The improvement of the reduced coordinate iterative procedure for simultaneous linear equations
by Timothy Mark Hodges

A thesis submitted in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE in Mechanical Engineering
Montana State University

Abstract:
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APPROVAL

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Timothy Mark Hodges

This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

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ABSTRACT

An iterative method for solving simultaneous linear equations is considered. Previously, the trial solution generation routine depended on knowledge of the physical system. A method is introduced to generate the trial solutions based on the system of equations. The new method is used to solve several differential equations with various boundary conditions, so as to show that the method is independent from the physical system. Examples of two steady-state problems and a transient problem, with 1000 unknowns, demonstrate the performance as compared to existing techniques. Results indicate that the method is superior in cases where the boundary conditions are more complicated, and the method is comparable in all other cases.

CHAPTER I

INTRODUCTION

Many numerical methods have been developed to accurately solve a large system of simultaneous equations. A method is presented here that has the possibility of being faster and more accurate than these methods.

The Reduced Coordinate Iterative Procedure (RCIP)[1] uses a linear combination of trial solutions, with each trial solution having an unknown weighting coefficient. Each weighting coefficient is determined by minimizing an error function. This minimizing process produces a smaller set of equations which can be solved directly for these weighting coefficients. Therefore, in solving a large set of equations, the RCIP method reduces the number of equations to be solved.

The method of least squares can be related to the RCIP method, because of the minimization of the squared residual in the RCIP method.[3] The least squares method finds the equation of a curve that passes through scattered points. This curve is found so that the sum of the square distances from each point to the curve is a minimum. In the RCIP method the sum of the squares of the residuals from the equation set, the variance, is minimized. The least squares method is used for curve fitting and not for the solution of simultaneous equations as RCIP is.

Another method that RCIP is related to is the Conjugate Direction Method [4,8]. The RCIP and Conjugate Direction methods both use the minimization of an error function in the solution of the problem. The Conjugate Direction Method minimizes the error along orthogonal directions. The Conjugate Direction Method will converge in less iterations than there are unknowns. The methods have been shown to be different [1].

To evaluate the performance of the RCIP method, two existing techniques were used: The first was a Gauss-Seidel iterative scheme using successive over-relaxation (SOR) [5], and the second was the Alternating Direction Explicit Procedure (ADEP)[6]. Both are excellent methods for comparation, because of their speed of convergence and their accuracy.

The RCIP method previously has had a systematic procedure for the generation of trial solutions. However, the trial solution generation was dependent on knowledge of the physical system which the equation represented. The trial solution generation should be independent of the knowledge of the physical system.

It is the purpose of this paper to present a trial solution generation scheme that is independent of the knowledge of the physical system while continuing to produce rapid convergence and be competitive with existing methods.

CHAPTER II

DESCRIPTION OF THE METHOD

The RCIP method uses a linear combination of trial solution vectors with each trial solution having an unknown scalar weighting coefficient. The weighting coefficients are obtained by solving a reduced set of equations which are obtained by minimizing the variance. The reduced set is solved using an exact or direct method. The trial solutions are different at each iteration and the iterative process continues until the desired convergence is reached [2]. The formulation of RCIP proceeds as follows:

Let

$$AX=F \tag{1}$$

be a system of n equations in n unknowns. Initially choose m vectors (m is the number of trial solutions, $G(k,m)_0$ n in length, the subscript 0 indicates the first iteration)

$$G(k,1)_0, G(k,2)_0, \ldots, G(k,m)_0, \qquad k=1,\ldots,n \tag{2}$$

where $m < n$, and set

$$X_0 = \sum_{j=1}^{m} \alpha_j \; G(k,j)_0 \qquad k=1,\ldots,n \tag{3}$$

where the scalars $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ are to be determined.

Let

$$V_0(\alpha_1,\ldots,\alpha_m) = \langle AX_0-F, AX_0-F\rangle = ||AX_0-F||^2, \tag{4}$$

be the inner product and the square of the norm of the residual ($R_0=AX_0-F$), respectively. Find the solution ($\alpha_{0,1},\ldots,\alpha_{0,m}$) to the system

$$\frac{\partial V_0}{\partial \alpha_p} = 0 \qquad p=1,\ldots,m. \qquad (5)$$

Designate

$$\overline{V}_0=V(\alpha_{0,1},\ldots,\alpha_{0,m}) \qquad (6)$$

and let

$$X(1,1)=\sum_{j=1}^{m} \alpha_{0,j}\ G(k,j)_0 \qquad k=1,\ldots,n. \qquad (7)$$

Then

$$V_1(\alpha_1,\ldots,\alpha_m)=||AX_1-F||^2 \qquad (8)$$

solve the system

$$\frac{\partial V_1}{\partial \alpha_p} = 0 \qquad p=1,\ldots,m\ , \qquad (9)$$

and if ($\alpha_{1,1},\ldots,\alpha_{1,m}$) is the solution, designate

$$\overline{V}_1=V_1(\alpha_{1,1},\ldots,\alpha_{1,m}) \qquad (10)$$

and let

$$X(2,1)=\sum_{j=1}^{m} \alpha_{1,j}\ G(k,j)_1 \qquad k=1,\ldots,n\ . \qquad (11)$$

At the ith stage,

$$X(i,1)=\sum_{j=1}^{m} \alpha_{i-1,j}\ G(k,j)_{i-1} \qquad k=1,\ldots,n\ . \qquad (12)$$

Select vectors $G(k,2)_i,\ldots,G(k,m)_i$, and let

$$X_i = \sum_{j=1}^{m} \alpha_j\ G(k,j)_i \qquad k=1,\ldots,n\ . \qquad (13)$$

Then

$$V_i(\alpha_1,\ldots,\alpha_m)=||AX_i-F||^2 \quad . \qquad (14)$$

Again solve the system

$$\frac{\partial V_i}{\partial \alpha_p} = 0 \qquad p=1,..,m. \qquad (15)$$

If this solution is $(\alpha_{i,1},\ldots,\alpha_{i,m})$, then

$$\overline{V}_i = V_i(\alpha_{i,1},\ldots,\alpha_{i,m}) \quad . \qquad (16)$$

The algorithm continues with

$$X(i+1,1)=\sum_{j=1}^{m} \alpha_{i,j} \; G(k,j)_i \qquad k=1,\ldots,n \; , \qquad (17)$$

a selection of $G(k,2)_{i+1},\ldots,G(k,m)_{i+1}$,

$$X_{i+1}=\sum_{j=1}^{m} \alpha_j \; G(k,j)_{i+1} \qquad k=1,\ldots,n \; , \qquad (18)$$

and a subsequent minimization of

$$V_{i+1}(\alpha_1,\ldots,\alpha_m)=||AX_{i+1}-F||^2. \qquad (19)$$

The solution of the original system is derived from

$$X = \text{Lim}_{i\rightarrow +\infty} G(i,1). \qquad (20)$$

To illustrate the theory of the above section an example problem is presented in Appendix A.

CHAPTER III

DESCRIPTION OF NEW TRIAL SOLUTION GENERATION

Previously the trial solution generation process was based on knowledge of the physical system represented. The new trial solution selection process successfully makes the RCIP method independent of the knowledge of the physical system. The new method utilizes only the system of equations and information resulting from other trial solutions to obtain sets of trial solutions.

At the beginning of the first iteration, an initial trial solution set is selected. On successive iterations the trial solution set is composed of the last best solution, and a sequence of special Gauss-Seidel solutions. Equations to be solved by Gauss-Seidel for trial solutions are formed from the same coefficient matrix as the original problem and a new right hand side. For the second trial solution the new right hand side is the residual from the equation set using the last best solution. The initial guess used for the Gauss-Seidel solution is zero. Three Gauss-Seidel iterations are performed to obtain trial solution two. The third trial solution is similar to trial solution two. The difference is that another new right hand side is formed from the residual of the second trial solution. Again a Gauss-Seidel solution scheme is used to solve for the solution to this new set. An initial guess of zero is used, with three iterations performed. The rest of the trial solutions are calculated in the same fashion until m number of trial solutions are reached. An example, of m equal to three trial solutions, is shown below.

Let

$$A*X=F \tag{21}$$

be an nxn system of equations. For the first iteration the first trial solution is

$$G(i,1) = \text{initial estimate} \qquad i=1,...,n. \tag{22}$$

Now find the residual vector for the first trial solution (or T.S. 1)

$$R_i(T.S.\ 1) = A*G(i,1) - F_i \qquad i=1,...,n. \tag{23}$$

Form the new equation set for trial solution two

$$A*G(i,2) = R_i(T.S.\ 1) \qquad i=1,...,n. \tag{24}$$

Solve using Gauss-Seidel with an initial guess of zero. Perform three iterations to obtain G(i,2). Now find the residual vector for the second trial solution

$$R_i(T.S.\ 2)= A*G(i,2) - R_i(T.S.\ 1) \qquad i=1,...,n. \tag{25}$$

Form the new equation set for trial solution three

$$A*G(i,3) = R_i(T.S.\ 2) \qquad i=1,...,n. \tag{26}$$

Solve this set using Gauss-Seidel with an initial guess of zero. Perform three iterations to obtain G(i,3). The three trial solutions for the first iteration have been found. For the second through the nth iterations the only change is that the first trial solution is the last best solution.

The basis for this trial solution selection process is based on the following:

$$A*G(i,1) - F - R_i(T.S.\ 1) = 0 \tag{27a}$$

$$A*G(i,2) - R_i(T.S.\ 1) - R_i(T.S.\ 2) = 0 \tag{27b}$$

$$A*G(i,3) - R_i(T.S.\ 2) = 0 \tag{27c}$$

subtract (27c) from (27b)

$$A*(G(i,2) - G(i,3)) - R_i(T.S.\ 1) = 0 \tag{28}$$

subtract (28) from (27a)

$$A*(G(i,1) - G(i,2) + G(i,3)) - F = 0\ . \tag{29}$$

Note, if G(i,3) is an exact solution to equation (27c) and

$$X_i = G(i,1) - G(i,2) + G(i,3). \tag{30}$$

then X_i is an exact solution to equation (21). Note that G(i,3) is approximate therefore, X_i is approximate.

The numerical algorithm for the generation of the trial solutions is as follows. The first trial solution for the first iteration is

$$G(i,1) = \text{initial estimate} \qquad i=1,\ldots,n. \tag{31}$$

The first trial solution for the second through the nth iterations is

$$G(i,1) = X_i \qquad i=1,\ldots,n. \tag{32}$$

The second through the mth trial solution is calculated as follows

$$R_i = F_i \qquad i=1,...,n \tag{33}$$

$$i=1,...,n \qquad kk=2,m \qquad p=1,..,3 \tag{34a}$$

$$R_i(kk) = (A(i,k)*G(k,kk-1)) - R_i(kk-1) \tag{34b}$$

$$G(i,kk) = 0 \tag{34c}$$

$$G(i,kk)^p = R_i(kk) \ / \ A(i,i) - \sum_{k=1}^{n} A(i,k)*G(k,kk) \ / \ A(i,i) \ . \tag{34d}$$

The above algorithms will calculate m number of trial solutions. An illustrative problem is solved and included in Appendix B.

CHAPTER IV

APPLICATIONS AND RESULTS

This chapter is devoted to the solution of problems by the RCIP method and a comparison with some of the existing techniques. In particular the SOR and ADEP methods are compared with the RCIP method. The RCIP method is intended to solve linear systems of algebraic equations. Therefore, to solve the following differential equations a finite difference technique is used to model each differential equation as a system of algebraic equations.

In each problem the convergence limit used is always that the variance be less than or equal to some convergence criterion. The variance of the equation set is defined as the sum of the residuals squared, for all the equations. When an exact solution is known an average percent error can be obtained which is defined as the average of the differences between the exact solution and the approximate solution divided by the exact solution.

PROBLEMS ONE THROUGH FOUR

The following four ordinary differential equations are solved to show the generality of the new method. Various boundary conditions are employed to illustrate a spectrum of problems. Problem one follows: Solve

$$\frac{d^2y}{dx^2} + y = 0 \tag{35}$$

for the region $0 <= x <= \pi/2$ with the derivative boundary conditions

$$y'(0) = 0 \quad , \quad y'(\pi/2) = 1 \quad . \tag{35a}$$

To illustrate the finite difference technique the above governing differential equation and boundary conditions will be modeled in algebraic form. The central difference approximation of the second order derivative is

$$\frac{d^2y}{dx^2} = \frac{y_{i+1} - 2\ y_i + y_{i-1}}{\Delta x^2} \tag{36a}$$

equation (35) becomes

$$\frac{y_{i+1} - 2\ y_i + y_{i-1}}{\Delta x^2} + y_i = 0 \quad . \tag{36b}$$

The central difference approximation of the first order derivative is

$$\frac{dy}{dx} = \frac{y_{i+1} - y_{i-1}}{2\Delta x} \tag{36c}$$

equations (35a) become

$$y_{i+1}(0) = y_{i-1}(0) \tag{36d}$$

$$y_{i+1}(\pi/2) - y_{i-1}(\pi/2) = 2\Delta x \quad . \tag{36e}$$

The above finite difference model approximates the differential equation as a system of linear equations.

