the author would yield the same equations of motion obtained by finite difference method and apparently that is why the solutions agree closely.

Benjamin B. D'Ewart, Jr. et al. [5] use the one-dimensional finite element model with lumped mass representation for the approximate computation of the velocity of sound in air. The study is essentially the same as that of Newton, as reported earlier. The resulting model is identical to the finite difference model. As expected, computed values of velocities are smaller than the true value.

Lysmer, together with several other authors [28,29,30] have published several papers on the application of finite elements to wave propagation studies. In Ref. [29], they represent a semi-infinite medium by a finite element model with energy absorbing boundaries by relating the velocities and stresses at node points in the infinite medium. A lumped mass representation is used. They compute the displacement response of a footing resting on a semi-infinite half space under a vertical harmonic force applied to the footing. The study is limited to the case of long waves, with wavelengths greater than twelve times the element size. Good agreement with the exact solution is obtained for this case as the effect of discretization-dispersion is not significant for long waves.

In Ref. [28] Lysmer uses the finite element model to determine the physical dispersion relation of Rayleigh wave propagation in a layered medium. He uses the displacement compatible rectangular element to derive the stiffness matrix and supplements it with a

lumped mass representation. In Ref. [30], Lysmer and Waas investigate the dispersion of horizontally polarized shearwaves (Love waves) with the same finite element, modified for out-of-plane deformation. This study is an extension of the previous study of dispersion of Rayleigh waves. The authors intended to study the physical dispersion of Rayleigh and Love waves but in fact the study is done on the dispersion of the finite element model rather than that of the continuum. Since the dispersion study done by Lysmer relates to the study of discretization-dispersion of the finite element to be presented later in this investigation, it is worthwhile to review Lysmer's work.

Fig. 2.8-1 shows the model used by Lysmer in his study of dispersion of Rayleigh waves. He considers the propagation of a plane harmonic wave in the x-direction with phase velocity c and frequency ν (cps) and assumes the equations of motion at the nodal points as:

$$\delta_x = u_x(y) \exp [2\pi i(\nu t - ax)]$$

and

(2.8-1)

$$\delta_y = u_y(y) \exp [2\pi i(\nu t - ax)]$$

where a is the wavenumber, $\frac{1}{\lambda}$. All finite elements are of the same length in the horizontal direction, equal to h , but no such restriction is made in the vertical direction. Adjacent nodes in a

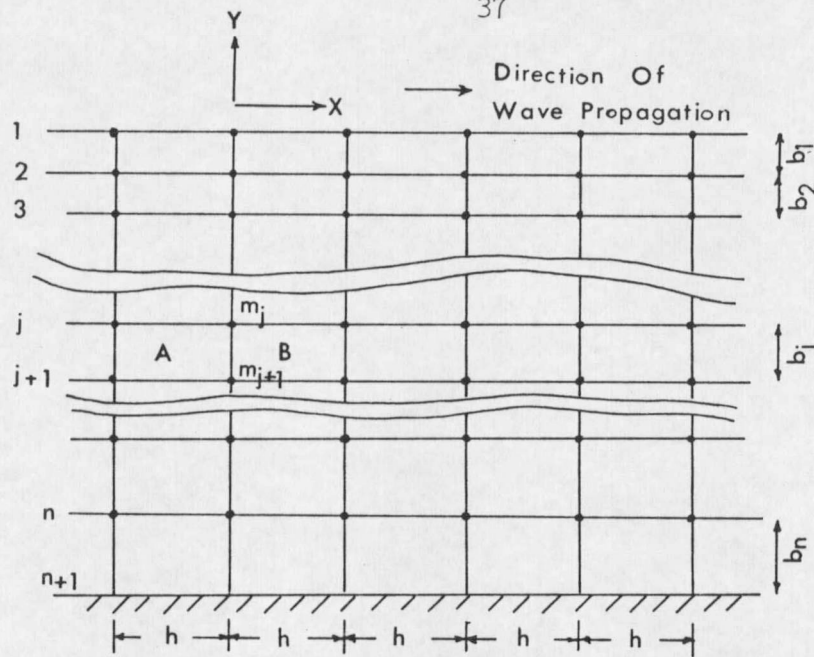


Fig. 2.8-2

Lumped Mass Model For Layered System

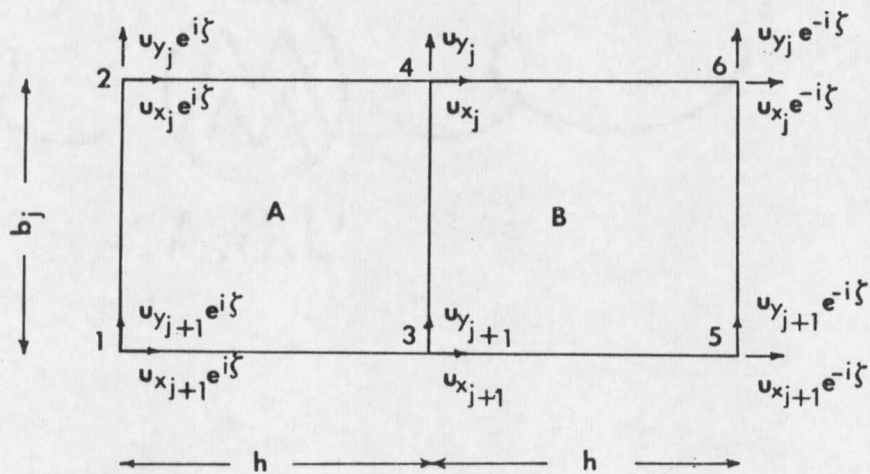


Fig. 2.8-3

Displacements of Elements A and B

horizontal row have the same phase difference of $2\pi ah$, say equal to ζ . Further, the displacement at any one node can be expressed in terms of the displacements of the neighboring nodes, as the phase difference is known (Fig. 2.8-2).

Shown are two elements in a row in the j th layer. The displacements at nodes on the common boundary can be written as a vector:

$$\{u\} = \begin{Bmatrix} u_{x_j} \\ u_{y_j} \\ u_{x_{j+1}} \\ u_{y_{j+1}} \end{Bmatrix} \quad (2.8-2)$$

The corresponding nodal force vector is:

$$\{Q\} = \begin{Bmatrix} Q_{x_j} \\ Q_{y_j} \\ Q_{x_{j+1}} \\ Q_{y_{j+1}} \end{Bmatrix} \quad (2.8-3)$$

The vectors $\{u\}$ and $\{Q\}$ are related through the matrix $[L]$ which Lysmer designates as the layer stiffness matrix.

$$\{Q\} = [L]\{u\} \quad (2.8-4)$$

The layer stiffness matrix includes the elastic coupling between the neighboring nodes on the same horizontal row and its terms are functions of phase parameter ζ , and hence also a function of the wavenumber a . It is to be noted that Lysmer does not include the

elastic coupling of all the neighboring nodes (e.g., considering node 3 in Fig. 2.8-2, its elastic interaction with nodes 5 and 1 only are included whereas the interactions with nodes 2, 4 and 6 are not considered) which is in error. When all the nodal displacements and nodal forces along a vertical line are considered, they are related by a stiffness matrix obtained by superimposing the individual layer stiffness matrices, say $[\bar{K}]$ such that:

$$\{\bar{Q}\} = [\bar{K}]\{\bar{u}\}, \quad (2.8-5)$$

where:

$$\{\bar{u}\} = \begin{Bmatrix} u_{x_1} \\ u_{y_1} \\ u_{x_2} \\ u_{y_2} \\ \vdots \\ u_{x_{n+1}} \\ u_{y_{n+1}} \end{Bmatrix} \quad \text{and} \quad \{\bar{Q}\} = \begin{Bmatrix} Q_{x_1} \\ Q_{y_1} \\ Q_{x_2} \\ Q_{y_2} \\ \vdots \\ Q_{x_{n+1}} \\ Q_{y_{n+1}} \end{Bmatrix}$$

where $[\bar{K}]$ is a $(2n+2) \times (2n+2)$ matrix. Equations of motion of free oscillations of all the nodes along a vertical line are written as:

$$[\bar{M}]\ddot{\{\bar{u}\}} + [\bar{K}]\{\bar{u}\} = \{0\} \quad (2.8-6)$$

where $[\bar{M}]$ is the diagonal lumped mass matrix. The eigenvalue problem is then defined by

$$|[\bar{K}] - \omega^2[\bar{M}]| = 0 \quad (2.8-7)$$

The eigenvalues so determined are function of the parameter ζ and hence functions of the wavenumber a . They are frequencies, which when divided by the wavenumber, yield the phase velocity values. Hence the dispersion relation of the phase velocity to the wavelength is determined. Lysmer compares his dispersion relation to an already available solution by Tolstoy and Usdin [59] and found good agreement for two modes. These are the fundamental modes of oscillation of the nodes along and perpendicular to the direction of propagation of the wavefront.

An important factor to be noted in Lysmer's investigation is that he made the horizontal length of the element equal to zero, which makes the parameter $\zeta = 0$, i.e., there is no phase difference between adjacent nodes in a horizontal row. This forces the wavelengths involved to be of infinite magnitude, for which computed phase-velocities are in close agreement with continuum velocities.

As will be shown in the following investigation, the limit of wavelengths to twelve times the grid dimension as suggested by Lysmer, avoids the region of strong discretization-dispersion. No indication is given by Lysmer that the dispersion he encounters is from the finite element modelling, rather than from natural causes. This distinction and quantification of the finite element

discretization-dispersion is the central topic of this thesis.

III. PROPAGATIONAL CHARACTERISTICS OF VARIOUS FINITE ELEMENT MODELS IN ONE AND TWO DIMENSIONS

3.1 Introduction

In the classical theory of wave propagation, the term dispersion is associated with the dependence of the phase velocity of a wavefront to its wavelength. This dispersion arises from the energy absorption characteristics in the propagating medium or from the reflection of wave fronts from the boundaries. It is characterized by the distortion of an input signal while propagating through the medium because of various frequency components travelling at different velocities, attenuating at different rates, or due to some complex pattern of reflection from the boundaries.

A real medium may exhibit dispersion and absorption (with or without dissipation) depending on the interaction between the wave and the medium. The information concerning this can be obtained by comparing the response function with the input signal. The dispersion relation is commonly expressed as the relationship between the phase velocity and the wave number which is the inverse of the wavelength. A plane wave is represented by the expression:

$$C \exp\{2\pi i a(ct-x)\}$$

where C is a constant, the amplitude of the wavefront,

$$a \text{ is the wavenumber, } = \frac{1}{\lambda},$$

c is the phase velocity, and

t and x are the independent time and space variables. When the phase velocity c is expressed as a function of the wavenumber a , it is known as the dispersion relation. In the previous chapter, dependence of phase velocity on wavelength and the dispersion relations were discussed in detail for longitudinal and flexural waves.

The dependence of phase velocity on the wavelength causes the wavefronts of different frequencies to travel at different velocities causing a signal containing a packet of waves either to change its shape, attenuate, or both. Because of dispersion, the group velocity of a given packet of waves differs from the phase velocities of the individual wavefronts. The group velocity represents the velocity of propagation of energy contained in a packet of waves while the phase velocity represents the velocity of an individual wavefront. By a wavepacket is meant a signal of finite duration in time whose frequency spectrum is quasimonochromatic. To be specific, if ω_0 is the driving frequency, the signal is a packet of frequencies of a narrow bandwidth with a central peak of ω_0 .

The dispersion relation of waves obtained by the Pochhammer-Chree theory, discussed in the previous chapter, is dependent on the geometry rather than the physical properties of the propagating medium. That is, the velocity of the wavefront varies as the ratio

of a characteristic cross-sectional dimension to the wavelength. This is the property of a bounded medium, arising from the complex pattern of reflection from the boundaries. If the propagation takes place in an infinite medium where no reflections are involved, the internal friction plays a significant role in dispersion. All real media exhibit some amount of internal friction. When the magnitude of the internal friction is frequency dependent causing the wavefronts to attenuate at different rates, it is always associated with strong dispersion.

In a bounded medium with no internal friction and where the effect of reflections from the boundaries is not significant, an input signal containing a quasimonochromatic wave packet travels without any appreciable distortion. A finite element model representing such a medium should exhibit the same "nondispersive" property, i.e., propagating the signal without any appreciable distortion.

A finite element not exhibiting the above property is called a dispersive element. The dispersion in the finite element model is not to be confused with the dispersion in the continuum due to reflections or internal friction, discussed before. The dispersive character of the finite element is the inability of the model to represent the dynamic behaviour of the continuum truly. It is an artificial dispersion resulting from the effective local variations of the elastic and inertial properties associated with the discretization

and, hence, is termed as discretization-dispersion. It is characterized by the element propagating wavefronts of different frequencies at different velocities. Many finite elements which give good convergent solutions in elastostatic problems are dispersive when used in wave propagation problems.

In the following sections of this chapter, several finite elements are evaluated for their dispersive properties in wave propagation modelling. Some of the elements which exhibit discretization-dispersion in wave propagation have been in use for elastostatic problems and have been found to yield good results as the discretization-dispersion does not have any influence on the static analysis. The finite elements considered are: (i) one-dimensional displacement-compatible element with one degree-of-freedom per node, (ii) one-dimensional first-order strain-compatible element with two degrees-of-freedom per node, (iii) two-dimensional displacement-compatible element with two degrees-of-freedom per node and (iv) two-dimensional first-order strain-compatible element with six degrees-of-freedom per node. In both one and two-dimensional formulations, first-order strain-compatible elements do not exhibit any discretization-dispersion. The study on one-dimensional elements was first done by Lang [25], and documented in his unpublished papers.

3.2 One-Dimensional Displacement-Compatible Finite Elements and Its Dispersive Characteristics

The element is sketched in Fig. 3.2-1. It has two degrees-of-

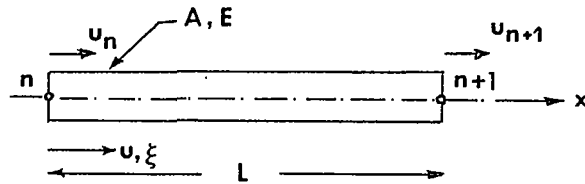


Fig. 3.2-1. One-dimensional Displacement Compatible Finite Element.

freedom, one per node. A linearly varying displacement function (which is a Lagrange interpolation polynomial) is used in the derivation of mass and stiffness matrices. The element is used for solving problems where the strain energy is a positive-definite function of the spatial derivatives, i.e., longitudinal elastic bar, suspended elastic cable or similar problems. The element satisfies the three criteria normally specified [64], i.e., ability to exhibit rigid body motion without storing strain energy, exhibit the state of constant strain when the continuum it represents is under such a state and the continuity of displacements across the interfaces between the elements. In static analysis, it yields solutions uniformly convergent in the nodal displacements with decreasing size of elements.

The element has the following geometric and physical properties:

Length of the element = L

Modulus of Elasticity = E

Area of cross-section = A

Mass density = ρ

n and n+1 are the node numbers.

Since the formulation is one-dimensional, Poissonic effect need not be considered. The stiffness and mass-matrices are presented below:

$$[K] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (3.2-1.2)$$

and consistent mass matrix:

$$[M] = \rho AL \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} = m \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \quad (3.2-1.b)$$

Consider a series of elements assembled in a row representing a longitudinal homogeneous elastic bar. The motion of the bar under the influence of external forces is given by the differential equation:

$$\rho(x) \frac{\partial^2}{\partial t^2} [u(x,t)] + EA \frac{\partial^2}{\partial x^2} [u(x,t)] = F(x,t), \quad 0 < x < L, \quad (3.2-2)$$

where $u(x,t)$ is the displacement function, and $F(x,t)$ is the function representing the external forces. When discretized by a finite element approximation, equation (3.2-2) takes the following form:

$$\begin{aligned}
 m \begin{bmatrix} \cdot & 1/6 & \cdot & \cdot & 0 \\ 1/6 & 2/3 & 1/6 & \cdot & \cdot \\ \cdot & 1/6 & 2/3 & 1/6 & \cdot \\ 0 & \cdot & 1/6 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{Bmatrix} \ddot{u}_{n-1} \\ \ddot{u}_n \\ \ddot{u}_{n+1} \\ \vdots \end{Bmatrix} + k \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & 0 \\ -1 & 2 & -1 & \cdot & \cdot \\ \cdot & -1 & 2 & -1 & \cdot \\ \cdot & \cdot & -1 & 2 & -1 \\ 0 & \cdot & \cdot & -1 & \cdot \end{bmatrix} \begin{Bmatrix} u_{n-1} \\ u_n \\ u_{n+1} \\ \vdots \end{Bmatrix} \\
 = \begin{Bmatrix} \vdots \\ F_{n-1} \\ F_n \\ F_{n+1} \\ \vdots \end{Bmatrix} \quad (3.2-3)
 \end{aligned}$$

The equation of motion of node n , when considered alone, is:

$$m \left[\frac{\ddot{u}_{n-1}}{6} + \frac{2}{3} \ddot{u}_n + \frac{\ddot{u}_{n+1}}{6} \right] + k [-u_{n-1} + 2u_n - u_{n+1}] = F_n \quad (3.2-4)$$

where $(\ddot{})$ represents the second order time derivative.

Phase velocity of a wavefront is the product of its frequency and wavelength,

$$V = \frac{v}{a} \quad (3.2-5)$$

where V is the phase velocity,

v is the frequency of the wavefront in cycles per second, and

a is the wavenumber, the inverse of the wavelength.

Consider a longitudinal wave of frequency v cycles/second,

spanning over several node points. Displacement of the node n may be

written as a harmonic function:

$$u_n = A_n \exp[2\pi i(vt - ax)] = A_n \exp[2\pi i(vt - anL)] \quad (3.2-6)$$

The amplitude $A_n = A$ is the same for all the nodes for a wave travelling without any attenuation.

Equation (3.2-6) also yields the phase relations between the adjacent nodes:

$$\left. \begin{aligned} u_{n+1} &= A \exp\{2\pi i[vt - a(n+1)L]\} \\ u_{n-1} &= A \exp\{2\pi i[vt - a(n-1)L]\} \end{aligned} \right\} \quad (3.2-7)$$

which reduce to:

$$\left. \begin{aligned} u_{n+1} &= u_n \exp[-i2\pi aL] \\ \text{and } u_{n-1} &= u_n \exp[i2\pi aL] \end{aligned} \right\} \quad (3.2-8)$$

The harmonic nature of the motion requires

$$\left. \begin{aligned} \ddot{u}_{n+1} &= -4\pi^2 v^2 u_{n+1} \\ \text{and } \ddot{u}_{n-1} &= -4\pi^2 v^2 u_{n-1} \end{aligned} \right\} \quad (3.2-9)$$

Consider the free oscillations of the node n with no external forces acting on it, $F_n = 0$. Equation (3.2-8) and (3.2-9) are substituted into (3.2-4), the latter takes the following form:

$$\left. \begin{aligned} \frac{m}{6} 4\pi^2 v^2 [4 + 2 \cos 2\pi aL] - 2k[1 - \cos 2\pi aL] &= 0 \\ \text{or } v^2 &= \frac{12k}{m} \frac{1 - \cos 2\pi aL}{4\pi^2 (4 + 2 \cos 2\pi aL)} \end{aligned} \right\} \quad (3.2-10)$$

Define a parameter, $\xi = \pi aL$. Velocity $V = \frac{v}{a}$, (as in eqn. 3.2-5).

Substituting in eqn. (3.2-10),

$$V^2 = \frac{v^2}{a^2} = L^2 \frac{k}{m} \frac{3\sin^2 \xi}{\xi^2(1+2\cos^2 \xi)} \quad (3.2-11)$$

The velocity of propagation of a wavefront of infinitely long waves is obtained when $\xi \rightarrow 0$, i.e., wavelength $\Lambda \rightarrow \infty$, as:

$$V_\infty^2 = \lim_{\xi \rightarrow 0} \left\{ L^2 \frac{k}{m} \frac{3\sin^2 \xi}{\xi^2(1+2\cos^2 \xi)} \right\} = L^2 \frac{k}{m}$$

($k = \frac{EA}{L}$; $m = \rho AL$; $\therefore V_\infty^2 = \frac{E}{\rho}$ which is the same expression as obtained in the classical formulation).

The phase velocity for an arbitrary wave with parameters ξ can be written in terms of V_∞ as:

$$V^2 = V_\infty^2 \frac{3\sin^2 \xi}{\xi^2[1+2\cos^2 \xi]}, \quad 0 \leq \xi \leq \frac{\pi}{2} \quad (3.2-12)$$

The range of the value of ξ can be explained physically in that an assemblage of finite elements with successive nodal spacing L cannot support the propagation of waves with wavelengths shorter than twice the length of the element.

A plot of the ratio V/V_∞ as a function of the parameter $\xi (= \pi aL)$ is shown in Fig. 3.2-2. Below the cutoff frequency, the element exhibits a variation of 20% in the velocity from its value of that for infinitely long waves, contrary to the constant velocity

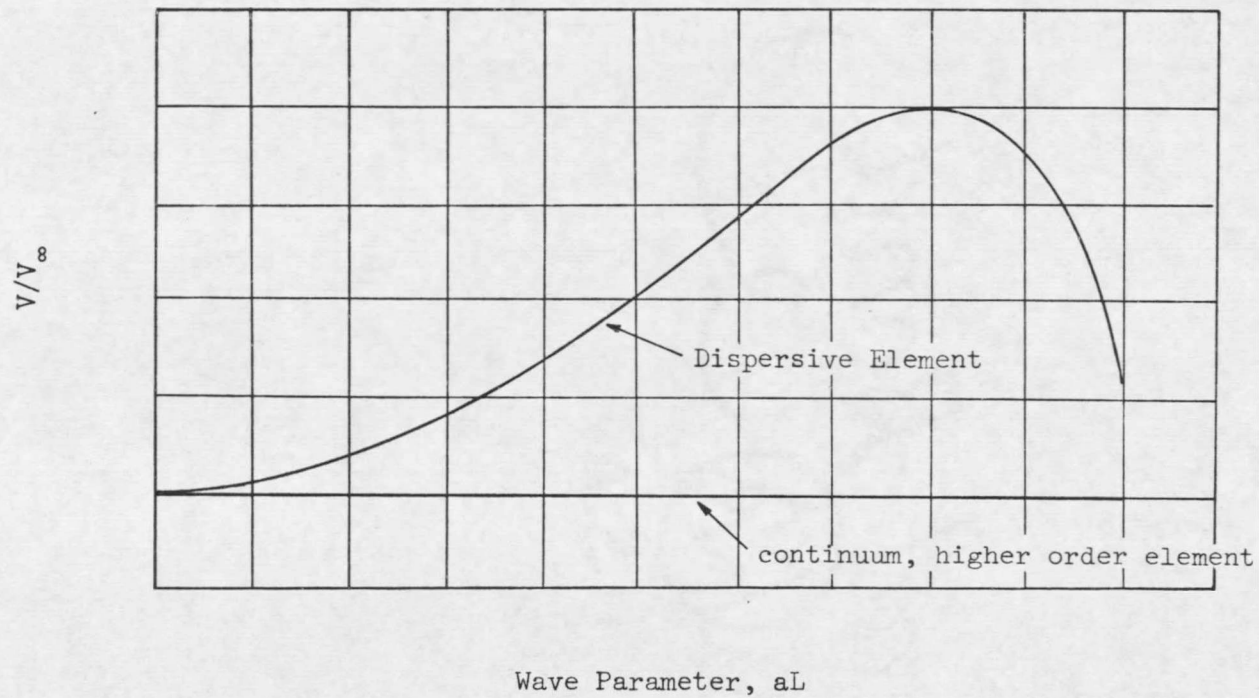


Fig. 3.2-2. Discretization-Dispersion Curves For One-Dimensional Finite Elements.

characteristics of the continuum. Thus the finite element representation of a longitudinal bar so derived results in significant discretization-dispersion. In a wave group, all except very long waves travel faster than the continuum velocity, resulting in an artificial distortion of the input signal. The finite element models are band limited, acting as low pass filters so that the components with shorter wavelengths will attenuate rapidly [52]. This is a property inherent in all the discrete formulations. Also it is dependent on the element size which can be adjusted according to the requirements of the problem.

3.3 One-Dimensional First Order Strain Compatible Finite Element and Its Dispersive Characteristics

The higher order finite elements are characterized by the inclusion of higher order spatial derivatives of displacement components as generalized coordinates in addition to the displacements. The advantages are that continuity of the spatial derivatives can be achieved across the interelement boundaries resulting in a rapid convergence of solutions. In elasticity problems, these derivatives are the normal and shear strain components. The continuity of the strain components across the interelement boundaries may be expected to improve the element characteristics. Interpolation polynomials are advantageously used for the development of elements with continuity of the displacement functions across the interelement

boundaries [7]. Hermitian interpolation polynomials are used when higher order compatibility is desired. A brief discussion on the theory of Hermitian polynomials and their use in the development of compatible stiffness matrices are presented later in the chapter.

For the element under discussion, the normal strain (first order spatial derivative of the displacement) continuity condition is imposed and hence first order Hermitian interpolation polynomials are used to represent the displacement configuration. The stiffness and consistent mass matrices and their detailed derivation are presented in Appendix A.

For a uniform longitudinal bar modeled by an assembly of first order finite elements, the equations of motion of a node, say n , are given by the following:

$$\frac{m}{420} \begin{bmatrix} 54 & 13L & 312 & 0 & 54 & 13L \\ -13L & -3L^2 & 0 & 8L^2 & 13L & -3L^2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{n-1} \\ \ddot{u}_{,x_{n-1}} \\ \ddot{u}_n \\ \ddot{u}_{,x_n} \\ \ddot{u}_{n+1} \\ \ddot{u}_{,x_{n+1}} \end{Bmatrix}$$

$$+k \begin{bmatrix} \frac{-6}{5} & \frac{-L}{10} & \frac{12}{5} & 0 & \frac{-6}{5} & \frac{L}{10} \\ \frac{L}{10} & \frac{-L^2}{30} & 0 & \frac{4L^2}{5} & \frac{-L}{10} & \frac{-L^2}{30} \end{bmatrix} \begin{Bmatrix} u_{n-1} \\ u, x_{n-1} \\ u_n \\ u, x_n \\ u_{n+1} \\ u, x_{n+1} \end{Bmatrix} = \begin{Bmatrix} F_n \\ \phi_n \end{Bmatrix} \quad (3.3-1)$$

The constants m and k have the same meaning as in eqn. (3.2-1).

For free oscillations with zero external forces $F_n = \phi_n = 0$.

Considering a longitudinal wave spanning over several node points with a frequency of ν cycles/second, displacement and normal strain at the node n can be expressed as the following periodic functions:

$$\left. \begin{aligned} u_n &= A \exp\{2\pi i(\nu t - anL)\} \\ \text{and} \\ u, x_n &= B \exp\{2\pi i(\nu t - anL)\} \end{aligned} \right\} \quad (3.3-2)$$

the notation used being the same as in eqn. (3.2-7). Correspondingly the adjacent nodes will oscillate with a phase difference of $2\xi = 2\pi aL$ radians as described in the previous section. Hence, the nodal strains and displacements at $n-1$ and $n+1$ can be expressed in terms of those at node n . When these relationships are substituted into eqn. (3.3-1), it reduces to the following:

$$[-4\pi^2 v^2 [\mathfrak{m}] + [\mathfrak{K}]] \begin{Bmatrix} A \\ B \end{Bmatrix} = \{0\} \quad (3.3-3)$$

$[\mathfrak{m}]$ and $[\mathfrak{K}]$ are 2x2 matrices derived from eqn. (3.3-1) with imposed phase differences.

Simultaneous vanishing of A and B would mean a trivial solution. So the determinant of the matrix in (3.3-3) must equal zero, which yields

$$\begin{vmatrix} (105-54 \sin^2 \xi) \pi^2 v^2 \frac{m}{k} - 126 \sin^2 \xi & (26\pi^2 v^2 \frac{m}{k} + 21) \sin \xi \cos \xi \\ (26\pi^2 v^2 \frac{m}{k} + 21) \sin \xi \cos \xi & 2\pi^2 v^2 \frac{m}{k} [1+6 \sin^2 \xi] - 7(3+2\sin^2 \xi) \end{vmatrix} = 0 \quad (3.3-4)$$

The determinant may be written as a quadratic equation:

$$[\pi^2 v^2 \frac{m}{k}]^2 \alpha_1 + [\pi^2 v^2 \frac{m}{k}] \alpha_2 + \alpha_3 = 0 \quad (3.3-5)$$

and the phase velocity is given by:

$$v^2 = \frac{v^2}{a^2} = \frac{k}{m} \frac{L^2}{\xi^2} \frac{\alpha_2 \pm \sqrt{\alpha_2^2 - 4\alpha_1 \alpha_3}}{2\alpha_1} \quad (3.3-6)$$

where:

$$\begin{aligned} \alpha_1 &= 210 + 476 \sin^2 \xi + 28 \sin^4 \xi \\ \alpha_2 &= -2205 - 1680 \sin^2 \xi + 336 \sin^4 \xi \\ \text{and } \alpha_3 &= 2205 \sin^2 \xi + 2205 \sin^4 \xi \end{aligned}$$

For waves of infinite wavelength, velocity $V_\infty = \sqrt{\frac{k}{m}} L$. In eqn. (3.3-6) the sign before the discriminant is taken as negative to assure V_∞^2 to be greater than zero and having a smaller value.

$$\therefore V^2 = V_\infty^2 \left\{ \frac{\alpha_2 - \sqrt{\alpha_2^2 - 4\alpha_1\alpha_3}}{2\xi^2\alpha_1} \right\} \quad (3.3-7)$$

Within the range of the cutoff frequency, a plot of the ratio of V/V_∞ , as a function of the parameter $\xi(=\pi aL)$ is shown in Fig. 3.2-2, labelled "first order element." It is seen that $V = V_\infty$ for all wavelengths within the range, which is the property exhibited by the continuum. So the element is non-discretization-dispersive.

3.4 Two-Dimensional Displacement Compatible Rectangular Plane-Stress Element and Its Dispersive Characteristics

This element has four nodal points and each node has two degrees-of-freedom, i.e., displacements in two perpendicular directions. The element is sketched in Fig. 3.4-1.

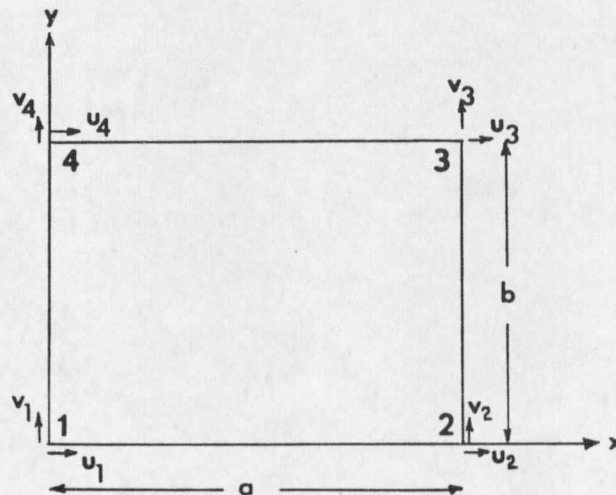


Fig. 3.4-1. Two-dimensional Displacement Compatible Rectangular Element.

The displacement functions chosen are:

$$\left. \begin{aligned} u &= C_1 + C_2 x + C_3 y + C_4 xy \\ \text{and} \\ v &= C_5 + C_6 x + C_7 y + C_8 xy \end{aligned} \right\} \quad (3.4-1)$$

The above displacement functions may be derived from the products of Lagrange interpolation polynomials in x and y-directions as given below:

$$u(x,y) = \sum_{i=1}^2 \sum_{j=1}^2 u_{ij} L_i(x) L_j(y) \quad (3.4-2)$$

The incorporation of Lagrange Interpolation polynomials ensures compatibility of displacements across interfaces between the elements. This element was first published in the literature in 1956 [60] and has been in use for solving two dimensional problems. Johnson and McLay [20] give a rigorous proof that this element yields results which are uniformly convergent for nodal displacements and convergent in the mean sense for nodal stresses and strains. The stiffness and consistent mass matrix derivations are found in several texts [44].

Consider an assembly of two-dimensional elements as shown in Fig. 3.4-2. For simplicity, we limit the consideration to a uniform grid of square elements while the same arguments are valid for nonuniform grids as well.

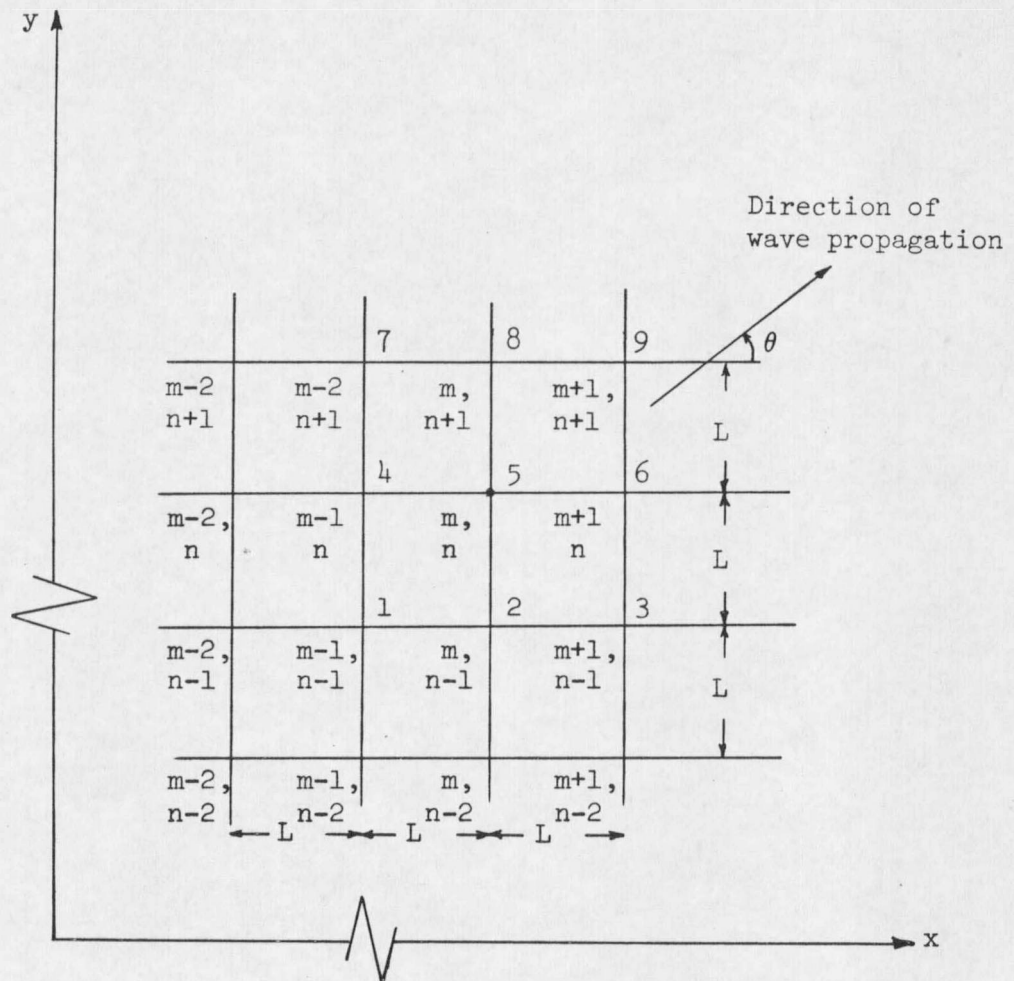


Fig. 3.4-2. Two-Dimensional Grid of Elements For Dispersion Study.

In the above grid, m represents the number of node points along the x -axis and n represents the number along the y -axis. The nodal displacements (degrees-of-freedom) at node 5 are elastically and inertially coupled with the nodal displacements at all the surrounding nodes. For a model having the following geometrical and physical properties:

Modulus of Elasticity, $E = 1$,

Poisson ratio, $\mu = 0.3$,

Mass Density of the material, $\rho = 1.$,

Thickness of the element, $h = 1$ and

Length and width of the element, $L = 1.$,

the equations of motion of the node 5 are given in the following matrix form:

$$[M]\{\ddot{\delta}\} + [K]\{\delta\} = \{F\} \quad (3.4-3)$$

where $[M]$ and $[K]$ are the following 2×18 matrices exhibiting inertial and elastic couplings between node 5 and the adjacent nodes.

$$[M] = \begin{bmatrix} .0278 & .000 & .0556 & .000 & .0278 & .000 & .1111 & .0556 & .4444 \\ .000 & .0278 & .000 & .111 & .000 & .0278 & .000 & .1111 & .000 \\ .000 & .1111 & .000 & .0278 & .000 & .1111 & .000 & .0278 & .000 \\ .4444 & .000 & .111 & .000 & .0278 & .000 & .1111 & .000 & .0278 \end{bmatrix}$$

and

$$[K] = \begin{bmatrix} -.24275 & -.17857 & .05494 & .000 & -.24275 & .17857 & -.60439 \\ -.17857 & -.24725 & .01373 & -.60439 & .17857 & -.24725 & .000 \\ .54956 & 1.97802 & .000 & -.60439 & .000 & -.24725 & .17857 \\ .09615 & .000 & 1.97802 & .000 & .10989 & .17857 & .24725 \\ .0989 & .000 & -.24725 & -.17857 \\ .000 & -.60439 & -.17857 & -.24725 \end{bmatrix}$$

The vector $\{\delta\}$ is the vector of the nodal displacements of node 5 and the eight adjacent nodes. $\{F\}$ is the 2×1 nodal force vector at node 5 and is equal to $\{0\}$ in the case of free oscillations.

Consider a wave spanning many elements, travelling in a direction shown in Fig. 3.4-2, without any significant attenuation or distortion. The motion of the point (x,y) can be represented by the following expressions:

$$\left. \begin{aligned} u &= A \exp\{2\pi i[\nu t - a_x x - a_y y]\} \\ \text{and} \\ v &= B \exp\{2\pi i[\nu t - a_x x - a_y y]\} \end{aligned} \right\} \quad (3.4-4)$$

where $a_x = a \cos\theta$; $a_y = a \sin\theta$ and a is the wave number.

Taking x and y -coordinate axes to some reference origin at a node $m=0$ and $n=0$, eqn. (3.4-4) may be rewritten as:

$$\left. \begin{aligned} u &= A \exp\{2\pi i[\nu t - a_x m L - a_y n L]\} \\ \text{and} \\ v &= B \exp\{2\pi i[\nu t - a_x m L - a_y n L]\} \end{aligned} \right\} \quad (3.4-5)$$

Motions of the other nodal points are also of the same form but they will be out-of-phase from that of node 5. The phase difference between adjacent nodes along the x -axis is equal to $2\pi a_x L$ and that along the y -axis is $2\pi a_y L$. Hence the motions of all the adjacent eight nodes can be written in terms of those at node 5. Further, the harmonic nature of the oscillations yield the following

relationship between displacements and accelerations:

$$\ddot{u}_{m,n} = -4\pi^2 v^2 u_{m,n}; \quad \ddot{v}_{m,n} = -4\pi^2 v^2 v_{m,n} \quad (3.4-6)$$

When the above substitutions are made in eqn. (3.4-5) (with $\{F\} = \{0\}$), it reduces to:

$$[-4\pi^2 v^2 [\mathfrak{m}] + [\mathfrak{K}]] \begin{Bmatrix} A \\ B \end{Bmatrix} = \{0\} \quad (3.4-7)$$

where $[\mathfrak{m}]$ and $[\mathfrak{K}]$ are, respectively, the inertial and elastic stiffness matrices with wave parameters substituted in. The vanishing of the right hand side vector requires either the identical vanishing of the constants A and B or a vanishing of the determinant of the coefficient matrix on the left hand side. Thus the problem reduces to an eigenvalue problem of the coefficient matrix on the left hand side of eqn. (3.4-7). The computed eigenvalues are the frequencies of node 5 which are functions of the wavenumber a , and, hence of the wavelength. The phase velocity of the wavefront is the product of the frequency (one of the computed eigenvalues) and the wavelength.

There are two eigenvalues obtained from the formulation; hence, two phase velocities, one corresponding to a dilatational wave and the other to a shear wave, as observed from the eigenvectors. The eigenvector corresponding to the dilatational wave is characterized by motion of the node in the direction of the wave, while that corresponding to the shear wave is characterized by motion of the node in the direction perpendicular to the propagation.

The element represents the plane stress behaviour of a two dimensional continuum with the following stress-strain relationship:

$$\sigma_{ij} = \bar{\lambda} u_{i,i} \delta_{ij} + 2G u_{i,j}; \quad i = 1, 2 \quad (3.4-8)$$

$\bar{\lambda} = \frac{\mu E}{1-\mu}$ is the modified Lamé's constant, and G is the rigidity modulus. Such a continuum propagates dilatational waves at a velocity $V_p' = \sqrt{\frac{\bar{\lambda} + 2G}{\rho}}$ and shear waves at a velocity $V_s' = \sqrt{\frac{G}{\rho}}$. The continuum represented by the finite element has the properties: $E = 1$; $\rho = 1$ and $\mu = 0.3$, resulting in values of $V_p' = 1.05$ and $V_s' = .62$. These values, together with the computed phase velocities of the finite element for different wavelengths are plotted in Figures 3.4-3 through 3.4-5 as functions of the parameter $\pi a L$ (a is the wave number and L is the length of the element, which here, is unity). The three figures show the discretization-dispersion relationship for waves travelling in three different directions, at angles $\theta = 0, .5$ and 1.0 radians to the x -axis.

An observation of Figures (3.4-3) through (3.4-5) shows that infinitely long shear waves travel at continuum velocities while long dilatational waves travel slightly faster than continuum velocities. Reduction in wavelength causes an increase in the velocity of propagation. The magnitude of discretization-dispersion changes with the direction of propagation with respect to the coordinate axes. This may be caused by the relative distance between nodes changing with

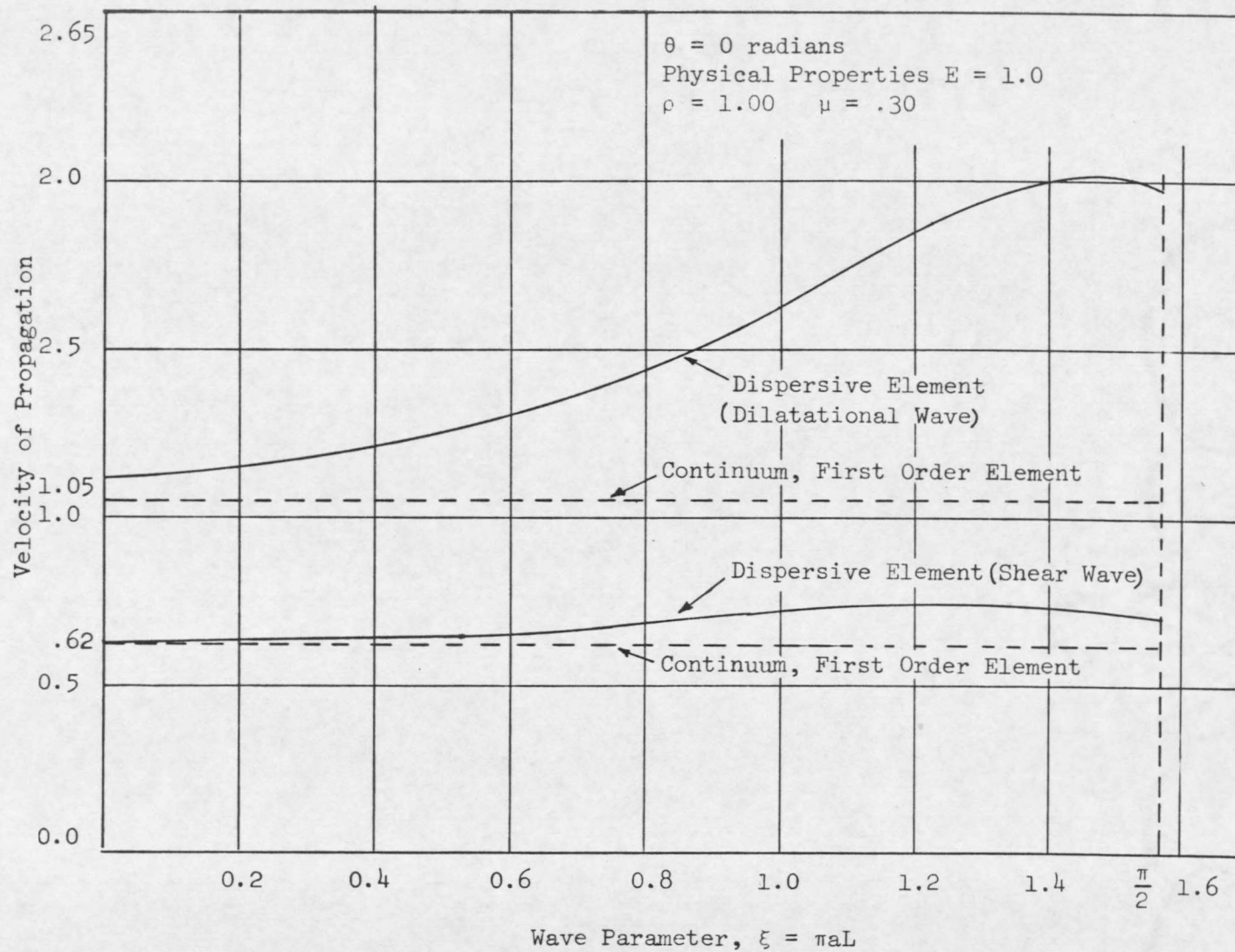


Fig. 3.4-3. Discretization-Dispersion Curves for Two Dimensional Finite Elements.

