Basic heat transfer using EAI TR-10 electronic analog computer
by Michael Epifanio Hilario

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE in Mechanical Engineering
Montana State University

Abstract:
The electronic analog computer is a very useful problem solving tool to the engineering student. The Montana State University Engineering Department has two analog computers for the use of the faculty and students. These are the EAI (Electronic Associates, Incorporated) TR - 48 and the smaller EAI TR - 10. The Mechanical Engineering Department has 4 TR -10 patchboards and the necessary wires and resistors to set up computer problems. Heat transfer is one field of engineering, among others, in which this department will use the analog computer for problem analysis at the undergraduate and graduate level.

An introductory course in analog computer programming should enable the student to use the computer for various problem solutions even without a strong background in electronics or servomechanisms. This thesis is presented as a guide to the student wishing to utilize the analog computer for heat transfer work. It demonstrates general problem analysis and some necessary procedures for obtaining solutions for some of the basic heat transfer equations.

The literature search in preparation of this thesis left the impression that there has not been much work in heat transfer using the analog computer at the undergraduate level. Rather, it appears that analog-simulation systems; such as the resistance analog, conductive-solid analog, and conductive-liquid analog, have been used more often for analysis.

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Abstract

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CHAPTER I

INTRODUCTION

There are two types of electronic computers in general use in industry for purposes of research, data processing, control systems, to name a few. These are the analog computer and the digital computer. The digital computer is able to store problem data and instructions. Through various related units, the digital finally supplies the results of computations. Generally, the digital computer performs only addition and there is no relationship between the computing time and the problem variables. On the other hand, analog computers do not count in terms of numbers, but use continuously variable data. Problem solution by the analog computer is accomplished by analogy; the computer is programmed so that its circuit equations have the same mathematical form as the equations of the problem. Thus voltages represent various physical quantities such as distance, temperature, force, displacement, and so on. The voltage magnitudes are related to the physical quantity magnitudes by a scale factor. The analog computer performs its operations in a continuous process with respect to time; therefore, there is a relationship between the actual computer run time and the problem time or some other independent variable.

Analog computers make it possible to simulate entire system problems, components, or subsystems. It may be that parameters of a system would be difficult, expensive, or time consuming to actually change in real equipment. However, if simulated on an analog computer, paramenters could be varied by simply using a dial in a well designed analog circuit. Process effects that take hours or even days can be simulated on an analog computer so that data can be obtained in a matter of seconds.

Although a knowledge of electronics is very valuable to a person utilizing the analog computer, it is not necessary. An introductory course in circuit analysis will suffice. The engineering student will find that using an analog computer is fairly easy and very applicable to engineering problems. For example, mechanical vibration differential equations have time as the independent variable and the analog computer produces solutions to differential equations with the independent variable being time. This then involves the direct solution of the equations. Many heat transfer problems also have time as the independent variable and the analysis and solution of these problems is fairly straightforward.

Any discussion of heat transfer solutions would not be complete without including the solution of partial differential equations. The mathematical description of any continuous media usually involves partial differential equations, i.e., differential equations having derivatives with respect to more than one independent variable. Since the electronic analog computer can integrate with respect to only one variable, namely time, it can only be used to solve ordinary differential equations. Thus it is necessary to convert a partial differential equation to one or more ordinary differential equations in order to solve it on the analog computer. There are two popular techniques for doing this: (1) the method of separation of variables, leading to an eigenvalue problem; (2) the finite difference method wherein derivatives with respect to all variables but one are replaced by finite-difference approximations. The first method will only be discussed and the second method will be demonstrated by analysis and analog computer solution.

The basis of analog computer operation is presented in the Appendix. No attempt is made to prove or explain any of the computer operations on the basis of electronic theory. The reader is instead referred to one of the texts listed in the literature consulted. The operation of the electronic analog computer itself is presented in Chapter 2. The EAI TR-10 analog computer was used exclusively. This is a relatively small computer and thus is somewhat limited in the size of problems which can be solved. The limiting factor is the number of operational amplifiers available for the problem solution.

The answers from the computer are obtained in the form of observations or recordings of continuous time variables. If the problem is static, that is, if the answers have one and only one value when the solution has been reached, all the variables will have ceased to change and their values can be read with a simple voltmeter. In most cases, however, the problem solution involves variables that change continuously with time and are recorded continuously in time. The problem solutions presented herein were recorded on a two-dimensional X -Y plot on paper measuring 10 inches x 15 inches then re-plotted on standard 8 1/2 inch x 11 inch paper. Regarding the solution of an equation or a set of equations, it should be understood that very seldom is just one numerical answer sought. Rather a better understanding of the situation or system being studied is more important.

A logical question arises as to when to use a digital computer and when to use an analog computer. Basically, the following statements concerning when to use a digital computer could be stated.

1. Problems involving large amounts of tabulated data in discrete-number form.

2. Problems with precise input data requiring precise solutions.

Similarly, the following statements apply concerning when to use an electronic analog computer.

1. Solution of simultaneous differential equations.

2. When an analogy of a mechanical, or other type of system is required. That is, a mathematical model will represent the physical problem, thus solution of the equations will be analogous to the physical system.

CHAPTER 2

EAI TR - 10 ELECTRONIC ANALOG COMPUTER

The TR - 10 patchboard contains all of the computer component connections which are explained in the Appendix. The patchboard is removable from the computer so that full use can be made of the computer. Problems can be "patched" or wired before putting the patchboard on the computer; then potentiometer settings can be made when the operator is ready for the problem solution.

The control panel of the TR-10 contains the following control switches.

Mode Control Switch; positions are RESET, HOLD, and OPERATE.

Reset — The programmed problem is at the conditons corresponding to time zero.

Hold — This position will stop and hold the problem solution instantly for observation.

Operate — This will start the problem solution. Operation can start from the reset or hold position.

Meter Mode Selector Switch; positions are POT BUS, NULL, V.M., AMP, BAL.

Pot Bus — This position is used to set values on pots. This is a precision potentiometer which is bridged so that when the pot which is being set reaches the same value as the precision potentiometer, the reading on the meter will show a null indication.

Null — A position used for setting potentiometers other than the ones which are found on the Montana State University analog computer.

V.M. This connects the meter for use as a voltimeter. The V.M. switch was not used for any problems.

Amp This connects the meter as a voltmeter to the amplifier selector switch so that the output voltage of each amplifier can be read on the meter. This is used as a check to determine what amplifier is overloaded (> 10 volts) when there is an overload indication.

Bal Used to balance, to zero, the stabilizer outputs of all amplifiers.

The important switch positions for problem solution are the pot bus and amp settings. The pot bus is necessary in order to obtain accurate pot settings. The amp setting enables the operator to read the voltage output of each amplifier. The computer will hum distinctly audibly when there is an amplifier which is overloaded.

This is a brief presentation of the more important control components of the TR-10 analog computer. A more extensive explanation of controls can be found in the operator's handbook which is always located in the computer room. The operator's handbook will also illustrate how to make the patch-board connections required from the computer diagram. The connections are very easy to make; the handbook being easy to follow and also, the operator soon learns the basic connections. The most important aspect of using the analog computer is to obtain the correct computer diagram.

CHAPTER 3

STEADY-STATE HEAT CONDUCTION: UNIFORM CROSS-SECTION FIN

Consider a fin of uniform circular cross-section, Figure 1, whose base is attached to a plane section which is maintained at the constant temperature T_b. The fin exchanges heat along its surface with a fluid at a bulk temperature T_a. A heat balance for a differential element of the fin must consider heat flow into the element by conduction, heat generated by the element, heat flow out of the element by conduction, and the heat capacity of the element. In steady-state heat transfer the heat capacity of the element is unimportant and if the fin is highly conducting the temperature along the fin is a function of the axial distance alone. Now a heat balance for a differential element in such a fin will be considered in order to derive an equation for the temperature distribution along its length. Heat flows by conduction into one face of the element while heat flows out of the opposite face by conduction and from the surface by convection. In the absence of energy generation within the element and under steady-state conditions:

Rate of heat flow by conduction into element at x	=	Rate of heat flow by conduction out of element at (x + dx)	+	Rate of heat flow by convection from surface

$$-kA\frac{dT_x}{dx} = \left[-kA\frac{dT_x}{dx} + \frac{d}{dx}\left(-kA\frac{dT_x}{dx}\right)dx + \ldots\right] + hA_s\ (T_x - T_a)$$

$$-kA\frac{dT_x}{dx} = -\ kA\frac{dT_x}{dx} - kA\frac{d^2T_x}{dx^2}\,dx + h(Pdx)\ (T_x - T_a)$$

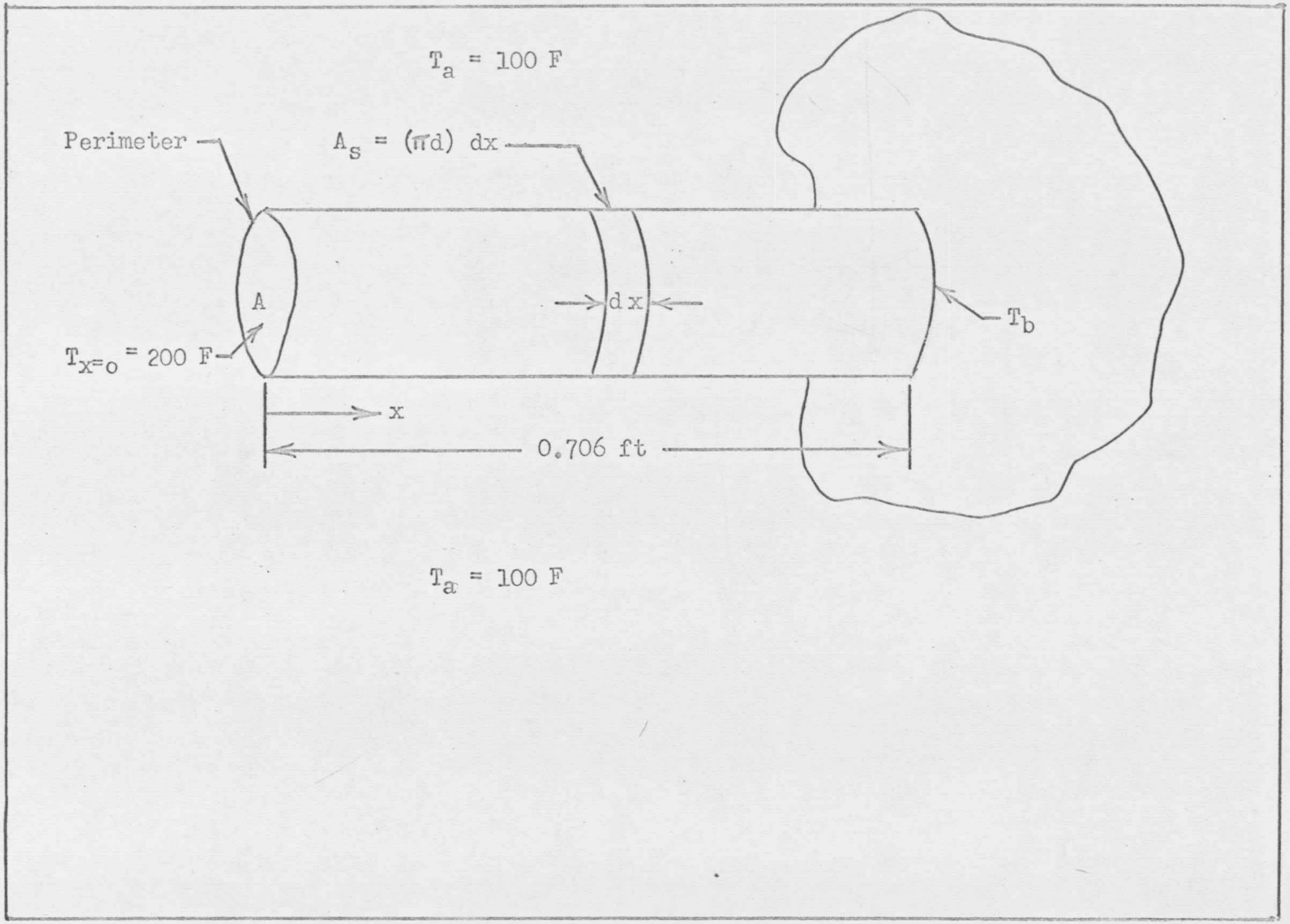


Figure 1. Uniform Cross-Section Rod Fin

$$kA \frac{d^2T_x}{dx^2} = hP(T_x - T_a)$$

(1) $$\frac{d^2T_x}{dx^2} = \frac{hP}{kA}(T_x - T_a)$$

Where q = Rate of heat flow

k = Thermal conductivity

h = Heat transfer coefficient

A_s = Surface area

A = Area of fin cross-section

P = Perimeter

Let T = $(T_x - T_a)$

T_a = Constant

Therefore

$$\frac{d^2T_x}{dx^2} = \frac{d^2T}{dx^2}$$

and eq. 1 can be written

(2) $$\frac{d^2T}{dx^2} = \frac{hP}{kA} T$$

Independent Variable Other Than Time

The independent variable of eq. 2 is distance rather than time. This does not mean that is is impossible to solve it on the analog computer, it does mean a transformation between time and distance must be defined. The degree of the correspondence between the distance x and the time t will depend on the units in which the equation is written. If distance x is in inches, a possible correspondence may be 1 inch = 1 second or if x is in feet, the correspondence may be 1 foot = 1 second, and so forth. Considering eq. 2, the units of p, h, k, and A must be consistent with the units of x. The relation of time and distance for the solution of the rod fin heat transfer equation, eq. 2, can be written:

$$x \text{ (ft)} = a \cdot t \text{ (sec)}$$

$$\frac{dx}{dt} = a$$

also

$$x = f(t)$$

$$T = g(x)$$

By using the chain rule twice and substituting

$$\frac{dT}{dt} = \frac{dx}{dt} \cdot \frac{dT}{dx}$$

$$\frac{dT}{dt} = a \frac{dT}{dx}$$

$$\frac{d^2T}{dt^2} = a\left[\frac{d}{dx}\left(\frac{dT}{dx}\right)\right] \quad \frac{dx}{dt}$$

$$\frac{d^2T}{dt^2} = a \cdot \frac{d^2T}{dx^2} \cdot \frac{dx}{dt}$$

$$\frac{d^2T}{dt^2} = a^2 \frac{d^2T}{dx^2}$$

and

$$\frac{d^2T}{dx^2} = \frac{1}{a^2} \frac{d^2T}{dt^2} \tag{3}$$

<u>Analog Computer Solution of a Steady-State Heat Conduction Problem</u>

Analysis of the uniform cross-section rod fin will continue by considering an example problem.

Problem: An engine is air-cooled by circular-rod fins. The surrounding ambient air temperature is 100 F. The temperature of the end of the fin that is entirely in the air is 200F. Determine the temperature distribution along the length of the fin. The problem is illustrated in Figure 1. Let

$h = 5 \frac{Btu}{hr\ ft^2\ F}$ $\qquad$ d = 1 inch diameter

$L = 0.706$ ft $\qquad$ $k = 25 \frac{Btu}{hr\ ft\ F}$

Equation 2 has been found to be the equation describing the heat flow for the rod fin. Let

$$m = \frac{Ph}{kA} = \frac{(\pi d)\ h}{K(\pi/4\ d^2)} = \frac{(\pi/12\ ft)\left(5\ \frac{BTU}{hr \cdot FT^2 \cdot F}\right)}{\left(25\ \frac{BTU}{hR \cdot FT \cdot F}\right)\left(\frac{\pi}{4} \cdot \frac{1}{144}\ ft^2\right)} = 9.6\ fT^{2^{-1}}$$

Then

$$\frac{d^2T}{dx^2} = mT = 9.6T\left(\frac{F}{fT^2}\right) \tag{4}$$

A substitution must be made since the equation involves an independent variable other than time. The method is explained previously. Equation 3 will be substituted into Eq. 4.

$$\frac{d^2T}{dx^2} = mT \tag{4}$$

$$\frac{1}{a^2}\frac{d^2T}{dt^2} = mT$$

or

$$\frac{d^2T}{dt^2} = a^2mT \tag{5}$$

In order to obtain a numerical value for a, t is arbitrarily selected as 1 second of problem time.

$$a = \frac{x}{t} = \frac{0.706\ ft}{1\ second}$$

$$a^2 = (0.706)^2 = 0.50$$

Using dot notation for the derivatives so that $\frac{d^2T}{dt^2}$ and $\frac{dT}{dt}$ will be $\ddot{T}$ and and $\dot{T}$ respectively results in:

(6) $\ddot{T} = a^2mT = 0.50\ (9.6)\ T = 4.80\ T$

This then is the equation which must be solved on the analog computer. The only boundary condition is:

$$T_{x=0} = (200 - 100)F = 100F$$

Three steps will be followed before wiring the computer patchboard. These are: (1) magnitude scaling, (2) time scaling, (3) draw a complete computer diagram. From knowledge of the problem, it is possible to estimate maximum temperatures and problem duration. It should be kept in mind that should an estimate be wrong, it is an easy thing to change the scale factors by changing the pot settings. For this problem, it will be assumed that T_{max} = 1000F. Therefore, on a "per volt" basis, the scale factor will be:

$$10v = 1000F$$

and the scaled variable is:

$$\left[0.01\ \frac{v}{F}\right] \qquad 1\ \text{volt} = 100F$$

The computer time constant is:

$$B = \frac{t_c}{t}$$

where t_c = computer time

t = actual problem time

For the rod fin problem, a satisfactory value of computer run time is 10 seconds, so that B = 10.