The exact solution for problem one is

$$y_{exact} = - \text{COS } x \quad . \tag{36f}$$

The finite difference model for problem 1 is used to illustrate the SOR algorithm used. The algorithm follows:

$$Y_i(r+1) = (1-\lambda)*Y_i(r) + \lambda * \frac{Y_{i+1}(r+1) + Y_{i-1}(r+1)}{2 - \Delta X^2} \tag{37}$$

the variable r indicates the previous iteration, the value r+1 indicates the present iteration, and the subscript i indicates the nodal point. The variable λ is the SOR factor, which is between 1 and 2 for over relaxation and between 0 and 1 for under relaxation. The value of λ=1.0 is just the Gauss-Seidel routine. The remaining problems were modeled in the same manner as problem 1. And the SOR factor used in the following problems was the same as above. [5]

Problem two follows: Solve

$$\frac{d^2y}{dx^2} + y = 0 \tag{38}$$

for the region $0 <= x <= \pi/2$ with the composite boundary conditions

$$y'(0) + y(0) = 1 \tag{39}$$

$$y'(\pi/2) - y(\pi/2) = -1 \quad . \tag{40}$$

The exact solution for equations (38-40) is

$$y_{exact} = (\text{SIN } x + \text{COS } x)/2 \quad . \tag{41}$$

Problem three follows: Solve

$$x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = x^3 \tag{42}$$

for the region $1 <= x <= 2$ with the mixed boundary conditions

$$y(1) = 0 \quad , \quad y'(2) = 0. \tag{43}$$

The exact solution for equations (42-43) is

$$y_{exact} = \frac{4}{3} x - \frac{5.5}{3} x^2 + \frac{1}{2} x^3 \ . \tag{44}$$

Problem four follows: Solve

$$(1-x^2) \frac{d^2y}{dx^2} - 6x \frac{dy}{dx} - 4y = 0 \tag{45}$$

for the region $0 <= x <= 0.5$ with the Dirichlet boundary conditions

$$y(0) = -4.5 \ , \ y(0.5) = 2 \ . \tag{46}$$

The exact solution for equations (45-46) is

$$y_{exact} = \frac{27}{6} (3x-x^2/(1-x^2)^2) - \frac{27}{6} (1/(1-x^2)^2) \ . \tag{47}$$

Standard finite difference techniques are used to algebraically approximate the above problems. In each of the first four problems a nodal network is chosen so that there is 50 spaces for each region, which requires a different number of equations for each problem. The different number of equations resulted from the different boundary conditions. A uniform initial estimate of 0.5 is chosen to start the methods.

In problem one the convergence limit was that the variance in the 51 equation set be less than or equal to 0.1. The results of the RCIP method for various numbers of trial solutions are shown in Table 1. The SOR method, with over and then under relaxation factors, did not obtain the convergence criterion in 10,000 iterations with a user

Table 1. Results of 1-D Ordinary Differential Equation Problem 1, with Derivative Boundary Conditions, for Various Numbers of Trial Solutions.

NUMBER OF TRIAL SOLUTIONS	NUMBER OF ITERATIONS	AVERAGE % ERROR	USER EXECUTION TIME (SEC)
2,3,4	DID NOT CONVERGE AT 100 ITERATIONS.		
5	11	0.3208	12.97
6	11	1.1182	15.59
7	8	1.1586	13.40
8	6	0.7707	11.63
9	5	0.2624	11.13

The SOR Method Did Not Converge at 10,000 Iterations.

execution time of 660 seconds. Therefore, the RCIP method is superior to SOR in that RCIP obtains a solution and SOR does not. When the number of trial solutions was less than 5 the convergence criterion was not obtained with the RCIP method, it did not converge at 100 iterations with a user time of 240 seconds. This is because the smaller number of trial solutions does not contain enough information to obtain rapid convergence.

Problem 2 is very similar to problem 1. The results of the solution of 51 equations are shown in Table 2. When the number of trial solutions is less than 8 the convergence criterion was not obtained with the RCIP method , this was because of the smaller number of trial solutions not containing enough information to obtain rapid convergence. The trial solutions 2-7 did not converge at 100 iterations with a time of approximately 240 seconds. The SOR method did not obtain the convergence criterion at 10,000 iterations with a time of 660 seconds. Several relaxation factors were utilized, over and then under relaxation, with no convergence being obtained. The RCIP method again, produces a solution and SOR does not. The limit of convergence for this problem is that the variance be less than or equal to 0.001.

Problem 3 was solved and the results are shown in Table 3 for the RCIP method and in Table 4 for the SOR method. A graph of the user execution time versus the number of trial solutions for the RCIP method, and over relaxation factor for the SOR method is included (Fig. 1). Clearly, as the graph shows, the convergence to a solution requires less time for the RCIP method than for the SOR method. The

Table 2. Results of 1-D Ordinary Differential Equation Problem 2, with Composite Boundary Conditions, for Various Numbers of Trial Solutions.

NUMBER OF TRIAL SOLUTIONS	NUMBER OF ITERATIONS	AVERAGE % ERROR	USER EXECUTION TIME (SEC)
2-7	DID NOT CONVERGE AT 100 ITERATIONS.		
8	22	.2306	41.29
9	10	.1739	21.69
10	4	.2227	9.93
11	4	.1333	10.95

The SOR Method did not Converge at 10,000 Iterations.

Table 3. Results of 1-D Ordinary Differential Equation Problem 3, with Mixed Boundary Conditions, for Various Numbers of Trial Solutions.

NUMBER OF TRIAL SOLUTIONS	NUMBER OF ITERATIONS	AVERAGE % ERROR	USER EXECUTION TIME (SEC)
2	DID NOT CONVERGE AT 100 ITERATIONS.		
4	20	.0761	17.77
6	8	.0686	10.99
8	6	.0088	11.19
10	5	.0181	11.84
12	4	.0324	11.54

Table 4. Results of 1-D Ordinary Differential Equation Problem 3, with Mixed Boundary Conditions, for Various SOR Factors.

SOR FACTORS	NUMBER OF ITERATIONS	AVERAGE % ERROR	USER EXECUTION TIME (SEC)
1.0-1.3	DID NOT CONVERGE AT 1000 ITERATIONS.		
1.35	837	.1216	52.21
1.4	347	.1594	21.92
1.5	648	.0629	40.50
1.6	680	.0634	42.52

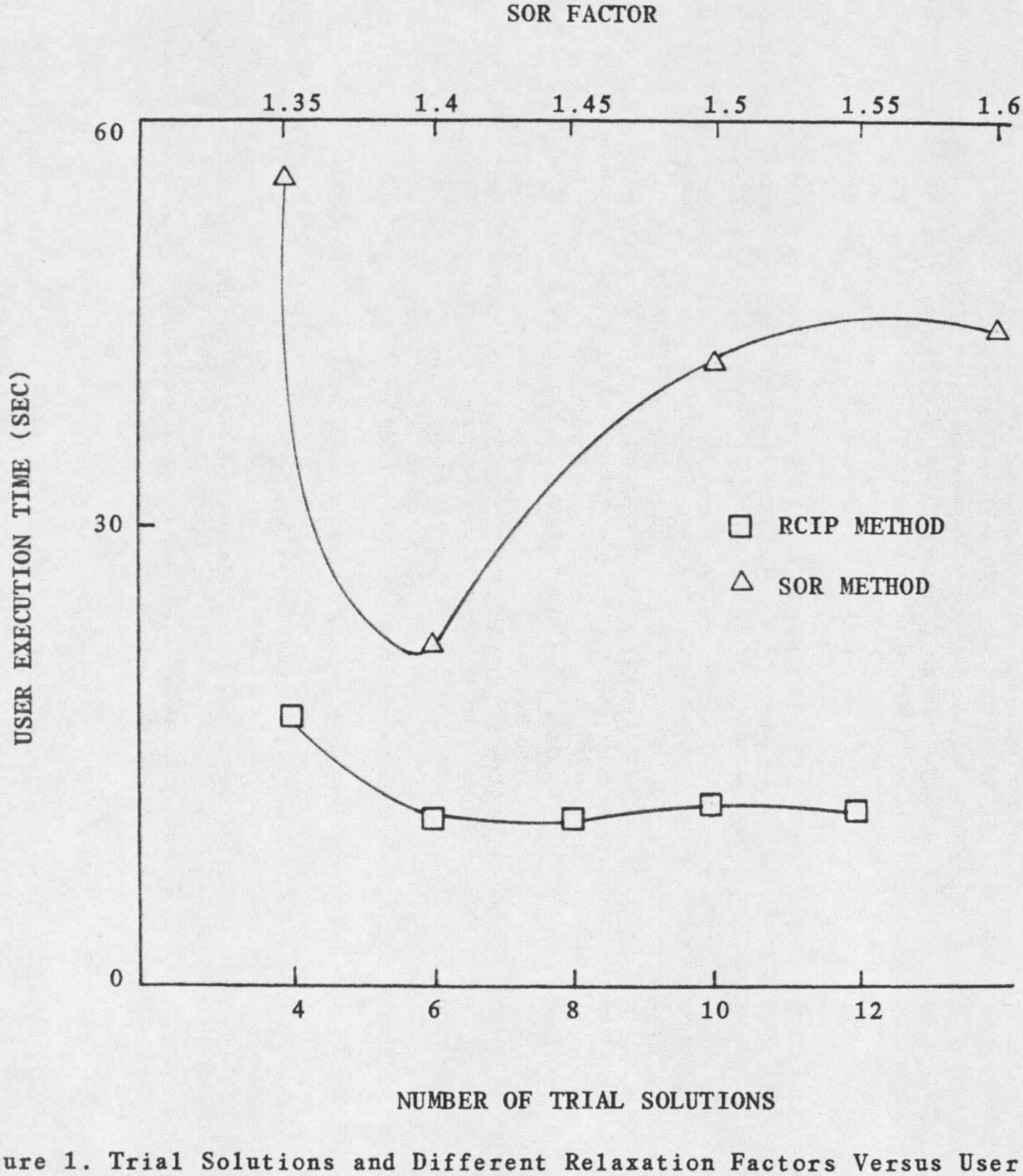

Figure 1. Trial Solutions and Different Relaxation Factors Versus User Execution Time for 1-D Ordinary Differential Equation Problem 3, with Mixed Boundary Conditions

graph of the trial solutions versus the execution time for the RCIP method is flat indicating about the same execution time for any number of trial solutions, where the graph of the SOR factor versus execution time for the SOR method has a dip indicating the importance of the optimum SOR factor. The solution to problem 3 is for 50 equations and a variance less than or equal to 0.001.

Problem 4 was solved and the results are found in Table 5 for the RCIP method and Table 6 for the SOR method. A graph of the user execution time versus the number of trial solutions for the RCIP method, and the over relaxation factors for the SOR method is included (Fig. 2). In Figure 2, the SOR and the RCIP method both have approximately the same results. The RCIP method was faster, by approximately 10 seconds, in the execution time than the SOR method, until the SOR method approaches its optimum relaxation factor where the SOR method was faster. For this problem the two methods are comparable. The methods solved 49 equations for a variance less than or equal to 0.001.

PROBLEM FIVE

Laplace's equation was solved in a rectanglar region of length one in the y direction and two in the x direction. Each side of the rectangle is maintained at a different temperature.

The problem is formally stated as follows: Solve

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad (48)$$

for the region

$$0 \leq x \leq 2 \ , \ 0 \leq y \leq 1 \qquad (49)$$

Table 5. Results of 1-D Ordinary Differential Equations Problem 4, with Dirichlet Boundary Conditions, for Various Numbers of Trial Solutions.

NUMBER OF TRIAL SOLUTIONS	NUMBER OF ITERATIONS	AVERAGE % ERROR	USER EXECUTION TIME (SEC)
2-3	DID NOT CONVERGE AT 100 ITERATIONS.		
4	51	7.5369	42.83
6	20	7.5286	25.89
8	10	7.5297	17.77
10	6	7.5252	13.58
11	5	7.5354	12.58

Table 6. Results of 1-D Ordinary Differential Equation Problem 4, with Dirichlet Boundary Conditions, for Various SOR Factors.