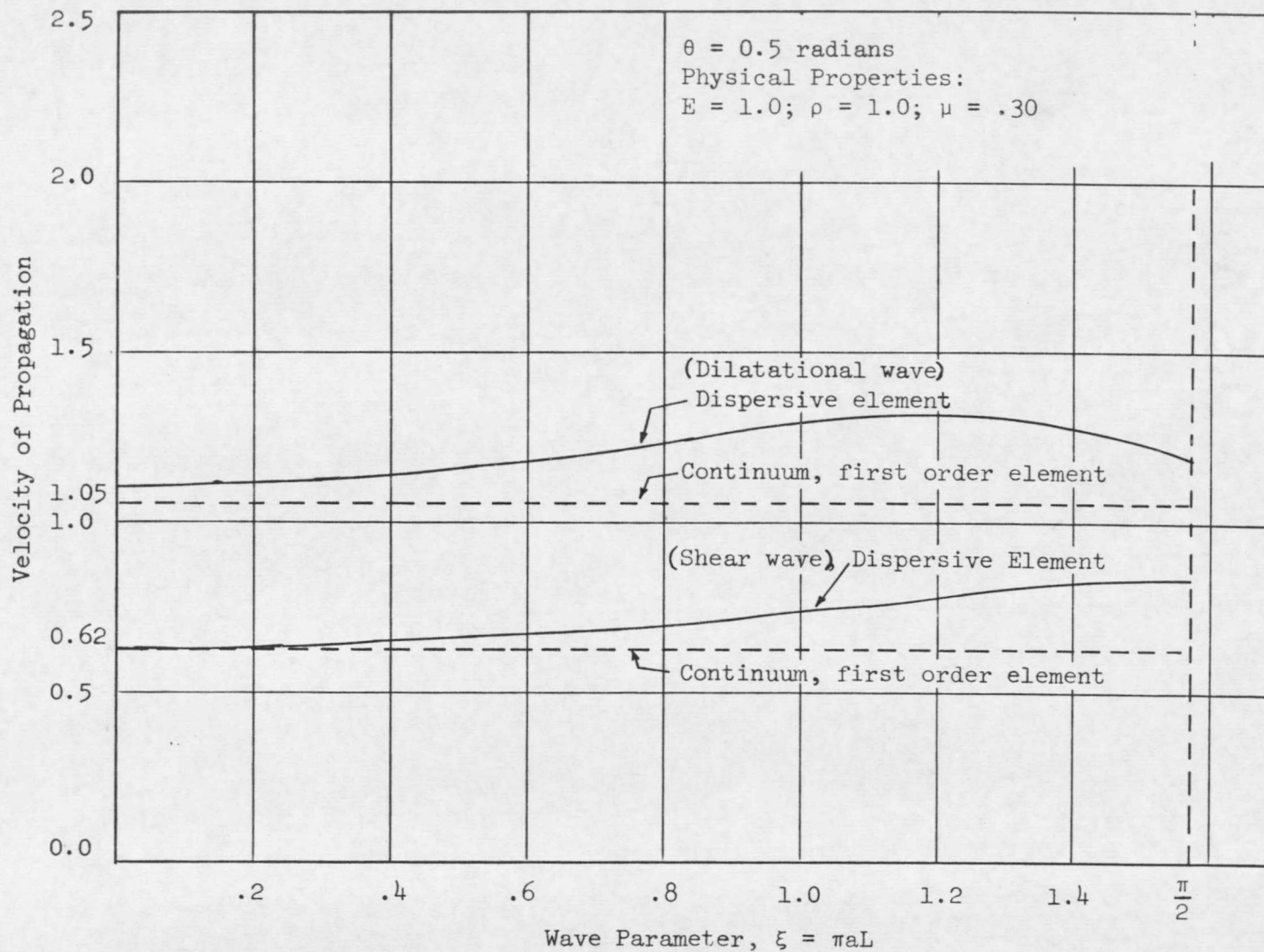


Fig. 3.4-4. Discretization-Dispersion Curves For Two-Dimensional Finite Elements.

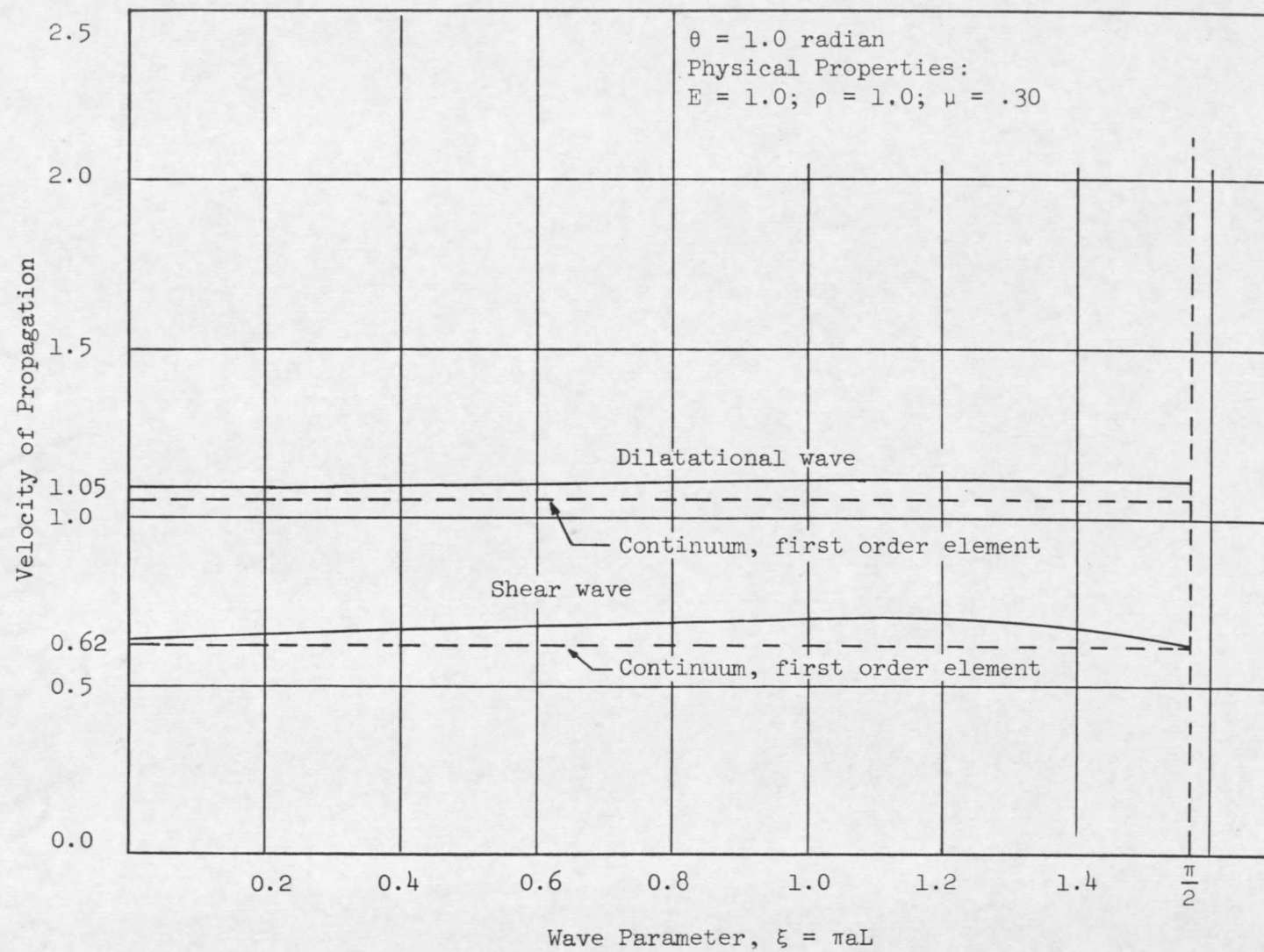


Fig. 3.4-5. Discretization-Dispersion Curves For Two Dimensional Finite Elements

the angle between the direction of propagation and the reference axis. Thus, the two-dimensional displacement compatible finite element yields solutions with considerable discretization-dispersion.

3.5 Interelement Compatibility in the Finite Element Method. Hermitian Interpolation Polynomials - Higher Order Elements.

The finite element method is an approximate variational formulation of minimizing a functional, commonly known as the Ritz method. It requires the assumption of a solution of a family of functions dependent on several unknown parameters and making the differential of the functional with respect to these parameters zero to achieve the minimization. In the finite element method these unknown parameters are the nodal displacements, or related to them through certain linear transformations. The conditions to be satisfied by the chosen family of functions are that: (i) they are continuous and piecewise differentiable to certain desired degree depending on the nature of the problem, and (ii) the displacement boundary conditions are satisfied [61].

The first condition calls for the conformity of the displacement functions and hence interelement compatibility. In plane-stress problems, the strain energy is a positive-definite function of the first order spatial derivatives; hence, the necessary compatibility is that of the displacement function. In plate bending problems,

the strain energy is a positive-definite function of the second order derivatives of the displacement function; hence, the compatibility required is that of the first order derivative of the displacement function, commonly known as the "normal slope compatibility."

Interelement compatibility conditions the finite element procedure as a Ritz formulation. Since the Ritz method has been studied extensively [21,32] for convergence and error estimation, it can be advantageously used in the finite element solution procedure.

Interelement compatibility is obtained easily by the use of interpolation polynomials as displacement functions. A polynomial is said to interpolate a function at certain points inside a domain including the boundary if it assumes the value of the function at those points. This concept can be generalized by including the interpolation of the derivatives of the functions (to any desired order) in addition to the value of the functions themselves at the interpolating points.

Commonly used in the finite element method are the Hermitian polynomials. These can be generated to interpolate functions and their derivatives to any desired order. A function $Y(x)$, interpolated in the domain $a \leq x \leq b$ at n points, $a \leq x_1 < x_2 < x_3 \cdots < x_n \leq b$, to m^{th} order, is

$$Y(x) \approx \sum_{i=1}^n \{Y(x_i) H_{oi}^{(m)}(x) + Y'(x_i) H_{li}^{(m)}(x) + \dots + Y^{(m)}(x_i) H_{mi}^{(m)}(x)\} \quad (3.5-1)$$

where $H_{oi}^{(m)}(x)$, $H_{li}^{(m)}(x)$, ... are the m^{th} order Hermitian interpolation polynomials. From the definition given in (3.4-1), these polynomials have the following properties:

(0):

$$H_{oi}^{(m)}(x_j) = \delta_{ij}$$

$$\frac{d}{dx} [H_{oi}^{(m)}(x_j)] = 0$$

$$\frac{d^2}{dx^2} [H_{oi}^{(m)}(x_j)] = 0$$

$$\frac{d^m}{dx^m} [H_{oi}^{(m)}(x_j)] = 0 \quad (3.5-2)$$

(1):

$$H_{li}^{(m)}(x_j) = 0$$

$$\frac{d}{dx} [H_{li}^{(m)}(x_j)] = \delta_{ij}$$

$$\frac{d^2}{dx^2} [H_{li}^{(m)}(x_j)] = 0$$

$$\frac{d^m}{dx^m} [H_{li}^{(m)}(x_j)] = 0 \quad (3.5-3)$$

(m-1):

$$H_{m-1}^{(m)}(x_j) = 0$$

$$\frac{d}{dx} [H_{m-1,i}^{(m)}(x_j)] = 0$$

$$\dots\dots\dots$$

$$\frac{d^{m-1}}{dx^{m-1}} [H_{m-1,i}^{(m)}(x_j)] = \delta_{ij}$$

$$\frac{d^m}{dx^m} [H_{m-1,i}^{(m)}(x_j)] = 0 \quad (3.5-4)$$

(m):

$$H_{m,i}^{(m)}(x_j) = 0$$

$$\frac{d}{dx} [H_{m,i}^{(m)}(x_j)] = 0$$

$$\dots\dots\dots$$

$$\frac{d^m}{dx^m} [H_{m,i}^{(m)}(x_j)] = \delta_{ij} \quad (3.5-5)$$

where the x_j 's are the argument values and the symbol δ_{ij} represents the Kronecker delta,

$$\begin{aligned} \delta_{ij} &= 1, & i &= j \\ &= 0, & i &\neq j \end{aligned}$$

Taking $x_1=0$, $x_2=a$ and $m=0$, we have:

$$H_{01}^{(0)}(0) = 1 ; H_{01}^{(0)}(a) = 0 \quad (3.5-6)$$

$$H_{01}^{(0)}(a) = 0 ; H_{02}^{(0)}(a) = 1$$

A pair of polynomials having these properties are:

$$H_{01}^{(0)}(x) = \frac{-(x-a)}{a}$$

and

$$H_{02}^{(0)}(x) = \frac{x}{a}$$

(3.5-7)

These are the ordinary Lagrange interpolation polynomials and are sketched in Fig. 3.4-1.

When $x_1=0$, $x_2=a$ and $m=1$, the requirements are:

$$H_{01}^{(1)}(0) = 1$$

$$\frac{d}{dx} H_{01}^{(1)}(0) = 0$$

$$H_{01}^{(1)}(a) = 0$$

$$\frac{d}{dx} H_{01}^{(1)}(a) = 0$$

$$H_{02}^{(1)}(0) = 0$$

$$\frac{d}{dx} H_{02}^{(1)}(0) = 0$$

$$H_{02}^{(1)}(a) = 1$$

$$\frac{d}{dx} H_{02}^{(1)}(a) = 0$$

$$H_{11}^{(1)}(0) = 0$$

$$\frac{d}{dx} H_{11}^{(1)}(0) = 1$$

$$H_{11}^{(1)}(a) = 0$$

$$\frac{d}{dx} H_{11}^{(1)}(a) = 0$$

$$H_{12}^{(1)}(0) = 0$$

$$\frac{d}{dx} H_{12}^{(1)}(0) = 0$$

$$H_{12}^{(1)}(a) = 0$$

$$\frac{d}{dx} H_{12}^{(1)}(a) = 1$$

(3.5-8)

The set of polynomials satisfying the above conditions are:

$$H_{01}^{(1)}(x) = \frac{1}{a^3} \{2x^3 - 3ax^2 + a^3\}$$

$$H_{02}^{(1)}(x) = -\frac{1}{a^3} \{2x^3 - 3ax^2\}$$

$$H_{11}^{(1)}(x) = \frac{1}{a^2} \{x^3 - 2ax^2 + a^2x\} \quad (3.5-9)$$

and

$$H_{12}^{(1)}(x) = \frac{1}{a^2} \{x^3 - ax^2\}$$

These are sketched in Fig. 3.4-2.

The above interpolation polynomials, when applied to finite element geometries yield element representations wherein the first order spatial derivatives of the displacement functions are continuous across the interelement boundaries. In one dimension, they yield the strain compatible one-dimensional element developed in Sec. 3.3. When extended to two dimensions they yield a rectangular element with strain compatibility across the boundaries. Based on this concept, the refined plane stress element is developed in the next section.

Though completeness of the displacement function is believed to be the most important condition for convergence [17] all the elements satisfying compatibility of displacements across boundaries yield convergent solutions. By imposing additional continuity of normal and shear strains, one may expect to improve the rate of convergence. Displacement compatible elements yield solutions uniformly convergent

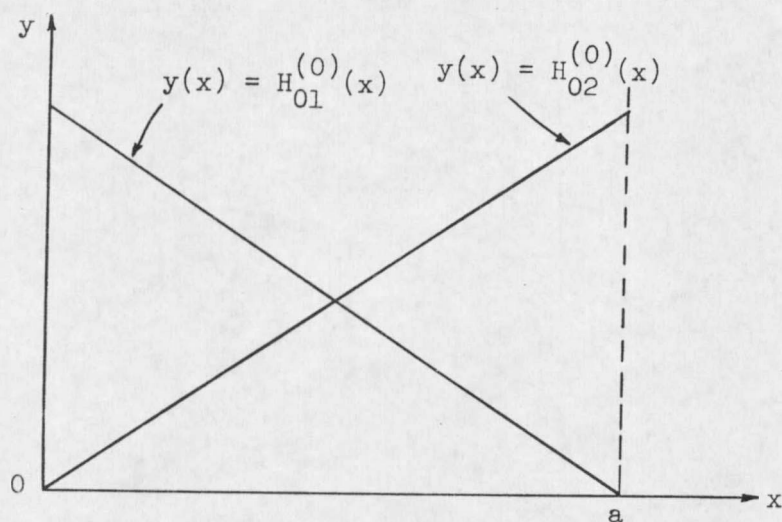


Fig. 3.5-1. Lagrange Interpolation Polynomials or Hermitian Interpolation Polynomials of Order Zero.

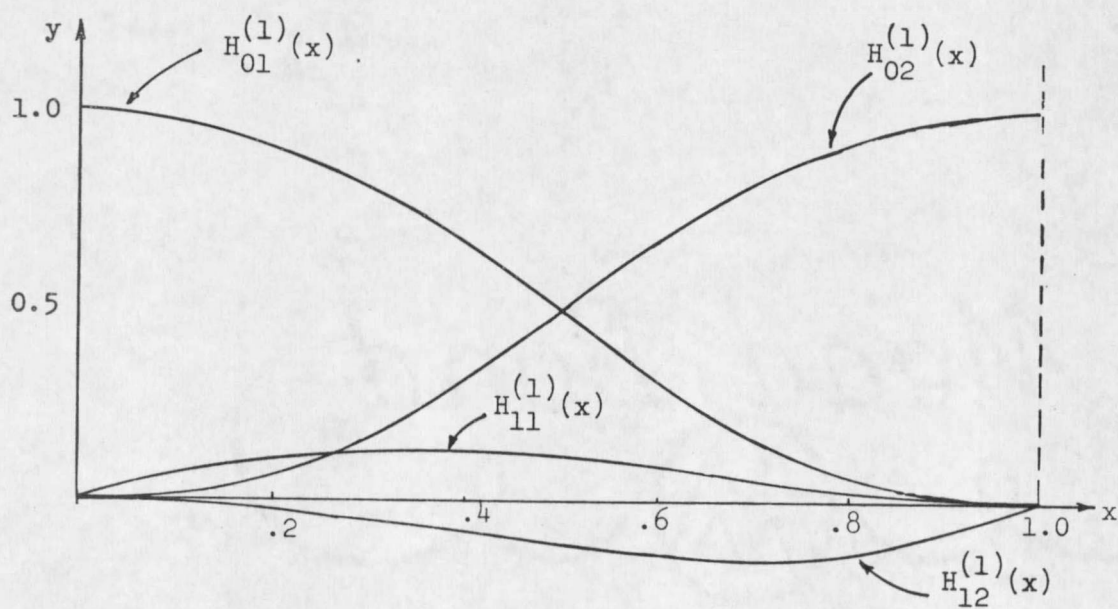


Fig. 3.5-2. First Order Hermitian Interpolation Polynomials.

for nodal displacements. By imposing additional continuity of normal and shear strains (which is done by adding four more degrees-of-freedom to each node in two-dimensional problems), one may expect faster convergence with a comparatively coarse mesh.

This method of including the derivatives of displacements as generalized coordinates has been termed by Zienkiewicz [62] as the use of "excessive" nodal continuities. He also states that they may become truly excessive when abrupt changes in the elastic properties of the material occur across the boundaries which cause discontinuities in strain. If the nodal points are located on the boundary, this will at most prevent uniform convergence of nodal strains only along the boundary. In real cases such a strain discontinuity will be finite and in the limit we can expect nodal strains to assume mean values in the neighborhood of the boundary as in the limit the strain energy stored in the elements will converge to the actual strain energy in the mean sense.

Finite elements which include derivatives of displacements as generalized coordinates are called higher order elements. Several higher order elements have been reported recently. Felippa [18] and Bergan [6] independently developed triangular elements using area coordinates with six nodal degrees of freedom, the nodes being taken at the apices of the triangle. Tocher and Hartz [58] developed a similar triangular element with displacement functions made up of

polynomial terms, with a modification of the triangular plate bending element developed by Clough and Tocher [12].

Advantages of the higher order elements compared to the ordinary elements are: (i) concentrated moments can be considered at a node without the use of concentrated equal and opposing forces and (ii) in-plane nodal rotations can be included in the analysis. A common example is the in-plane analysis of a circular disk (or membrane) under the action of a concentrated normal couple vector.

3.6 Development of a Refined First Order Strain Compatible Rectangular Plane-Stress Element - Its Dispersive Characteristics

The rectangular element shown in Fig. 3.5-1 has twenty-four degrees-of-freedom, six at each node. They are two displacements, u and v in perpendicular directions and four displacement derivatives

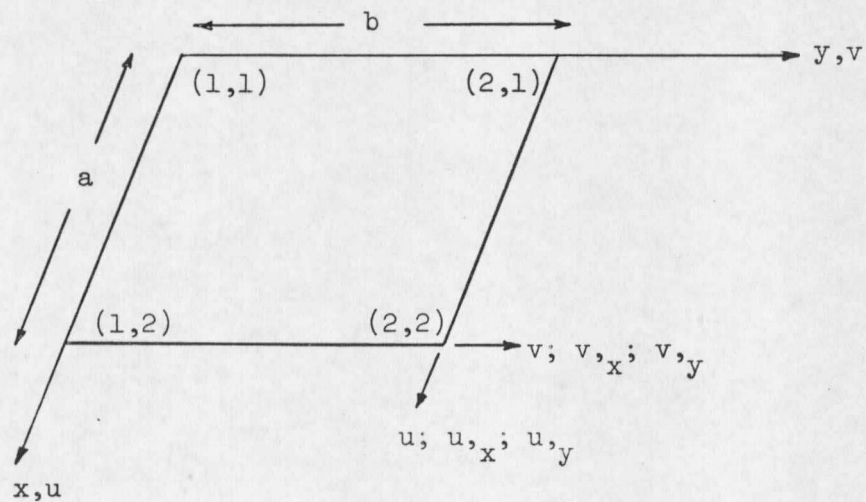


Fig. 3.6-1. Two-Dimensional First Order Strain Compatible Rectangular Finite Element.

with respect to the two independent variables, u_x , u_y , v_x and v_y .

The displacement functions u and v are represented by the products of first order Hermitian interpolation polynomials in the x and y directions as follows:

$$u(x,y) = \sum_{i=1}^2 \sum_{j=1}^2 [u_{ij} H_{0i}(x) H_{0j}(y) + u_{x_{ij}} H_{1i}(x) H_{0j}(y) + u_{y_{ij}} H_{0i}(x) H_{1j}(y)] \quad (3.6-1)$$

and

$$v(x,y) = \sum_{i=1}^2 \sum_{j=1}^2 [v_{ij} H_{0i}(x) H_{0j}(y) + v_{x_{ij}} H_{1i}(x) H_{0j}(y) + v_{y_{ij}} H_{0i}(x) H_{1j}(y)] \quad (3.6-2)$$

where the interpolation polynomials are:

$$\begin{aligned} H_{01}(x) &= \frac{1}{a^3} [2x^3 - 3ax^2 + a^3] \\ H_{02}(x) &= \frac{-1}{a^3} [2x^3 - 3ax^2] \\ H_{11}(x) &= \frac{1}{2a} [x^3 - 2ax^2 + a^2x] \\ H_{12}(x) &= \frac{1}{a^2} [x^3 - ax^2] \end{aligned} \quad (3.6-3)$$

and similar equations in the y -direction. In the form given above,

the displacement functions are expressed as functions of generalized nodal displacements.

The generalized nodal displacement vector is:

$$\{\delta\}^T = [u_{11} \ u_{,x_{11}} \ u_{,y_{11}} ; u_{12} \ u_{,x_{12}} \ u_{,y_{12}} ; u_{21} \ u_{,x_{21}} \ u_{,y_{21}} ; \\ u_{22} \ u_{,x_{22}} \ u_{,y_{22}} ; v_{11} \ v_{,x_{11}} \ v_{,y_{11}} ; v_{12} \ v_{,x_{12}} \ v_{,y_{12}} ; \\ v_{21} \ v_{,x_{21}} \ v_{,y_{21}} ; v_{22} \ v_{,x_{22}} \ v_{,y_{22}}]$$

Expressions for $u_{,x}$, $u_{,y}$, $v_{,x}$ and $v_{,y}$ are obtained by appropriately differentiating the expressions (3.6-1) and (3.6-2).

The stiffness and consistent mass matrix terms are derived using Lagrange's equations of motion from the strain energy and kinetic energy of the deformed element expressed in terms of the generalized nodal displacements. The strain energy and kinetic energy in a rectangular plane-stress element during in-plane deformation are given by the following expression:

strain energy:

$$U = \frac{Eh}{2(1-\mu^2)} \int_{x=0}^a \int_{y=0}^b \{ [u_{,x}^2 + 2\mu u_{,x} v_{,y} + v_{,y}^2] \\ + \left(\frac{1-\mu}{2} \right) [u_{,y}^2 + 2u_{,y} v_{,x} + v_{,x}^2] \} dx dy \quad (3.5-4)$$

and the kinetic energy:

$$T = \frac{\rho h}{2} \int_{x=0}^a \int_{y=0}^b [\dot{u}^2 + \dot{v}^2] dx dy \quad (3.5-5)$$

where E is the modulus of elasticity of the material,

μ is the Poisson's ratio,

h is the thickness of the element,

ρ is the mass density and

u and v are displacements in the x and y-directions. A subscript after a comma represents a first order partial derivative with respect to that subscript. Stiffness and mass matrix coefficients are determined by differentiating the strain energy and kinetic energy expressions with respect to appropriate nodal displacements, that is:

$$k_{ij} = \frac{\partial^2 U}{\partial \delta_i \partial \delta_j} \quad \text{and} \quad m_{ij} = \frac{\partial^2 T}{\partial \delta_i \partial \delta_j} \quad (3.6-6)$$

All the above computations for the generation of stiffness and mass matrix terms are performed in the computer. A flow-chart of the program is presented in Appendix B.

Dispersion calculations are done similar to the procedure for the two-dimensional displacement compatible element, presented in Sec. 3.4. A grid made up of an assembly of uniform square elements, shown in Fig. 3.6-2, is considered. Consider a wave front, spanning

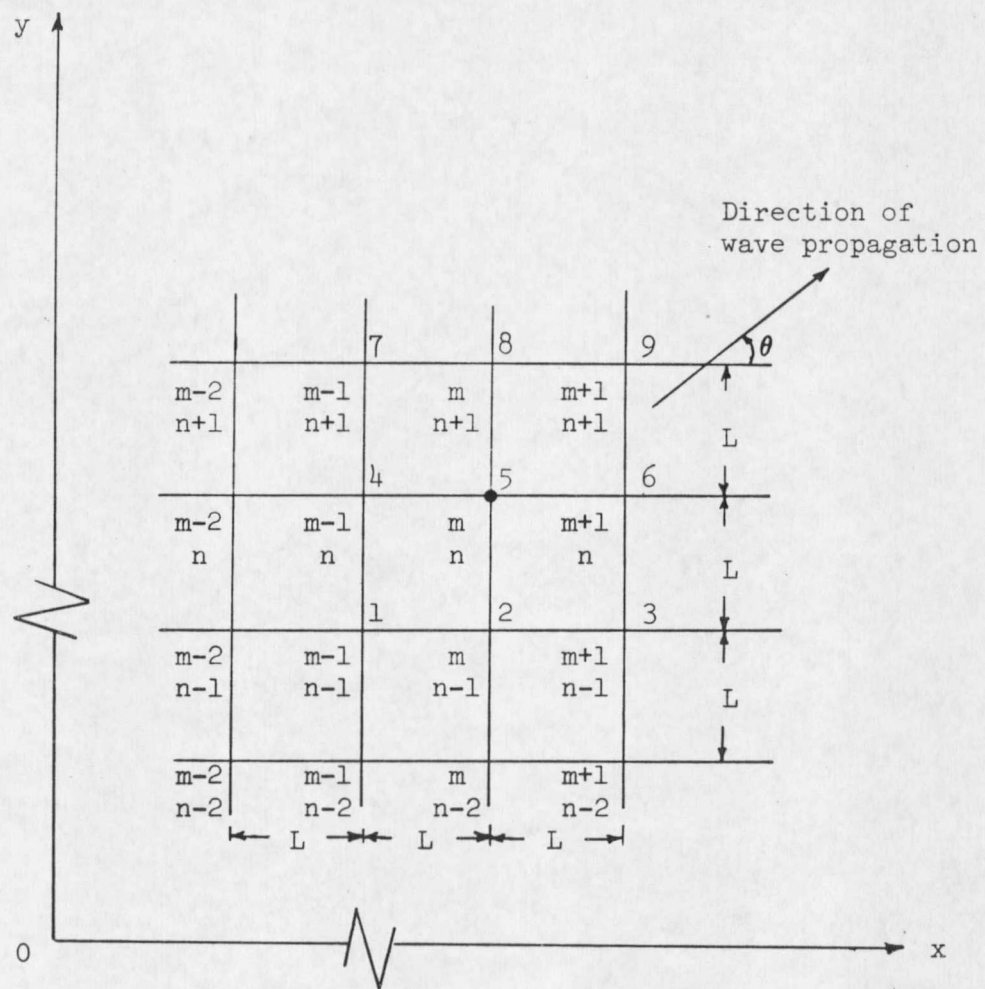


Fig. 3.6-2. Two-Dimensional Grid of Elements For Dispersion Study.

several elements, travelling in the direction shown. Oscillations of the nodes can be expressed as harmonic functions, eqn. (3.4-4). The generalized displacements at any one node (m,n) are expressed as functions of time as follows:

$$\left. \begin{aligned} u_{m,n} &= A_1 \exp\{2\pi i[\nu t - a_x m L - a_y n L]\} \\ v_{m,n} &= A_2 \exp\{2\pi i[\nu t - a_x m L - a_y n L]\} \\ u_{x_{m,n}} &= A_3 \exp\{2\pi i[\nu t - a_x m L - a_y n L]\} \\ u_{y_{m,n}} &= A_4 \exp\{2\pi i[\nu t - a_x m L - a_y n L]\} \\ v_{x_{m,n}} &= A_5 \exp\{2\pi i[\nu t - a_x m L - a_y n L]\} \\ v_{y_{m,n}} &= A_6 \exp\{2\pi i[\nu t - a_x m L - a_y n L]\} \end{aligned} \right\} \quad (3.6-7)$$

For a given wavelength, the phase differences between adjacent nodes are known. Considering node 5 in the Fig. 3.6-2, motion of all the surrounding eight nodes are expressible in terms of the motion of node 5. Further, the harmonic nature of motion yields a relationship between the displacements and accelerations.

The equations of motion of node 5 in Fig. 3.6-2 under the influence of zero external forces are written as:

$$[M]\{\ddot{\delta}\} + [K]\{\delta\} = \{0\} \quad (3.6-7)$$

where $[M]$ and $[K]$ are 6×54 matrices, respectively, representing the elastic and inertial interactions between the generalized displacements

at node 5 and those at the surrounding eight nodes. $\{\delta\}$ is a 54×1 vector comprising the generalized displacements of the nine nodes. As the generalized displacements of the surrounding eight nodes are expressible in terms of those at node 5 through phase relations, eqn (3.6-7) reduces to:

$$[m]\{\ddot{\delta}_5\} + [K]\{\delta_5\} = \{0\} \quad (3.6-8)$$

where $[m]$ and $[K]$ are 6×6 reduced inertial and elastic stiffness matrices whose coefficients are functions of the wavelength and $\{\delta_5\}$ is a 6×1 vector of generalized displacements at node 5. The terms in the generalized displacement vector are products of harmonic functions and nonvanishing arbitrary constants. As before, a nontrivial solution demands the identical vanishing of the determinant:

$$|[K] - 4\pi^2 v^2 [m]| = 0$$

where v is the frequency of wavefront in cycles per second.

In the present case the eigenvalue problem yields six eigenvalues. These eigenvalues are functions of the wavelength as the terms in the matrices $[m]$ and $[K]$ are functions of the wavelength. They are the frequencies of the wavefront, and when multiplied by wavelength, yield the phase velocities.

The computed phase velocity values are listed in Table 3.6-1, together with the physical properties and wave velocities of the continuum. Of the three values of phase velocity corresponding to the

TABLE 3.6-1

Physical properties of the continuum:

Modulus of Elasticity $E = 1$ Poisson's ratio $\mu = 0.3$ Mass density $= \rho = 1$ Modulus of rigidity $= G = \frac{E}{2(1+\mu)}$

Modified Lamé's constant (plane stress):

$$\bar{\lambda} = \frac{\mu E}{1-\mu} = .323$$

Dilatational velocity in continuum $= \left(\frac{\bar{\lambda}+2G}{\rho}\right)^{1/2} = 1.05$ Shear wave velocity in continuum $= \left(\frac{G}{\rho}\right)^{1/2} = 0.62$

Ratio of element length to wavelength L/ λ	Dilatational Wave Velocity			Shear Wave Velocity		
	V_1	V_2	V_3	V_1	V_2	V_3
.05	1.0483	21.2159	21.6345	0.6202	12.5517	12.8445
.10	1.0483	10.0876	10.8340	0.6202	5.9679	6.5007
.15	1.0483	6.2841	7.2418	0.6202	3.7177	4.4224
.20	1.0483	4.3731	5.4523	0.6202	2.5874	3.4113
.25	1.0484	3.2395	4.3840	0.6203	1.9165	2.8269
.30	1.0486	2.4958	3.6762	0.6204	1.4760	2.4531
.35	1.0489	1.9732	3.1730	0.6206	1.1673	2.1933
.40	1.0496	1.5864	2.7953	0.6209	0.9385	1.9943
.45	1.0505	1.2886	2.4974	0.6212	0.7624	1.8220
.50	1.0490	1.0552	2.2520	0.6206	0.6243	1.6563

dilatational wave, the lowest remains constant for all wavelengths and is equal to the dilatational wave velocity in the continuum. Similarly, of the three values of phase velocity corresponding to shear waves, the lowest remains constant for all wavelengths and is equal to the shear wave velocity in the continuum. Hence the refined element is free from discretization-dispersion.

Figures 3.6-3 and 3.6-4 show the dispersion curves for the refined finite element, for dilatational and shear wave propagation respectively. For the lowest mode, the velocity of propagation remains constant and is equal to the continuum velocity for all wavelengths for both dilatational and shear waves. The dispersion curves for the higher modes appear similar to the dispersion curves of higher modes in the theories of confined media. Since the element is free from discretization-dispersion, these dispersion curves represent the physical dispersion present in a two-dimensional continuum under plane stress conditions. For the second mode, the wavefront travels at faster velocities for long wavelengths and travels at approximately continuum velocities for short wavelengths. In the third mode, the phase velocity for all wavelengths is greater than that of the second mode while for short wavelengths, it is considerably greater than the continuum velocity. Such dispersion curves for higher modes are not obtainable for the ordinary displacement compatible element as it has fewer degrees-of-freedom per node.

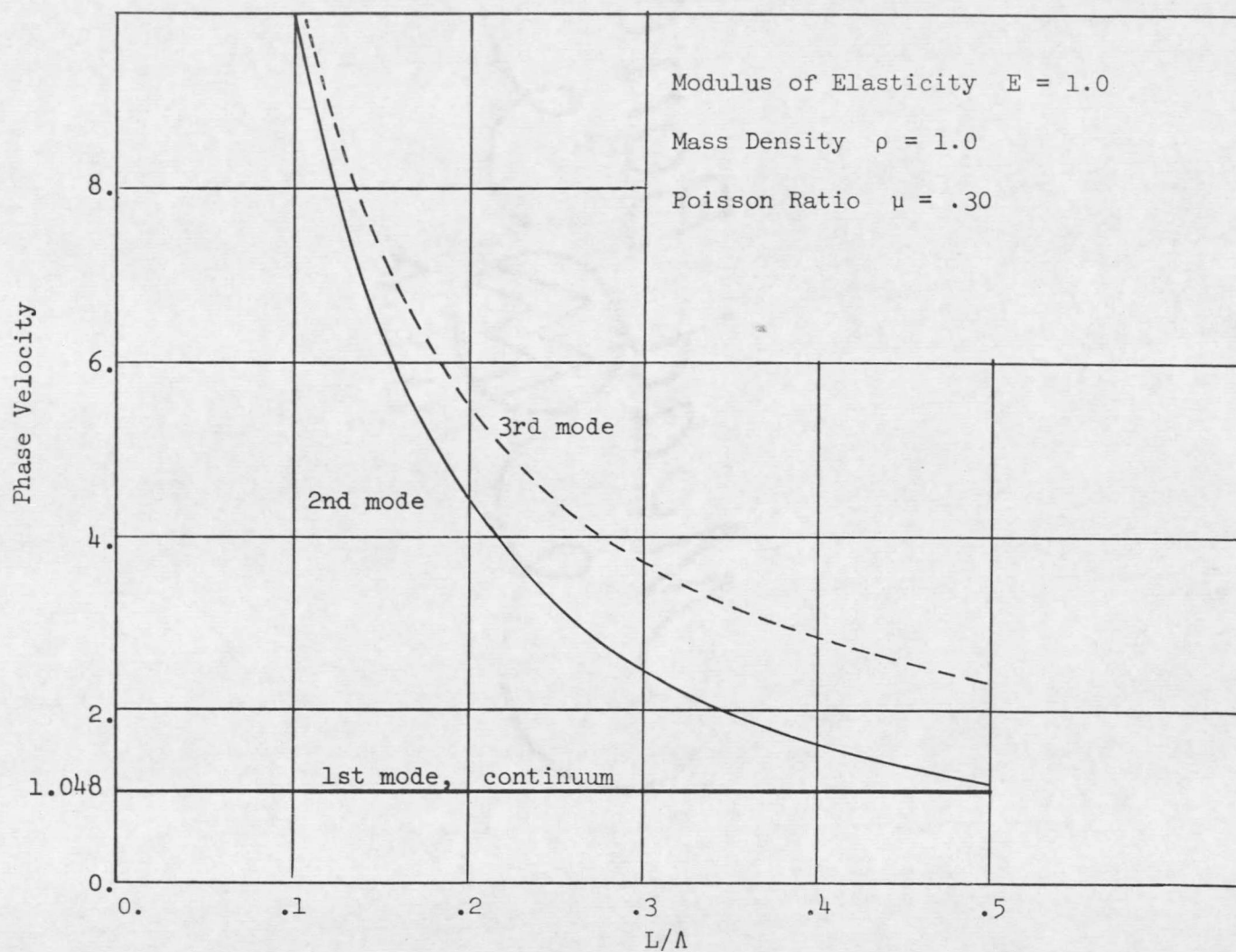


Fig. 3.6-3. Dispersion Curves For Two-Dimensional First Order Strain Compatible Finite Element Dilatational Waves.

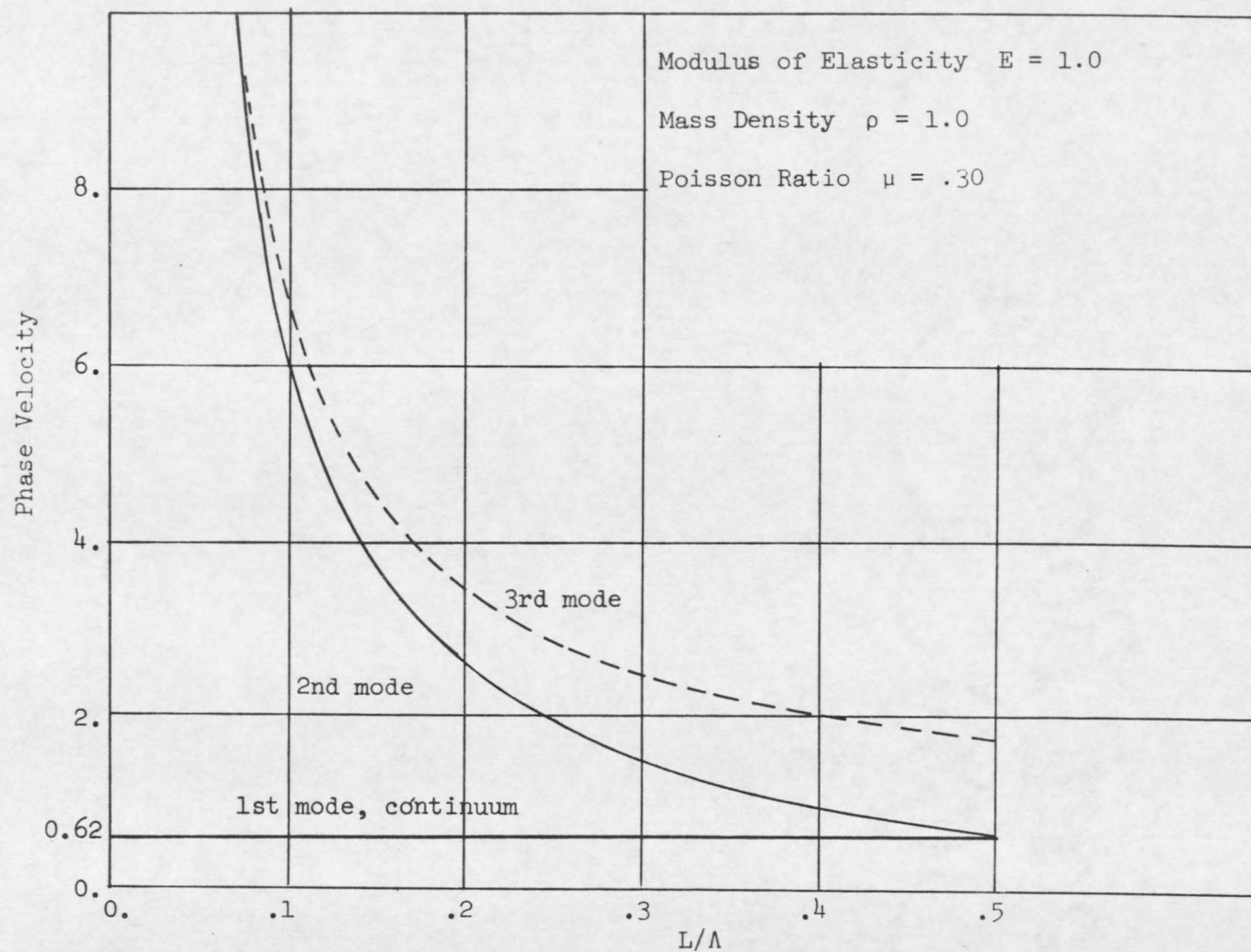


Fig. 3.6-4. Dispersion Curves For Two-Dimensional First Order Strain Compatible Finite Element Shear Waves.

The test procedure developed here for determining the discretization-dispersion associated with the finite elements may be used as a standard test on any element to determine its suitability of being used in dynamic and wave propagation analyses. The greater the discretization-dispersion in the finite element, the less accurate it is for such analyses. Of the large number of plane stress and strain elements now used in finite element programs, only those checked for non-discretization-dispersion should be used for accurate wave propagation modelling.

IV. PROPAGATIONAL CHARACTERISTICS OF TWO-DIMENSIONAL DISPLACEMENT
COMPATIBLE AND STRAIN COMPATIBLE RECTANGULAR FINITE ELEMENTS
IN TRANSMITTING DILATATIONAL AND SHEAR WAVES

4.1 Introduction

The discretization-dispersion associated with the two-dimensional finite elements in wave propagation modelling is discussed in detail in Chapter III. It is observed that both one and two dimensional finite elements satisfying strain compatibility across interfaces (developed with displacement functions derived from the first order Hermitian interpolation polynomials) are free from discretization-dispersion. Thus, the element transmits waves of different frequencies at continuum velocities. In this chapter, special problems encountered in the use of the first order elements are examined. Problems are formulated and solved for the propagation of dilatational and shear waves caused by harmonic and nonharmonic force excitations, using ordinary and refined finite elements, and the results are compared with existing closed form classical solutions.

4.2 Finite Element Model Boundary Conditions

When the differential equations (1.2-1) through (1.2-6), developed for the continuum, are discretized by a finite element approximation, they take the following form:

$$[M]\{\ddot{\delta}\} + [K]\{\delta\} = \{F\} \text{ in } c, t > t_0 \quad (4.2-1)$$

$[M]$ and $[K]$ are the mass and stiffness matrices, $\{\delta\}$ is the nodal displacement vector, with the following displacement boundary conditions:

$$\{\delta^B\} = \{W\}, \quad \{\delta^B\} \in S_I \quad (4.2-2)$$

and the initial conditions on displacements:

$$\{\delta(t)\} = \{W_0\}, \quad t = t_0 \quad (4.2-3)$$

and on velocities:

$$\{\dot{\delta}(t)\} = \{V_0\}, \quad t = t_0 \quad (4.2-4)$$

It is to be noted that boundary conditions are not yet specified on the boundaries S_{II} and S_{III} corresponding to equations (1.2-3) and (1.2-4). In a continuous system, these are the boundary conditions on the first order spatial derivatives of displacement functions, i.e., strain components. Strains are related to the stress components, which, when scalar multiplied by the outward drawn unit normal vector, are in equilibrium with the surface tractions. In a zeroth order (displacement compatible) finite element formulation, strain components are not part of the generalized nodal displacements, and hence, these boundary conditions cannot and need not be incorporated. In a higher order finite element formulation, strain components are among the generalized coordinates and, hence, the natural boundary conditions are to be explicitly stated. A rational approach is required in specifying these boundary conditions.

