



New solution generating transformations for the stationary axially-symmetric Einstein-Maxwell equations
by Terry Lee Lemley

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Physics
Montana State University
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Abstract:

Exact solution generating methods for the stationary axially-symmetric Einstein-Maxwell equations are investigated in this thesis. In particular, the attempt has been made to extend Cosgrove's vacuum symmetry groups, Q and Q_{\sim} , to the electrovac case. The primary method used is that of invariant groups, but due to the close relationship to Backlund transformations, a brief review of the latter is given.

The symmetry groups Q and Q_{\sim} are found to exist for the electrovac case, but as of yet their application has met with minimal success.

It is still felt by the author that more of the vacuum results pertaining to Q and Q_{\sim} can be extended to the electrovac case.

NEW SOLUTION GENERATING TRANSFORMATIONS
FOR THE STATIONARY AXIALLY-SYMMETRIC
EINSTEIN-MAXWELL EQUATIONS

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A thesis submitted in partial fulfillment
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Date May 28, 1983

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ABSTRACT

Exact solution generating methods for the stationary axially-symmetric Einstein-Maxwell equations are investigated in this thesis. In particular, the attempt has been made to extend Cosgrove's vacuum symmetry groups, Q and \tilde{Q} , to the electrovac case. The primary method used is that of invariant groups, but due to the close relationship to Bäcklund transformations, a brief review of the latter is given. The symmetry groups Q and \tilde{Q} are found to exist for the electrovac case, but as of yet their application has met with minimal success. It is still felt by the author that more of the vacuum results pertaining to Q and \tilde{Q} can be extended to the electrovac case.

CHAPTER 1

INTRODUCTION

This thesis investigates a method of finding exact solutions for the stationary axially-symmetric Einstein-Maxwell equations. The adjectives preceding the words Einstein-Maxwell imply that a system with axial-symmetry, uniformly rotating about its axis of symmetry, is being investigated. The words Einstein-Maxwell reveal that this thesis belongs to the subject areas of gravitation and electrodynamics with their related equations.

Being nonlinear and coupled, the Einstein-Maxwell equations developed in Chapter 2 are very difficult to solve. Before special methods were introduced for finding new solutions, only a handful of solutions were known. Two methods which have been found useful in finding new solutions are the methods of group transformations and Backlund transformations. These are reviewed in Chapters 3 and 4, respectively.

The case where no exterior electromagnetic fields are present has been investigated by numerous workers in recent years. There is a growing consensus among these workers that this problem is nearly solved. A brief review of this vacuum case is given in Chapter 5.

Allowance for electromagnetic fields with the given symmetries introduces a source term into the Einstein equations. Chapter 6 reviews results found previous to this thesis for this case.

Chapter 7 extends the Q and \tilde{Q} transformations found by Cosgrove (1979a) for vacuum to the case where stationary axially-symmetric electromagnetic fields are present. This is the first chapter in which original material is presented.

Chapter 8 uses the new transformations in an attempt to generate a new solution. The results of this thesis are given in Chapter 9.

CHAPTER 2

THE FIELD EQUATIONS

At the turn of the century only two forces in nature were known, the gravitational force and the electromagnetic force. Both were beautifully portrayed by two physical theories, the former by Newton's Law of Universal Gravitation and the latter by Maxwell's field equations. Einstein founded the theory of special relativity on what has amounted to the choosing of Maxwell's field equations over Newton's laws of motion. Including the gravitational field eventually led Einstein to the theory of general relativity. Thus, both classical fields played important roles in the development of general relativity.

General relativity consists partially in the solving of the equations governing the gravitational field, the Einstein equations. If electromagnetic fields are present, then general relativity modifies the Maxwell equations to include the effects of curved spacetime. The field equations are written in the deceptively simple tensor form:

$$R_{ik} - 1/2 g_{ik} R = (-8\pi G/c^4) T_{ik} \quad (2.1)$$

are the Einstein equations and

$$F^{ik}_{;k} = -4\pi s^i \quad ; \quad F_{ik,l} + F_{li,k} + F_{kl,i} = 0 \quad (2.2,3)$$

are the Maxwell equations. Together they are called the Einstein-Maxwell equations.

The Einstein-Maxwell equations are usually thought of in the following manner. On the left-hand side of Equations (2.1) are the

terms related to the curvature of spacetime. This curvature has as its source the energy-momentum tensor T_{ik} which stands on the right-hand side of Equations (2.1). This source term includes the effects of all the matter fields, excluding gravity, and is basically the statement of mass-energy equivalence in the generation of the gravitational field. In particular, it will contain the energy and momentum of the electromagnetic fields present.

The Maxwell equations are modified directly by the covariant derivative which contains the effects of the gravitational field. (As is usual in the literature, a comma represents partial differentiation, and the semi-colon represents covariant differentiation.) Indirect effects of the gravitational field are contained in the current density term s^i on the right-hand side of Equations (2.2).

Taken together, the Einstein-Maxwell equations are a very difficult set of equations to solve. They are highly coupled and nonlinear.

In this thesis, several simplifying conditions will be added. First, the regions of space considered will be exterior to the sources; that is, the current density s^i will be zero, and the energy-momentum tensor T_{ik} will contain terms which depend only on the free electromagnetic fields. This situation is called electrovac. Second, the fields will be assumed stationary. Third, the fields will be assumed axially-symmetric. Finally, the fields will be assumed invariant under the simultaneous transformation of the coordinates, $(t, \phi) \rightarrow (-t, -\phi)$.

The first condition simplifies the field equations to

$$R_{ik} = (-8\pi G/c^4)T_{ik} \quad (2.1a)$$

and

$$F^{ik}_{;k} = 0 ; F_{ik,l} + F_{li,k} + F_{kl,i} = 0. \quad (2.2a,3)$$

Axial-symmetry means the existence of an axis of symmetry around which any rotation leaves the fields invariant. Mathematically, this defines an azimuthal coordinate ϕ which measures the degree of rotation about the axis.

Stationary means that all of the sources have reached a uniform state of motion such that all the fields are independent of time. Mathematically, this defines the time coordinate t of which all the fields are independent.

Finally, the symmetry transformation $(t, \phi) \rightarrow (-t, -\phi)$ has the following physical meaning. If the object producing the fields is rotating about its axis, then the above transformation brings the velocity distribution back into itself. This means that the energy-momentum tensor and current density go into themselves. Since nothing has changed within the source under the transformation, the external fields produced by the source should be the same.

This last condition greatly simplifies the form of the metric tensor g_{ik} . Before adding the last condition, the line element had the form

$$\begin{aligned} ds^2 &= g_{ik} dx^i dx^k \\ &= g_{11} dt dt + 2g_{12} dt d\phi + 2g_{13} dt dx^3 + 2g_{14} dt dx^4 + \\ &\quad g_{22} d\phi d\phi + 2g_{23} d\phi dx^3 + 2g_{24} d\phi dx^4 + g_{33} dx^3 dx^3 + \\ &\quad 2g_{34} dx^3 dx^4 + g_{44} dx^4 dx^4. \end{aligned} \quad (2.4)$$

(Here $t = x^1$, $\phi = x^2$, and $g_{ik} = g_{ik}(x^3, x^4)$.)

Imposing the last condition rids the g_{jk} of the 13, 14, 23, and 24 components. The line element can now be written in the box diagonal form

$$ds^2 = f_{AB} dx^A dx^B - h_{MN} dx^M dx^N; \quad A, B = 1, 2; \quad (2.5)$$

$$M, N = 3, 4.$$

Notation

A brief review of the notation to be used in the development of the field equations will now be given. This notation follows Kinnersley (1977a).

To raise indices in a two-dimensional space with metric f_{AB} , either (f_{AB}) 's inverse matrix or ϵ^{AB} may be used. ($\epsilon^{12} = \epsilon_{12} = -\epsilon^{21} = -\epsilon_{21} = 1$; $\epsilon^{11} = \epsilon_{22} = \epsilon^{22} = \epsilon_{11} = 0$.) This can be seen by the following argument.

For a general two by two matrix,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The inverse is easily shown to be

$$A^{-1} = \frac{1}{(ad-bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{(ad-bc)} \epsilon A^\dagger \epsilon^\dagger, \quad (2.6)$$

with

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and the superscript \dagger indicating matrix transposition. In the case where the matrix (f_{AB}) is symmetric, the inverse matrix is given by

$$(f_{AB})^{-1} = \frac{1}{\det(f_{AB})} \epsilon (f_{AB}) \epsilon^\dagger, \quad (2.7)$$

or in index notation by

$$f^{\tilde{A}\tilde{B}} \equiv (f_{AB})^{-1} = -\rho^{-2} \epsilon^{AC} \epsilon^{BD} f_{CD}; \quad -\rho^2 = \det(f_{AB}). \quad (2.7a)$$

Therefore, defining f^{AB} by

$$f^{AB} = \epsilon^{AC} \epsilon^{BD} f_{CD} = -\rho^2 f^{\tilde{A}\tilde{B}}, \quad (2.8)$$

it is seen that

$$f^{AB} f_{BC} = -\rho^2 f^{\tilde{A}\tilde{B}} f_{BC} = -\rho^2 \delta^A_C. \quad (2.9)$$

Strictly speaking, since ϵ^{AC} is a tensor density of weight minus one, f^{AB} is a tensor density of weight minus two (Weinberg, 1972: 99). For coordinate transformations which have the Jacobian equal to one, they are strict tensors. This detail will be ignored as it does not affect the following development.

Given the vector V_A , raising the index is defined by

$$V^B \equiv \epsilon^{BA} V_A. \quad (2.10)$$

It is important to note that the contracted index is the second index in the ϵ^{BA} . With this convention and the identity

$$\epsilon_{AC} \epsilon^{CB} = \epsilon^{BC} \epsilon_{CA} = -\delta^B_A = -\delta^B_A, \quad (2.11)$$

it is seen that to lower the index on V^B requires multiplying by ϵ_{BD} .

That is,

$$V_D = \epsilon_{BD} V^B. \quad (2.12)$$

This is consistent with the raising operation leading back to the original vector.

Return to the stationary axially-symmetric metric of Equation (2.5). The indices will be raised using ϵ^{AB} in the (t, ϕ) coordinate block and the inverse of (h_{MN}) in the (x^3, x^4) coordinate block. If, instead of raising indices in this manner, the inverse of (f_{AB}) and ϵ^{MN} are used, then the raised indices will be marked with a superscript tilde. For example,

$$V^{\tilde{M}} = \epsilon^{MN} V_N, \quad (2.13)$$

with $\epsilon^{34} = \epsilon_{34} = -\epsilon^{43} = -\epsilon_{43} = 1$ and $\epsilon^{33} = \epsilon_{33} = \epsilon^{44} = \epsilon_{44} = 0$.

The two-dimensional covariant derivative associated with h_{MN} will be designated ∇ . The divergence of a vector field $V = (V_3, V_4)$ is given by

$$\nabla \cdot V = h^{-1/2} (h^{1/2} h^{MN} V_M)_{,N}, \quad (2.14)$$

h being the determinant of h_{MN} . The expression $h^{1/2} h^{MN}$ in two-dimensions is conformally invariant. That is, if the metric is transformed to \bar{h}^{MN} , with $\bar{h}_{MN} = e^{-2\Gamma} h_{MN}$, then $\bar{h}^{-1/2} \bar{h}^{MN} = h^{1/2} h^{MN}$. This transformation is not a coordinate transformation; under this transformation $V_M \rightarrow V_M$ and $V^M \rightarrow e^{2\Gamma} V^M$. Therefore, if $\nabla \cdot V = 0 = (h^{1/2} h^{MN} V_M)_{,N}$, then $\bar{\nabla} \cdot \bar{V} = \bar{h}^{-1/2} (\bar{h}^{1/2} \bar{h}^{MN} \bar{V}_M)_{,N} =$

$\bar{h}^{-1/2} (h^{1/2} h^{MN} V_M)_{,N} = 0$. Since every two-dimensional metric space is conformally related to the Euclidean two-dimensional space, (Eisenhart, 1909:92-93), it is possible to pick $\bar{h}_{MN} = \delta_{MN}$. In this case, the covariant derivative ∇ reduces to the ordinary gradient in rectangular coordinates. The line element is now written

$$ds^2 = f_{AB} dx^A dx^B - e^{2\Gamma} \delta_{MN} dx^M dx^N; \quad A, B = 1, 2; \\ M, N = 3, 4. \quad (2.5a)$$

Another common form of the line element, Lewis (1932), is related to the above by the relations

$$f_{11} = f, \quad f_{12} = -f\omega, \quad f_{22} = f\omega^2 - \rho^2 f^{-1}, \\ e^{2\Gamma} = f^{-1} e^{2\Upsilon}. \quad (2.15)$$

In this parameterization, the line element is written

$$ds^2 = f(dt - \omega d\phi)^2 - f^{-1} e^{2\Upsilon} ((dx^3)^2 + (dx^4)^2) + f^{-1} \rho^2 d\phi^2. \quad (2.16)$$

Again, $-\rho^2 = \det(f_{AB})$ and is a function of x^3 and x^4 .

The Maxwell Equations

Using the line element of Equation (2.5a), the field equations will now be written out, beginning with the Maxwell equations (2.2a,3).

Introducing the electromagnetic potential A_i by the equation $F_{ik} = A_{k,i} - A_{i,k}$, Equations (2.3) are automatically satisfied. The F_{ik} are independent of (t, ϕ) , and the A_i will also be assumed independent of (t, ϕ) . Hence, the only surviving components left are $F_{AM} = -A_{A,M}$ and $F_{MN} = A_{N,M} - A_{M,N}$. The Maxwell equations (2.2a) may be written, (Weinberg, 1972:125),

$$(\sqrt{-g} F^{ik})_{,k} = 0. \quad (2.2b)$$

These fall into two sets of equations. The first set is

$$0 = (\sqrt{-g} F^{MN})_{,N} = (\rho e^{2\Gamma} F^{MN})_{,N} = (\rho F^M_N)_{,N}, \quad (2.17)$$

which has solutions

$$F^{MN} = C \rho^{-1} \epsilon^{MN}. \quad (2.18)$$

This corresponds to a magnetic field in the azimuthal direction falling off as ρ^{-1} , which would be produced by an infinite line current along the symmetry axis. Since this solution is not physical for a bounded source the constant C will be set to zero.

The final set of equations is

$$0 = (\sqrt{-g} F^{AM})_{,M} = \nabla \cdot (\rho f^{\tilde{A}\tilde{B}} \nabla A_B) = \nabla \cdot (\rho^{-1} f^{AB} \nabla A_B). \quad (2.19)$$

Define $\tilde{\nabla}$ by $\tilde{\nabla} = (\partial/\partial x^4, -\partial/\partial x^3)$, then one has the following identity.

For all scalar functions U , $\nabla \cdot (\tilde{\nabla} U) = 0$. Conversely, if $\nabla \cdot V = 0$, then there exists a scalar potential U such that $V = \tilde{\nabla} U$. Therefore,

Equations (2.19) indicate the existence of scalar fields B_A such that

$$\tilde{\nabla} B_A = \rho^{-1} f_A^B \nabla A_B. \quad (2.20)$$

Using the relations $\tilde{\nabla} = -\nabla$ and $f^{AB} f_{BC} = -\rho^2 \delta^A_C$, an inverse relation can be derived yielding

$$\tilde{\nabla} A_A = -\rho^{-1} f_A^B \nabla B_B. \quad (2.20a)$$

Forming the complex combination $\phi_A = A_A + iB_A$, the ϕ_A are seen to satisfy

$$\nabla \phi_A = -i \rho^{-1} f_A^B \tilde{\nabla} \phi_B. \quad (2.21)$$

The Einstein Equations

The Einstein equations (2.1a) are next considered. It can be shown that

$$2R^A_C = \rho^{-1} \nabla \cdot (\rho^{-1} f^{AD} \nabla f_{DC}) \quad (2.22)$$

and

$$4\pi T^A_B = \rho^{-1} \nabla \cdot (\rho^{-1} (f^{AC} A_B \nabla A_C - 1/2 \delta^A_B f^{CD} A_C \nabla A_D)) . \quad (2.23)$$

Hence, the Einstein equations are, setting $G = c = 1$,

$$\nabla \cdot (\rho^{-1} (f^{AC} \nabla f_{BC} - 4f^{AC} A_B \nabla A_C + 2\delta^A_B f^{CD} A_C \nabla A_D)) = 0. \quad (2.24)$$

Using the identity

$$\nabla^A_B - \nabla_B^A = \delta^A_B \nabla^C_C, \quad (2.25)$$

this can be written

$$\nabla \cdot (\rho^{-1} (f^{AC} \nabla f_{BC} - 2f^{AC} A_B \nabla A_C - 2f^C_B A^A \nabla A_C)) = 0. \quad (2.26)$$

This, in turn, implies the existence of the scalar potentials ψ_{AC} such that

$$\tilde{\nabla} \psi_{AC} = \rho^{-1} (f^A_B \nabla f_{BC} - 2f^B_A A_C \nabla A_B - 2f^B_C A^A \nabla A_B) . \quad (2.27)$$

One notes that

$$\begin{aligned} \tilde{\nabla} \psi^A_A &= \rho^{-1} (f^{AB} \nabla f_{AB}) = 1/2 \rho^{-1} \nabla (f^{AB} f_{AB}) \\ &= -\rho^{-1} \nabla \rho^2 = -2\nabla \rho \end{aligned} \quad (2.28)$$

or

$$\tilde{\nabla} \psi^A_A = -2\nabla \rho . \quad (2.28a)$$

This implies that

$$\nabla \cdot \nabla \rho = 0. \quad (2.29)$$

As a consequence of Equation (2.29), one can choose new coordinates

$x^{3'} = \rho$, $x^{4'} = z$, where z is defined by

$$\nabla \rho = \tilde{\nabla} z , \quad (2.30)$$

and then the line element (2.16) becomes

$$ds^2 = f(dt - \omega d\phi)^2 - f^{-1} e^{2\gamma'} (d\rho^2 + dz^2) + f^{-1} \rho^2 d\phi^2 \quad (2.31)$$

with $e^{2\gamma'} = e^{2\gamma}/(\rho_3^2 + \rho_4^2)$. This form of the line element is known as the line element in Weyl canonical coordinates.