SOR FACTORS	NUMBER OF ITERATIONS	AVERAGE % ERROR	USER EXECUTION TIME (SEC)
1.0-1.3	DID NOT CONVERGE AT 1000 ITERATIONS.		
1.4	925	7.5231	55.71
1.5	709	7.5234	42.78
1.6	517	7.5237	31.25
1.7	342	7.5248	20.84
1.8	181	7.5463	11.22
1.9	148	7.5367	9.23

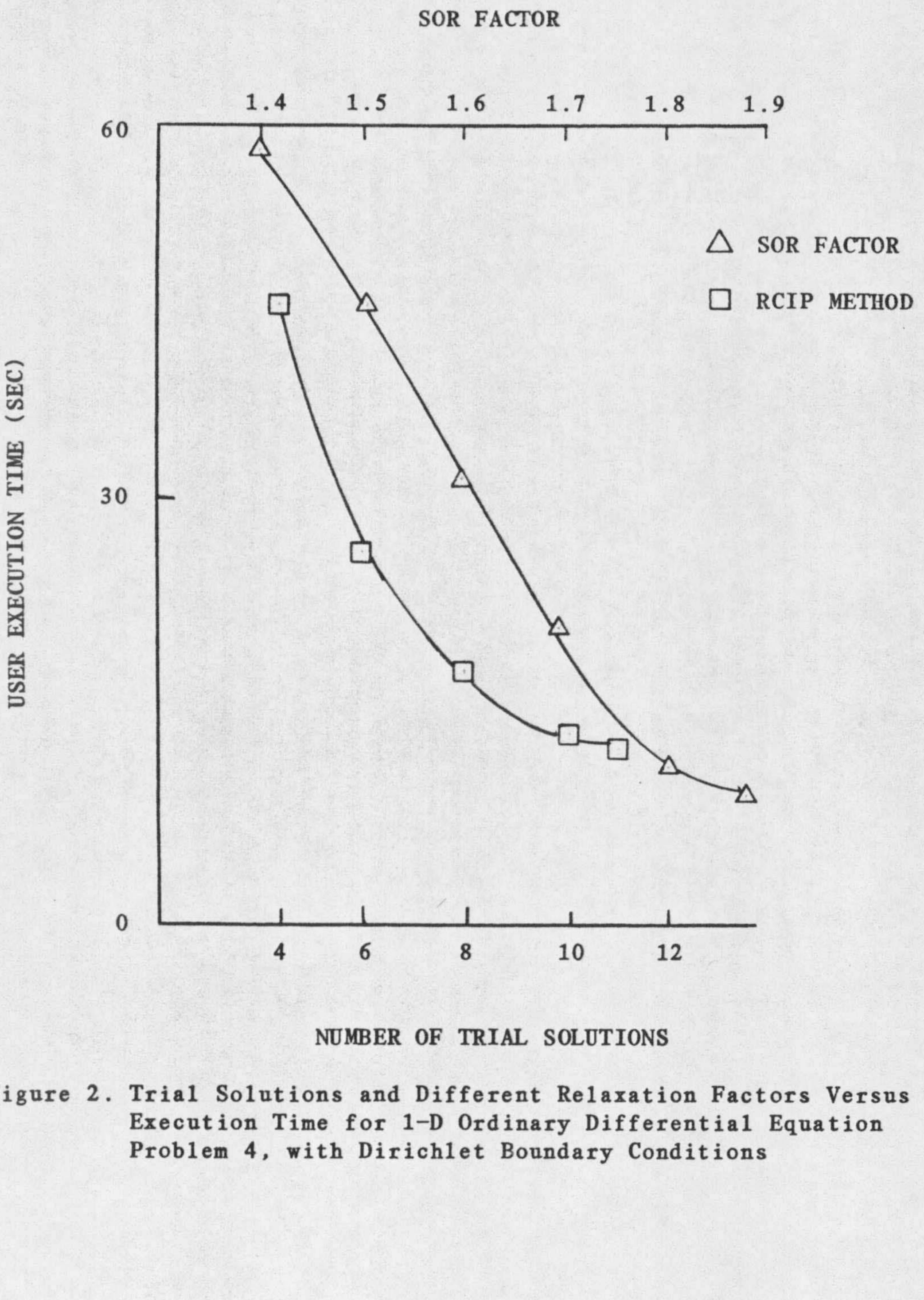

Figure 2. Trial Solutions and Different Relaxation Factors Versus User Execution Time for 1-D Ordinary Differential Equation Problem 4, with Dirichlet Boundary Conditions

with the boundary conditions

$$T(x,0) = 200 \tag{50a}$$

$$T(x,2) = 400 \tag{50b}$$

$$T(0,y) = 300 \tag{51a}$$

$$T(1,y) = 100 \quad . \tag{51b}$$

A standard finite difference technique was used to approximate the equation in (48). A nodal network of 7 nodes in the y direction and 15 nodes in the x direction, was selected. This results in Δx and Δy being 0.125.

All solution attempts began with an initial estimate of a uniform temperature distribution equal to unity, the results are compared to the analytical solution [7]. The convergence limit for the solution of the 105 equations was set at the variance being less than or equal to 0.000001. Data from the solutions using different numbers of trial solutions is presented in Table 7 for the RCIP method, and data from the solutions using different relaxation factors is presented in Table 8 for the SOR method.

The results for the RCIP method are compared to the results of the SOR method. A graph of execution time versus trial solutions for the RCIP method, and SOR factor for the SOR method will indicate the effectiveness of the RCIP method in speed and accuracy.(Fig. 3)

Figure 3 indicates that the SOR method has a steady decrease in execution time for each successive SOR factor until the optimum value of 1.53 is reached. After the optimum is reached an increase is

Table 7. 2-D Laplace's Equation Results for Various Numbers of Trial Solutions

NUMBER OF TRIAL SOLUTIONS	NUMBER OF ITERATIONS	AVERAGE % ERROR	USER EXECUTION TIME (MIN)
2	23	7.0529	1.3430
3	9	7.0528	.9422
4	6	7.0527	.8967
5	4	7.0528	.7912
6	3	7.0527	.7387
7	3	7.0527	.8490

Table 8. 2-D Laplace's Equation Results for Various SOR Factors

SOR FACTOR	NUMBER OF ITERATIONS	AVERAGE % ERROR	USER EXECUTION TIME (MIN)
1.0 (Gauss-Seidel)	129	7.0529	1.6250
1.2	84	7.0529	1.0895
1.4	51	7.0529	.6958
1.5	35	7.0528	.5023
1.6	35	7.0527	.5010
1.7	48	7.0527	.6588
1.8	75	7.0527	.9808

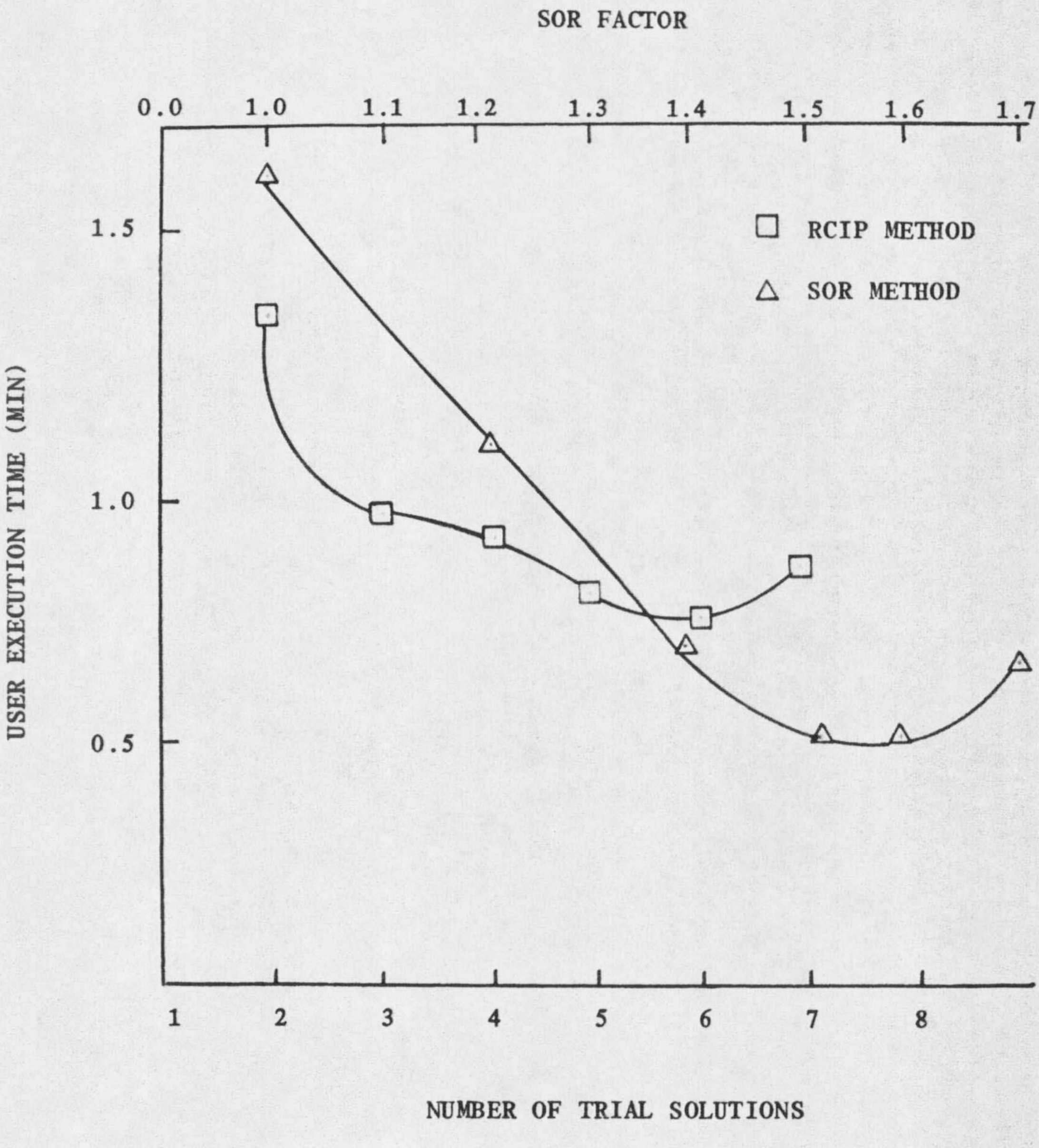

Figure 3. Trial Solutions and Different Relaxation Factors Versus User Execution Time for 2-D Laplace's Equation

observed. In the RCIP method there is a large decrease in the execution time from 2 trial solutions to 3 trial solutions, but thereafter until 7 trial solutions the execution time decreases by only 9%. The decrease in execution time from 3 to 4 trial solutions is only 0.046 minutes, where the average decrease from 3 to 6 trial solutions is 0.068 minutes per trial solution. This small decrease is due to the next to last iterations variance being just above the convergence limit this causes the method to perform one more iteration with a 98% decrease in the variance, $0.2X10^{-9}$, below the convergence limit.

The SOR methods solution had a smaller execution time at its optimum value of 1.53 than any of the solutions of the RCIP method. The RCIP method is flat, only 9% difference from 3 to 6 trial solutions, in its solutions for different numbers of trial solutions. This shows that any choice of trial solutions, with the possible exception of 2, is going to take approximately the same amount of time to obtain a solution. Where the SOR method has a large difference , 69% from Gauss-Seidel to the optimum SOR factor, in its execution time. A accurate solution with acceptable execution time is only obtained close to the optimum value.

PROBLEM SIX

The elliptical partial differential equation to be solved is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \qquad (52)$$

where T, x, y, and z are non-dimensional variables for temperature and three spatial coordinates. The geometry is a cube of face length 0.4

located concentrically in a large cube of face length one. The inner cube is maintained at a temperature of unity while the surface of the outer cube is set at zero temperature. By using the symmetry of the problem only a sixteenth of the composite cube need be considered. (Fig. 4)

The problem is formally stated as: Solve

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \qquad (53)$$

for the region

$$0 <= x <= z \quad , \quad 0 <= y <= 1 \quad , \quad 0 <= z <= 1 \qquad (54)$$

with the boundary conditions

$$T(x,y,1) = 0 \qquad (55a)$$

$$T(x,1,z) = 0 \qquad (55b)$$

$$T(x,y,0.4) = 1 \qquad (55c)$$

$$T(x,0.4,z) = 1 \qquad (55d)$$

$$\frac{\partial T}{\partial x}(z,y,z) = 0 \quad , \quad \frac{\partial T}{\partial y}(x,0,z) = 0 \qquad (55e)$$

$$\frac{\partial T}{\partial z}(z,y,z) = 0 \quad , \quad \frac{\partial T}{\partial x}(0,y,z) = 0 \quad . \qquad (55f)$$

A standard finite difference technique was used to approximate the second order derivatives in equation (53). A central difference approximation was used for the derivatives in equations (55e) and (55f). A nodal network of 10 nodes per side was selected resulting in 10 nodal spaces, therefore, the number of equations is 475 with $\Delta x = \Delta y = \Delta z = 0.1$.

