



Students Use of Multiple Representations in Mathematical Problem Solving
by James William Ballard

A dissertation submitted in partial fulfillment of the requirement for the degree of Doctor of Education
Montana State University
© Copyright by James William Ballard (2000)

Abstract:

This study investigated students use of mathematical representations and translations during probability problem solving. The purpose of the representation research was to identify the representations used by students, when they chose to use a representation, and how frequently and successfully they used the representations. The research looked for patterns of behavior that characterized students' successful and unsuccessful use of the representations. The research also identified the translations between the representations. The purpose of the translation research was to identify patterns of movement between representations that characterize the students' use of translations and identify the time spent using any particular representation. A qualitative research design was employed. The subjects ($n = 21$) were selected from a finite mathematics course. The data gathering included an interview to familiarize each volunteer with the research process, a problem-solving session in which each attempted to solve five probability problems while vocalizing the process, and a post-session interview in which students explained their solution method.

This research demonstrates that there are distinct differences in the manner that successful and error-prone students use translations and representations. Successful problem solvers are able to analyze the problems and discover a solution method before translating, know when and how to use Venn diagrams, and use symbolic algebra competently. Error-prone problem solvers are often unable to discover a solution method and hence will initially use a Venn diagram (or other representations) to "discover" a solution method. Error-prone problem solvers do not separate their use of translations and representations as distinctly (as do the successful solvers) and often do not finish a translation before attempting another translation. This indicates that the error-prone are unable to arrive at a conclusion on how to solve problems and do not understand (a) how a representation will clarify a problem, (b) the purpose of representations, and (c) which representation to use.

Recommendations include the following: students need extensive practice translating between representations, need to understand what the various representations reveal, and need practice using many different representations. Further research is needed to identify when and how students should receive instruction with translations and representations.

STUDENTS' USE OF MULTIPLE REPRESENTATIONS IN
MATHEMATICAL PROBLEM SOLVING

by

James William Ballard

A dissertation submitted in partial fulfillment
of the requirement for the degree

of

Doctor of Education

MONTANA STATE UNIVERSITY-BOZEMAN
Bozeman, Montana

October, 2000

D378
B2125

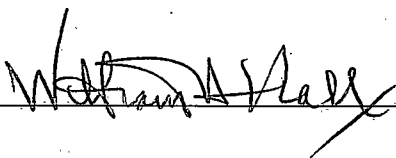
APPROVAL

of a dissertation submitted by

James William Ballard

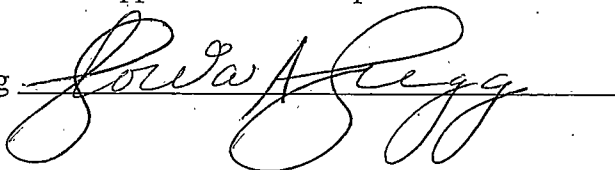
This dissertation has been read by each member of the dissertation committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

Dr. William Hall

October 18, 2000
Date

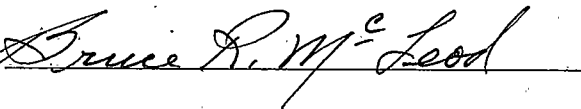
Approved for the Department of Education

Dr. Gloria Gregg

Oct. 23, 2000
Date

Approved for the College of Graduate Studies

Dr. Bruce McLeod

10-24-00
Date

STATEMENT OF PERMISSION TO USE

In presenting this dissertation in partial fulfillment of the requirements for a doctoral degree at Montana State University, I agree that the Library shall make it available to borrowers under rules of the Library. I further agree that copying of this dissertation is allowable only for scholarly purposes, consistent with "fair use" as prescribed in the U.S. Copyright Law. Requests for extensive copying or reproduction of this dissertation should be referred to Bell & Howell Information and Learning, 300 North Zeeb Road, Ann Arbor, Michigan 48106, to whom I have granted "the exclusive right to reproduce and distribute my dissertation in and from microform along with the non-exclusive right to reproduce and distribute my abstract in any format in whole or in part."

Signature James W. Ballard

Date 20 OCTOBER 2000

ACKNOWLEDGMENTS

Special acknowledgment and gratitude is given to the co-chairperson of my doctoral committee, Dr. Ted Hodgson, who was a significant factor in helping me obtain my degree. He routinely went out of his way to assist me in any problems that were encountered with this dissertation. Ted's consistent encouragement and quiet guidance were always a great blessing and a source of strength.

I also wish to express appreciation and thanks to the members of my committee; Dr. William Hall (chairperson), Dr. Jana Noel, Dr. Eric Strohmeier, and Dr. Lyle Andersen. I thank you each for giving of your time and helping me on so many occasions. A special thanks is given to Dr. Hall for making sure the dissertation was done correctly.

I also wish to thank Marye Hefty for her editorial skills and for taking the time to do the many readings of this dissertation.

I would like to thank my extended family for their encouragement and patience through these years. Thank you John, Jackie, Janey, and Stan. A very special thanks is extended to my two moms, Floss and Lois, for their love and support. I express my gratitude to them both.

Finally, a very special thanks is given to my wife, Kathie Lang, for all of her love and emotional support. I give her special thanks for being so understanding and providing so much encouragement. This dissertation is dedicated to her.