In this problem, T is a function of x so that there is no problem time involved in this steady-state condition, but a value of B must be selected to give a satisfactory computer time.

The computer diagram is shown in Figure 2. The initial condition, or known boundary value, is at the exposed end of the rod fin where $x = 0$. The computer diagram is a solution to Equation 6. The output Y is shown on the computer diagram and represents T. A voltage must be generated to represent distance x or time t on the X axis. The voltage is generated by integrating a voltage as is shown in Figure 2. A voltage of 10 volts will be generated in 10 seconds. An arm scale setting on the X - Y plotter of 1 volt/inch will generate 10 inches on the plotting paper in 10 seconds.

In order to show the computer solution (volts and time) and the conversion to problem solution (temperature and distance), both graphs are shown as Figure 3 and Figure 4 respectively. The conversion factors used:

1 volt = 100 F (per volt basis)

1 second of problem time = 0.706 feet

10 seconds of computer time = 1 second of problem time

The computer time was 10 seconds; therefore, the problem time was 1 second. This 1 second represented 0.706 feet of rod fin length.

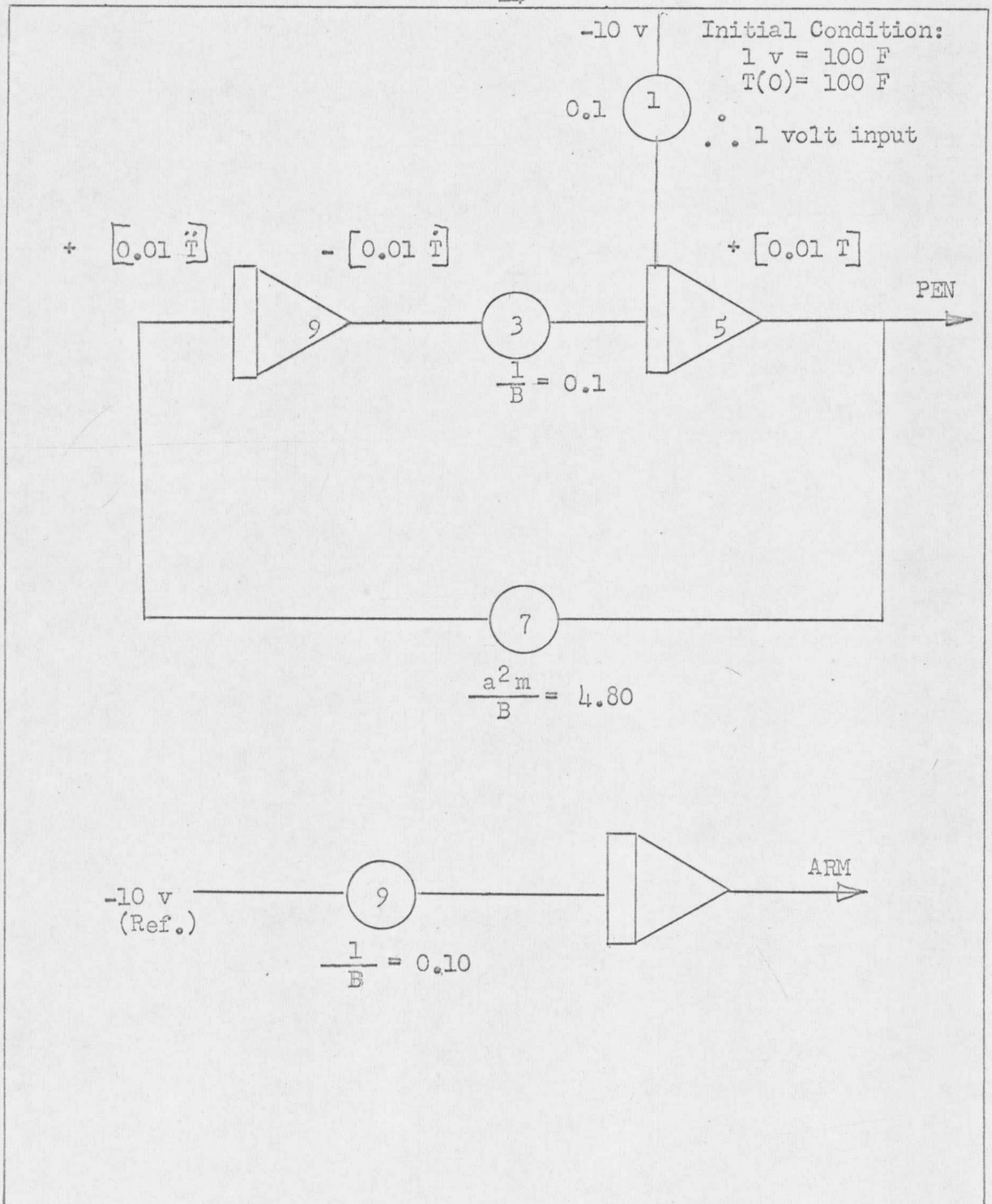


Figure 2. Analog Computer Diagram for Rod Fin Problem

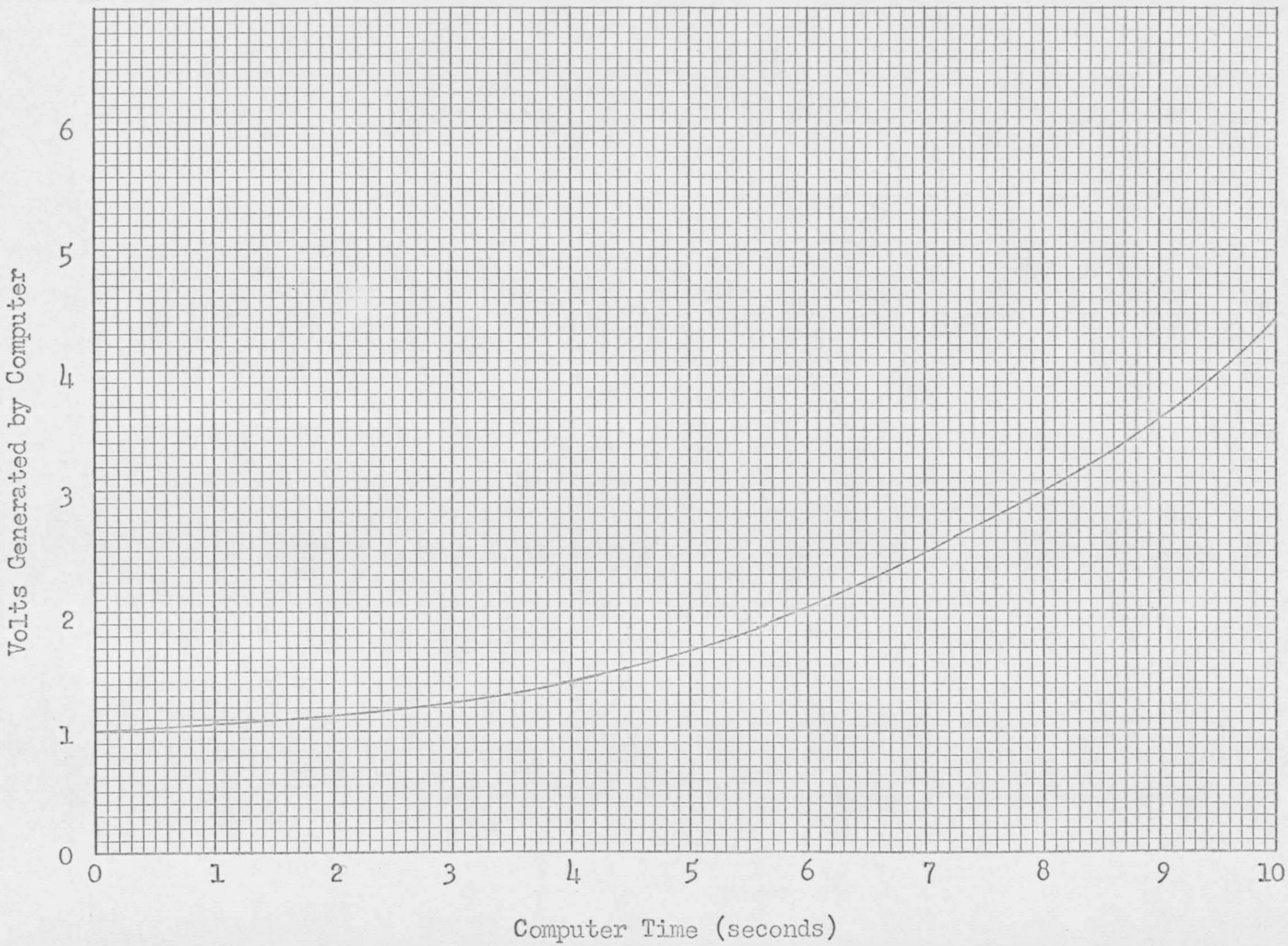
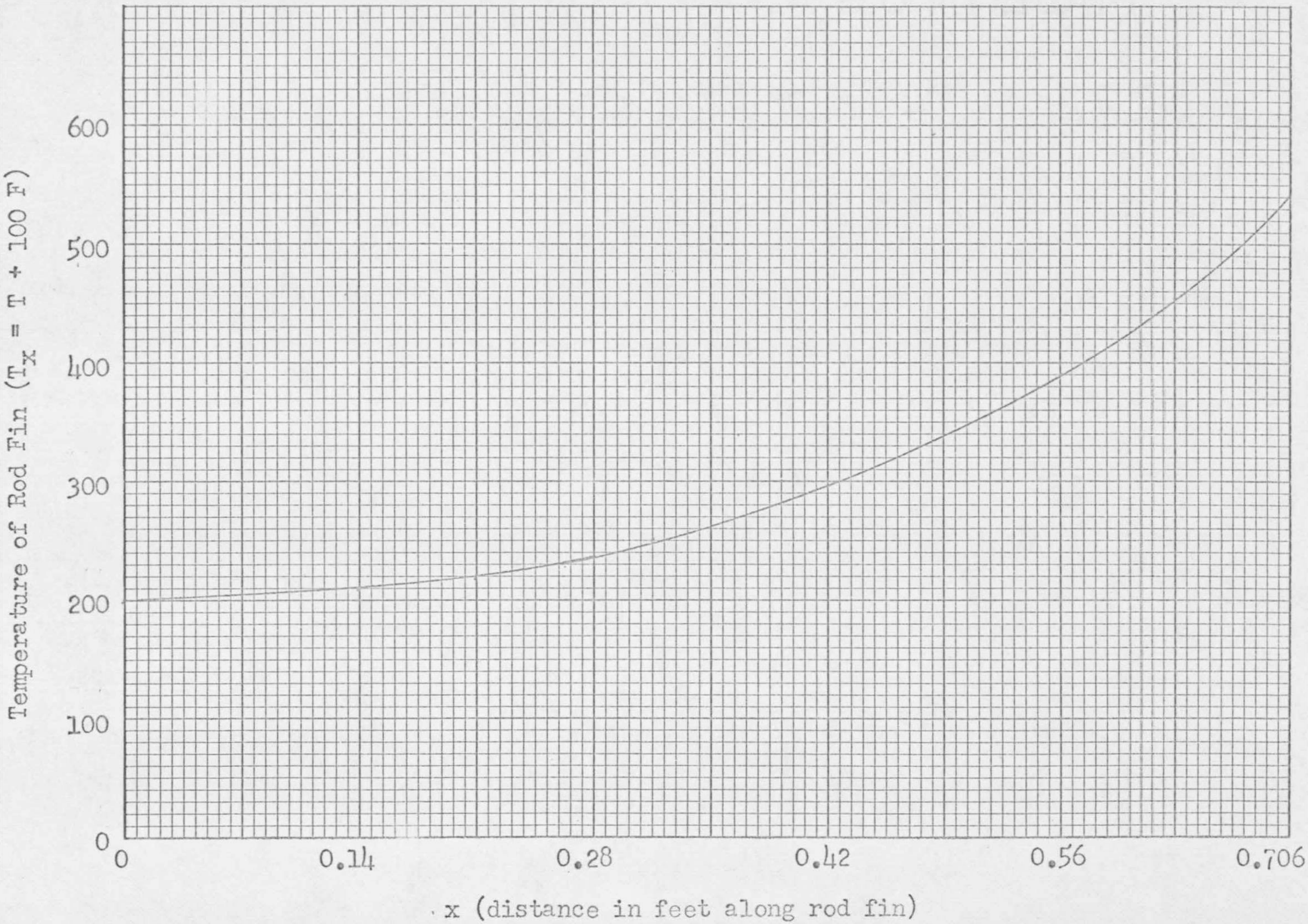


Figure 3. Computer Solution of Rod Fin Problem



$x = a \cdot t, \quad t = t_c/B \; \therefore \; x = (0.0706) \cdot t_c$

Figure 4. Solution of Rod Fin Problem

CHAPTER 4

HEAT TRANSFER BY THE TECHNIQUE OF LUMPED PARAMETERS

This is probably the simplest technique available for the treatment of the transient exchange of thermal energy between a solid and the fluid in which it is immersed. In this approach the temperature gradients on the interior of the solid are neglected and the solid is assumed to be at a uniform temperature. Such a condition is approached when a solid of high thermal diffusivity transfers heat to a fluid through a film that presents a high resistance to heat flow. This situation is sometimes characterized as one in which the film resistance is controlling. Under these conditions, the rate at which the internal energy of the body decreases may be equated to the rate at which energy is transferred across the boundary of the solid. In Figure 5 a detailed derivation of this type is presented as an example. Generally the equation which results from this technique shows the rate of cooling of the solid is proportional to the difference between the mean temperature of the solid and the bulk temperature of the surrounding fluid:

$$\frac{dT_1}{dt} = -K(T_1 - T_2) \tag{7}$$

where T_1 = mean temperature of the conducting solid

T_2 = bulk temperature of the ambient fluid

K = a constant of proportionality.

Introducing the excess temperature, $T = T_1 - T_2$, the solution to this equation is:

$$T(t) = T(0)\, e^{-Kt} \tag{8}$$

The transient behavior of the mean temperature of the solid will depend on the constant K which in turn is related to such geometric factors as the surface area and volume and to such heat transfer parameters as the film coefficient and the specific heat of the solid.

A problem in which Equation 7 will apply may be stated in order to illustrate the computer solution.

Problem: A metal-casting of volume V and surface area A_s, initially at 900 F is suddenly placed in an atmosphere of 0F. Using K values of 0.3 and 0.7, let us determine the temperature of the casting as a function of time.