An inverse relation can be found to Equation (2.27) and is

$$\nabla f_{AC} = \rho^{-1} f_A^B (\tilde{\nabla} \psi_{BC} + 2A_C \tilde{\nabla} B_B + 2A_B \tilde{\nabla} B_C) \quad (2.32)$$

with the symmetry condition on the f_{AC} requiring

$$f^{AB} (\nabla \psi_{BA} + 2A_A \nabla B_B + 2A_B \nabla B_A) = 0. \quad (2.33)$$

Let

$$\Omega_{AC} = \psi_{AC} + 2A_A B_C \quad (2.34)$$

and define

$$H_{AC} = f_{AC} + i\Omega_{AC} - \Phi_A^* \Phi_C + \epsilon_{AC} K, \quad (2.35)$$

with

$$\nabla K = \Phi_X^* \nabla \Phi^X. \quad (2.36)$$

Then it has been shown, Kinnersley (1977a), that

$$\nabla H_{AC} = -i\rho^{-1} f_A^B \tilde{\nabla} H_{BC}, \quad (2.37)$$

which is the same form as Equations (2.21).

Defining $G = H_{11}$ and $\phi = \phi_1$, after some algebra, the following equations result.

$$f_{11} \nabla^2 \phi = (\nabla G + 2\phi^* \nabla \phi) \cdot \nabla \phi, \quad (2.38)$$

$$f_{11} \nabla^2 G = (\nabla G + 2\phi^* \nabla \phi) \cdot \nabla G. \quad (2.39)$$

These equations are the Ernst equations, Ernst (1968b), and contain all the information necessary for the solution of the Einstein-Maxwell equations. (∇^2 is the Laplacian operator associated with the three-dimensional metric $(dx^3)^2 + (dx^4)^2 + \rho^2 d\phi^2$; that is, $\partial^2/\partial\rho^2 + \partial^2/\partial z^2 + 1/\rho \partial/\partial\rho$.) Equation (2.29) must also be added to Equations (2.38,39) if noncanonical coordinates are chosen.

Nothing thus far has been said about the R_{AM} and R_{MN} field equations. The first set of these are identically zero. The R_{MN} equations allow for the γ in Equation (2.16) to be found by integrating the following two equations.

$$\begin{aligned} \rho^{-1} \gamma_{,3} = & \frac{f^{-2}}{4} [G_{,3} G^*_{,4} + G_{,4} G^*_{,3} + 2\bar{\Phi} (G_{,3} \bar{\Phi}^*_{,4} + G_{,4} \bar{\Phi}^*_{,3}) + \\ & 2\bar{\Phi}^* (G^*_{,3} \bar{\Phi}_{,4} + G^*_{,4} \bar{\Phi}_{,3}) + 4\bar{\Phi}^* \bar{\Phi} (\bar{\Phi}_{,3} \bar{\Phi}^*_{,4} + \bar{\Phi}_{,4} \bar{\Phi}^*_{,3})] - \\ & f^{-1} (\bar{\Phi}_{,3} \bar{\Phi}^*_{,4} + \bar{\Phi}^*_{,3} \bar{\Phi}_{,4}), \end{aligned} \quad (2.40)$$

and

$$\begin{aligned} \rho^{-1} \gamma_{,4} = & -\frac{f^{-2}}{4} [G_{,4} G^*_{,4} - G_{,3} G^*_{,3} + 2\bar{\Phi} (G_{,4} \bar{\Phi}^*_{,4} - G_{,3} \bar{\Phi}^*_{,3}) + \\ & 2\bar{\Phi}^* (G^*_{,4} \bar{\Phi}_{,4} - G^*_{,3} \bar{\Phi}_{,3}) + 4\bar{\Phi}^* \bar{\Phi} (\bar{\Phi}_{,4} \bar{\Phi}^*_{,4} - \bar{\Phi}_{,3} \bar{\Phi}^*_{,3})] + \\ & f^{-1} (\bar{\Phi}_{,4} \bar{\Phi}^*_{,4} - \bar{\Phi}^*_{,3} \bar{\Phi}_{,3}). \end{aligned} \quad (2.41)$$

If solutions to Equations (2.21,37) or (2.38,39) are known, then the right-hand sides of Equations (2.40,41) are known. Hence, Equations (2.21,37) or (2.38,39) are solved first, and then Equations (2.40,41) are integrated.

The field equations have now been introduced. Other forms of the equations exist, but these shall be introduced only where necessary. Equations (2.21,37) or (2.38,39) are very elegant and suggest that the gravitational field and electromagnetic field have much in common.

The next two chapters will develop some mathematical machinery useful in finding new solutions to these equations.

CHAPTER 3

TRANSFORMATION GROUPS

In this chapter, transformation groups will be reviewed and a method presented by example to find them for a given set of partial differential equations. The presentation given in this chapter follows closely that given by Hamermesh (1962: 279-321) and Cosgrove (1979a: 37-64) and is sufficient for the understanding of this thesis. A more detailed exposition concerning transformation groups is given in Eisenhart (1933).

Properties of Transformation Groups

Consider the transformation on the variables x^i , $i = 1, 2, \dots, n$, given by

$$x^i \rightarrow \bar{x}^i = f^i(x^1, x^2, \dots, x^n; a^1, a^2, \dots, a^t) \quad (3.1)$$

or symbolically by

$$x^i \rightarrow \bar{x}^i = f^i(x; a). \quad (3.1a)$$

The functions f^i are analytic in the parameters a^α , with $\alpha = 1, 2, \dots, t$; that is, they are expandable in a convergent Taylor series. In order for the transformations (3.1) to form a group, the f^i must satisfy certain conditions.

Group closure requires that if two transformations with parameters a^α and b^α are performed in succession, then there exists parameters c^α such that

$$f^i(x;c) = f^i(\bar{x};b) = f^i(f(x;a);b). \quad (3.2)$$

This means that the c^α may be expressed in terms of the a^α and b^α as

$$c^\alpha = \phi^\alpha(a;b). \quad (3.3)$$

The ϕ^α will be assumed analytic in its arguments. Using Equations (3.3), Equations (3.2) may be written as

$$f^i(x, \phi(a;b)) = f^i(f(x;a);b). \quad (3.4)$$

Existence of the group identity implies the existence of the unique parameter set e^α such that

$$\bar{x}^i = f^i(x;e) = x^i \quad (3.5)$$

for all x .

Each transformation must have an inverse. This implies that for each parameter set a^α , there is a parameter set \bar{a}^α such that

$$\bar{x}^i = f^i(\bar{x};\bar{a}) = f^i(f(x;a);\bar{a}) = x^i. \quad (3.6)$$

This implies that the transformations (3.1) are solvable for the x^i in terms of the \bar{x}^i ; the condition on the f^i being that the Jacobian of the transformation must be different from zero. Taken with Equations (3.3), Equations (3.6) implies that

$$e^\alpha = \phi^\alpha(a;\bar{a}), \quad (3.7)$$

which in turn implies that the \bar{a}^α can be written in terms of the a^α , since the e^α are fixed; that is,

$$\bar{a}^\alpha = \Omega^\alpha(a). \quad (3.8)$$

The $\Omega^\alpha(a)$ will be assumed analytic. The analyticity requirements are used to make the group a Lie group.

Using Equations (3.3,8), it is possible to show that the a^α may be expressed in terms of the \bar{a}^α and c^α . In other words, the

Jacobians $|\partial\phi^\alpha/\partial a^\beta|$ and $|\partial\phi^\alpha/\partial b^\beta|$ are nonzero, implying the matrices $\partial\phi^\alpha/\partial a^\beta$ and $\partial\phi^\alpha/\partial b^\beta$ are invertible matrices.

Attention will now be shifted from the finite transformations (3.1) to the infinitesimal transformations associated with Equations (3.1). Infinitesimal transformations are transformations close to the identity; their parameters are close to the identity parameters e^α .

Let a^α be the parameters that take the points x^i into \bar{x}^i . The neighboring parameters $a^\alpha + da^\alpha$ will take the points x^i into the points $\bar{x}^i + d\bar{x}^i$, since the f^i are analytic in the a^α . Parameters that are close to the identity, $e^\alpha + \delta a^\alpha$, can be found which will take the points \bar{x}^i into the points $\bar{x}^i + d\bar{x}^i$. Two different paths from x^i to $\bar{x}^i + d\bar{x}^i$ are thus possible and are given by

$$\bar{x}^i + d\bar{x}^i = f^i(x; a + da) \quad (3.9)$$

or the path

$$\bar{x}^i = f^i(x; a), \text{ and } \bar{x}^i + d\bar{x}^i = f^i(\bar{x}, e + \delta a). \quad (3.10)$$

Expanding the last equation in a Taylor series yields

$$d\bar{x}^i = (\partial f^i(\bar{x}; a) / \partial a^\beta)_a = 0 \delta a^\beta \equiv F^i_\beta(\bar{x}) \delta a^\beta. \quad (3.11)$$

Equation (3.3) yields

$$a^\alpha + da^\alpha = \phi^\alpha(a; e + \delta a) \quad (3.12)$$

so that by expanding the last equation yields

$$da^\alpha = (\partial\phi^\alpha(a; b) / \partial b^\beta)_b = e^\alpha \delta a^\beta \equiv \theta^\alpha_\beta(a) \delta a^\beta. \quad (3.13)$$

At $a^\alpha = e^\alpha$, $\theta^\alpha_\beta = \delta^\alpha_\beta$.

Since $\theta^\alpha_\beta(a)$ is invertible, Equations (3.13) are solvable for the δa^α in terms of the da^α .

$$\delta a^\alpha = \psi^\alpha_\beta(a) da^\beta, \quad (3.14)$$

with the $\psi_{\beta}^{\alpha}(a)$ defined by

$$\psi_{\eta}^{\alpha} \ominus_{\beta}^{\eta} = \delta_{\beta}^{\alpha} \text{ and } \psi_{\eta}^{\alpha}(e) = \delta_{\eta}^{\alpha}. \quad (3.15)$$

Substituting Equations (3.14) into Equations (3.11) yields

$$d\bar{x}^i = F_{\beta}^i(\bar{x}) \psi_{\alpha}^{\beta}(a) da^{\alpha} \quad (3.16)$$

or, equivalently,

$$\partial \bar{x}^i / \partial a^{\alpha} = F_{\beta}^i(\bar{x}) \psi_{\alpha}^{\beta}(a). \quad (3.17)$$

If the $F_{\beta}^i(\bar{x})$ and $\psi_{\alpha}^{\beta}(a)$ are known, then the transformations (3.1) may be found.

The infinitesimal transformations (3.10) change the function $A(x)$ as follows,

$$\begin{aligned} dA &= (\partial A / \partial x^i) dx^i = (\partial A / \partial x^i) F_{\beta}^i(x) \delta a^{\beta} \\ &= \delta a^{\beta} F_{\beta}^i(x) (\partial A / \partial x^i) = \delta a^{\beta} X_{\beta} A \end{aligned} \quad (3.18)$$

with the operator X_{β} defined by

$$X_{\beta} = F_{\beta}^i(x) (\partial / \partial x^i). \quad (3.19)$$

The X_{β} are called infinitesimal operators of the group because the operator $(1 + X_{\beta} \delta a^{\beta})$ is close to the identity. Using $(1 + X_{\beta} \delta a^{\beta})$ on the x^i yields Equations (3.11).

Defining the commutator of the infinitesimal operators by

$$[X_{\alpha}, X_{\beta}] = X_{\alpha} X_{\beta} - X_{\beta} X_{\alpha}, \quad (3.20)$$

it is shown in Hamermesh (1962: 299-301) that the commutators satisfy

$$[X_{\alpha}, X_{\beta}] = c_{\alpha\beta}^{\kappa} X_{\kappa}, \quad (3.21)$$

with the $c_{\alpha\beta}^{\kappa}$ independent of the a^{α} ; and they are called the structure constants. It has been shown, see Cohn (1957: 94-106), that if the structure constants are known, then the group transformations can be found via the equations

$$F_{\sigma}^j (\partial F_{\tau}^i / \partial x^j) - F_{\tau}^j (\partial F_{\sigma}^i / \partial x^j) = c_{\tau\sigma}^k F_{\kappa}^i, \quad (3.22)$$

and

$$(\partial \psi_{\mu}^{\kappa} / \partial a^{\lambda}) - (\partial \psi_{\lambda}^{\kappa} / \partial a^{\mu}) = c_{\tau\sigma}^{\kappa} \psi_{\mu}^{\tau} \psi_{\lambda}^{\sigma}. \quad (3.23)$$

Needless to say, the equations above are difficult to solve. If they can be solved, then Equations (3.17) are then used to find f^i . The important point is that if the $c_{\tau\sigma}^k$ are found and correspond to those of a known group, the transformations are determined near the identity.

Applications

Transformation groups are used to generate new solutions from a given solution of the Einstein-Maxwell equations. For example, starting with the flat space metric, it is possible to generate the Schwarzschild metric; that is, a transformation exists which has the physical property of producing mass. Although in the classical theory this is only a mathematical curiosity, in a quantum theory it could have major importance.

In its most general sense, the symmetry transformation changes both the dependent and the independent variables while leaving the field equations invariant. Clarification is needed as to the meaning of invariance and it is worthwhile to review this before examining methods of finding symmetry transformations.

Take the Laplace equation in three dimensions as the first example. It is written in manifestly covariant form

$$\nabla^2 \phi = 0. \quad (3.24)$$

This equation holds true in all coordinates; however, the partial

differential equation it represents has different forms in different coordinates. In Cartesian coordinates, the partial differential equation is

$$(\partial^2\phi/\partial x^2) + (\partial^2\phi/\partial y^2) + (\partial^2\phi/\partial z^2) = 0, \quad (3.25)$$

in spherical coordinates,

$$\frac{1}{r^2} \left(\frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) \right) + \frac{1}{r^2 \sin \theta} \left(\frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \phi^2} = 0, \quad (3.26)$$

and in cylindrical coordinates, its form is

$$\frac{1}{\rho} \left(\frac{\partial}{\partial \rho} \left(\rho \frac{\partial \phi}{\partial \rho} \right) \right) + \frac{1}{\rho^2} \frac{\partial^2 \phi}{\partial \phi^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \quad (3.27)$$

It is clear that the form of the partial differential equation depends on the coordinates chosen, even though Equation (3.24) is covariant. Invariance of form is precisely what is meant when the equations are said to be invariant under a given symmetry transformation.

The transformation of coordinates which do leave Equations (3.25) invariant are the rotations and inversions. For instance, under the rotation of coordinates, (x,y,z) transforms into $(\bar{x},\bar{y},\bar{z})$ by an orthogonal transformation, and Equation (3.25) becomes

$$(\partial^2\phi/\partial \bar{x}^2) + (\partial^2\phi/\partial \bar{y}^2) + (\partial^2\phi/\partial \bar{z}^2) = 0. \quad (3.28)$$

This means that if $\phi(x,y,z)$ is a solution to Equation (3.25), then $\phi(\bar{x},\bar{y},\bar{z})$ is also a solution to Equation (3.25) when $(\bar{x},\bar{y},\bar{z})$ are expressed in terms of (x,y,z) .

As an example of a noncoordinate transformation, take the equation to be the vacuum Ernst equation, see Equation (2.39),

$$(G+G^*)\nabla^2 G = 2\nabla G \cdot \nabla G. \quad (3.28)$$

The transformation, called the Ehler's or gravitational duality transformation, is given by

$$G \rightarrow \bar{G} = G/(1+i\gamma G); \quad (3.29)$$

γ is a real parameter. It is simple to verify that \bar{G} satisfies Equation (3.28) and is a new solution.

How are transformations like the rotations and the Ehler's transformation (3.29) found? Obviously, the rotations are encountered early in a physicist's career and have a clear physical meaning. Many other transformations in physics are modeled after the rotations and have a similar intuitive feeling. Ehler's transformation (3.29), however, is not intuitive at first encounter. A method is needed independent of intuition whereby such transformations can be found. Fortunately, such a method exists and consists first in finding the infinitesimal transformation which leaves the partial differential equations invariant, and then the integration of Equations (3.22,23,17) may be performed to yield the finite transformation. Although this method of finding symmetry transformations is complex, it again has the advantage of not relying on intuition. (Intuition is good to have, but seldom is it proved without effort.)

A Method

The vacuum Ernst equation (3.28) will be used to demonstrate this method. In particular, the Ehler's transformation (3.29) will be one of the symmetry transformations found.

Consider a finite transformation taking G into \bar{G} with the transformation parameter s . Restrict \bar{G} to be an explicit function of

only G , G^* , and s , written out as

$$\overline{G \rightarrow G} = \overline{G}(G, G^*; s). \quad (3.30)$$

Similarly for G^* ,

$$\overline{G^* \rightarrow G^*} = \overline{G^*}(G, G^*; s). \quad (3.31)$$

\overline{G} is not assumed to be the complex conjugate of $\overline{G^*}$. Expand Equations (3.30,31) in a McClaurin's series with the boundary conditions $\overline{G}(s = 0)$ and $\overline{G^*}(s = 0) = G^*$. To first order in s ,

$$\overline{G} = G + sA(G, G^*), \quad A = \left. \frac{d\overline{G}}{ds} \right|_{s=0}, \quad (3.32)$$

and

$$\overline{G^*} = G^* + sB(G, G^*), \quad B = \left. \frac{d\overline{G^*}}{ds} \right|_{s=0}. \quad (3.33)$$

It is necessary for G to satisfy Equation (3.28) and for G^* to satisfy the complex conjugate equation

$$(G + G^*) \nabla^2 G^* = 2 \nabla G^* \cdot \nabla G^*. \quad (3.34)$$

Using the chain rule of calculus,

$$\nabla \overline{G} = \nabla G + s(A_G \nabla G + A_{G^*} \nabla G^*) \quad (3.35)$$

and

$$\begin{aligned} \nabla^2 \overline{G} = \nabla^2 G + s(A_G \nabla^2 G + A_{G^*} \nabla^2 G^* + A_{GG} \nabla G \cdot \nabla G + \\ 2A_{GG^*} \nabla G \cdot \nabla G^* + A_{G^*G^*} \nabla G^* \cdot \nabla G^*), \end{aligned} \quad (3.36)$$

with $A_G = \partial A / \partial G$, $A_{GG^*} = \partial^2 A / \partial G \partial G^*$, etc. Also,

$$\overline{(G + G^*)} = (G + G^*) + s(A + B). \quad (3.37)$$

Similar expressions for ∇G^* and $\nabla^2 G^*$ can also be found. Putting these results into Equations (3.28,34) and using the fact that G and G^*

already satisfy these equations, the first order terms in s are left equal to zero. The resulting equations are

$$\begin{aligned} & (2(A+B)/(G+G^*) - 2A_G + (G+G^*)A_{GG})\nabla G \cdot \nabla G + \\ & (2(G+G^*)A_{GG^*} - 4A_{G^*})\nabla G \cdot \nabla G^* + \\ & ((G+G^*)A_{G^*G^*} + 2A_{G^*})\nabla G^* \cdot \nabla G^* = 0, \end{aligned} \quad (3.38)$$

and

$$\begin{aligned} & (2(A+B)/(G+G^*) - 2B_{G^*} + (G+G^*)B_{G^*G^*})\nabla G^* \cdot \nabla G^* + \\ & (2(G+G^*)B_{GG^*} - 4B_G)\nabla G \cdot \nabla G^* + ((G+G^*)B_{GG} + 2B_G)\nabla G \cdot \nabla G = 0. \end{aligned} \quad (3.39)$$

The coefficients of $\nabla G \cdot \nabla G$, $\nabla G^* \cdot \nabla G^*$, and $\nabla G \cdot \nabla G^*$ are set equal to zero.