TABLE OF CONTENTS

1. INTRODUCTION TO THE STUDY.....	1
Introduction to Representations and Mathematical Connections.....	1
Need for the Study.....	7
Introduction to Data Collection and Methodology.....	8
Research Questions:	10
2. A REVIEW OF RELATED LITERATURE.....	12
Introduction	12
Mathematical Problems, Translations, and Representations	13
Translations and Representations as Problem Solving Tools.....	19
Visualization and Analytic Strategies during Problem Solving.....	23
3. METHODOLOGICAL DETAIL OF THE STUDY.....	28
Introduction	28
Pilot Study.....	29
Participant Selection.....	30
Participant Description.....	31
Data Collection	32
Pre-Session Interview	32
Videotaped Problem-Solving Sessions	33
Post-Session Interview.....	34
Fieldnotes	35
Data Analysis.....	37
Research Questions:	45
4. REVIEW AND ANALYSIS OF THE DATA.....	46
Overview	46
Introduction to the Data	46
Performance-Based Definitions.....	49
Solutions Data by Subject Class.....	50
Characteristics of Competent Problem Solvers	54
Examples of Competent Characteristics:	55
Characteristics of Error-Prone Problem Solvers	69
Examples of Error-prone Characteristics:	72
Time-Protocol Comparison of Problem Solving Behavior.....	103
Time-line Tables: Competent Problem Solvers	106
Time-Line Tables: Error-Prone with Solutions	116
Time-Line Tables: Error-Prone "Did Not Finish".....	125

5. IMPLICATIONS, DISCUSSION, AND RECOMMENDATIONS	136
Overview of the Study	136
Summary and Implications	139
Recommendations for Further Research	148
REFERENCES CITED	150
APPENDICES	153
APPENDIX A: ANALYSIS OF PROBLEM-SOLVING PROTOCOLS	154
A Framework for the Macroscopic Analysis of Problem-Solving Protocols	155
APPENDIX B: AN EXAMPLE OF PROTOCOL ANALYSIS AND PARSING	160
An Example of Protocol Analysis and Parsing	161
APPENDIX C: SAMPLE QUESTIONS FOR PRE-SESSION INTERVIEW AND SAMPLE INTERVIEW	170
Pre Problem-Solving Interview Questions	171
Sample Interview Transcription	171
APPENDIX D: CONSENT FORM	174
Subject Consent Form For Participation In Human Research	175
APPENDIX E: PROBLEM SET	178
Problem 1	179
Problem 2	180
Problem 3	181
Problem 4	182
Problem 5	183

LIST OF TABLES

Table 3.1 Translation Processes.....	17
Time-Line 3.1	44
Table 4.1. Answers Given by Problem and Student.....	48
Table 4.2: Types of Answers Given by Competent Students	51
Table 4.3: Types of Answers Given by Error-Prone Students	52
Table 4.4: Answers Given by Competent Students	53
Table 4.5: Answers Given by Error-Prone Students	53
Time-Line 4.1	107
Time-Line 4.2	108
Time-Line 4.3	109
Time-Line 4.4	111
Time-Line 4.5	113
Time-Line 4.6	114
Time-Line 4.7	117
Time-Line 4.8	118
Time-Line 4.9	119
Time-Line 4.10	123
Time-Line 4.11	124
Time-Line 4.12	126
Time-Line 4.13	128
Time-Line 4.14	129
Time-Line 4.15	131
Time-Line 4.16	133
Time-Line 4.17	134

Time-line B.....	168
------------------	-----

LIST OF FIGURES

Figure 1.1 Connections.....	5
Figure 1.2: Translations.....	7
Protocol 3.1	40
Protocol 4.1	57
Protocol 4.2:	62
Protocol 4.3	73
Protocol 4.4	81
Protocol 4.5	87
Protocol 4.6	94
Protocol 4.7	98
Figure 4.1: Translations and Representations	139
Protocol B.....	162

ABSTRACT

This study investigated students use of mathematical representations and translations during probability problem solving. The purpose of the representation research was to identify the representations used by students, when they chose to use a representation, and how frequently and successfully they used the representations. The research looked for patterns of behavior that characterized students' successful and unsuccessful use of the representations. The research also identified the translations between the representations. The purpose of the translation research was to identify patterns of movement between representations that characterize the students' use of translations and identify the time spent using any particular representation. A qualitative research design was employed. The subjects ($n = 21$) were selected from a finite mathematics course. The data gathering included an interview to familiarize each volunteer with the research process, a problem-solving session in which each attempted to solve five probability problems while vocalizing the process, and a post-session interview in which students explained their solution method.

This research demonstrates that there are distinct differences in the manner that successful and error-prone students use translations and representations. Successful problem solvers are able to analyze the problems and discover a solution method before translating, know when and how to use Venn diagrams, and use symbolic algebra competently. Error-prone problem solvers are often unable to discover a solution method and hence will initially use a Venn diagram (or other representations) to "discover" a solution method. Error-prone problem solvers do not separate their use of translations and representations as distinctly (as do the successful solvers) and often do not finish a translation before attempting another translation. This indicates that the error-prone are unable to arrive at a conclusion on how to solve problems and do not understand (a) how a representation will clarify a problem, (b) the purpose of representations, and (c) which representation to use.

Recommendations include the following: students need extensive practice translating between representations, need to understand what the various representations reveal, and need practice using many different representations. Further research is needed to identify when and how students should receive instruction with translations and representations.

CHAPTER 1

INTRODUCTION TO THE STUDY

Introduction to Representations and Mathematical Connections

Throughout the past century, the mathematics and mathematics education communities have noted the importance of multiple representations as tools for teaching and learning. In the early years of the twentieth century, for instance, the *Reorganization of Mathematics in Secondary Education Report* (National Committee on Mathematical Requirements of the Mathematics Association of America, 1923) recommended that students should develop the ability to use algebra, understand and interpret graphical representations, and have familiarity with geometric forms and the elementary properties and relations of these forms. Throughout the 1920's and 1930's (due, in part, to the influence of the 1923 MAA publication), teacher education journals contained numerous articles advocating and demonstrating the uses of multiple representations to solve geometric and algebraic problems (e.g. Nyberg, 1925; Bradshaw, 1925; or Haertter, 1931).