Since the differential equation is already a function of time, Equation 7 can be scaled and a computer diagram drawn. The maximum values which can be assumed for magnitude scaling are

$$T_{max} = 900 \text{ F}$$

$$\dot{T}_{max} = 300 \text{ F/second}$$

Therefore, the scaled variables on the basis that 10 volts = 900 F will be

$$\left[\frac{T}{900}\right], \left[\frac{\dot{T}}{300}\right]$$

The temperature drop is very rapid with the K values which are used, therefore, 2 seconds of problem time will be considered. The computer time can be extended to 10 seconds by using a time scale factor.

$$t_c = Bt$$

$$B = \frac{t_c}{t} = \frac{10 \text{ seconds}}{2 \text{ seconds}} = 5$$

The input to the integrator must, however, be divided by 5. The initial condition is T(0) = 900F. Since 10 volts equal 900F, 10 volts input to the integrator is used. The equation for the computer diagram can be found as follows:

$$\mathring{T} = -K\ T$$

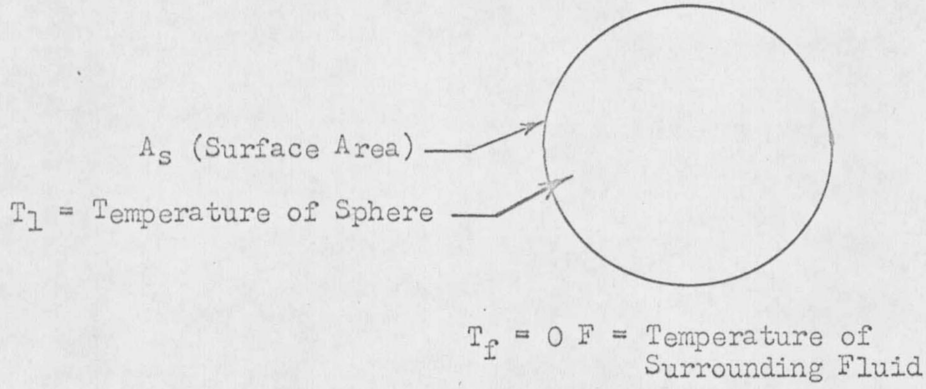
$$300\left[\frac{\mathring{T}}{300}\right] = -K\ (900)\left[\frac{T}{900}\right]$$

$$\left[\frac{\mathring{T}}{300}\right] = -3K\left[\frac{T}{900}\right]$$

(9) For K = 0.7 $\left[\frac{\mathring{T}}{300}\right] = -2.1\left[\frac{T}{900}\right]$

(10) For K = 0.3 $\left[\frac{\mathring{T}}{300}\right] = -0.9\left[\frac{T}{900}\right]$

The complete computer diagram is shown in Figure 6, along with the computer diagram which will generate the time function. The computer solution is shown in Figure 7.



$$q_{mat} = q_{surf}$$

$$-c\rho V\left(\frac{dT_1}{dt}\right) = hA_s(T_1 - T_f)$$

Let $\quad T = (T_1 - T_f)$

$\quad T_f = \text{Constant}$

and $\quad \frac{dT}{dt} = \frac{dT_1}{dt}$

Then

$$-c\rho V\left(\frac{dT}{dt}\right) = hA_sT$$

$$\frac{dT}{dt} = \frac{hA_s}{c\rho V}T = -KT$$

Where:

$K = \frac{hA_s}{c\rho V}$ $\qquad \rho$ = Density

h = Heat Transfer Coefficient $\qquad A_s$ = Surface area

c = Specific heat $\qquad$ V = Volume

Figure 5. One-Dimensional Heat Transfer Across a Surface -- Cooling of a Sphere

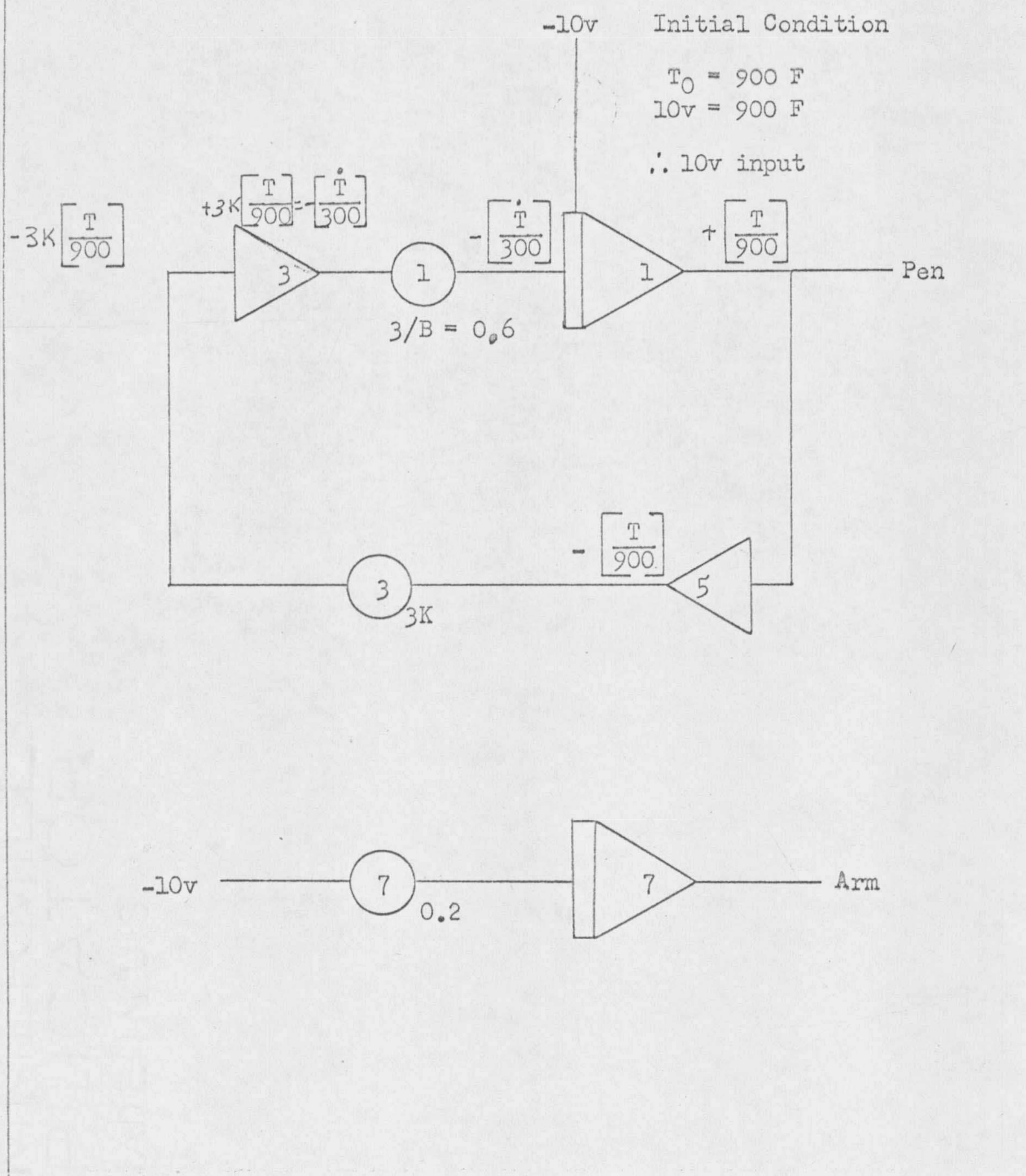


Figure 6. Analog Computer Diagram for Cooling Sphere Problem

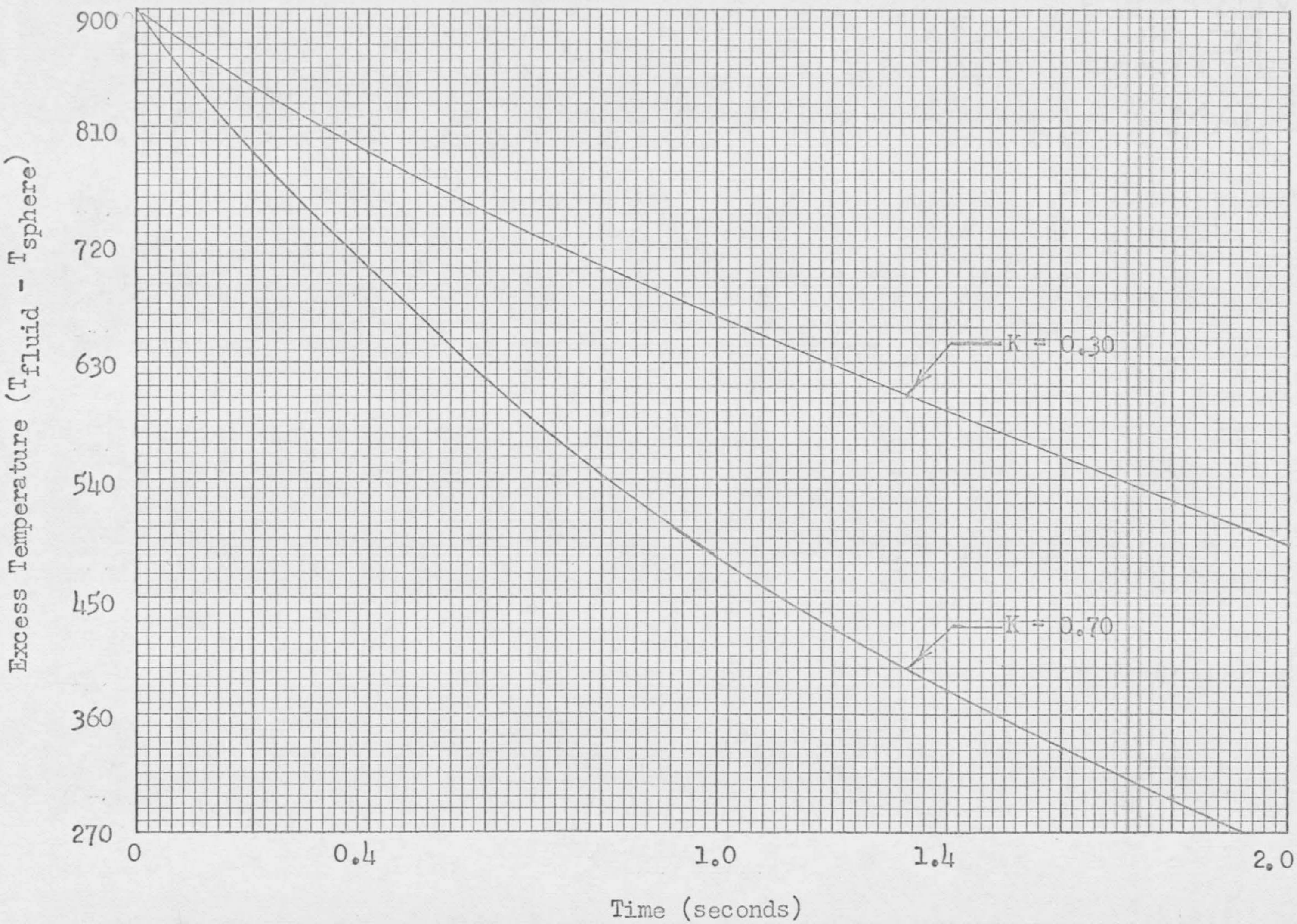


Figure 7. Computer Solution of Cooling Sphere Problem

CHAPTER 5

TWO TECHNIQUES FOR THE SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

Many heat transfer problems require the solution of partial differential equations. The solution must satisfy the partial differential equations and the pertinent boundary conditions defining the problem. The electronic analog computer can solve partial differential equations only by converting them to ordinary differential equations. Generally, there are two methods of accomplishing this conversion. They are; (1) separation of variables leading to an eigenvalue problem, and (2) replacing the partial derivatives by finite differences.

Of the two approaches, the finite difference method is most often used. The other method is not always applicable since the variables sometimes cannot be separated; this is especially true with non-linear equations. Also, the summation of all of the eigenfunction products is involved and this requires many analog computer components.

For the purpose of obtaining the one-dimensional heat conduction equation, consider the slab of uniform cross section which is shown in Figure 8a. The slab is insulated at all surfaces except the ends so that heat flow is in the x direction alone. If a heat balance is written for the differential element of volume A dx, the one-dimensional heat conduction equation may be derived. The total heat flowing into the elemental volume either increases the average temperature of the volume or flows through the element:

$$-kA \left.\frac{\partial T}{\partial x}\right|_{x} = c\rho\, dv \left(\frac{\partial T}{\partial t}\right) - kA \left.\frac{\partial T}{\partial x}\right|_{x+dx}$$

$$-kA\frac{\partial T}{\partial x}\Big|_{x} + kA\frac{\partial T}{\partial x}\Big|_{x+\Delta x} = c\rho\ (A\,dx)\frac{\partial T}{\partial t}$$

$$-KA\frac{\partial T}{\partial x} + \left[kA\frac{\partial T}{\partial x} + \frac{d}{dx}\left(kA\frac{\partial T}{\partial x}\right)dx\right] = c\rho\,(A dx)\frac{\partial T}{\partial t}$$

$$-kA\frac{\partial T}{\partial x} + kA\frac{\partial T}{\partial x} + kA\frac{\partial^2 T}{\partial x^2}dx = c\rho\,(A dx)\frac{\partial T}{\partial t}$$

$$kA\frac{\partial^2 T}{\partial x^2}dx = c\rho\ A dx\frac{\partial T}{\partial t}$$

$$kA\frac{\partial^2 T}{\partial x^2} = c\rho A\frac{\partial T}{\partial t}$$

or:

(11) $$\frac{\partial^2 T}{\partial x^2} = \frac{c\beta}{k}\frac{\partial T}{\partial t}$$ one-dimensional heat conduction equation

Finite-Difference Approximation

It can be seen that Equation 11 contains two independent variables--location and time. Therefore, an exact solution on the electronic analog computer is not possible, however, instead of computing temperature at all x values, the temperature T can be computed at certain stations along x, Figure 8b. This is possible by using the finite difference technique and will reduce Equation 11 to a system of first-order ordinary differential equations. When solved simultaneously, they will provide an approximate solution.

It must be pointed out that it is possible to divide the right side, $\partial T/\partial t$, into discrete intervals, Δt, however, since the analog computer integrates with respect to time, it is advantageous to divide distance x into discrete intervals and integrate with respect to time.

Central finite-differences will be used in transforming $\partial^2 T/\partial x^2$ into finite-difference form. Consider the temperature - distance curve of Figure 8b, with the Δx increments shown. First the respective slopes of the general curve at the stations (n + 1/2) and (n - 1/2).

$$\left.\frac{\partial T}{\partial x}\right|_{n+1/2} = \frac{T_{n+1} - T_n}{\Delta x}$$

$$\left.\frac{\partial T}{\partial x}\right|_{n-1/2} = \frac{T_n - T_{n-1}}{\Delta x}$$

Since $\frac{\partial^2 T}{\partial x^2}$ is the rate of change of slope, it follows that:

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_n = \frac{\left.\frac{\partial T}{\partial x}\right|_{n+1/2} - \left.\frac{\partial T}{\partial x}\right|_{n-1/2}}{\Delta x}$$

substituting

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_n = \frac{\frac{T_{n+1} - T_n}{\Delta x} - \frac{T_n - T_{n-1}}{\Delta x}}{\Delta x}$$

$$(12) \quad \left.\frac{\partial^2 T}{\partial x^2}\right|_n = \frac{T_{n+1} - 2T_n + T_{n-1}}{(\Delta x)^2}$$

Substituting Equation 12 into Equation 11 yields a general equation for the n^{th} station.

$$(13) \quad \left.\frac{dT}{dt}\right|_n = \frac{k}{c\rho} \left[\frac{T_{n+1} - 2T_n + T_{n-1}}{(\Delta x)^2}\right]$$

Now, the time derivative of T_n is a total derivative since x is fixed at ($n \cdot \Delta x$) for T_n. Equation 13 contains only one independent variable (time) and is in a form which can be solved on the analog computer. The equation represents an array of simultaneous equations; the size of the array is equal to the number of increments which may be arbitrarily selected along the x axis.

Separation of Variables

As an example of the solution of Equation 11, let us consider the problem of the slab of thickness ℓ when both surfaces are maintained at zero temperature and the initial timperature distribution is f(x). That is, the problem where

$$\frac{\partial T}{\partial t} = \frac{k}{c\rho} \frac{\partial^2 T}{\partial x^2} \quad \text{and} \quad \begin{aligned} T(o,t) &= 0 \\ T(\ell,t) &= 0 \\ T(x,o) &= f(x) \end{aligned} \tag{11}$$

To separate the variables let us assume that

$$T(x,t) = F(x)\, G(t)$$

and by substitution in the partial differential equation, it may be shown that the functions must satisfy the following ordinary differential equations:

$$F''(x) + p^2 F(x) = 0 \tag{14}$$

$$G'(t) + \frac{kp^2}{\rho C} G(t) = 0 \tag{15}$$

The solutions of these equations, after applying the boundary conditions are,

$$F_n(x) = A_n \sin px \qquad \text{where } p = \frac{n\pi}{\ell} \text{ at } n = 1,2,3,\ldots.\infty$$

$$G_n(t) = B_n e^{-V_n^2 t} \qquad \text{where } V_n = p\sqrt{\frac{k}{\rho C}}$$

Hence for any value of n, the following expression satisfies both the transient heat conduction equation and the boundary conditions of the example problem. Letting $C_n = A_n B_n$

$$T_n(x,t) = C_n \sin\frac{n\pi x}{\ell} \cdot e^{-V_n^2 t}$$

Not only is this expression a solution of equation 11 and of the boundary conditions, but the sum over all values of n of expressions of this type is also a solution.

$$T(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{\ell} \cdot e^{-V_n^2 t} \qquad (17)$$

The solution must satisfy the initial condition $T(x, 0) = f(x)$, therefore, it follows that

$$T(x,0) = f(x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{\ell} = \sum_{n=1}^{\infty} B_n F_n$$

where F_n is the eigenfunction corresponding to the eigenvalue n. At this point, the expression may be recognized to be the Fourier sine series expansion and

$$C_n = \frac{2}{\ell}\int_0^{\ell} f(x) \sin\frac{n\pi}{\ell}x\, dx$$

It may be possible to solve the transient heat conduction equation by using the separation of variables technique on the analog computer. Without seeking a numerical solution, but considering the problem only in terms of the steps necessary to achieve a computer solution, we may proceed as follows. Set up computer diagrams, Figures 9a and 9 b, which will give solutions to Equations 14 and 15. The boundary value F(0), Figure 9a, is placed on the second integrator. The p^2 potentiometer is adjusted until the integrator output satisfies the boundary conditions. A number of eigenvalues determined by the values of the p potentiometer settings may be found and each one that will satisfy the boundary conditions yields an eigenfunction F_n. To determine the B_n values, a number of circuits such as Figure 9c are patched at the same time and the previously determined eigenfunctions F_n multiplied by potentiometers representing the values of B_n. These products are then summed and the B_n potentiometers adjusted by trial and error until the output of the summer approximates the initial temperature distribution, f(x); Figure 9c.