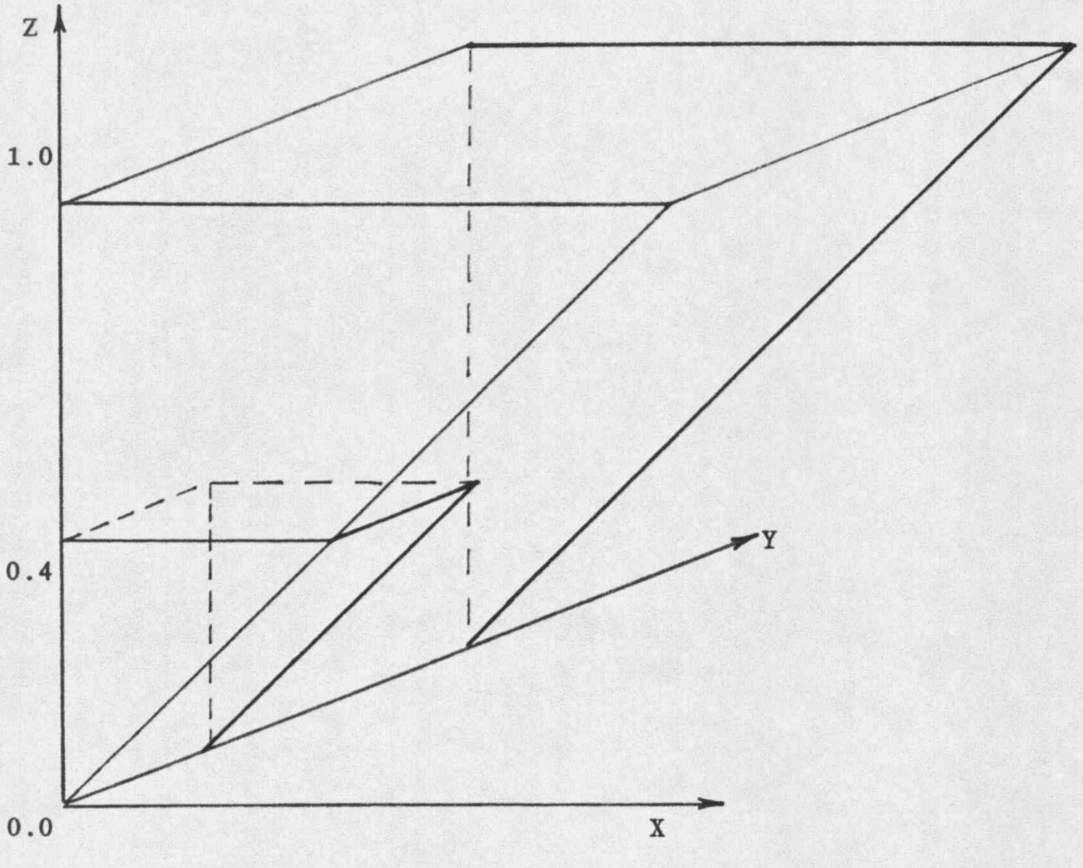

Figure 4. Cube-Cube Geometry for 3-D Steady-State Heat Conduction Problem

To determine a measure of the accuracy of the solutions, heat fluxes were calculated for the inner cube surfaces, and for the outer cube surfaces. To non-dimensionalize these quantities, they were divided by the heat flux of concentric spheres with the same boundary conditions (T_i inner cube, T_o outer cube), and radii equal to the face lengths of the inner (r_i) and outer (r_o) cubes. The value T_2 is the boundary condition acting over the area A. The value T_1 is the inner parallel nodal point associated with T_2, with a distance of Δx between them. This non-dimensional heat flux (q') is as follows

$$q' = \frac{-A(T_2-T_1)(r_o-r_i)}{\Delta x \; 4\pi \; r_i r_o (T_i-T_o)} \tag{56}$$

$$T_i = 1 \; , \; T_o = 0 \; , \; r_i = .4 \; , \; r_o = 1 \tag{57a}$$

$$\Delta x = .1 \quad , \; A = .1 \times .1 \tag{57b}$$

$$q' = \frac{-0.15(T_2-T_1)}{4\pi} \quad . \tag{58}$$

Different numbers of trial solutions, for the RCIP method, were used to obtain solutions to this problem. An acceptable solution was reached, with a variance less than or equal to 0.0000001, for each trial solution set. The solutions using two or three trial solutions were slower to converge than the solutions using 4, 5, 6, or 7 trial solutions, which all take approximately the same amount of time. The average rate increase from 5 to 7 trial solutions is 0.045 seconds per trial solution. The increase of 1.51 seconds in execution time from 5 to 6 trial solutions is higher than the average rate increase. This is due to the next to last iterations variance being just above the convergence limit. This causes the method to continue with another iteration. This continuation to the last iteration gives a variance, $0.1X10^{-10}$, 100% lower than the convergence limit.

The results from the RCIP method (Table 9) were compared to the results obtained by solving the same problem using the SOR method. The same convergence limit was used as for the RCIP method, variance less than or equal to 0.0000001, to obtain comparable results. Over relaxation factors ranging from 1.0 (which makes the routine identical to Gauss-Seidel) to a value of 1.8 were used in the SOR method to obtain solutions. The optimum value of the over relaxation factor was found to be 1.6.

A uniform temperature distribution of 0.5 was used as the initial estimate in each method. The results from the various SOR runs are shown in Table 10. The results are also shown on a graph (Fig. 5) of execution time versus the number of trial solutions in the RCIP method, and the relaxation factor in the SOR method. The graph of the results of various SOR factors and execution time decreases at an approximate rate of 1.89 seconds per 0.1 increase in the SOR factor. While the graph of the RCIP methods results decreases at an approximate rate of 2.6 seconds per trial solution from 2 to 4 trial solutions. Thereafter, the curve is flat with a 2% increase for trial solutions 4, 5, 6, and 7.

The two curves cross with the SOR having the smallest execution time at the optimum. If the optimum SOR factor is known then the SOR method has a significant advantage. The disadvantage of the SOR method

Table 9. Results for the 3-D Steady-State Heat Conduction Problem, Cube-Cube Geometry, for Various Numbers of Trial Solutions.

NUMBER OF TRIAL SOLUTIONS	NUMBER OF ITERATIONS	HEAT FLUX INNER CUBE	HEAT FLUX OUTER CUBE	% DIFFERENCE IN HEAT BALANCE	USER EXECUTION TIME (SEC)
2	18	.09527	.09530	.03845	12.29
3	7	.09527	.09529	.02601	8.55
4	4	.09527	.09527	.00304	7.09
5	3	.09527	.09529	.01795	7.15
6	3	.09527	.09527	.00110	8.66
7	2	.09527	.09528	.00943	7.24

Table 10. Results of the 3-D Steady-State Heat Conduction Problem, Cube-Cube Geometry, for Various SOR Factors.

SOR FACTOR	NUMBER OF ITERATIONS	HEAT FLUX INNER CUBE	HEAT FLUX OUTER CUBE	% DIFFERENCE IN HEAT BALANCE	USER EXECUTION TIME (SEC)
1.0 (Gauss-Seidel)	104	.09526	.09531	.05595	16.55
1.2	69	.09526	.09531	.05081	11.44
1.4	44	.09526	.09530	.03516	7.86
1.5	33	.09527	.09529	.02392	6.18
1.6 (optimum)	26	.09527	.09527	.00129	5.23
1.7	33	.09527	.09527	.00141	6.19
1.8	51	.09527	.09527	.00073	8.82

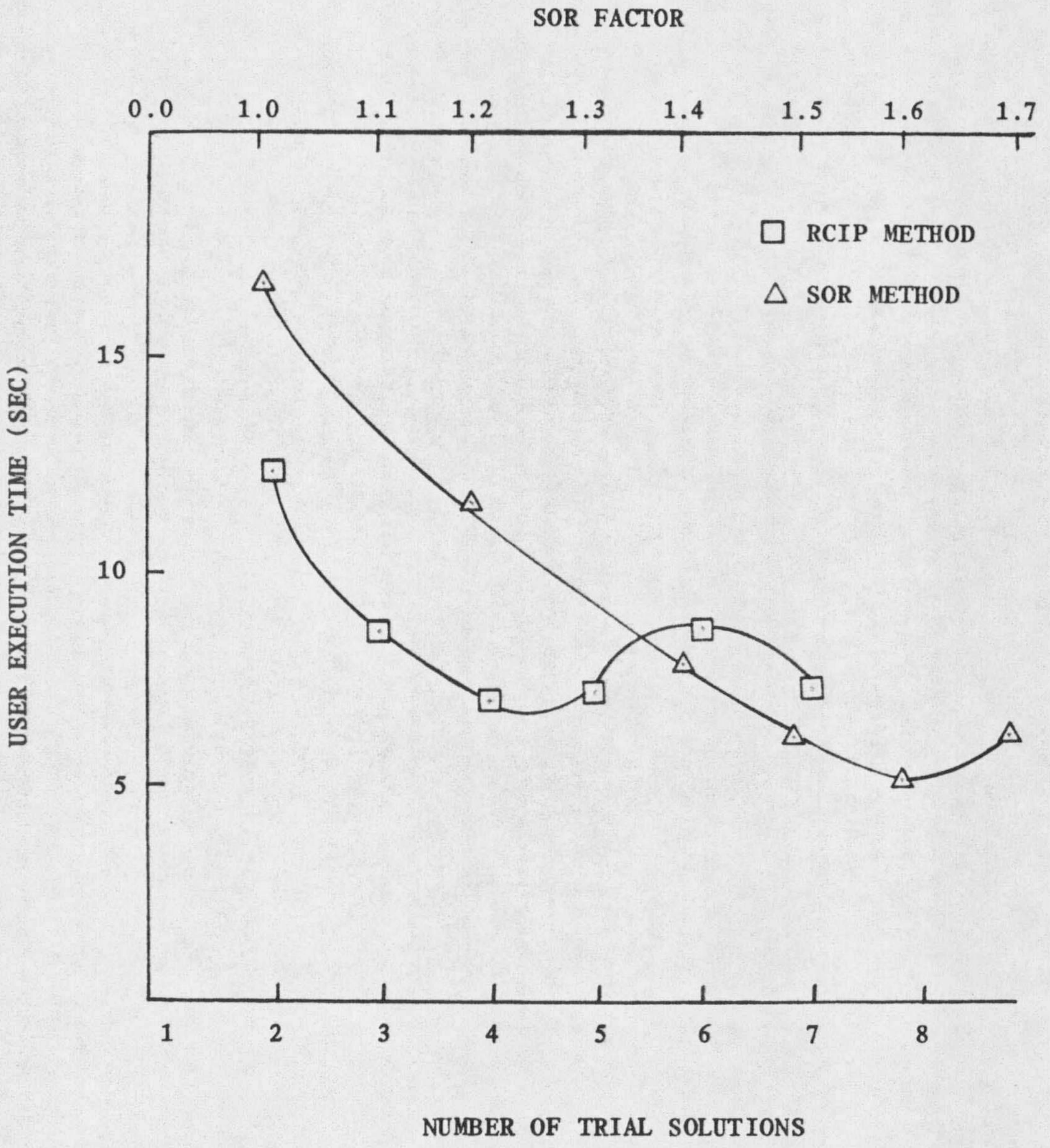

Figure 5. Trial Solutions and Different Relaxation Factors Versus User Execution time for 3-d Steady-State Heat Conduction Problem Cube-Cube Geometry

was that a trial and error process must be used to obtain the optimum value of the relaxation factor. The RCIP method will be more efficient for the trial solutions 4, 5, 6, and 7, because of the small variation in the execution times. The choice of 2 or 3 trial solutions was not as efficient as 4 to 7 trial solutions, but 2 or 3 still produce excellent results.

PROBLEM SEVEN

The parabolic partial differential equation to be solved is

$$\frac{\partial T}{\partial \tau} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \qquad (59)$$

for the cube region 0 <= x <= 1, 0 <= y <= 1, and 0 <= z <= 1, where T, τ, x, y, and z are all non-dimensional variables representing temperature, time, and three spatial coordinates, respectively. The boundary conditions for (59) are:

$$T = 0 \text{ at } x=1 \text{ or } y=1 \text{ or } z=1 \qquad (60a)$$

$$\frac{\partial T}{\partial x} = 0 \text{ at } x=0 \qquad (60b)$$

$$\frac{\partial T}{\partial y} = 0 \text{ at } y=0 \qquad (60c)$$

$$\frac{\partial T}{\partial z} = 0 \text{ at } z=0 \qquad (60d)$$

and the initial condition is

$$T = 1 \text{ at } \tau = 0 \quad . \qquad (61)$$

The algebraic approximation of the parabolic equation (59) was accomplished by using the Crank Nicolson method [5], and a central difference was used to approximate the first order derivative in (60b-60d). The cube geometry is approximated by a nodal network of 10X10X10 nodes which gives 1000 equations and 1000 unknowns. This results in 10 spaces on each face of the cube, with a mesh size of $\Delta x = \Delta y = \Delta z = 0.1$. Results were determined for three elapsed times, τ=0.1, 0.5, and 1.0. The effect of using different values of the time increment was also studied.

The ADEP method was compared to the RCIP method for this problem. A uniform temperature distribution, equal to unity, was used as the initial estimate to start the solution process for both the RCIP and ADEP methods. Tables 11 and 12, and Figures 6, 7, and 8 illustrate the relation between the execution time and accuracy of each method.

A value of 4 trial solutions was used to obtain the results shown in Table 11 for the RCIP method. The number of trial solutions was chosen from the results of the previous 3-D steady-state problem, in which 4 trial solutions produced the smallest execution time (7.09 seconds).

For the elapsed time of τ=0.1 Table 12 and Fig. 6 shows that ADEP produces an average percent error of more than .26% for all the time steps except for Δt=0.01, which proved too large to maintain accuracy. The average percent error increases from .26% to .28% as the time step gets smaller. This increasing error is due to the large number of iterations and the error in the algebraic approximation.