Recently, the recommendations of the National Council of Teachers of Mathematics (NCTM, 1989, 1991, 1995, 2000) have prompted renewed interest in mathematical representations. In particular, there appears to be recognition of a relationship between students' facility using multiple representations and their success at problem solving. As an example, the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) state that the use of multiple representations empower students as problem solvers:

Students who are able to apply and translate among different representations of the same problem situation or of the same mathematical concept will have at once a powerful, flexible set of tools for solving problems and a deeper appreciation of the consistency and beauty of mathematics. (p. 146)

In support of this belief, the *Standards* include a Mathematical Connections standard at each of the grade bands: K-4, 5-8, 9-12. In a departure from recommendations

that focus on traditional mathematics content, the 9-12 Mathematical Connections standard recommends that representations become an object of explicit study:

In grades 9 - 12, the mathematics curriculum should include investigation of the connections and interplay among various mathematical topics and their applications so that all students can --

- recognize equivalent representations of the same concept;
 - relate procedures in one representations of the same concept;
 - use and value the connections among mathematical topics;
 - use and value the connections between mathematics and other disciplines.
- (NCTM, 1989, p. 148)

Moreover, the connections concept is woven into nearly all of the various other standards.

The 9-12 Functions standard, for example, recommends that the mathematics curriculum include the continued study of functions so that all students can “represent and analyze relationships using tables, verbal rules, equations, and graphs [and] translate among tabular, symbolic, and graphical representations of functions (NCTM, 1989, p.154).”

More recently, a focus on representations is embodied in the problem-solving recommendations of the newest set of standards, the *Principles and Standards for School Mathematics*¹ (NCTM, 2000). For example, the Standards 2000 document states:

Of the many descriptions of problem-solving strategies, some of the best known can be found in the work of Polya (1957). Frequently cited strategies include using diagrams, looking for patterns, listing all possibilities, trying special values or cases, working backwards, guessing and checking, creating an equivalent problem, and creating a simpler problem. (p.53-54)

Notice that each of the aforementioned strategies entails a translation and a “representation” of the problem. That is, each strategy involves the construction of an alternative representation of the problem information. As a teacher, the obvious question that comes to mind is; how should these strategies for representing mathematics problems be taught? In particular, should the construction of alternative representations of mathematics

¹ Herein the NCTM Standards documents are collectively referred to as the *Standards* or the *NCTM Standards*. However, when referring to a particular Standard the document will be so designated, e.g. the *Principles and Standards for School Mathematics* is called *Standards 2000*.

information receive explicit attention, and, if so, how should this content be integrated into the mathematics curriculum?

In addition to implicit recommendations regarding the teaching of multiple representations, the *Standards 2000* document takes a very explicit stand on the subject. Specifically, *Standards 2000* includes a separate Representations standard at each of the four grade bands (K-2, 3-5, 6-8, 9-12), rather than as part of the Connections standard as was done in the 1989 *Standards* document. It should be noted, however, that *Standards 2000* recognizes that connections and representations are interrelated concepts and associates these two standards throughout its curricular recommendations².

To provide teachers with models of the classroom uses of mathematical connections and multiple representations, all versions of the *Standards* documents contain example problems and classroom vignettes. The 9-12 Representations standard of *Standards 2000*, for instance, examines typical student solutions to the following problem:

A flight from SeaTac Airport near Seattle, Washington, to LAX Airport in Los Angeles has to circle LAX several times before being allowed to land. Plot a graph of the distance of the plane from Seattle against time from the moment of takeoff until landing. (Hughes-Hallet et al. 1994, p. 6)

In its discussion of problem-solving strategies, *Standards 2000* maintains that part of the teacher's role is to help students connect their personal images to more conventional representations:

One very useful window into students' thinking is student-generated representations. To illustrate this point, consider the...problem [stated immediately above] that might be presented to a tenth-grade class: Students could work individually or in pairs to produce distance-versus-time graphs for this problem, and teachers could ask them to present and defend those graphs to the classmates. Graphs produced by this class, or perhaps by student in other classes, could be handed out for careful critique and comment. When they perform critiques, students get a considerable amount of practice in communicating mathematics as well as in

² In the mathematics education literature, representations are often represented as one branch of mathematical connections. *Standards 2000* recognizes this relationship, but explicitly differentiates between the two for teaching, learning, and research.

constructing and improving on representations, and the teacher gets information that can be helpful in assessment (p. 363).

Through the use of examples, such as the one above, the *Standards* suggest that connections between the descriptive (contextual), algebraic, graphical (geometric), and tabular formats facilitate concept development and mathematical problem solving.

Overall, it can be concluded that the Mathematical Connections and Representations standards focus on two primary areas: (1) the connections that can be made among mathematical topics and (2) connections that can be made between mathematics and other disciplines.

In addition, two classes of connections are generally discussed in the research literature. The first is the connections that are made between real-life problems and mathematics (often called "mathematical modeling"). The second class of connections are those that are made between two or more mathematical representations of a problem. Both types of connections are illustrated in Figure [1.1].

With the newfound curricular interest in multiple representations, there is newfound research interest in the topic. Research by Skemp (1987), Janvier (1987), and Janvier, Girardon, and Morand (1987) refer to each form of representation (e.g., contextual, algebraic, geometric) as a "mode" and the process of transferring between two modes as a "translation." The representations research will be reviewed and presented in the subsequent review of literature chapter, as will formal definitions of the terminology used in the research.