It is obvious that the separation of variables technique using the analog computer involves many steps and becomes quite impractical even if the series is severely truncated. This approach has other severe limitations, particularly where the order of the equation is high and involves more than two variables. Much equipment is needed for this method of solution and considerable operator judgment is required. The EAI TR-10 would not be an adequate analog computer for the separation of variables method.

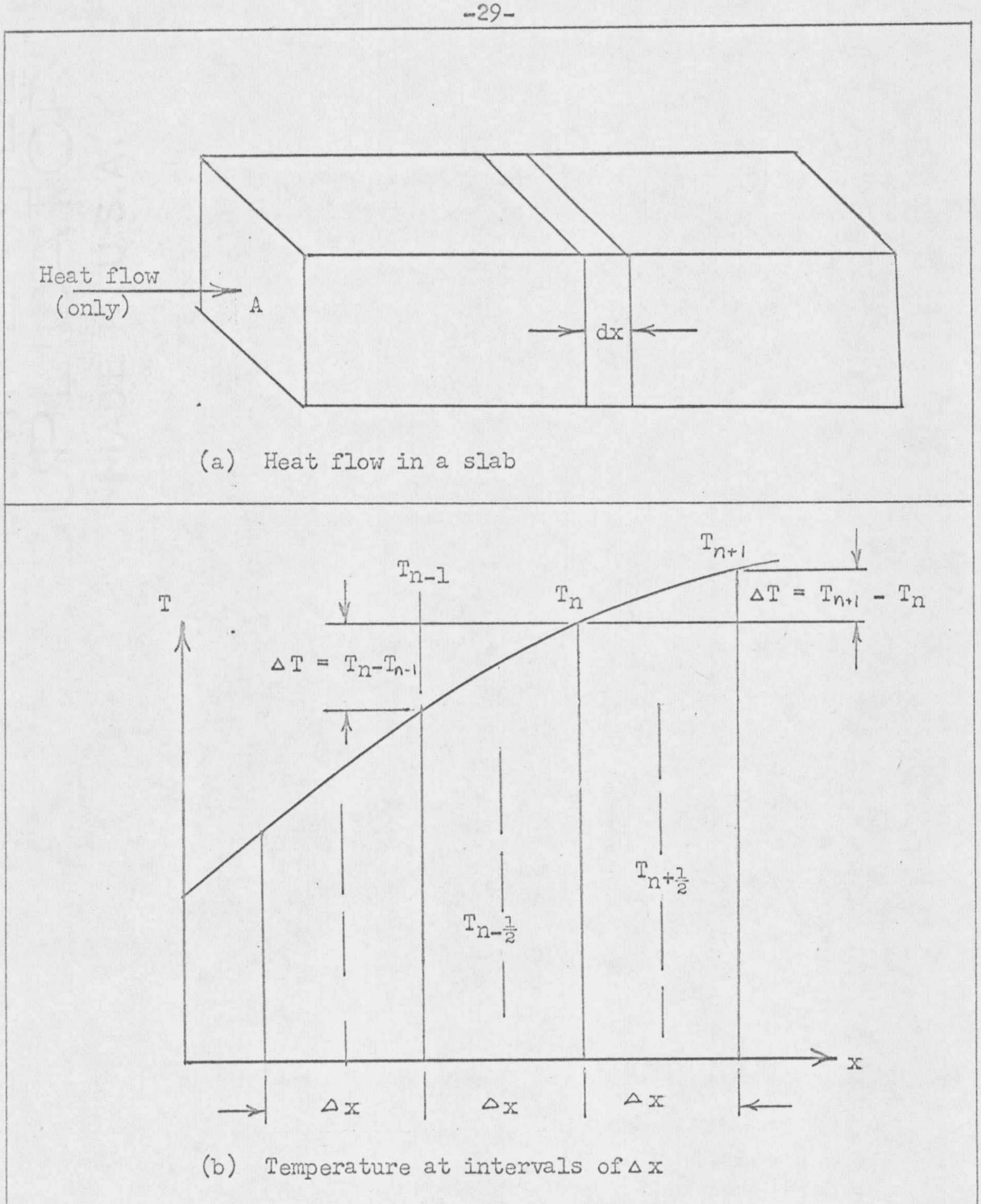
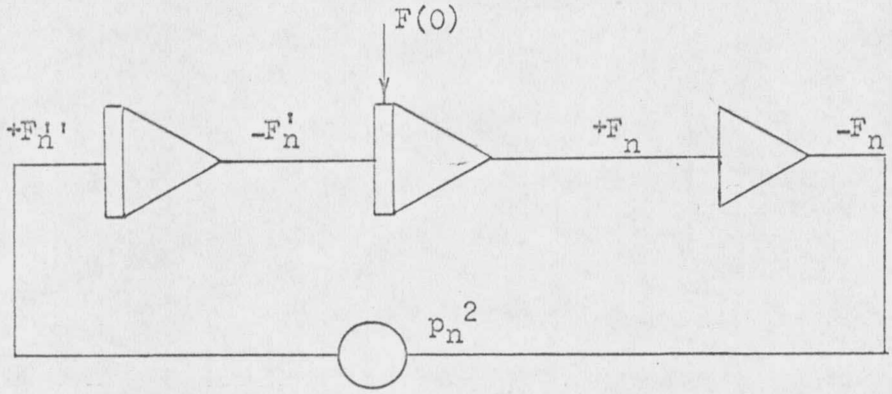
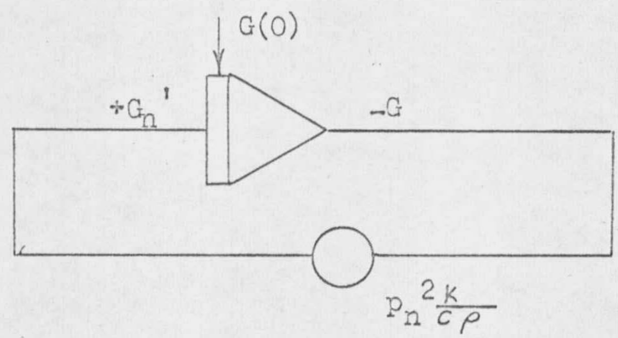


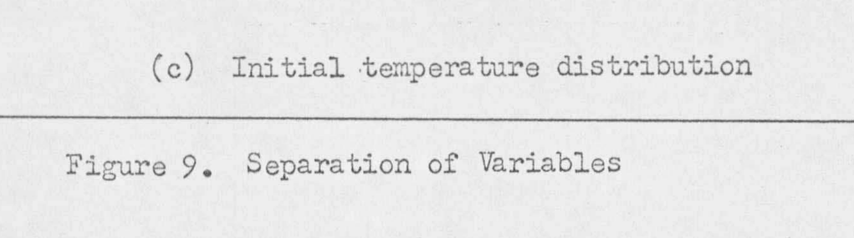
Figure 8. Finite-Differences--Heat Flow in a Slab



(a) Solution of $F_n'' + p^2F_n = 0$



(b) Solution of $G' + \frac{\kappa}{c\rho}p^2G = 0$



(c) Initial temperature distribution

Figure 9. Separation of Variables

CHAPTER 6

ANALOG SOLUTION OF AN UNSTEADY-STATE HEAT CONDUCTION PROBLEM

The example problem which follows will demonstrate the use of finite-differences and the solution of the unsteady-state one dimensional heat equation, Equation 11.

Problem: A steel slab initally at a temperature of 100 F is suddenly exposed to surroundings at a temperature such that the faces of the slab are at a constant temperature of 0F. The slab is considered to be infinite between the two planes and the temperature varies with distance x as shown in Figure 10. Determine the temperature distribution along the x direction.

The unsteady state heat transfer in the slab can be expressed by Equation 11.

$$\frac{\partial T}{\partial t} = \frac{k}{c\rho} \frac{\partial^2 T}{\partial x^2}$$

A finite-difference substitution, Equation 14, is made so that

$$\frac{dT_n}{dt} = \frac{k}{C\rho\,(\Delta x)^2} \quad (T_{n+1} - 2T_n + T_{n-1})$$

The following values will be used for the problem solution.

$$k = 26 \ \frac{\text{Btu}}{\text{hr ft F}}$$

$$\rho = 489 \ \text{lb/ft}^3$$

$$C = 0.12 \ \text{Btu/lb F}$$

$$x = L = 8 \ \text{inches}$$

$$x = 1 \ \text{inch} = 1/12 \ \text{ft.}$$

Let $$m = \frac{k}{C\rho(\Delta x)^2}$$

Then

$$m = \frac{26 \frac{\text{Btu}}{\text{hr ft F}}}{\left(0.12 \frac{\text{Btu}}{\text{lb F}}\right)\left(489 \frac{\text{lb}}{\text{ft}^3}\right)\left(\frac{1}{144} \text{ ft}^2\right) \left(3600 \frac{\text{sec}}{\text{hr}}\right)}$$

$$m = 0.0177 \text{ sec}^{-1}$$

The slab will be divided into 8 increments (Δx) of 1 inch. Since both faces of the slab are exposed to the same temperature, by symmetry as shown in Figure 10:

$$T_0 = T_8 = 0 \text{ F}$$

$$T_1 = T_7$$

$$T_2 = T_6$$

$$T_3 = T_5$$

$$T_4 = \text{center of slab}$$

Therefore, the problem solution need only provide the temperatures T_1, T_2, T_3, and T_4, at locations X_1, X_2, X_3, and X_4 respectively. These temperatures will provide a temperature profile as a function of time throughout the slab. The boundary conditions are, for $T = T(x,t)$

$$T(x,0) = T_n \qquad 0 < x \leq L$$

$$T(0,t) = 0 \qquad t > 0$$

$$T(L,t) = 0 \qquad x = L$$

The maximum value of T is 100 F; therefore, the system equations will be scaled so that 10 volts output will equal 100F. The scaled system equations will be

$$\left[\frac{\dot{T}_1}{100}\right] = 0.0177 \left(\left[\frac{T_2}{100}\right] -2 \left[\frac{T_1}{100}\right]\right) \qquad T_0 = 0F$$

$$\left[\frac{\dot{T}_2}{100}\right] = 0.0177\left(\left[\frac{T_3}{100}\right] - 2\left[\frac{T_2}{100}\right] + \left[\frac{T_1}{100}\right]\right)$$

$$\left[\frac{\dot{T}_3}{100}\right] = 0.0177\left(\left[\frac{T_4}{100}\right] - 2\left[\frac{T_3}{100}\right] + \left[\frac{T_2}{100}\right]\right)$$

$$\left[\frac{\dot{T}_4}{100}\right] = 0.0177\left(\left[\frac{T_5}{100}\right] - 2\left[\frac{T_4}{100}\right] + \left[\frac{T_3}{100}\right]\right)$$

$$\left[\frac{T_3}{100}\right] = \left[\frac{T_5}{100}\right]$$

$$\left[\frac{\dot{T}_4}{100}\right] = 0.0177\left(2\left[\frac{T_3}{100}\right] - 2\left[\frac{T_4}{100}\right]\right)$$

The initial condition (time zero) at each station is $T_n = 100$ F. Since 10 volts = 100 F, each integrator must have 10 volts input (100 F), as shown in Figure 11.

In some cases, it may be necessary to use a trial and error method to time scale the problem. For this problem, it was found that 2000 seconds were required before the temperature at any location in the slab nearly reaches 0 F. Therefore, the time scale for the problem was

5 seconds computer time = 500 seconds problem time

$$B = \frac{tc}{t} = \frac{5}{500} = 0.01$$

This means that the computer solution takes place 100 times faster than the actual solution.

Pots 3, 5, 9, 13, (Figure 11) must be divided by B (0.01). This applies whenever the problem time is scaled to a computer time (Ref. Appendix). The input multiplier is

$$\frac{0.0177}{0.01} = 1.77$$

The pots can only be set to a value equal to or less than unity, so the pots must be used with a 10 multiplier. The pot is set at 0.177 and this is multiplied by 10 to obtain 1.77.

The time function generator, Figure 11, for generating the time on the X-Y plotter is set to produce an output of 0.1t. Thus, if t = 20 seconds, 0.1 t = 2 volts. In other words, 2 volts are generated in 20 seconds. By setting the "arm scale" setting of the X-Y plotter at 0.5 volts/inch, the 2 volts will be plotted over a distance of 4 inches and since the time scale factor is 100, the 4 inches which are plotted in 20 seconds will represent 2000 seconds of actual problem time.

In dropping from a temperature of 100 F to 0 F, the magnitude scaling is such that a range of 10 volts is covered. The "pen scale" setting of the X-Y plotter is set at 1 volt/inch so that 10 inches are plotted for the 10 volts (100 F) of the problem. Thus, for the Y-axis

1 volt = 1 inch = 10 F

The computer solution is shown in Figure 12a and Figure 12 b.