Fig. 4.2-1 shows a two-dimensional domain with four different types of boundaries, i.e., a fixed boundary (AB) where no

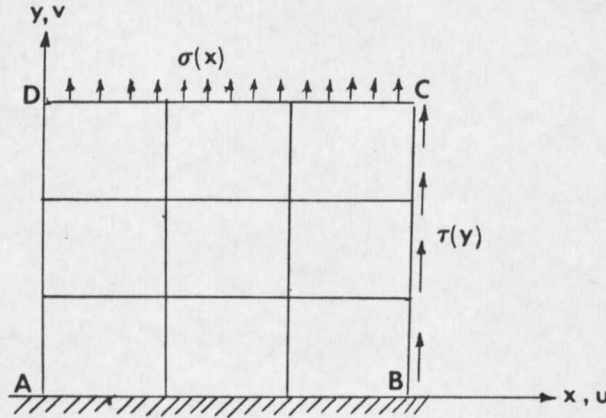


Fig. 4.2-1.

displacements are permitted, a free boundary (AD) subjected to no surface tractions, a boundary subjected to shearing tractions (DC) and a boundary subjected to normal tractions (BC). If u and v are the displacements in the x and y -directions, under plane-stress conditions, the boundary value problem is defined by:

$$\left. \begin{aligned} \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 v}{\partial x \partial y} + \frac{1-\mu}{2} \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} \right) &= 0 \\ \frac{\partial^2 v}{\partial y^2} + \mu \frac{\partial^2 u}{\partial x \partial y} + \frac{1-\mu}{2} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} \right) &= 0 \end{aligned} \right\} \quad (4.2-5)$$

inside ABCD, and the following boundary conditions

$$\left. \begin{aligned} (i) \quad \frac{E}{1-\mu^2} \left(\frac{\partial u}{\partial x} + \mu \frac{\partial v}{\partial y} \right) &= \sigma(x) \\ \text{and} \quad \frac{E}{2(1+\mu)} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) &= 0 \end{aligned} \right\} \quad \text{on CD} \quad (4.2-6)$$

$$\begin{aligned}
 & \text{(ii)} \quad \frac{E}{1-\mu^2} \left(\frac{\partial v}{\partial y} + \mu \frac{\partial u}{\partial x} \right) = 0 \\
 & \text{and} \quad \frac{E}{2(1+\mu)} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \tau(y)
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} & \text{(ii)} \quad \frac{E}{1-\mu^2} \left(\frac{\partial v}{\partial y} + \mu \frac{\partial u}{\partial x} \right) = 0 \\ & \text{and} \quad \frac{E}{2(1+\mu)} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \tau(y) \end{aligned}} \right\} \text{on BC} \quad (4.2-7)$$

$$\begin{aligned}
 & \text{(iii)} \quad \frac{E}{(1-\mu^2)} \left(\frac{\partial v}{\partial y} + \mu \frac{\partial u}{\partial x} \right) = 0 \\
 & \text{and} \quad \frac{E}{2(1+\mu)} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = 0
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} & \text{(iii)} \quad \frac{E}{(1-\mu^2)} \left(\frac{\partial v}{\partial y} + \mu \frac{\partial u}{\partial x} \right) = 0 \\ & \text{and} \quad \frac{E}{2(1+\mu)} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = 0 \end{aligned}} \right\} \text{on AD} \quad (4.2-8)$$

and

$$\text{(iv)} \quad u = v = 0 \text{ on AB} \quad (4.2-9)$$

This is a boundary value problem on displacements rather than on strains. When the above boundary conditions are applied to a zeroth order finite element model where strains are not among the generalized coordinates, only boundary condition (4.2-9) is incorporated and it is consistent with the Ritz method. In a first order finite element model, strain components are among the generalized coordinates and, hence, the boundary conditions (4.2-6) to (4.2-8) are to be incorporated at the node points along the respective boundaries. These boundary conditions consist of two equations each, and hence, at these nodes, the number of degrees-of-freedom is reduced from six to four.

On the fixed boundary AB, eqn. (4.2-9) imposes constraints on displacement components u and v to be zero. Since u and v are zero along the boundary, their derivatives with respect to x , i.e.

$\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ are also equal to zero at nodes on the boundary. The above four constraints reduce the number of degrees-of-freedom from six to two at each node.

Representation of proper boundary conditions at the junction of a free boundary and a fixed boundary is difficult because of the development of stress singularities at the corners or junctions. For example, at node 1 in Fig. 4.2-1, on the fixed boundary, $u = 0$. But at a small distance away from the fixed boundary along the y-axis on the free boundary, $u \neq 0$ resulting in an infinite value of $\frac{\partial u}{\partial y}$ at A. This boundary condition is simulated in several ways but the differences are negligible.

Nodal forces are computed consistently, based on the deformation of the element along the boundary, where external forces are distributed. If $f(s)$ represents the surface tractions on a boundary S and the displacement on that boundary is given by $u(s)$, then, the work done by the traction forces is equal to:

$$W = \int_S f(s) u(s) dS = \{F\}^T \{\delta\}$$

where $\{\delta\}$ is the vector of nodal displacements on S and $\{F\}$ is the corresponding generalized force vector, which is to be evaluated.

4.3 Solution Procedures

Equation (4.2-1) is a set of simultaneous ordinary differential equations which approximately represents the dynamic behavior of the continuum, represented by the finite element model. There are many numerical procedures available for solving approximately this set of differential equations. Two such methods used in this study are the so-called acceleration-pulse method [37] and the fourth order Runge-Kutta method with modification due to Gill [45]. The acceleration-pulse method is found to be a hundred times faster than the Runge-Kutta method when coded in the computer, and is used for most calculations. Both methods are briefly explained below.

4.3.1 Acceleration-Pulse Method

The term originates from the basic assumption in the derivation that the area under an acceleration curve may be replaced by a series of equivalent pulses at regular time intervals, say Δt . The magnitude of the pulse at t_n is given by:

$$I(t_n) = \ddot{x}_n \Delta t \quad (4.3-1)$$

This is shown in Fig. 4.3-1(b). Since a pulse is applied at t_n there is a discontinuity in the velocity at t_n . During the time interval Δt , the velocity is constant and the displacement varies linearly with respect to time. The velocity and displacement curves so obtained are shown in Fig. 4.3-1(c) and (d).

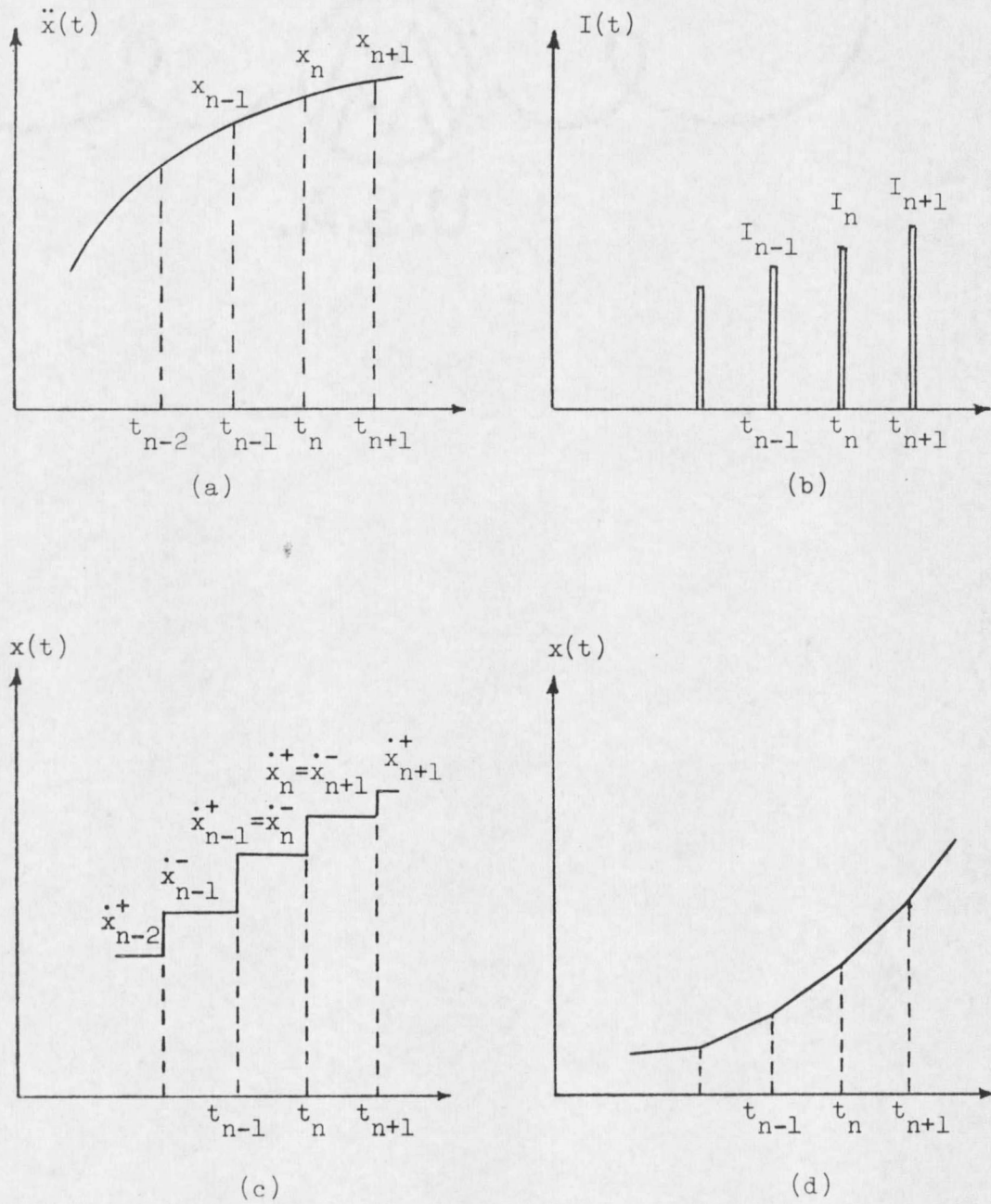


Fig. 4.3-1. Acceleration-Pulse Extrapolation Method: (a) Given Acceleration Curve, (b) Acceleration-Pulses to Replace Given Acceleration Curve, (c) Velocity, (d) Displacement.

Suppose t_n^- and t_n^+ are the times immediately before and after the pulse at t_n and let $\dot{x}_n^-$ and $\dot{x}_n^+$ indicate respectively the velocity at t_n^- and t_n^+ . The two velocities are related by the following equation:

$$\dot{x}_n^+ = \dot{x}_n^- + \ddot{x}_n \Delta t \quad (4.3-2)$$

The relation between x_{n-1} and x_n and between x_n and x_{n+1} are given by:

$$\begin{aligned} x_n - x_{n-1} &= \dot{x}_n^- \Delta t \\ x_{n+1} - x_n &= \dot{x}_n^+ \Delta t \end{aligned} \quad (4.3-3)$$

Combining equations (4.3-2) and (4.3-3), the three successive displacements are related by:

$$x_{n+1} = 2x_n - x_{n-1} + \dot{x}_n (\Delta t)^2 \quad (4.3-4)$$

This is the basic recurrence formula for acceleration-pulse extrapolation. Evaluation of x_1 is by the equation:

$$x_1 = 2x_0 - x_{-1} + \ddot{x}_0 (\Delta t)^2 \quad (4.3-5)$$

where x_0 is given by the initial condition, which when zero, x_{-1} is also taken as zero.

4.3.2 Fourth Order Runge-Kutta Method

Runge-Kutta methods are algorithms of one-step procedures involving the evaluation of first order derivatives of object functions while producing results equivalent in accuracy to the higher order Taylor formulas. Solution by direct Taylor expansion of object

function is generally not practicable if the derivatives of higher order than the first order are retained. Approximations of second, third and fourth order require the estimation of the first order derivative of the object function at two, three and four values respectively, of the independent variable t in the closed interval $t_i < t < t_{i+1}$. Fourth order Runge-Kutta formulas are of the form:

$$x_{i+1} = x_i + \Delta t [ak_1 + bk_2 + ck_3 + dk_4] \quad (4.3-5)$$

where $a, b, c, d, k_1, k_2, k_3$ and k_4 are parameters derived in the development of the algorithm. The method used in this study is credited to Gill in which eqn. (4.3-5) takes the following form:

$$x_{i+1} = x_i + \frac{\Delta t}{6} \left\{ k_1 - 2\left(1 - \frac{1}{\sqrt{2}}\right) k_2 + 2\left(1 - \frac{1}{\sqrt{2}}\right) k_3 + k_4 \right\} \quad (4.3-6)$$

where

$$k_1 = \frac{d}{dt} [x_i, t_i],$$

$$k_2 = \frac{d}{dt} \left[x_i + \frac{\Delta t}{2} k_1, t_i + \frac{\Delta t}{2} \right]$$

$$k_3 = \frac{d}{dt} \left[x_i + \left(-\frac{1}{2} + \frac{1}{\sqrt{2}}\right) \Delta t \cdot k_1 + \left(1 - \frac{1}{\sqrt{2}}\right) \Delta t \cdot k_2, t_i + \frac{\Delta t}{2} \right]$$

and

$$k_4 = \frac{d}{dt} \left[x_i - \frac{1}{\sqrt{2}} \Delta t \cdot k_2 + \left(1 + \frac{1}{\sqrt{2}}\right) \Delta t \cdot k_3, t_i + \Delta t \right]$$

4.4 Scope of the Study

A homogeneous prismatic elastic bar, 54 in. x 6 in. x 1 in. is modelled by an assembly of nine square elements, as shown in Fig. 4.4-1.

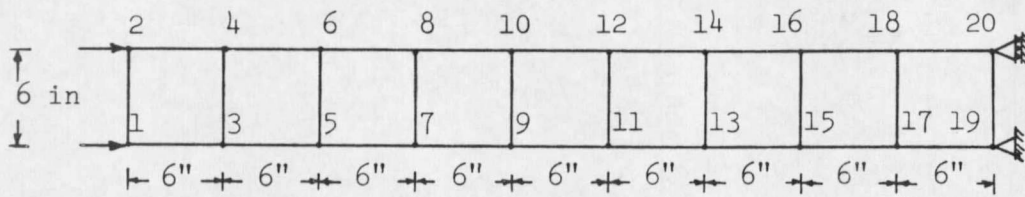
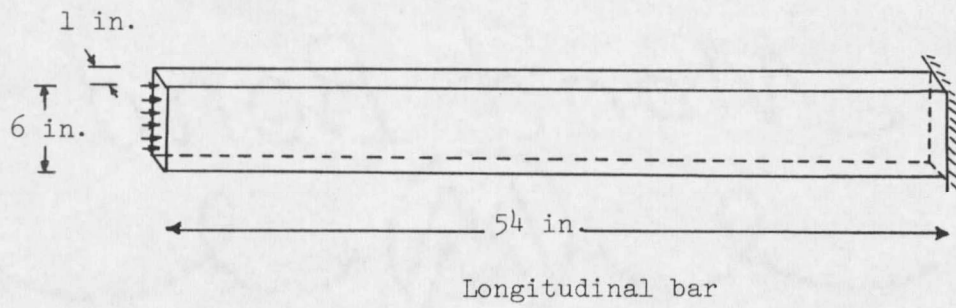


Fig. 4.4-1.

Physical properties of the continuum considered are given in Table 4.4-1. One end of the bar is fixed to a rigid base, and hence, in the finite element model, the corresponding nodes 19 and 20 are constrained not to have any displacement in the x and y-directions.

The remainder of the surfaces are free, not subjected to any displacement constraints and appropriate boundary conditions as discussed before are imposed at the nodes on these boundaries.

Harmonic and nonharmonic force inputs are applied at the free end, causing dilatational and shear wave propagation to take place. Velocity responses at several nodes are computed for various kinds of force excitations in both dispersive and nondispersive element formulations. Influence of discretization-dispersion on the velocity of propagation and on the distortion of the propagating wave front are studied in the dispersive element model. Nondispersive element model responses are computed and studied to ascertain the absence of discretization-dispersion. The force excitations considered include low and high frequency harmonic excitations, harmonic excitations above the cutoff frequency, a forcing function which is a sum of several harmonic excitations of different frequencies, and a step-rise force input. A finite element model is formed by an assembly of nonsquare elements and studied for its responses under the above force excitations. This is done to evaluate the stronger attenuation characteristics present in the nonsquare elements to filter high

TABLE 4.4-1

Material Properties:

Modulus of Elasticity, $E = 29 \times 10^6$ psiMass Density, $\rho = 7.53 \times 10^{-4}$ slugs/in³Poisson's Ratio, $\mu = 0.29$

General Three Dimensional Case:

Lamé's Constants:

$$\lambda = \frac{\mu E}{(1+\mu)(1-2\mu)} = 15.5 \times 10^6 \text{ psi}$$

$$G = \frac{E}{2(1+\mu)} = 11.25 \times 10^6 \text{ psi}$$

$$\text{Dilatational Wave Velocity} = \left(\frac{\lambda + 2G}{\rho} \right)^{1/2} = 2.25 \times 10^5 \text{ in/sec.}$$

$$\text{Shear Wave Velocity} = \left(\frac{G}{\rho} \right)^{1/2} = 1.221 \times 10^5 \text{ in/sec.}$$

Two Dimensional Plane Stress Case:

Lamé's Constants:

$$\bar{\lambda} = \frac{\mu E}{(1-\mu^2)} = 9.16 \times 10^6 \text{ psi}$$

$$G = \frac{E}{2(1+\mu)} = 11.25 \times 10^6 \text{ psi}$$

$$\text{Dilatational Wave Velocity} = V_p = \sqrt{\frac{\bar{\lambda} + 2G}{\rho}} = 2.05 \times 10^5 \text{ in/sec.}$$

$$\text{Shear Wave Velocity} = V_s = \sqrt{\frac{G}{\rho}} = 1.221 \times 10^5 \text{ in/sec.}$$

$$\text{Bar Velocity}^* = V_b = \sqrt{\frac{E}{\rho}} = 1.962 \times 10^5 \text{ in/sec.}$$

*Bar Velocity is the velocity of propagation of a dilatational wave when there is no Poissonic effect.

frequency oscillations.

In the case of propagation of dilatational waves, contact between the bar and the rigid base is assumed to be smooth so as to avoid the possible development of shearing tractions. It is possible to represent such a boundary condition by leaving the displacements in the y-direction at nodes 19 and 20 to be arbitrary. This results in a possible rigid body translation of the model in the y-direction resulting in a singular stiffness matrix. The singularity is avoided by arresting the lateral displacements at one of the two nodes. The other boundary conditions imposed at nodes 19 and 20 are that the shearing strain component $\frac{\partial u}{\partial y}$ is made equal to zero. Shearing strain component $\frac{\partial v}{\partial x}$ is left arbitrary. Normal strain components $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ are left arbitrary.

4.5 Propagation of Dilatational Waves Caused by Harmonic Force Excitations

Harmonic axial force excitations of low, medium and high frequencies are applied at the free end of the bar. The force is applied uniformly over the free end of the bar which causes the generation and propagation of dilatational waves. The size of an individual element is 6 in. x 6 in. Cut off frequency is that frequency corresponding to the wavelength, equal to twice the length of the element, which is 12 in. in the present case. The corresponding

frequency for dilatational waves is 17,050 cycles/second. Force excitations considered here are of frequencies 3800, 10,000, 15,000 and 25,000 cycles/second.

4.5-1 Low Frequency Harmonic Axial Force Input - 3800 cycles/second

Velocity responses at several node points are computed using the acceleration-pulse extrapolation method up to 2000 microseconds in steps of 5 microseconds each. The wavefront travels through one element length of continuum in approximately 29.3 microseconds. So, the propagation of the wavefront across an element is carried out in about six time increments in the computation. There is no significant error accumulation in the numerical integration scheme over the time span of the computation.

Computed velocity responses at several node points including the free end for both nondispersive and dispersive element models are presented in Figures 4.5-1 through 4.5-4. As expected, velocity response at a given node is harmonic until the time when the incident wave returns to that node after being reflected at the fixed end, and interferes with the incident wave front.

The following methods are used to determine the velocity of propagation of the wave front from the velocity response curves of different node points: (i) from the time taken by the wave front to travel from a given node to the fixed end, reflect from the fixed

Nondispersive Element Model
 Uniform Grid of Square Elements
 Harmonic Force Input at Free End
 3800 cycles/sec. See Table 4.4-1
 for Material Properties

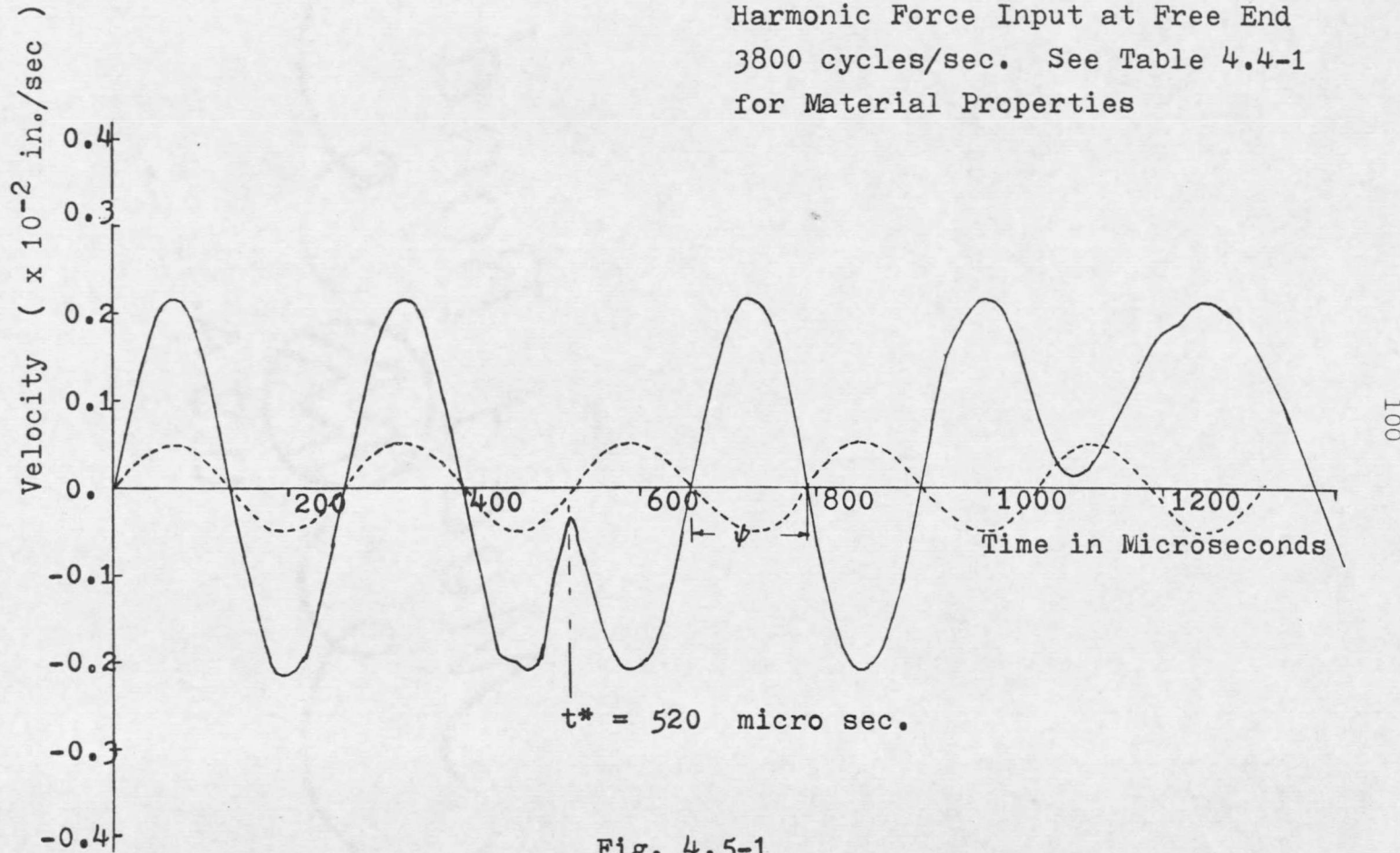


Fig. 4.5-1
 Velocity Response at the Free End

Nondispersive Element Model
 Uniform Grid of Square Elements
 Harmonic Axial Force Input at
 Free End - 3800 cycles/sec.
 See Table 4.4-1 for Material Properties

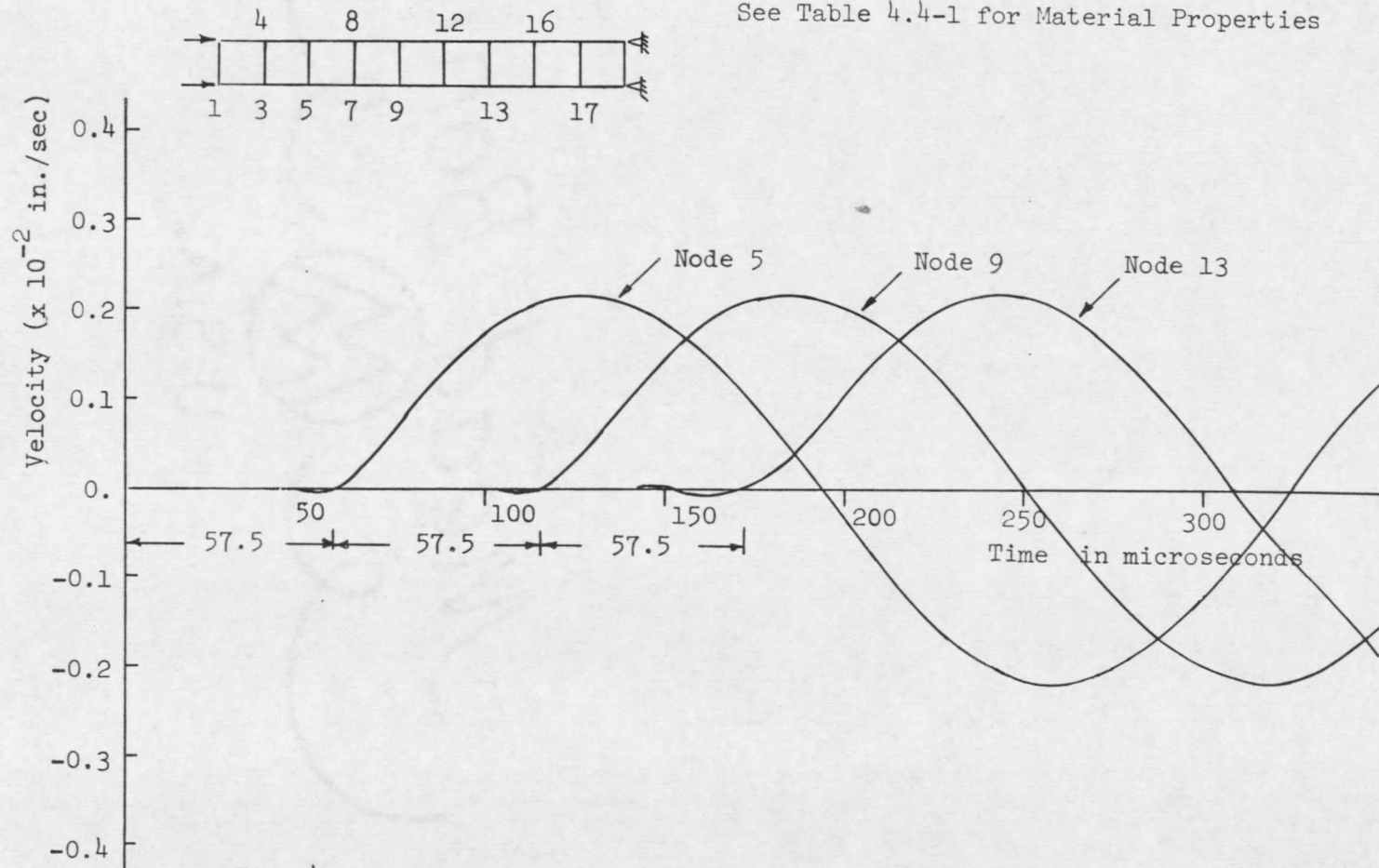


Fig. 4.5-2. Velocity Response at Interior Nodes.

Dispersive Element Model
 Uniform Grid of Square Elements
 Harmonic Axial Force Input at
 Free End - 3800 cycles/sec.
 See Table 4.4-1 for Material
 Properties

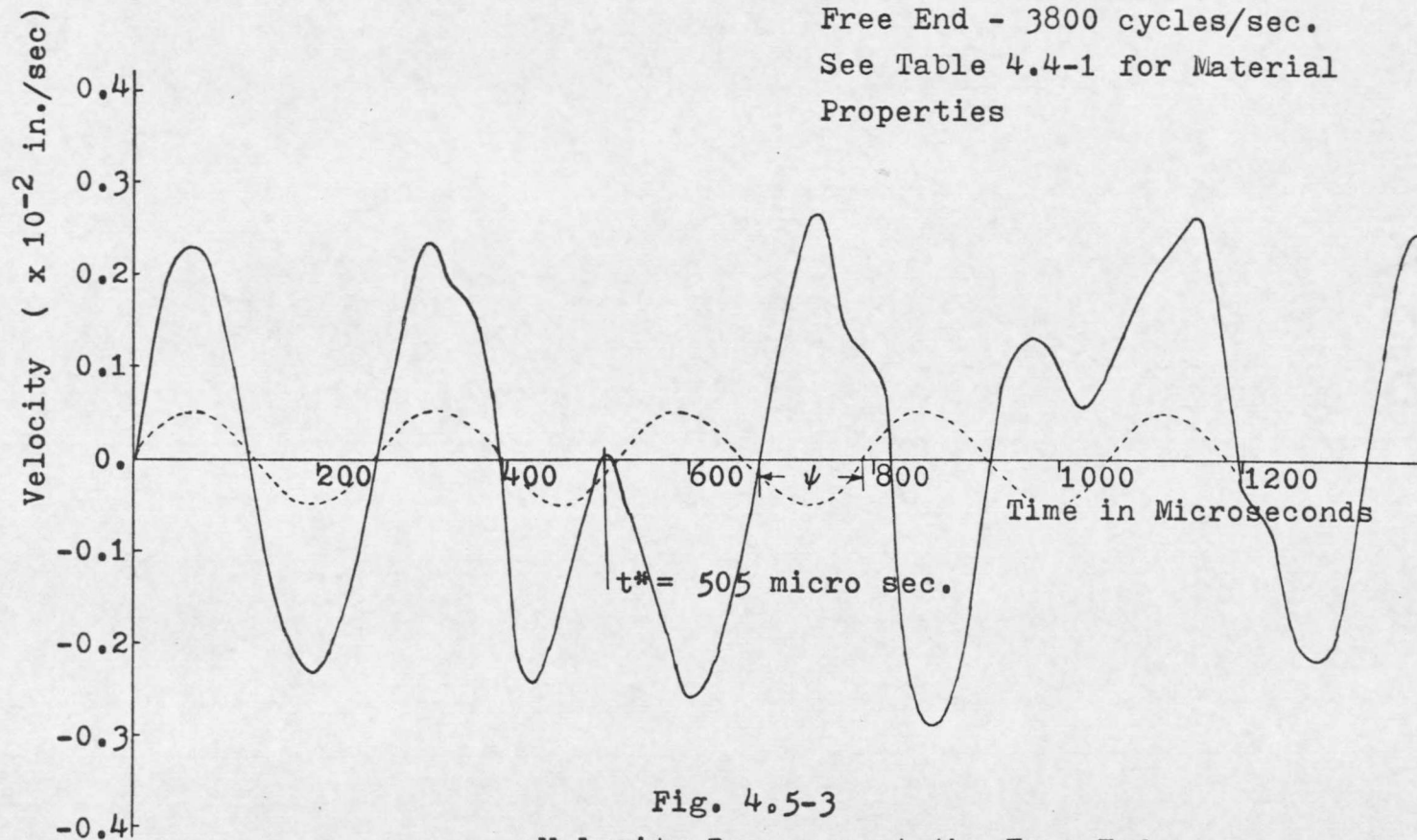


Fig. 4.5-3
 Velocity Response at the Free End

Dispersive Element Model
 Uniform Grid of Square Elements
 Harmonic Axial Force Input at
 Free End - 3800 cycles/sec
 See Table 4.4-1 for Material
 Properties

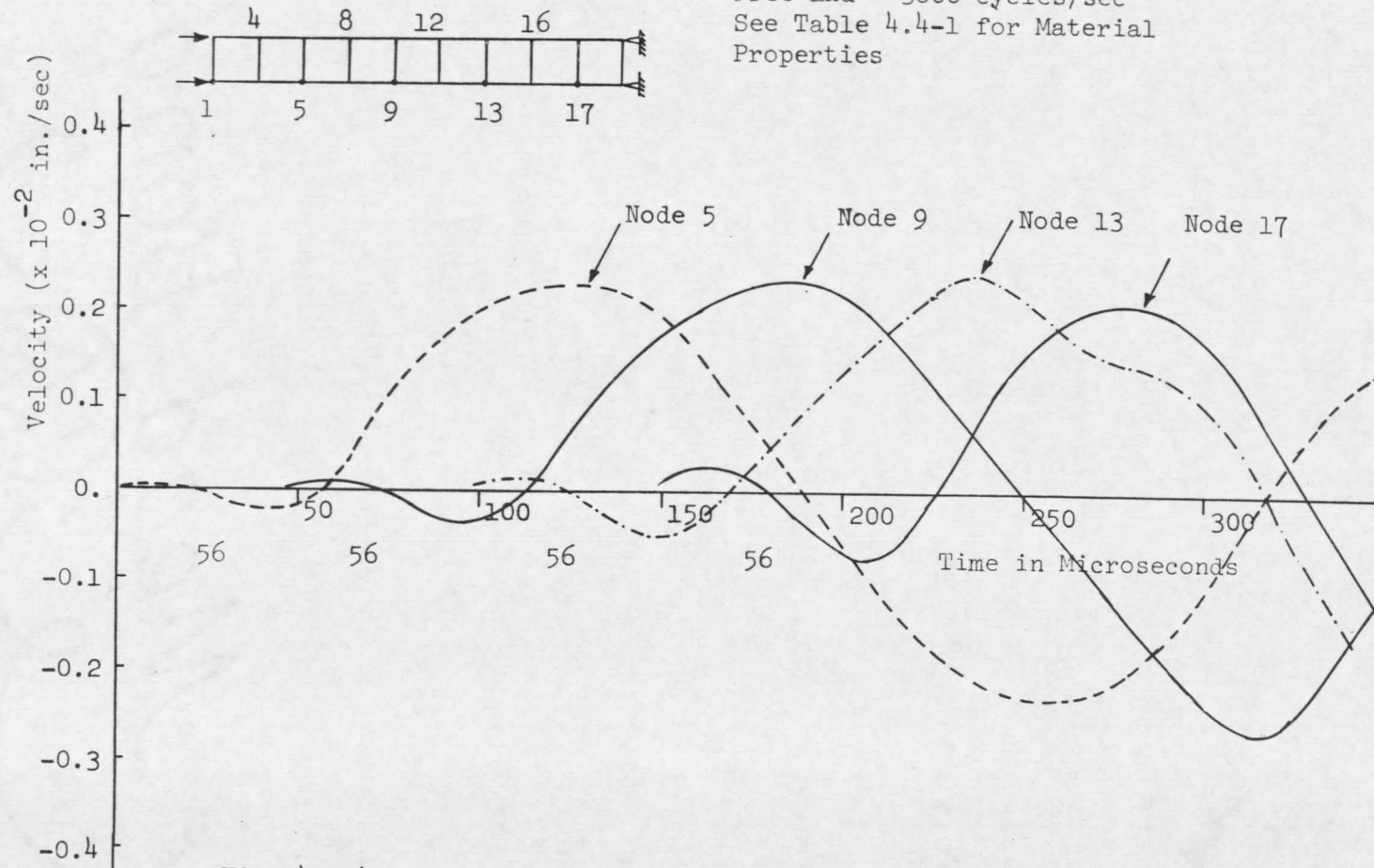


Fig. 4.5-4. Velocity Responses at Interior Nodes.

end and return to the node of interest, (ii) from the time lag of the response between two given nodes, as this would be equal to the time taken by the wave front to travel the distance between the two nodes, and (iii) from the phase difference between the input force excitation and the velocity response at the free end at the time the incident wave front has returned to the free end.

Fig. 4.5-1 shows the velocity response at the free end for the nondispersive element model. The incident wave returns to the free end at time $t^* = 520$ microseconds (marked in the figure). The length of the bar is 54 inches, the wave front travels twice the length of the bar in 520 microseconds, so the velocity of propagation is equal to 2.07×10^5 in/sec., compared with a continuum velocity of 2.05×10^5 in/sec. The error is about 1% which is reasonable for a numerical computation.

Fig. 4.5-2 shows the velocity responses of nodes 5, 9 and 13. These nodes are separated by 12 inches, and node 5 is located 12 inches from the free end. Time lag between the responses at nodes 5 and 9, and 9 and 13 are 57.5 microseconds, as noted in the figure. Node 5 starts to oscillate 57.5 microseconds after the harmonic force is applied to the free end. This means that the wavefront travels 12 inches distance in 57.5 microseconds, implying a velocity of propagation of 2.08×10^5 in/sec. which compares well with the previously computed value.

In Fig. 4.5-1, velocity response after the time t^* is that of the free end after the incident wave has returned to the free end after its first reflection. The input signal has a period of 263 microseconds and is shown in Fig. 4.5-1 in dotted lines. The free end velocity response lags the input signal by 132 microseconds, which is half the period, i.e., the phase difference is π radians. The response can be expressed as:

$$\dot{u} = A \sin (\omega t + \psi)$$

where ω is the input angular frequency,

ψ is the phase angle, equal to π radians, and

A is the amplitude magnification, which in this case is unity.

This response is the superposition of three components of velocities:

- (i) due to the input force function,
- (ii) due to the wavefront reflected at the fixed end, travelling towards the free end, and
- (iii) due to the wave reflected from the free end, travelling away from the free end. When we consider a point in the close neighborhood of the free end, in the limit, the latter two responses will be in phase with each other. Let the phase difference between the input forcing function and the latter two components be ϕ . The sum of the three components becomes:

$$\sin \omega t - \sin (\omega t + \phi) - \sin (\omega t + \phi) = A \sin (\omega t + \psi) \quad (4.5-1)$$

where $A = 1.00$ and $\psi = \pi$ radians.

Equation 4.5-1 may be rewritten as:

$$\sin \omega t - 2 \sin (\omega t + \phi) = \sin (\omega t + \pi) = -\sin \omega t \quad (4.5-2)$$

or $2 \sin (\omega t + \phi) = \sin \omega t$ or $\phi = 0$ radians. Hence the reflected wavefront arrives at the free end at the end of the second period, i.e., after 526 microseconds. This is equivalent to a velocity of propagation of 2.055×10^5 in/sec. which is in good agreement with the continuum velocity and the previously computed values.

Figures 4.5-3 and 4.5-4 show the velocity responses at the free end and interior nodes for the dispersive element model. When compared with the free end velocity response for the nondispersive element model (Fig. 4.5-3) it shows large perturbations with increasing time span. The incident wave front returns to the free end after 505 microseconds implying a velocity of propagation equal to 2.14×10^5 in/sec., resulting in an error of 4.5%. Fig. 4.5-4 shows the velocity responses of nodes 5, 9, 13 and 17. Time lag between the responses at these nodes are 56 microseconds. The corresponding velocity of propagation is 2.14×10^5 in/sec. which is the same as computed above.

Free end velocity response of the dispersive element model in Fig. 4.5-3 after time $t = 505$ microseconds, lags the input excitation with a phase difference of $.864\pi$ radians. Amplitudes

are magnified by a factor of 1.19. Proceeding as before, the velocity of the free end is obtained from the equation:

$$\sin \omega t - 2 \sin (\omega t + \phi) = 1.17 \sin (\omega t - .864\pi)$$

Solving the above equation, $\phi = .46$ radians, which is equivalent to a time lead of $(46 \times \frac{263}{2\pi}) \approx 20$ microseconds. So, the initial incident wavefront returns to the free end 20 microseconds before the end of the second period of the input pulse, i.e., after 506 microseconds. This is equivalent to velocity of propagation of 2.14×10^5 in/sec., this value being the same as computed above.

An important difference observed between the response curves in Fig. 4.5-2 and 4.5-4 is that in the case of the dispersive element model, interior nodes start oscillating before the arrival of the wavefront. It is an artificial farfield coupling inherent in the dispersive element. The artificial far field coupling is absent in the nondispersive element as seen in Fig. 4.5-3 wherein the nodes do not oscillate before the arrival of the wavefront. It is believed that the artificial far field coupling is absent in the nondispersive element because of the refinements introduced by the incorporating of stronger compatibility conditions.

4.5-2 Medium Frequency Harmonic Axial Force Input - 10,000 cycles/sec.

Computations were carried out on both nondispersive and dispersive element models, similar to the methods explained in the previous

section. The acceleration-pulse extrapolation method was used with a time increment of 5 microseconds. Velocity responses at the free end and several interior nodes are computed and presented in Figures 4.5-5 to 4.5-8. Velocity of propagation in the nondispersive element model as determined from the free end response (Fig. 4.5-5) and from the responses of interior nodes (Fig. 4.5-5) is 2.08×10^5 in/sec. The error is 1.5% which is the same as that for the low frequency wave (3800 cps). Velocity of propagation in the dispersive element model as determined from the free end velocity response (Fig. 4.5-7) and from the responses of interior nodes (Fig. 4.5-8) is 2.18×10^5 in/sec. resulting in an error of 6.5% which is greater than that for the low frequency waves (3800 cps).

The dispersion study done in Chapter III is applied to a continuum and does not apply to the bar geometry considered here. Hence the velocities computed here cannot be compared with those computed in the dispersion study.

Again, a significant factor observed in the dispersive element model response is the increased artificial far field coupling in the responses of interior nodes (Fig. 4.5-8) compared with the responses of low frequency waves (Fig. 4.5-4).

Nondispersive Element Model
 Uniform Grid of Square Elements
 Harmonic Axial Force input at Free end
 10,000 cps. See Table 4.4-1 for Material
 Properties

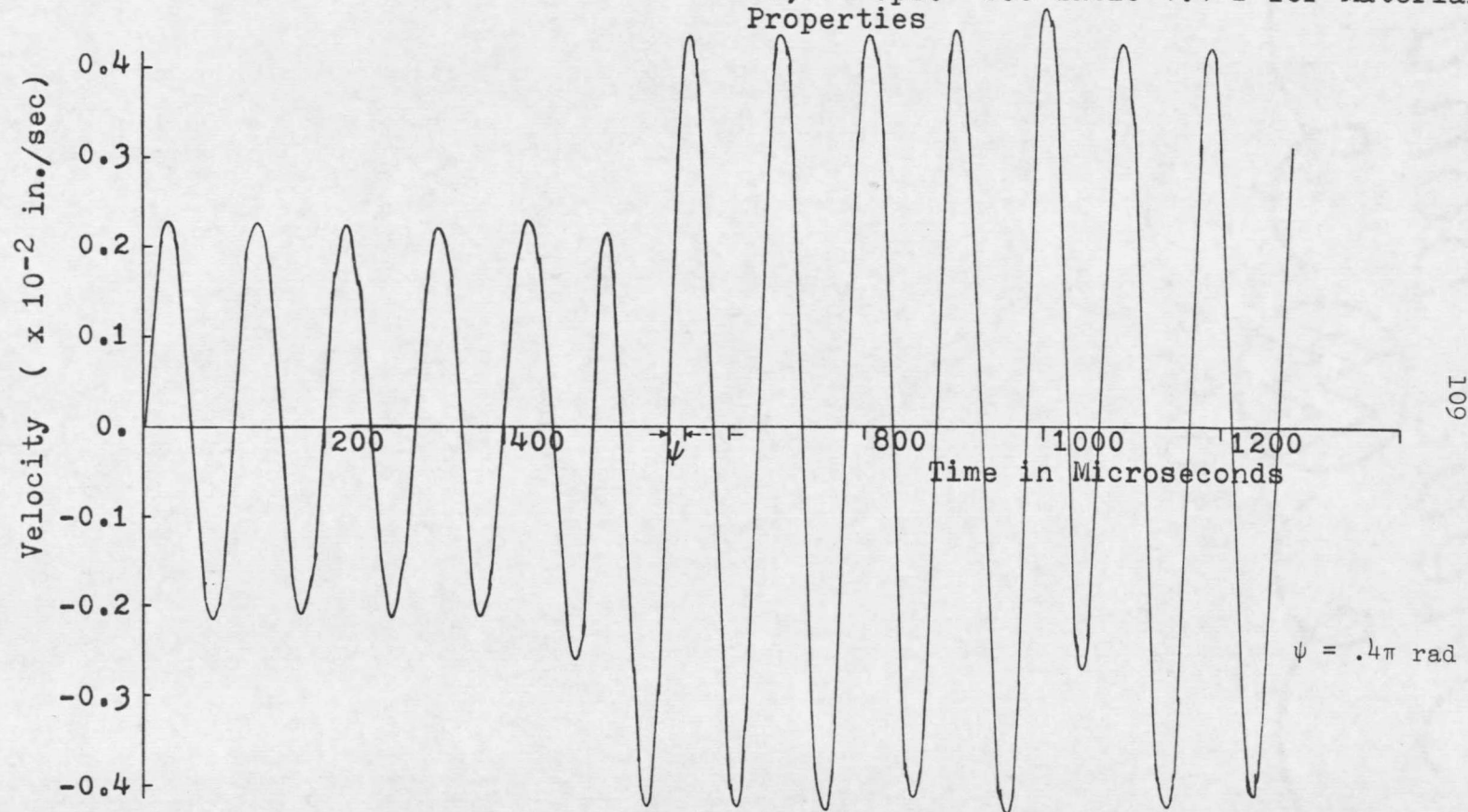


Fig. 4 .5-5 Velocity Response at the Free End

Nondispersive Element Model
 Uniform Grid of Square Elements
 Harmonic Axial Force Input at
 Free End - 10,000 cycles/sec
 See Table 4.4-1 for Material Properties

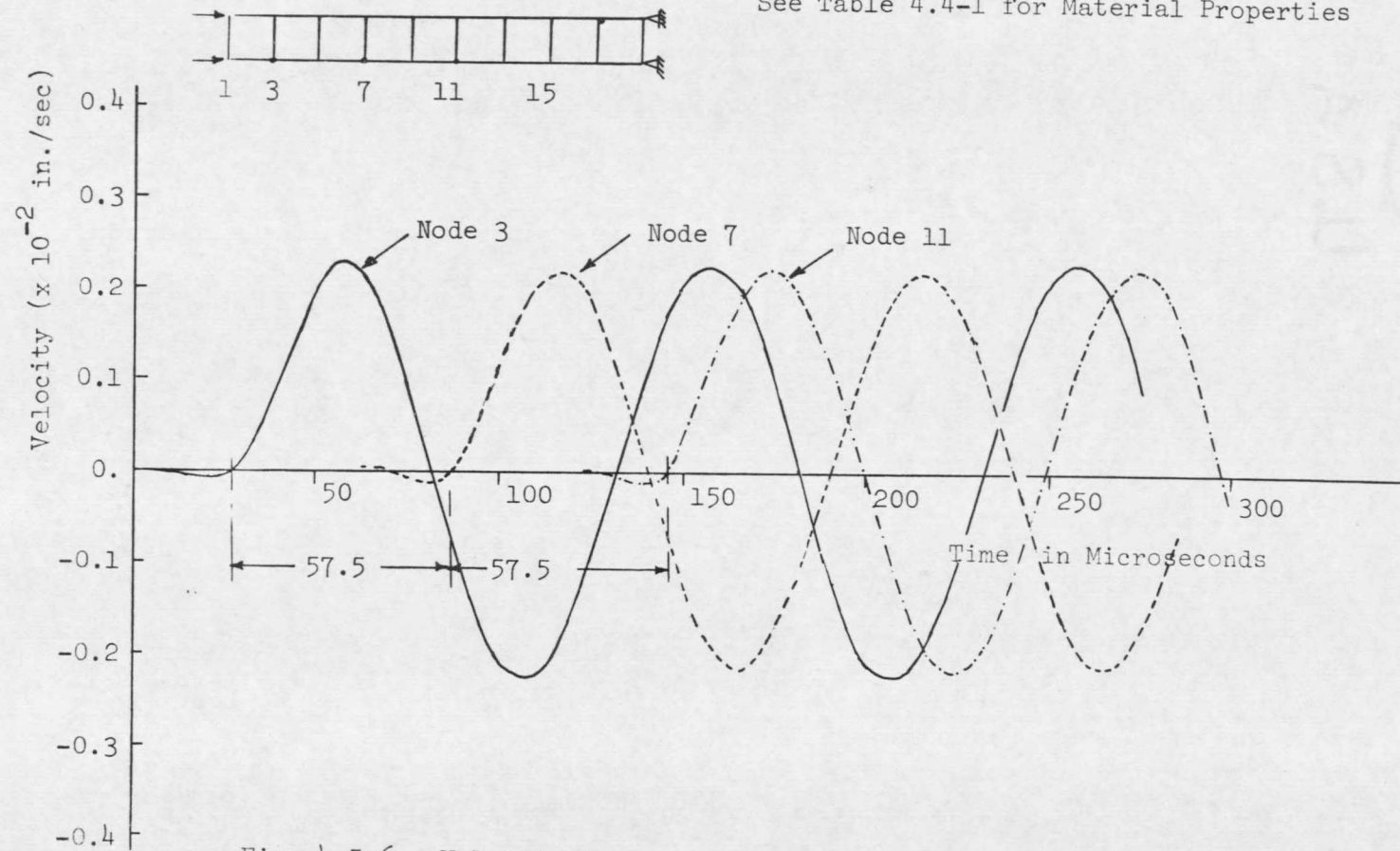


Fig. 4.5-6. Velocity Responses at Interior Nodes.