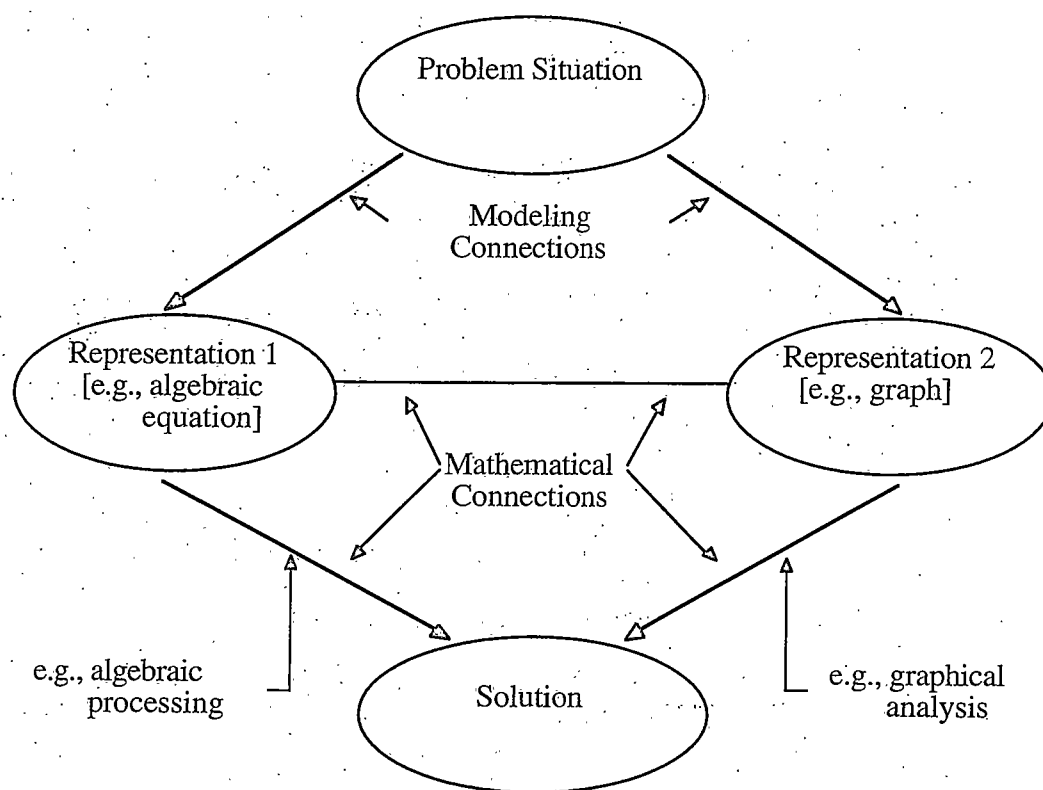


Figure 1.1 Connections

Two general types of connections (NCTM, 1989 p. 148)

Although the *Standards* have helped to focus attention on the translations and representations, the *Standards* naturally oversimplifies the processes by which students construct and use representations. Several related studies (Behr, Lesh, Post, & Silver, 1983; Janvier, 1987; Lesh, Landau, & Hamilton, 1983) provide comprehensive descriptions of many of the representation and translation concepts. Lesh, Landau, and Hamilton (1983), for example, examined the development and use of student's conceptual models in the setting of investigating children's rational-number understanding and problem-solving behaviors in realistic problem-solving situations. Lesh and his colleagues define the conceptual model as

an adaptive structure consisting of (a) within-concept networks of relations and operations that the student must coordinate in order to make judgments concerning the concept; (b) between-concept systems that link and/or combine within-concept networks; (c) systems of representations (e.g. written symbols, pictures, and concrete materials), together with coordinated systems of translations among and transformations within modes; and (d) systems of modeling processes; that is, dynamic mechanisms that enable the first three components to be used, or to be modified or adapted to fit real situations (p. 264).

The research then focuses on the manner in which middle-grade students solved problems that were presented in a variety of formats. Toward this end, the researchers report that in applied problem-solving, important translation and/or modeling processes include

- (a) simplifying the original problem situation by ignoring 'irrelevant' characteristics in a real situation in order to focus on other characteristics;
- (b) establishing a mapping between the problem situation and the conceptual model(s) used to solve the problem;
- (c) investigating the properties of the model in order to generate information about the original situation; and
- (d) translation (or mapping) the predictions from the model back into the original situation and checking whether the results 'fit' (p. 270).

The models presented by Lesh and his colleagues (1983) describe how students use representation and translation. However, Lesh et al. made no attempt to (a) explain what effect the use of inappropriate representations or incorrect translations has on problem solving success, (b) examine how more mathematically mature students use translations and representations during problem solving, or (c) illustrate the types of mistakes made by the subjects as they used representations and translate between representations.

In the dissemination of his doctoral study, Hodgson (1993) describes aspects and patterns in students' errors in translation between representation, in particular, the procedural nature of the translation tasks. The translation Hodgson investigated is illustrated in Fig 1.2 by the arrow pointing from the Algebraic format to the Visual or Graphical format.

Hodgson's research identifies and characterizes the systematic procedural errors students make as they translate algebraic expressions into Venn diagrams. Note that Hodgson investigated translation tasks and characterized errors as the students translated in

one direction, from algebra to Venn diagrams. Hodgson made no attempt to investigate students efforts to translate in the other direction (from the Venn diagrams back to the algebraic representation) or between other representations.