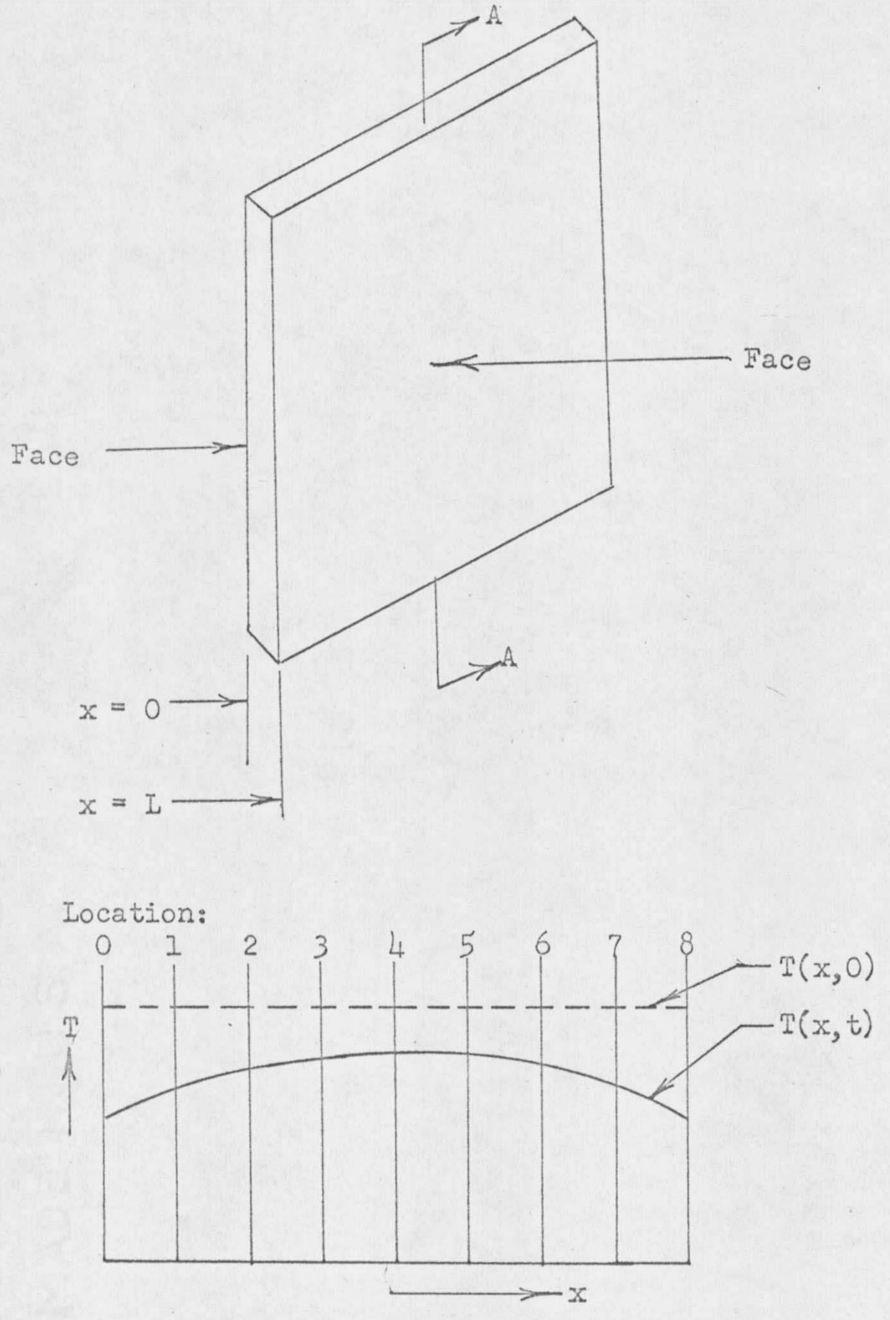


Figure 10. Unsteady-State Heat Conduction

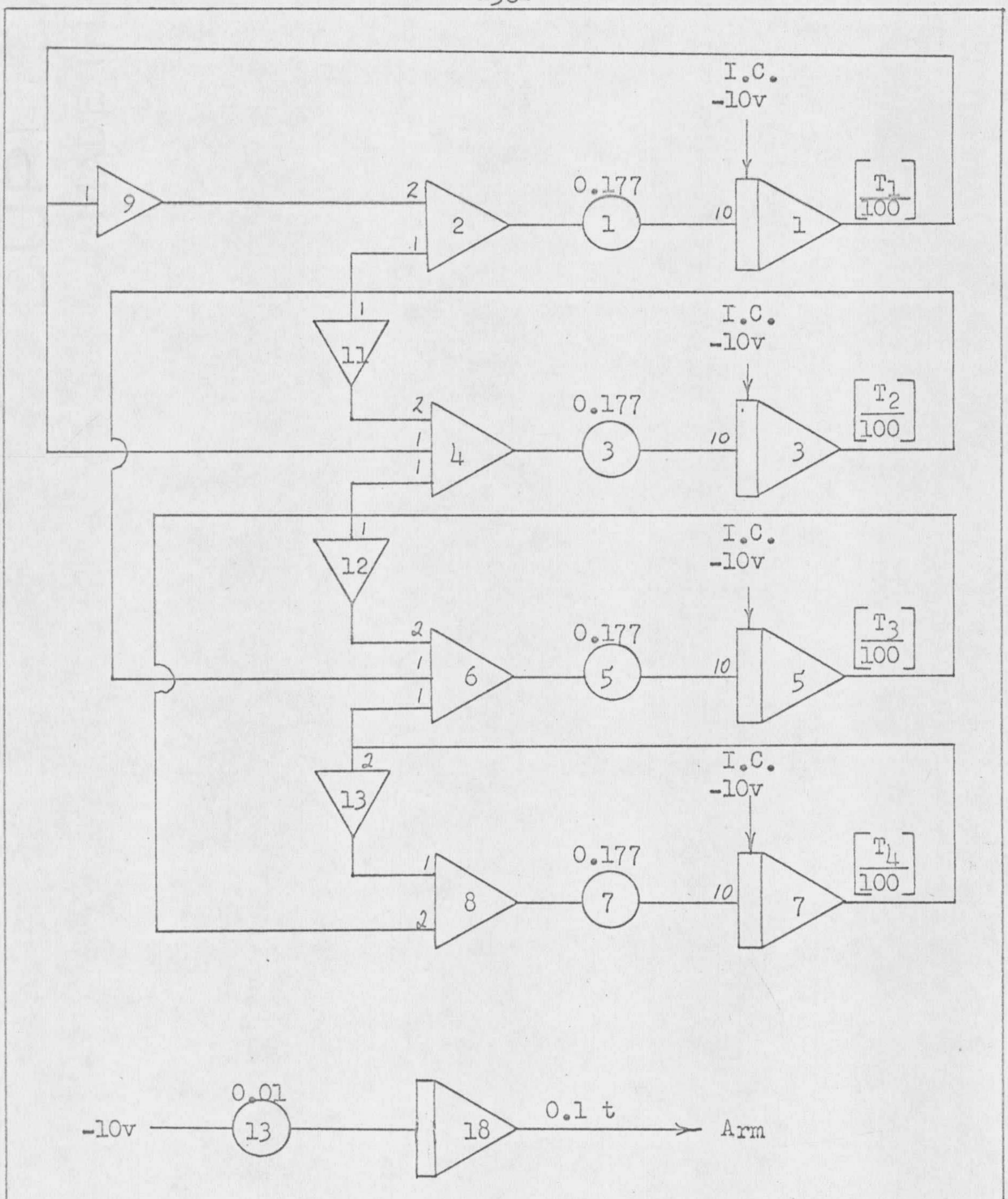


Figure 11. Analog Computer Diagram for Unsteady-State Heat Conduction in a Slab

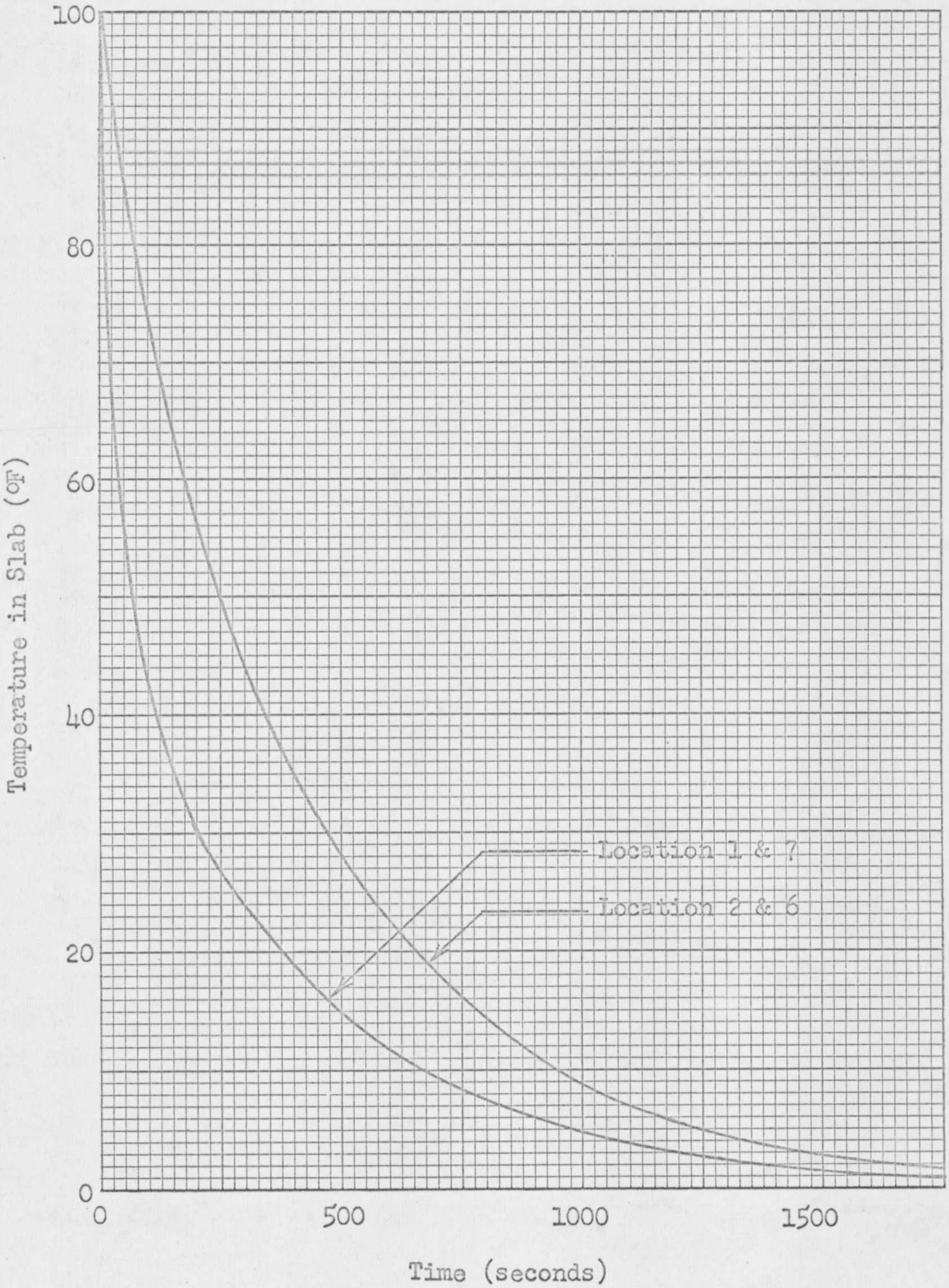


Figure 12a. Temperature-Time. Heat Conduction in Slab

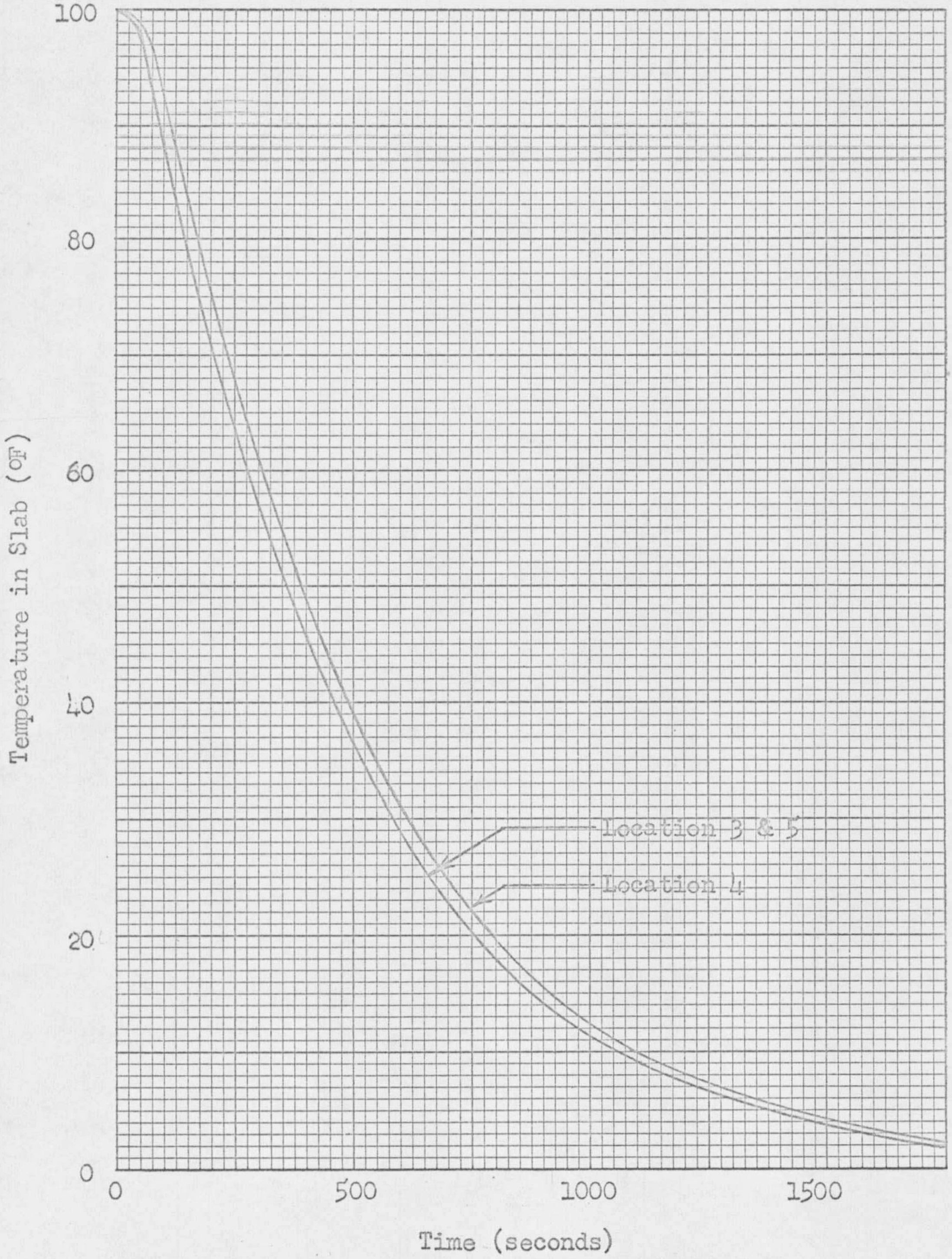


Figure 12b. Temperature-Time. Heat Conduction in Slab

CHAPTER 7

ONE DIMENSIONAL, UNSTEADY-STATE HEAT FLOW PROBLEM

A one dimensional, unsteady-state heat conduction problem can serve as an example in order to again demonstrate the method of finite-differences in solving partial differential equations. One dimensional heat flow can be defined by the equation

$$\frac{\partial T}{\partial t} = \frac{k}{c\rho} \frac{\partial^2 T}{\partial x^2} \qquad (11)$$

The aluminum rod, Figure 13, will be divided into 4 increments so that Δx will equal 1/4 feet. This will provide 4 stations in the rod at which temperatures will be found. The rod is initially in thermal equilibrium at 100 F and is assumed to be perfectly insulated except at the left end. The unsteady heat-flow state in the rod will be introduced by reducing the temperature of the exposed end to 0 F. The parameters of the rod are given in Figure 13. Note that k is in seconds rather than in hours. This will facilitate the time scaling.

Let

$$M = \frac{k}{c\rho(\Delta X)^2}$$

$$M = 0.0154 \text{ sec.}^{-1}$$

The substitution of Equation 12 into Equation 11 gives the equation

$$\frac{dT_n}{dt} = \frac{k}{c\rho\,(\Delta x)^2}\,(T_{n+1} - 2\,T_n + T_{n-1})$$

The maximum value of T_n is 100 F, therefore, the scaled variables are

$$\left[\frac{\dot{T}_n}{100}\right], \left[\frac{T_n}{100}\right]$$

For the 4 stations being considered, the scaled system equations which are to be solved simultaneously are

$$\left[\frac{\dot{T}_1}{100}\right] = M\left(\left[\frac{T_2}{100}\right] - 2\left[\frac{T_1}{100}\right]\right)$$

$$-\left[\frac{\dot{T}_2}{100}\right] = M\left(2\left[\frac{T_2}{100}\right] - \left[\frac{T_1}{100}\right] - \left[\frac{T_3}{100}\right]\right)$$

$$\left[\frac{\dot{T}_3}{100}\right] = M\left(\left[\frac{T_4}{100}\right] - 2\left[\frac{T_3}{100}\right] + \left[\frac{T_2}{100}\right]\right)$$

$$-\left[\frac{\dot{T}_4}{100}\right] = M\left(2\left[\frac{T_4}{100}\right] - 2\left[\frac{T_3}{100}\right]\right)$$

Notice that alternate equations have alternate signs. This is so that no sign-changing amplifiers are required, since the phase of each integrating amplifier output is correct for the respective inputs to a preceding or following amplifier.

It was found that to time scale the solution,

$$10 \text{ sec computing time} = (B)\ 1000 \text{ sec problem time}$$

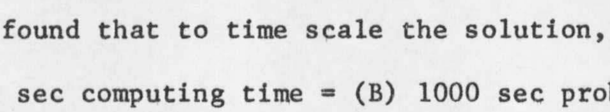
$$B = 0.01$$

The initial conditions are

$$T(x,\ 0) = 100\ F$$

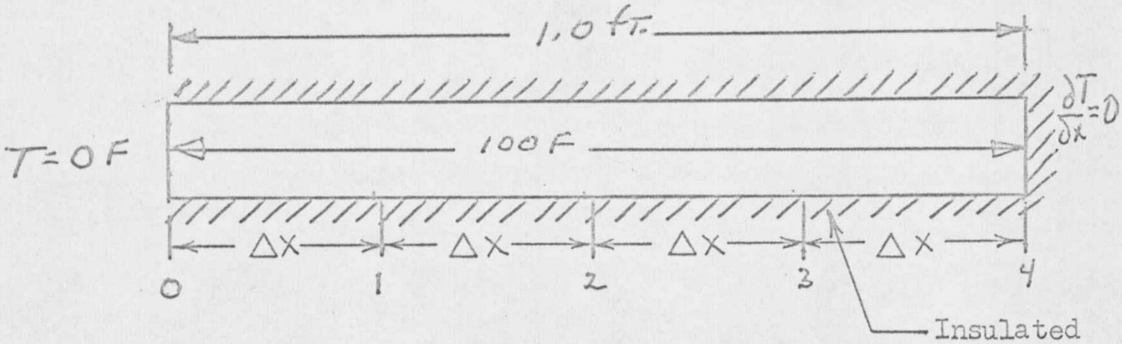
therefore, since 10 volts = 100 F, 10 volts input must be the initial condition at all 4 integrators.

The computer diagram, along with the time generating diagram, is shown in Figure 14. Notice that a multiplier of 2 is required at many inputs. This can be obtained without a pot preceding a 10 multiplier by this technique:



In other words, 2 inputs from the same source result in a multiplication of two.