(G and G^* are assumed functionally independent; therefore, $(G+G^*)$ is not a function of $(G-G^*)$.) The following coupled set of linear partial differential equations results:

$$\begin{aligned} & (G+G^*)^2 A_{GG} - 2(G+G^*)A_G + 2A = -2B, \\ & (G+G^*)A_{GG^*} - 2A_{G^*} = 0, \\ & (G+G^*)A_{G^*G^*} + 2A_{G^*} = 0, \\ & (G+G^*)^2 B_{G^*G^*} - 2(G+G^*)B_{G^*} + 2B = -2A, \\ & (G+G^*)B_{G^*G} - 2B_G = 0, \\ & (G+G^*)B_{GG} + 2B_G = 0. \end{aligned} \quad (3.40a-f)$$

First solve Equation (3.40c) using the integrating factor

$(G+G^*)$. This yields the solution

$$A = -f(G)/(G+G^*) + g(G).$$

From Equation (3.40b) it is found that

$$\frac{d}{dG}(\ln f) = 4/(G+G^*).$$

Since f is a function of G only, the above implies that $f = 0$. (Do not confuse the f here with the metric function introduced in Chapter 2.)

The only solution possible is then $A = g(G)$. In a similar manner, using Equations (3.17f,e), it is shown that $B = h(G^*)$. Two coupled ordinary differential equations now result from Equations (3.40a,d). They are

$$(G+G^*)^2 g_{GG} - 2(G+G^*)g_G + 2g = -2h$$

and

$$(G+G^*)^2 h_{G^*G^*} - 2(G+G^*)h_{G^*} + 2h = -2g.$$

To solve these two equations, use the power series expansions

$$g = \sum_{i=0}^{\infty} g_i G^i \quad \text{and} \quad h = \sum_{i=0}^{\infty} h_i (G^*)^i,$$

with g_i, h_i constants. The solution is

$$A = g_0 + g_1 G + g_2 G^2, \quad (3.41a)$$

$$B = -g_0 + g_1 G^* - g_2 (G^*)^2, \quad (3.41b)$$

with g_0, g_1, g_2 arbitrary complex constants. If the added requirement of $\overline{G^*} = (G)^*$ is given, then the g_0 and g_2 are purely imaginary, and g_1 is real. The infinitesimal transformations are now written

$$\overline{G \rightarrow G} = G + s(g_0 + g_1 G + g_2 G^2), \quad (3.42a)$$

$$\overline{G^* \rightarrow G^*} = G^* + s(-g_0 + g_1 G^* - g_2 (G^*)^2). \quad (3.42b)$$

To work only with real parameters, define $a^1 = isg_0$, $a^2 = sg_1$, and $a^3 = isg_2$. Then the infinitesimal transformations may be written in the form of Equation (3.11) with

$$\overline{G} = G - ia^1 + a^2 G - ia^3 G^2, \quad (3.43a)$$

$$\overline{G^*} = G^* + ia^1 + a^2 G^* + ia^3 (G^*)^2. \quad (3.43b)$$

The infinitesimal operators are, comparing Equations (3.43a,b) with Equations (3.19),

$$\begin{aligned}
 X_1 &= i\partial/\partial G^* - i\partial/\partial G, & X_2 &= G\partial/\partial G + G^*\partial/\partial G^*, \\
 X_3 &= i(G^*)^2\partial/\partial G^* - iG^2\partial/\partial G,
 \end{aligned}
 \tag{3.44}$$

with the commutators

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = -2X_2, \quad [X_2, X_3] = X_3. \tag{3.45}$$

Equations (3.45) yield the structure constants, and these may be used to integrate the group. Instead, define the following operators by

$$J_1 = 1/2(X_1 + X_3), \quad J_2 = X_2, \quad J_3 = 1/2(X_1 - X_3). \tag{3.46}$$

Then the new commutators are given by

$$[J_1, J_2] = J_3, \quad [J_2, J_3] = -J_1, \quad [J_3, J_1] = -J_2. \tag{3.47}$$

These are the same commutation relations as for the Lorentz transformations in three dimensions, Hamermesh (1962: 307), and define the group $SO(2,1)$. Thus, the symmetry group found above is locally isomorphic to $SO(2,1)$.

One Parameter Subgroups

One parameter subgroups of Equations (3.43a,b) will now be examined. An important point to note on one parameter groups is that they are all locally isomorphic to the translation group in one dimension, Hamermesh (1962: 295). This means that for one parameter groups, the ϕ^α of Equations (3.3) satisfy

$$\phi(a;b) = a+b. \tag{3.48}$$

This implies that the $\psi(a)$ in Equations (3.17) is equal to one, meaning Equation (3.17) can be written

$$\frac{d\bar{x}^i}{da} = F^i(\bar{x}), \tag{3.49}$$

and to get the transformation functions f^i , it is only necessary to solve Equation (3.49).

The one parameter subgroups of Equations (3.43a,b) satisfy the following differential equations:

$$\frac{\bar{dG}}{da^1} = -i, \quad \frac{\overline{dG^*}}{da^1} = i, \quad a^1 = 1, \quad a^2 = a^3 = 0; \quad (3.50a)$$

$$\frac{\bar{dG}}{da^2} = \bar{G}, \quad \frac{\overline{dG^*}}{da^2} = \overline{G^*}, \quad a^2 = 1, \quad a^1 = a^3 = 0; \quad (3.50b)$$

$$\frac{\bar{dG}}{da^3} = -i\bar{G}^2, \quad \frac{\overline{dG^*}}{da^3} = i(\overline{G^*})^2, \quad a^3 = 1, \quad a^1 = a^2 = 0. \quad (3.50c)$$

These equations are easily solved and have solutions

$$\bar{G} = G - ia^1, \quad (3.51a)$$

$$\bar{G} = e^{a^2} G, \quad (3.51b)$$

$$\bar{G} = G / (1 + ia^3 G). \quad (3.51c)$$

As promised, the Ehler's transformation has been reproduced in Equation (3.51c) as a one parameter subgroup of the symmetry group of form given by Equations (3.30,31). The other two subgroups are explained as follows. Equation (3.51a) results from the definition of ψ_{AB} in Equations (2.27) up to an additive constant, and Equation (3.51b) corresponds to the change of coordinates $(t, \phi) \rightarrow (e^{-1/2a^2} t, e^{1/2a^2} \phi)$.

The whole group of transformations is represented by a real two by two matrix with determinant equal to one;

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1, \quad (3.52)$$

such that

$$\bar{G} = (dG - ic) / (a + ibG). \quad (3.53)$$

See Cosgrove (1979b: Equation 2.3)) for example. If two transformations are performed in succession, the final transformed G will have the same

form given in Equation (3.53), but the elements will be those of the two representation matrices multiplied together. This group is named H by Kinnersley (1977a:1531) and (P) by Cosgrove (1979a:56).

The Neugebauer-Kramer Mapping

Only continuous transformations have been considered up to this point. At this point, a very important discrete symmetry transformation will be reviewed.

Define

$$2f = G+G^* \text{ and } 2i\psi = G-G^*. \quad (3.54)$$

Using the vacuum Ernst equations yields

$$\nabla_3 \cdot (f^{-1} \nabla f + f^{-2} \psi \nabla \psi) = 0, \quad (3.55)$$

$$\nabla_3 \cdot (f^{-2} \nabla \psi) = 0, \quad (3.56)$$

and these are equivalent to the Ernst equation.

Equation (3.56) implies the existence of a potential ω such that

$$\nabla \omega = \rho f^{-2} \nabla \psi. \quad (3.57)$$

Eliminating ψ in favor of ω yields the following two equations,

$$\nabla_3 \cdot (f^{-1} \nabla f + \rho^{-2} f^2 \omega \nabla \omega) = 0, \quad (3.58)$$

$$\nabla_3 \cdot (\rho^{-2} f^2 \nabla \omega) = 0. \quad (3.59)$$

(Equations (3.58,59) are restatements of Equations (2.26) for vacuum in the Lewis parameterization. Likewise, Equations (3.55,56) are restatements of Equations (2.27).)

It was noted that there is a mapping between Equations (3.55,56) and (3.58,59), given by

$$f \rightarrow \rho f^{-1}, \quad \omega \rightarrow i\psi, \quad \psi \rightarrow -i\omega, \quad (3.60)$$

which sends one set of equations into the other and vice versa. This mapping is the Neugebauer-Kramer mapping and will be used in latter chapters.

The concept of group transformations has now been reviewed. A method has been presented by which symmetry groups may be found for given partial differential equations. Using this method for the vacuum Ernst equation, a known symmetry group has been reproduced. This symmetry group will reappear in Chapter 5 where the vacuum case will be briefly reviewed.

CHAPTER 4

BÄCKLUND TRANSFORMATIONS

Another transformation which has proved useful in generating new solutions from old solutions is the Bäcklund transformation, see Eisenhart (1909:284). The variables to be transformed are now the first partial derivatives. These, in turn, may be integrated to yield new solutions. In this chapter, Bäcklund transformations are introduced and then demonstrated by an example. The presentation in this chapter follows closely that given by Lamb (1974) and Neugebauer (1979).

Properties of the Bäcklund Transformation

The notation used in this section is: $z = z(x,y)$, $\bar{z} = \bar{z}(\bar{x},\bar{y})$, $p = \partial z/\partial x$, $\bar{p} = \partial \bar{z}/\partial \bar{x}$, $q = \partial z/\partial y$, $\bar{q} = \partial \bar{z}/\partial \bar{y}$, $r = \partial^2 z/\partial x^2$, $\bar{r} = \partial^2 \bar{z}/\partial \bar{x}^2$, $s = \partial^2 z/\partial x \partial y$, $\bar{s} = \partial^2 \bar{z}/\partial \bar{x} \partial \bar{y}$, $t = \partial^2 z/\partial y^2$, $\bar{t} = \partial^2 \bar{z}/\partial \bar{y}^2$. Subscripts indicate the partial derivative with respect to the subscripted variable.

The Bäcklund transformation to be considered is written

$$p = f(\bar{x}, \bar{y}, z, \bar{z}, \bar{p}, \bar{q}), \quad q = h(\bar{x}, \bar{y}, z, \bar{z}, \bar{p}, \bar{q}), \quad (4.1)$$

and

$$\bar{x} = x, \quad \bar{y} = y. \quad (4.1a)$$

In order that the integrability condition $p_y = q_x$ holds, f and h must satisfy

$$\begin{aligned}
W &= p_y - q_x \\
&= f_{\bar{y}} - h_{\bar{x}} + f_{\bar{z}q} - h_{\bar{z}p} + f_{\bar{z}q} - h_{\bar{z}p} + \\
&\quad (f_{\bar{p}} - h_{\bar{q}})s + f_{\bar{q}t} = h_{\bar{p}r} = 0.
\end{aligned} \tag{4.2}$$

This equation may be satisfied identically in which case $f_{\bar{p}} - h_{\bar{q}} = f_{\bar{q}} = h_{\bar{p}} = 0$ and $f_{\bar{y}} - h_{\bar{x}} + f_{\bar{z}q} - h_{\bar{z}p} + f_{\bar{z}q} - h_{\bar{z}p} = 0$. Equation (4.2) may also be considered as a second-order partial differential equation for \bar{z} . If this equation is of a form known as Monge-Ampere, Forsythe (1959:200), then the transformations (4.1) are known as Bäcklund transformations.

In practice, it is the partial differential equations (4.2) which are started with and the Bäcklund transformations (4.1) which are sought.

Neugebauer's Bäcklund Transformation

The example will again be the vacuum Ernst equation (3.28). In the case of noncanonical coordinates, the field equations also include Equation (2.29). Written out, the field equations are

$$\begin{aligned}
(G+G^*)[G_{,3,3}+G_{,4,4} + \frac{1}{\rho}(\rho_{,3}G_{,3}+\rho_{,4}G_{,4})] \\
= 2[(G_{,3})^2 + (G_{,4})^2],
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
(G+G^*)[G^*_{,3,3}+G^*_{,4,4} + \frac{1}{\rho}(\rho_{,3}G^*_{,3}+\rho_{,4}G^*_{,4})] \\
= 2[(G^*_{,3})^2 + (G^*_{,4})^2],
\end{aligned} \tag{4.4}$$

and

$$\rho_{,3,3} + \rho_{,4,4} = 0. \tag{4.5}$$

It is convenient to change variables. First, the independent variables are changed to $x_1 = x^3 + ix^4$ and $x_2 = x^3 - ix^4$. Equations (4.3-5) become

$$(G+G^*)[G_{,1,2} + \frac{1}{2\rho}(\rho_{,1}G_{,2} + \rho_{,2}G_{,1})] = 2G_{,1}G_{,2}, \quad (4.3a)$$

$$(G+G^*)[G^*_{,1,2} + \frac{1}{2\rho}(\rho_{,1}G^*_{,2} + \rho_{,2}G^*_{,1})] = 2G^*_{,1}G^*_{,2}, \quad (4.4a)$$

and

$$\rho_{,1,2} = 0. \quad (4.5a)$$

At this point, Neugebauer (1979) changes the order of the equations by introducing the following dependent variables:

$$M_1 = G_{,1}/(G+G^*), \quad M_2 = G^*_{,1}/(G+G^*), \quad M_3 = \rho_{,1}/\rho, \quad (4.6a)$$

$$N_1 = G^*_{,2}/(G+G^*), \quad N_2 = G_{,2}/(G+G^*), \quad N_3 = \rho_{,2}/\rho. \quad (4.6b)$$

The Equations (4.3a-5a) then become equivalent to the system of first order coupled partial differential equations

$$M_{i,2} = c_i^{k1} M_k N_1, \quad N_{i,1} = c_i^{k1} N_k M_1, \quad (4.7)$$

with $c_1^{11} = c_2^{22} = c_3^{33} = -c_1^{12} = -c_2^{21} = -1$, and $c_1^{32} = c_1^{13} = c_2^{31} = c_2^{23} = -1/2$. These equations already contain the integrability

conditions $G_{,1,2} = G_{,2,1}$ and $G^*_{,1,2} = G^*_{,2,1}$. Therefore, Equation (4.2) will be satisfied automatically if the transformed \bar{M}_i 's and \bar{N}_i 's satisfy Equations (4.7).

The Bäcklund transformation is assumed to have the following form.

$$M_i = A_i^k M_k, \quad N_i = (A^{-1})_i^k N_k, \quad (4.8)$$

with

$$(A_i^k) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \quad \alpha\beta = \gamma. \quad (4.9)$$

The requirement that \bar{M}_i and \bar{N}_i satisfy Equations (4.7) results in the following equations for α and γ .

$$d\gamma = \gamma(\gamma-1)M_3 dx_1 + (\gamma-1)N_3 dx_2, \quad (4.10)$$

$$d(\alpha\gamma^{-1/2}) = [-\gamma^{1/2}M_2 + (\alpha\gamma^{-1/2})(M_2 - M_1) + (\alpha\gamma^{-1/2})^2 M_1 \gamma^{1/2}] dx_1 + \\ [-\gamma^{-1/2}N_1 + (\alpha\gamma^{-1/2})(N_1 - N_2) + (\alpha\gamma^{-1/2})^2 N_2 \gamma^{-1/2}] dx_2. \quad (4.11)$$

The solution of these equations introduces integration constants. For example, the solution to Equation (4.10) is

$$\gamma = \frac{b - (\rho - iz)}{b + (\rho + iz)}, \quad (4.10a)$$

where b is an integration constant, ρ satisfies Equation (4.5a), and z is ρ 's harmonic conjugate. Each set of integration constants represents a different transformation and will be labelled $(I_1)_a$. The I_1 represents all the transformations of Equation (4.8,9), and the a represents the integration constants. For example, the matrix representing $(I_1)_a$ will be written in the same form as the matrix of Equation (4.9) with elements $\alpha_a, \beta_a, \gamma_a$. The α_a and γ_a must satisfy Equations (4.10,11). It will now be shown that the I_1 form a group.

Given two transformations $(I_1)_b$ and $(I_1)_a$, it must be shown that the product $(I_1)_a(I_1)_b$ is also an I_1 transformation. By definition, the transformation $(I_1)_a(I_1)_b$ is represented by the matrix

$$\begin{pmatrix} \alpha_b \alpha_a & 0 & 0 \\ 0 & \beta_b \beta_a & 0 \\ 0 & 0 & \gamma_b \gamma_a \end{pmatrix}, \quad (\alpha_b \alpha_a)(\beta_b \beta_a) = (\gamma_b \gamma_a). \quad (4.13)$$

Then it can be shown that $\alpha_b \alpha_a$ and $\gamma_b \gamma_a$ satisfy Equations (4.10,11), remembering that the $(I_1)_a$ transformation acts on the $(I_1)_b M_i$, $(I_1)_b N_i$.

Hence, the I_1 transformations exhibit the group closure property. Since the matrices representing the group transformations are diagonal, the transformations I_1 are also commutative.

Associativity, existence of the identity, and inverses for each transformation are then shown to follow. The major step in each of the above proofs is the step showing that the respective matrix element satisfies Equations (4.10,11).

The Neugebauer-Kramer mapping introduced in Chapter 3 now plays an important role. In the (M_i, N_i) realization, this mapping is represented by the matrices U_i^k and V_i^k , where

$$(U_i^k) = \begin{pmatrix} 0 & -1 & 1/2 \\ -1 & 0 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } (V_i^k) = \begin{pmatrix} -1 & 0 & 1/2 \\ 0 & -1 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}. \quad (4.14)$$

With this discrete group, to be called S , define a new group of transformations I_2 by SI_1S . I_2 is also a group of Bäcklund transformations given in the (M_i, N_i) realization by the matrices W_i^k and Z_i^k where

$$(W_i^k) = \begin{pmatrix} \beta & 0 & (\gamma - \beta)/2 \\ 0 & \alpha & (\gamma - \alpha)/2 \\ 0 & 0 & \gamma \end{pmatrix}, \quad (4.15)$$

and

$$(Z_i^k) = \begin{pmatrix} -\alpha^{-1} & 0 & (2\alpha\gamma)^{-1}(\alpha - \gamma) \\ 0 & \beta^{-1} & (2\beta\gamma)^{-1}(\beta - \gamma) \\ 0 & 0 & \gamma^{-1} \end{pmatrix}. \quad (4.16)$$

From these, for I_2 transformations, γ still satisfies Equation (4.10), but now

$$\begin{aligned}
d(\alpha\gamma^{-1/2}) = & [-\gamma^{1/2}(M_3/2-M_1)+(\alpha\gamma^{-1/2})(M_2-M_1)+ \\
& (\alpha\gamma^{-1/2})^2(M_3/2-M_2)\gamma^{1/2}]dx_1+ \\
& [-\gamma^{-1/2}(N_3/2-N_1)+(\alpha\gamma^{-1/2})(N_2-N_1)+ \\
& (\alpha\gamma^{-1/2})^2(N_3/2-N_2)\gamma^{1/2}]dx_2 .
\end{aligned} \tag{4.17}$$

It would seem that after each transformation, first order partial differential equations would have to be solved since the M_i , N_i are first derivatives. But Neugebauer has produced a very important theorem.