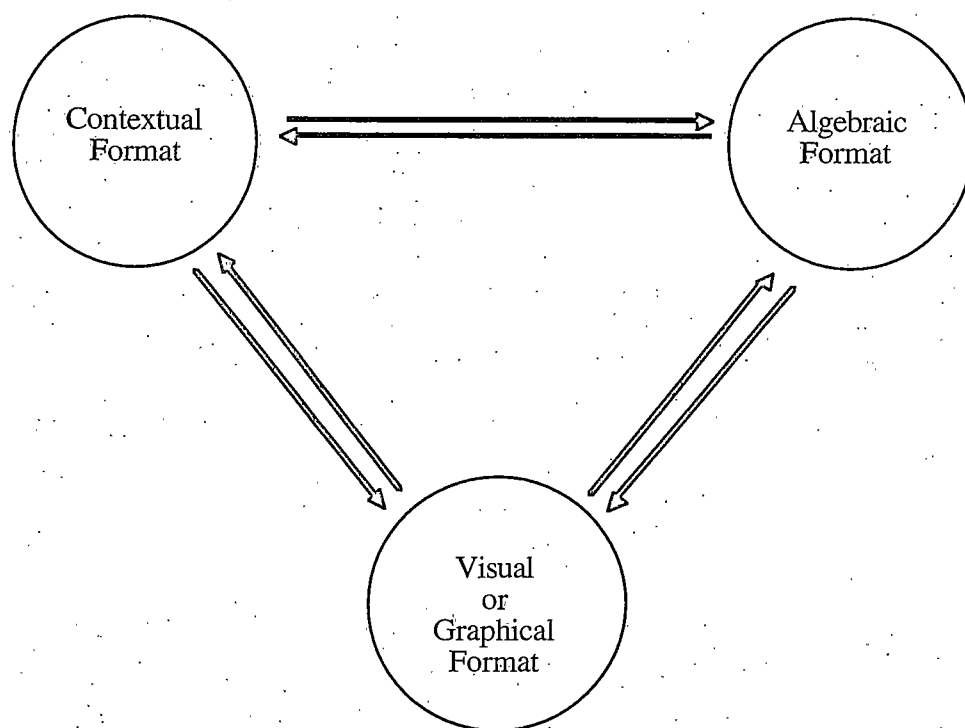


Figure 1.2: Translations

Directionality of Translations between Representations during Problem Solving and Concept Development Activities

Need for the Study

Probability problems provided an ideal context for examining students use of multiple representations because the students have opportunity to use graphical, symbolic algebra, and language representations. The research herein extends the work of Hodgson (1993) and other researchers in that it (a) identifies and characterizes the translation process between more than one representation and (b) investigates translations in both directions between the representations. This study differs from Hodgson's in that it does not attempt

to identify students' procedural errors. Rather, the study identifies the representations used, whether the student was able to successfully translate between representations, and why the student choose to translate. Figure 1.2 illustrates the modes and possible translation paths that were investigated. The author found no research that examines (in detail) translation processes between these three representations. Moreover, no existing study examines the patterns of behavior in the use of translations by student problem-solvers.

The present study, therefore, contributes to the existing research base in that it elaborates on and extends the existing problem-solving and multiple representation research (e.g., Gfeller, Niess, & Hamilton, 1999; Hodgson, 1993; Lesh, Landau, & Hamilton, 1983; Schoenfeld, 1985; and Janvier, 1987). It is believed that the study herein will contribute information that provides researchers with a better understanding of how students use multiple representations in problem solving, as well as information as to where and how problem solvers have difficulty using multiple representations. In doing so, the study will provide improved and informed direction for both teaching and curriculum design of instruction involving translations and representations in mathematical problem solving.

Introduction to Data Collection and Methodology

The data from the student interviews and probability problem solutions were gathered at Indiana University in Bloomington, Indiana. The analysis of this research took place at Montana State University-Bozeman and at Oregon Institute of Technology. The representation and translation data were collected from interviews with a group of student volunteers. For the study, five probability problems were administered to students ($n = 21$) from various sections of a finite mathematics course. Each of the probability problems can be solved using elementary algebra, Venn diagrams, or by using both of these representations. Each participant was video taped while attempting to solve the problems and

their solution methods were analyzed to determine their use of multiple representations and translations.

For the purpose of this research, the term *successful problem solver* describes an individual that was competent at solving the assigned probability problems, that is, the student consistently obtained correct solutions to probability problems that require translations between several representations.

Likewise, *unsuccessful problem solver* describes an individual that was not competent at solving the assigned probability problems, that is, the student did not consistently obtain correct solutions to probability problems that require translations between several representations.

The identification as to whether the student was a successful or unsuccessful problem solver was made post hoc, on the basis of whether the student was able to successfully solve a set of probability problems.

Each subject demonstrated his or her solution method(s) as he or she attempted to solve probability problems like that of Example 1. The participants were asked to verbalize their thinking and each solution attempt was video and audio recorded.

Example 1

At Giant State University a survey of students taking college mathematics found that 40% took college algebra, 30% took statistics, and 42% took neither course. If one of the students from the survey is chosen at random, what would be the probability that she or he took either algebra or statistics but not both courses?

The video taped "out loud" problem-solving sessions were analyzed to identify and characterize the subject's use of representations and translations. The methods of data collection and analysis are further described in the Methodology chapter. Appendix A

provides a description of the rubric used to identify (a) when a subject was using a particular representation and (b) when a subject was translating between representations. Appendix B provides an example of how the rubric was used to identify and characterize a subject's use of translations and representations

In particular, this research focused on (a) the subjects use of multiple representations and (b) on the cognitive links the subjects made between the representations (translations). This research identifies (a) what representations successful and unsuccessful students use, (b) when they choose to use representations, (c) how frequently they use each representation, and (d) how successful they are at using the representations. Further, this research identifies (e) patterns of movement between the representations that characterize the successful and unsuccessful students use of translations and (f) patterns in the time spent using each representation (between translations) that characterize the successful and unsuccessful students use of translations. The research questions are introduced in the following section.

For the purpose of this research (a) the contextual format is probability problems written in English; (b) the algebraic format is Symbolic Algebra at a college introductory level; and (c) the visual or graphical format is usually Venn diagrams, although two students attempted (unsuccessfully) to solve the problems using Tree diagrams.

Research Questions:

The researcher entered the study with four research questions, two that were considered "Primary" and two "Secondary" questions. Primary research questions focus on the mechanics of students' representation use, whereas secondary questions focus on the role of representation use in problem solving.