The computer solution is shown in Figure 15. Many variations of the basic problem may be accomplished very easily. The initial temperature distribution along the rod can be changed by adjusting the appropriate initial conditions (voltages). Other combinations of $k/c\rho$ values may be investigated by merely changing pot settings.



x = 0.25 feet

Aluminum Material

Properties:

$$k = 0.037 \frac{\text{Btu}}{\text{sec ft F}}$$

$$c = 0.212 \frac{\text{Btu}}{\text{lb F}}$$

$$\rho = 168 \text{ lb/ft}^3$$

Initial Conditions

$T(0,t) = 0$ F

$T(x,0) = 100$ F

Figure 13. One-Dimensional Heat Transfer in an Aluminum Rod

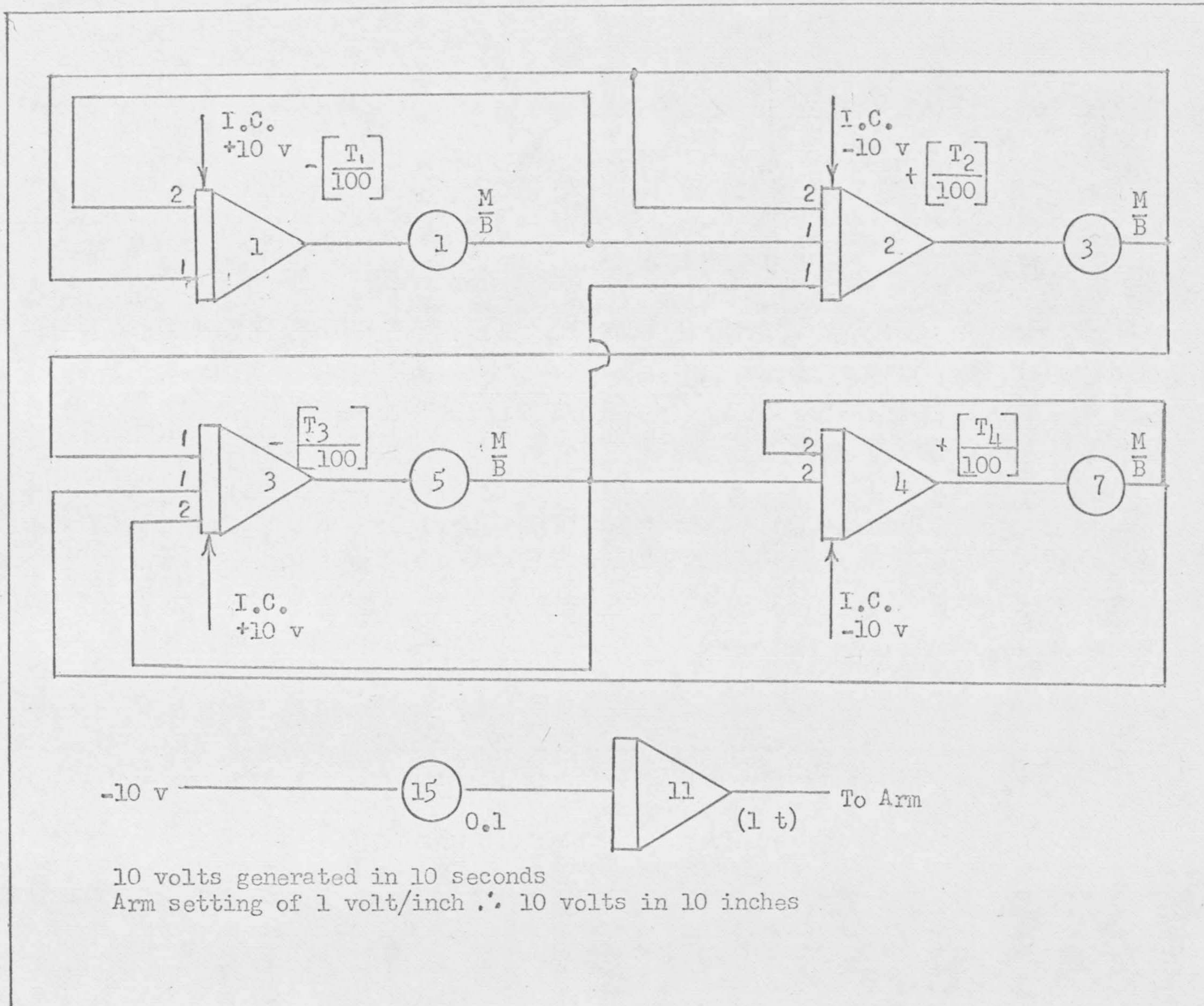


Figure 14. Computer Diagram for One-Dimensional Heat Flow in an Aluminum Rod

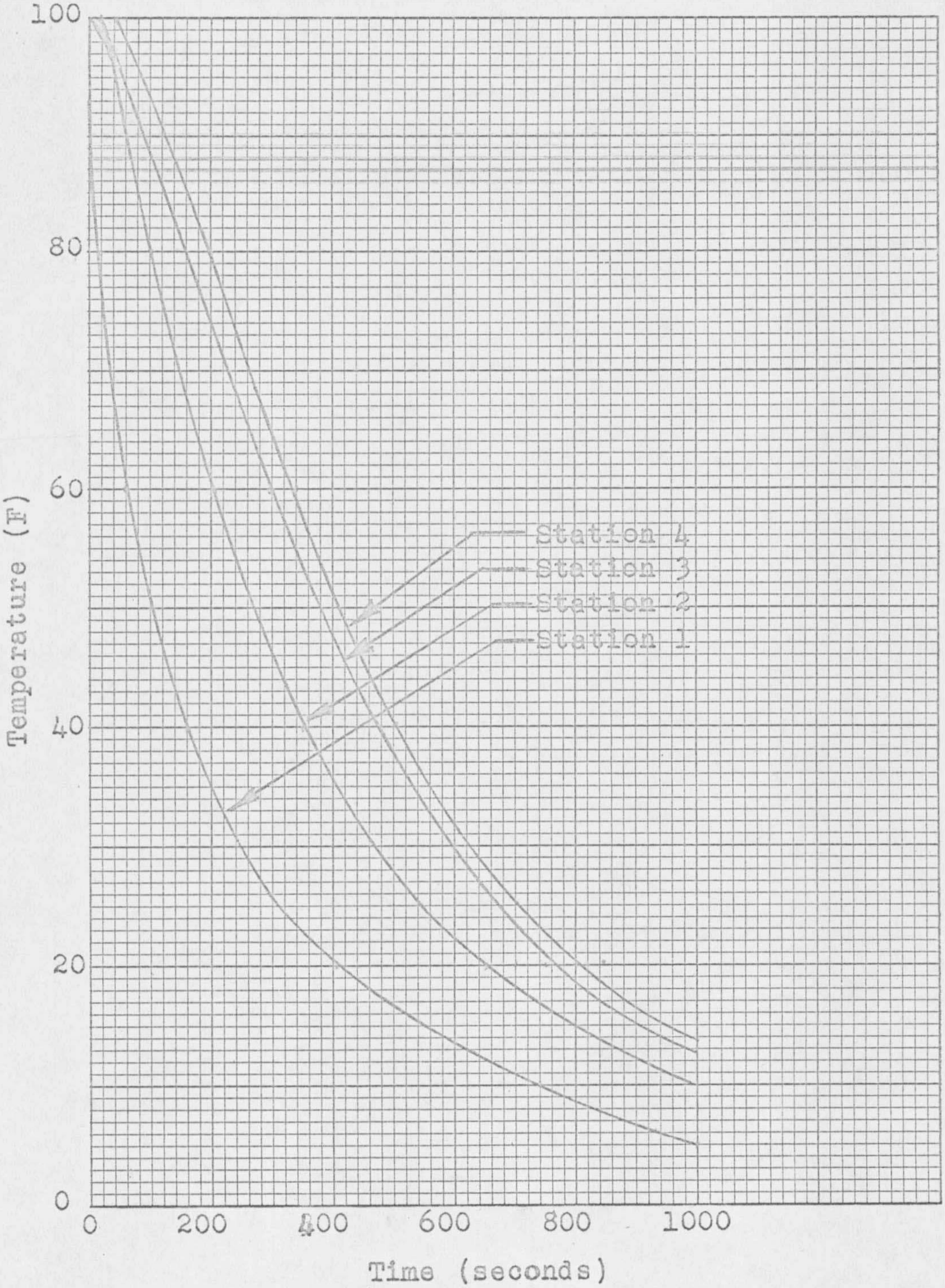


Figure 15. Solution of One-Dimensional Heat Flow in an Aluminum Rod

CHAPTER 8

ANALOG COMPUTER SIMULATION OF A PLUG FLOW SYSTEM

In developing mathematical models for combined fluid flow-heat transfer devices, it has been found, according to S. T. Ball, Instruments and Control Systems, that the plug flow concept yields better results than other approximations for turbulent flow in pipes or tubes. In the plug flow system, it is assumed that there is no mixing or heat conduction in the fluid in the direction of flow, and perfect mixing perpendicular to the flow path. If perfect mixing is assumed, computations may be made with average fluid temperatures rather than local values and the analysis is considerably simplified since ordinary differential equations describe the system.

S. T. Ball states that in liquid flow, the transport times or distance-velocity lags of the liquids are usually significant in determining the transient response characteristics. Usually solutions are for only one particular case because generalized solutions are impractical due to the large number of variables involved. In gas systems, the transport times are negligible compared to the thermal lags, so generalized solutions are obtained more readily.

To illustrate the use of the analog computer in problems of this type, a flow regime in a section of insulated pipe will be investigated. The response of the fluid outlet temperature to step changes in the inlet temperature to an insulated pipe will be simulated.

Consider the case of gas flowing through a section of insulated pipe which has significant heat capacity, Figure 16. Assume that all of the mass of the pipe is concentrated at a point which is at some mean temperature and that the transport time of the gas is negligible. If this is so, the section-length has to be small. Also, in developing the system equations, the gas mean temperature is assumed to be equal to 1/2 of the sum of the inlet gas temperature and outlet gas temperature. Thus, the length has to be small in order that the value of the gas mean temperature be an approximately accurate assumption. The system equations are

Gas heat balance

$$W_g c_g (T_{go} - T_{gi}) = hA(T_p - T_g)$$

$$T_g = 1/2 \, (T_{gi} + T_{go})$$

Pipe heat balance

$$W_p C_p \frac{dT_p}{dt} = hA \, (T_g - T_p)$$

Adding

$$(T_{go} - T_{gi}) = hA/W_g c_g (T_p - T_g)$$

$$\underline{(T_{go} + T_{gi}) = 2T_g}$$

$$2T_{go} = 2T_g + hA/W_g c_g \, (T_p - T_g)$$

$$T_{go} = T_g + hA/2W_g c_g (T_p - T_g)$$

Subtracting

$$(T_{go} - T_{gi}) = hA/W_g c_g \, (T_p - T_g)$$

$$\underline{(T_{go} - T_{gi}) = 2T_g}$$

$$T_{gi} = T_g - hA/2W_g c_g (T_p - T_g)$$

And

$$T_g = T_{gi} + \frac{hA}{2W_g c_g}(T_p - T_g)$$

The preceding equations can be related and connected according to the computer diagram of Figure 17. There will not be any time or magnitude scaling, rather the potentiometers will be set merely to show the response of the system to changes of inlet gas temperature.

Note that the setting of pot 1 represents the reciprocal time constant for pipe temperature changes and is independent of pipe length because the heat transfer area A and pipe mass M are both proportional to length. The outputs of pots 2 and 3 each represent one-half of the total change in gas temperature, which is proportional to the difference in the mean gas and pipe temperatures. The coefficients of pots 2 and 3 are proportional to pipe length. The transient characteristics of the system can be completely described when just two quantities are specified

$$t_p = \frac{M\ c_p}{hA} \qquad \text{time-constant } t_p \text{ in seconds}$$

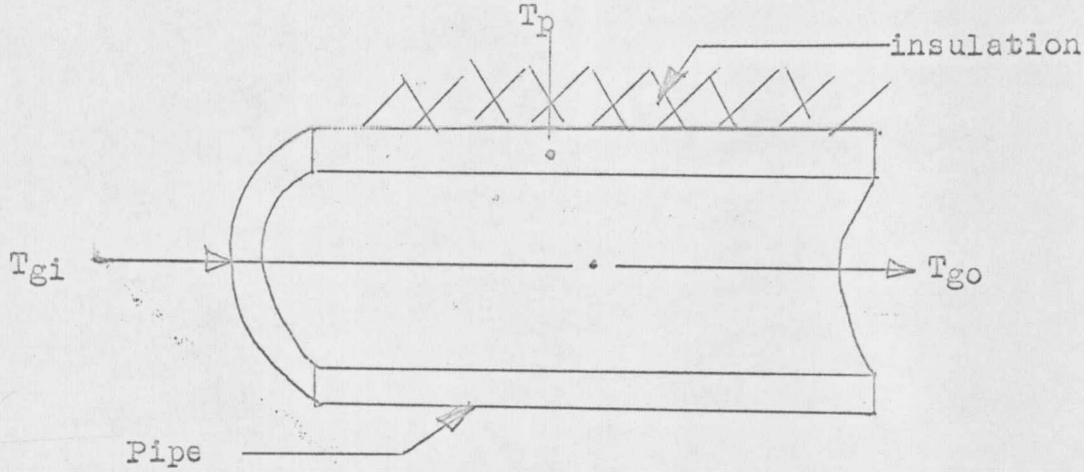
$$n = \frac{1}{2}\left(\frac{hA}{W\ c_g}\right) \qquad \text{dimensionless}$$

It was found that the value of the section length affected the results considerably. The value $n = 0.50$ was selected for all of the computer runs.

The pipe time constant was set at a value of $t_p = 1.0$. A decrease of this value resulted in a slower problem response. This is because the decrease could only be affected by the heat transfer coefficient. This means that no change in flow velocity was considered since a change in velocity would affect the heat transfer coefficient h.

The outlet gas temperature T_{go} was affected by a change in the inlet gas temperature T_{gi}. This can be seen in Figure 18. The 100%, Y axis, of Figure 18 represents the maximum response of T_{go} at 100% input; on the TR-10, this would be 10 volts input at T_{gi}. The other curves represent 2, 4, 6, 8 volts input. With numerical values and proper scaling, a numerical solution could be found. The computer diagram would remain the same.

This problem demonstrates how systems of related equations can be modeled on the analog computer.