Table 11. Results of 3-D Transient Heat Conduction Problem, Cube Geometry, Using RCIP.

TIME STEP Δt	NUMBER OF TIME STEPS	ELAPSED TIME, τ	AVERAGE % ERROR	VARIANCE	USER EXECUTION TIME (MIN)
0.02	5	0.1	6.2355	.1739E-07	.4635
0.0125	8	0.1	.6703	.1978E-10	.6825
0.01	10	0.1	.4398	.8791E-06	.7838
0.005	20	0.1	.2852	.6876E-09	.8332
0.004	25	0.1	.2811	.4648E-10	1.0180
0.0025	40	0.1	.2808	.8269E-13	1.5627
0.002	50	0.1	.2814	.3033E-14	1.9273
0.025	20	0.5	2.4292	.9586E-10	1.6240
0.02	25	0.5	.5827	.6116E-06	1.8158
0.0125	40	0.5	.1233	.1627E-07	2.1403
0.01	50	0.5	.0303	.2478E-08	2.2113
0.005	100	0.5	.1010	.2145E-11	3.7417
0.004	125	0.5	.1164	.1476E-12	4.6422
0.002	250	0.5	.1368	.9767E-17	9.2708

Table 11. (continued)

0.05	20	1.0	1540.0572	.1800E-08	1.9290
0.04	25	1.0	237.1156	.1974E-06	2.0658
0.025	40	1.0	1.7589	.1723E-08	2.4108
0.02	50	1.0	.4782	.3728E-09	2.6940
0.0125	80	1.0	.3681	.1043E-10	3.5928
0.01	100	1.0	.5620	.1529E-11	4.0158
0.005	200	1.0	.8218	.1343E-14	7.3598

Table 12. Results of 3-D Transient Heat Conduction Problem, Cube Geometry, Using ADEP.

TIME STEP Δt	NUMBER OF TIME STEPS	ELAPSED TIME, τ	AVERAGE % ERROR	USER EXECUTION TIME (MIN)
0.01	10	0.1	24.2707	.1210
0.001	100	0.1	.2672	.3162
0.0005	200	0.1	.2605	.5322
0.0002	500	0.1	.2784	1.1833
0.0001	1000	0.1	.2818	2.2705
0.001	500	0.5	1.2731	1.1302
0.0005	1000	0.5	.4254	2.2125
0.00025	2000	0.5	.2140	4.3562
0.0002	2500	0.5	.1887	5.4307
0.000125	4000	0.5	.1612	8.6435
0.0001	5000	0.5	.1549	10.7778
0.002	500	1.0	11.1638	1.1108

Table 12. (Continued)

0.001	1000	1.0	3.4006	2.1768
0.0005	2000	1.0	1.5261	4.3140
0.00025	4000	1.0	1.0616	8.5410

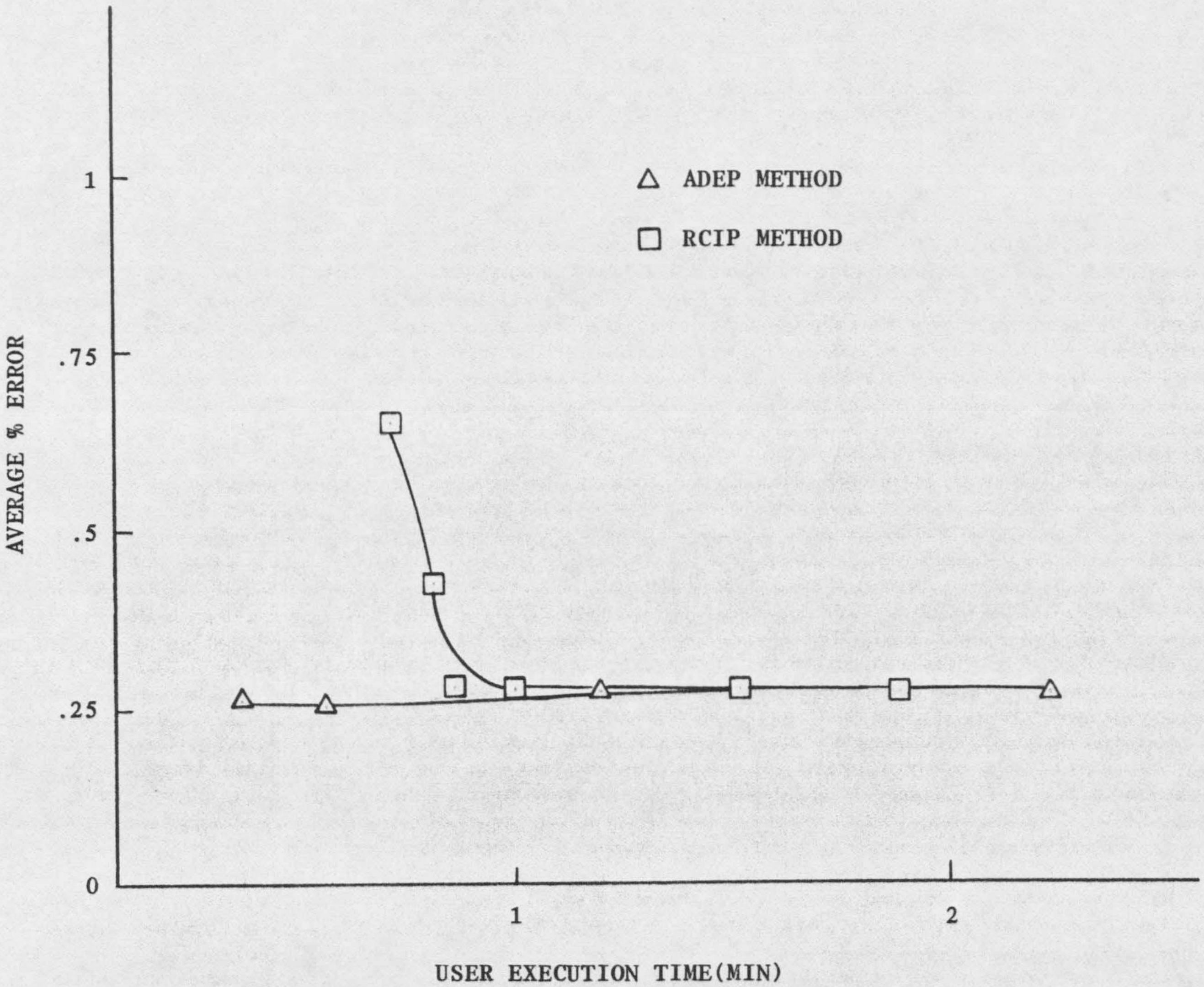

Figure 6. User Execution Time Versus Average Percent Error. RCIP Versus ADEP for the 3-D Transient Heat Transfer Problem with an Elapsed Time $\tau=0.1$.

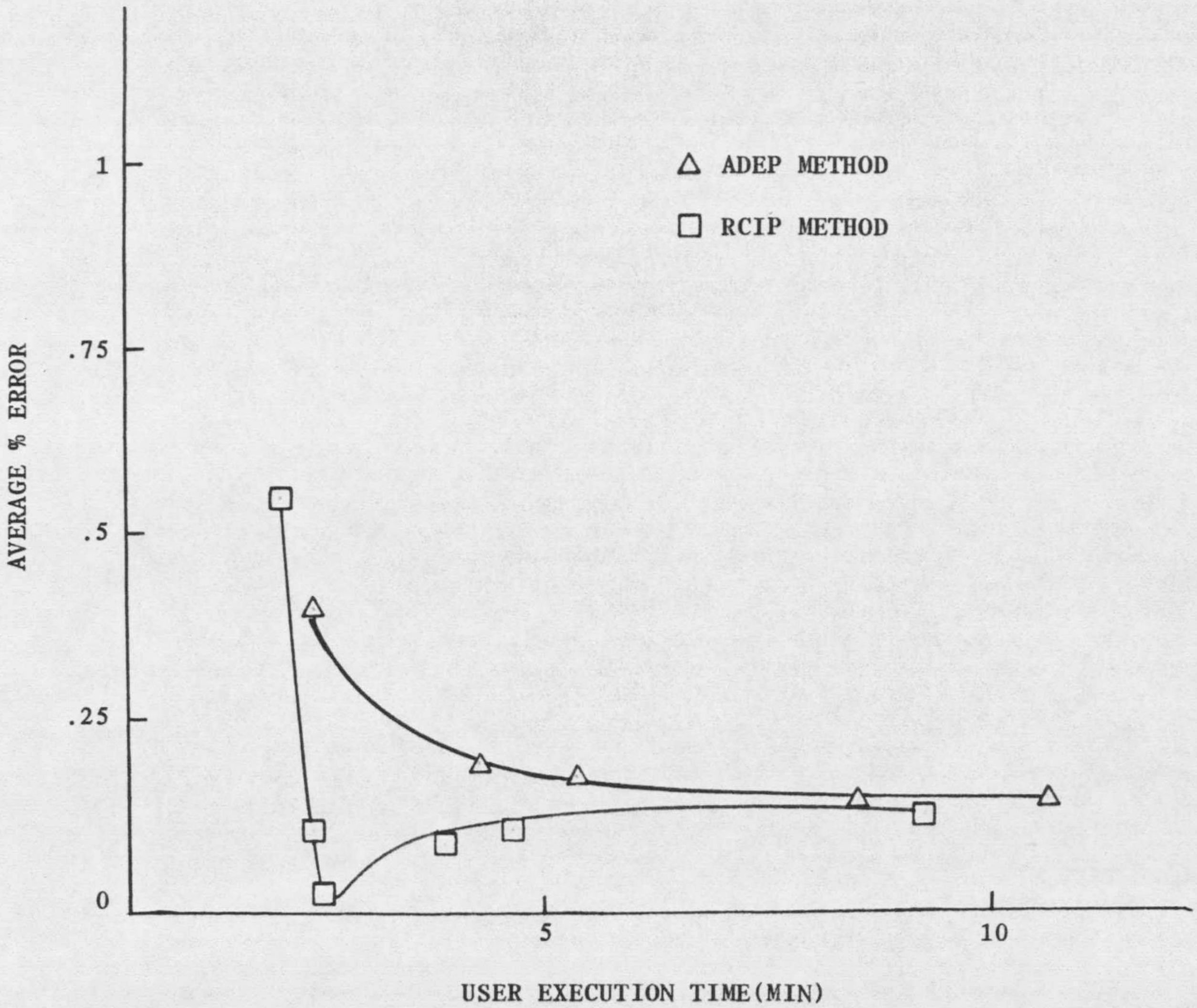

Figure 7. User Execution Time Versus Average Percent Error. RCIP Versus ADEP for the 3-D Transient Heat Transfer Problem with an Elapsed Time $\tau=0.5$.

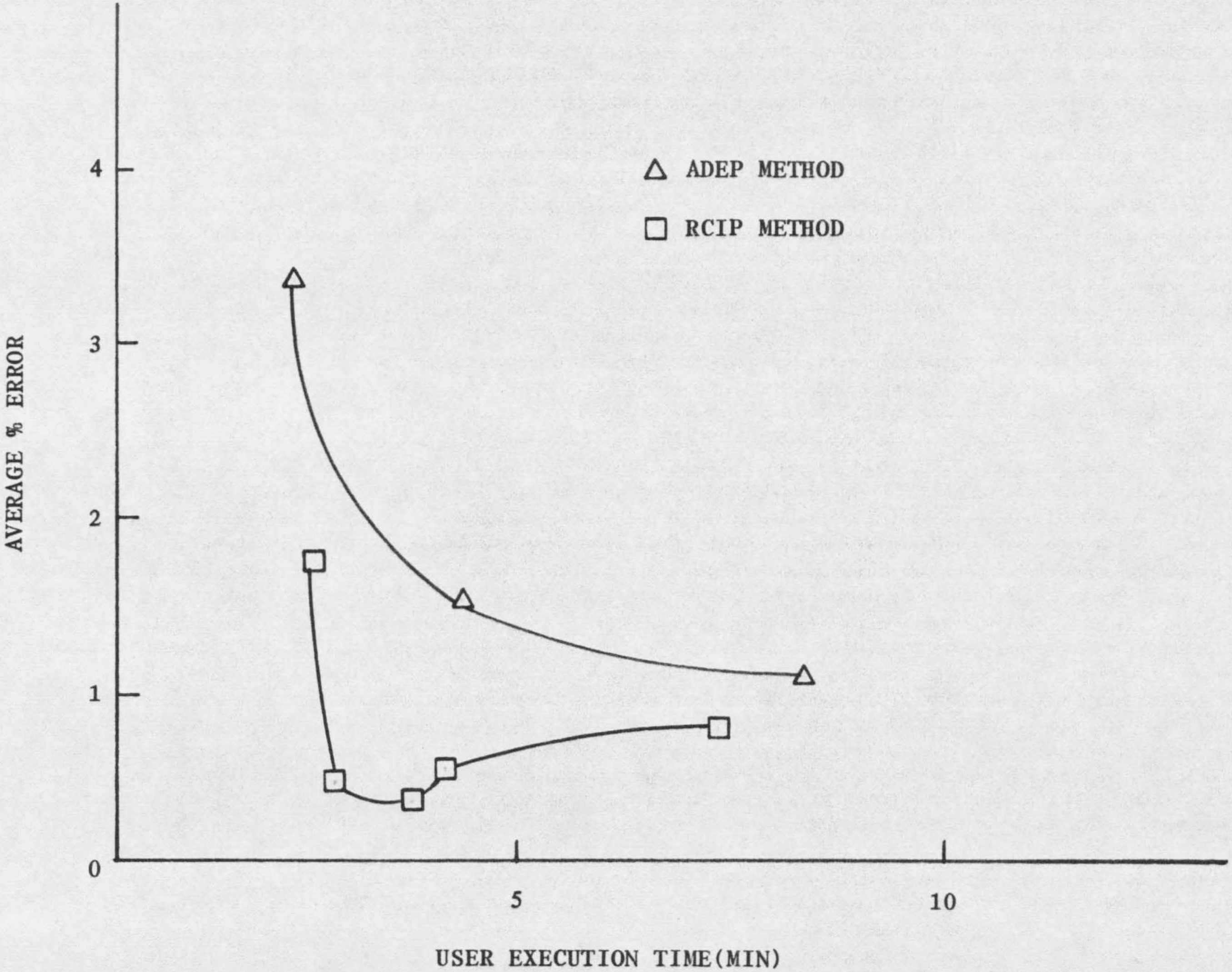

Figure 8. User Execution Time Versus Average Percent Error. RCIP Versus ADEP for the 3-D Transient Heat Transfer Problem with an Elapsed Time $\tau=1.0$.