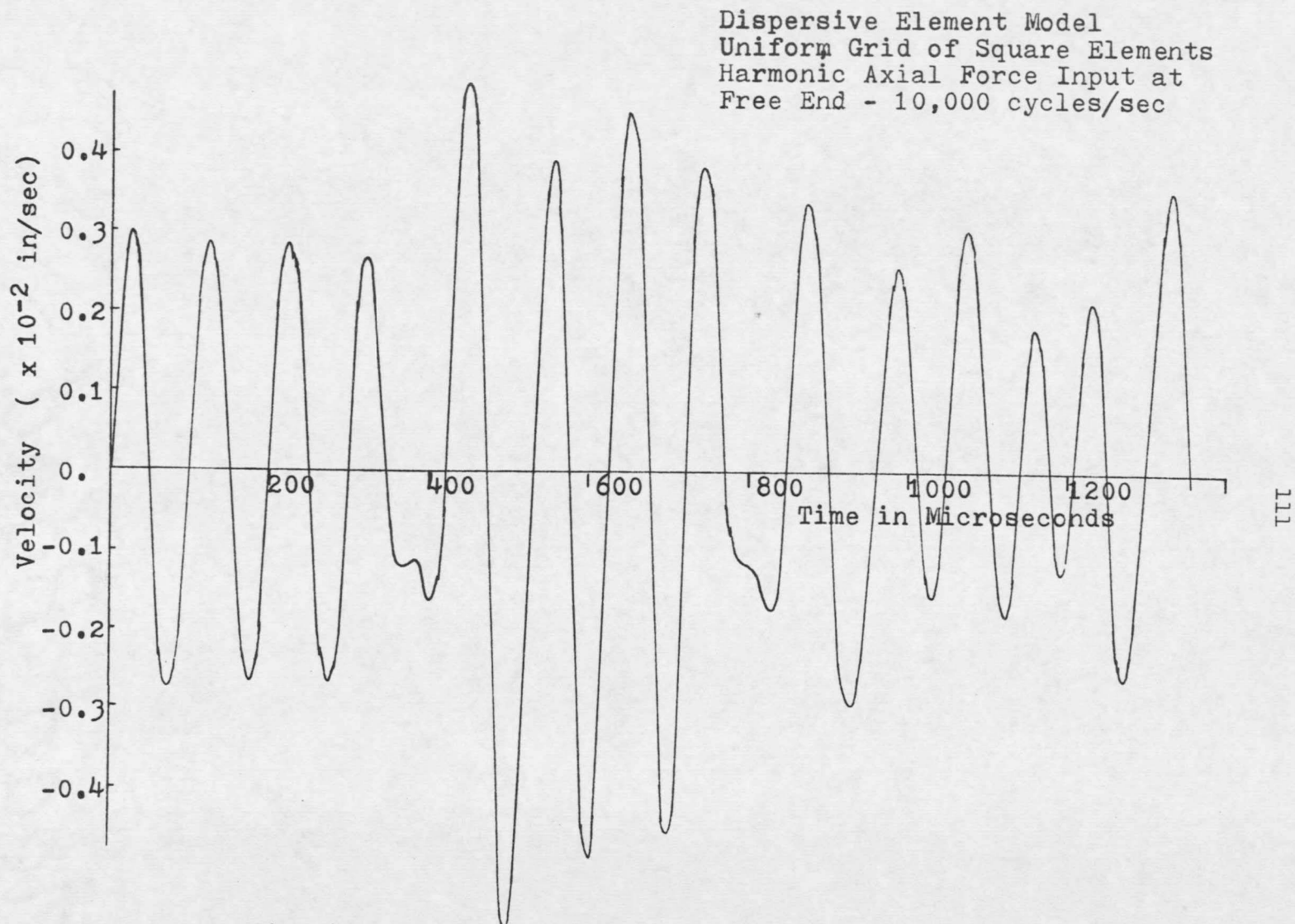


Fig. 4.5-7 Velocity Response at Free End

Dispersive Element Model
 Uniform Grid of Square Elements
 Harmonic Axial Force Input at
 Free End - 10,000 cycles/sec

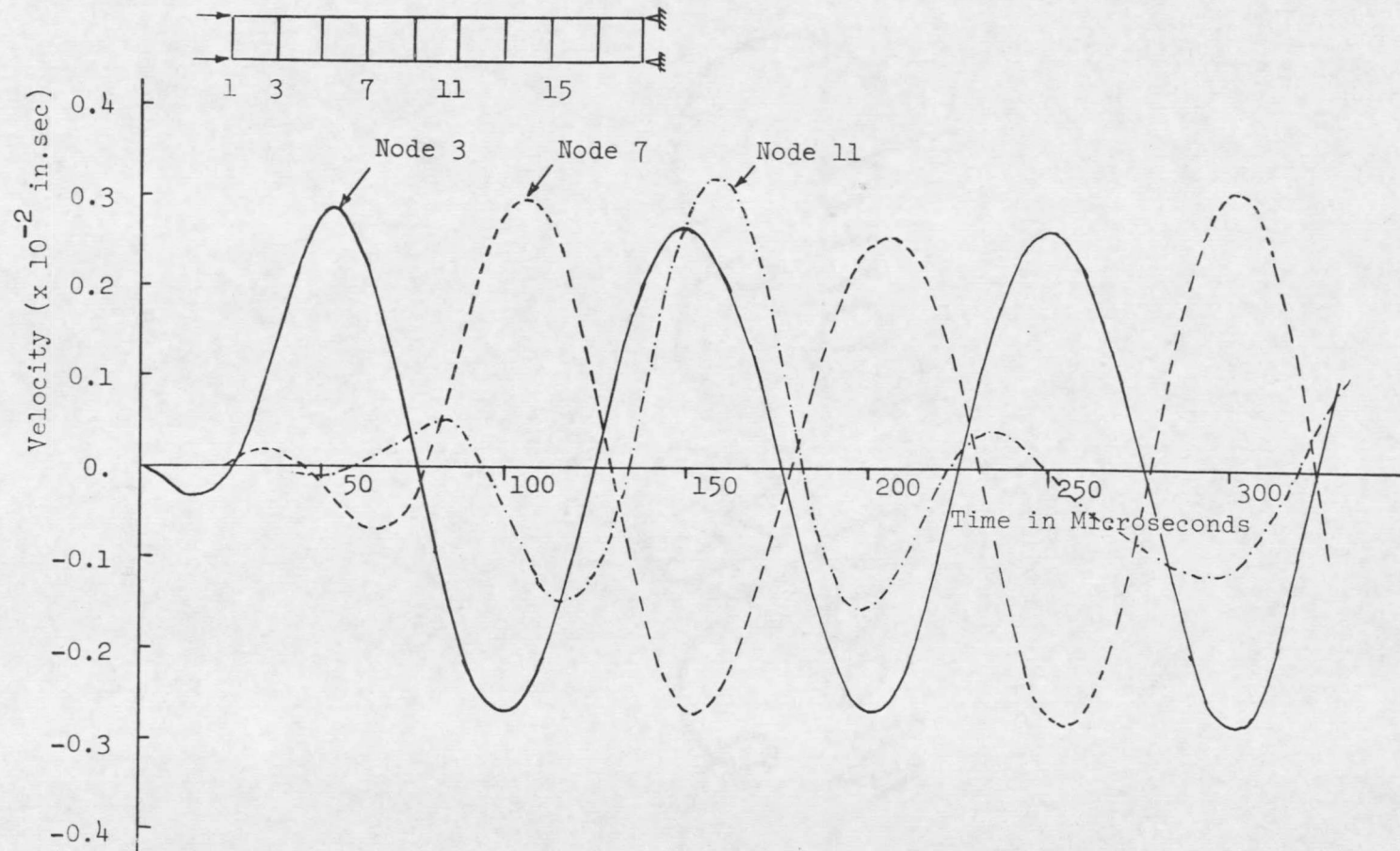


Fig. 4.5-8. Velocity Responses at Interior Nodes.

4.5.2 High Frequency Harmonic Axial Force Excitations - 15,000 and 25,000 cycles/sec.

Frequencies of 15,000 and 25,000 cps are selected because the former is in the neighborhood of the cutoff frequency (17,050 cps) while the latter is above the cutoff frequency. For the 15,000 cps harmonic excitation, the velocity responses at the free end and several interior node points are shown in Figures 4.5-9 to 4.5-12. For the nondispersive element model, the velocity of propagation is 2.08×10^5 in/sec., in agreement with the previously computed values. Velocity of propagation in the dispersive element model is 2.40×10^5 in/sec., resulting in an error of 17.5%, larger than those for the low frequency (3800 cps) and medium frequency (10,000) cps waves. Velocity responses of interior nodes (Fig. 4.5-12) show greater artificial farfield coupling than that for low and medium frequency waves (Figures 4.5-4 and 4.5-8). One difference between the free end responses of the nondispersive and dispersive element models is the apparent early arrival of the reflected wavefront in the dispersive element model. The wave front travelling with a velocity of 2.40×10^5 in/sec. arrives at the free end after 450 microseconds. But in the free end response (Fig. 4.5-11), it arrives some time between 360 and 390 microseconds. This apparent early arrival, inconsistent with the computed velocity, is attributed to the artificial far field coupling which is significantly increased for high frequency waves.

Nondispersive Element Model
Harmonic Axial Force Input at
Free End - 15,000 cycles/sec

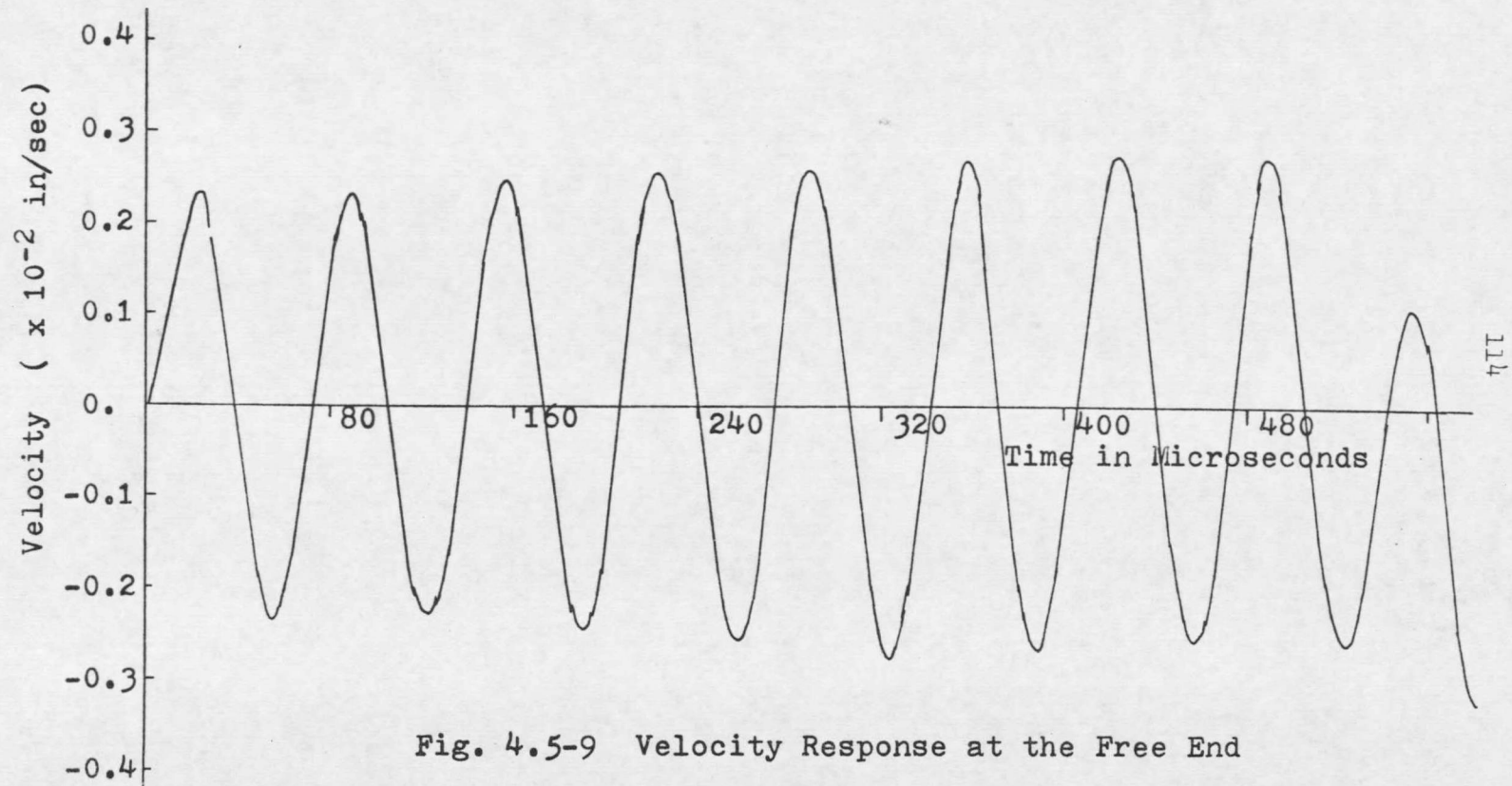


Fig. 4.5-9 Velocity Response at the Free End

Nondispersive Element Model
Uniform Grid of Square Elements
Harmonic Axial Force Input at
Free End - 15000 cycles/sec

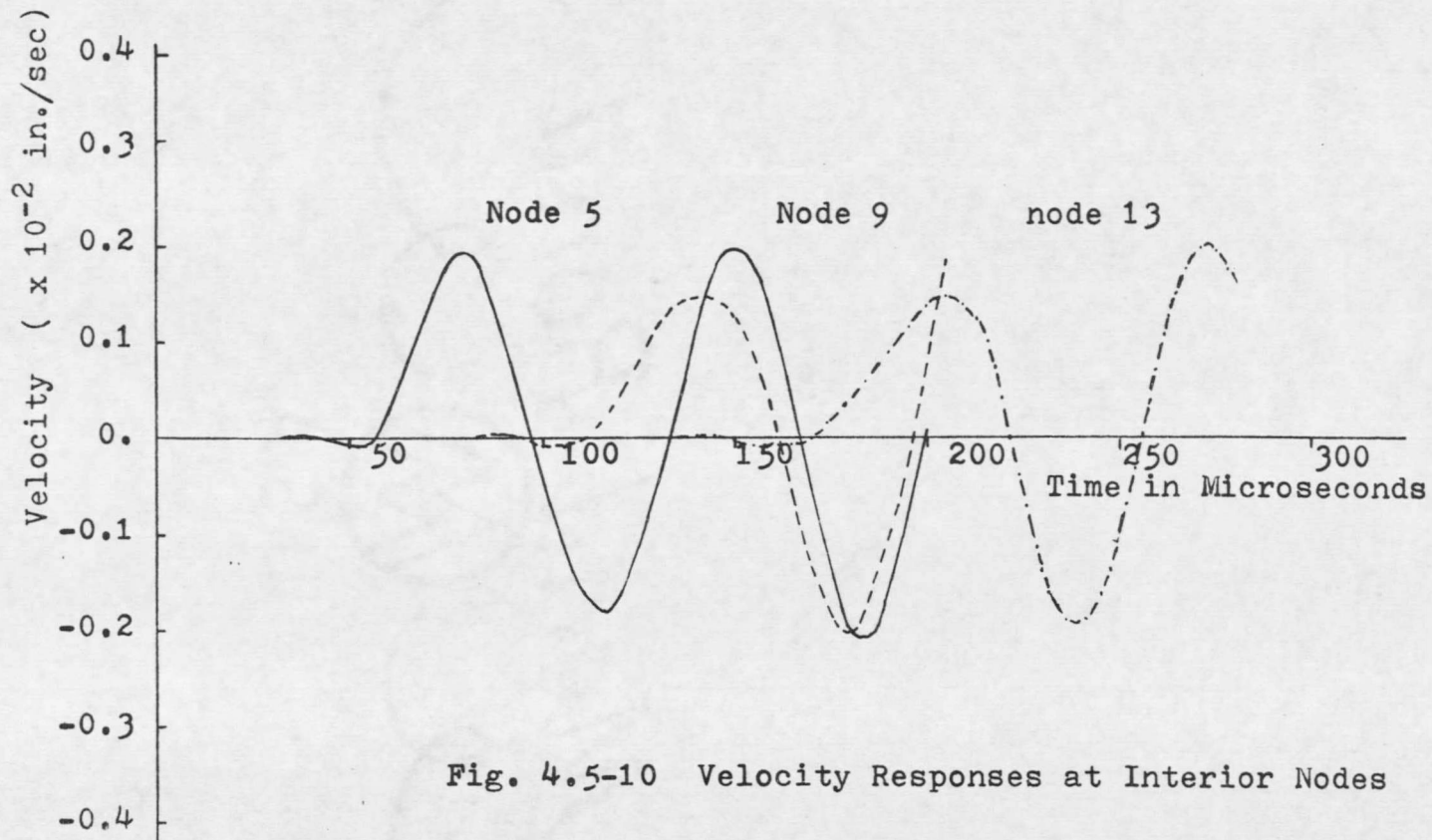


Fig. 4.5-10 Velocity Responses at Interior Nodes

Dispersive Element Formulation
15,000 CPS Harmonic Force Input
Velocity Response at Free End
Uniform Grid of Square Elements

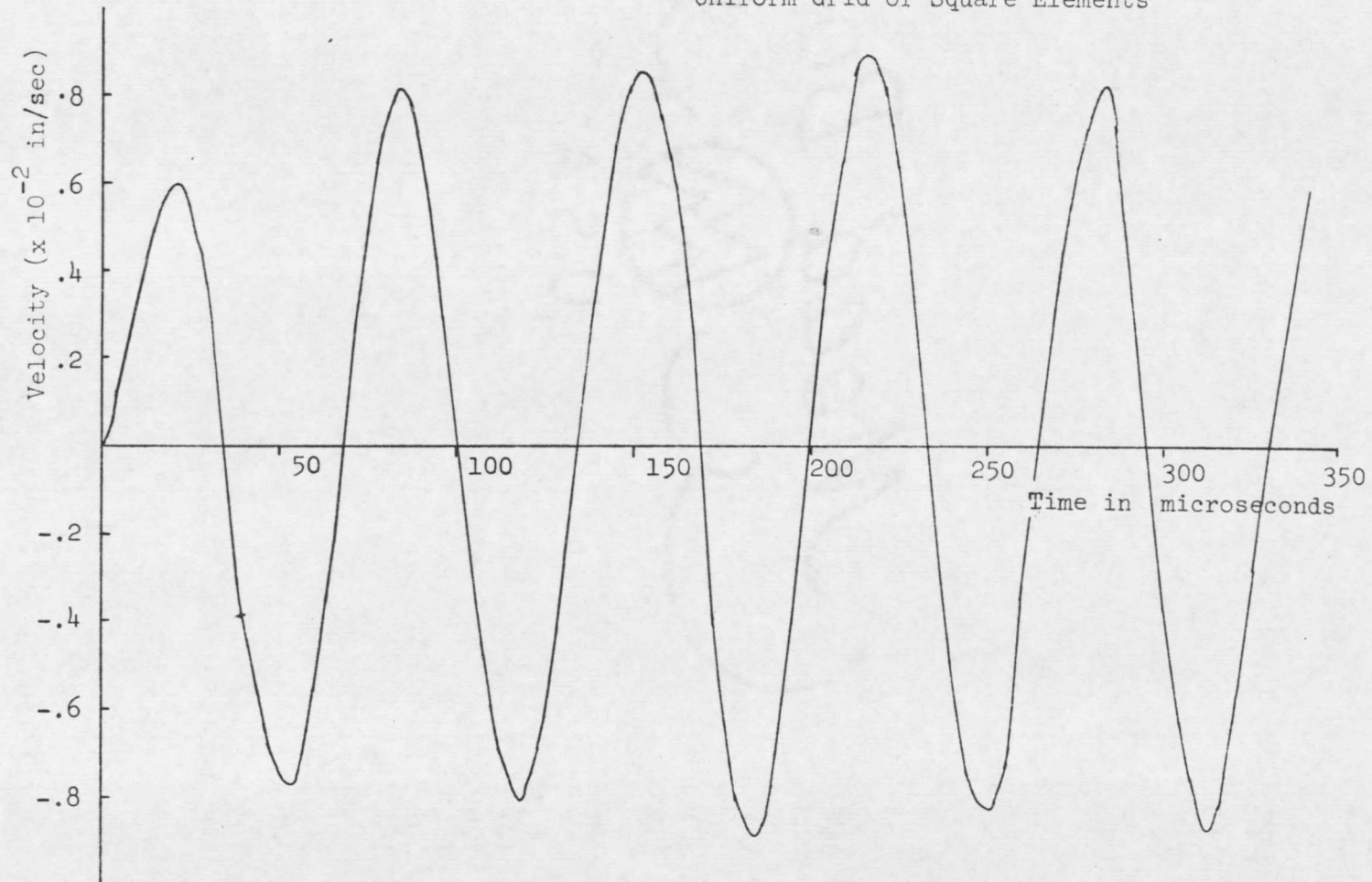


Fig. 4.5-11. Velocity Response at Free End.

Dispersive Element Model
 Harmonic Axial Force Input at
 Free End - 15,000 cycles/sec

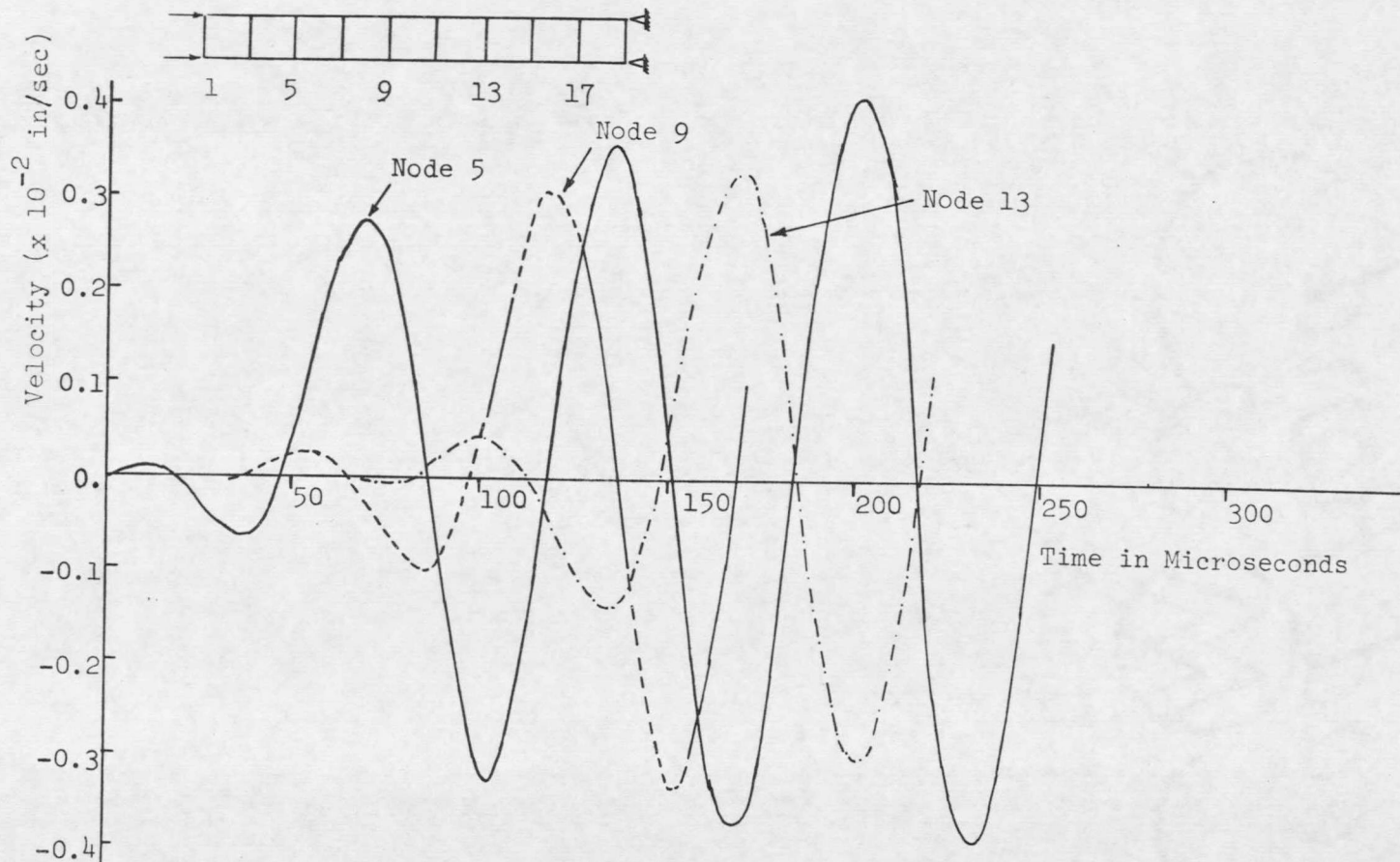


Fig. 4.5-12. Velocity Responses at Interior Nodes.

The harmonic excitation of 25,000 cps frequency was applied to determine the model response above the cut off frequency. Velocity responses at several node points in the non-dispersive and the dispersive element models are presented in Figures 4.5-13 and 4.5-14. The significant feature is the attenuation of the wavefront at the interior nodes. Both the nondispersive and dispersive element models have the characteristic of filtering high frequency waves.

4.5-4 Low Frequency Harmonic Waves in Models Made up of Nonsquare Elements

In the cases described above, the physical body was approximated by an assembly of square elements. Here we evaluate the properties of a model made up of nonsquare elements in transmitting low frequency harmonic waves. Longitudinal bars, 5 $\frac{1}{4}$ in. x 4in. x 1in. and 5 $\frac{1}{4}$ in. x 3in. x 1in. were represented by nondispersive and dispersive element models respectively as shown in Fig. 4.5-15 and 4.5-16. Harmonic axial force excitation with a frequency of 3800 cps was applied at the free end and the velocity responses were computed. Velocity responses at the free ends are shown in Figures 4.5-15 and 4.5-16. The responses are identical to their counterparts in models made up of square elements (Fig. 4.5-1 and 4.5-3) except for increased amplitude in the present model due to a reduced area of cross-section. Hence the models made up of nonsquare elements retain the properties of those of square elements.

Nondispersive Element Model
 Harmonic Axial Force Input at
 Free End - 25,000 cycles/sec
 (Above the Cut off Frequency)

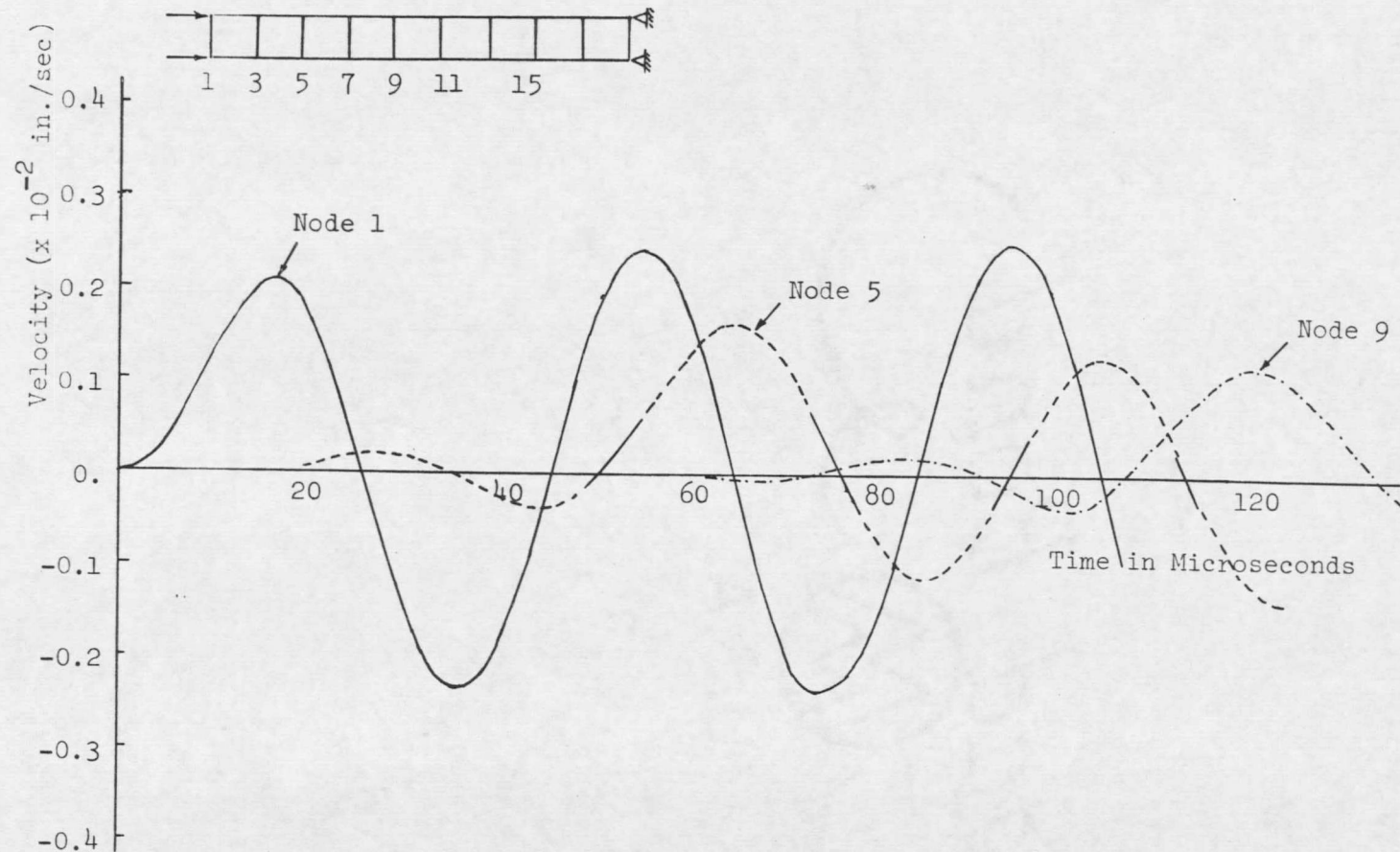


Fig. 4.5-13. Velocity Responses at Several Nodes.

Dispersive Element Model
 Harmonic Axial Force Input at
 Free End - 25,000 cycles/sec
 (Above the cut off frequency)

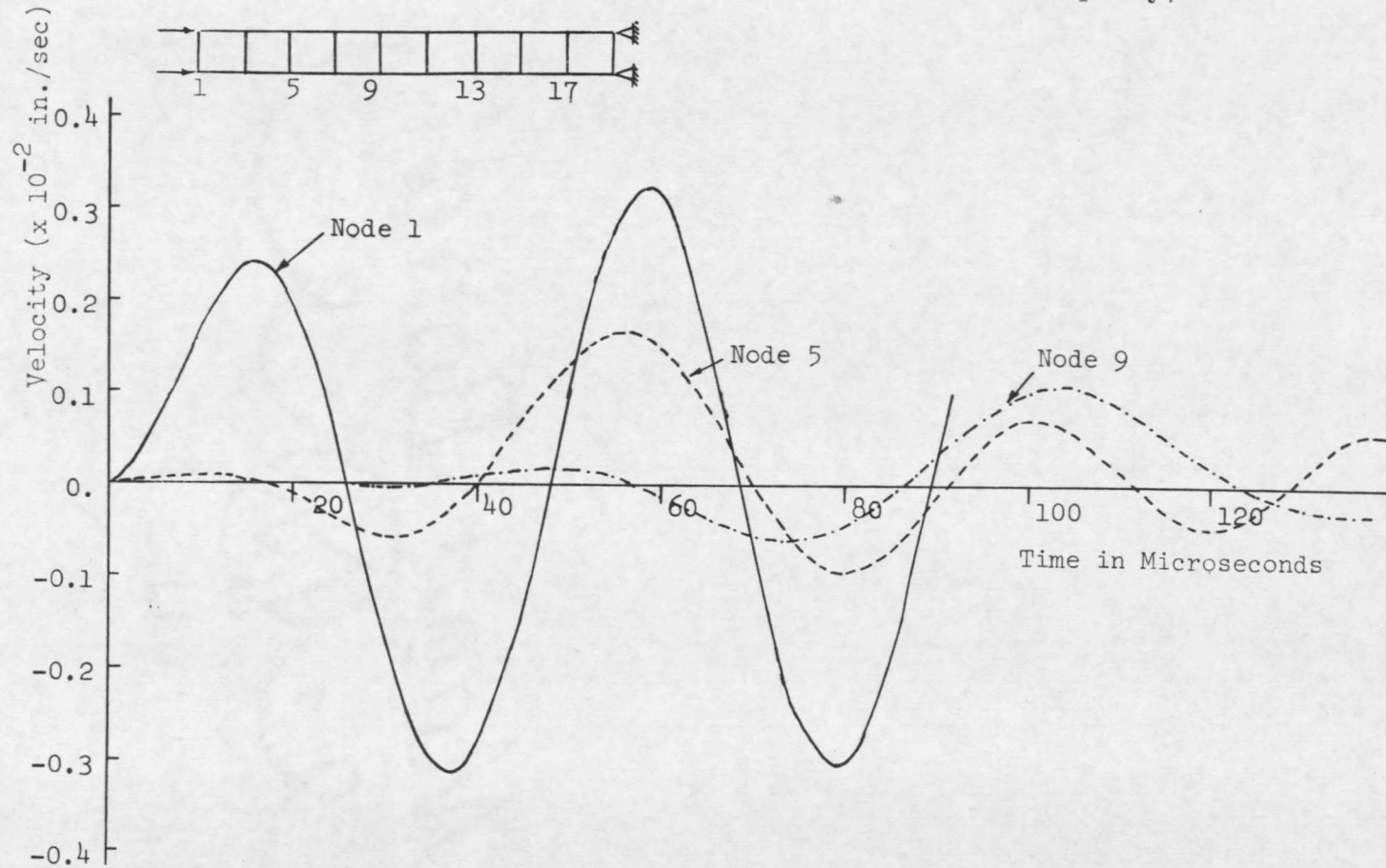


Fig. 4.5-14. Velocity Responses at Several Nodes.

Nondispersive Element Model
Nonsquare Elements (6 in x 4 in)
Harmonic Axial Force at Free End
Frequency: 3800 cycles/sec

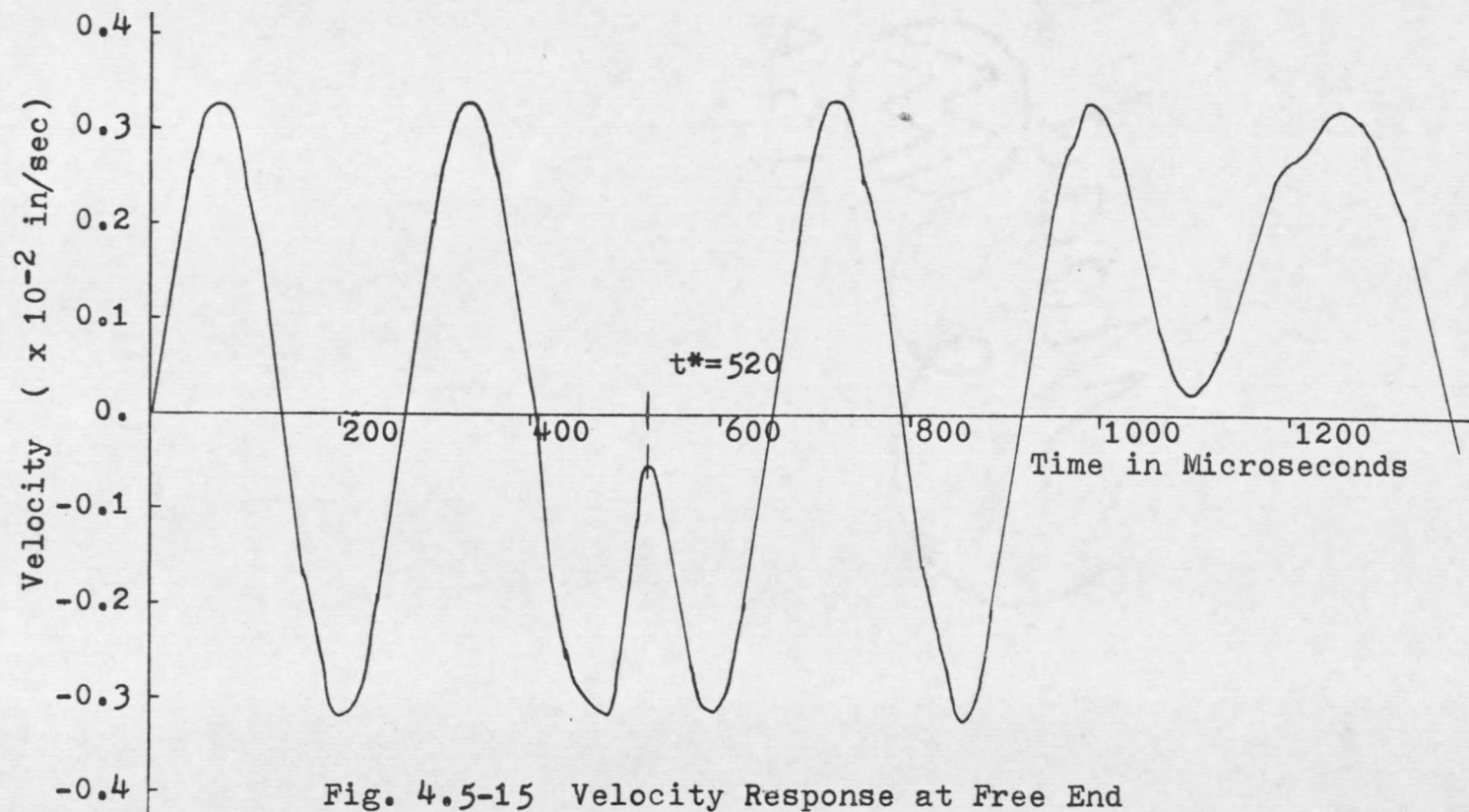


Fig. 4.5-15 Velocity Response at Free End

Dispersive Element Model
Nonsquare Elements (6 in x 3 in)
Harmonic Axial Force Input at
Free End - 3800 cycles/sec

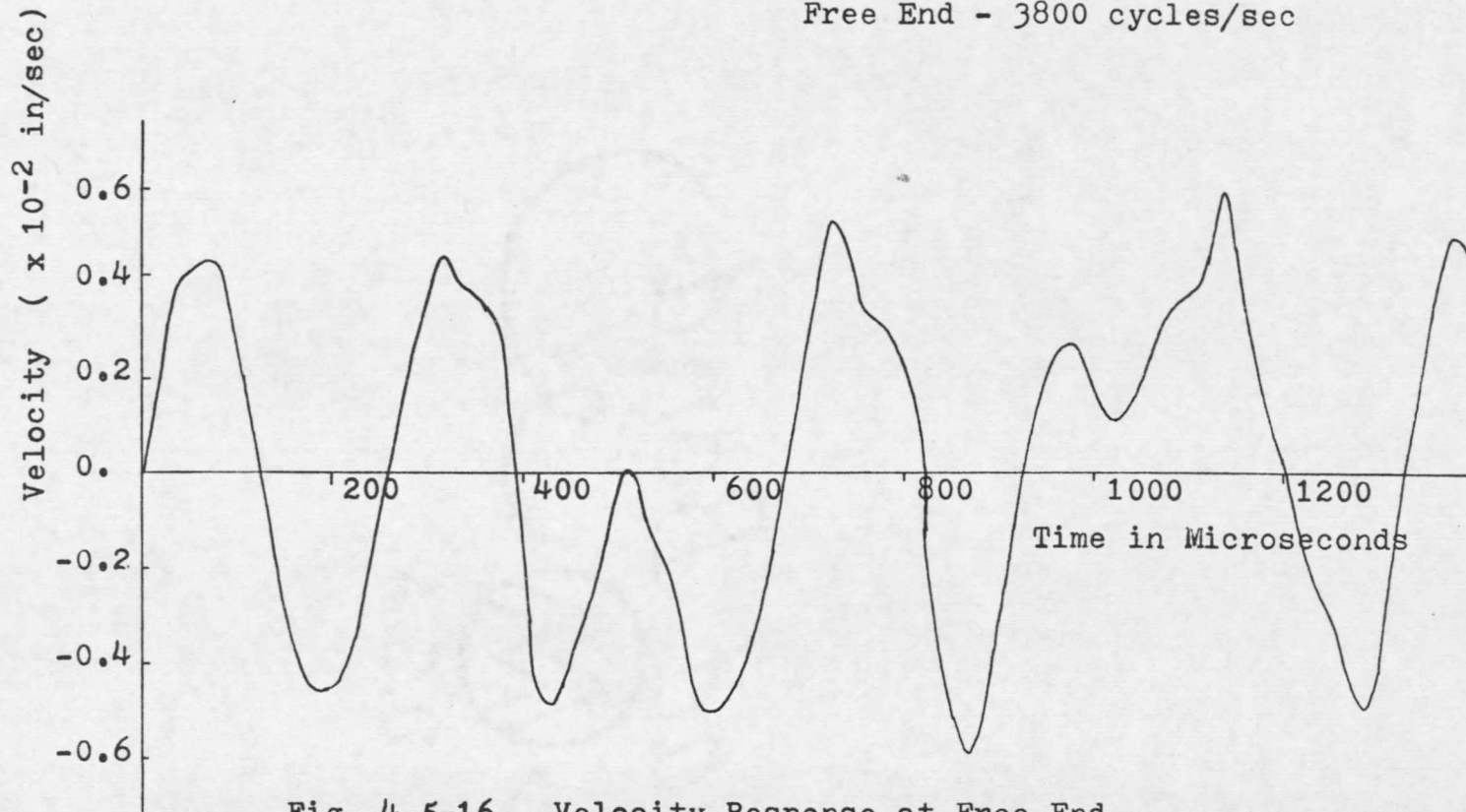
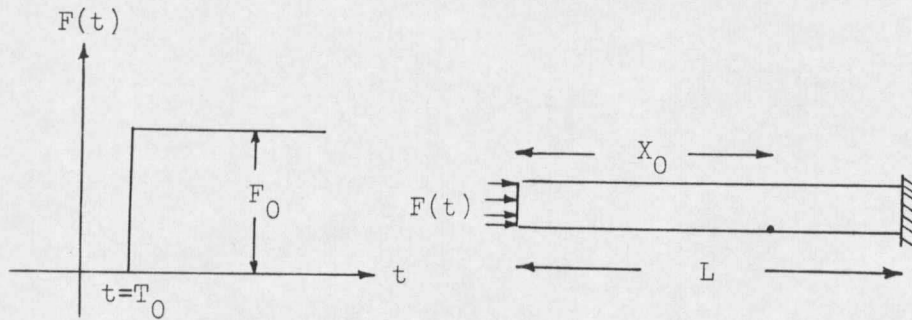


Fig. 4.5-16 Velocity Response at Free End

4.6 Nonharmonic Force Excitation - A Step Rise Force Input

An axial step force is applied at the free end of the bar shown in Fig. 4.4-1 and the propagation of the waves generated is studied with nondispersive and dispersive finite elements. A step function is sectionally continuous with a finite discontinuity. When expanded in a Fourier series, the step function may be considered as a superposition of an infinite number of harmonic functions. Most of these functions have frequencies above the cut off frequency and the amplitudes of the high frequency functions may (not necessarily) be small. These high frequency components will undergo dispersion and attenuation in both the zeroth and first order finite element models. So, the finite element solution is not expected to be close to the exact solution as in the case of harmonic excitations. The exact solution for the velocity response at the free end and at an interior point of the bar during the application of a step force is shown in Fig. 4.6-1. The velocity responses themselves are sectionally continuous functions.