Neugebauer's Commutation Theorem

Given an initial solution M_i , N_i and an arbitrary Bäcklund transformation $(I_1)_a$, then there are always three nontrivial Bäcklund transformations $(I_2)_b$, $(I_1)_c$, $(I_2)_d$, such that the product transformation $(I_2)_d(I_1)_c(I_2)_b(I_1)_a$ is the identity transformation. The corresponding α 's and γ 's are

$$\begin{aligned}
\alpha_b = \frac{(\alpha_a - \gamma_a)}{(\gamma_a(\alpha_a - 1))}, \quad \alpha_c = \frac{\gamma_a}{\alpha_a}, \quad \alpha_d = \frac{(\alpha_a - 1)}{(\alpha_a - \gamma_a)}, \\
\gamma_b = \frac{1}{\gamma_a}, \quad \gamma_c = \gamma_a, \quad \gamma_d = \frac{1}{\gamma_a}.
\end{aligned} \tag{4.18}$$

Here the α_a and α_c satisfy Equation (4.11) and the α_b and α_d satisfy Equation (4.17). All the γ 's satisfy Equation (4.10).

This is a very important theorem. It considerably simplifies generating new solutions from old solutions without having to integrate partial differential equations at each step.

Now that two major methods of solution generation have been reviewed, their use with the vacuum equations will be summarized in the next chapter.

CHAPTER 5

REVIEW OF THE VACUUM CASE

In the last two chapters, two methods were presented for generating new solutions of the vacuum Ernst equations. In Chapter 2, both the electromagnetic and gravitational fields satisfy similar equations. This similarity has been used to extend vacuum results to electrovac; therefore, it is natural to review the vacuum case first. The notation to be used in this chapter follows that given by Kinnersley and Chitre (1977a,b; 1978a,b).

Fields, Potentials, and Generating Functions

Defining N_{AB} by

$$\nabla N_{AB} = H_{XA} * \nabla H_{XB}^X \quad (5.1)$$

it is possible to show that

$$H_{AB}^2 = i(N_{AB} + H_{AX} H_{BX}^X) \quad (5.2)$$

satisfies the field Equations (2.37). Continuing in this way, define

$$\nabla^{mn} N_{AB} = H_{XA}^m * \nabla H_{XB}^n; \quad (5.3)$$

then it is possible to show that

$$H_{AB}^{n+1} = i(N_{AB}^{1n} + H_{AX}^{1n} H_{BX}^n) \quad (5.4)$$

satisfies Equations (2.37). All of the H_{AB}^n satisfy

$$\nabla H_{AB}^n = -i \rho^{-1} f_A^X \nabla H_{XB}^{X-n} \quad (5.5)$$

Defining

$$H_{AB} = i\epsilon_{AB}, \quad (5.6)$$

implies that

$$\text{On } N_{AB}^n = -iH_{AB}^n, \quad (5.7)$$

and the fields H_{AB}^n are incorporated into the hierarchy of potentials N_{AB}^{mn} . This extends the range of m and n to all integer values.

Kinnersley and Chitre (1977b) have given certain relations satisfied by the N_{AB}^{mn} . These are

$$N_{AB}^{mn} - N_{BA}^{nm} = H_{XA}^m * H_B^{nX} + \epsilon_{AB} \delta^m_0 \delta^n_0 \quad (5.8)$$

and

$$N_{AB}^{m,n+1} - N_{AB}^{m+1,n} = iN_{AX}^{m1} H_B^{nX} - 3\epsilon_{AB} \delta^m_{-1} \delta^n_0 \quad (5.9)$$

It has proven useful to define generating functions in terms of the potentials. The two generating functions used in the vacuum case are, somewhat changing the notation of Kinnersley and Chitre (1978b),

$$N_{AB}(s,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} s^m t^n N_{AB}^{mn} \quad (5.10)$$

and

$$F_{AB}(t) = \sum_{n=0}^{\infty} t^n H_{AB}^n = iN_{AB}(0,t). \quad (5.11)$$

There are algebraic relationships between these two generating functions, such that when the $F_{AB}(t)$ are known, the $N_{AB}(s,t)$ can be found.

The Symmetry Groups

In Chapter 3, the group H was found starting with the vacuum Ernst equation. There are other groups beside H and these are now reviewed.

The Groups G and H

Kinnersley (1977a) starts with the (t, ϕ) coordinate transformation group G, isomorphic to the group of 2 by 2 real matrices with unit determinant $SL(2, R)$. G's three generators are chosen as

$$t' = t + a\phi, \quad \phi' = \phi, \quad (5.12)$$

$$t' = t, \quad \phi' = \phi + bt, \quad (5.13)$$

$$t' = ct, \quad \phi' = c^{-1}\phi, \quad (5.14)$$

named $(G_1)_a$, $(G_2)_b$, $(G_3)_c$, respectively. The finite action of these three generators on the f_{AB} is given by

$$\begin{aligned} f_{11}' &= f_{11}, \quad f_{12}' = f_{12} - af_{11}, \\ f_{22}' &= f_{22} - 2af_{12} + a^2 f_{11}; \end{aligned} \quad (5.15)$$

$$\begin{aligned} f_{22}' &= f_{22}, \quad f_{12}' = f_{12} - bf_{22}, \\ f_{11}' &= f_{11} - 2bf_{12} + b^2 f_{22}; \end{aligned} \quad (5.16)$$

$$f_{11}' = c^{-2}f_{11}, \quad f_{12}' = f_{12}, \quad f_{22}' = c^2 f_{22}. \quad (5.17)$$

Being a group of coordinate transformations, G does not physically alter the spacetime. However, by defining the group $H \equiv SGS$ with S being the Neugebauer-Kramer map, H is found to be the group for Equations (3.55,56). Its three generators, $(H_1)_a = S(G_1)_{-ia}S$,

$(H_2)_b = S(G_2)_{ib}S$, $(H_3)_c = S(G_3)_c^{-1}S$, produce the following action on f and ψ respectively.

$$f' = f, \psi' = \psi - a; \quad (5.18)$$

$$f' = \frac{1}{(1-b\psi^2+b^2f^2)}, \psi' = \frac{\psi-b(f^2+\psi^2)}{(1-b\psi^2+b^2f^2)}; \quad (5.19)$$

$$f' = c^2f, \psi' = c^2\psi. \quad (5.20)$$

These three finite transformations are recognized as the one parameter subgroups found in Chapter 3.

The Group K

Next, the infinitesimal action of G (H) is found on the ψ_{AB} (f_{AB}). This allows the infinitesimal action of G and H to be found on the H_{AB} , and then on the N_{AB}^{mn} .

Forming commutators with the different infinitesimal operators of G and H , an infinite-dimensional Lie algebra is generated. The operators thus defined will be named T_{XY}^k and define the group K . Their

action on the N_{AB}^{mn} is given by

$$T_{XY}^k N_{AB}^{mn} = \epsilon_A(X^N Y^B)^{m+k, n} - N_A(X^\epsilon Y)^{m, n+k} + \sum_{s=1}^k N_A(X^N Y^B)^{ms, k-s, n}, \quad (5.21)$$

where the parentheses indicate symmetrization with respect to the

enclosed indices. The operators T_{XY}^0 are the infinitesimal operators of

G , and the $(T_{11}^{-1}, T_{12}^0, T_{22}^1)$ are the infinitesimal operators of H . The

commutators by which the T_{XY}^k were originally defined are

$$[T_{XY}^k, T_{AB}^1] = \epsilon_B(X^T Y^A)^{k+1} + \epsilon_A(X^T Y^B)^{k+1}. \quad (5.22)$$

For example, the commutator of T_{11}^0 and T_{22}^1 defines T_{12}^1 .

The Groups Q and \tilde{Q}

Another group, Q, lying outside the K group was found by Cosgrove (1979a). This group changes not only the potentials, but also the coordinates (ρ, z) . The infinitesimal action of Q on the coordinates (ρ, z) is

$$Q\rho = \rho z, \quad Qz = 1/2(z^2 - \rho^2), \quad (5.23)$$

and on the potentials N_{AB}^{mn} is

$$QN_{AB}^{mn} = 1/4[(m+A-1) N_{AB}^{m+1,n} + (n+B-1) N_{AB}^{m,n+1} + i N_{A2}^{m1} H_{1B}^n]. \quad (5.24)$$

Given the Neugebauer-Kramer map S, a conjugate group $\tilde{Q} \equiv SQS$ is defined.

Its infinitesimal action on (ρ, z) is identical to Equation (5.23), and

its infinitesimal action on N_{AB}^{mn} is

$$\tilde{Q}N_{AB}^{mn} = 1/4[(m+1) N_{AB}^{m+1,n} + n N_{AB}^{m,n+1}]. \quad (5.25)$$

From Equations (5.24,25) it is seen that

$$Q - \tilde{Q} = -1/4 T_{12}^1 \quad (5.26)$$

holds for these infinitesimal operators.

Applications

The problem of generating new solutions from a given solution now separates into three parts. Given a solution; for example flat spacetime, the potentials N_{AB}^{mn} must be found. Second, the integration of the infinitesimal operators must be performed. Third, only those transformations which produce physical solutions should be used. By physical solution, it is usually meant that as the quantity $\rho^2 + z^2$ approaches infinity, the metric approaches the flat spacetime metric.

Finding the generating functions $N_{AB}(s,t)$ and $F_{AB}(t)$ has proved the most effective way in finding the potentials N_{AB}^{mn} . An example of a generating function which has been found is the flat spacetime

generating function $F_{AB}(t)$, Kinnersley and Chitre (1978a),

$$F_{AB}(t) = \begin{pmatrix} \frac{t}{S(t)} & \frac{i}{S(t)} \\ \frac{-i(1-2tz+S(t))}{S(t)} & \frac{(1-2tz-S(t))}{2tS(t)} \end{pmatrix}, \quad (5.27)$$

with

$$S(t)^2 = (1-2tz)^2 + (2t\rho)^2. \quad (5.28)$$

Other generating functions which have been found for a given spacetime are listed below.

1. Zipoy-Voorhees; Kinnersley and Chitre (1978b)
2. Weyl; Hoenselaers, Kinnersley, and Xanthopoulos (1979)
3. Kerr-NUT; Hauser and Ernst (1979a)

Generating functions have proved useful in the integration of infinitesimal transformations to finite transformations. For example, Cosgrove (1980a) has successfully integrated the infinitesimal Q and \tilde{Q} transformations to the finite $(Q)_{4s}$ and $(\tilde{Q})_{4s}$ transformations using $N_{AB}(s,t)$ and $F_{AB}(t)$. Some of these finite transformations are

$$(\tilde{Q})_{4s} N_{AB}(t_1, t_2) = \frac{t_2}{s+t_2} N_{AB}(s+t_1, s+t_2) + \frac{s}{s+t_2} \epsilon_{AB}, \quad (5.29)$$

$$(Q)_{4s} F_{11}(t) = \frac{itF_{11}(s+t)}{(s+t)F_{12}(s)}, \quad (5.30a)$$

$$(Q)_{4s} F_{12}(t) = \frac{iF_{12}(s+t)}{F_{12}(s)}. \quad (5.30b)$$

More relations of this type are found in the last reference cited. In these, it must be remembered that (ρ, z) also changes.

Other finite transformations which have been found are listed below.

1. The null parameter case; that is when the parameters q_{xy}^k in the operator $q_{xy}^k T_{xy}^k$ satisfies $q_{xy}^k q_{xy}^k = 0$. Kinnersley and Chitre (1978a)

2. For static metrics, T_{12}^k has been integrated to yield all the static spacetimes. Kinnersley and Chitre (1978a)

3. The B subgroup, having infinitesimal operators $\beta = T_{11}^k + T_{22}^{k+2}$, $k = 0, 1, 2, \dots$, preserves asymptotic flatness. This subgroup has been used to generate new solutions. For example, Kerr-Nut was first generated from the Schwarzschild potentials using this subgroup. Its applicability does depend on the initial solution. Kinnersley and Chitre (1978b).

4. The HKX transformations may be used to generate asymptotically flat spacetime with arbitrary multipole moments. Hoenselaers, Kinnersley, and Xanthopoulos (1979).

Bäcklund Transformations

In Chapter 4, the Bäcklund transformation found by Neugebauer was described. Cosgrove (1979b) has shown that I_1 and I_2 are equivalent to the two groups HQ and \tilde{GQ} . (HQ simply means the group consisting of products of the elements in the groups H and Q. Since these two groups commute, order is no problem.) Neugebauer (1980a,b) has applied I_1 and I_2 and the commutation theorem to generate arbitrary multipole moments for asymptotically flat spacetime. Cosgrove (1980a), using both Harrison's Bäcklund transformation and Neugebauer's commutation theorem,

has also been able to generate asymptotically flat spacetimes with arbitrary multipole moments.

Summary

Other methods exist besides the group transformation and the Backlund transformation. Belinsky and Sakharov (1978,1979) use an inverse-scattering technique. Hauser and Ernst (1979a,b; 1980a,b) use an integral equation method. Both are capable of generating asymptotically flat spacetimes with arbitrary multipole moments. With so many methods capable of generating equivalent results, see Cosgrove (1980a), it is no wonder that these workers feel the vacuum case to be essentially solved.

CHAPTER 6

REVIEW OF THE ELECTROVAC CASE

Many of the results from the vacuum case have been extended to the electrovac case. These electrovac results will be reviewed in this chapter. The notation again follows that given by Kinnersley and Chitre (1977a,b; 1978a,b).

Fields, Potentials, and Generating Functions

The fields H_{AB}^n and the potentials N_{AB}^{mn} were introduced in Chapter 5. In a like manner, the fields H_{AB}^n and ϕ_A^n and the potentials N_{AB}^{mn} , M_A^{mn} , L_B^{mn} , K^{mn} are introduced for electrovac. They are defined recursively by the following relationships:

$$\phi_A^1 = \Phi_A^1, \quad H_{AB}^1 = H_{AB}^1, \quad (6.1)$$

$$\nabla K^{mn} = \phi_X^m * \nabla \phi^X, \quad \nabla L_B^{mn} = \phi_X^m * \nabla H_B^X, \quad (6.2)$$

$$\nabla M_A^{mn} = H_{XA}^m * \nabla \phi^X, \quad \nabla N_{AB}^{mn} = H_{XA}^m * \nabla H_B^X, \quad (6.3)$$

$$\phi_A^{n+1} = i(M_A^{n+1} + 2\phi_A^{nK} + H_{AX}^n \phi^X), \quad (6.4)$$

$$H_{AB}^{n+1} = i(N_{AB}^{n+1} + 2\phi_A^{nL_B} + H_{AX}^n H_B^X). \quad (6.5)$$

All the fields ϕ_A^n and H_{AB}^n satisfy the same field equations.

$$\nabla\phi_A^n = -i\rho^{-1}f_A^X \chi_{\phi X}^{-n}, \quad (6.6)$$

$$\nabla H_{AB}^n = -i\rho^{-1}f_A^X \chi_{H_{XB}}^{-n}. \quad (6.7)$$

Defining

$$K = -1/2i, \quad H_{AB}^0 = i\epsilon_{AB}, \quad (6.8)$$

implies

$$M_A^n = -i\phi_A^n, \quad N_{AB}^n = -iH_{AB}^n, \quad (6.9)$$

and that the original fields are contained in the hierarchy of potentials. The relations found by Kinnersley and Chitre (1977b) are

$$K^{mn} - K^{nm*} = \phi_X^m \phi_X^{n*}, \quad (6.10)$$

$$L_B^{mn} - M_B^{nm*} = \phi_X^m H_B^{n*}, \quad (6.11)$$

$$N_{AB}^{mn} - N_{BA}^{nm*} = H_{XA}^m H_B^{n*} + \epsilon_{AB} \delta_0^m \delta_0^n, \quad (6.12)$$

and

$$K^{m,n+1} - K^{m+1,n} = 2iK^{ml} K^{ln} + iL_X^{ml} \phi_X^{n*} + (i\delta_0^m \delta_0^n)/2, \quad (6.13)$$

$$L_B^{m,n+1} - L_B^{m+1,n} = 2iK^{ml} L_B^{ln} + iL_X^{ml} H_B^{n*}, \quad (6.14)$$

$$M_A^{m,n+1} - M_A^{m+1,n} = 2iM_A^{ml} K^{ln} + iN_{AX}^{ml} \phi_X^{n*}, \quad (6.15)$$

$$N_{AB}^{m,n+1} - N_{AB}^{m+1,n} = 2iM_A^{ml} L_B^{ln} + iN_{AX}^{ml} H_B^{n*} - 3\epsilon_{AB} \delta_0^m \delta_0^n. \quad (6.16)$$

Generating functions are defined by

$$K(s,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} s^m t^n K^{mn}, \quad K(t) = \sum_{n=0}^{\infty} t^n K^n, \quad (6.17)$$

$$L_A(s,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} s^m t^n L_A^{mn}, \quad L_A(t) = \sum_{n=0}^{\infty} t^n L_A^n, \quad (6.18)$$

$$M_A(s,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} s^m t^n M_A^{mn}, \quad \phi_A(t) = \sum_{n=0}^{\infty} t^n \phi_A^n, \quad (6.19)$$

$$N_{AB}(s,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} s^m t^n N_{AB}^{mn}, \quad F_{AB}(t) = \sum_{n=0}^{\infty} t^n H_{AB}^n. \quad (6.20)$$

Taking the gradient of Equations (6.4,5) leads to the equations

$$\nabla F_{AB}(t) = \frac{it}{S^2(t)} \left[((1-2tz)\nabla H_{AX} - 2t\rho\tilde{\nabla} H_{AX}) F_B^X(t) + 2((1-2tz)\nabla\phi_A - 2t\rho\tilde{\nabla}\phi_A) L_B(t) \right], \quad (6.21)$$

$$\nabla\phi_A(t) = \frac{it}{S^2(t)} \left[((1-2tz)\nabla H_{AX} - 2t\rho\tilde{\nabla} H_{AX}) \phi^X(t) + 2((1-2tz)\nabla\phi_A - 2t\rho\tilde{\nabla}\phi_A) K(t) \right], \quad (6.22)$$

$$S^2(t) = (1-2tz)^2 + (2t\rho)^2. \quad (6.23)$$

The first integral of the above equations is

$$2K(t)\text{Det}(F(t)) - 2\phi^X(t)L^Y(t)F_{XY}(t) = \frac{i}{S(t)}. \quad (6.24)$$

Important algebraic relationships can be derived from Equations (6.10-16). Some of these are:

$$N_{AB}(s,t) = \frac{s}{s-t} \epsilon_{AB} + \frac{2itS(s)}{(s-t)} \left[K(s)F_{XA}(s)F_B^X(t) + (L_A(s)F_{XB}(t) - L_B(t)F_{XA}(s))\phi^X(s) \right], \quad (6.25)$$

$$M_A(s,t) = \frac{2itS(s)}{(s-t)} \left[K(s)\phi^X(t)F_{XA}(s) - K(t)\phi^X(s)F_{XA}(s) - L_A(s)\phi_X(s)\phi^X(t) \right], \quad (6.26)$$

$$L_B(s,t) = \frac{stS(s)}{(s-t)} [\text{Det}(F(t))L_B(t) - L^Y(s)F_{XY}(s)F_B^X(t)], \quad (6.27)$$

$$K(s,t) = \frac{i(st-s^2-t^2)}{2(s-t)} + \frac{stS(s)}{(s-t)} [K(t)\text{Det}(F(t)) - L^Y(s)F_{XY}(s)\phi^X(t)], \quad (6.28)$$

$$K^*(s) = -\phi^X \phi_X^*(s) - iS(s)\text{Det}(F(s))/2, \quad (6.29)$$

$$L_A(s) = -\phi^X F_{XA}^*(s) + S(s)\phi_X(s)F_A^X(s)/s, \quad (6.30)$$

$$F_{CA}^*(s) = -2iS^{-1}(s)((1-2sz)\delta_C^B + 2isf_C^B) \times \\ (K(s)F_{BA}(s) - \phi_B(s)L_A(s) + \phi_B^*\phi_X(s)F_A^X(s)), \quad (6.31)$$

$$\phi_C^*(s) = sS^{-1}(s)((1-2sz)\delta_C^B + 2isf_C^B) \times \\ (F_{BX}(s)L^X(s) - \phi_B^*\text{Det}(F(s))). \quad (6.32)$$

In the above, $\text{Det}(F(s))$ is the determinant of $F_{AB}(s)$.