The graph of the execution time versus accuracy for the RCIP method (Fig. 6) indicates that when the smallest percent error (0.28%) was reached an optimum value of the time step had also been reached. Therefore, further refinement of the time step would require a larger amount of time to produce the same accuracy. The RCIP method was able to reach the accuracy of the ADEP method, but ADEP was superior in execution time for a specified accuracy over the RCIP method. This is due to the small elapsed time τ=0.1. Because of this small time the solution process resembles that of the steady-state problem. However, noting that as the elapsed time is increased larger time steps can be used. The ability of RCIP to use larger time steps, as compared to the smaller time steps that ADEP requires, enables RCIP to obtain the required accuracy in less time.

For the elapsed time of τ=0.5, Tables 11, 12 and Figure 7, indicate that the RCIP method produces more accurate results in less time than the ADEP method. The RCIP method obtained an average percent error equal to 0.0303% in 2.1403 minutes of execution time, with a time step of 0.0125 and 40 iterations. The ADEP method reached its best results (0.1612%) at Δt=0.000125 and 4000 iterations, with an execution time of 8.6435 minutes. The graph, Fig. 7, indicates that when the smallest percent error 0.0303% occurs the RCIP method has reached an optimum time step. The further refinement of the time step produced poorer accuracy (0.1164%) and larger execution times (4.6422 minutes).

For the elapsed time of τ=1.0, Table 11, 12, and Fig. 8, the graph of the results of the RCIP method is entirely below the graph of ADEP's results. At each time step the RCIP method is much faster. The use of the larger time steps results in the RCIP method having the advantages of less iterations with less error, and a faster execution time. If larger elapsed times are used ADEP will perform numerous iterations, therefore the ADEP method will be slower than the RCIP method and less accurate.

All computer runs were accomplished on the Honeywell CP-6 computer in double precision. A flow chart of the computer program is included in Appendix C.

CHAPTER V

DEVELOPMENT OF RCDP WITH APPLICATION AND RESULTS

Many direct methods have been developed to solve sets of linear equations. These methods have been restricted to solving small sets of equations usually less than 40 [5]. A new method is presented here which could prove to be much faster for the solution of large sets of tridiagonal equations.

This new method, called the Reduced Coordinate Direct Procedure (RCDP), uses two trial solutions to solve for the weighting coefficients in the RCIP method. The trial solutions generated consist of one trial solution due to the coefficient matrix and one trial solution due to the boundary conditions. These two trial solutions successfully solve the problem in one iteration. An attempt was made to expand this method to a general routine to solve non-tridiagonal systems. A problem was discovered with this expansion. For non-tridiagonal systems more than two trial solutions are necessary to solve the problem. In the development of the extra trial solutions and the subsequent reduced matrix to solve for the weighting coefficients the problem was found. When more than two trial solutions is required the reduced matrix becomes closer and closer to being singular as the number of trial solutions is increased. Therefore the method would give erroneous results.

The algorithms to form the trial solutions for RCDP are as follows. Let

$$A*X = F \tag{62}$$

be a system of n equations and n unknowns. To calculate the trial solution due to the boundary conditions set

$$G(i,m) = 0 \qquad i=1,...,m-1 \tag{1}$$

then

$$G(i+m-1,m) = F(i)/A(i,i+m-1) - \sum_{j=1}^{i+m-2} A(i,j)*G(j,m)/A(i,i+m-1) \tag{2}$$

$$i=1,...,n-m+1 \quad .$$

To calculate the coefficient matrix trial solution set

$$G(i,i) = 1 \qquad i=1,...,m-1 \tag{3}$$

$$G(j,i) = 0 \qquad j=1,...,m-1 \;,\; j \mathrel{/=} i \;,\quad i=1,...,m-1 \tag{4}$$

then

$$G(i+m-1,k) = - \sum_{j=1}^{i+m-2} A(i,j)*G(j,k)/A(i,i+m-1) \tag{5}$$

$$i=1,...,n-m+1 \;,\quad k=1,...,m-1 \quad . \tag{6}$$

The algorithms are written in general form, although for a tridiagonal system m is equal to two. A comparision of RCDP with an existing technique, Gaussian-Elimination [5], were made to determine the validity of the method. An illustrative problem is solved and included in Appendix D.

The four one-dimensional problems stated in Chapter IV pages 11 and 12 are to be solved by each method. A number of different mesh sizes were chosen to obtain results for small to large equation sets.

The results of the RCDP and the Gaussian-Elimination methods for various numbers of equations are shown in the following eight tables. Comparisons are made for each problem on a graph of execution time versus average percent error.

The results of the RCDP method and the results of the Gaussian-Elimination method show that at larger mesh sizes the two methods are comparable. As the mesh size gets smaller, or larger equation sets, the RCDP method produces accurate results in less time than the Gaussian-Elimination method. This is due to the number of calculations performed by each method. The Gaussian-Elimination method requires n^3 calculations to produce a solution, and the RCDP requires $n^{2.8}$ calculation to produce a solution.

In problem 4 the graph Figure 12 shows that the RCDP method produces much more accurate results than the Gaussian-Elimination method in less time.

Table 13. Results of RCDP for the Ordinary Differential Equation, with Derivative Boundary Conditions, for Various Numbers of Equations.

NUMBER OF EQUATIONS	AVERAGE % ERROR	USER EXECUTION TIME (SEC)
11	.3099	.18
21	.0772	.30
51	.0123	.80
101	.0031	2.20
201	.0008	7.53

Table 14. Results of Gaussian-Elimination for the Ordinary Differential Equation, with Derivative Boundary Conditions, for Various Numbers of Equations.

NUMBER OF EQUATIONS	AVERAGE % ERROR	USER EXECUTION TIME (SEC)
11	.3099	.16
21	.0772	.30
51	.0123	1.53
101	.0031	8.94
201	.0008	66.02

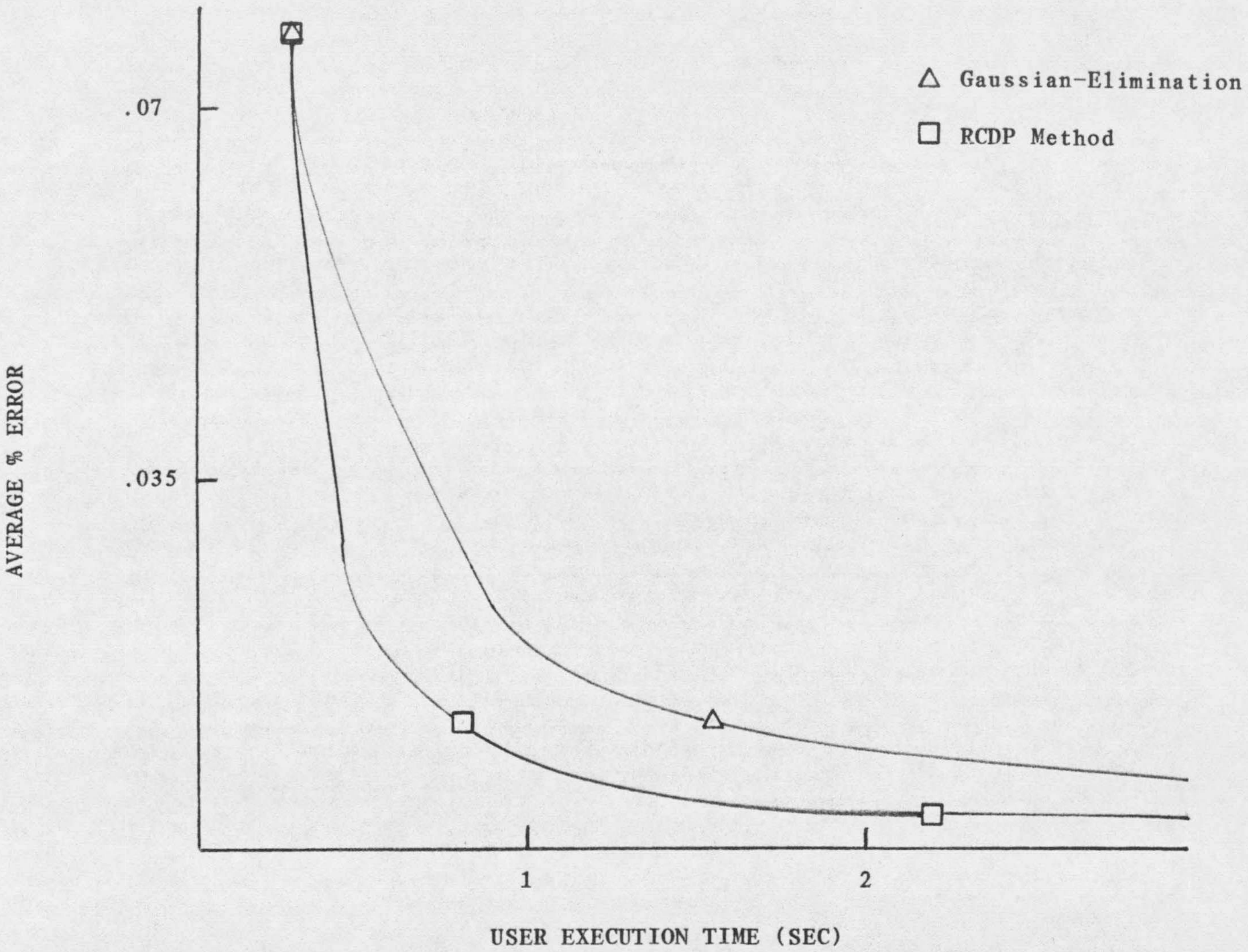

Figure 9. User Execution Time Versus Average Percent Error for RCDP Versus Gaussian-Elimination of 1-D Ordinary Differential Equation Problem 1, Derivative Boundary Conditions.

Table 15. Results of RCDP for the Ordinary Differential Equation, with Composite Boundary Conditions, for Various Numbers of Equations.

NUMBER OF EQUATIONS	AVERAGE % ERROR	USER EXECUTION TIME (SEC)
11	.1246	.17
21	.0318	.29
51	.0052	.80
101	.0013	2.19
201	.0003	7.40

Table 16. Results of Gaussian-Elimination for the Ordinary Differential Equation, with Composite Boundary Conditions, for Various Numbers of Equations.