Primary Research Questions:

1. Is there a difference in successful and unsuccessful students' use of multiple representations to solve probability problems?
2. Is there a difference in the patterns exhibited by successful and unsuccessful students with regard to their translation between representations?

Secondary Research Questions:

3. With regard to the problem solving process as a whole, what benefits (if any) do students gain through the use of multiple representation?
4. What difficulties (if any) do students encounter in their use of multiple representations?

CHAPTER TWO

A REVIEW OF RELATED LITERATURE

Introduction

Through a detailed investigation of students' solutions to probability problems (and their solution processes), this study examined students' use of mathematical representations and their translation between representations. Specifically, the students' use of Venn diagrams, symbolic algebra, and contextual (English) language, and translations between these representation modes, were analyzed in reference to two primary (Questions 1 and 2) and two secondary (Questions 3 and 4) research questions:

1. Is there a difference in successful and unsuccessful students' use of multiple representations to solve probability problems?
2. Is there a difference in the patterns exhibited by successful and unsuccessful students with regard to their translation between representations?
3. With regard to the problem solving process as a whole, what benefits (if any) do students gain through the use of multiple representation?
4. What difficulties (if any) do students encounter in their use of multiple representations?

In this chapter, research related to this investigation is reviewed in order to lay a theoretical foundation for an examination of students' use of representations and translations during problem solving. Namely, the fundamental elements of translation and representation activities are presented and reviewed in light of existing research literature. Secondly, theoretical and empirical considerations are examined regarding the use of translations and representations. In particular, this review seeks to identify the utility and limitation of translations and representations during problem solving. The chapter concludes

with a brief review of the research literature regarding learning styles, with a particular focus on problem solvers' use of visual and analytic strategies.

Mathematical Problems, Translations, and Representations

Problem solving, a topic that has long been of interest to educators and psychologists, served as the basis of this study. According to Lester and Charles (1985), problem-solving situations are those in which an individual or group (a) must solve some type of problem; (b) the problem may be presented either as a "real life" situation or in some other mode, such as an English or mathematical question; and (c) the solver does not have ready access to a schemata or algorithm to apply to the problem or question. That "the problem solver does not have a solution schemata" does not imply that he or she doesn't possess the necessary cognitive and manipulative tools to solve the problem. However, answering the question is only an "exercise" if the individual possesses a solution schemata, algorithm, or knows immediately how to "do" the problem.

In the study described herein, students were administered five probability problems. The problems were "true" problems in that they met Lester and Charles (1985) criteria of problem solving, as mentioned above. However, many of the students, especially the more mathematically mature students, were able to solve the problems. Yet, the method of solution was not immediately apparent nor did a solution algorithm exist. Rather, students solved the problems through the use of a broad class of tools called mathematical "connections."

In Chapter One, it was noted that two classes of mathematical connections are discussed in the research literature. The first class of connections involves mathematical modeling, or the translation of information (generally relational in nature) observed in the "real world" into some alternative expression. In the setting of mathematics education, the "expression" of the event is generally made in context (i.e., in the spoken or written language native to the individual expressing the relationship), as a set of algebraic symbols

and expressions, as a Cartesian or data graph, or as some combination of these representations.

The second class of connections includes those that can be made between two or more expressions. That is, this class of connections includes those connections that can be made between contextual descriptions, graphics, or algebraic expressions--or any other of the several descriptors of real-world events. Specific descriptors may include geometric figures, Venn diagrams, tree diagrams, Cartesian graphs, tables, lists, flow charts, letters and numerals in descriptive patterns, written descriptions, of any mix of these (Bell, 1976). This research focused on this second class of connections: those that can be made between the various models or representations of real-world events, but not on the real-world events.

As a word, representation is open to many interpretations. In the context of this study, as in most mathematics education research, representation refers to any observable embodiment that symbolizes or expresses an event or relationship. It should be noted that this definition represents a departure from that employed by Bell (1976), Janvier (1987), and some other researchers. Namely, this study did not restrict its focus to expressions directly related to real-world events.

An additional issue concerning representation research, and an important issue that needs to be addressed, is the need to observe the representation. Mental imagery is an essential component of thought. As such, Lesh, Post, and Behr (1987) remark that it is both naive and restrictive to require a representation to be observable. In particular, these researchers maintain that a representation may be an internal conceptualization and, therefore, unobservable to the researcher. The research literature (e.g., Behr, Lesh, Post, & Silver, 1983; Lesh, 1981; Lesh, Landau, & Hamilton, 1983) identifies at least six distinct types of representations used by problem solvers. These include (a) manipulative models such as arithmetic blocks and fraction bars; (b) graphics, pictures, and diagrams; (c) experience-based "scripts" wherein students' knowledge is organized around "real-

world" events or experiences that serve as context to describe and solve other problems; (d) specialized forms of spoken languages, such as used by mathematicians and logicians; (e) spoken or written language as used in context by lay persons, i.e., non-scientists and non-mathematicians; and (f) written symbols and phrases--including algebraic equations or set-logic expressions. Note that this study focused on three of the aforementioned representations--graphs; spoken language, and algebra--and the translations and transformations among and between these three modes of representation.

The six representation modes listed above, together with the translation processes that link them, are essential features of the steps in mathematical modeling. Lesh and his colleagues (1987) identify these modeling steps as (a) simplifying the original situation to reveal the most important aspects of the model, (b) building a map (i.e., a translation) between the original situation and the model, (c) investigating the model to understand the original situation, (d) translating the relevant information from the model back to the original situation, and (e) checking that the translated information is a sensible representation of some observable elements of, and relevant to, the original situation. With regard to this research, the original situation presented to each research subject was a probability problem in written English. That is, the "context" of the problem was written English (as opposed to spoken English). For most of the subjects, the representation model chosen was a Venn diagram representation of the problem information, symbolic algebra, or some combination of the two.