W = gas mass flowrate lb/sec

C_g = gas specific heat Btu/lb F

T_{go} = gas outlet temperature F

T_{gi} = gas inlet temperature F

h = heat transfer coefficient Btu/sec ft^{2}F

A = heat transfer area ft^2

T_p = pipe mean temperature F

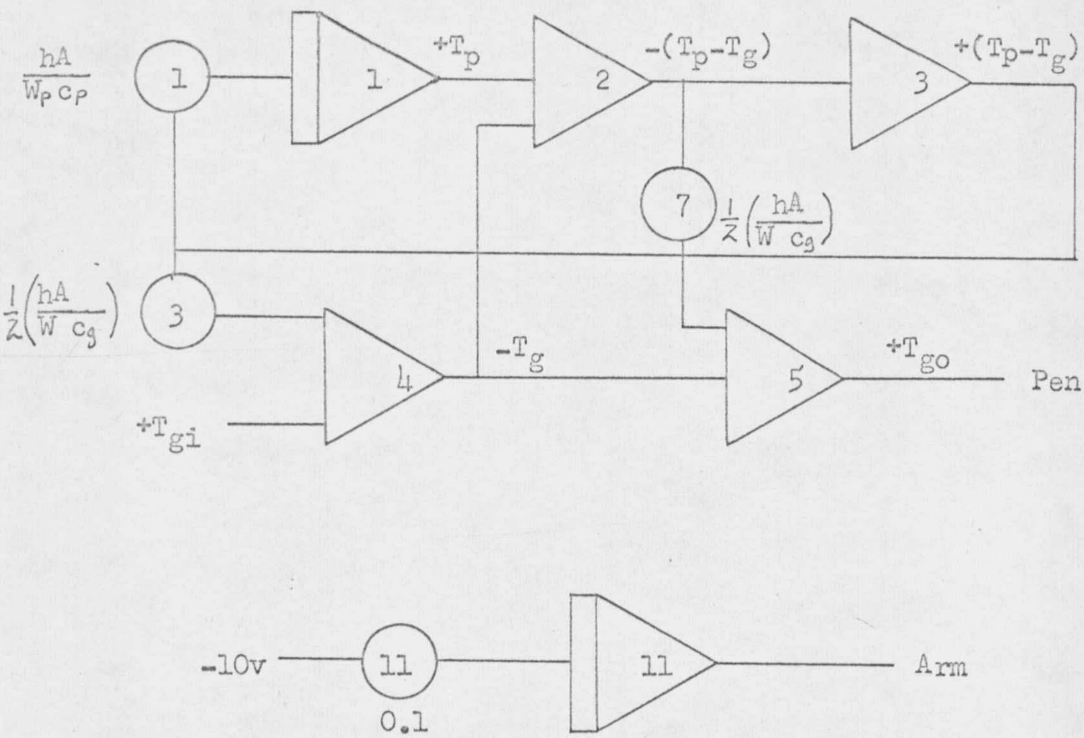
T_g = gas mean temperature F

W_p = mass of pipe lb

C_p = pipe specific heat Btu/lb F

t = time

Figure 16. Gas Flowing in an Insulated Pipe



$$T_{go}-T_{gi} = \frac{hA}{Wc_g}(T_p-T_g)$$

$$T_{gi}+T_{go} = 2\ T_g$$

$$+T_{go} = T_g + \frac{hA}{2W c_g}(T_p-T_g)$$

$$+T_{gi} = T_g - \frac{hA}{2W c_g}(T_p-T_g)$$

$$T_g = T_{gi} + \frac{hA}{2W c_g}(T_p-T_g)$$

Figure 17. Computer Diagram for Plug-Flow Simulation

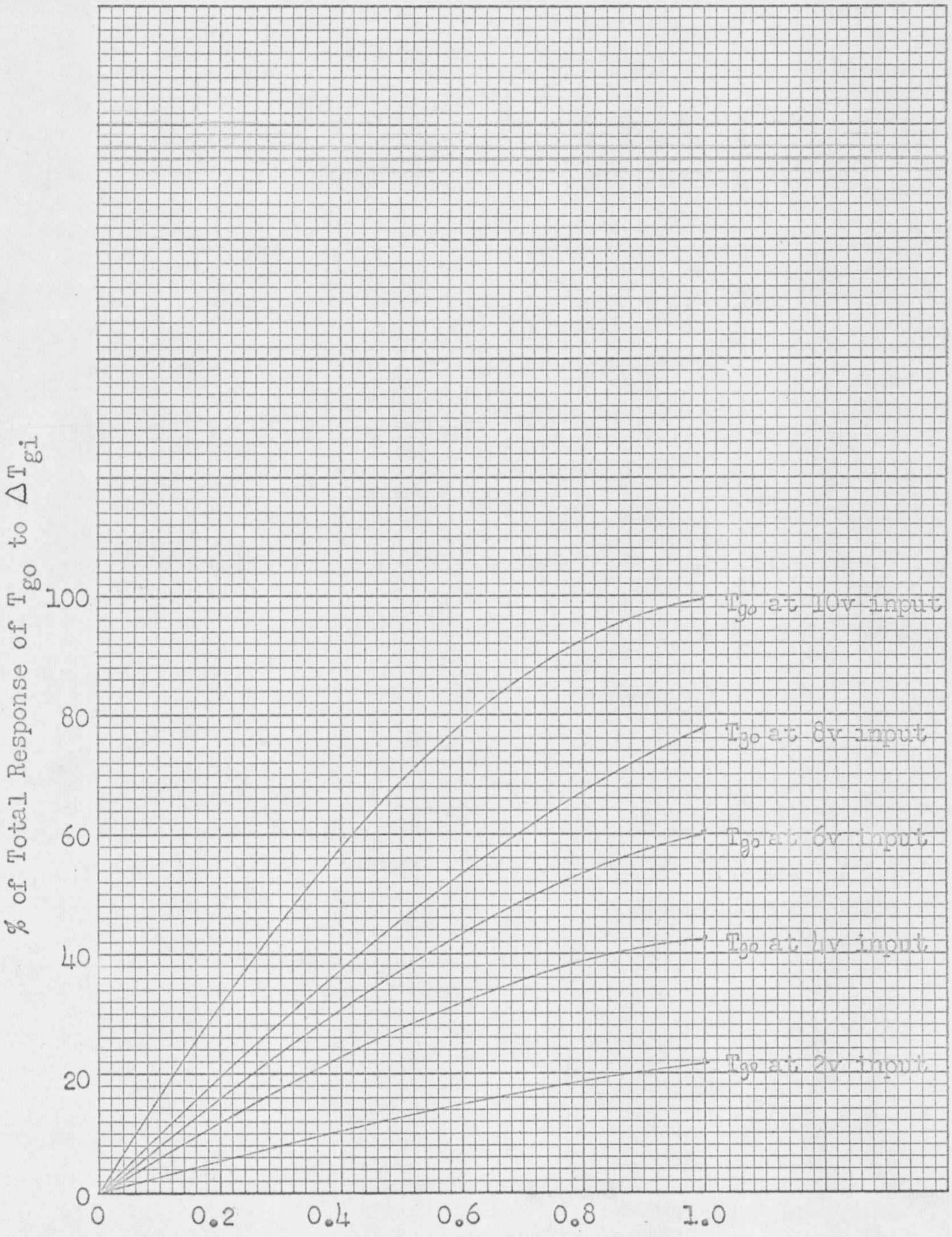


Figure 18. Plug-Flow Simulation - Response of T_{go} to Changes in T_{gi}

APPENDIX

APPENDIX

Basically, the electronic analog computer solves equations since it performs mathematical operations which result in solutions of the equations used to represent actual systems. To solve these equations, it is necessary for the electronic analog computer to be able to perform specific mathematical operations on direct voltages and to interconnect these various operations.

It is the purpose of this section to present the elements of the analog computer and their operation to the extent that the operator will be able to set up computer diagrams. The obvious operations which are necessary for the solution of a differential equation are:

1. to multiply
2. to integrate
3. to obtain the sum of 2 or more variables.

The basic electronic components which perform these operations are the multiplier or attenuator, the integrator network, and the operational amplifier. The operational amplifier is the most versatile computing unit within the computer. There are other computer components which perform more specialized operations. For example the X^2 diode function generator, the log diode function generator, the variable diode function generator, and others. These are described as needed in the solution of problems.

OPERATIONAL AMPLIFIER

The basic computer component is the amplifier. By a suitable choice of resistors and connections to other components, many relations can be obtained.

1. Sign changing

The output voltage will always be opposite in sign to the input. Whenever an amplifier is used, the sign will always change. The computer symbol is shown in Figure 19 (a).

2. Multiplication by 10

If the feedback resistor has ten times the value of the input resistor, then the output will be opposite in sign and ten times the input voltage. The computer symbol is shown in Figure 19 (b).

3. Summation

The amplifier can have two or more input voltages and the output will be the sum of the input voltages. The computer symbol is shown in Figure 19 (c).

4. Subtraction

By using two amplifiers, one as a sign changer and the second as a summer, two voltages can be subtracted. The method and computer symbol are shown in Figure 19 (d).

ATTENUATOR OR POTENTIOMETER

The attenuator provides the capability to multiply a variable voltage by a constant whose value lies between zero and unity. The computer symbol for the attenuator or pot is shown in Figure 20 (a). The number inside of the pot symbol represents the number of the computer potentiometer which was used.

1. Multiplication by a constant C between -1 and -10

This necessitates the combined use of a pot and an amplifier. The computer symbol, Figure 20 (b), shows that the pot is set to C/10 of the desired value. The denominator or 10 is due to the multiplication factor of 10 which is set on the amplifier.

2. Division by a constant B

This computer operation can be treated as multiplication by 1/B.

INTEGRATING NETWORK

The integrator network is connected across a high gain d-c amplifier and the necessary input resistors are applied to that network. If a second order differential equation is to be solved, two integrators are required and if only a first order differential equation is to be solved, only one integrator is needed. The computer symbol for an integrating network is shown in Figure 20 (c). The pot preceding the integrator may or may not be needed.

REFERENCE VOLTAGE

At various points in a problem, constant voltages of different values are required. The EAI TR-10 analog computer uses plus and minus 10 volts as sources for all computer signal voltages. These can be found on the patch panel. To produce constant voltages other than plus or minus 10 volts, the pots are used along with a reference voltage source. For example, if a reference voltage of 5 volts is required, use plus 10 volts reference and then connect this to a pot which is set to 0.50. This is the same as multiplying 10 volts x 0.50 = 5 volts.

The reference voltage is always used as an initial condition or as a forcing function.

The basic elements of the computer have been listed along with the computer diagram symbols. A more complete list of symbols used to represent computer devices can be found in the computer operator's handbook.

The potentiometer or attenuator, the summer, and the summing integrator network are the three basic linear elements required for the solution of systems of linear differential equations. Solving non-linear equations requires the additional components of multiplication, function generation, limiting and switching. However, the present discussion will consider only linear differential equations.

Computer Diagram Example

A simple linear differential equation example can best illustrate the method of obtaining a computer diagram using computer symbols. No magnitude scaling or time scaling will be used in this example. It must be remembered that no familiarity with the computer is required for the representation of the computer setup by means of a computer diagram.

Consider the differential equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0$$

Using dot notation

$$\ddot{x} + \dot{x} + x = 0$$

Step 1. Solve for the highest-order derivative

$$\ddot{x} = -\dot{x} - x$$

Sometimes a "block diagram", Figure 21 (a), will be helpful for preliminary analysis. This diagram ignores all signs and values and is used to show the "flow" of the problem.

Step 2. Integrate the highest-order derivative and all subsequent deriva-

derivatives. Figure 21 (b). Notice the sign changes.

Step 3. Multiply each term by the constants given in the equation, Figure 21 (c). A pot is placed before each integrator.

Step 4. Add the terms as given in the equation. At this point, $-(\dot{x} + x)$ must be added. A summer is used. Figure 21 (d).

Step 5. Satisfy the equation by closing the loop as given by the equation. In Figure 21 (d), $\ddot{x}$ has been found to be the output of summer B. Therefore, it can be connected into integrator C. This closes the loop.

Magnitude Scaling

It is important to remember that the computer will establish mathematical relations not between the original variables of a problem, but between voltages simulating these variables. Since the limiting value of the TR-10 analog computer is $\pm$ 10 volts, the voltages must be arranged so that they do not exceed these values.

Magnitude scaling will set up a relationship between the voltage observed as an output and the problem variable. There are various methods of magnitude scaling; however, only the method used in the problems is presented here.

If in a problem involving some variable T, the value of T varies from 0 F to 1000 F, then 1 volt on the computer could represent 100 F. If this is so, the problem variable F will vary between 0 F and 1000 F and the computer voltage will vary between 0 volts and +10 volts. It is desirable to have the computer voltage reach its maximum voltage when the variable F reaches its maximum. If the computer can cover the maximum voltage range ($\pm$ 10 volts), it will minimize the effects of noise, resolution errors, and computer drift.

The maximum values of the physical variable can be scaled on a "per volt" basis so that

$$\frac{10 \text{ volts}}{\text{max. value of physical variable}}$$

This is called the magnitude scale factor. In the example cited, the magnitude scale factor would be

$$\frac{10 \text{ v}}{1000 \text{ F}} = 0.01 \frac{\text{V}}{\text{F}} \qquad 1 \text{ volt} = 100 \text{ F}$$

Another way of obtaining a magnitude scale factor would be to write the scale factor as such $\left[\frac{F}{1000}\right]$ $\qquad 10 \text{ v} = 1000 \text{ F}$

This simply means that 10 volts = 1000 F. It is a matter of keeping track of the voltage - variable correspondence.

The following examples show how to determine the magnitude scale factors:

Physical Variable	Scaled Variable-"per volt"	Scaled Variable
velocity $\dot{x}_{max} = 200$ ft/sec	$\frac{10\ \dot{x}}{200} = \left[\frac{\dot{x}}{20}\right]$	$\left[\frac{\dot{x}}{200}\right]$
Rate of temperature change $\dot{T}_{max} = 100$ F/sec	$\frac{10\ \dot{T}}{100} = \left[\frac{\dot{T}}{10}\right]$	$\left[\frac{\dot{T}}{100}\right]$
Temperature $T_{max} = 800$ F	$\frac{10\ T}{1000} = \left[\frac{T}{100}\right]$	$\left[\frac{T}{1000}\right]$

Although the maximum temperature is expected to be 800 F, a maximum of 1000 F was used to obtain a scale factor. This is permissible and merely gives a number (1000) which is easier to work with.

The scaled variables which will appear as voltages in the computer circuit are identified by using square brackets about them.

GAIN FACTOR - EXAMPLE PROBLEM 1

In order to construct the scaled computer diagram, the "gain factor" must be included at the input to each of the integrators. This gain factor is necessary and results from the difference in scale factors which are used. The following hypothetical computer diagram analysis can illustrate the gain factor as well as the method of obtaining the diagram. Suppose that the maximum values of the variables of a problem are:

$$\ddot{X}_{max} = 32 \text{ ft/sec}^2$$

$$\dot{X}_{max} = 8 \text{ ft/sec}$$

$$X_{max} = 2 \text{ ft}$$

The equation is

$$\ddot{X} + 3\,\dot{X} + 16\,X = 0$$

The magnitude scale factors are

$$\left[\frac{\ddot{X}}{32}\right], \left[\frac{\dot{X}}{8}\right], \left[\frac{X}{2}\right]$$

and the scaled equation is

$$32\left[\frac{\ddot{X}}{32}\right] + (3)(8)\left[\frac{\dot{X}}{8}\right] + (16)(2)\left[\frac{X}{2}\right] = 0$$

In order to maintain the equation, the terms of the equation are multiplied, or divided, by the same value as is used in obtaining the scale factors. Solution of the equation for the highest-order differential:

$$\left[\frac{\ddot{X}}{32}\right] = -\,0.75\left[\frac{\dot{X}}{8}\right] - \left[\frac{X}{2}\right]$$

The computer diagram, Figure 22 (a), shows the multiplication by a factor of 4 at the input to each integrator. Consider the equation to be solved by the first integrator: $-\dot{x} = -\int_0^t \ddot{x}\,dt + \dot{x}(0)$

Using scale factors

$$-8\left[\frac{\dot{x}}{8}\right] = -\int_0^t 32\left[\frac{\ddot{x}}{32}\right]dt - 8\left[\frac{\dot{x}(0)}{8}\right]$$

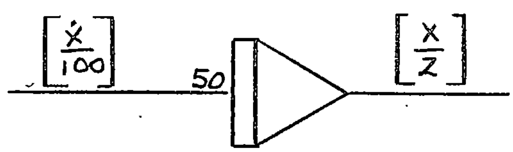
or

$$-\left[\frac{\dot{x}}{8}\right] = -4\int_0^t \left[\frac{\ddot{x}}{32}\right]dt - \left[\frac{\dot{x}(0)}{8}\right]$$

A general method to find the value of the "gain" needed is:

Scaled coefficient of the (n - 1) order differential $\div$ Scaled coefficient of the n^{th} order differential

Example:



$$\frac{1}{2} \div \frac{1}{100} = 50$$

The gain factor which must be used as an input to the integrator is 50. This gain factor value <u>does not</u> consider the time scale factor which will be discussed in the next section. It will be seen that a large gain factor can be reduced with the appropriate time scale factor.

<u>Time Scaling</u>

It will often happen that the rate at which a process simulation runs is unsuitable; it may be too fast for the recording device to follow or too slow for the convenience of the person interested in the answer. In either case, it is necessary to alter the rate at which the simulation runs; this is done by a technique called time scaling.

The ideal duration of a single run on the computer is somewhere from 10 to 60 seconds. Too long a solution time may introduce some error due to integrator drift. Also the computer operator desires results in a reasonable time. The capacity of the recording equipment to follow the solution places a lower limit on the solution time. To remain within these limitations, a change in time can be introduced by relating physical or real time to analog computer time by a time scale factor B, so that

$$t_c = Bt \qquad \begin{array}{l} t_c = \text{computer time in seconds} \\ t = \text{real time in seconds} \end{array}$$

If $B = 1$, the computer is operating in real time.

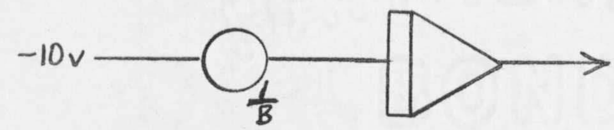
If $B < 1$, the computer solution is speeded up; things happen faster on the computer than in the physical system.

If $B > 1$, the computer solution is slowed down; things happen slower on the computer than in the physical system.

Through mathematical deduction, it can be proven that time scaling can be accomplished by multiplying every input to an integrator by $1/B$.

<u>X - Y Plotter</u>

The output of the problem may be observed on an oscilloscope or recorded on a strip-chart recorder or on an X-Y plotter. In this report, the X-Y plotter was used exclusively. The X-Y plotter uses a moving pen and stationary paper. This device has two voltage inputs, enabling one to plot one varying voltage against another. The "Arm" output moves along a horizontal line and the "Pen" moves vertically depending on voltage for its deflection. The input for the Pen is the output of the dependent variable of the problem. The output of the dependent variable can come from any location in the computer circuit. That is, it can be the output of any amplifier. The input to the Arm will determine the rate of horizontal travel and will represent the independent variable time. The voltage representing time is generated by an integrator and drives the Arm on the X-Y plotter a distance proportional to time. For example



$$t = \int_0^{t_c} \frac{10}{B}\, dt = \frac{10}{B} t$$

If B = 10, then

$$t = \int_0^{t_c} 1\, dt = 1\, t_c$$

If t_c = 10 seconds, then 10 volts will be generated in 10 seconds of actual computer operating time. The "Arm Scale" setting on the plotter will control the length of travel of the pen. If the setting is set to 1 volt/inch, the plotter will generate the above 10 volts in 10 inches and 10 seconds.

<u>Summary-Example Problem 2</u>

To summarize the techniques of obtaining a complete analog computer diagram, a second-order differential equation will be considered as an example.