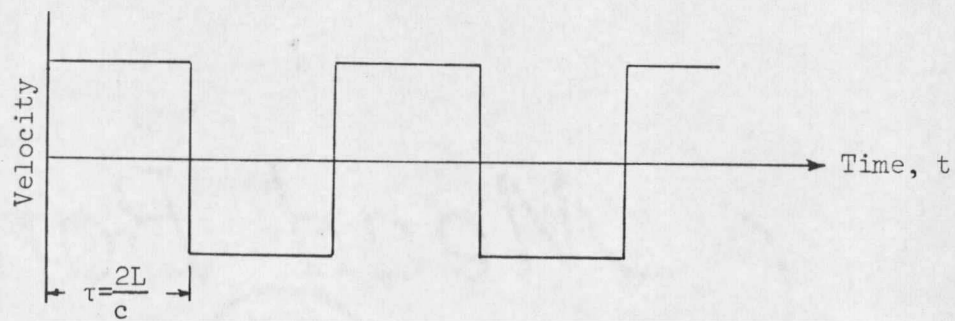
Using the finite element model with both nondispersive and dispersive elements, the velocity responses are computed and shown in Figures 4.6-2 to 4.6-5. Figures 4.6-2 and 4.6-4 show the velocity at the free end while Figures 4.6-3 and 4.6-5 show the velocity responses at an interior point, at a distance of 24 in. away from the free end. For the dispersive element model (Fig. 4.6-4) the computed



Input Forcing Function

Longitudinal Bar

$$\frac{X_0}{L} = \frac{2}{3}$$



Velocity Response at Free End

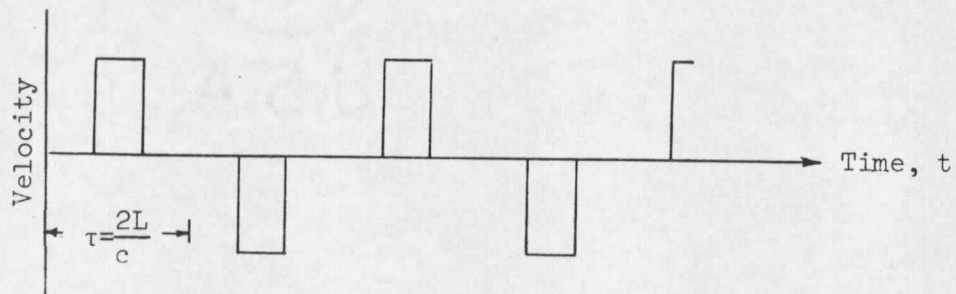
Velocity Response at X_0 :

Fig. 4.6-1.

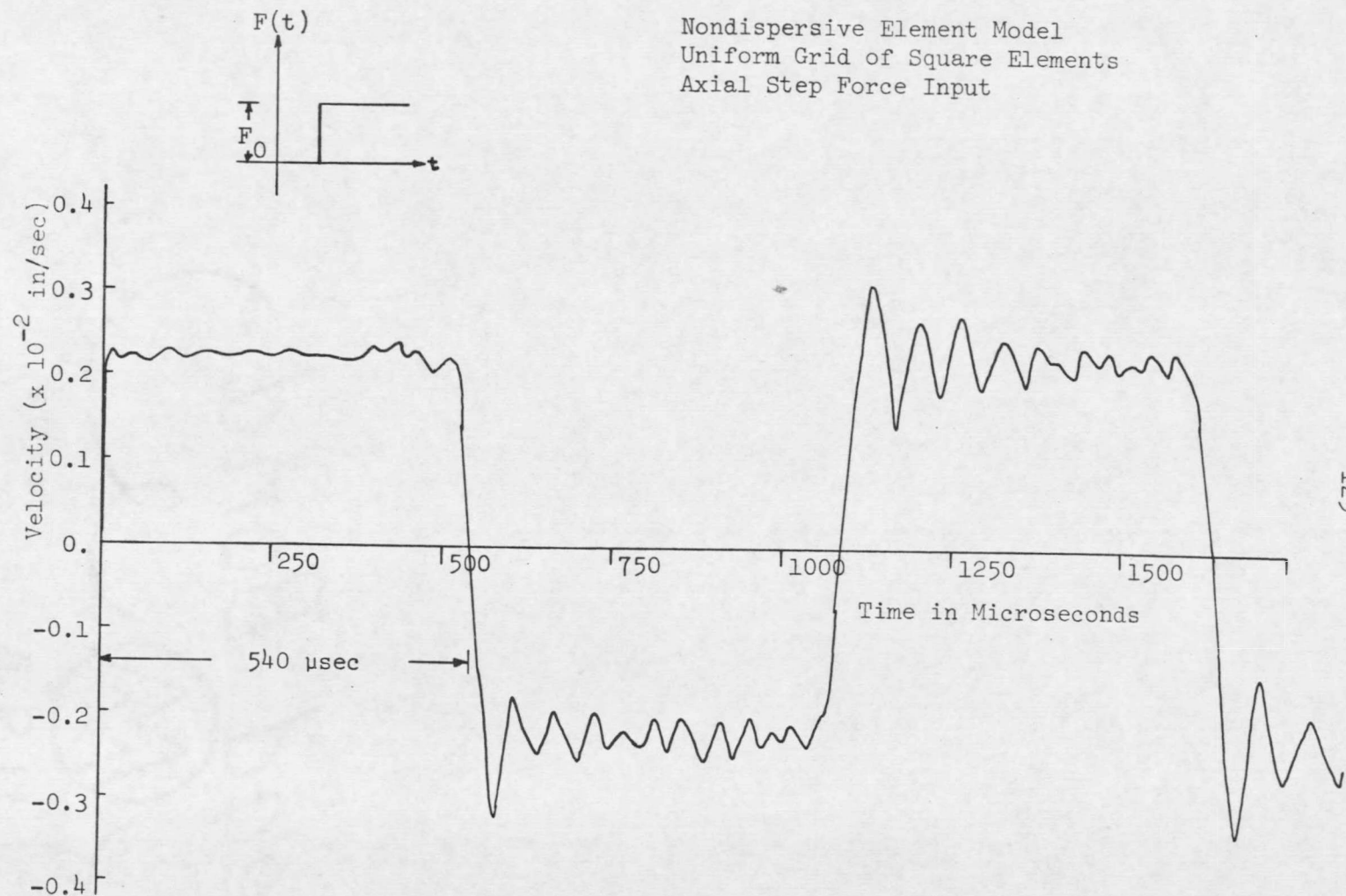


Fig. 4.6-2. Velocity Response at Free End.

Nondispersive Element Model
 Uniform Grid of Square Elements
 Axial Step Force Input

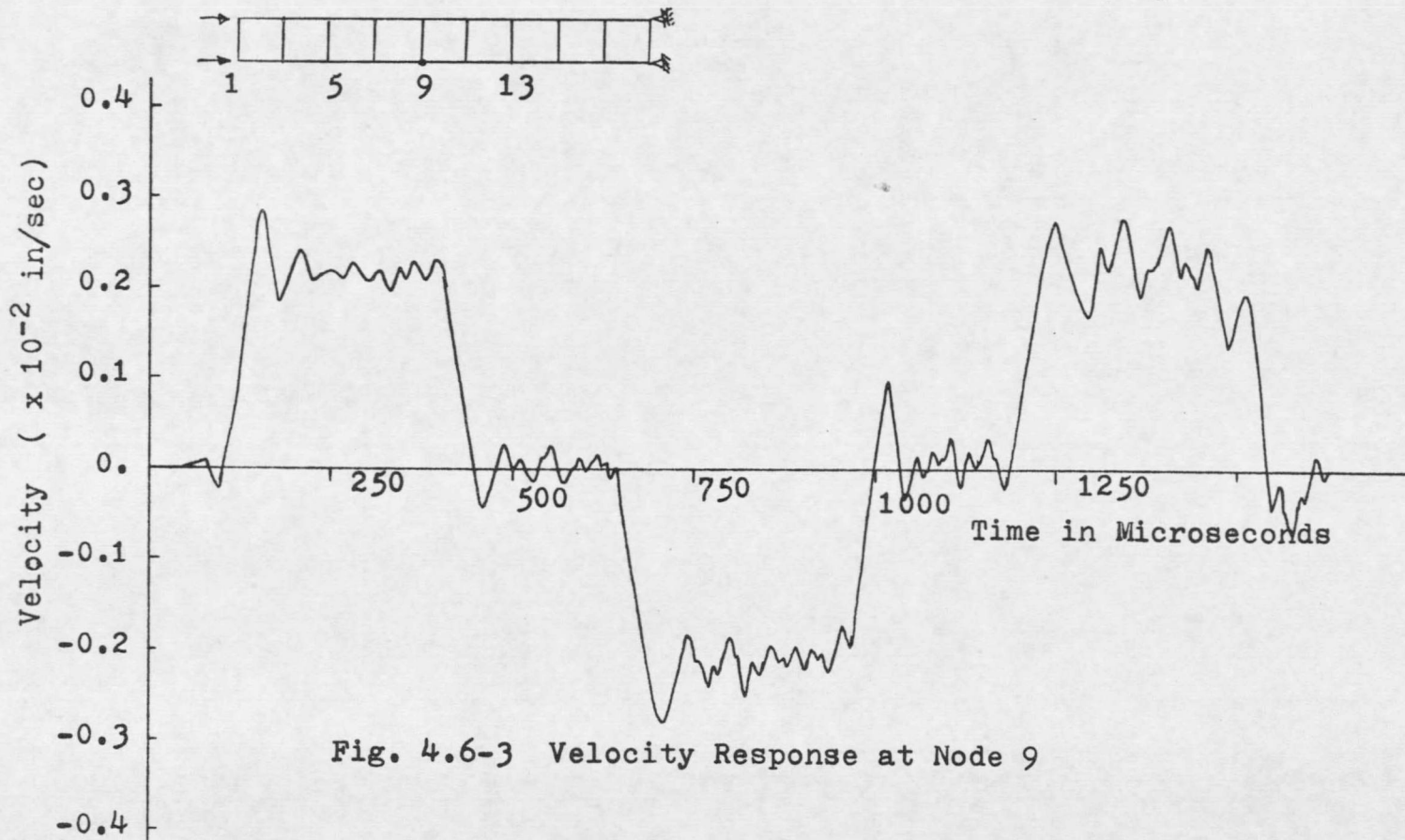


Fig. 4.6-3 Velocity Response at Node 9

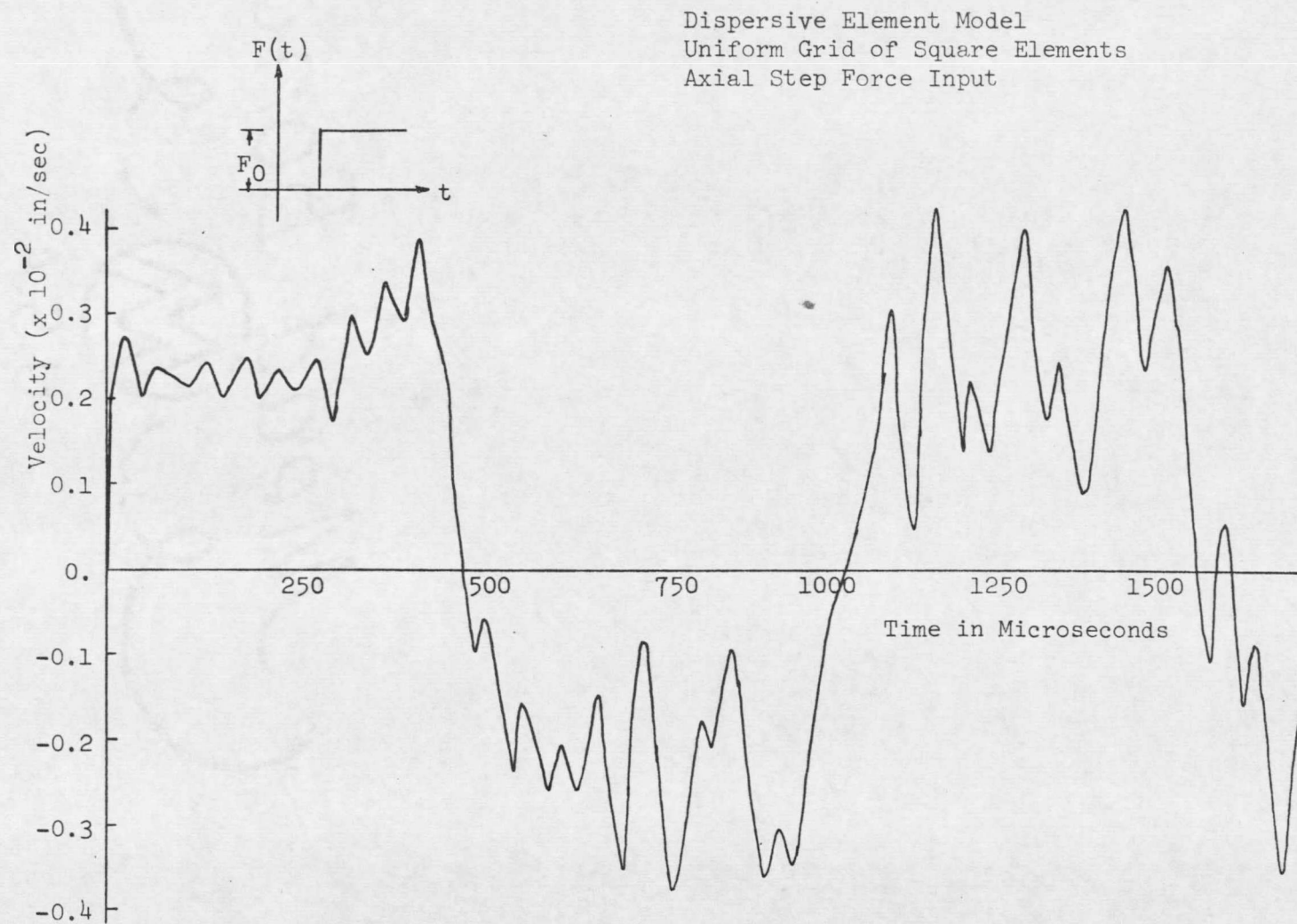


Fig. 4.6-4. Velocity Response at Free End.

Dispersive Element Model
Uniform Grid of Square Elements
Axial Step Force Input

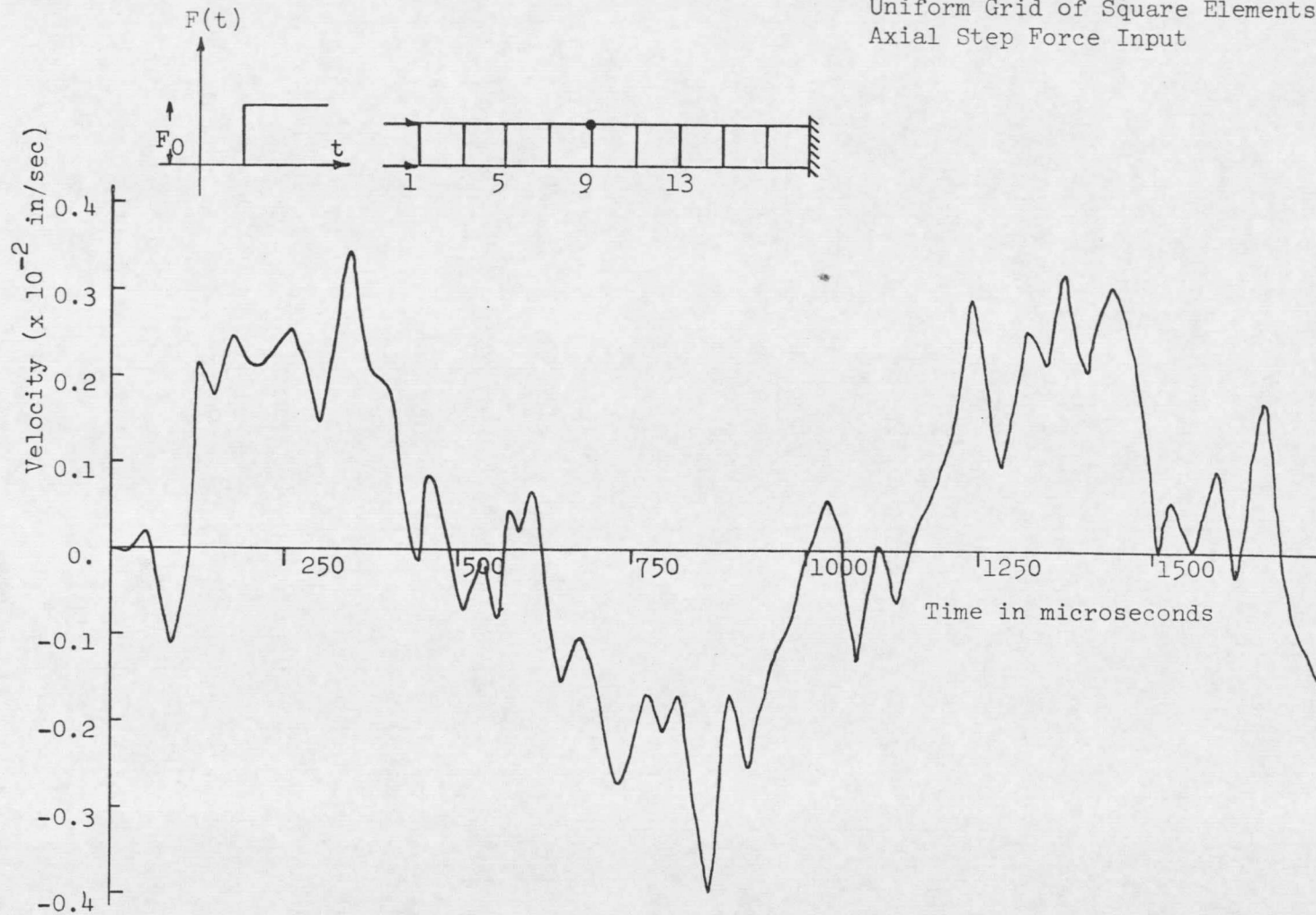


Fig. 4.6-5. Velocity Response at Node 9.

solutions show large oscillations deviating from the exact solution. These oscillations grow with increasing time so that the response is badly distorted after the first half period. The response solutions obtained with the nondispersive element model (Fig. 4.6-2 and 4.6-3) are closer to the exact solutions than the dispersive element solutions (Fig. 4.6-4 and 4.6-5).

The sources of error in these solutions are: (i) influence of the discretization-dispersion on the low-frequency components, (ii) effect of the artificial farfield coupling, (iii) errors in the numerical integration scheme for solving the differential equations and (iv) the attenuation and dispersion of the high frequency components above the cut off frequency. The solution computed using the nondispersive element model is free from the defect of discretization-dispersion and the artificial far field coupling, but the latter two errors are not eliminated. The errors in the numerical integration scheme cannot be eliminated entirely but can be reduced substantially by the use of an improved solution procedure. Such a solution procedure is the fourth order Runge-Kutta method with the modification due to Gill, explained in Sec. 4.3.3. This method was used for the computation of velocity responses for nondispersive and dispersive element models. Computed responses are plotted in Figures 4.6-6 and 4.6-7.

The solution obtained with the nondispersive element model (Fig. 4.6-6) is closer to the exact solution with less error than that

Nondispersive Element Model
Uniform Grid of Square Elements
Axial Step Force Input
Integration by 4th order Runge-
Kutta Method

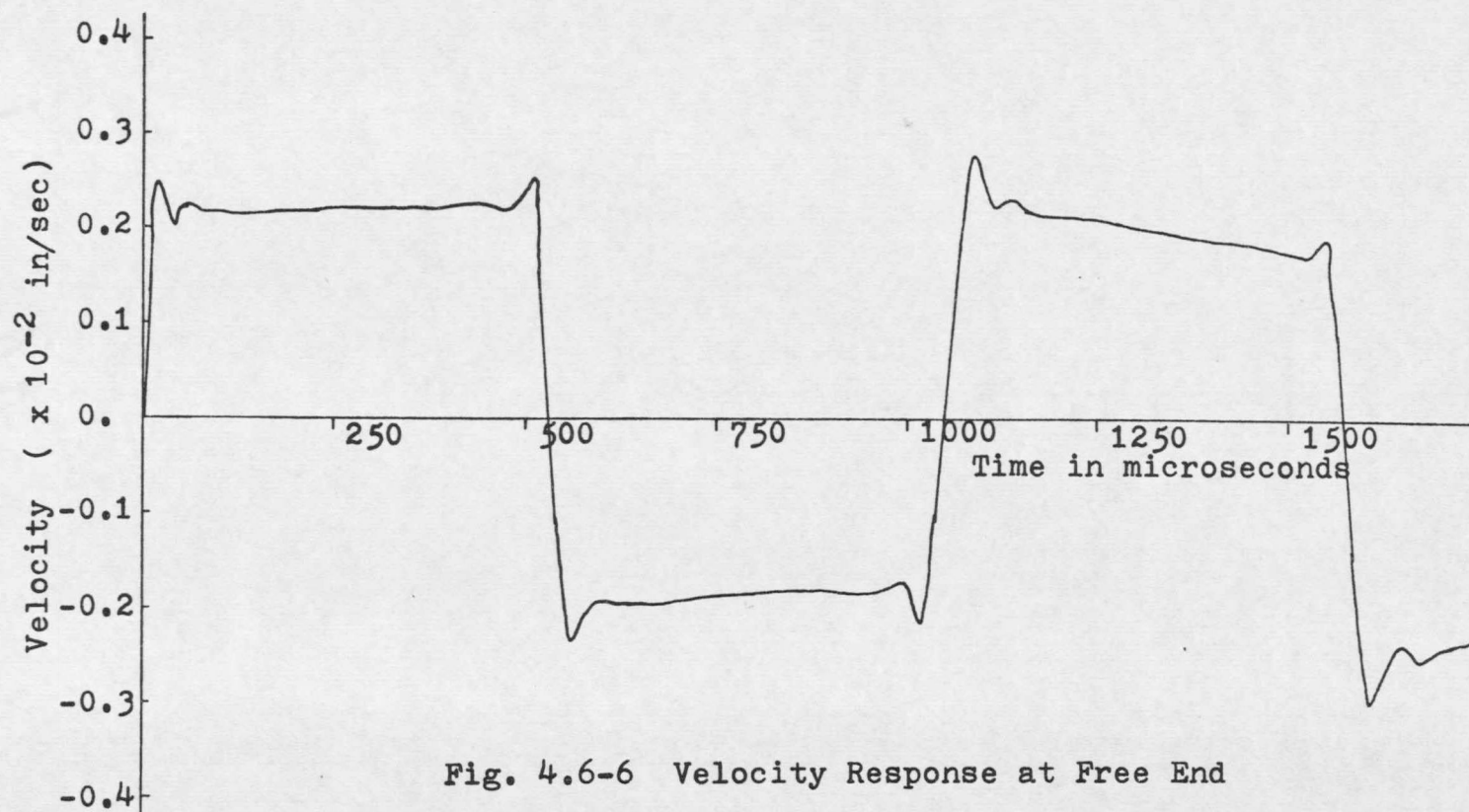


Fig. 4.6-6 Velocity Response at Free End

Dispersive Element Model
Uniform Grid of Square Elements
Axial Step Force Input
Integration by 4th order Runge-Kutta
Method

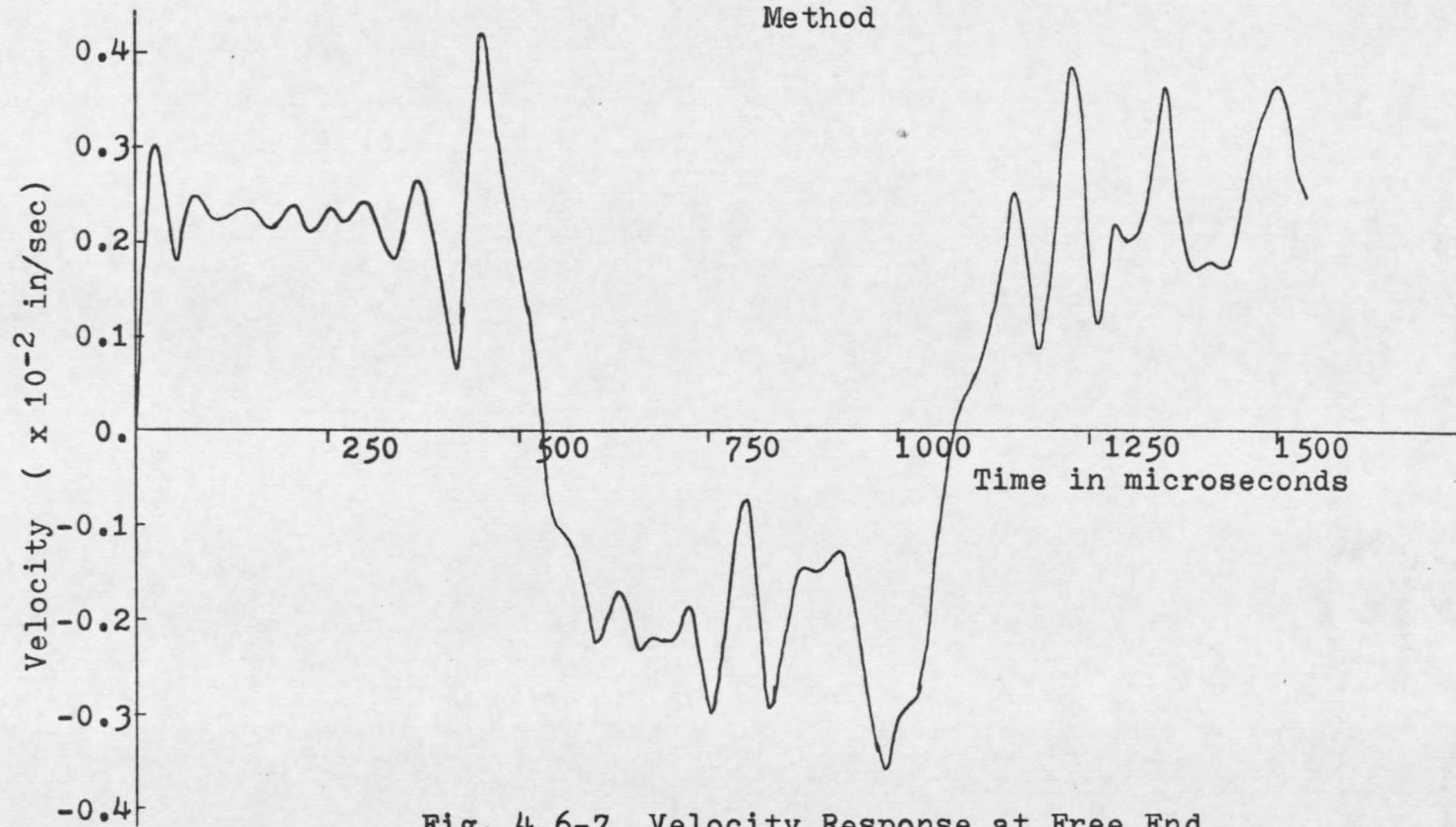


Fig. 4.6-7 Velocity Response at Free End

computed with the acceleration-pulse procedure (Fig. 4.6-2). The wave profile shows slight attenuation with increasing time. This is attributed to the attenuation of the high frequency components above the cut off frequency which gradually subtract from the waveform. This error cannot be eliminated because of the nature of the input forcing function. Solution for the dispersive element model (Fig. 4.6-7) shows large oscillations and progressive deterioration with increasing time.

In the response curves showing the velocity at the free end, one half-period is equal to the time taken by the wave front to travel through twice the length of the bar. For the nondispersive element model, the response curve obtained with the fourth order Runge-Kutta method (Fig. 4.6-6) has a half period of 530 microseconds resulting in a wave velocity of 2.04×10^5 in/sec. The response curve obtained with the acceleration-pulse procedure (Fig. 4.6-2) has a half-period of 540 microseconds, equivalent to a velocity of propagation of 2.00×10^5 in/sec. (compared to the continuum velocity of 2.05×10^5 in/sec.).

To determine the influence of an assembly of nonsquare elements on the propagation of waves, a 54in. x 3in. x 1in. bar is approximated by an assembly of nine elements of size 6in. x 3in. x 1in. in place of the previously used square elements of dimension, 6in. x 6in. x 1in. An axial step force is applied at the free end of the bar and the

propagation of the wavefront is evaluated using only the nondispersive element model. The acceleration-pulse extrapolation method was used for integrating the differential equations. The free end velocity is presented in Fig. 4.6-8. Compared to the response determined from the square element model (Fig. 4.6-2), a conspicuous difference is the attenuation of the response with increasing time. This is due to the fact that nonsquare elements have stronger filtering characteristics than the square elements in attenuating high frequency waves. The cut-off frequency for the present model is the same as that for the model made up of square elements as the element dimension has not yet changed in the direction of propagation. There is no significant change in the velocity of propagation.

4.7 Axial Force Excitations Made Up of Several Different Frequencies

As seen before, discretization-dispersion present in the displacement compatible finite element causes the wavefronts of different frequencies to propagate at different velocities while the nondispersive element propagates all wavefronts at continuum velocities. So, a wave group made up of several different frequencies, when applied to a model made up of dispersive elements, should undergo distortion while propagating due to the fact that the individual wave fronts travel at different velocities. In a nondispersive element model, because of the absence of discretization-dispersion, such a wave

Nondispersive Element Model
Uniform Grid of Nonsquare Elements
(6 in. x 3 in)
Axial Step Force Input

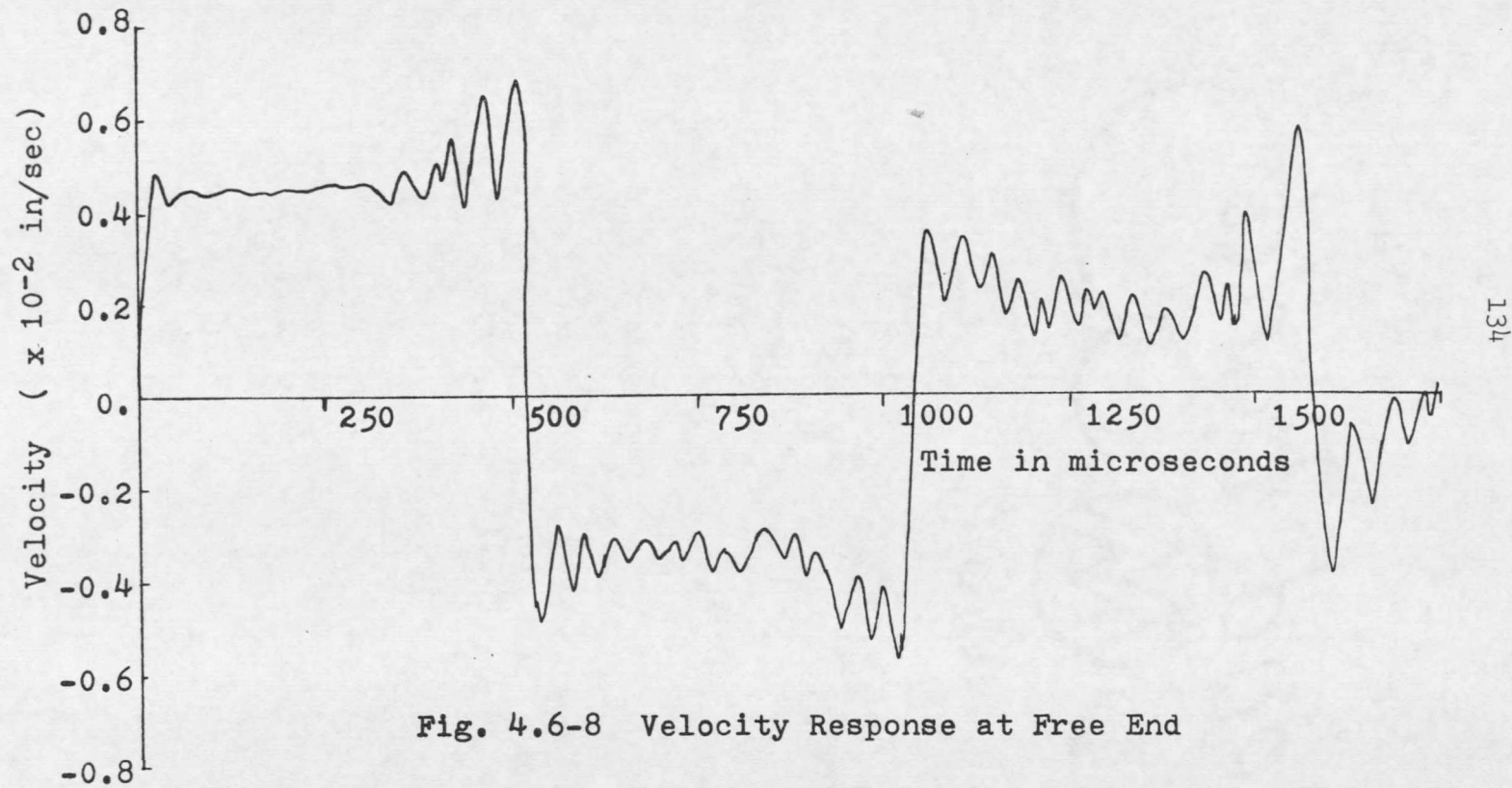


Fig. 4.6-8 Velocity Response at Free End

group should travel without any distortion. To test the validity of the above conclusions, an axial force excitation which is a sum of three harmonic functions of frequencies 2500, 5000 and 7500 cycles/sec. with equal amplitudes, was applied to the free end of the models made up of nondispersive and dispersive elements, and the velocity responses are computed.

Figure 4.7-1 shows the exciting force as a function of time. Figures 4.7-2 to 4.7-5 show the velocity responses at various nodes of the nondispersive element model, as indicated in the figures. After the arrival of the incident wavefront at a given node, velocity response of the node is identical to the response at the free end, with a time lag equivalent to the time taken by the wavefront to travel from the free end to the given node. The time of arrival of the reflected wavefront is marked in the figures as t^* . The velocity of propagation computed in this calculation matches the velocity of propagation of the nondispersive element models considered previously.

Corresponding velocity responses for the dispersive element model are shown in Figures 4.7-6 to 4.7-9. Free end velocity response (Fig. 4.7-6) matches that of the nondispersive element model until the time of arrival of the reflected wavefront. But in the dispersive element model, the reflected wavefront arrives earlier than in the nondispersive element model due to the discretization-dispersion present. Velocity responses at interior nodes are distorted

Axial Input Forcing Function - Combination
of Three Harmonic Waves - 2500 + 5000 +
7500 cycles/sec., Equal Amplitudes
Applied At Free End

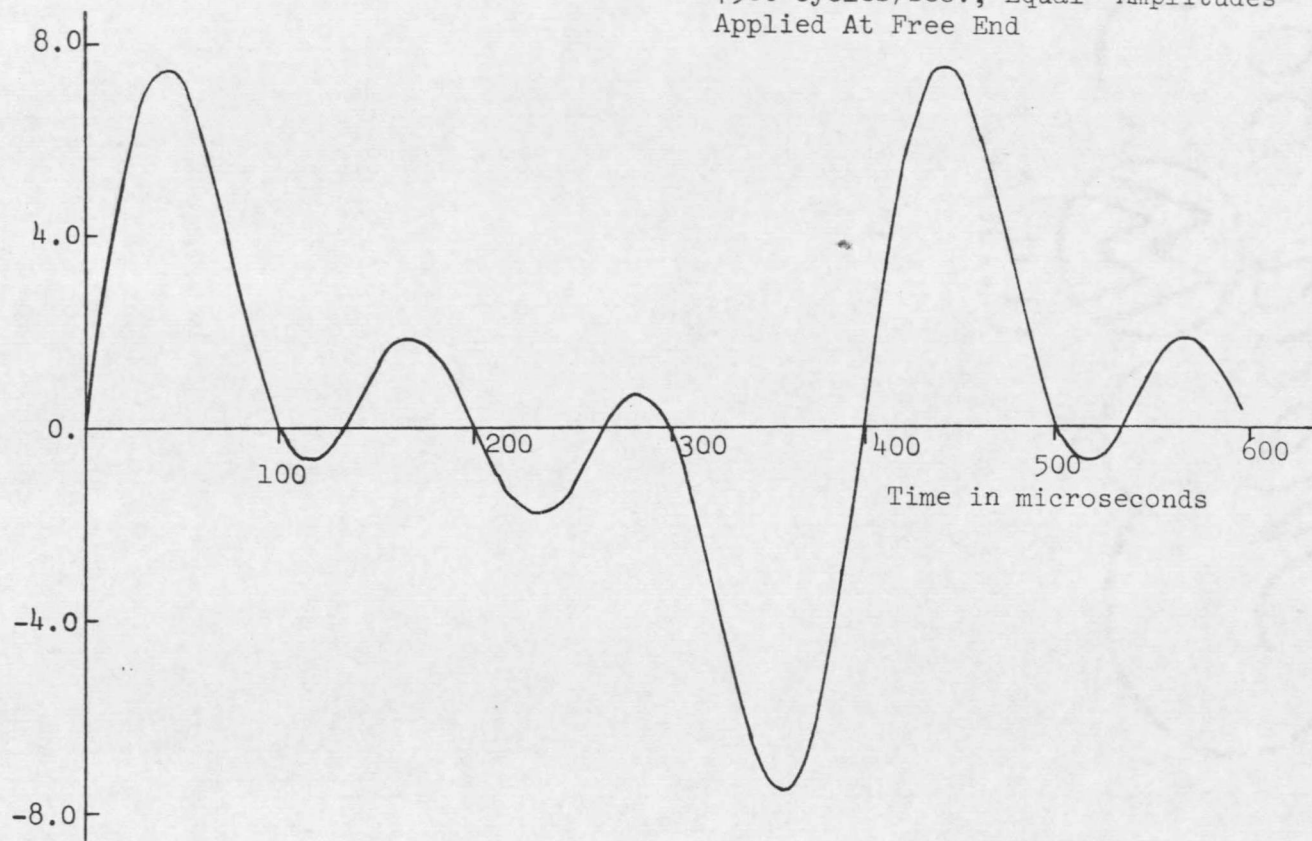


Fig. 4.7-1. Input Forcing Function.

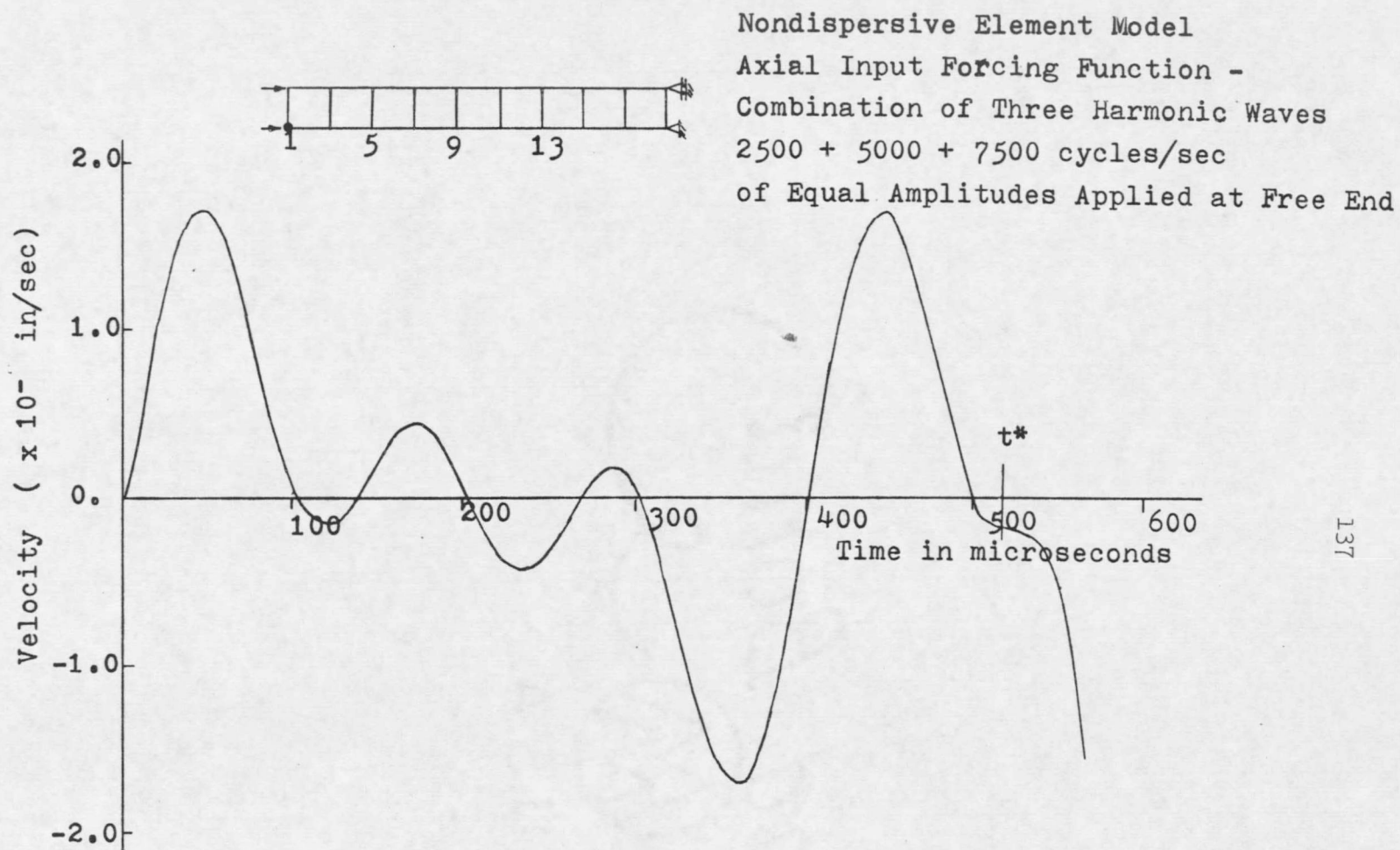
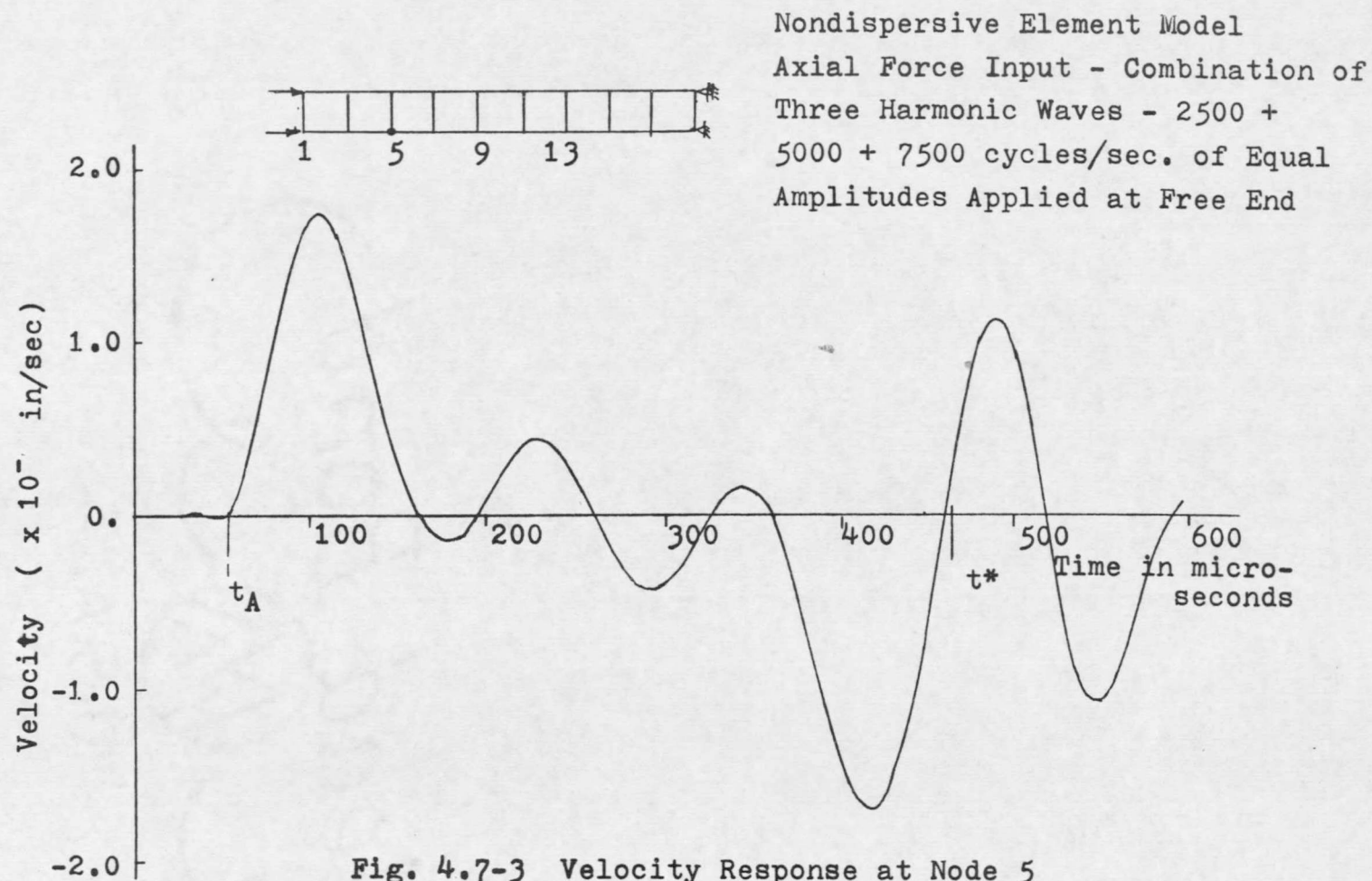