Lesh and colleagues (1987) suggest that the modeling action (i.e., those representations reached through acts of translation and transformation) tend to be *plural*, *unstable*, and *evolving*. Moreover, these three attributes play important roles in the evolution of concepts and representations during the course of problem-solving sessions.

An act of representation is *plural* in that problem solvers commonly use more than one representational system (with appropriate acts of translation) to attain the necessary

understanding of the problem. In particular, Lesh and colleagues found that problem solvers typically employ several different representations when solving a problem, and the various representations used may be incompatible with one another. However, the researchers noted that each of the representations help the student to understand or solve some aspect of the problem.

The modeling action may also be plural in the sense that the problem solver may use a connected series of representations while attempting to solve a problem. For example, with regard to a probability problem, a student might translate contextual information into a Venn diagram, solve the Venn by translating information into a symbolic algebra mode, and then translate information back to the Venn and into the contextual mode. Each of the various representations may depict only a portion of the modeled system, and several serial acts of translation may be needed before the problem solver gains a good understanding of the problem. Furthermore, Lesh and his colleagues (1987) assert that good problem solvers tend to be sufficiently flexible in their use of a variety of relevant representational systems that they instinctively switch to the most convenient representation to emphasize at any given point in the solution process. It is interesting to note that the present study does not support the assertion that good problem solvers instinctively switch between representations.

Acts of representation are said to be *evolving* when the representational mode is used by the problem solver not only to represent and solve some aspect of a problem, but also as a tool to gain understanding of the basic nature of the problem. Through the use of representations, a problem solver may discover and understand relationship inherent in the problem that were previously not accessible or not noticed.

The act of representation is *unstable* in that the problem solver may focus on a particular representation of the problem and, in so doing, forget the "big picture" or other important aspects of the problem. Alternatively, problem solvers that attempt to focus on the "big picture" may forget to account for important aspects of the problem.

Lesh and his colleagues (1987) also remark that not only are the distinct types of representations systems important, but translations among the representations and transformations within them, are also important. A *translation* identifies “the psychological processes involved in going from one mode of representation to another” (Janvier 1987). For example, one translation event is the process whereby a written English description of a relationship is re-expressed as a set of algebraic equations. Also, the process of expressing a set of algebraic equations into written English is another translation act. Note that these two examples are inverse translations with respect to one another and are said to be *complementary*. Any translation may involve one, two, or several representation modes. Also, any translation identifies a complementary translation.

Janvier (1987) describes the translation process with respect to four modes of representation, as in Table 3.1.

Table 3.1 Translation Processes
(Janvier 1987)

To: From:	Situations, Verbal Description	Tables	Graphs	Formulae
Situations, Verbal Description		Measuring	Sketching	Modeling
Tables	Reading		Plotting	Fitting
Graphs	Interpretation	Reading off		Curve fitting
Formulae	Parameter Recognition	Computing	Sketching	
Note, Each non-diagonal region of the table indicates a mode (representation). Translation is the process of making a cognitive and representational shift between two or more modes.				

Each shaded region along the diagonal of the table represents a transformation (or transposition) within a mode. By the definition used herein, a transposition is not a

translation. An example of a set of transpositions is the *process* by which an equation is being simplified or solved. Although each algebraic step does present a new representation of the initial relationship, there is no new mode of representation because each of the representations is in symbolic algebra. Hence, the algebraic steps are not defined as translations. Note that the equations are not transpositions. Rather, a transposition is the process that changes one equation into another.

As Janvier (1987) notes, not all translations can be represented by this simple two-dimensional table. In particular, some translations are indirect and include the use of one or several intermediate modes. As an example, the translation "table $\rightarrow$ formula" is often processed as "table $\rightarrow$ graph $\rightarrow$ formula." Likewise, the complementary translation "formula $\rightarrow$ table" is often processed as "formula $\rightarrow$ graph $\rightarrow$ table." Note that direct translations are substantially different from indirect translations. From a research perspective, a subject's use of an indirect translation may be much more difficult to identify than a direct translation and, because of internalized mental representations, may be mistaken for a direct translation.

As noted above, a translation must involve at least two modes of representation. To complete a translation, therefore, Janvier (1987) maintains that one must transform the source 'target wise.' In other words, one must examine the source (the representation to be translated) in the context of the target (the representation to be constructed by the translation). As an example, to translate an English statement into an algebraic relation, the translator must (a) frame a cognitive correspondence between each relational identifier (e.g., English nouns) and an appropriate algebraic symbol, (b) identify any relationships between and among the identifiers in the English statement of the problem, (c) quantify each relationships to be translated, (d) represent the relational quantifiers in algebraic notation, and (e) make an algebraic equation using the symbols and quantifiers. Only when each of

these tasks is achieved is the contextual to symbolic algebra translation completed. Moreover, by performing each of the five steps, the translator has identified the source relationship in the context of the target. Note that a wrong identification between a symbol and a identifier or a quantification error could produce an inaccurate representation and might make the translation unattainable.