NUMBER OF EQUATIONS	AVERAGE % ERROR	USER EXECUTION TIME (SEC)
11	.1246	.16
21	.0318	.30
51	.0052	1.53
101	.0013	8.86
201	.0003	65.39

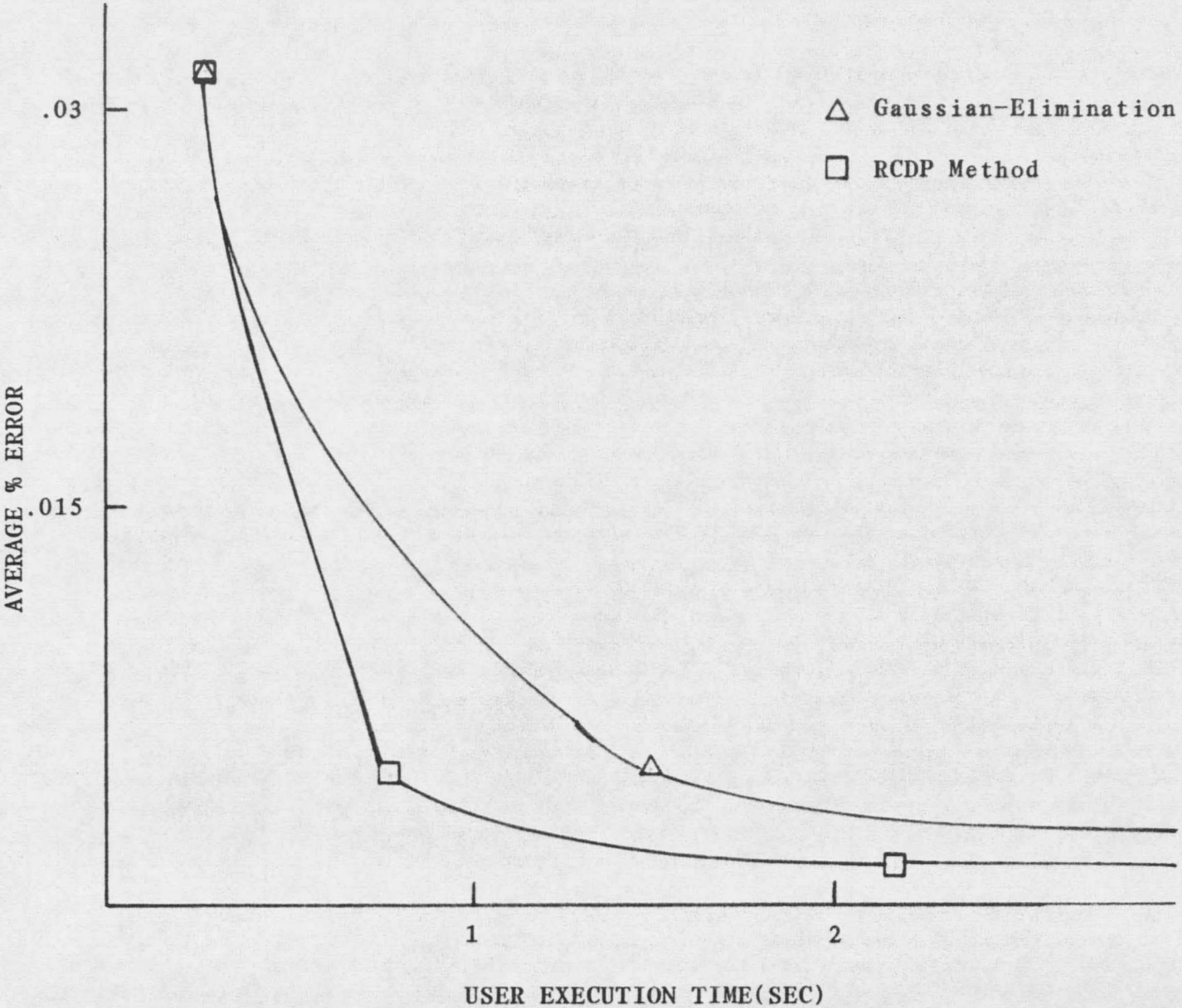

Figure 10. User Execution Time Versus Average Percent Error of RCDP Versus Gaussian-Elimination for 1-D Ordinary Differential Equation Problem 2, Composite Boundary Conditions.

Table 17. Results of RCDP for the Ordinary Differential Equation, with Mixed Boundary Conditions, for Various Numbers of Equations.

NUMBER OF EQUATIONS	AVERAGE % ERROR	USER EXECUTION TIME (SEC)
10	.7485	.17
20	.1857	.28
50	.0296	.77
100	.0074	2.18
200	.0018	7.59

Table 18. Results of Gaussian-Elimination for the Ordinary Differential Equation, with Mixed Boundary Conditions, for Various Numbers of Equations.

NUMBER OF EQUATIONS	AVERAGE % ERROR	USER EXECUTION TIME (SEC)
10	.7485	.15
20	.1857	.28
50	.0296	1.46
100	.0074	8.77
200	.0018	67.00

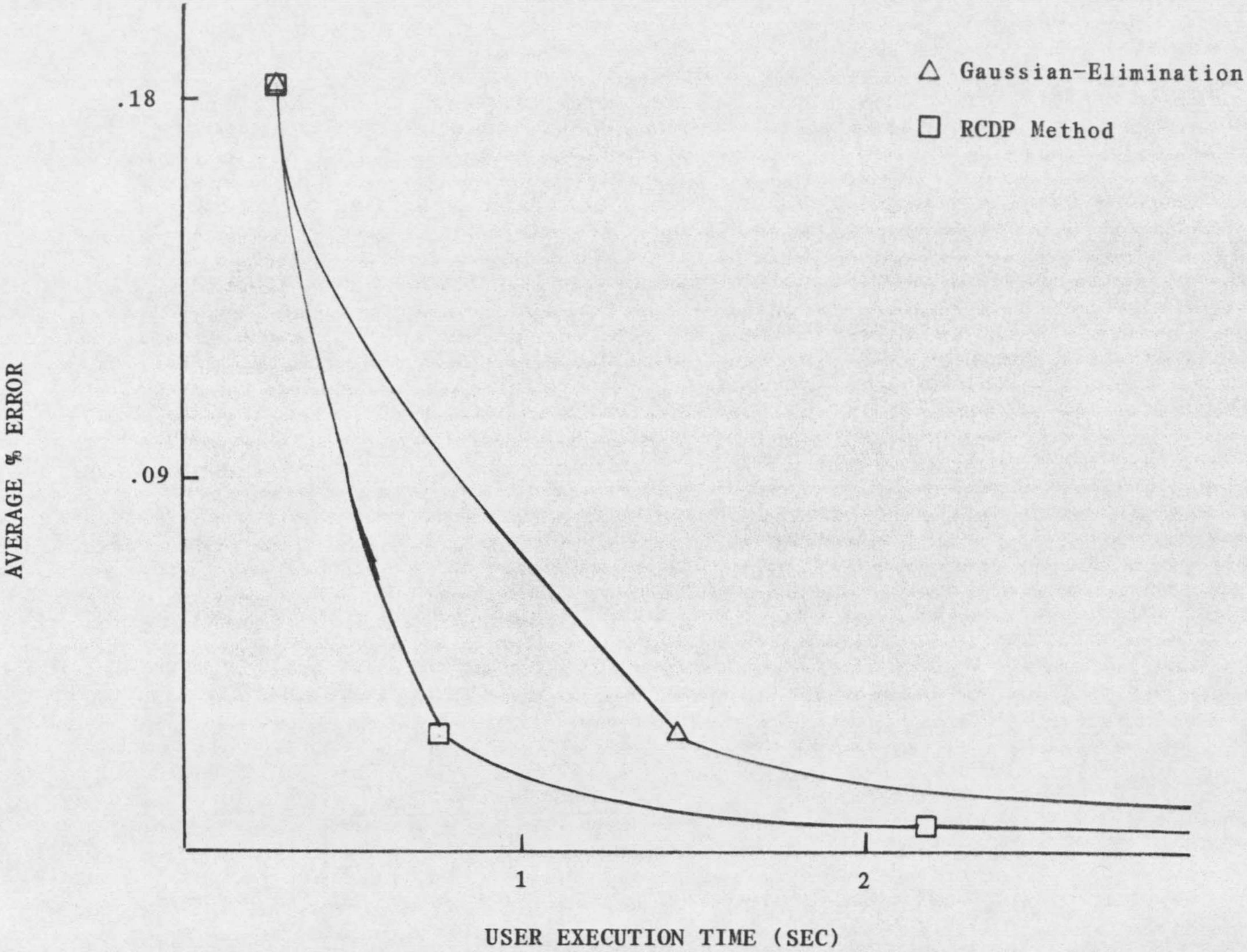

Figure 11. User Execution Time Versus Average Percent Error for RCDP Versus Gaussian-Elimination of 1-D Ordinary Differential Equation Problem 1, Mixed Boundary Conditions.

Table 19. Results of RCDP for the Ordinary Differential Equation, with Dirichlet Boundary Conditions, for Various Numbers of Equations.

NUMBER OF EQUATIONS	AVERAGE % ERROR	USER EXECUTION TIME (SEC)
9	2.6465	.15
19	5.9274	.26
49	7.2303	.74
99	4.4752	2.13
199	.3946	7.28

Table 20. Results of Gaussian-Elimination for the Ordinary Differential Equation, with Dirichlet Boundary Conditions, for Various Numbers of Equations.

NUMBER OF EQUATION	AVERAGE % ERROR	USER EXECUTION TIME (SEC)
9	19.0485	.14
19	94.2719	.26
49	123.7314	1.37
99	62.9538	8.38
199	27.2497	63.46

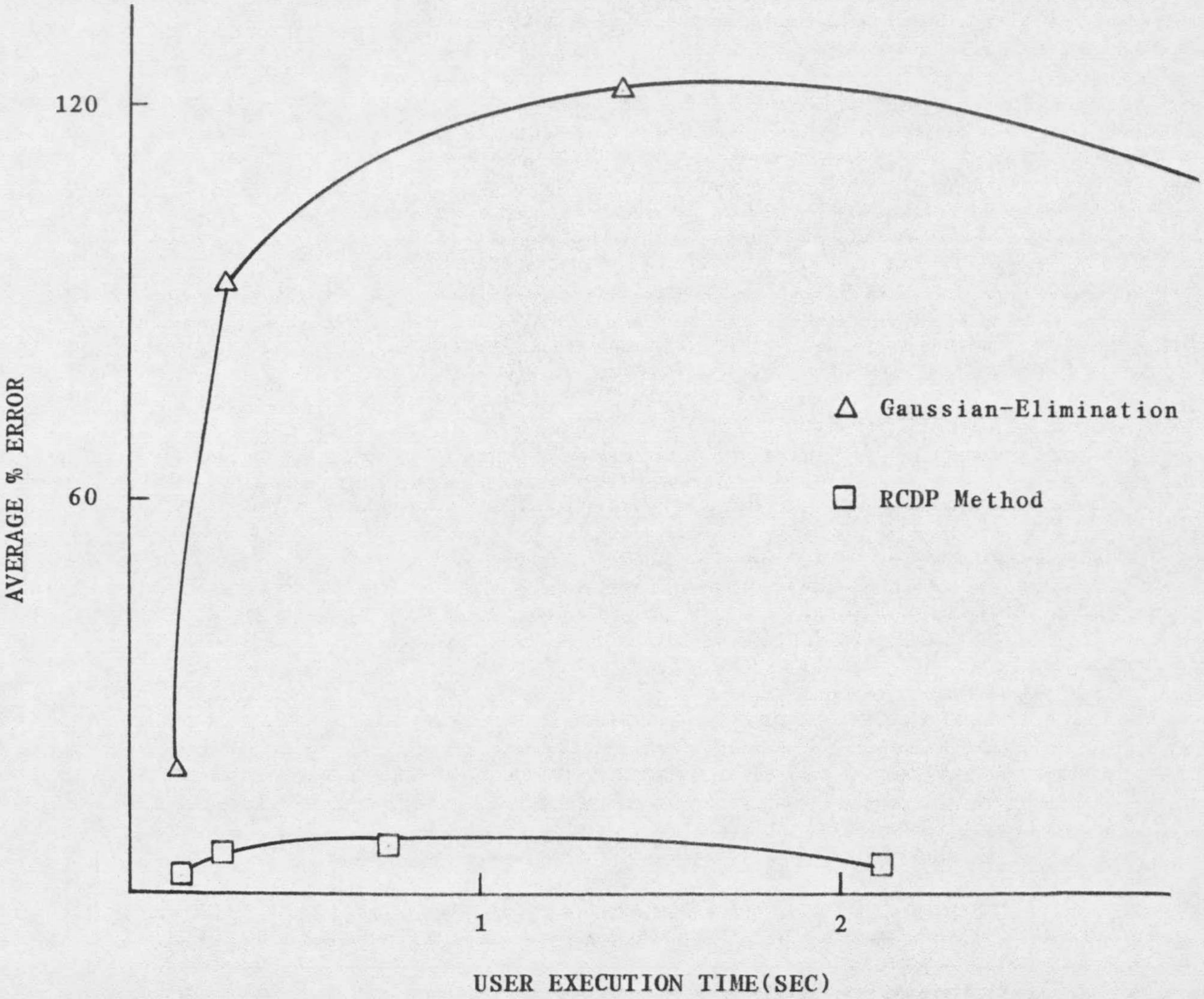

Figure 12. User Execution Time Versus Average Percent Error of RCDP Versus Gaussian-Elimination for 1-D Ordinary Differential Equation Problem 4, Dirichlet Boundary Conditions

CHAPTER VI

CONCLUSIONS

The results show that the RCIP method is an excellent solution routine for linear equation sets. The RCIP method proved to be most useful on problems with complicated boundary conditions. The first two 1-D problems having mixed and composite boundary conditions were solved by the RCIP method. The SOR method however, did not produce a solution. Conclusion, try the RCIP method when convergence problems are encountered. The RCIP method and SOR method with optimum over-relaxation factor solution produces comparable results for the 2-D and 3-D steady-state problems. If the optimum SOR factor is not known then a trial and error search must be accomplished to determine the optimum relaxation factor. The RCIP method does not require a search for a optimum number of trial solutions, choosing any number of trial solutions, except one, results in fast accurate solutions. In the solution to the transient problem RCIP has some definite advantages over ADEP, which has the ablity to operate with larger time steps and rapid convergence of the RCIP method. The number of trial solutions to pick if a single solution attempt is sought is six trial solutions. This number of trial solutions produced excellent results.