Fig. 4.7-2 Velocity Response at Node 1



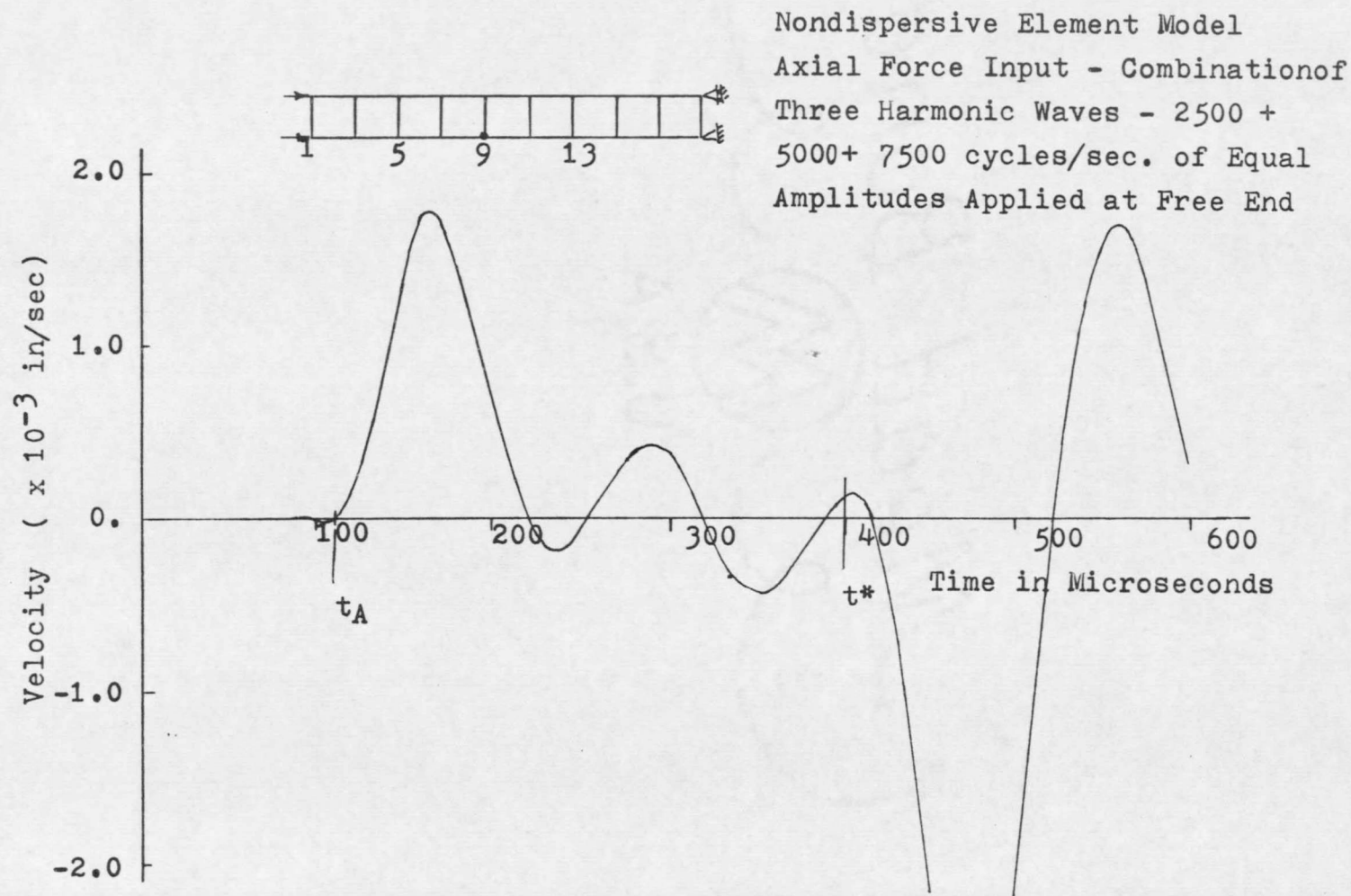


Fig. 4.7-4 Velocity Response at Node 9

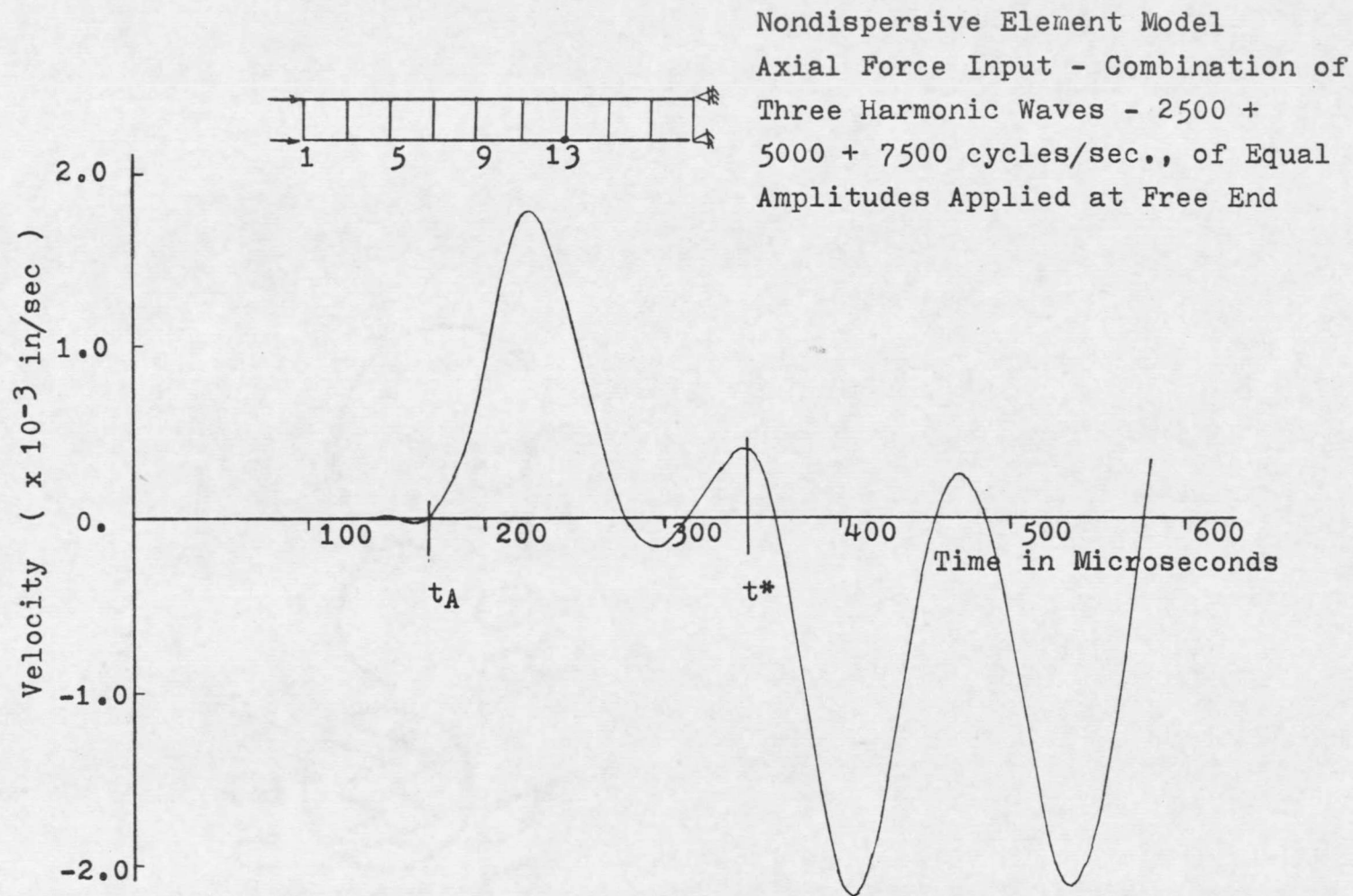


Fig. 4.7-5 Velocity Response at Node 13

Dispersive Element Model

Axial Force Input - Combination of

Three Harmonic Waves - 2500 +

5000 + 7500 cycles/sec. of Equal
Applied at Free End

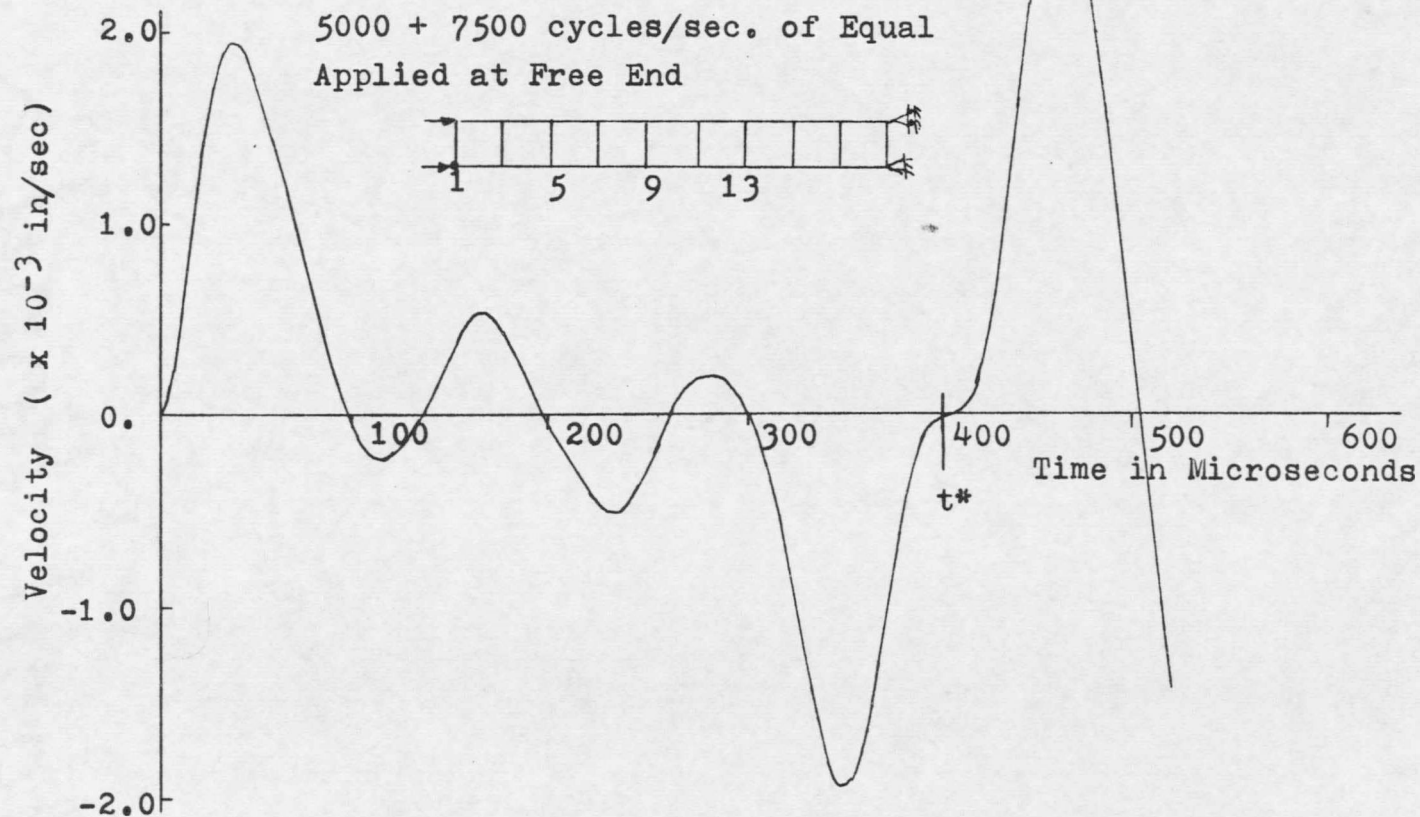


Fig. 4.7-6 Velocity Response at Node 1

Dispersive Element Model

Axial Force Input - Combination of
Three Harmonic Waves - 2500 +
5000 + 7500 cycles/sec. of Equal
Amplitudes Applied at Free End

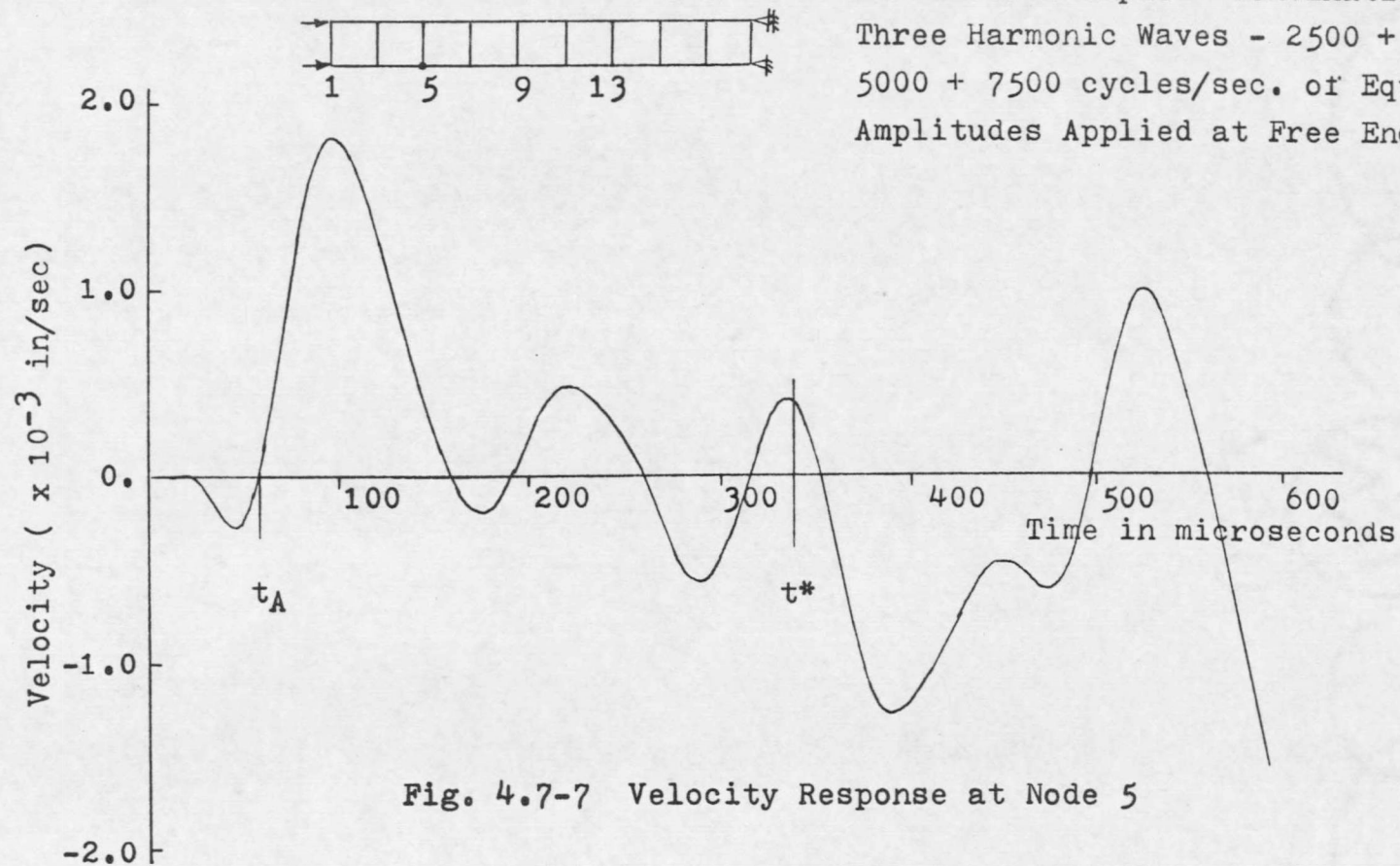


Fig. 4.7-7 Velocity Response at Node 5

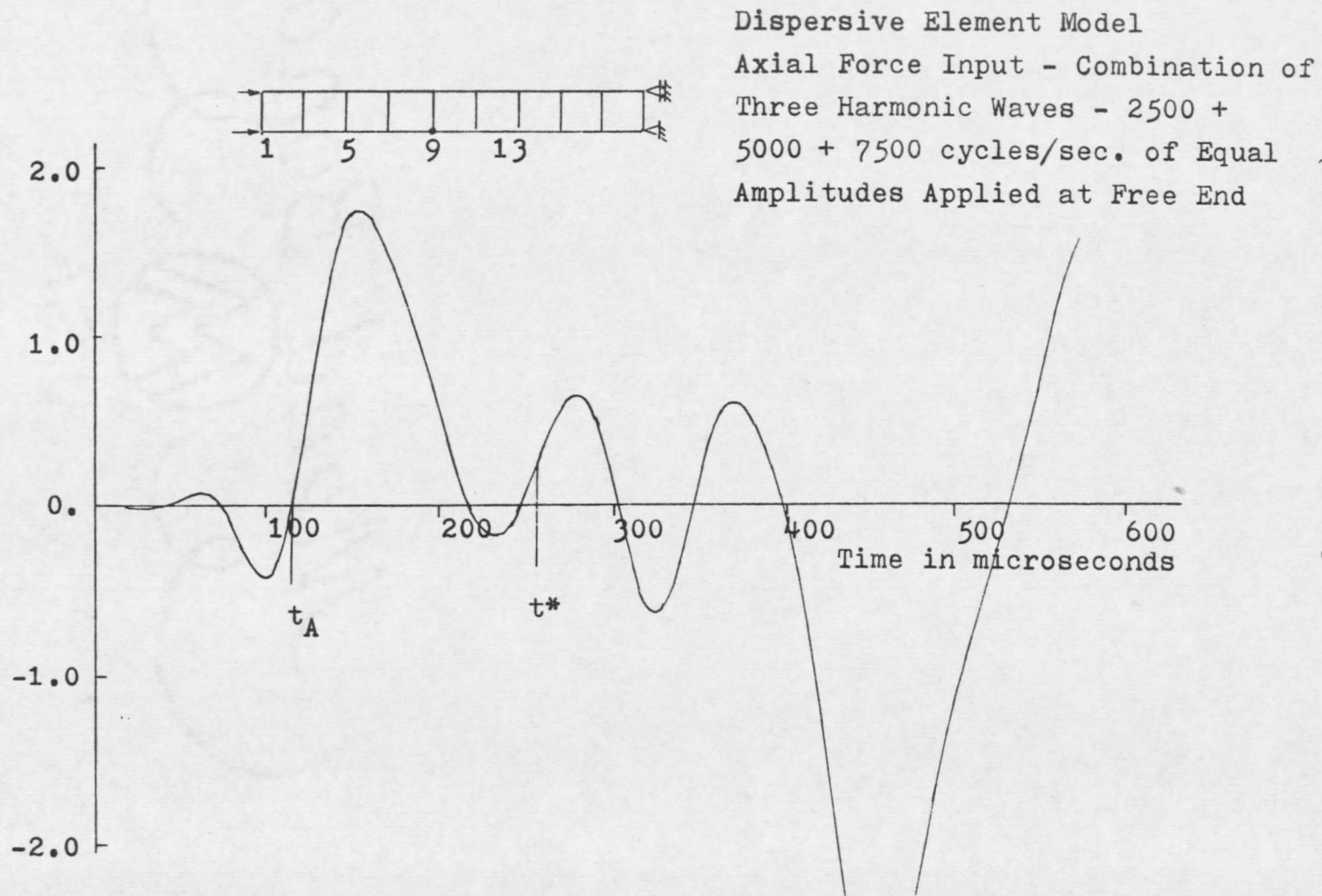


Fig. 4.7-8 Velocity Response at Node 9

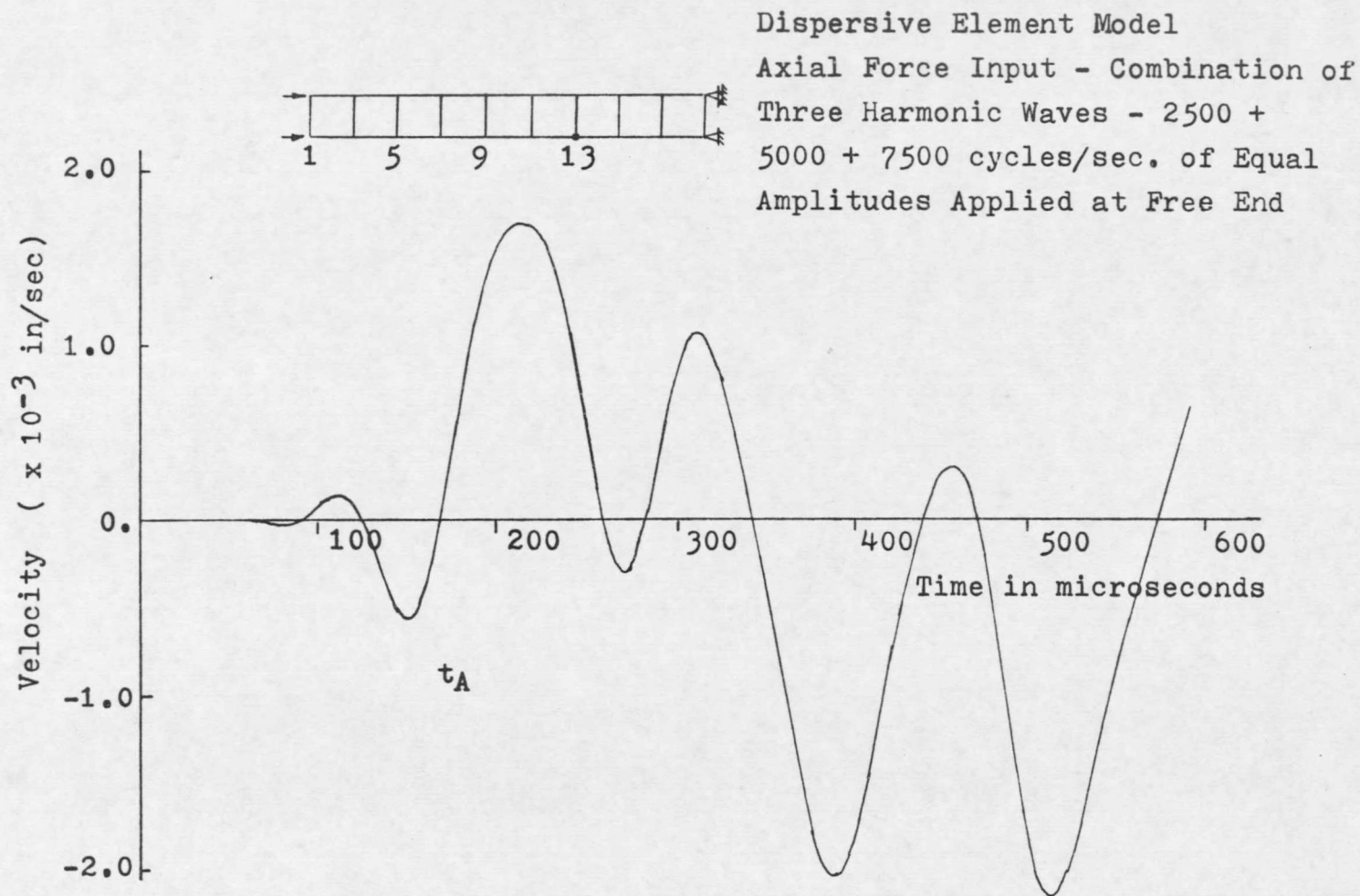


Fig. 4.7-9 Velocity Response at Node 13

considerably (Figures 4.7-7 to 4.7-9) from the input forcing function. Further, the artificial farfield coupling present in the element causes nodes to oscillate before the arrival of the wavefront. The above computations show the ability of the refined element to propagate a wave group containing waves of several different frequencies at continuum velocities.

4.8 Propagation of Shear Waves

In this section results are reported of the one-dimensional square element model tested for shear wave propagation. Harmonic lateral force inputs of 2000 and 6000 cycles per second are considered. Shear wave velocity in the continuum, represented by the finite element model, is 1.221×10^5 in/sec. (Table 4.4-1). The finite element model considered has a cut off frequency of 10,180 cycles/sec. Lateral forces are applied at the free end of the bar to generate shear waves. In applying lateral forces, flexural waves are also generated, effects of which are difficult to distinguish from that of the shear waves. Generation of pure shear waves demand some modification in the boundary conditions at the free and fixed ends. At the free end, nodes 1 and 2 (Fig. 4.8-1) are treated as if they are on a free boundary except that the axial displacements are arrested. At the fixed end, unlike the model for dilatational waves, axial and lateral displacements are arrested at both nodes 19 and 20,

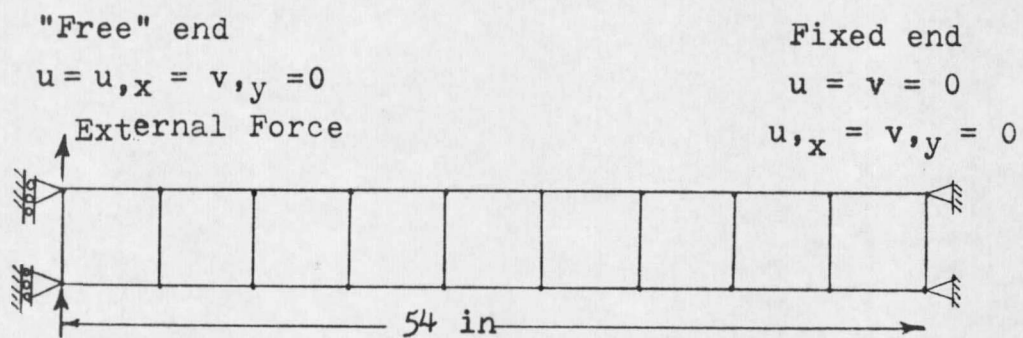


Fig. 4.8-1. The Finite Element Model
Modified for Shear Wave Propagation

and normal strain coordinates $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ are made equal to zero. Shear strain coordinates are left arbitrary.

Lateral velocity responses are computed at various nodes. The acceleration-pulse extrapolation method is used to solve the differential equations in time increments of 5 microseconds for both nondispersive and dispersive element models.

Figures 4.8-2 and 4.8-3 show the lateral velocity responses at the free end under a harmonic lateral force excitation with a frequency of 2000 cps for the nondispersive and the dispersive elements models respectively. Fig. 4.8-1 shows that for the nondispersive element model the initial incident wave front returns to the free end after 890 microseconds. This is equivalent to a velocity of propagation of 1.212×10^5 in/sec. compared with a continuum velocity of 1.221×10^5 in/sec., an error of less than one percent. Fig. 4.8-3 shows that with the dispersive element model, the initial wavefront returns to the free end after 635 microseconds. This is equivalent to a velocity of propagation of 1.86×10^5 in/sec. which is in error by 52.5% over the continuum velocity.

Velocity response at an interior node, node 3, at a distance 6 in. from the free end are shown in Fig. 4.8-4 and 4.8-5 for the nondispersive and dispersive element models respectively. With the nondispersive element, the velocity response at node 3 lags that at the free end by 50 microseconds. This implies that the wave front

Shear Wave Propagation
Nondispersive Element Model
Lateral Harmonic Force Applied
at Free End - 2000 cycles/sec

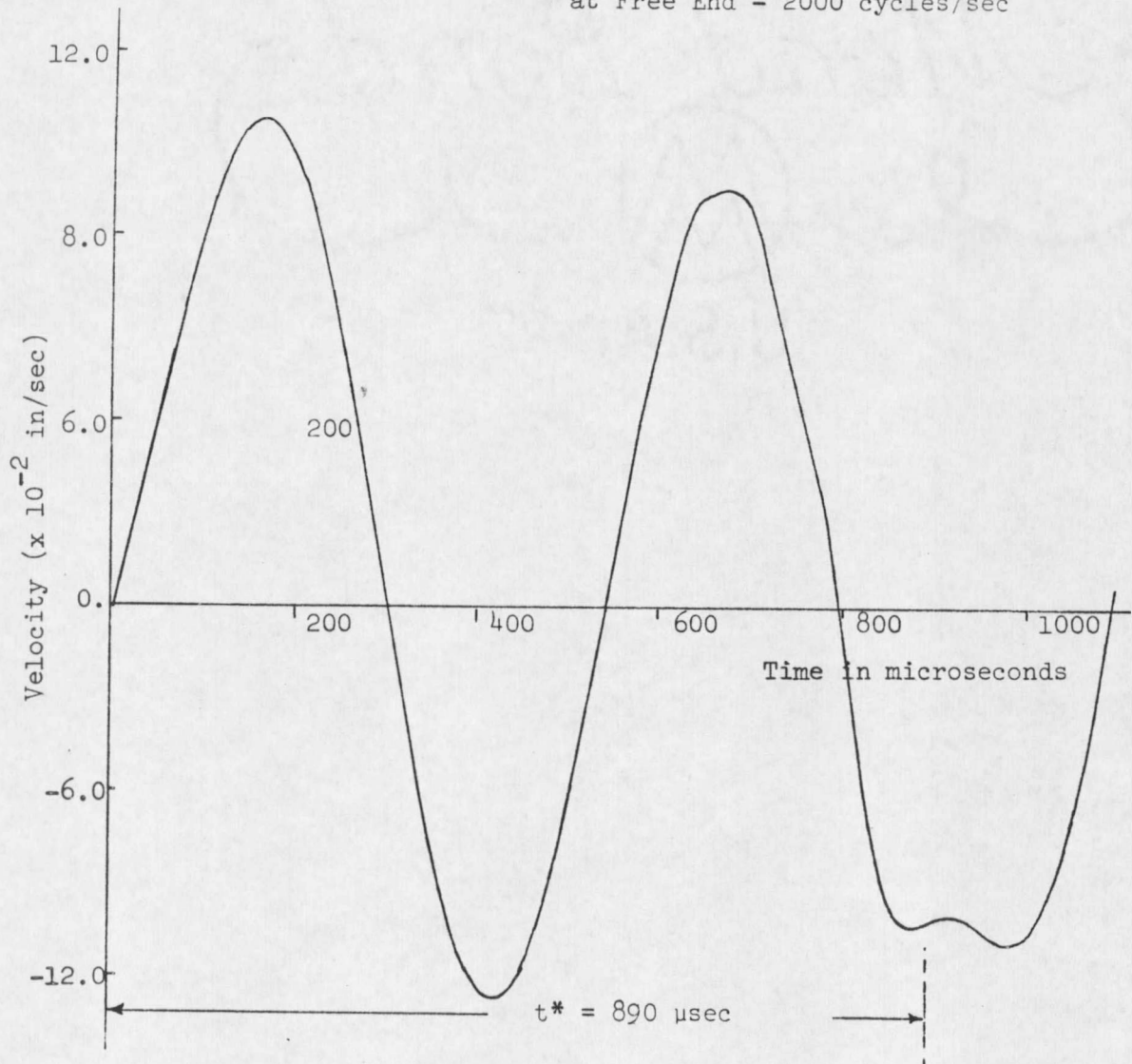


Fig. 4.8-2. Velocity Response at Free End.

Shear Wave Propagation
Dispersive Element Model
Lateral Harmonic Force Applied
At Free End - 2000 cycles/sec

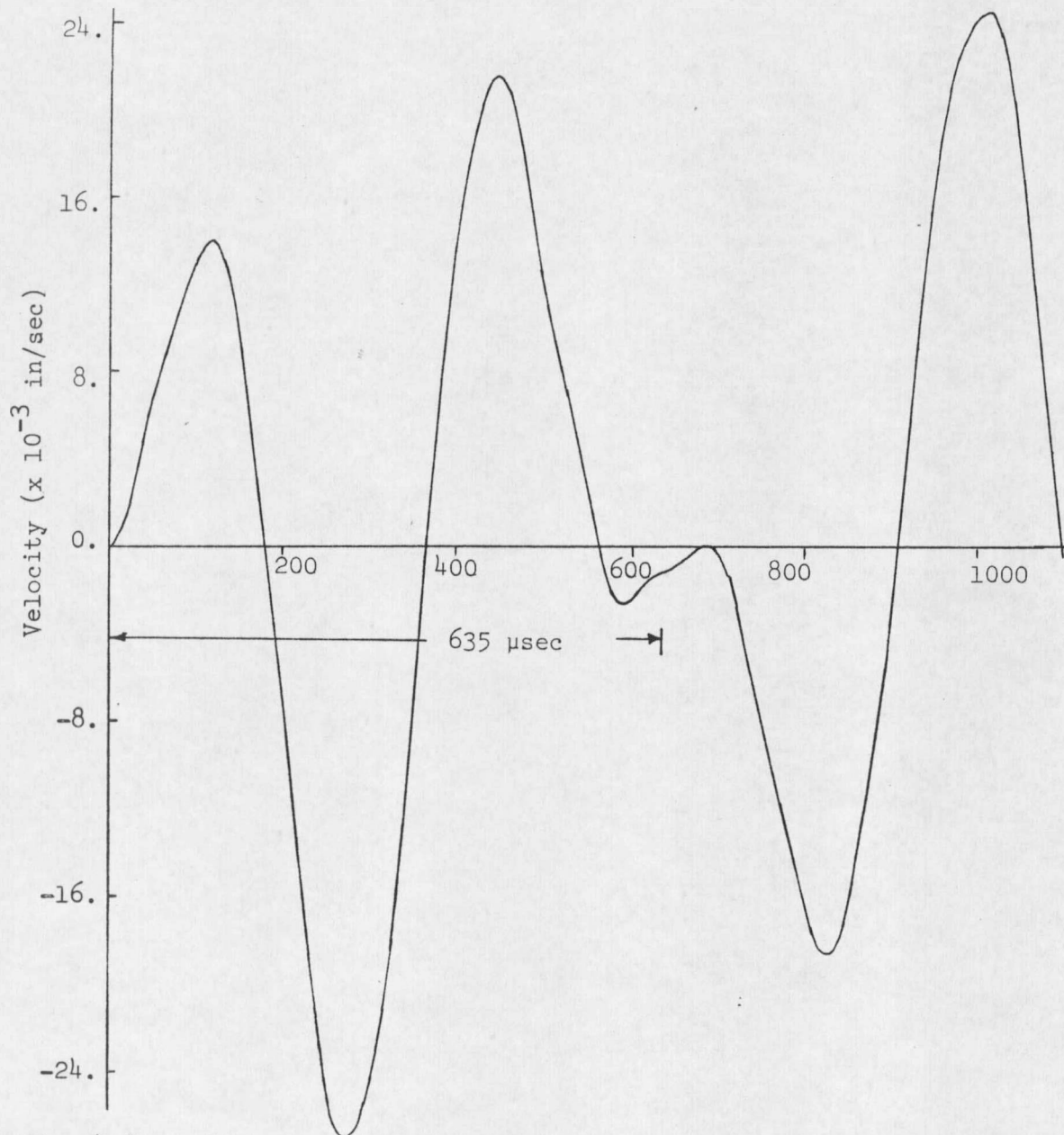


Fig. 4.8-3. Velocity Response at Free End.

Shear Wave Propagation
Nondispersive Element Model
Lateral Harmonic Force Applied
At the Free End - 2000 cycles/sec

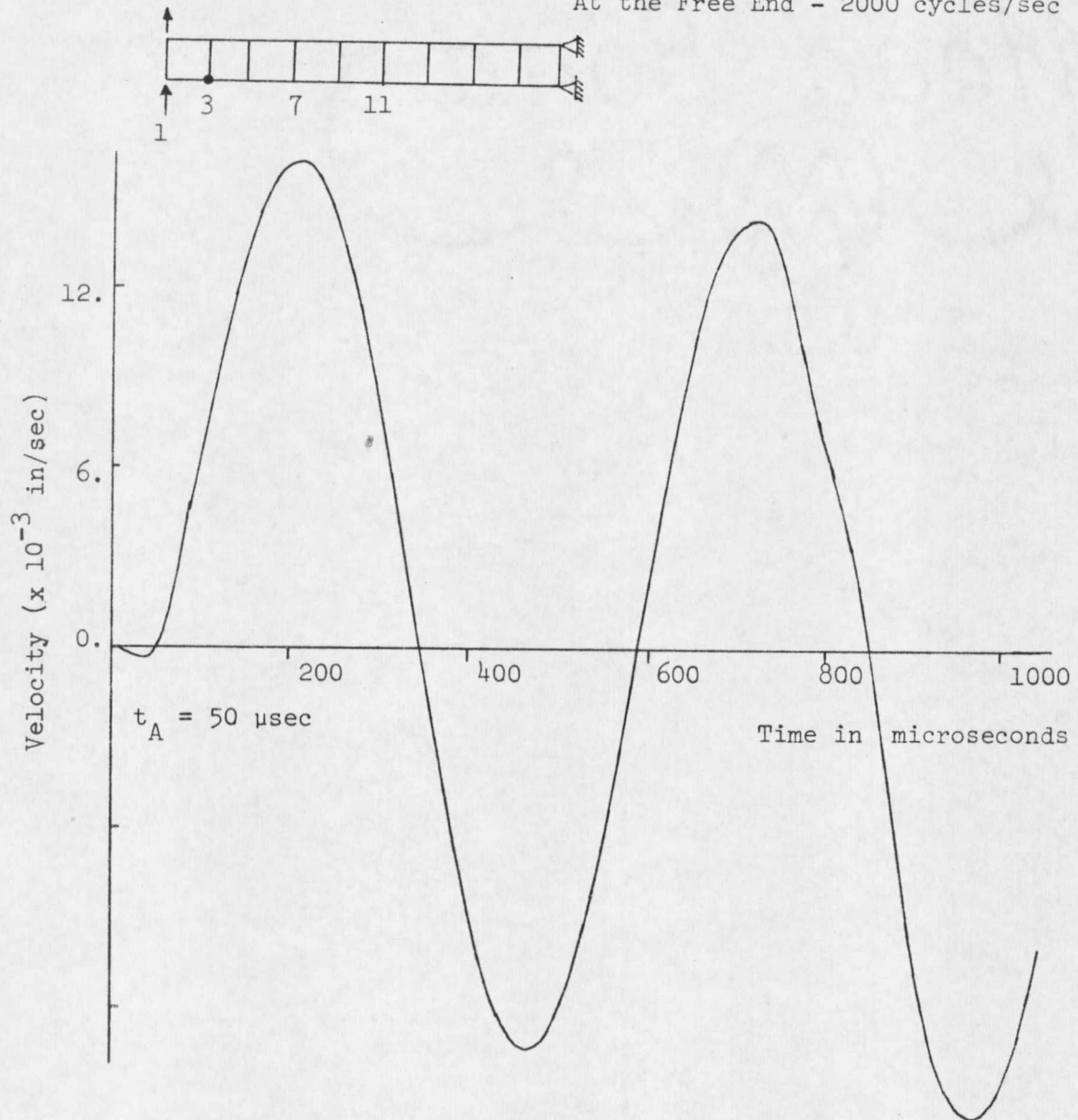


Fig. 4.8-4. Velocity Response at Node 3.

Shear Wave Propagation
Dispersive Element Model
Lateral Harmonic Force Applied
At the Free End - 2000 cycles/sec

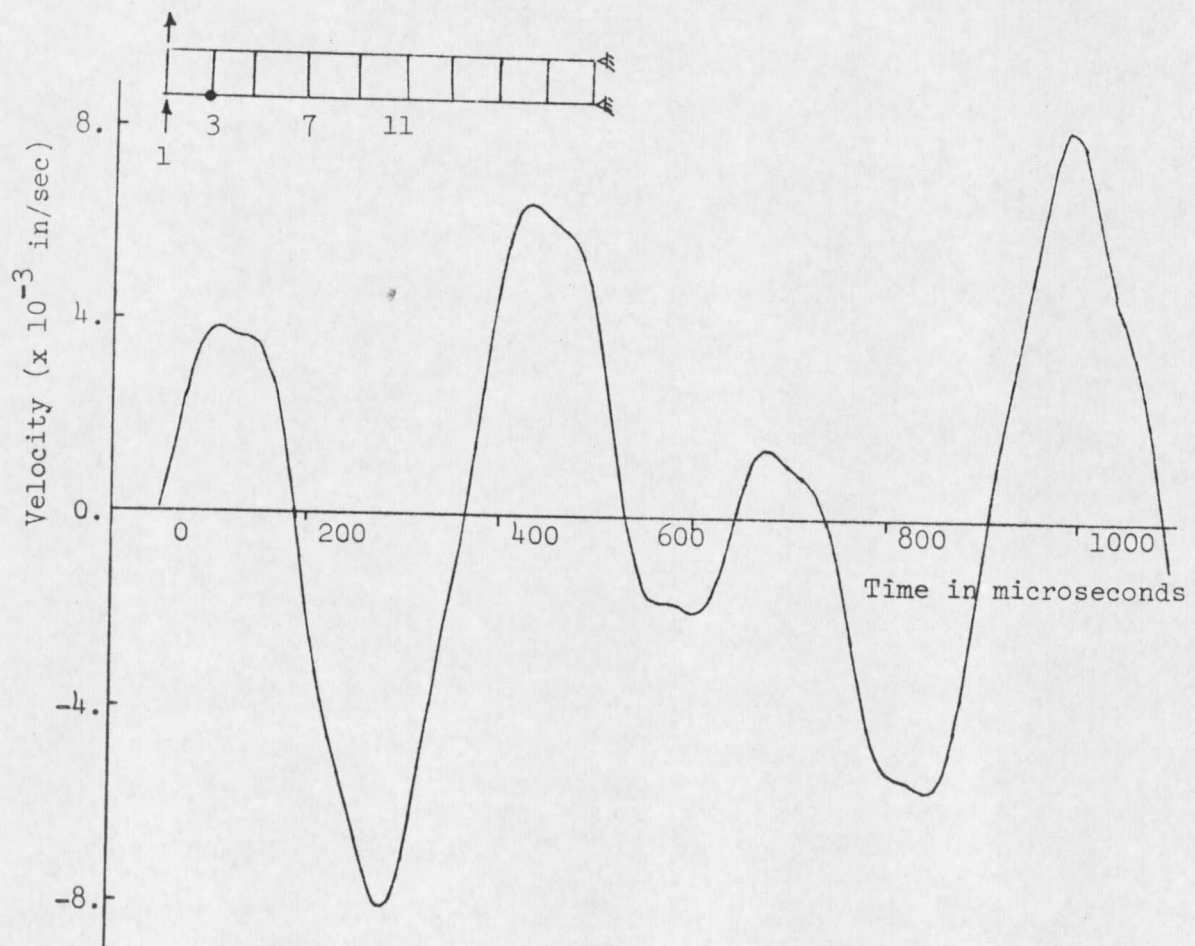


Fig. 4.8-5. Velocity Response at Node 3.

travels 6 in. distance in 50 microseconds, resulting in a velocity of propagation of 1.20×10^5 in/sec. which compares well with the previously computed value. Significant behaviour observed in the dispersive element model is the serious distortion of the wave front, even for the low frequency excitation. Also, pronounced attenuation is observed in that the amplitudes are reduced in an irregular pattern.

Figures 4.8-6 and 4.8-7 show the lateral velocity responses at the free end for the nondispersive and dispersive element models respectively under a harmonic excitation of 6000 cycles/sec. In the nondispersive element model, the incident wave returns to the free end after 885 microseconds, equivalent to a wave velocity of 1.22×10^5 in/sec. which compares well with the velocity for the 2000 cycles/sec. wave and the continuum velocity. In the dispersive element model, the incident wavefront returns to the free end after 390 microseconds, equivalent to a velocity of propagation of 2.76×10^5 in/sec. which is in error by 125% over the continuum velocity.

Thus, the models built with the first order finite elements are free from discretization-dispersion in propagating shear waves while the ordinary displacement compatible element models exhibit discretization-dispersion which is more pronounced than in the case of dilatational wave propagation.

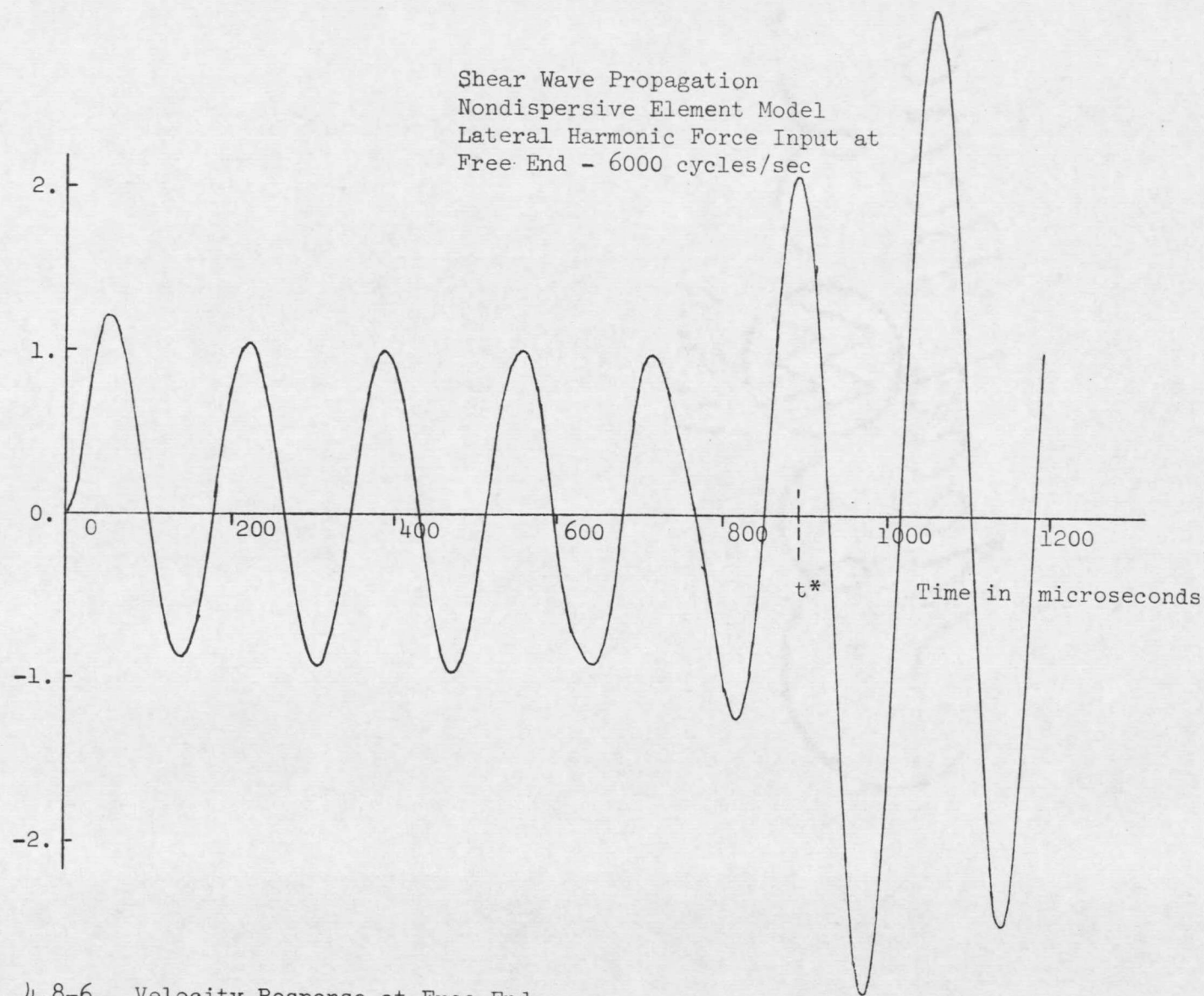


Fig. 4.8-6. Velocity Response at Free End.