The Symmetry Groups

The vacuum groups can be enlarged to the electrovac case. The groups G' and H' are defined and used to find an enlarged symmetry group K' .

The Groups G' and H'

The group G' is comprised of the coordinate transformations given in Equations (5.12-14) and the electromagnetic and gravitational gage transformations

$$\phi_A \rightarrow \phi_A + a_A, \quad H_{AB} \rightarrow H_{AB} - \phi_A^* a_B - \phi_A a_B^* - a_A^* a_B + i\alpha_{AB}. \quad (6.33)$$

These gage transformations have a_B complex and α_{AB} real and symmetric; they do not change the metric f_{AB} or the electromagnetic field F_{AM} . This now defines G' which is an eight parameter group.

The group H' is also an eight parameter group leaving Equations (2.38,39) invariant. The generators of H' are the gravitational and electromagnetic gage transformations:

$$\phi_1 \rightarrow \phi_1, H_{11} \rightarrow H_{11} + i\alpha, \quad (6.34)$$

$$\phi_1 \rightarrow \phi_1 + a, H_{11} \rightarrow H_{11} - 2a^* \phi_1 - a^* a; \quad (6.35)$$

the generalized Ehler's transformation

$$\phi_1 \rightarrow \frac{\phi_1}{(1 + \gamma H_{11})}, H_{11} \rightarrow \frac{H_{11}}{(1 + i\gamma H_{11})}; \quad (6.36)$$

the Harrison (1968) mixing transformation

$$\phi_1 \rightarrow \frac{\phi_1 + c H_{11}}{(1 - 2c^* \phi_1 - c^* c H_{11})}, \quad (6.37)$$

$$H_{11} \rightarrow \frac{H_{11}}{(1 - 2c^* \phi_1 - c^* c H_{11})}; \quad (6.38)$$

and the scaling and electromagnetic duality transformation

$$\phi_1 \rightarrow \beta e^{i\epsilon} \phi_1, H_{11} \rightarrow \beta^2 H_{11}. \quad (6.39)$$

In the above, $\alpha, \beta, \gamma, \epsilon$ are real constants while a and c are complex constants. There is no simple way to get H' from G' since the Neugebauer-Kramer mapping's action on the metric f_{AB} has not been found; however, the mapping's action on the group generators has been found and will be described later in this chapter.

The Group K'

Again, by forming all possible commutators of the infinitesimal operators of G' and H' , an infinite-dimensional parameter group is generated. This group is called K' and is the electrovac generalization

of K . The infinitesimal operators are $T_{xy}^k, C_x^k, C_x^{*k}, \Sigma$, and their action is given in Kinnersley and Chitre (1977b:1540) and reproduced here. (In the above reference, the action is given with the group parameters already implied. For example, what Kinnersley and Chitre refer to

as c_A^k is the infinitesimal operator $c_X^k C_x^k + c_X^{*k} C_x^{*k}$. To avoid

confusion, $\gamma_{xy}^k, C_x^k, C_x^{*k}, \sigma$ will be the parameters and $T_{xy}^k, C_x^k, C_x^{*k}, \Sigma$

will be the operators.)

$$T_{XY}^{k mn} N_{AB} = \epsilon_A^{m+k, n} N_{A(Y^N X)B} - N_A^{m, n+k} (X^E Y)B + \sum_{s=1}^k N_A^{ms} N_{A(X^N Y)B}^{k-s, n} \quad (6.40)$$

$$T_{XY}^{k mn} M_A = \epsilon_A^{m+k, n} (Y^M X) + \sum_{s=1}^k N_A^{ms} N_{A(X^M Y)}^{k-s, n} \quad (6.41)$$

$$T_{XY}^{k mn} L_B = \epsilon_B^{m, n+k} (Y^L X) + \sum_{s=1}^k L_{(X^N Y)B}^{ms} N_{A(X^N Y)B}^{k-s, n} \quad (6.42)$$

$$T_{XY}^{k mn} K = \sum_{s=1}^k L_{(X^M Y)}^{ms} N_{A(X^M Y)}^{k-s, n} \quad (6.43)$$

$$C_X^{k mn} N_{AB} = 2i \epsilon_{XA}^{m+k, n} L_B - 2i \sum_{s=1}^k N_{AX}^{ms} L_B^{k-s, n} \quad (6.44)$$

$$C_X^{k mn} M_A = 2i \epsilon_{XA}^{m+k, n} K + N_{AX}^{m, n+k-1} - 2i \sum_{s=1}^{k-1} N_{AX}^{ms} K^{k-s, n} \quad (6.45)$$

$$C_X^{k mn} L_B = -2i \sum_{s=1}^k L_X^{ms} L_B^{k-s, n} \quad (6.46)$$

$$C_X^{k,mn} = L_X^{m,n+k-1} - 2i \sum_{s=1}^{k-1} L_X^{ms} K^{k-s,n}, \quad (6.47)$$

$$C_X^{*N}_{AB} = -2i \epsilon_{XB} M_A^{m,n+k} + 2i \sum_{s=1}^k M_A^{ms} N_{XB}^{k-s,n}, \quad (6.48)$$

$$C_X^{*M}_A = 2i \sum_{s=1}^k M_A^{ms} M_X^{k-s,n}, \quad (6.49)$$

$$C_X^{*L}_B = -2i \epsilon_{XB} K^{m,n+k} + N_{XN}^{m+k-1,n} + 2i \sum_{s=1}^k K^{ms} N_{XB}^{k-s,n}, \quad (6.50)$$

$$C_X^{*K} = M_X^{m+k-1,n} + 2i \sum_{s=1}^k K^{ms} M_X^{k-s,n}, \quad (6.51)$$

$$\sum N_{AB}^{k,mn} = -2 \sum_{s=1}^k M_A^{ms} L_B^{k-s+1,n}, \quad (6.52)$$

$$\sum M_A^{k,mn} = -i M_A^{m,n+k} - 2 \sum_{s=1}^k M_A^{ms} K^{k-s+1,n}, \quad (6.53)$$

$$\sum L_B^{k,mn} = i L_B^{m+k,n} - 2 \sum_{s=1}^k K^{ms} L_B^{k-s+1,n}, \quad (6.54)$$

$$\sum K^{k,mn} = i K^{m+k,n} - i K^{m,n+k} - 2 \sum_{s=1}^{m+1} K^{ms} K^{k-s+1,n} - 1/2 \delta_0^m \delta_0^n \delta^k 1, \quad (6.55)$$

with k running over all integer values.

Kinnersley (1980) has found eight-parameter subgroups $(G^{m,n})$ of K' isomorphic to $SU(2,1)$. They are

$$G^{(m,n)} = \{-m, 0, m, n, (m+n), (1-m-n), (1-n), 0, \sigma\}. \quad (6.56)$$

Also found were involutive automorphisms which map the subgroups into one another. These are

$$\begin{pmatrix} r,s \\ I \end{pmatrix} : K' \rightarrow K', \quad \begin{pmatrix} r,s \\ I \end{pmatrix} : \begin{pmatrix} m,n \\ G \end{pmatrix} \rightarrow \begin{pmatrix} r-m,s-n \\ G \end{pmatrix}, \quad (6.57)$$

$$\begin{pmatrix} r,s \\ I \end{pmatrix} \gamma_{11} \begin{pmatrix} r,s \\ I \end{pmatrix} = i \gamma_{22}, \quad (6.58)$$

$$\begin{pmatrix} r,s \\ I \end{pmatrix} \gamma_{12} \begin{pmatrix} r,s \\ I \end{pmatrix} = -\gamma_{12}, \quad (6.59)$$

$$\begin{pmatrix} r,s \\ I \end{pmatrix} \gamma_{22} \begin{pmatrix} r,s \\ I \end{pmatrix} = -i \gamma_{11}, \quad (6.60)$$

$$\begin{pmatrix} r,s \\ I \end{pmatrix} c_1 \begin{pmatrix} r,s \\ I \end{pmatrix} = c_2^*, \quad (6.61)$$

$$\begin{pmatrix} r,s \\ I \end{pmatrix} c_2 \begin{pmatrix} r,s \\ I \end{pmatrix} = i c_1^*, \quad (6.62)$$

$$\begin{pmatrix} r,s \\ I \end{pmatrix} c_1^* \begin{pmatrix} r,s \\ I \end{pmatrix} = -i c_2, \quad (6.63)$$

$$\begin{pmatrix} r,s \\ I \end{pmatrix} c_2^* \begin{pmatrix} r,s \\ I \end{pmatrix} = c_1, \quad \begin{pmatrix} r,s \\ I \end{pmatrix} \sigma \begin{pmatrix} r,s \\ I \end{pmatrix} = -\sigma. \quad (6.64)$$

The subgroups $\begin{pmatrix} 0,0 \\ G \end{pmatrix}$ and $\begin{pmatrix} 1,0 \\ G \end{pmatrix}$ are G' and H' respectively. The automorphism $\begin{pmatrix} 1,0 \\ I \end{pmatrix}$ is the Neugebauer-Kramer map. In vacuum, its action on the potentials has been found, but this does not hold true in the electrovac case.

This completes the review of electrovac, the next chapter will extend Cosgrove's vacuum Q and \tilde{Q} transformations to the electrovac case.

CHAPTER 7

THE EXTENDED Q AND \tilde{Q} GROUPS

In this chapter, the vacuum transformation groups Q and \tilde{Q} will be extended to include the electromagnetic field described in Chapter 2. First Q will be extended and then Equation (5.26) will be used for the initial ansatz of the infinitesimal \tilde{Q} transformation. This ansatz will be modified allowing the finite transformation to be integrated, bringing agreement with the vacuum relation

$$\tilde{Q} = \begin{matrix} (1,0) & (1,0) \\ I & Q & I \end{matrix} ;$$

$\begin{matrix} (1,0) \\ I \end{matrix}$ being the discrete automorphism described in Chapter 6.

Associated with \tilde{Q} is its complex conjugate group \tilde{Q}^* found in this chapter to be equivalent to $\begin{matrix} (1,1) & (1,1) \\ I & Q & I \end{matrix}$. This \tilde{Q}^* transformation merges with \tilde{Q} in the vacuum case and is; therefore, a new feature belonging to the electrovac case.

The Extended Q Group

The vacuum transformation group Q was first extended to the electrovac case by Cosgrove (1980b) during a bout with the flu. This group's action on the fields and potentials was first found by myself in a roundabout manner. A simplified version is presented in this thesis.

Cosgrove (1980) has found the finite action of the vacuum Q transformation on the generating functions $F_{AB}(t) = iN_{AB}(0,t)$. From his results, the infinitesimal action of Q is found to be

$$QH_{11}^n = 1/4[n H_{11}^{n+1} + iH_{12}^n H_{11}^n], \quad (7.1)$$

$$QH_{21}^n = 1/4[n H_{21}^{n+1} + iN_{21}^n + iH_{22}^n H_{11}^n], \quad (7.2)$$

$$QH_{12}^n = 1/4[(n+1) H_{12}^{n+1} + iH_{12}^n H_{12}^n], \quad (7.3)$$

$$QH_{22}^n = 1/4[(n+1) H_{22}^{n+1} + iN_{22}^n + iH_{22}^n H_{12}^n], \quad (7.4)$$

These relations are assumed to hold unaltered. The only choice remaining is the infinitesimal action of Q on ϕ_A^n . Since the field equations for the ϕ_A^n are similar to the field equations for H_{A1}^n , the ansatz is made which transforms ϕ_A^n like H_{A1}^n . Hence,

$$Q\phi_1^n = 1/4[n \phi_1^{n+1} + iH_{12}^n \phi_1^n], \quad (7.5)$$

$$Q\phi_2^n = 1/4[n \phi_2^{n+1} + iM_2^n + iH_{22}^n \phi_1^n]. \quad (7.6)$$

The consistency of Equations (7.1-6) is shown in Appendix A. The infinitesimal action on the ρ and z coordinates is assumed the same as in the vacuum case; that is, $Q\rho = \rho z$, $Qz = 1/2(z^2 - \rho^2)$. Using Equations (6.2,3; 7.1-6), the infinitesimal action of Q on the potentials can be determined. The action on the potentials is given by

$$QK^{mn} = 1/4[(m+n) K^{m,n+1} - imL_1^{m1} \phi_2^n + i(m+1)L_2^{m1} \phi_1^n - 2imK^{m1} K^{1n}], \quad (7.7)$$

$$QM_1^{mn} = 1/4[(m+n) M_1^{m,n+1} - imN_{11}^{m1} \phi_2^n + i(m+1)N_{12}^{m1} \phi_1^n - 2imM_1^{m1} K^{1n}], \quad (7.8)$$

$$QM_2^{mn} = 1/4[(m+n+1) M_2^{m,n+1} - 2i(m+1)M_2^{m1} K^{1n} + i(m+2)N_{22}^{m1} \phi_1^n - i(m+1)N_{21}^{m1} \phi_2^n]. \quad (7.9)$$

$$QL_1^{mn} = 1/4[(m+n) L_1^{m,n+1} - 2imK L_1^{ml} L_n^{ln} - imL_1^{ml} H_{21}^{ln} + i(m+1)L_2^{ml} H_{11}^{ln}], \quad (7.10)$$

$$QL_2^{mn} = 1/4[(m+n+1) L_2^{m,n+1} - 2imK L_2^{ml} L_n^{ln} - imL_1^{ml} H_{22}^{ln} + i(m+1)L_2^{ml} H_{12}^{ln}], \quad (7.11)$$

$$QN_{11}^{mn} = 1/4[(m+n) N_{11}^{m,n+1} - 2imM_1 L_1^{ml} L_n^{ln} - imN_{11}^{ml} H_{21}^{ln} + i(m+1)N_{12}^{ml} H_{11}^{ln}], \quad (7.12)$$

$$QN_{12}^{mn} = 1/4[(m+n+1) N_{12}^{m,n+1} - 2imM_1 L_2^{ml} L_n^{ln} + i(m+1)N_{12}^{ml} H_{12}^{ln} - imN_{11}^{ml} H_{22}^{ln}], \quad (7.13)$$

$$QN_{21}^{mn} = 1/4[(m+n+1) N_{21}^{m,n+1} - 2i(m+1)M_2 L_1^{ml} L_n^{ln} + i(m+2)N_{22}^{ml} H_{11}^{ln} - i(m+1)N_{21}^{ml} H_{21}^{ln}], \quad (7.14)$$

$$QN_{22}^{mn} = 1/4[(m+n+2) N_{22}^{m,n+1} - 2i(m+1)M_2 L_2^{ml} L_n^{ln} + i(m+2)N_{22}^{ml} H_{12}^{ln} - i(m+1)N_{21}^{ml} H_{22}^{ln}]. \quad (7.15)$$

As an example of how the above relations were obtained, Equation (7.12) is derived in Appendix B.

Finally, the infinitesimal action of Q on the metric function γ (see Equation (2.15,16)) is shown in Appendix C to be

$$Q\gamma = 0 \quad (7.16)$$

The Finite Q Transformation

The finite transformations are found by using the generating functions $F_{AB}(t)$ and $\phi_A(t) = iM_A(0,t)$. For example, the infinitesimal action on $\phi_1(t)$ and $H_{12}(t)$ is found from Equations (7.3,5) to be

$$Q\phi_1(t) = 1/4[D_1\phi_1(t) - \phi_1(t)/t + iH_{12}\phi_1(t)], \quad (7.17)$$

$$QF_{12}(t) = 1/4[D_1F_{12}(t) + iH_{12}F_{12}(t)], \quad (7.18)$$

with $D_1 = \partial/\partial t$. These equations are integrated in Appendix D to yield

the finite transformations. The other finite transformations are found in a similar manner; all of the finite transformations are listed below.

$$(Q)_{4s} F_{11}(t) = \frac{itF_{11}(s+t)}{(s+t)F_{12}(s)}, \quad (7.19)$$

$$(Q)_{4s} F_{12}(t) = \frac{iF_{12}(s+t)}{F_{12}(s)}, \quad (7.20)$$

$$(Q)_{4s} F_{21}(t) = \frac{it}{(s+t)} \left[N_{21}(s,s+t) - \frac{N_{22}(s,s)F_{11}(s+t)}{F_{12}(s)} \right] - \frac{is}{(s+t)} \quad (7.21)$$

$$(Q)_{4s} F_{22}(t) = i \left[N_{22}(s,s+t) - \frac{N_{22}(s,s)F_{12}(s+t)}{F_{12}(s)} \right], \quad (7.22)$$

$$(Q)_{4s} \phi_1(t) = \frac{it\phi_1(s+t)}{(s+t)F_{12}(s)}, \quad (7.23)$$

$$(Q)_{4s} \phi_2(t) = \frac{it}{(s+t)} \left[M_2(s,s+t) - \frac{N_{22}(s,s)\phi_1(s+t)}{F_{12}(s)} \right], \quad (7.24)$$

Since the action of the infinitesimal Q transformation was obtained directly from the vacuum case, it should be no surprise that the results here are similar.