The RCDP method and Gaussian-Elimination are comparable methods for the solution of small equation sets. The RCDP has an advantage of producing accurate results much faster than the Gaussian-Elimination routine for equation sets larger than 40.

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APPENDICES

APPENDIX A

ILLUSTRATIVE EXAMPLE OF RCIP

Consider the following example.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 3 \\ 1 \end{Bmatrix} .$$

For an initial estimate select

$$G(0,1) = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \quad \text{and} \quad G(0,2) = \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

thus n=3 and m=2. Then equation (3) becomes

$$X_0 = \alpha_1 \; G(0,1) + \alpha_2 \; G(0,2) = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_2 \end{Bmatrix}$$

and

$$A \; X_0 - F = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_2 \end{Bmatrix} - \begin{Bmatrix} 2 \\ 3 \\ 1 \end{Bmatrix}$$

$$= \begin{bmatrix} \alpha_1 + \alpha_2 - 2 \\ \alpha_1 + 2\alpha_2 - 3 \\ \alpha_2 - 1 \end{bmatrix}$$

$$\langle AX_0-F, \; AX_0-F \rangle = (\alpha_1+\alpha_2-2)^2 + (\alpha_1+2\alpha_2-3)^2 + (\alpha_2-1)^2$$

which is equal to the $V_0(\alpha_1,\alpha_2)$ in equation (4). Forming the partial derivatives according to equation (5) yields

$$\frac{\partial V_0}{\partial \alpha_1} = 2\alpha_1 + 3\alpha_2 - 5$$

and

$$\frac{\partial V_0}{\partial \alpha_2} = \alpha_1 + 2\alpha_2 - 3 .$$

Thus

$$\frac{\partial V_0}{\partial \alpha_1} = 0 \quad \text{and} \quad \frac{\partial V_0}{\partial \alpha_2} = 0$$

yeilds the system

$$2\alpha_1 + 3\alpha_2 = 5$$

$$\alpha_1 + 2\alpha_2 = 3$$

with the solution

$$\alpha_1 = 1 \quad , \quad \alpha_2 = 1 \quad .$$

That is $(\alpha(0,1),\alpha(0,2)) = (1,1)$ and thus

$$X_0 = G(0,1) + G(0,2) = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

For the second iteration, equation (7) becomes

$$G(1,1) = X_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

select arbitrarily

$$G(1,2) = \begin{Bmatrix} 1 \\ 2 \\ 2 \end{Bmatrix} .$$

Therefore following equations (8)-(12) we have

$$X_1 = \alpha_1 G(1,1) + \alpha_2 G(1,2) = \begin{bmatrix} \alpha_1 + \alpha_2 \\ \alpha_1 + 2\alpha_2 \\ \alpha_1 + 2\alpha_2 \end{bmatrix},$$

$$AX_1 - F = \begin{bmatrix} 2\alpha_1 + 3\alpha_2 - 2 \\ 3\alpha_1 + 5\alpha_2 - 3 \\ \alpha_1 + 2\alpha_2 - 1 \end{bmatrix},$$

$$V_1(\alpha_1,\alpha_2) = \langle AX_1 - F, AX_1 - F \rangle$$

$$= (2\alpha_1 + 3\alpha_2 - 2)^2 + (3\alpha_1 + 5\alpha_2 - 3)^2 + (\alpha_1 + 2\alpha_2 - 1)^2 ,$$

$$\frac{\partial V_1}{\partial \alpha_1} = 14\alpha_1 + 23\alpha_2 - 14$$

$$\frac{\partial V_1}{\partial \alpha_2} = 23\alpha_1 + 38\alpha_2 - 23 ,$$

and the equations to be solved are

$$14\alpha_1 + 23\alpha_2 = 14$$

$$23\alpha_1 + 38\alpha_2 = 23$$

with solution

$$\alpha_1 = 1 , \alpha_2 = 0 .$$

Thus

$$X_1 = G(1,1) = X_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

the algorithm terminates with the solution

$$X_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} .$$

APPENDIX B

SOLUTION OF A SIMPLE BOUNDARY VALUE PROBLEM USING RCIP

The problem is as follows

$$\frac{d^2u}{dx^2} - u + x = 0 \quad \text{on} \quad 0 < x < 1 ,$$

with

$$u(0) = 0 \quad \text{and} \quad u(1) = 0 \; , \; (u_{exact} = x - \sinh(x)/\sinh(1)) \; .$$

Use a central difference approximation, and a step size of .1, to obtain the following equation set to be solved,

$$\begin{bmatrix} -201 & 100 & 0 & . & . & . & . & 0 \\ 100 & -201 & 100 & . & . & . & . & 0 \\ . & . & . & . & & & & . \\ . & . & . & & . & & & . \\ . & . & . & & & . & & . \\ 0 & 0 & 0 & . & . & . & 100 & -201 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ u_9 \end{Bmatrix} = \begin{Bmatrix} -.1 \\ -.2 \\ . \\ . \\ . \\ -.9 \end{Bmatrix}$$

<u>ITERATION 1</u>

We have n = 9 and pick m = 3.

For the initial estimate a constant value of one is chosen.

$$G(0,1) = \begin{Bmatrix} 1. \\ 1. \\ 1. \\ 1. \\ 1. \\ 1. \\ 1. \\ 1. \\ 1. \end{Bmatrix} , \quad G(0,2) = \begin{Bmatrix} .692 \\ .461 \\ .299 \\ .191 \\ .121 \\ .076 \\ .171 \\ .399 \\ .697 \end{Bmatrix} , \quad G(0,3) = \begin{Bmatrix} -.088 \\ -.131 \\ -.139 \\ -.142 \\ -.174 \\ -.257 \\ -.272 \\ -.207 \\ -.103 \end{Bmatrix}$$

$$V_0 = .849$$

Avg. % error = 4.763

$$\alpha_1 = .0695$$

$$\alpha_2 = -.0663$$

$$\alpha_3 = .0483$$

ITERATION 2

$$G(1,1) = \begin{Bmatrix} .0193 \\ .0326 \\ .0429 \\ .0499 \\ .0530 \\ .0520 \\ .0450 \\ .0330 \\ .0183 \end{Bmatrix}, \quad G(1,2) = \begin{Bmatrix} .00415 \\ .00385 \\ .00303 \\ .00209 \\ .00105 \\ -.000237 \\ -.00361 \\ -.00691 \\ -.00604 \end{Bmatrix}, \quad G(1,3) = \begin{Bmatrix} -.000864 \\ -.00127 \\ -.00122 \\ -.000544 \\ .00079 \\ .00218 \\ .00260 \\ .00204 \\ .00102 \end{Bmatrix}$$

$$V_1 = .00612$$

Avg. % error = 1.218

$$\alpha_1 = 1.04$$

$$\alpha_2 = -1.01$$

$$\alpha_3 = 1.28$$

ITERATION 3

At this iteration the

$$V_2 = .0000873$$

and

Avg. % error = .399 .

ITERATION 5

At this iteration the

$$V_5 = .000000000722$$

and the

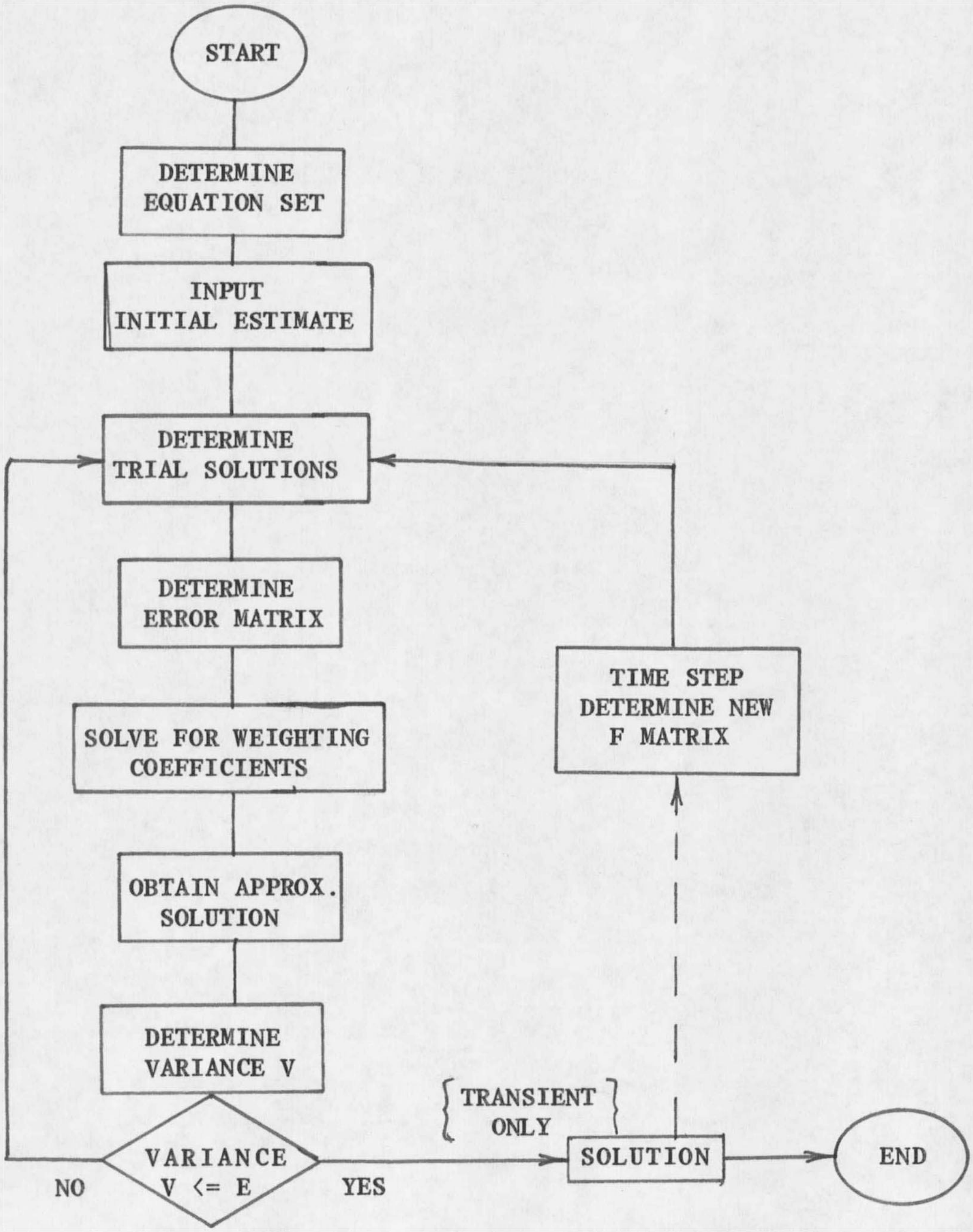
Avg. % error = .0762 .

The method has converged to the specified limit that the variance be less than or equal to .00000001. The solution to the problem is as follows:

$$X_5 = \begin{Bmatrix} .01475 \\ .02866 \\ .04085 \\ .05044 \\ .05655 \\ .05822 \\ .05447 \\ .04426 \\ .02650 \end{Bmatrix} .$$

APPENDIX C

FLOW CHART OF RCIP COMPUTER PROGRAM



E = CONVERGENCE LIMIT

APPENDIX D

SOLUTION OF A SIMPLE BOUNDARY VALUE PROBLEM USING RCDP

The same problem solved in Appendix B is solved here using the RCDP method. The problem is as follows

$$\frac{d^2u}{dx^2} - u + x = 0 \quad \text{on} \quad 0 < x < 1 \ ,$$

with

$$u(0) = 0 \quad \text{and} \quad u(1) = 0 \ , \ (u_{exact} = x - \sinh(x)/\sinh(1)).$$

The coefficient trial solution is

$$G_1 = \begin{Bmatrix} 1.00 \\ 2.01 \\ 3.04 \\ 4.10 \\ 5.20 \\ 6.36 \\ 7.57 \\ 8.87 \\ 10.25 \end{Bmatrix}$$

and the boundary condition trial solution is

$$G_2 = \begin{Bmatrix} 0.0000 \\ -0.0010 \\ -0.0040 \\ -0.0101 \\ -0.0202 \\ -0.0356 \\ -0.0573 \\ -0.0866 \\ -0.1247 \end{Bmatrix} .$$

The weighting coefficients are

$$\alpha_1 = 0.0147$$

$$\alpha_2 = 1.00 \quad ,$$

and the variance and average percent error are

Variance = 0.22E-32

Average % Error = 0.0756 .

The solution is

$$u_i = \begin{Bmatrix} 0.0148 \\ 0.0287 \\ 0.0408 \\ 0.0505 \\ 0.0566 \\ 0.0582 \\ 0.0545 \\ 0.0443 \\ 0.0265 \end{Bmatrix}.$$

Hodges, T. M.
The improvement of the reduced coordinate iterative procedure ...

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