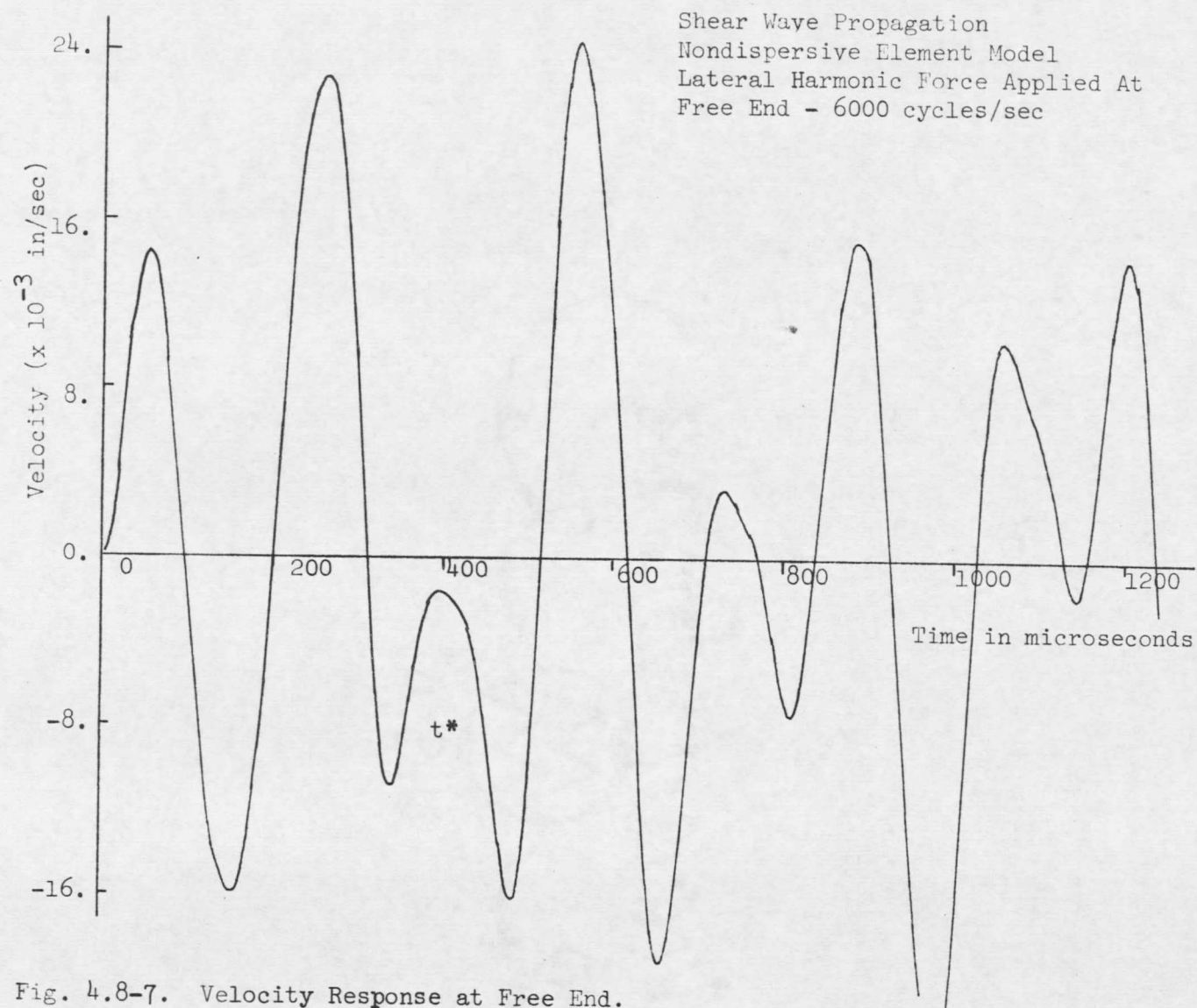


Fig. 4.8-7. Velocity Response at Free End.

V. TWO-DIMENSIONAL PROBLEM

5.1. Introduction

In this chapter, a two-dimensional elastic continuum is modelled by an assembly of the refined finite elements and the propagation of dilatational and shear waves inside the continuum are studied. The geometry of the elastic continuum considered is shown in Fig. 5.1-1. The boundaries on the sides AF and AB are fixed while the boundaries on the sides BC, CD, DE and EF are free. Normal and tangential surface tractions of uniform intensity, varying harmonically with time, are applied on the side EF. Of these externally applied forces, the normal tractions generate dilatational waves and the tangential tractions generate shear waves, which propagate in the negative x-direction. During the propagation, they come across a step change in elevation, which will cause part of the wave fronts to be reflected and part to be transmitted around the corner.

Previous studies done of a similar problem are the analysis and experiment of the propagation of a Rayleigh wave past a step change in elevation. The analytical study was done by the Green's function method of approximation 31 . It was done for the case of a step change in elevation far smaller than the wave length, wherein the wavefront did not undergo significant distortion. The experimental study was carried out by the photoelastic method[15]. Dilatational, shear and surface waves were generated by an explosion in the model. The propagation of the surface waves was studied by dynamic photo-

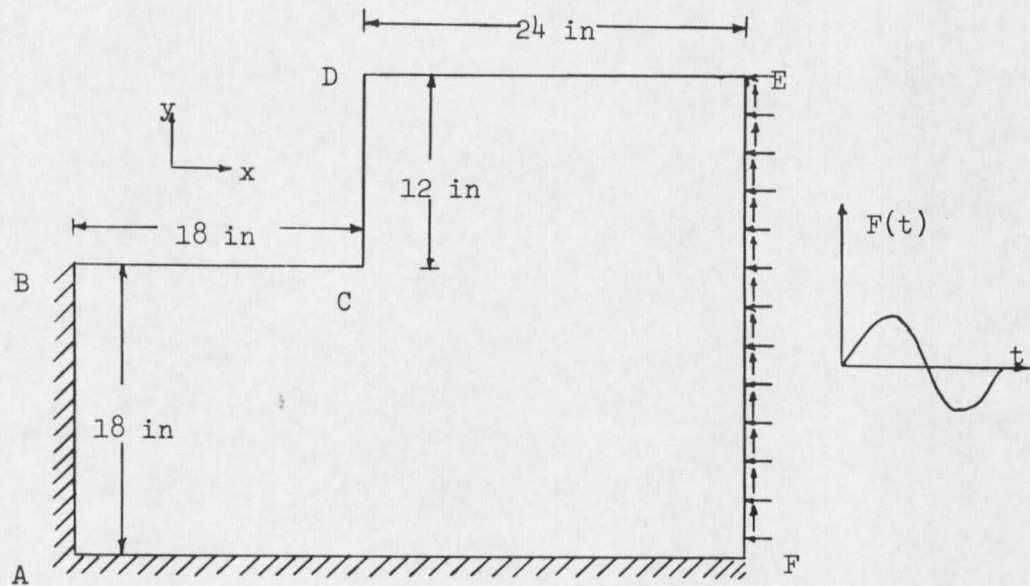


Fig. 5.1-1. Two-Dimensional Elastic Continuum

Physical Properties:

Mass Density, $\rho = 7.530 \times 10^{-4} \text{ slugs/in}^3$

Lamé's Constants, $\lambda = 15.50 \times 10^6 \text{ psi}$

$G = 11.25 \times 10^6 \text{ psi}$

Velocity of Propagation:

Dilatational waves, $V_p = 2.250 \times 10^5 \text{ in/sec}$

Shear Waves, $V_s = 1.221 \times 10^5 \text{ in/sec}$

elasticity.

The study here is mainly intended to verify the reflection and transmission characteristics of the finite element model on the dilatational and shear waves. The physical properties of the elastic continuum considered are:

Mass density, $\rho = 7.53 \times 10^{-4}$ slugs/in.³, and

Lamé's Constants, $\lambda = 15.5 \times 10^6$ psi

and $G = 11.25 \times 10^6$ psi .

The corresponding velocities of propagation are 2.25×10^5 in./sec for dilatational waves and 1.221×10^5 in./sec for shear waves.

5.2. Finite Element Model.

The finite element model is shown in Fig. 5.2-1, along with the node numbers. The model has 182 degree of freedom. The boundary conditions are imposed as described in the last chapter. For the nodes on the fixed boundary, the number of degrees of freedom are reduced from six to two, while for the nodes on the free boundary, four degrees of freedom are retained. The nodes 18 and 42 (corners D and E) have two degrees of freedom each, i.e., two displacements in perpendicular directions, while node 16 (corner C) has six degrees of freedom. Node 1 (corner A) is a junction of two fixed boundaries, and has four degrees of freedom. Nodes 4 and 37 (corners B and F) are junctions of fixed and free boundaries, and have two degrees of freedom each.

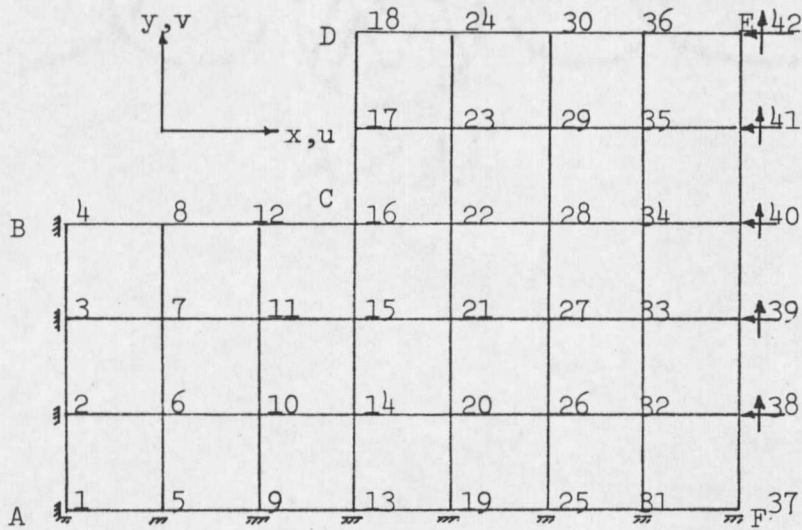


Fig. 5.2-1. Finite Element Model

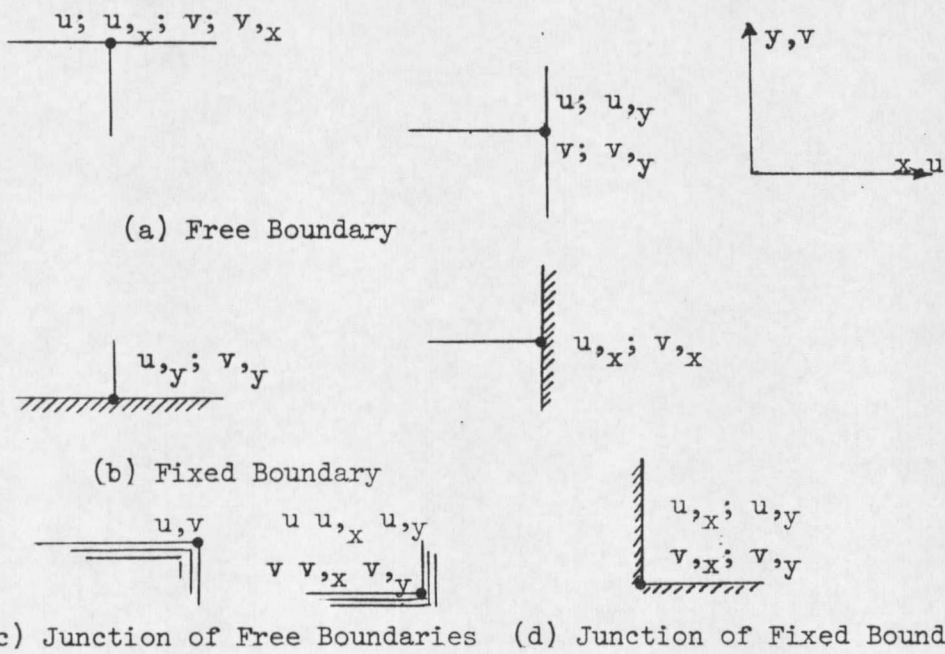


Fig. 5.2-2. Various Boundary Conditions

The nodal forces are computed consistently, based on the deformation of the elements along the boundary, where the external forces are acting. For this grid, the cut off frequencies are 18,750 cps for dilatational waves and 10,175 cps for shear waves. The harmonic surface tractions considered in this problem have a frequency of 7000 cps, well below the cut off frequency. The acceleration-pulse extrapolation method, explained in Sec. 4.3.1, is used for integrating the resulting differential equations. The integration is carried out in equal time steps of 5 microseconds. The dilatational waves travel across one element length in a little more than five time steps, while the shear waves travel the same distance in about ten time steps.

5.3. Results.

The velocity responses computed for several nodes shown in Figures 5.3-1 through 5.3-10. Of interest are the nodes in the neighborhood of the step, as there will be natural dispersion of the wavefronts in this region due to the geometry. In the response curves, the solid curves represent the velocity of the nodes in the x-direction, and the dashed curves represent the velocity in the y-direction. The notations t_{AP} and t_{AS} stand for the initial arrival times of the dilatational and shear wave fronts respectively. The notations t_{RP} and t_{RS} stand for the time of arrival of the reflected dilatational and shear waves respectively.

The dilatational waves travel faster than shear waves, and hence, arrive at a given node earlier than the arrival of the shear waves. In the neighborhood of the free boundary, the Rayleigh effect interferes with the incident wavefronts. The Rayleigh effect is the generation of surface waves in the vicinity of free surfaces. The surface waves cause large oscillation of particles in both longitudinal and lateral directions and travel at 0.92 times the velocity of shear waves. Surface wave effects reduce rapidly with increasing distance from the free boundary. In the response curves for nodes 30, 18, 17 and 16 (Figures 5.3-1 through 5.3-4), there is significant velocities in the lateral directions during the time before the arrival of the shear wave, but after the arrival of the dilatational wave. These nodes are located on the free surface, and the generation of surface waves on the incidence of dilatational waves cause the motion in the lateral directions.

The wavefronts get reflected at the step and this is seen in the response curves for nodes 23 and 30 (Figures 5.3-1 and 5.3-9). In both cases, the arrival of the reflected dilatational wave is distinctly seen, while, for the shear wave, it is not as distinct. This occurs because the shear waves travel slower, and the natural dispersion of the dilatational waves in the neighborhood of the corner D (Fig. 5.3-2) shows amplified velocities. The phenomenon is due to the Rayleigh effect and due to the natural dispersion of the

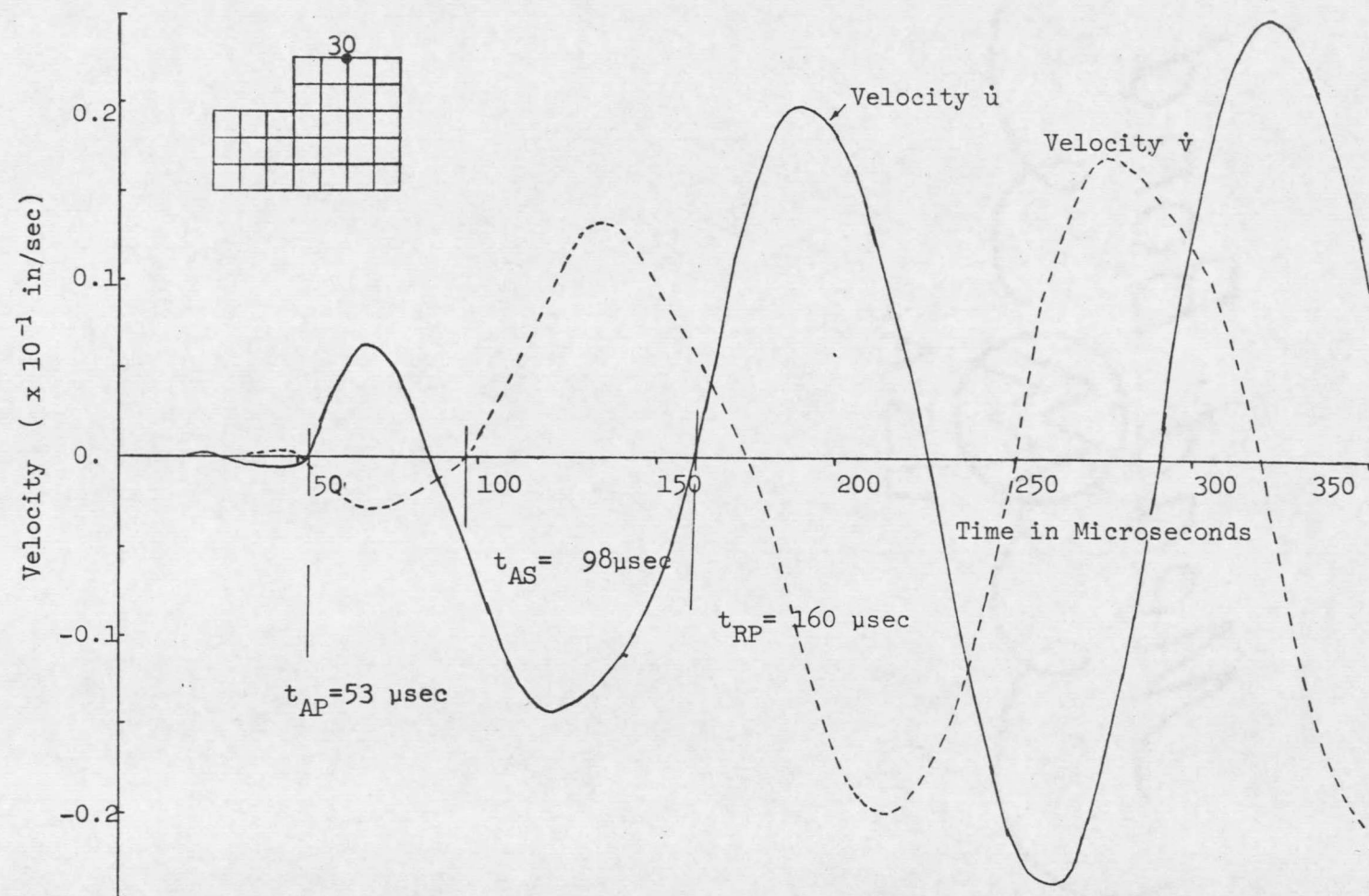


Fig. 5.3-1. Velocity Response at Node 30

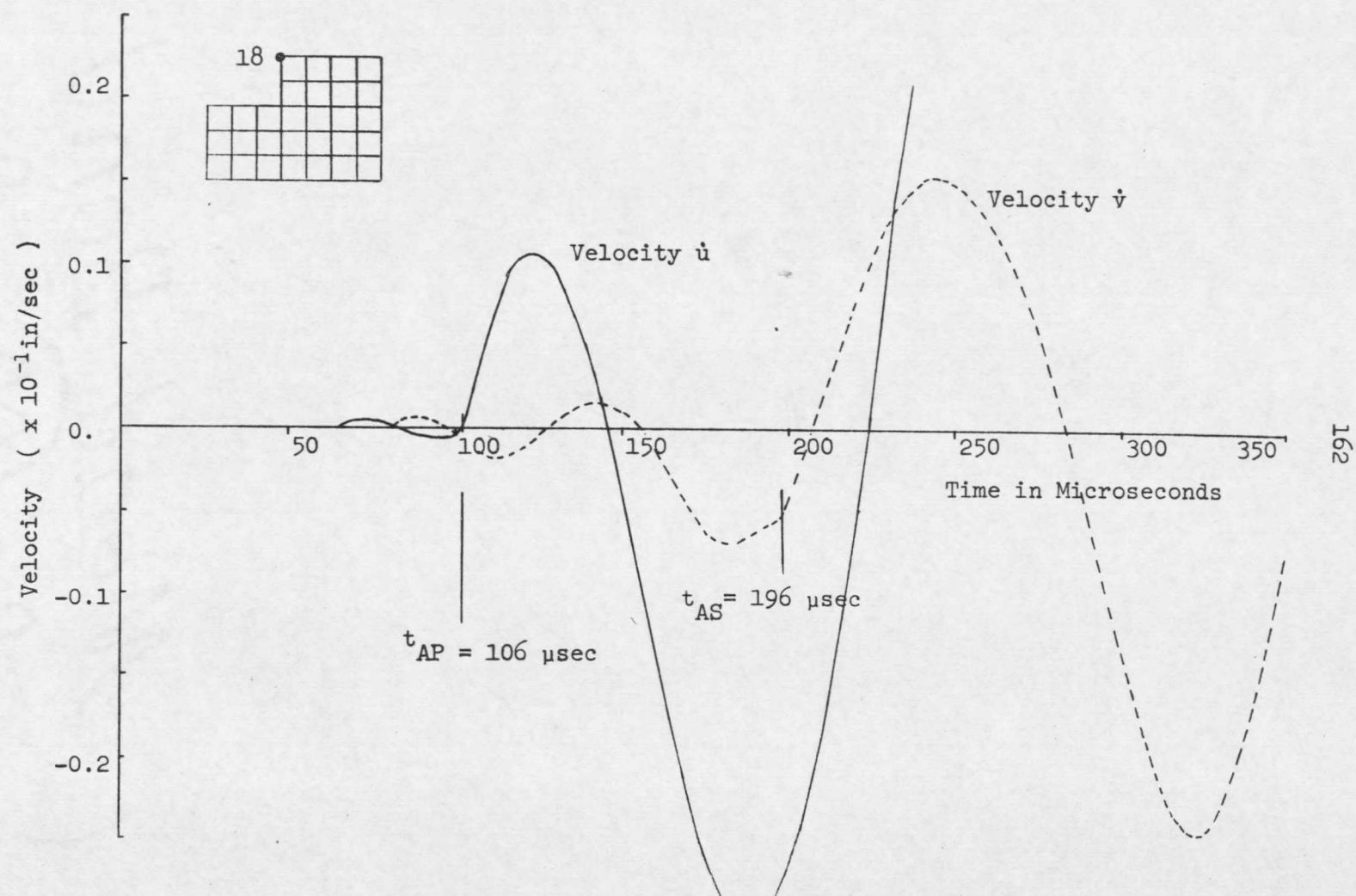


Fig. 5.3-2. Velocity Response at Node 18

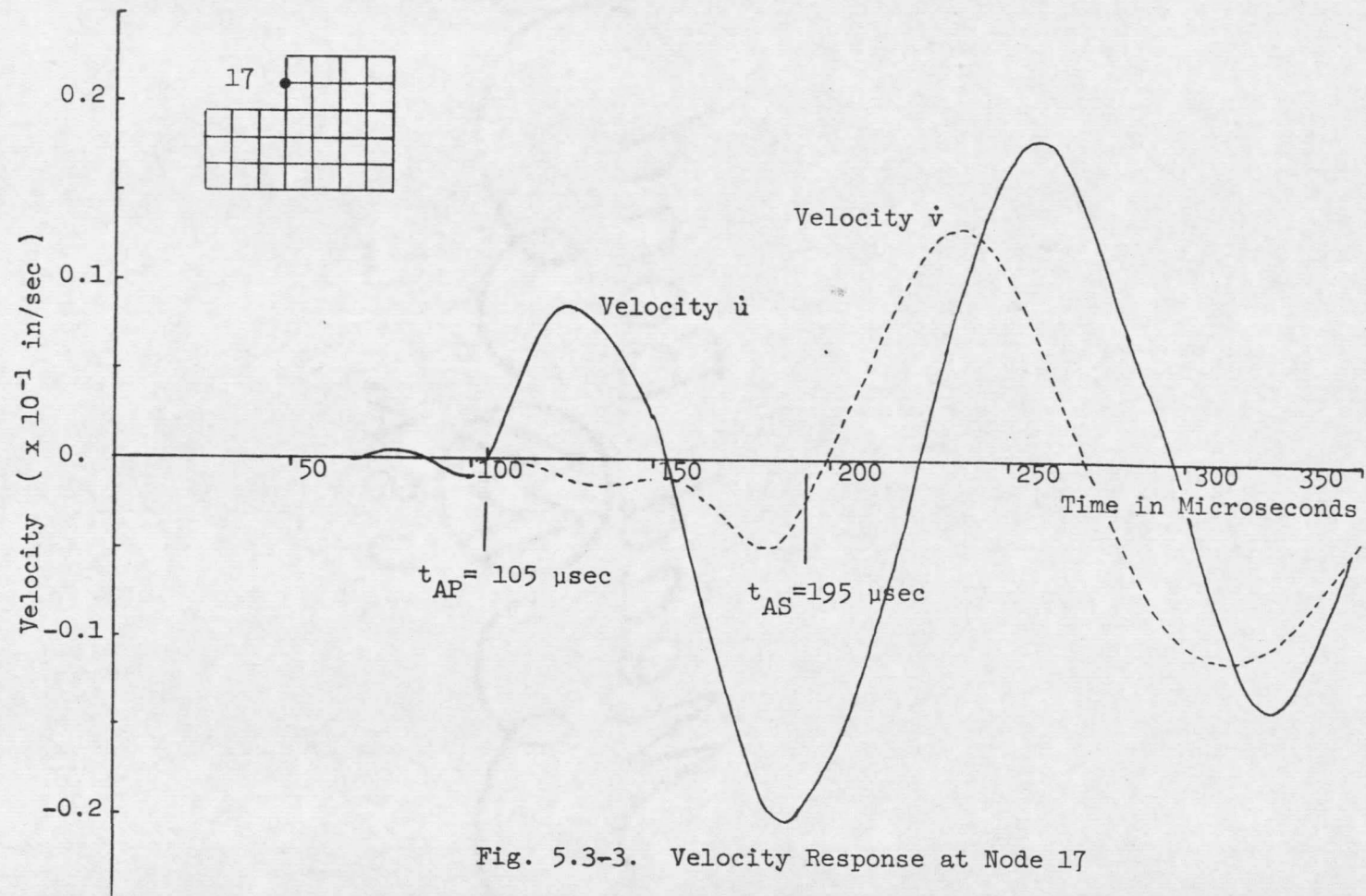


Fig. 5.3-3. Velocity Response at Node 17

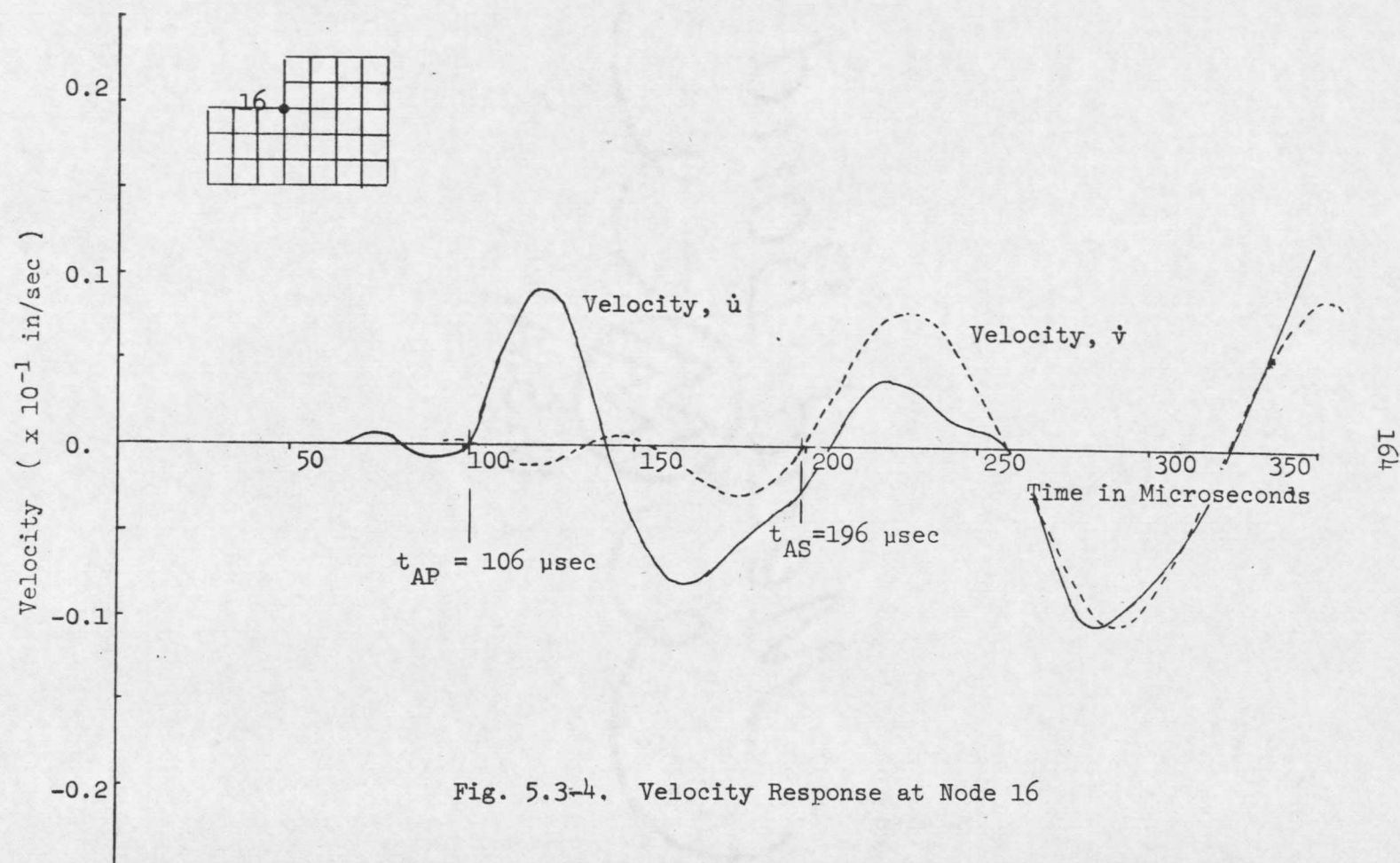


Fig. 5.3-4. Velocity Response at Node 16

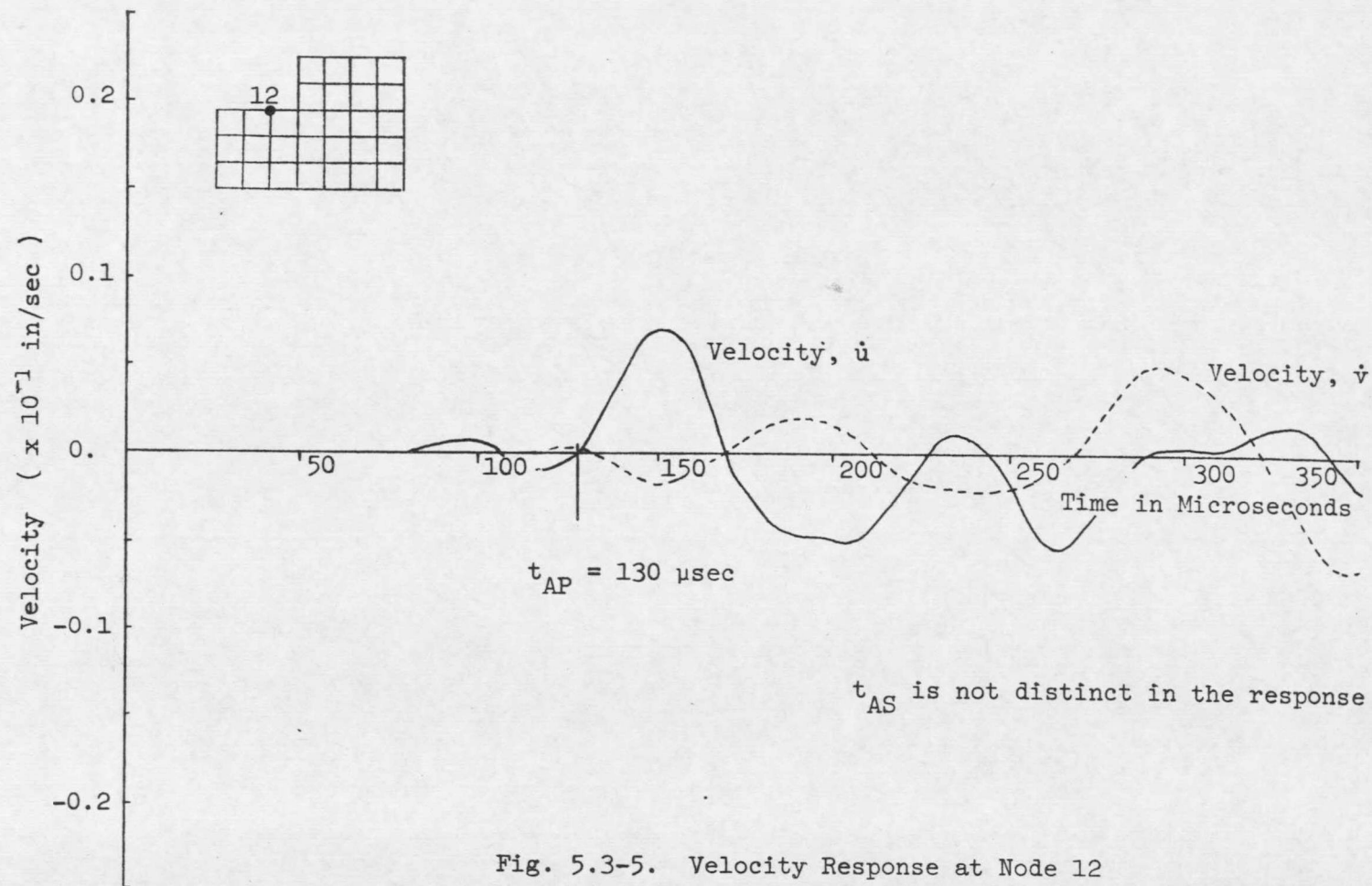


Fig. 5.3-5. Velocity Response at Node 12

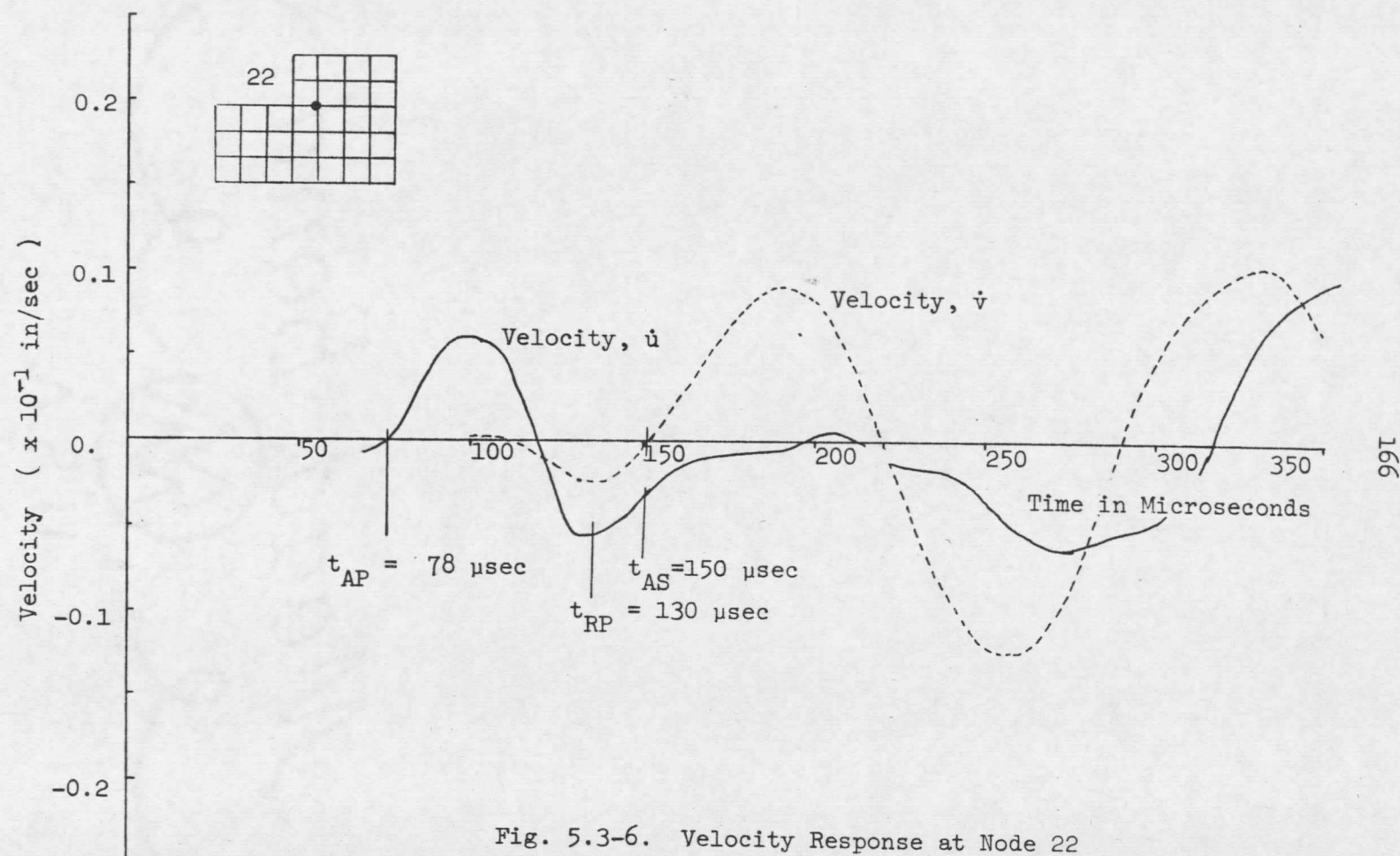


Fig. 5.3-6. Velocity Response at Node 22

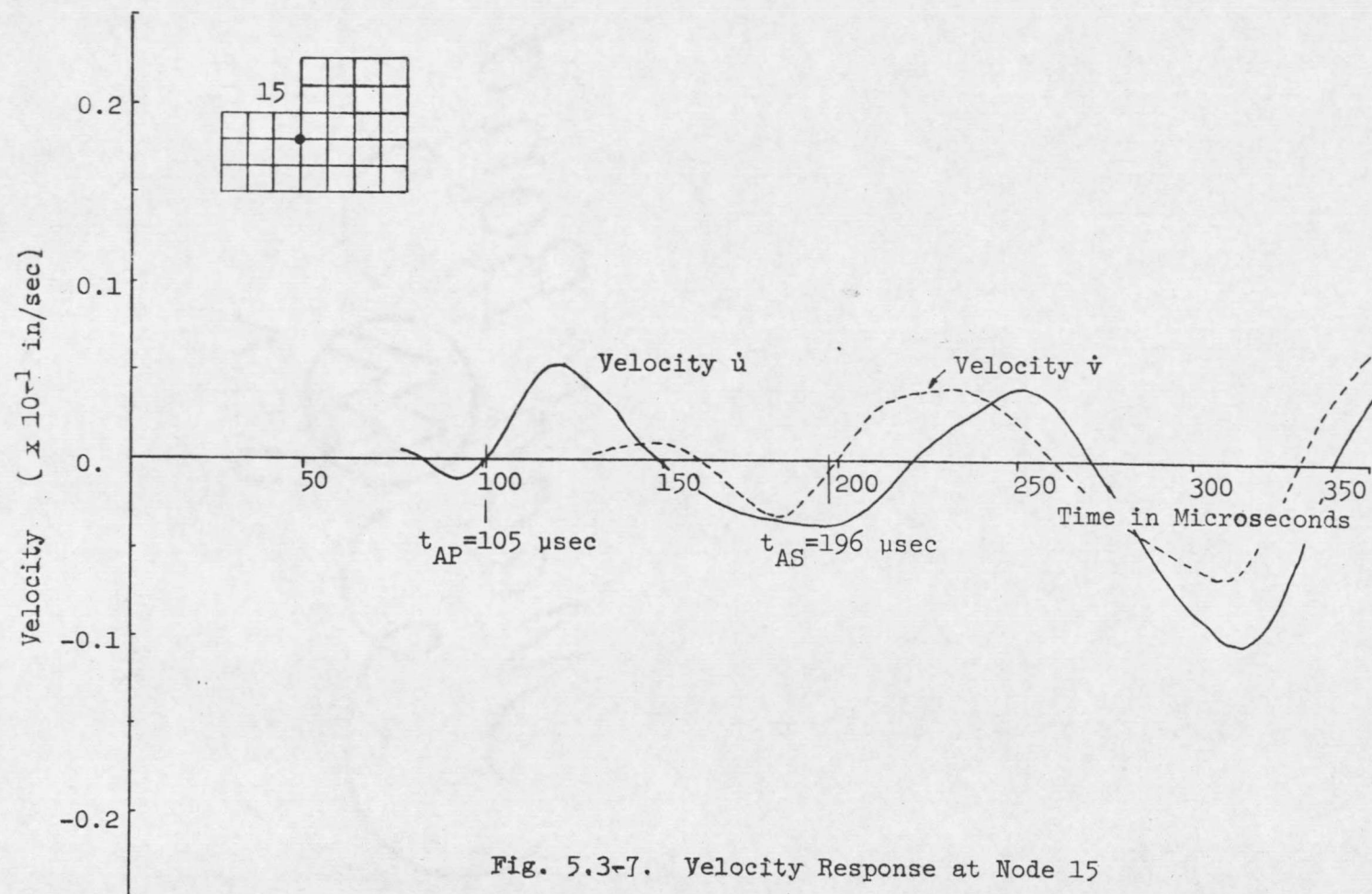


Fig. 5.3-7. Velocity Response at Node 15

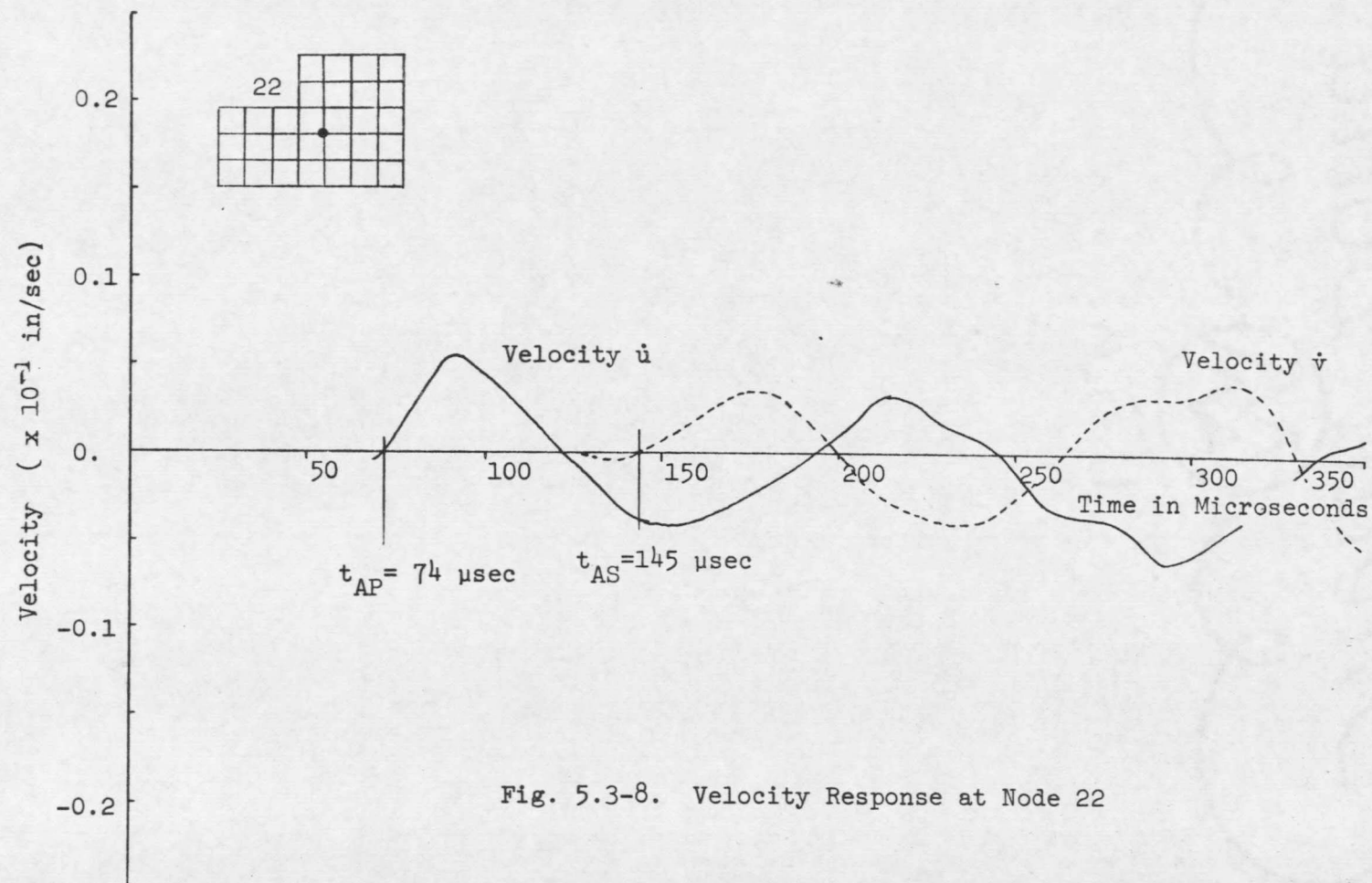


Fig. 5.3-8. Velocity Response at Node 22

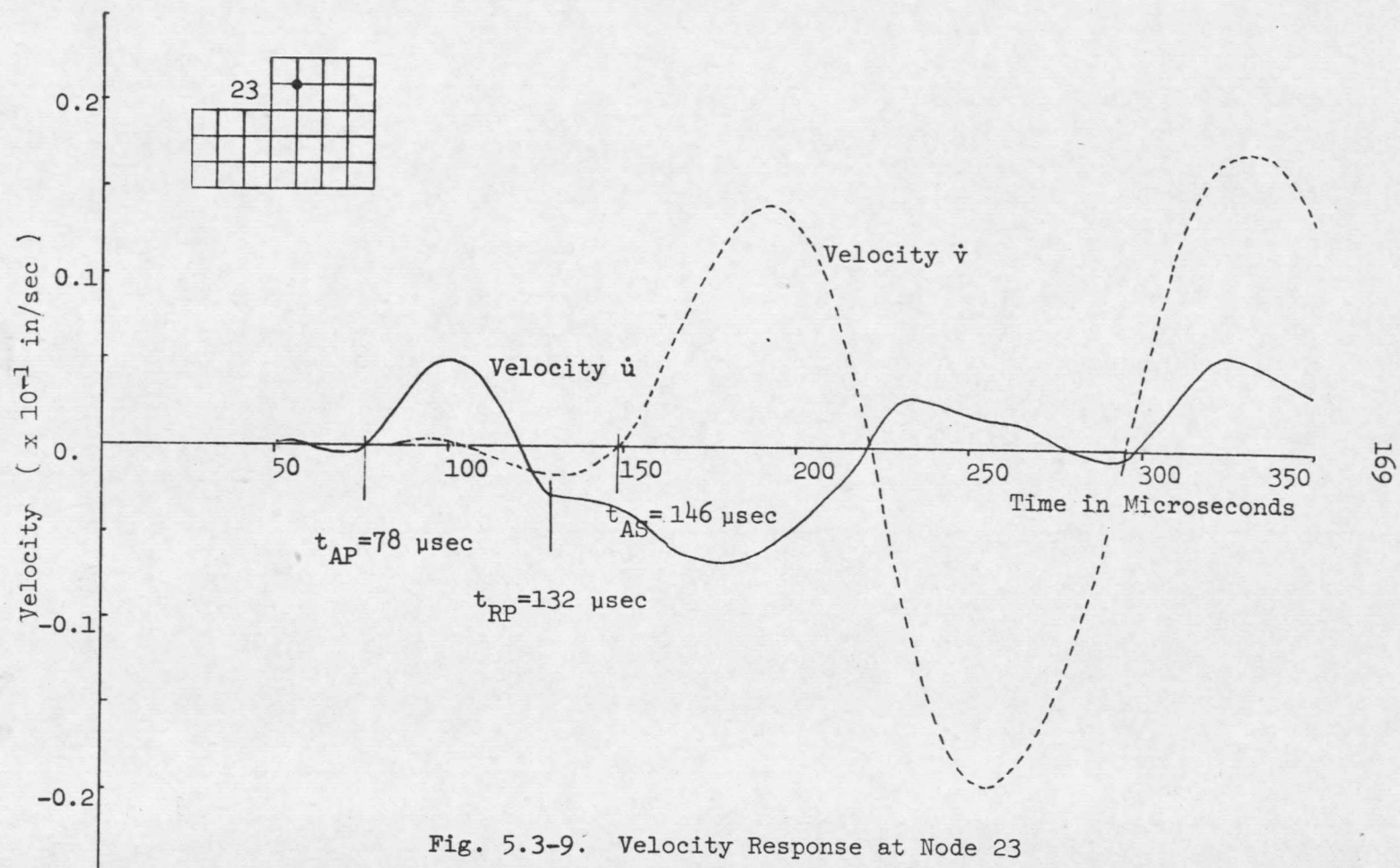


Fig. 5.3-9. Velocity Response at Node 23

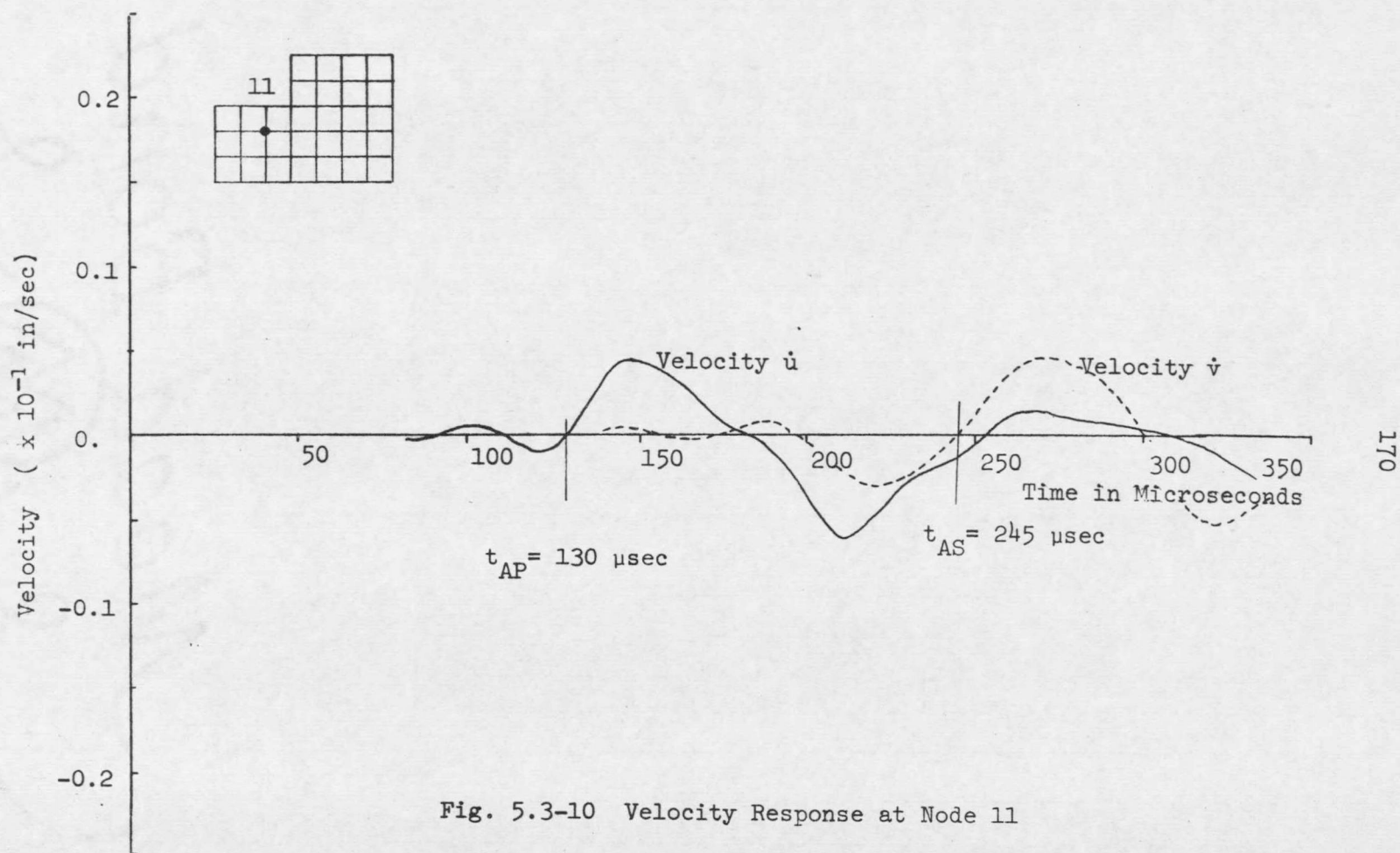


Fig. 5.3-10 Velocity Response at Node 11

dilatational and shear waves at the corner. Further, the surface waves travelling along the side DE are forced to change direction of travelling, causing additional dispersion.