The vacuum result $\gamma'(\rho', z') = \gamma(\rho, z)$ is seen to follow immediately from Equation (7.16). This means that the new metric function $\gamma'(\rho, z)$ equals $\gamma(\bar{\rho}, \bar{z})$, where $\bar{\rho} = (Q)_{-4s}\rho$ and $\bar{z} = (Q)_{-4s}z$.

Search for \tilde{Q}

In Equation (5.26), the relationship between the vacuum infinitesimal transformations Q and \tilde{Q} is given. This equation will be taken as the starting ansatz to find \tilde{Q} for the electrovac case. A further requirement will be that the \tilde{Q} group should be a conjugate group

SQS where S is one of the involutive automorphisms $I^{(r,s)}$. To begin

with, define an operator \tilde{Q}_0 by

$$Q - \tilde{Q}_0 = -1/4 T_{12}^1. \quad (7.25)$$

\tilde{Q}_0 has the following action on the potentials.

$$\tilde{Q}_0 N_{AB}^{mn} = 1/4 [(m+1) N_{AB}^{m+1,n} + n N_{AB}^{m,n+1} + i M_A^{ml} L_B^{ln}], \quad (7.26)$$

$$\tilde{Q}_0 K^{mn} = 1/4 [(m K^{m+1,n} + n K^{m,n+1} - 1/2 i L_X^{ml} \phi^{nX}], \quad (7.27)$$

$$\tilde{Q}_0 M_A^{mn} = 1/4 [(m+1/2) M_A^{m+1,n} + n M_A^{m,n+1} - 1/2 i N_{AX}^{mn} \phi^{X}], \quad (7.28)$$

$$\tilde{Q}_0 L_A^{mn} = 1/4 [(m+1/2) L_A^{m+1,n} + n L_A^{m,n+1} + i K^{ml} L_A^{ln}]. \quad (7.29)$$

Equation (7.26) is very similar to the expression found by Cosgrove

(1980: Equation (2.14)) for vacuum. The difference lies in the term

$i M_A^{ml} L_B^{ln}$. It is this term and the similar terms in Equations (7.26-29)

which prevent a simple integration to the finite transformation. An

alternative definition for \tilde{Q} is; therefore, sought.

At this stage, the Σ transformation of Equations (6.52-55) was noticed. In particular,

$$\Sigma K^{mn} = L_X^{ml} \phi^{nX}, \quad (7.30)$$

$$\Sigma L_B^{mn} = i L_B^{m+1,n} - 2K^{ml} L_B^{ln}, \quad (7.31)$$

$$\Sigma M_b^{mn} = -i M_B^{m+1,n} + N_{BX}^{ml} \phi^{nX}, \quad (7.32)$$

$$\Sigma N_{AB}^{mn} = -2M_A^{ml} L_B^{ln}. \quad (7.33)$$

Defining \tilde{Q} as

$$\tilde{Q} = \tilde{Q}_0 + i\Sigma/8, \quad (7.34)$$

the action on the potentials is much simpler.

$$\tilde{Q}K = 1/4[m K^{m+1,n} + n K^{m,n+1}], \quad (7.35)$$

$$\tilde{Q}M_A = 1/4[(m+1) M_A^{m+1,n} + n M_A^{m,n+1}], \quad (7.36)$$

$$\tilde{Q}L_A = 1/4[m L_A^{m+1,n} + n L_A^{m,n+1}], \quad (7.37)$$

$$\tilde{Q}N_{AB} = 1/4[(m+1) N_{AB}^{m+1,n} + n N_{AB}^{m,n+1}]. \quad (7.38)$$

Unfortunately, due to the i in Equation (7.34), this \tilde{Q} is not Hermitian.

\tilde{Q} and $\tilde{Q}^* = \tilde{Q}_0 - i\Sigma/8$ must be regarded as two different infinitesimal operators. For this reason, $(\tilde{Q}A)^* \neq \tilde{Q}A^*$, but rather $(\tilde{Q}A)^* = \tilde{Q}^*A^*$. This is an essential complication, since the transforms of K^* , L_A^* , M_A^* , N_{AB}^* are needed. Using the complex conjugates of Equations (7.30-33) and

Equations (6.13-16) yield

$$\tilde{Q}K^* = 1/4[m K^{*m+1,n} + n K^{*m,n+1} + iL_X^{*m,n} \chi^*], \quad (7.39)$$

$$\tilde{Q}M_A^* = 1/4[m M_A^{*m+1,n} + n M_A^{*m,n+1} + iN_{AX}^{*m,n} \chi^*], \quad (7.40)$$

$$\tilde{Q}L_A^* = 1/4[m L_A^{*m+1,n} + (n+1) L_A^{*m,n+1} + iL_X^{*m,n} H_A^* \chi^*], \quad (7.41)$$

$$\tilde{Q}N_{AB}^* = 1/4[m N_{AB}^{*m+1,n} + (n+1) N_{AB}^{*m,n+1} + iN_{AX}^{*m,n} H_B^* \chi^* + 3\epsilon_{AB} \delta^m_{-1} \delta^n_0]. \quad (7.42)$$

Equations (7.35-42) and their complex conjugates (\tilde{Q} replaced by \tilde{Q}^*) may be solved using generating functions. In terms of the generating functions, Equations (7.35-38) are

$$\tilde{Q}K(t_1, t_2) = 1/4[D_1 + D_2 - (t_1 + t_2)/(t_1 t_2)]K(t_1 t_2) + i(t_2/t_1 - t_1/t_2)/8, \quad (7.43)$$

$$\tilde{Q}M_A(t_1, t_2) = 1/4[D_1 + D_2 - 1/t_2]M_A(t_1, t_2), \quad (7.44)$$

$$\tilde{Q}L_A(t_1, t_2) = 1/4[D_1 + D_2 - (t_1 + t_2)/(t_1 t_2)]L_A(t_1, t_2), \quad (7.45)$$

$$\tilde{Q}N_{AB}(t_1, t_2) = 1/4[D_1 + D_2 - 1/t_2]N_{AB}(t_1, t_2) + \epsilon_{AB}/t_2, \quad (7.46)$$

where $D_1 = \partial/\partial t_1$, $D_2 = \partial/\partial t_2$. Equations (7.40,42), in terms of the generating functions $F_{AB}^*(t) = -iN_{AB}^*(0, t)$ and $\phi_A^*(t) = -iM_A^*(0, t)$ are

$$\tilde{Q}\phi_A^*(t) = 1/4[D\phi_A^*(t) - \phi_A^*(t)/t + iH_{AX}^* \phi^X(t)^*], \quad (7.47)$$

$$\tilde{Q}F_{AB}^*(t) = 1/4[DF_{AB}^*(t) + iH_{AX}^* F_B^X(t)^*], \quad (7.48)$$

where $D = \partial/\partial t$. Equations (7.46,48) are integrated in Appendix E as examples. The results are the finite transformations:

$$\begin{aligned} (\tilde{Q})_{4s} K(t_1, t_2) &= \frac{t_1 t_2}{(s+t_1)(s+t_2)} K(s+t_1, s+t_2) + \\ &\frac{is(t_2 - t_1)(s+t_1+t_2)}{2(s+t_1)(s+t_2)}, \end{aligned} \quad (7.49)$$

$$(\tilde{Q})_{4s} L_A(t_1, t_2) = \frac{t_1 t_2}{(s+t_1)(s+t_2)} L_A(s+t_1, s+t_2) \quad (7.50)$$

$$(\tilde{Q})_{4s} M_A(t_1, t_2) = \frac{t_2}{(s+t_2)} M_A(s+t_1, s+t_2), \quad (7.51)$$

$$(\tilde{Q})_{4s} N_{AB}(t_1, t_2) = \frac{t_2}{(s+t_2)} N_{AB}(s+t_1, s+t_2) + \frac{\epsilon_{AB}}{(s+t_2)}, \quad (7.52)$$

$$(\tilde{Q})_{4s} F_{AB}^*(t) = i(F^{*-1}(s))_{AX} F_B^X(s+t), \quad (7.53)$$

$$(\tilde{Q})_{4s} \phi_A^*(t) = \frac{it}{(s+t)} (F^{*-1}(s))_{AX} \phi^X(s+t). \quad (7.54)$$

The right-hand sides of these equations contain the original generating functions. The transformations of $(\tilde{Q}^*)_{4s}$ may be found by taking the complex conjugate of Equation (7.49-54).

It remains to be shown that $(\tilde{Q})_{4s}$ and $(\tilde{Q}^*)_{4s}$ are related to

$(Q)_{4s}$ by one of the discrete groups $I^{(r,s)}$. This is done by looking at the commutators of the infinitesimal operators. Using Equations (7.26-29) and Equations (6.40-55), the following commutators are obtained.

$$[\tilde{Q}_0, T_{XY}] = k T_{XY}/4, \quad [\tilde{Q}_0, \Sigma] = k \Sigma /4,$$

$$[\tilde{Q}_0, C_X] = (2k-1) C_X/8, \quad [Q_0, C_X^*] = (2k-1) C_X^*/8. \quad (7.55)$$

Using Equation (7.25), it is also found that

$$[Q, T_{11}] = (k-1) T_{11}/4, \quad [Q, T_{22}] = (k+1) T_{22}/4,$$

$$[Q, T_{12}] = k T_{12}/4, \quad [Q, C_1] = (k-1) C_1/4,$$

$$[Q, C_2] = k C_2/4, \quad [Q, C_1^*] = (k-1) C_1^*/4,$$

$$[Q, C_2^*] = k C_2^*/4, \quad [Q, \Sigma] = k \Sigma /4. \quad (7.56)$$

If an operator J is to be a conjugate operator SQS, J's commutators must be conjugate to Q's commutators. That is,

$$[J, A] = S[Q, A]S, \quad (7.57)$$

where S is to be one of the $I^{(r,s)}$. It is found that for no combination of integer (r,s) is this true concerning the operators \tilde{Q}_0 and Q.

Defining the operators

$$\tilde{Q}_n = \tilde{Q}_0 + in\Sigma/8, \quad (7.58)$$

with n ranging over all integer values, using the nonvanishing commutators of Σ in Equations (7.55,56) and in

$$[\Sigma, C_X] = i C_X, \quad [\Sigma, C_X^*] = -i C_X^*, \quad (7.59)$$

then $\tilde{Q}_n = \begin{pmatrix} (r,s) \\ I & Q & I \end{pmatrix}$ for the following integer values of $(n,r,s) = (2m+1, 1, m+1); m=0, 1, 2, 3, \dots$. It is seen that the operators \tilde{Q}^* and \tilde{Q} defined previously in the chapter are $\begin{pmatrix} (1,0) \\ I & Q & I \end{pmatrix}$ and $\begin{pmatrix} (1,1) \\ I & Q & I \end{pmatrix}$, respectively. (In finding this result, it must be noted that Equations (6.57-60) hold for the parameters of the transformation. It is also possible to write this in terms of the operators; for example, in Equation (6.60), $T_{22}^k \rightarrow T_{11}^{k+r}$.) Now that \tilde{Q} and \tilde{Q}^* have been found, the next step is to apply them.

CHAPTER 8

APPLICATION OF THE Q GROUP

In this chapter, the finite transformation $(Q)_{4S}$'s action will be found on the metric components f_{11} and f_{12} . It will then be applied to the Reissner-Nordstrom metric and potentials found by T.C. Jones (1980).

Action on the Metric Components f_{11} and f_{12}

Equations (6.24, 29-32) imply the following useful relations.

$$(1-2tz)\epsilon_{AB} + 2itf_{AB} = S^2(t)[F_{AX}^*(t)F_B^X(t) + \frac{2i}{t}\phi_A^*(t)\phi_B(t)]. \quad (8.1)$$

For the metric components f_{11} and f_{12} , this implies

$$2itf_{11} = S^2(t)(F_{11}^*(t)F_{12}(t) - F_{12}^*(t)F_{11}(t) + \frac{2i}{t}\phi_1^*(t)\phi_1(t)), \quad (8.2)$$

$$2itf_{12} = S^2(t)(F_{11}^*F_{22}(t) - F_{12}^*(t)F_{21}(t) + \frac{2i}{t}\phi_1^*(t)\phi_2(t)) - (1-2tz) \quad (8.3)$$

The finite action of $(Q)_{4S}$ on f_{11} will be derived here. Due to the length of the calculation, the finite action of $(Q)_{4S}$ on f_{12} will only be stated.

The following useful relations will be used.

$$(Q)_{4S}\rho = \rho S^{-2}(s), \quad (8.4)$$

$$(Q)_{4S}z = -\frac{1}{2s} - (z - \frac{1}{2s})S^{-2}(s), \quad (8.5)$$

$$(Q)_{4S}S(t) = S(s+t)/S(s). \quad (8.6)$$

Here and elsewhere in this chapter,

$$S^2(t) = (1-2tz)^2 + (2t\rho)^2. \quad (8.7)$$

Using Equation (8.2) in the form

$$2itf_{11} = \overline{S^2(t)}(\overline{F_{11}^*(t)}\overline{F_{12}(t)} - \overline{F_{12}^*(t)}\overline{F_{11}(t)}) + \frac{2i}{t} \overline{\phi_1^*(t)}\overline{\phi_1(t)},$$

where the overhead bars indicate transformed quantities, and using

Equations (7.19,20,22), it is possible to show that

$$2itf_{11} = \frac{tS^2(s+t)}{(s+t)S^2(s)F_{12}^*(s)F_{12}(s)} \times \\ [F_{11}^*(s+t)F_{12}(s+t) - F_{12}^*(s+t)F_{11}(s+t) + \\ \frac{2i}{(s+t)} \phi_1^*(s+t)\phi_1(s+t)]. \quad (8.8)$$

Comparing Equation (8.7) with Equation (8.2); it follows that

$$(Q)_{4s} f_{11} = \frac{f_{11}}{S^2(s)F_{12}^*(s)F_{12}(s)}. \quad (8.9)$$

This is exactly the form found for the vacuum case by Cosgrove (1980:

Equation (3.1c)). It can also be shown that the following equation is true.

$$(Q)_{4s} f_{12} = \frac{i}{2s} + \frac{i(z-1/2s)}{S^2(s)} - \frac{N_{22}(s,s)f_{11}}{S^2(s)F_{12}^*(s)F_{12}(s)} + \\ \frac{2}{F_{12}^*(s)} [(f_{12}+iz)[F_{12}(s)\det(F^*(s)) + \\ \frac{2i}{s} (\phi_1^*(s)F_{22}^*(s) - \phi_2^*(s)F_{12}^*(s))\phi_1(s)] - \\ f_{11}[F_{22}(s)\det(F^*(s)) + \frac{2i}{s} (\phi_1^*(s)F_{22}^*(s) - \\ \phi_2^*(s)F_{12}^*(s))\phi_2(s)]]. \quad (8.10)$$

Application of Q to Reissner-Nordstrom

T.C. Jones (1980: Equations (6.44-47) found the following results for Reissner-Nordstrom.

$$\phi_1(t) = \frac{tq\beta}{(x+\beta)} [1-4t^2]^{-1/2}, \quad (8.11)$$

$$F_{11}(t) = \frac{t}{(x+\beta)} \left[\frac{(x-2ty)}{S(t)} - \beta \right] [1-4t^2]^{-1/2}, \quad (8.12)$$

$$F_{12}(t) = \frac{i}{(x+\beta)} \left[\frac{(x-2ty)}{S(t)} + \beta \right] [1-4t^2]^{-1/2}, \quad (8.13)$$

$$f_{11} = (x^2-1)/(x+\beta)^2. \quad (8.14)$$

In these equations, β is a constant associated with the charge-to-mass ratio, x and y are prolate spheroidal coordinates defined by

$$x = \frac{1}{2k} \left[\sqrt{(z+k)^2 + \rho^2} + \sqrt{(z-k)^2 + \rho^2} \right], \quad (8.15)$$

$$y = \frac{1}{2k} \left[\sqrt{(z+k)^2 + \rho^2} - \sqrt{(z-k)^2 + \rho^2} \right], \quad (8.16)$$

which may be written

$$x = 1/2[S(-1/2k)+S(1/2k)], \quad (8.17)$$

and

$$y = 1/2[S(-1/2k)-S(1/2k)]. \quad (8.18)$$

From Equation (8.8),

$$\bar{f}_{11}(x,y) = \frac{f_{11}(x,y)}{S^2(x,y;s)F_{12}^*(x,y;s)F_{12}(x,y;s)}. \quad (8.19)$$

This is equivalent to

$$\bar{f}_{11}(x,y) = \frac{f_{11}(x',y')}{S^2(s',y';s)F_{12}^*(x',y';s)F_{12}(x',y';s)}, \quad (8.20)$$

with $(x',y') = ((Q)_{-4s}x, (Q)_{-4s}y)$. Using

$$S(x',y';s) = (Q)_{-4s}S(x,y;s) = 1/S(-s), \quad (8.21)$$

and using Equations (8.10-12,17,18,and 20), it follows that

$$\bar{f}_{11}(x,y) = \frac{(1-4s)^2[(S(-s-1/2k)+S(-s+1/2k))^2-4S^2(-s)]}{[S(-s-1/2k)(1-2s)+S(-s+1/2k)(1+2s)+2\beta]^2}. \quad (8.22)$$

Use of Equation (8.10) is not needed in calculating $(Q)_{4s}$'s action on f_{12} , since for Reissner-Nordstrom f_{12} is zero. Another way of seeing this is to begin with Equation (7.9) in Kinnersley (1977a) reproduced here as

$$\nabla\omega = \rho f^{-2}(\tilde{\nabla}\Omega_{11} + i(\phi_1^* \tilde{\nabla}\phi_1 - \phi_1 \tilde{\nabla}\phi_1^*)). \quad (8.23)$$

The fact that $F_{11}(t)$ and $\phi_1(t)$ are real and $F_{12}(t)$ is pure imaginary for Reissner-Nordstrom, combined with Equations (7.19,20,23), implies that the transformed $\bar{F}_{11}(t)$, $\bar{\phi}_1(t)$ are real. This in turn implies that the right-hand side of Equation (8.23) is zero. This means that the transformed ω is a constant, which we are free to choose as zero. Thus, it is seen that $(Q)_{4s}$ leaves the Reissner-Nordstrom spacetime static.