(18) $x'' + 8x' + 100x = f(t)$

$x(0) = 20$ inches

$x'(0) = -5$ inch/second

Assume maximum values of

$x_m = 2$ inches

$x'_m = 20$ inch/second

$x''_m = 200$ inch/second^2

The magnitude scale factors are

$\left[\frac{x}{2}\right]$ 10 volts = 2 inch

$\left[\frac{x'}{20}\right]$ 10 volts = 20 in/sec

$\left[\frac{x''}{200}\right]$ 10 volts = 200 in/sec^2

Solving Equation 18 for the highest order differential, substituting the magnitude scaled variables, and making proper adjustments to keep the equation true:

$$x'' = -8\,x' - 100\,x + f(t) \tag{18}$$

$$200\left[\frac{x''}{200}\right] = (-8)(20)\left[\frac{x'}{20}\right] - (100)(2)\left[\frac{x}{2}\right] + f(t)$$

$$\left[\frac{x''}{200}\right] = -0.80\left[\frac{x'}{20}\right] - \left[\frac{x}{2}\right] + \frac{f(t)}{200}$$

This equation can now be used to draw the computer diagram, Figure 22 (b). The time scale factor B can be neglected during this process but must be added in the computer circuit. To obtain the necessary gain factor between $\left[\frac{x''}{200}\right]$ and $\left[\frac{x'}{20}\right]$

$$\left[\frac{1}{20}\right] \div \left[\frac{1}{200}\right] = 10$$

and between $\left[\frac{x'}{20}\right]$ and $\left[\frac{x}{2}\right]$

$$\left[\frac{1}{2}\right] \div \left[\frac{1}{20}\right] = 10$$

This gain could be obtained by an integrator input multiplication of 10. A time scale factor of B = 10 would slow down the solution and also decrease the gain, Figure 22 (c).

It should be remembered that the important phase of problem solving with the analog computer is to obtain a correct diagram so that the patch board can be wired correctly. Of lesser importance is the estimation of maximum magnitudes and run duration since these values can usually be changed by merely changing the potentiometer settings.

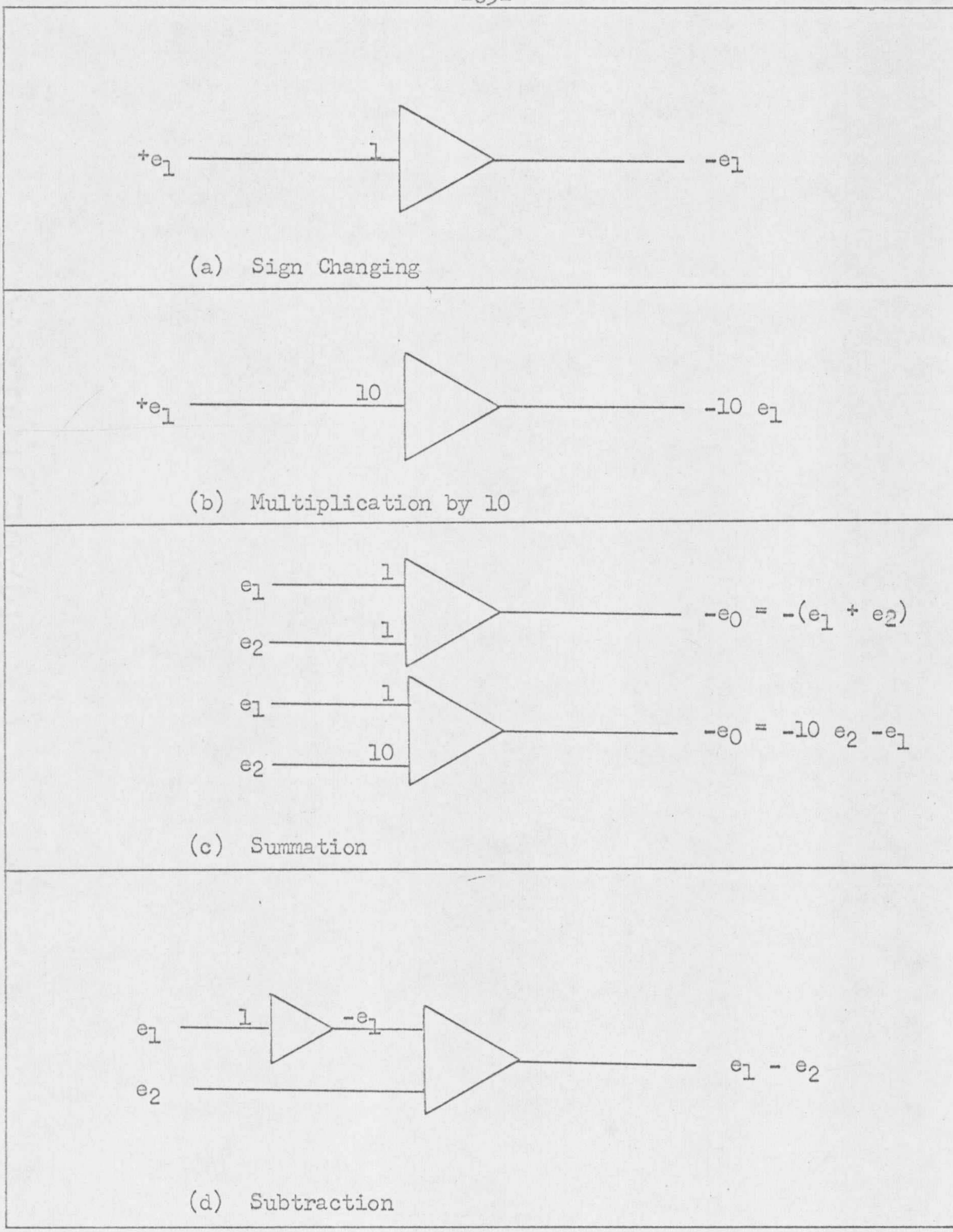


Figure 19. Basic Computer Symbols

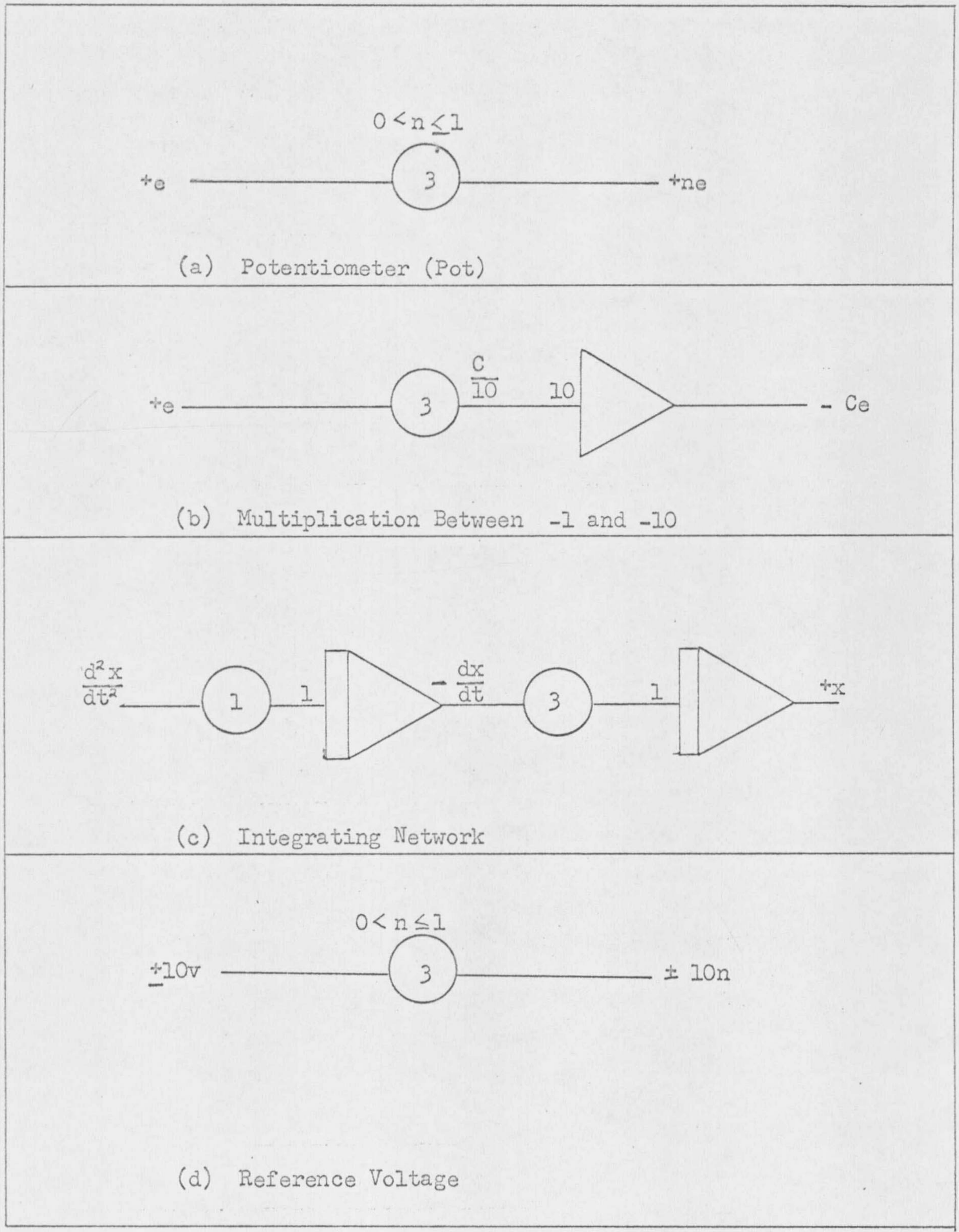
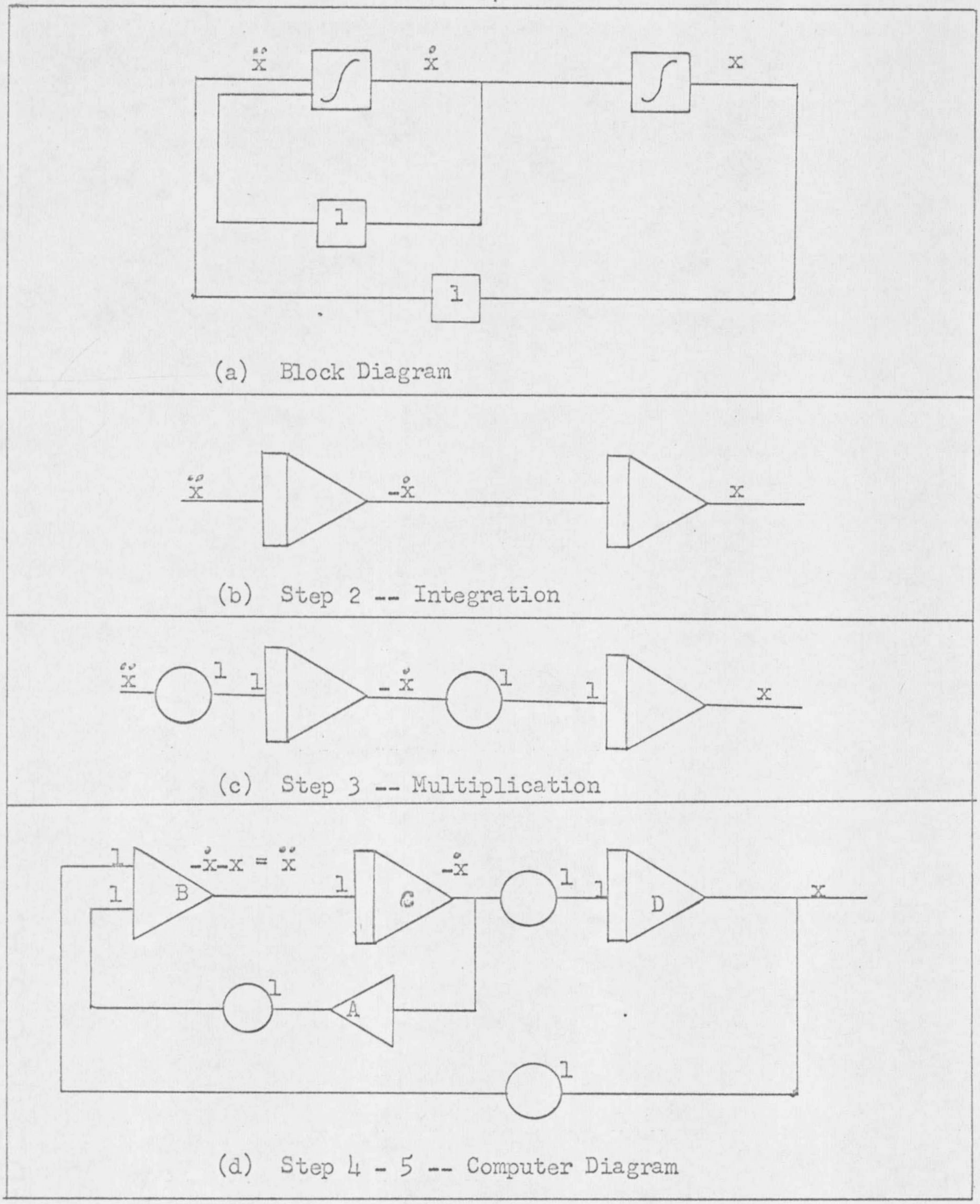


Figure 20. Basic Computer Symbols



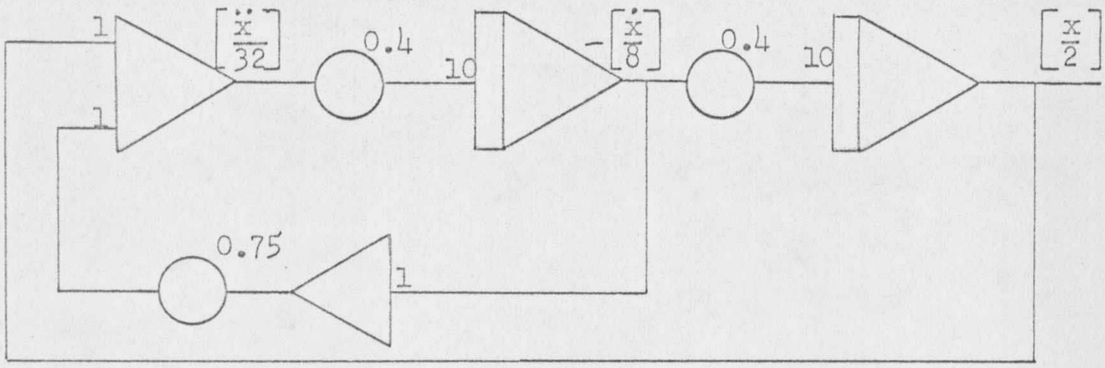
(a) Block Diagram

(b) Step 2 -- Integration

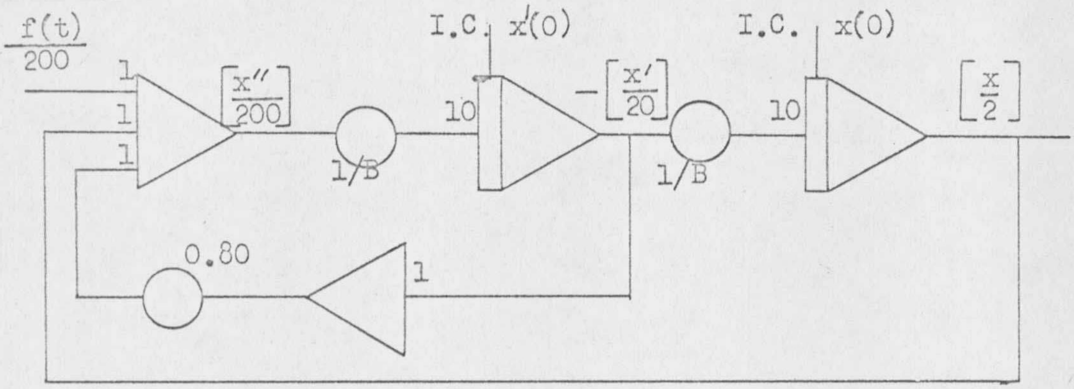
(c) Step 3 -- Multiplication

(d) Step 4 - 5 -- Computer Diagram

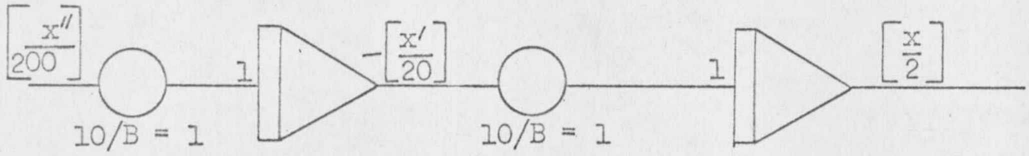
Figure 21. Solution of Example Problem



(a) Gain Factor -- Example Problem 1



(b) Computer Diagram for Example Problem 2



(c) Reducing Gain Factor by Time Scaling

Figure 22. Example Problems -- Diagramming Fundamentals

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