A comparison of the velocity responses at nodes 16 and 12 (Figures 5.3-4 and 5.3-5) shows that the amplitudes of velocities at node 12 are smaller than those at 16. This is because only part of the waves get transmitted across the node 16, while the other part is reflected. The reflection is observed in the velocity response at node 22 (Fig. 5.3-6). A comparison of the velocity responses at nodes 22 and 23 (Figures 5.3-6 and 5.3-9) confirms the partial reflection as the effect of the reflected waves is not as significant in the former as in the latter.

The response curve for node 22 shows a negligible amount of artificial farfield coupling in the model. The node is situated away from the step change in elevation and the effect of natural dispersion is not significant. The node does not show appreciable motion until the arrival of the appropriate wavefront. The nodes closer to the fixed boundary exhibit smaller amplitudes of motion than those situated away from the fixed boundary. This is to be expected as the body shows increased constraint in the neighborhood of the fixed boundaries.

Low artificial farfield coupling and accurate reflection characteristics make the refined element suitable for accurate wave propagation analysis. Greatest value of the refined element may well be in

the analysis of confined domains where a little complication in the geometry makes the classical approach of analysis all but impossible. The need is evident to extend this two-dimensional analysis to modeling a Rayleigh wave incident upon a step geometry. This will involve the use of a large model and the application of a dynamic excitation so that the surface waves generated travel free from the interference of dilatational and shear waves. The problem has application in the determination of the response of structures like dams during earthquake excitations.

VI. SUMMARY

Problems associated with the application of the finite element method to two dimensional elastic wave propagation in solids is investigated comprehensively. Several limitations on the application of the conventional finite elements to wave propagation modelling are noted, these limitations present also in other discretization methods such as the finite difference method. A refined finite element with stronger compatibility conditions across interelement boundaries is found to eliminate defects that occur with conventional finite elements in wave propagation modelling.

Two serious defects associated with the conventional finite elements are discretization-dispersion and artificial farfield coupling. The discretization-dispersion causes the wavefronts of different frequencies to propagate at different velocities with significant deviations from the velocity of propagation in the continuum. In this investigation, consistent mass representation is used and it is observed that the velocity of propagation in the finite element model is greater than the continuum velocities. Lumped mass representation has been used by several investigators [5,25,52], and it has been found that the finite element velocities are slower than the continuum velocities. When the wavelengths involved are long, the lumped mass models transmit waves approximately

at continuum velocities, whereas, for shorter waves, it results in serious discretization-dispersion. The finite difference models currently used have the same properties as the conventional zeroth order finite elements, and hence, the solutions derived with them are affected by discretization-dispersion.

The magnitude of the discretization-dispersion is also a function of the geometry of the model. A test procedure is developed to evaluate the magnitude of the discretization-dispersion in two-dimensional continuum for a given finite element. For the conventional displacement compatible finite elements, when representing a two-dimensional continuum, the maximum discretization-dispersion is of the order of 92% for dilatational waves and 20% for shear waves, below the cut off frequency. When used in modelling a longitudinal bar as in Chapter IV, less discretization-dispersion is noted for dilatational waves (6.5% at 10,000 cycles/sec) while more for shear waves (52.5% at 2000 cycles/sec). The refined element developed in Chapter III is free from discretization-dispersion and the models built with these elements yield responses closely approximating to those of the continua.

Artificial farfield coupling causes a given node to oscillate before the arrival of the wavefront at that node. Its magnitude increases with increasing frequency of the wave front and increasing distance of the node from the point of application of external force.

For example, in the model of the longitudinal bar considered in Chapter IV, while propagating dilatational waves, the effect of artificial farfield coupling at nodes 5 and 13 are respectively about 10% and 21% at a frequency of 3800 cycles/sec (Figures 4.5-2 and 4.5-4), while its effects are 17% and 40% at 10,000 cycles/sec (Figures 4.5-6 and 4.5-8), indicating a significant deviation from the continuum response. It cannot be related to the off-diagonal terms in the inverse of the mass matrix as the response caused is not instantaneous but delayed and grows with increasing time. The test procedure developed in Chapter III for the evaluation of discretization-dispersion does not include the effects of artificial farfield coupling. The artificial farfield coupling causes difficulty in predicting the true response when two or more wave fronts interfere with each other. Finite element models built with the nondispersive element do not show any significant artificial farfield coupling.

Finite element models act as low-pass filters, attenuating the wavefronts above the cut off frequency. This is inherent in all discretization formulations and cannot be removed. So, the element size must be adjusted to the type of wavefront involved. Refined nonsquare elements have the same nondispersive properties as the square elements. Nonsquare elements have stronger filtering characteristics than the square elements in attenuating the high frequency waves, and yield erroneous solutions when used for forcing

functions such as a step-rise force (Fig. 4.6-8).

When used for solving problems of wave propagation in confined domains, the discretization-dispersion in the conventional finite elements interferes with the physical dispersion, yielding solutions with serious error. Absence of discretization-dispersion in the refined finite element makes it especially useful for solving such problems. Further, the reflections at the boundaries are more accurately represented, leading to more accurate solutions. In infinite and semi-infinite domains where the physical dispersion is not significant, conventional finite element models can be used and some interpretation scheme can be developed to determine the right solutions.

This investigation is just the beginning of a rational approach to the application of the finite element method to elastic wave propagation in solids. The refined elements developed with first order strain compatibility are free from discretization-dispersion in both one and two dimensions. The two-dimensional finite element developed in this investigation is rectangular in shape. A triangular element, free from discretization-dispersion, would be useful for wave propagation studies in nonuniform geometries. There are several such triangular elements already available in the literature [6,18,58] with first order strain compatibility across interelement boundaries but their dispersive characteristics have not yet been evaluated.

If some of those elements are found to be free from discretization-dispersion, they can be used in wave propagation modelling.

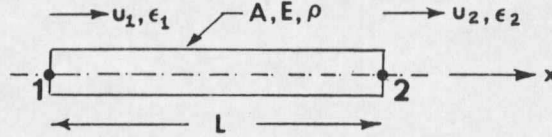
The test procedure for evaluation of discretization-dispersion developed in this study may be used to test other finite elements for their dispersive characteristics. As may be expected, different finite elements will exhibit different amounts of discretization-dispersion. When accuracy is demanded, it is recommended that the elements with zero or negligible discretization-dispersion be used for modelling.

APPENDICES

APPENDIX A

ONE-DIMENSIONAL FIRST ORDER STRAIN COMPATIBLE FINITE ELEMENT

1. Derivation of Stiffness Matrix:



u - displacement,

ϵ - normal strain,

L - length of the element,

E - Modulus of Elasticity

A - Area of cross-section

$$u(x) = \sum_{i=1}^2 u_i H_{0i}^{(1)}(x) + u_{,x_i} H_{1i}^{(1)}(x) \quad (\text{A.1-1})$$

where

$$H_{01}^{(1)}(x) = \frac{1}{L^3} [2x^3 + 3Lx^2 + L^3]$$

$$H_{02}^{(1)}(x) = -\frac{1}{L^3} [2x^3 - 3Lx^2]$$

$$H_{11}^{(1)}(x) = \frac{1}{L^2} [x^3 - 2Lx^2 + L^2x]$$

and

$$H_{12}^{(1)}(x) = \frac{1}{L^2} [x^3 - Lx^2]$$

(A.1-2)

Let $u_{,x_1} = \epsilon_1$ and $u_{,x_2} = \epsilon_2$.

$$\begin{aligned}
 \therefore u(x) &= \frac{u_1}{L^3} [2x^3 + 3L^2 + L^3] - \frac{u_2}{L^3} [2x^3 - 3Lx^2] \\
 &+ \frac{\epsilon_1}{L^2} [x^3 - 2Lx^2 + L^2x] + \frac{\epsilon_2}{L^2} [x^3 - Lx^2] \\
 &= u_1 + \epsilon_1 x + \frac{3(u_2 - u_1)}{L^2} x^2 - \frac{2\epsilon_1 + \epsilon_2}{L} x^2 + \frac{\epsilon_1 + \epsilon_2}{L^2} x^3 - \frac{2(u_2 - u_1)}{L^3} x^3
 \end{aligned}$$

or

$$u(x) \doteq \frac{1}{L^3} [1 \ x \ x^2 \ x^3] \begin{bmatrix} L^3 & 0 & 0 & 0 \\ 0 & L^3 & 0 & 0 \\ -3L & -2L^2 & 3L & -L^2 \\ 2 & L & -2 & L \end{bmatrix} \begin{Bmatrix} u_1 \\ \epsilon_1 \\ u_2 \\ \epsilon_2 \end{Bmatrix} \quad (\text{A.1-3})$$

or

$$u(x) = \frac{1}{L^3} [1 \ x \ x^2 \ x^3] [T] \{r\} \quad (\text{A.1-4})$$

Apply the principle of stationary value of potential energy:

$$\delta[W_E + W_I] = 0$$

$$\delta W_E = -\{R\}\{\delta r\} \text{ where } \{\delta r\} = \begin{Bmatrix} \delta u_1 \\ \delta \epsilon_1 \\ \delta u_2 \\ \delta \epsilon_2 \end{Bmatrix}$$

$$\text{and } \delta W_I = \int_L P(x) \delta\left(\frac{\partial u}{\partial x}\right) dx, \quad P(x) = AE \frac{\partial u}{\partial x}$$

$$\therefore \{\delta r\}^T \{R\} = \int_L AE \left(\frac{\partial u}{\partial x} \right) \delta \left(\frac{\partial u}{\partial x} \right) dx \quad (A.1-5)$$

$$\frac{\partial u}{\partial x} = [0 \ 1 \ 2x \ 3x^2] [T] \{r\}$$

$$\delta \left(\frac{\partial u}{\partial x} \right) = [0 \ 1 \ 2x \ 3x^2] [T] \{\delta r\} = \{\delta r\}^T [T]^T \begin{Bmatrix} 0 \\ 1 \\ 2x \\ 3x^2 \end{Bmatrix}$$

$$\{\delta r\}^T \{R\} = \{\delta r\}^T [T]^T \int_L AE \begin{Bmatrix} 0 \\ 1 \\ 2x \\ 3x^2 \end{Bmatrix} [0 \ 1 \ 2x \ 3x^2] [T] \{r\} dx \quad (A.1-6)$$

$$\{R\} = [T]^T [N] [T] \{r\}$$

where

$$[N] = AE \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & L & L^2 & L^3 \\ 0 & L^2 & \frac{4}{3}L^3 & \frac{3}{2}L^4 \\ 0 & L^3 & \frac{3}{2}L^4 & \frac{9}{5}L^5 \end{bmatrix}$$

The stiffness matrix is:

$$[K] = [T]^T [N] [T]$$

$$= \frac{AE}{L} \begin{bmatrix} \frac{6}{5} & \frac{L}{10} & \frac{-6}{5} & \frac{L}{10} \\ \frac{L}{10} & \frac{2L^2}{15} & \frac{-L}{10} & \frac{-L^2}{30} \\ \frac{-6}{5} & \frac{-L}{10} & \frac{6}{5} & \frac{-L}{10} \\ \frac{L}{10} & \frac{-L^2}{30} & \frac{-L}{10} & \frac{2L^2}{15} \end{bmatrix} \quad (\text{A.1-7})$$

2. Derivation of the Mass Matrix:

Virtual work of external and inertial body force:

$$\{\delta r\}^T \{R_E\} + \int_L \rho A \ddot{u} \delta(u) dx = \{\delta r\}^T \{R\}, \quad (\text{A.2-1})$$

ρ is the mass density, and $(\ddot{})$ indicating second order time derivative.

$$\ddot{u} = [1 \ x \ x^2 \ x^3] [T] \{\ddot{r}\}$$

$$\{\delta r\}^T \{R_E\} + \{\delta r\}^T \int_L \rho A [T]^T \begin{Bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{Bmatrix} [1 \ x \ x^2 \ x^3] [T] \{r\} dx = \{\delta r\}^T \{R\}$$

Hence:

$$\{R_E\} + \rho A [T]^T [N'] [T] \{\ddot{r}\} = \{R\} \quad (\text{A.2-2})$$

where

$$[N'] = \begin{bmatrix} L & \frac{L^2}{2} & \frac{L^3}{3} & \frac{L^4}{4} \\ \frac{L^2}{2} & \frac{L^3}{3} & \frac{L^4}{4} & \frac{L^5}{5} \\ \frac{L^3}{3} & \frac{L^4}{4} & \frac{L^5}{5} & \frac{L^6}{6} \\ \frac{L^4}{4} & \frac{L^5}{5} & \frac{L^6}{6} & \frac{L^7}{7} \end{bmatrix}$$

and the mass matrix is:

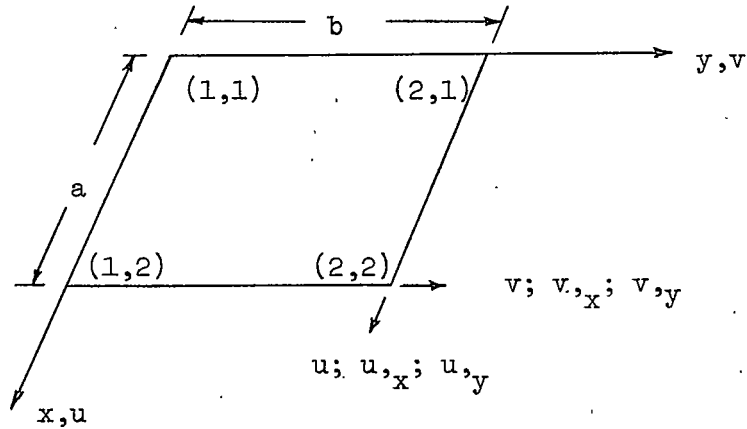
$$[M] = \rho A [T]^T [N'] [T] \quad (A.2-3)$$

$$= \frac{\rho A L}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \quad (A.2-4)$$

APPENDIX B

TWO-DIMENSIONAL FIRST ORDER STRAIN COMPATIBLE RECTANGULAR FINITE ELEMENT

B.1 Derivation of Stiffness and Mass Matrices.



Displacement functions are given by:

$$u(x,y) = \sum_{i=1}^2 \sum_{j=1}^2 [u_{ij} H_{oi}(x) H_{oj}(y) + u_{,x_{ij}} H_{li}(x) H_{oj}(y) + u_{,y_{ij}} H_{oi}(x) H_{lj}(y)] \quad (\text{B.1-1})$$

and

$$v(x,y) = \sum_{i=1}^2 \sum_{j=1}^2 [v_{ij} H_{oi}(x) H_{oj}(y) + v_{,x_{ij}} H_{li}(x) H_{oj}(y) + v_{,y_{ij}} H_{oi}(x) H_{lj}(y)] \quad (\text{B.1-2})$$

where u and v are the displacement components in x and y -directions.

The interpolation polynomials are:

$$H_{01}(x) = \frac{1}{a^3} [2x^3 - 3ax^2 + a^3]$$

$$H_{02}(x) = \frac{-1}{a^3} [2x^3 - 3ax^2]$$

(B.1-3)

$$H_{11}(x) = \frac{1}{a^2} [x^3 - 2ax^2 + a^2x]$$

$$H_{12}(x) = \frac{1}{a^2} [x^3 - ax^2]$$

and similar equations on y.

Equation (B.1-1) can be rewritten as:

$$u(x,y) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ \frac{-3x^2}{a^2} & \frac{-2x^2}{a} & \frac{3x^2}{a^2} & \frac{-x^2}{a} \\ \frac{2x^3}{a^3} & \frac{x^3}{a^2} & \frac{-2x^3}{a^3} & \frac{x^3}{a^2} \end{vmatrix} \begin{vmatrix} u_{11} & u_{,y11} & u_{12} & u_{,y12} \\ u_{,x11} & 0 & u_{,x12} & 0 \\ u_{21} & u_{,y21} & u_{22} & u_{,y22} \\ u_{,x21} & 0 & u_{,x22} & 0 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & \frac{-3y^2}{b^2} & \frac{2y^3}{b^3} \\ 0 & y & \frac{-2y^2}{b} & \frac{y^3}{b^2} \\ 0 & 0 & \frac{3y^2}{b^2} & \frac{-2y^3}{b^3} \\ 0 & 0 & \frac{-y^2}{b} & \frac{y^3}{b^2} \end{vmatrix}$$

(B.1-4)

$$= [HX]^T [U] [HY]$$

(B.1-5)

where [HX], [U] and [HY] are the matrices in equation (B.1-4).

Similarly,

$$v(x,y) = [HX]^T [V] [HY] \quad (B.1-6)$$

where

$$[V] = \begin{bmatrix} v_{11} & v_{,y_{11}} & v_{12} & v_{,y_{12}} \\ v_{,x_{11}} & 0 & v_{,x_{12}} & 0 \\ v_{21} & v_{,y_{21}} & v_{22} & v_{,y_{22}} \\ v_{,x_{21}} & 0 & v_{,x_{22}} & 0 \end{bmatrix}$$

The derivatives of the displacement functions are written as:

$$\frac{\partial}{\partial x} [u(x,y)] = u_{,x}(x,y) = [DHX]^T [U] [HY] \quad (B.1-7)$$

and

$$\frac{\partial}{\partial y} [u(x,y)] = u_{,y}(x,y) = [HX]^T [V] [DHY] \quad (B.1-8)$$

where

$$[DHX] = \frac{\partial}{\partial x} [HX] \text{ and } [DHY] = \frac{\partial}{\partial y} [HY]$$

Similar equations for $v_{,x}$ and $v_{,y}$ are:

$$v_{,x}(x,y) = [DHX]^T [V] [HY]$$

and

$$v_{,y}(x,y) = [HX]^T [V] [DHY]$$

Strain energy stored in the element during deformation:

$$U = \frac{Eh}{2(1-\mu^2)} \int_{x=0}^a \int_{y=0}^b \{ [u,^2_x + 2\mu u,^2_{x,y} + v,^2_y] + \frac{(1-\mu)}{2} [u,^2_y + 2\mu u,^2_{y,x} + v,^2_x] \} dx dy \quad (B.1-9)$$

where E = Modulus of elasticity,

μ = Poisson's ratio

and h = thickness of the element.

Equation (B.1-9) can be rewritten as:

$$U = \alpha \int_{x=0}^a \int_{y=0}^b \{ [DHX]^T [U] [HY] [DHY]^T [U] [HY] + 2\mu [DHY]^T [U] [HY] [HX]^T [V] [DHY] + [HX]^T [U] [DHY] [HX]^T [V] [DHY] \} + \frac{(1-\mu)}{2} \{ [HX]^T [U] [DHY] [HX]^T [U] [DHY] + 2\mu [HX]^T [U] [DHY] [DHY]^T [V] [HY] + [DHY]^T [V] [HY] [DHY]^T [V] [HY] \} dx dy \quad (B.1-10)$$

where

$$\alpha = \frac{Eh}{2(1-\mu^2)}$$

The displacement vector is given by:

$$\{\delta\}^T = [u_{11} \ u,_{x11} \ u,_{y11} ; u_{12} \ u,_{x12} \ u,_{y12} ; u_{22} \ u,_{x22} \ u,_{y22} ; u_{21} \ u,_{x21} \ u,_{y21} ; v_{11} \ v,_{x11} \ v,_{y11} ; v_{12} \ v,_{x12} \ v,_{y12} ; v_{22} \ v,_{x22} \ v,_{y22} ; v_{21} \ v,_{x21} \ v,_{y21}]$$

The stiffness matrix terms are given by:

$$K_{ij} = \frac{\partial^2 U}{\partial \delta_i \partial \delta_j} \quad (B.1-11)$$

Kinetic energy of the element during a motion:

$$T = \frac{\rho}{2} \int_{x=0}^a \int_{y=0}^b (\dot{u}^2 + \dot{v}^2) dx dy \quad (B.1-12)$$

where ρ is the mass density and the dots over u and v represent the first order time derivatives. Equation (B.1-12) is rewritten as:

$$T = \frac{\rho}{2} \int_{x=0}^a \int_{y=0}^b \{ [HX]^T [\dot{U}] [HY] [HX]^T [\dot{U}] [HY] + [HX]^T [\dot{V}] [HY] [HX]^T [\dot{V}] [HY] \} dx dy \quad (B.1-13)$$

Mass matrix terms are given by:

$$M_{ij} = \frac{\partial^2 T}{\partial \delta_i \partial \delta_j} \quad (B.1-14)$$

Define the following matrices:

$$HX_{ij} = XA_{ij} \quad x^{MX}_{ji}$$

where

$$[XA] = \begin{bmatrix} 1 & 0 & \frac{-3}{a^2} & \frac{2}{a^3} \\ 0 & 1 & \frac{-2}{a} & \frac{1}{a^2} \\ 0 & 0 & \frac{3}{a^2} & \frac{-2}{a^3} \\ 0 & 0 & -\frac{1}{a} & \frac{1}{a^2} \end{bmatrix}$$

and

$$[MX] = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix};$$

$$DHX_{ij} = DX_{ij} x^{MD_{ij}}$$

where

$$[DX] = \begin{bmatrix} 0 & 0 & \frac{-6}{a^2} & \frac{6}{a^3} \\ 0 & 1 & \frac{-4}{a} & \frac{3}{a^2} \\ 0 & 0 & \frac{6}{a^2} & \frac{-6}{a^3} \\ 0 & 0 & \frac{-2}{a} & \frac{3}{a^2} \end{bmatrix}$$

and

$$[MD] = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} ;$$

$$HY_{ij} = YB_{ij} y^{NY_{ij}}$$

where

$$[YB] = \begin{bmatrix} 1 & 0 & \frac{-3}{b^2} & \frac{2}{b^3} \\ 0 & 1 & \frac{-2}{b} & \frac{1}{b^2} \\ 0 & 0 & \frac{3}{b^2} & \frac{-2}{b^3} \\ 0 & 0 & \frac{1}{b^2} & \frac{1}{b} \end{bmatrix}$$

and

$$[NY] = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} ;$$

$$DHY_{ij} = DY_{ij} y^{ND_{ij}}$$

where

$$[DY] = \begin{bmatrix} 0 & 0 & \frac{-6}{b^2} & \frac{6}{b^3} \\ 0 & 1 & \frac{-4}{b} & \frac{3}{b^2} \\ 0 & 0 & \frac{6}{b^2} & \frac{-6}{b^3} \\ 0 & 0 & \frac{-2}{b} & \frac{3}{b^2} \end{bmatrix}$$

and

$$[ND] = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Define a matrix $[Q]$ such that:

$$Q_{ij} = \int_{x=0}^a \int_{y=0}^b x^{i-1} y^{j-1} dx dy .$$

B.2 Flow-chart for the Generation of the Stiffness and Mass Matrices.

DATA: E - Modulus of elasticity

PSSN - Poisson ratio

RHØ - mass density

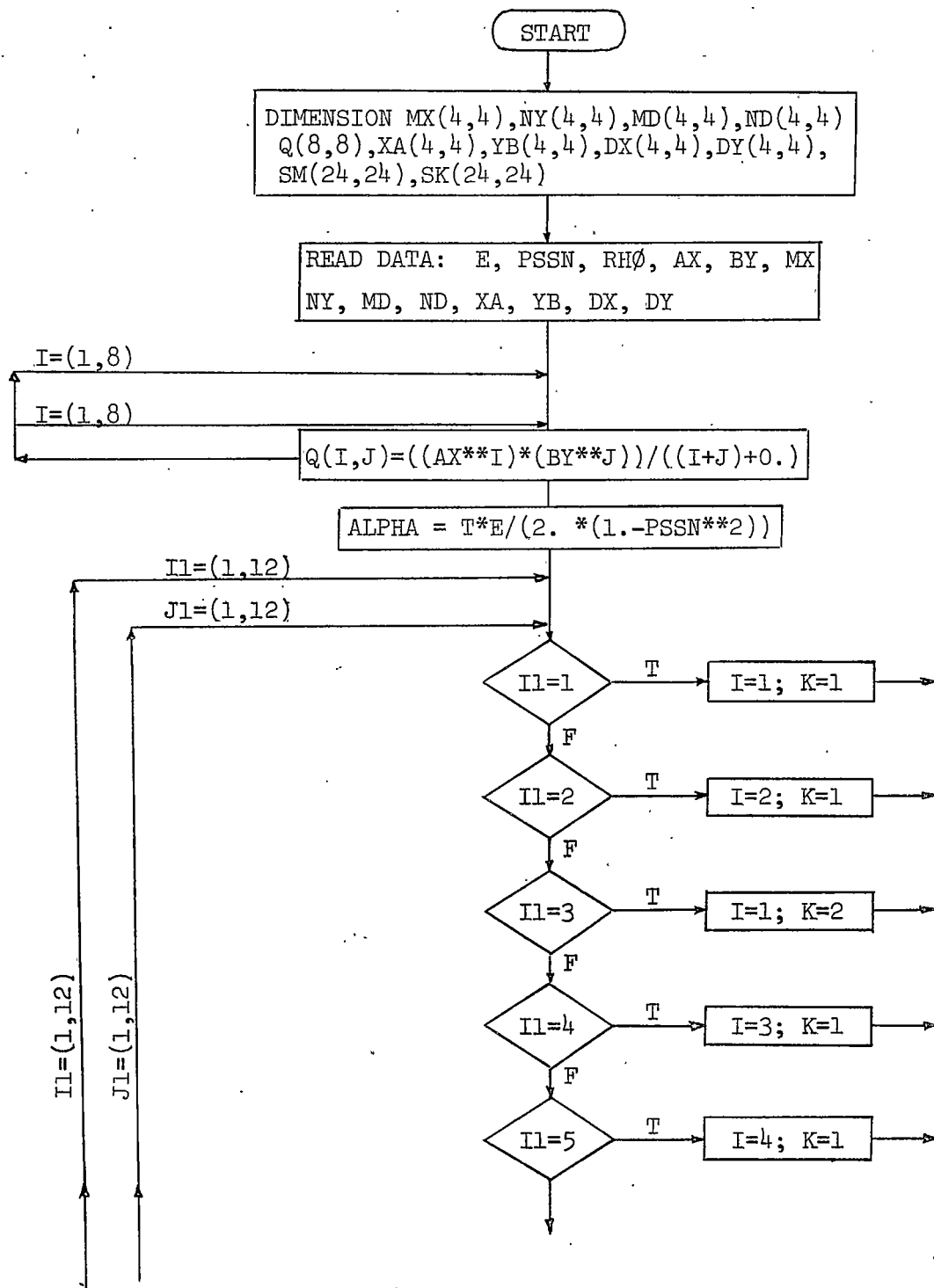
AX,BY - Lengths of the sides of the element

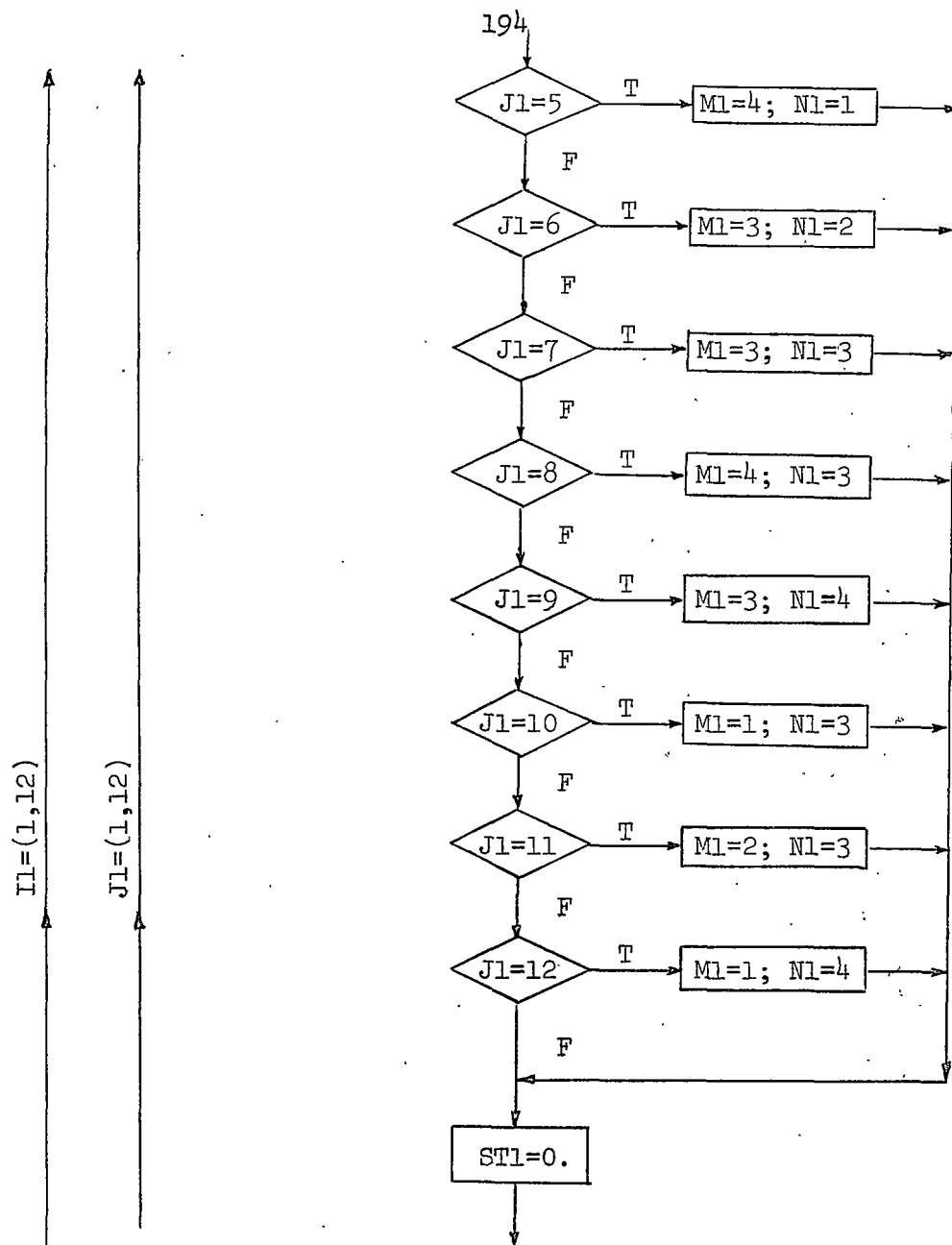
T - Thickness of the element

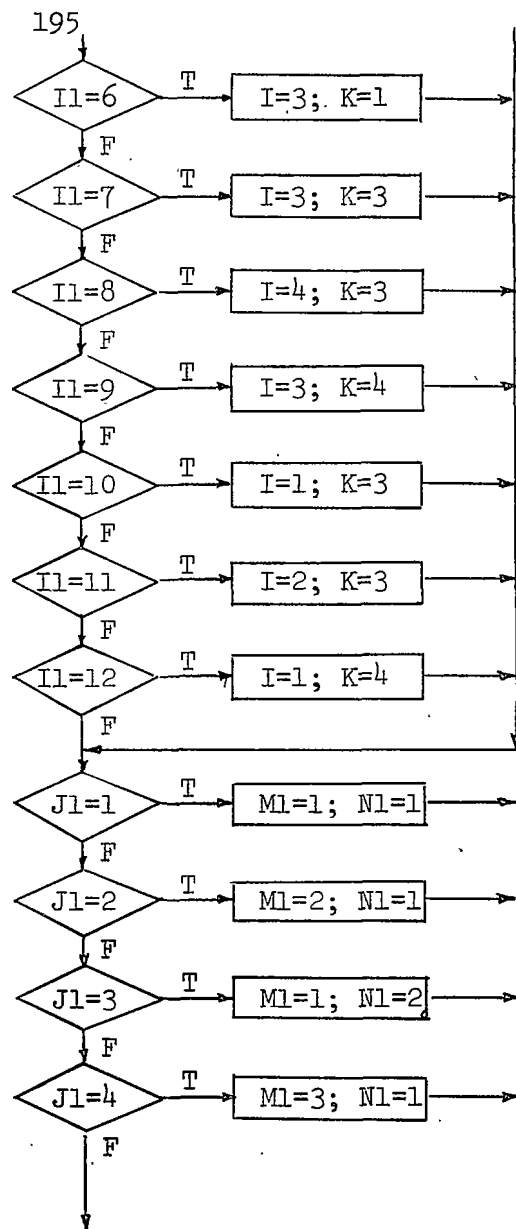
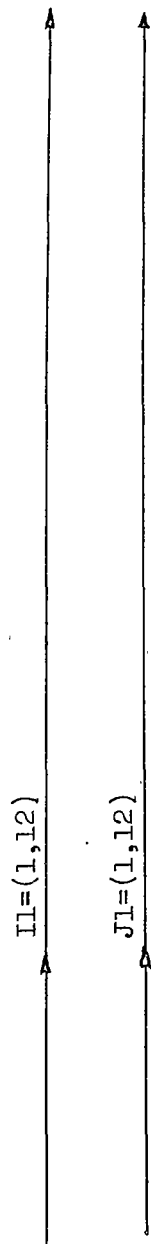
[SK] - Stiffness matrix

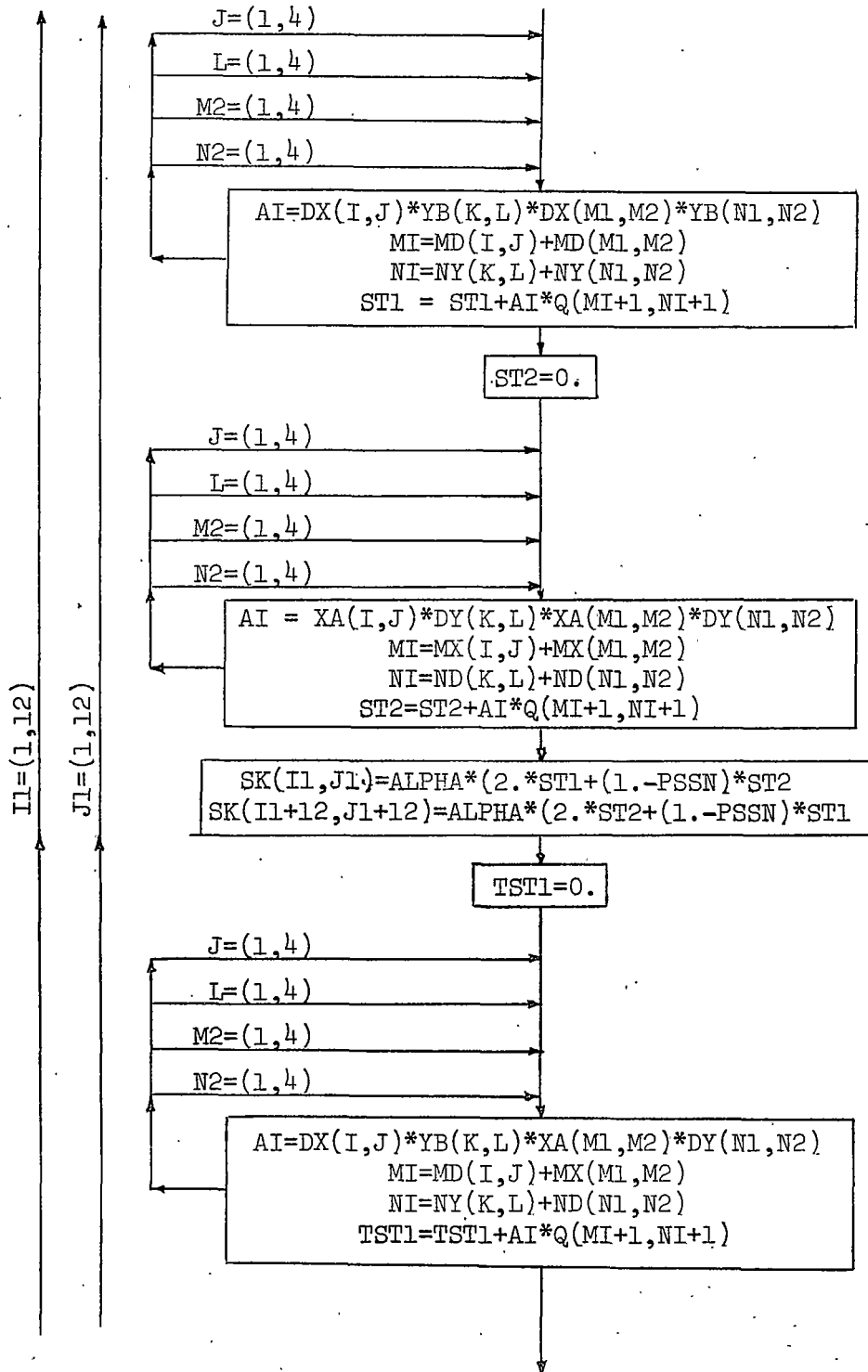
[SM] - Mass matrix

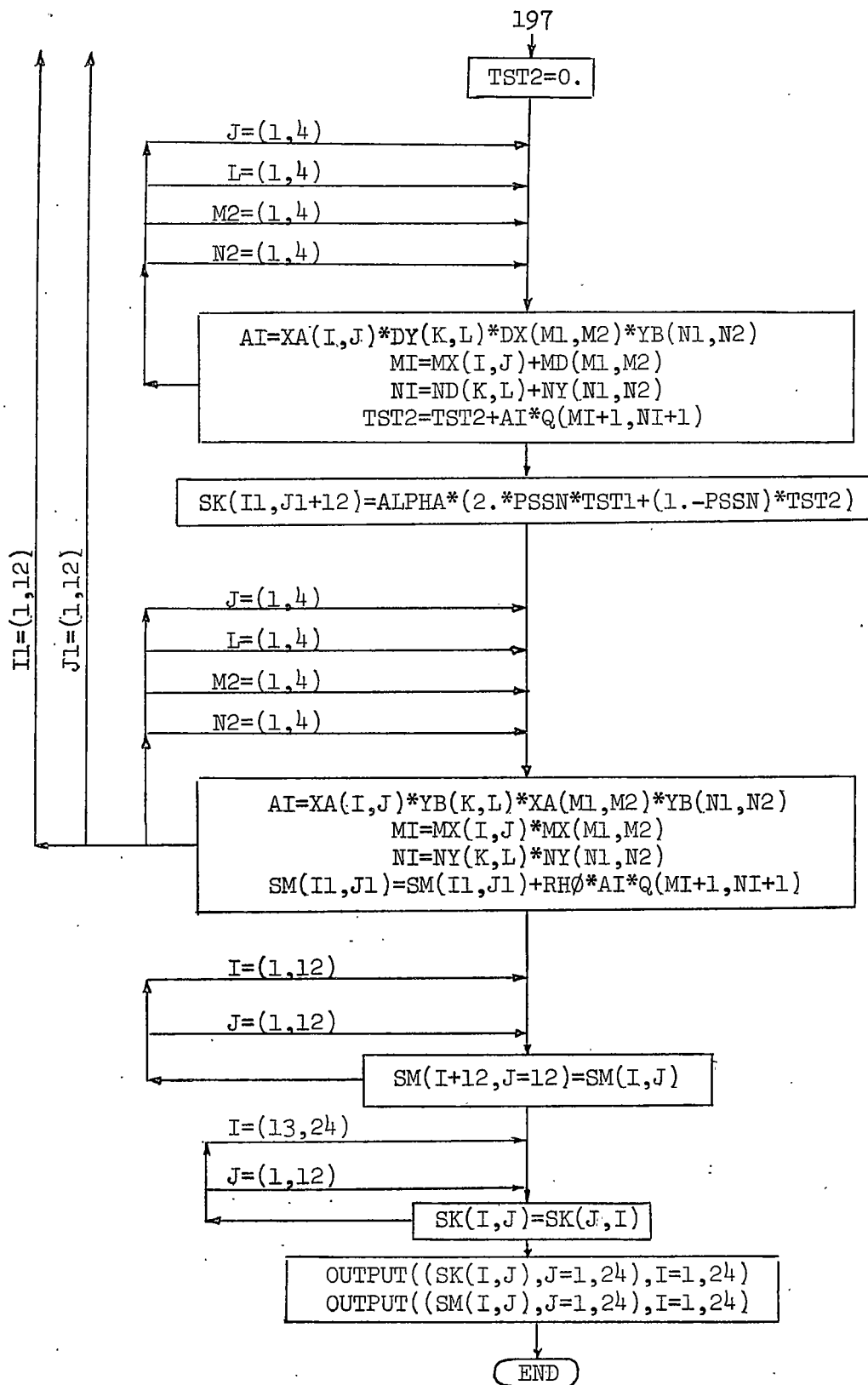
[MX],[NY],[MD],[ND],[Q],[XA],[YB],[DX],[DY] - matrices as
defined above.











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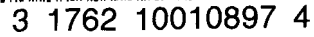
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