In an argument similar to that for finding $(Q)_{4s}f_{11}$, beginning with Equation (7.23), it is possible to show that

$$\bar{\phi}_1(x,y) = \frac{\sqrt{\beta^2-1}}{[S(-s-1/2k)(1-2s)+S(-s+1/2k)(1+2s)+2\beta]}. \quad (8.24)$$

Since this is real, it is equal to A_1 , and B_1 is equal to zero. Then Equations (6.6) imply, since $\omega = 0$,

$$\tilde{\nabla}\phi_2 = -i\rho f^{-1}\nabla\phi_1, \quad (8.25)$$

which may be integrated to yield ϕ_2 .

CHAPTER 9

SUMMARY

The main results of this thesis were found in Chapters 7 and 8. In Chapter 7, the vacuum groups found by Cosgrove were extended to the electrovac case. In Chapter 1, the \tilde{Q} transformation on an arbitrary metric was found. As an example, this was applied to the Reissner-Nordstrom metric; however, no new solution was found in this way. Up to this point, the extension of the vacuum case to electrovac has been successful.

A problem that has arisen resides in the application of the \tilde{Q} and Q^* found in Chapter 7. The careful avoidance of these transformations in Chapter 8 was no accident. They have been troublesome since first discovered.

First, the infinitesimal action of $Q(Q^*)$ on the metric function has not been found. Second, and most problematic, is the meaning of Equations (7.49-54). As an example to what is meant by the last sentence, consider the following. Equations (7.52,53) imply that $(\tilde{Q})_{4s}(H_{11}^*)$ is not equal to $((\tilde{Q})_{4s}H_{11})^*$. Now by definition,

$$2f_{11} = H_{11} + H_{11}^* + 2\phi_1^* \phi_1. \quad (9.1)$$

To calculate the transformed f_{11} , $(\tilde{Q})_{4s}(H_{11}^*)$ and $(\tilde{Q})_{4s}(\phi_1^*)$ must be used. What then is $((\tilde{Q})_{4s}H_{11})^*$? This problem was already mentioned in Chapter 7 and was forced upon the problem when the $i\Sigma/8$ term was added

to the transformation. It may be better to leave out this term if complex metrics are to be avoided.

The following argument suggests that the vacuum case may be further extended to electrovac. There are a group of theorems known as Baker-Campbell-Hausdorff theorems. The one of use is

$$\exp(sQ)\exp(-s\tilde{Q}) = \exp(sQ - s\tilde{Q} + \frac{s^2}{2}[Q, \tilde{Q}] + \frac{s^3}{12}[Q, [Q, \tilde{Q}]] + \frac{s^3}{12}[[Q, \tilde{Q}], \tilde{Q}] + \dots). \quad (9.2)$$

From Chapter 7,

$$[Q, \tilde{Q}] = 1/2[Q, T_{12}] + i[Q, \Sigma]/8 = T_{12}/16 + i\Sigma/32, \quad (9.3)$$

Similarly,

$$[Q, [Q, \tilde{Q}]] = T_{12}^3/32 + i\Sigma/64, \quad (9.4)$$

$$[[Q, \tilde{Q}]Q] = T_{12}^3/32 + i\Sigma/64. \quad (9.5)$$

What this seems to imply is that

$$\exp(sQ)\exp(-s\tilde{Q}) = \exp\left(\sum_{n=1}^{\infty} (a_n T_{12}^n + b_n \Sigma^n)\right). \quad (9.6)$$

Because all the T_{12}^k and Σ^k commute, and as long as the transformation (9.6) exists, the the transformations of the form $\exp(sQ)\exp(-sQ)$ will commute with each other. This may be the extension of Harrison's single Bäcklund transformation; see Cosgrove (1981).

The importance in studying the $\tilde{Q}(\tilde{Q}^*)$ transformation or their equivalent modes of expression in other methods will probably bring the electrovac case to the state of success of the vacuum case.

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APPENDICES

APPENDIX A

CONSISTENCY OF EQUATIONS (7.1-6)

The proof for Equations (7.1-6) is given here. The field equations are

$$\nabla \phi_A^n = -i\rho^{-1} f_A^X \tilde{\nabla} \phi_X^n \quad (6.6)$$

and

$$\nabla H_{AB}^n = -i\rho^{-1} f_A^X \tilde{\nabla} H_{XB}^n. \quad (6.7)$$

In order for Q to leave the field equations invariant, the transformed quantities must satisfy Equations (6.6,7). (It must not be forgotten that ∇ and $\tilde{\nabla}$ are also transformed.) In terms of the infinitesimal Q operator, this requires that

$$\nabla(Q\phi_A^n) = -i\rho^{-1} f_A^X \tilde{\nabla}(Q\phi_X^n) + i\rho^{-1} z f_A^X \tilde{\nabla} \phi_X^n - i\rho^{-1} (Qf_A^X) \tilde{\nabla} \phi_X^n \quad (A.1)$$

and

$$\nabla(QH_{AB}^n) = -i\rho^{-1} f_A^X \tilde{\nabla}(QH_{XB}^n) + i\rho^{-1} z f_A^X \tilde{\nabla} H_{XB}^n - i\rho^{-1} (Qf_A^X) \tilde{\nabla} H_{XB}^n. \quad (A.2)$$

It is seen that the quantities Qf_{AB} are needed. Using

$$f_{11} = 1/2(H_{11} + H_{11}^*) + \phi_1^* \phi_1, \quad (A.3)$$

$$f_{12} = 1/4(H_{12} + H_{12}^* + H_{21} + H_{21}^* + 2\phi_1^* \phi_2 + 2\phi_2^* \phi_1), \quad (A.4)$$

$$f_{22} = 1/2(H_{22} + H_{22}^*) + \phi_2^* \phi_2, \quad (A.5)$$

it can be shown that

$$Qf_{11} = \frac{-f_{11}\Omega_{12}}{2} + f_{11}z, \quad (A.6)$$

$$Qf_{12} = \frac{-f_{11}\Omega_{22}}{2} + f_{11}z, \quad (A.7)$$

and

$$Qf_{22} = \frac{f_{22}\Omega_{12}}{2} + f_{22}z - f_{12}\Omega_{22}. \quad (A.8)$$

Writing Equations (A.1,2) out in full yields the following equations.

$$\begin{aligned} & \nabla(Q\phi_1^n) + i\rho^{-1} f_{12} \tilde{\nabla}(Q\phi_1^n) \\ &= i\rho^{-1} [f_{11} \tilde{\nabla}(Q\phi_2^n) + z f_{12} \tilde{\nabla} \phi_1^n - z f_{11} \tilde{\nabla} \phi_2^n - (Qf_{12}) \tilde{\nabla} \phi_1^n + (Qf_{11}) \tilde{\nabla} \phi_2^n], \quad (A.9) \end{aligned}$$

$$\begin{aligned} & \nabla(Q\phi_2) - i\rho^{-1}f_{12}\tilde{\nabla}(Q\phi_2) \\ &= -i\rho^{-1}[f_{22}\tilde{\nabla}(Q\phi_1) - zf_{22}\tilde{\nabla}\phi_1 + zf_{21}\tilde{\nabla}\phi_2 + (Qf_{22})\tilde{\nabla}\phi_1 - (Qf_{21})\tilde{\nabla}\phi_2], \end{aligned} \quad (\text{A.10})$$

$$\begin{aligned} & \nabla(QH_{11}) + i\rho^{-1}f_{12}\tilde{\nabla}(QH_{11}) \\ &= i\rho^{-1}[f_{11}\tilde{\nabla}(QH_{21}) + zf_{12}\tilde{\nabla}H_{11} - zf_{11}\tilde{\nabla}H_{21} - (Qf_{12})\tilde{\nabla}H_{11} + (Qf_{11})\tilde{\nabla}H_{21}], \end{aligned} \quad (\text{A.11})$$

$$\begin{aligned} & \nabla(QH_{21}) - i\rho^{-1}f_{12}\tilde{\nabla}(QH_{21}) \\ &= -i\rho^{-1}[f_{22}\tilde{\nabla}(QH_{11}) - zf_{22}\tilde{\nabla}H_{11} + zf_{21}\tilde{\nabla}H_{21} + (Qf_{22})\tilde{\nabla}H_{11} - (Qf_{21})\tilde{\nabla}H_{21}], \end{aligned} \quad (\text{A.12})$$

$$\begin{aligned} & \nabla(QH_{12}) + i\rho^{-1}f_{12}\tilde{\nabla}(QH_{12}) \\ &= i\rho^{-1}[f_{11}\tilde{\nabla}(QH_{22}) + zf_{12}\tilde{\nabla}H_{12} - zf_{11}\tilde{\nabla}H_{22} - (Qf_{12})\tilde{\nabla}H_{12} + (Qf_{11})\tilde{\nabla}H_{22}], \end{aligned} \quad (\text{A.13})$$

$$\begin{aligned} & \nabla(QH_{22}) - i\rho^{-1}f_{12}\tilde{\nabla}(QH_{22}) \\ &= -i\rho^{-1}[f_{22}\tilde{\nabla}(QH_{12}) - zf_{22}\tilde{\nabla}H_{12} + zf_{21}\tilde{\nabla}H_{22} + (Qf_{22})\tilde{\nabla}H_{12} - (Qf_{21})\tilde{\nabla}H_{22}]. \end{aligned} \quad (\text{A.14})$$

All of the above equations fall into pairs of equations similar to

$$\begin{aligned} & \nabla(QA) + i\rho^{-1}f_{12}\tilde{\nabla}(QA) \\ &= i\rho^{-1}[f_{11}\tilde{\nabla}(QB) + zf_{12}\tilde{\nabla}A - zf_{11}\tilde{\nabla}B - (Qf_{12})\tilde{\nabla}A + (Qf_{11})\tilde{\nabla}B], \end{aligned}$$

and

$$\begin{aligned} & \nabla(QB) - i\rho^{-1}f_{12}\tilde{\nabla}(QB) \\ &= -i\rho^{-1}[f_{22}\tilde{\nabla}(QA) - zf_{22}\tilde{\nabla}A + zf_{21}\tilde{\nabla}B + (Qf_{22})\tilde{\nabla}A - (Qf_{21})\tilde{\nabla}B]. \end{aligned} \quad (\text{A.16})$$

Furthermore, all the A and B satisfy the field equations

$$\nabla A = -i\rho^{-1}(f_{12}\tilde{\nabla}A - f_{11}\tilde{\nabla}B) \quad (\text{A.17})$$

and

$$\nabla B = -i\rho^{-1}(f_{22}\tilde{\nabla}A - f_{21}\tilde{\nabla}B). \quad (\text{A.18})$$

Equations (7.1-6) imply that the A and B satisfy

$$QA = 1/4[nA + iH_{12}A], \quad (\text{A.19})$$

$$Q_B = 1/4[n_B + iM_2 + iH_{22}A] \text{ or } 1/4[n_B + iN_{21} + iH_{22}A] \quad (\text{A.20})$$

for the A consisting of ϕ_1 or H_{11} , and the B consisting of ϕ_2 or H_{21} ,

and

$$Q_A = 1/4[(n+1)A + iH_{12}A], \quad (\text{A.21})$$

$$Q_B = 1/4[(n+1)B + iN_{22} + iH_{22}A] \quad (\text{A.22})$$

when the A are H_{12} and the B are H_{22} .

The proof begins by examining the left-hand side of Equation (A.15),

$$\begin{aligned} \nabla(Q_A) + i\rho^{-1}f_{12}\tilde{\nabla}(Q_A) &= 1/4[(n+1)\nabla A + iH_{12}\nabla A + iA\nabla H_{12} \\ &\quad + i\rho^{-1}f_{12}[(n+1)\tilde{\nabla} A + iH_{12}\tilde{\nabla} A + iA\nabla H_{12}]], \end{aligned}$$

with $(n+1)$ equaling n if Equation (A.19) is used or $(n+1)$ if Equation (A.21) is used. Using Equation (A.17) and Equation (6.7), the left-hand side is now

$$\begin{aligned} &1/4[(n+1)(-i\rho^{-1}f_{12}\tilde{\nabla} A + i\rho^{-1}f_{11}\tilde{\nabla} B) \\ &\quad + iH_{12}(-i\rho^{-1}f_{12}\tilde{\nabla} A + i\rho^{-1}f_{11}\tilde{\nabla} B) \\ &\quad + iA(-i\rho^{-1}f_{12}\tilde{\nabla} H_{12} + i\rho^{-1}f_{11}\tilde{\nabla} H_{22}) \\ &\quad + i\rho^{-1}f_{12}[(n+1)\tilde{\nabla} A + iH_{12}\tilde{\nabla} A + iA\nabla H_{12}]] \\ &= \frac{i\rho^{-1}f_{11}}{4} [(n+1)\tilde{\nabla} B + iH_{12}\tilde{\nabla} B + iA\nabla H_{22}]. \end{aligned}$$

Now the right-hand side is examined.

$$\begin{aligned} &i\rho^{-1}[f_{11}\tilde{\nabla}(Q_B) + zf_{12}\tilde{\nabla}A - zf_{11}\tilde{\nabla}B - (Qf_{12})\tilde{\nabla}A + (Qf_{11})\tilde{\nabla}B] \\ &= i\rho^{-1}[f_{11}\tilde{\nabla} 1/4[(n+1)B + i(M_2, N_{21}, N_{22}) + iH_{22}A] \\ &\quad + zf_{12}\tilde{\nabla}A - zf_{11}\tilde{\nabla}B + (\frac{f_{11}\Omega_{22}}{2} - f_{12}z)\nabla A + (-\frac{f_{11}\Omega_{22}}{2} + f_{11}z)\nabla B] \end{aligned}$$

$$\begin{aligned}
&= \frac{i\rho^{-1}f_{11}}{4} [(n+1)\nabla^{\sim n+1} B + i(\nabla^{\sim n} M_2, \nabla^{\sim n} N_{21}, \nabla^{\sim n} N_{22}) + iH_{22}\nabla^{\sim n} A + iA\nabla^{\sim n} H_{22} \\
&\quad + 2\Omega_{22}\nabla^{\sim n} A - 2\Omega_{12}\nabla^{\sim n} B] \\
&= \frac{i\rho^{-1}f_{11}}{4} [(n+1)\nabla^{\sim n+1} B + i(H_{12}\nabla^{\sim n} \phi_2 - H_{22}\nabla^{\sim n} \phi_1, H_{12}\nabla^{\sim n} H_{21} - H_{22}\nabla^{\sim n} H_{11}, \\
&\quad H_{12}\nabla^{\sim n} H_{22} - H_{22}\nabla^{\sim n} H_{12}) + (iH_{22} + 2\Omega_{22})\nabla^{\sim n} A - 2\Omega_{12}\nabla^{\sim n} B + iA\nabla^{\sim n} H_{22}].
\end{aligned}$$

Since the quantities in parentheses all represent $H_{12}\nabla^{\sim n} B - H_{22}\nabla^{\sim n} A$,

$$\begin{aligned}
&= \frac{i\rho^{-1}f_{11}}{4} [(n+1)\nabla^{\sim n+1} B + i(H_{12}\nabla^{\sim n} B - H_{22}\nabla^{\sim n} A) + (iH_{22} + 2\Omega_{22})\nabla^{\sim n} A \\
&\quad - 2\Omega_{12}\nabla^{\sim n} B + iA\nabla^{\sim n} H_{22}] \\
&= \frac{i\rho^{-1}f_{11}}{4} [(n+1)\nabla^{\sim n+1} B + (iH_{12} - 2\Omega_{22})\nabla^{\sim n} B + (-iH_{22} + iH_{22} + 2\Omega_{22})\nabla^{\sim n} A \\
&\quad + iA\nabla^{\sim n} H_{22}] \\
&= \frac{i\rho^{-1}f_{11}}{4} [(n+1)\nabla^{\sim n+1} B + iH_{12}\nabla^{\sim n} B + iA\nabla^{\sim n} H_{22}],
\end{aligned}$$

since $iH_{12} - 2\Omega_{12} = iH_{12}$ and $0 = -iH_{22} + iH_{22} + 2\Omega_{22}$.

The left-hand side and right-hand side of Equation (A.15) match for all n . This implies Equations (A.9,11,13) are true for Equations (7.1-6). Next, Equation (A.16) must be checked. Since the procedure is similar, no explanation will be given. First examine the left-hand side.

$$\begin{aligned}
\nabla(QB) - i\rho^{-1} f_{12} \tilde{\nabla}(QB) &= 1/4 \{ \nabla[(n+1) B + i(M_2, N_{21}, N_{22}) + iH_{22} A] \\
&\quad - i\rho^{-1} f_{12} \tilde{\nabla}[(n+1) B + i(M_2, N_{21}, N_{22}) + iH_{22} A] \} \\
&= 1/4 [(n+1) \nabla B + i(H_{12} \nabla B - H_{22} \nabla A) + iH_{22} \nabla A + iA \nabla H_{22} \\
&\quad - i\rho^{-1} f_{12} [(n+1) \tilde{\nabla} B + i(H_{12} \tilde{\nabla} B - H_{22} \tilde{\nabla} A) + iH_{22} \tilde{\nabla} A + iA \tilde{\nabla} H_{22}]] \\
&= 1/4 \{ (n+1) (-i\rho^{-1} f_{22} \tilde{\nabla} A + i\rho^{-1} f_{21} \nabla B) \\
&\quad + iH_{12} * (-i\rho^{-1} f_{22} \tilde{\nabla} A + i\rho^{-1} f_{21} \tilde{\nabla} B) \\
&\quad - iH_{22} * (-i\rho^{-1} f_{12} \tilde{\nabla} A + i f_{11} \rho^{-1} \nabla B) \\
&\quad + iH_{22} (-i\rho^{-1} f_{12} \tilde{\nabla} A + i\rho^{-1} f_{11} \nabla B) + iH_{22} \nabla A + iA \nabla H_{22} \\
&\quad - i\rho^{-1} f_{12} [(n+1) \tilde{\nabla} B + i(H_{12} \tilde{\nabla} B - H_{22} \tilde{\nabla} A) + iH_{22} \tilde{\nabla} A + iA \tilde{\nabla} H_{22}] \} \\
&= 1/4 \{ -i\rho^{-1} f_{22} (n+1) \tilde{\nabla} A - i\rho^{-1} f_{22} (iH_{12} \tilde{\nabla} A) - i\rho^{-1} f_{12} (-2iH_{22} \tilde{\nabla} A) \\
&\quad - i\rho^{-1} f_{11} (iH_{22} \tilde{\nabla} B - i\rho^{-1} f_{12} (2iH_{22} \tilde{\nabla} A)) + (i\rho^{-1} f_{11} (iH_{22} \tilde{\nabla} B) \\
&\quad - i\rho^{-1} f_{22} (iA \nabla H_{12})) \} \\
&= \frac{-i\rho^{-1}}{4} \{ f_{22} (n+1) \tilde{\nabla} A + (if_{22} H_{12} * -2if_{12} H_{22} * + 2if_{12} H_{22}) \tilde{\nabla} A
\end{aligned}$$

$$+(if_{11}H_{22}^*-if_{11}H_{22})\tilde{\nabla}B+if_{22}A\tilde{\nabla}H_{12}$$

Next, the right-hand side.

$$\begin{aligned} & -i\rho^{-1}[f_{22}\tilde{\nabla}(QA)-zf_{22}\tilde{\nabla}A+zf_{21}\tilde{\nabla}B+(\frac{f_{22}\Omega_{12}}{2}+f_{22}z-f_{12}\Omega_{22})\tilde{\nabla}A \\ & +(\frac{f_{11}\Omega_{22}}{2}-f_{12}z)\tilde{\nabla}B] \\ & = -i\rho^{-1}\{\frac{f_{22}}{4}\tilde{\nabla}[(n+1)A+iH_{12}A] \\ & +(\frac{f_{22}\Omega_{12}}{2}-f_{12}\Omega_{22})\tilde{\nabla}A+\frac{f_{11}\Omega_{22}}{2}\tilde{\nabla}B\} \\ & = \frac{-i\rho^{-1}}{4}\{(n+1)f_{22}\tilde{\nabla}A+if_{22}H_{12}\tilde{\nabla}A+if_{22}A\tilde{\nabla}H_{12} \\ & + (2f_{22}\Omega_{12}-4f_{12}\Omega_{22})\tilde{\nabla}A+4f_{11}\Omega_{22}\tilde{\nabla}B\} \\ & = \frac{-i\rho^{-1}}{4}\{(n+1)f_{22}\tilde{\nabla}A+(if_{22}H_{12}+2f_{22}\Omega_{12}-4f_{12}\Omega_{22})\tilde{\nabla}A \\ & + 2f_{11}\Omega_{22}\tilde{\nabla}B+if_{22}A\tilde{\nabla}H_{12}\} \end{aligned}$$

One sees, after a little algebra, that the right-hand side again equals the left-hand side of Equation (A.16). This implies that Equations (A.10,12,14) are true for Equations (7.1-6). Therefore, Equations (7.1-6) are true.

APPENDIX B

EXAMPLE CALCULATION FOR EQUATION (7.12)

The object of this appendix is to show how Equation (7.12) was obtained.

Begin with the defining relation for N_{11}^{mn} ,

$$\nabla N_{11}^{mn} = H_{11}^m \nabla H_{21}^n - H_{21}^m \nabla H_{11}^n. \quad (B.1)$$

From this it can be shown that the infinitesimal operator Q acting on N_{11}^{mn} , QN_{11}^{mn} , must satisfy

$$\nabla(QN_{11}^{mn}) = (QH_{11}^m)^n \nabla H_{21}^m - (QH_{21}^m)^n \nabla H_{11}^m + H_{11}^m \nabla(QH_{21}^n) - H_{21}^m \nabla(QH_{11}^n). \quad (B.2)$$

The $QH_{11}^m = 1/4[mH_{11}^{m+1} - iH_{12}^m H_{11}^m]$, can be expressed as

$$\begin{aligned} & - i/4[mN_{11}^{m1} + 2m\phi_1^m M_1 - H_{11}^m (mH_{21} + 2m\phi_1^m \phi_2 + (m-1)H_{12}^m) \\ & + H_{21}^m (H_{11} + 2\phi_1^m \phi_1 + H_{11}^m)], \end{aligned} \quad (B.3)$$

after using Equations (6.4,5,10-11). A similar expression may be

obtained for QH_{21}^m , and is

$$\begin{aligned} & - i/4[(m+1)N_{12}^{m1} + 2m\phi_2^m M_1 - H_{11}^m ((m+1)H_{22} + 2m\phi_2^m \phi_2 + (m-1)H_{22}^m) \\ & + H_{21}^m ((m+1)H_{12} + 2m\phi_2^m \phi_1 + mH_{21}^m)]. \end{aligned} \quad (B.4)$$

Using these, it can be shown, after some algebra, that

$$\begin{aligned} \nabla(QN_{11}^{mn}) = & i/4[2(m+n)(H_{11}^m (f_{12} - iz) - H_{21}^m f_{11}) \nabla H_{21}^n + \\ & 2(m+n)(-H_{11}^m f_{22} + H_{21}^m (f_{12} + iz)) \nabla H_{11}^n + \\ & (-mN_{11}^{m1}) \nabla H_{21}^n + (m+1)N_{12}^{m1} \nabla H_{11}^n - 2mM_1 \nabla L_1 + 2nL_1 \nabla M_1 + \\ & nH_{21}^m \nabla N_{11}^n - (n-1)H_{11}^m N_{12}^n]. \end{aligned} \quad (B.5)$$

Using

$$\begin{aligned}
 \nabla N_{11}^{m,n+1} &= H_{11}^m \nabla H_{21}^{n+1} - H_{21}^{m,n+1} \nabla H_{11}^n \\
 &= i[2(H_{11}^m (f_{12} - iz) - H_{21}^{m,n+1} f_{11}^n) \nabla H_{21}^{n+1} \\
 &\quad + 2(-H_{11}^m f_{22}^{n+1} + H_{21}^{m,n+1} (f_{12} + iz)) \nabla H_{11}^n \\
 &\quad + 2L_1 \nabla M_1^{m,n+1} + H_{21}^{m,n+1} \nabla N_{11}^n - H_{11}^m \nabla N_{12}^n],
 \end{aligned}$$

it is seen that

$$\begin{aligned}
 \nabla(QN_{11})^{m,n} &= 1/4[(m+n) \nabla N_{11}^{m,n+1} - 2im \nabla(M_1^{m,n+1} L_1^n) - im \nabla(N_{11}^m H_{21}^n) \\
 &\quad + i(m+1) \nabla(N_{12}^m H_{11}^n)]
 \end{aligned} \tag{B.6}$$

which proves Equation (7.12).

APPENDIX C

CONSISTENCY OF EQUATION (7.16)

Equation (7.16) is derived in this appendix. Begin with Equations (2.40,41) and with the variable change

$$x_1 = \rho + iz, \quad x_2 = \rho - iz. \quad (\text{C.1})$$

Equations (2.40,41) then become

$$\begin{aligned} \gamma_{,i} = \frac{\rho f^{-2}}{2} [& G_{,i} G^*_{,i} + 2\phi_1 G_{,i} \phi_1^* + 2\phi_1^* G^*_{,i} \phi_1 + 4\phi_1^* \phi_1 \phi_1^*] \\ & - 2\rho f^{-1} \phi_1^*_{,i} \phi_1_{,i}, \end{aligned} \quad (\text{C.2})$$

with $i = 1, 2$. Under the Q transformation, the left-hand side of Equations (C.2) becomes

$$\gamma'_{,j} = \gamma_{,j} + ((Q\gamma)_{,j} \pm i x_j \gamma_{,j}) \quad (\text{C.3})$$

where the plus sign applies if $j=1$ and the minus sign applies if $j=2$.

For the transformation of the right-hand side of Equations (C.2), the following relations will be useful.

$$Q\rho = \rho z, \quad Qz = 1/2(z^2 - \rho^2), \quad Qx_1 = -ix_1^2/2, \quad Qx_2 = ix_2^2/2, \quad (\text{C.4})$$

$$\partial/\partial x_1' = (1 + i\epsilon x_1)\partial/\partial x_1, \quad \partial/\partial x_2' = (1 - i\epsilon x_2)\partial/\partial x_2, \quad (\text{C.5})$$

$$QG = 1/4(H_{11}^2 + iH_{12}G) = i(N_{11} + 2\phi_1 L_1 + G H_{21})/4, \quad (\text{C.6})$$

$$QG^* = 1/4(H_{11}^{*2} - iH_{12}^* G^*) = -i(N_{11}^* + 2\phi_1^* L_1^* + G^* H_{21}^*)/4, \quad (\text{C.7})$$

$$Q\phi_1 = 1/4(\phi_1^2 + iH_{12}\phi_1) = i(M_1 + 2\phi_1 K + G\phi_2)/4, \quad (\text{C.8})$$

$$Q\phi_1^* = 1/4(\phi_1^{*2} - iH_{12}^*\phi_1^*) = -i(M_1^* + 2\phi_1^* K^* + G^*\phi_2^*)/4. \quad (\text{C.9})$$

Then it is seen

$$\frac{\rho f^{-2}}{2} \rightarrow \frac{\rho f^{-2}}{2} + \frac{\epsilon \rho f^{-2}}{2} (\Omega_{12} - z) \quad (\text{C.10})$$

and

$$-2\rho f^{-1} \rightarrow -2\rho f^{-1} - \varepsilon \rho f^{-1} \Omega_{12} \quad (\text{C.11})$$

The bracketed term on the right-hand side of Equation (C.2) goes into

$$\begin{aligned} & [G_{,i} G^*_{,i} + 2\phi_{1,i} G_{,i} \phi_{1,i}^* + 2\phi_{1,i}^* G^*_{,i} \phi_{1,i} + 4\phi_{1,i}^* \phi_{1,i}^* \phi_{1,i}^*] + \\ & \varepsilon \{ [G_{,i} G^*_{,i} + 2\phi_{1,i} G_{,i} \phi_{1,i}^* + 2\phi_{1,i}^* G^*_{,i} \phi_{1,i} + 4\phi_{1,i}^* \phi_{1,i}^* \phi_{1,i}^*] \times \\ & \quad [-\Omega_{21} + i(K-K^*) \pm (2ix_i - i\rho)] + \\ & \quad i\phi_{2,i} f G_{,i} \phi_{1,i}^* - i\phi_{2,i}^* f G^*_{,i} \phi_{1,i} + 2(K-K^*) f \phi_{1,i}^* \phi_{1,i} \} \quad (\text{C.12}) \end{aligned}$$

where the \pm means the same as in Equation (C.3). Finally,

$$\begin{aligned} \phi_{1,i}^* \phi_{1,i} \rightarrow \phi_{1,i}^* \phi_{1,i} + \varepsilon \{ 1/2 \phi_{1,i}^* \phi_{1,i} [-\Omega_{21} + 2i(K-K^*) \pm (4ix_i - 2i\rho)] \\ + i(\phi_{2,i} G_{,i} \phi_{1,i}^* - \phi_{2,i}^* G^*_{,i} \phi_{1,i})/4 \} \quad (\text{C.13}) \end{aligned}$$

Combining terms on the right-hand side of Equations (C.2) then yields

$$\begin{aligned} & \pm ix_i \{ \text{the right-hand side of Equations (C.2)} \} \\ & = \pm ix_i \gamma_{,i} \quad (\text{C.14}) \end{aligned}$$

From Equations (C.3) and (C.14), it can be concluded that

$$(Q\gamma)_{,j} = 0 \quad Q\gamma = \text{a constant.} \quad (\text{C.15})$$

since $\gamma \rightarrow 0$ as $\rho^2 + z^2$ approaches infinity, the conclusion is

$$Q\gamma = 0. \quad (\text{C.16})$$

APPENDIX D

INTEGRATION OF EQUATIONS (7.17,18)

Equations (7.17,18) are integrated using Equation (3.49). They then become the partial differential equations

$$D_2 \phi_1(t,s) = [D_1 \phi_1(t,s) - \frac{1}{t} \phi_1(t,s) + iH_{12}(s)\phi_1(t,s)], \quad (D.1)$$

$$D_2 F_{12}(t,s) = [D_1 F_{12}(t,s) + i(D_1 F_{12}(0,s))F_{12}(t,s)], \quad (D.2)$$

with $D_1 = \partial/\partial t$, $D_2 = \partial/\partial s$, $Q_{4s} F_{12}(t) = F_{12}(t,s)$, and

$$(Q)_{4s} \phi_1(t) = \phi_1(t,s).$$

Equation (D.2) is solved by a separation of variables

$$F_{12}(t,s) = g(s)h(s+t). \quad (D.3)$$

Substituting this into Equation (D.2) yields

$$\frac{d}{ds} \left(\frac{1}{g} \right) = -i \frac{dh}{ds}, \quad (D.4)$$

whose solution is trivially

$$g(s) = \frac{1}{c - ih(s)}. \quad (D.5)$$

Therefore,

$$F_{12}(t,s) = \frac{h(s+t)}{c - ih(s)}. \quad (D.6)$$

The initial condition $F_{12}(t,0) = F_{12}(t)$ requires $c = 0$. Since $h(s)$ is arbitrary, $h(0)$ may be set equal to one implying

$$F_{12}(t,s) = \frac{iF_{12}(s+t)}{F_{12}(s)}, \quad (D.7)$$

which further implies

$$H_{12}(s) = \frac{i}{F_{12}(s)} \frac{dF_{12}}{ds}. \quad (D.8)$$

Equation (D.1) becomes

$$D_2 w(t,s) - D_1 w(t,s) = -\frac{1}{t} w(t,s), \quad (D.9)$$

where $w(t,s) = F_{12}(s)\phi_1(t,s)$. This is solved by separating variables to

$$w(t,s) = g(t)h(s+t). \quad (D.10)$$

Substituting into Equation (C.9) yields

$$\frac{dg}{g} = \frac{dt}{t} \rightarrow g = ct. \quad (D.11)$$

Therefore,

$$\phi_1(t,s) = \frac{ct h(s+t)}{F_{12}(s)}. \quad (D.12)$$

The initial condition $\phi_1(t,0) = \phi_1(t)$ requires

$$h(t) = \frac{i\phi_1(t)}{ct}, \quad (D.13)$$

or

$$\phi_1(t,s) = \frac{it\phi_1(s+t)}{(s+t)F_{12}(s)}. \quad (D.14)$$

APPENDIX E

INTEGRATION OF EQUATIONS (7.46,48)

Equation (7.46) represents the partial differential equation

$$\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2}\right) N_{AB}(t_1, t_2, s) = \frac{-1}{t_2} (N_{AB}(t_1, t_2, s) - \epsilon_{AB}), \quad (\text{E.1})$$

s being the group parameter.

This equation is solved by the method of characteristics, the equations being equivalent to

$$ds = -dt_1 = -dt_2 = \frac{-t_2 dN_{AB}}{N_{AB} - \epsilon_{AB}}. \quad (\text{E.2})$$

The solutions are

$$s + t_1 = c_1, \quad (\text{E.3})$$

$$s + t_2 = c_2, \quad (\text{E.4})$$

$$N_{AB} = g_{AB} t_2 + \epsilon_{AB}, \quad (\text{E.5})$$

where g_{AB} is determined by the initial condition

$$N_{AB}(t_1, t_2, 0) = N_{AB}(t_1, t_2). \quad \text{Set } s = 0, \text{ then}$$

$$g_{AB} = \frac{N_{AB}(c_1, c_2) - \epsilon_{AB}}{c_2}. \quad (\text{E.6})$$

Therefore, for $s \neq 0$,

$$\begin{aligned} N_{AB}(t_1, t_2, s) &= \frac{t_2}{s+t_2} (N_{AB}(s+t_1, s+t_2) - \epsilon_{AB}) + \epsilon_{AB} \\ &= \frac{t_2}{s+t_2} N_{AB}(s+t_1, s+t_2) + \frac{s}{s+t_2} \epsilon_{AB}. \end{aligned} \quad (\text{E.7})$$

Equation (7.48) represents the partial differential equation

$$\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right) F_{AB}^*(t, s) = i H_{AX}^*(s) F_B^X(t, s), \quad (\text{E.8})$$

or

$$\frac{\partial}{\partial s} - \frac{\partial}{\partial t} F_B^A(t, s) = i \frac{\partial}{\partial t} F_X^A(t, s) \Big|_{t=0} F_B^X(t, s), \quad (\text{E.9})$$

or in matrix notation

$$\frac{\partial}{\partial s} - \frac{\partial}{\partial t} F^*(t, s) = i \frac{\partial}{\partial t} F^*(t, s) \Big|_{t=0} F^*(t, s). \quad (\text{E.10})$$

The ansatz is made that

$$F^*(t, s) = g(s)h(s+t) \quad (\text{E.11})$$

where g and h are 2 by 2 matrices. Putting this into Equation (E.10), and being careful to maintain the matrix order, the equation becomes

$$\frac{dg}{ds} = ig \frac{dh}{ds} g, \quad (\text{E.12})$$

or

$$g^{-1} \frac{dg}{ds} g^{-1} = i \frac{dh}{ds}. \quad (\text{E.13})$$

Next, use the matrix identity

$$\frac{dA^{-1}}{ds} = -A^{-1} \frac{dA}{ds} A^{-1}, \quad (\text{E.14})$$

to find that

$$\frac{dg^{-1}}{ds} = -i \frac{dh}{ds}. \quad (\text{E.15})$$

Therefore,

$$g^{-1}(s) = C - ih(s), \quad (\text{E.16})$$

where C is a constant 2 by 2 matrix. This yields

$$F^*(t,s) = (C - ih(s))^{-1} h(s+t). \quad (\text{E.17})$$

Setting $t = 0$, and using the initial condition $F^*(0,s) = i$ requires

$C = 0$. Therefore,

$$F^*(t,s) = ih^{-1}(s)h(s+t). \quad (\text{E.18})$$

Set $s = 0$, use the initial condition $F^*(t,0) = F^*(t)$;

$$F^*(t) = ih^{-1}(0)h(t). \quad (\text{E.19})$$

Since $h(t)$ is thus far arbitrary, it is possible to set $h^{-1}(0) = 1$ so that

$$h(t) = -iF^*(t). \quad (\text{E.20})$$

Therefore,

$$\begin{aligned} F^*(t,s) &= i(-iF^*(s))^{-1} -iF^*(s+t) \\ &= iF^{*-1}(s)F^*(s+t) \end{aligned} \quad (\text{E.21})$$

$$F_B^A(t,s) = i(F^{*-1})_X^A(s)F_B^X(s+t). \quad (\text{E.22})$$