Translations and Representations as Problem Solving Tools

On any occasion wherein a "real-world" event is described, a translation has taken place and a mode of representation has been selected (by the translator). That is, the description of the event is not the same as the event itself. Rather, the description is a representation of the event in some mode chosen by the one making the description (i.e., the describer). The representation mode is chosen on the basis of the abilities, aptitudes, and/or the cognitive inventory of the describer. Consequently, a single real-world event may be described via music, song, poem, picture, objects such as sculpture or models, re-enactment, or conversational language. Also, the primary representation may be translated into a secondary mode of representation, as when spoken language is used to describe a picture. In any case, whatever the representation chosen, the

representational systems differ from one another because they emphasize or de-emphasize different aspects of the underlying structure of the concept. They also differ in generative power and in their ability to manipulate relevant ideas and data simply and economically in various situations. For example, sometimes a picture is worth a thousand words; sometimes language is clearer and more efficient. (Lesh, R., Landau, M., & Hamilton, E. 1983, p. 265)

In their summary of two research studies, the Applied Problem Solving project and the Rational Number Project, Lesh and colleagues (1983) reviewed students' use of representation and translations during problems solving. They report that:

(a) during the solution process, subjects frequently change the problem representation from one form to another (e.g., written symbols to spoken words, spoken words to pictures or concrete models, etc.); and (b) at any stage, two or more

representational systems may be used simultaneously, each illuminating some aspects of the situation while de-emphasizing or distorting others. (p. 264)

Interestingly, Lesh and his colleagues found that a number of subjects (approximately 17%) believed that their solution may depend on the type of representation(s) used while solving the problem. That is, these problem solvers believed that if they chose a particular representation to solve the problem they might find a correct solution. But, if they had chosen a different method of representing and solving the problem, a different, but equally correct, solution might have emerged.

During their investigation, Lesh and his colleagues (1983) asked students who had made errors while adding fractions to "act out" the incorrect problems using manipulatives. After each student found a correct answer with the manipulatives, he or she was then confronted with the wrong answers in their written work. After seeing both answers, about half of these students continued to maintain that their written work was correct, even though it disagreed with their second solution. Lesh and his colleagues report:

The discrepancy between the results obtained from the two different representations apparently did not trouble these children; several explicitly stated something like, "That's okay! These are pizzas and those are numbers -- they aren't the same." Such comments seem to indicate either a belief that mathematical computations (in symbolic form) need not agree with real-world observations (of clay pizzas) or that mathematics is simply unpredictable, so sometimes one obtains one answer and sometimes another for the same problem.

The presence of concrete materials in the follow-up problem did not enhance performance. Whereas 30 children had obtained the correct answer to the work problem (all of them using written symbols), a total of only 20 children arrived at the correct answer when the same problem was posed using concrete materials. Three students attempted to solve the problem using the concrete materials; they all obtained incorrect results. All 20 children who were correct were among the 27 children who persisted in using written symbols to obtain an answer. This means the 7 subjects who had previously obtained the correct answer using written symbols were no longer able to solve the problem using written symbols after it was presented in concrete form. They became confused and reverted to adding both numerators and denominators rather than using the LCD [lowest common denominator] approach as before. *This outcome runs counter to the widespread belief that materials make a problem easier to solve because it is more meaningful and real.* (Italics added) ... Attending to the concrete materials somehow contributed to the breakdown to an effective symbolic method for solution. (1983, p. 280-281)

Lesh et al. (1983) argues that, although a concrete model may provide a representation that allows students to investigate some or many aspects of a situation, it may obscure other portions of the relationships. Therefore, it may be difficult or impossible for the problem solver to carry out the entire solution process with a concrete model. Lesh and his colleagues conclude that good students eventually learn to select an appropriate representational system to fit each particular part of a problem situation or each specific stage in the overall solution. Moreover, this understanding builds slowly and requires coordinating complex within-concept networks and between-concept systems.

In his research on university students efforts to solve routine and non-routine problems, Schoenfeld (1985) found that problem solving is influenced by four aspects of knowledge and behavior: resources, heuristics, control, and belief systems. Of these, Schoenfeld maintains that control is the most essential. Successful students, Schoenfeld found, engage in *periodic monitoring and assessment* of solutions as they evolve and the curtailing of attempts that are unfavorably assessed. That is, the successful problem solver continually monitored her or his translations and representation modes so as to verify their validity and usefulness in solving the problem. It is seen then that Lesh's immature problem solvers--those that arrived at different solutions via different representation and did not see a conflict--were unable to monitor and assess their solutions as they evolved. That is, they were unable to make a strong connection between the various representations, and they did not understand that the various representations were nothing more than looking at the same situation or relationship from a different point of view.

According to Schoenfeld (1985), a review of the research concerning students' flawed resources and consistent error patterns serves to document three points:

1. It is easy to underestimate the complexity of ostensibly simple procedures, especially after one has long since mastered them. A complete delineation of the skills required to perform an operation as simple as subtracting one three-digit number from another is much more complex than one might expect. It also points to many places where one can go wrong.

2. A large number of mistakes in simple procedures may be the result of mislearning. That is, students' flawed attempts at implementing a procedure may not be, as one might naively assume, the result of their "not having gotten it yet." Rather, the students may be implementing the same incorrect procedure over and over again.
3. Part of the support structure for resources consists of *knowledge representation*. The ways in which information is represented (structured, perceived) by an individual may determine how successful the individual can be in using that information. (p. 61)

Recent research indicates the validity and importance of item three of Schoenfeld's conclusions. In a study of students' representational style in CAS-assisted problem solving, Heid et al. (1997) report that many students show a marked preference for a particular representation in which to test conjectures. Further, some subjects rely on the preferred representation as the initial arbiter for the truth of their conjectures. That is, a subject's use of a particular representation not only effected how he or she attempted to solve a problem, it also influenced his or her determination of whether their solution was correct. Heid and her fellow researchers found that students generated multiple-representations for various purposes, including (a) to have a manipulable representation (e.g., algebraic), (b) to serve as a "place holder" for a concept or idea to which the student can return, and (c) as a conceptual or memory aid. The representations used as conceptual aids were often student generated or non-standard symbolic representations. Their purpose seemed to be to allow the student to mentally hold onto aspects of the concept that he or she was trying to depict or understand--but not as a representation to be manipulated or refined.

Heid and her colleagues report that a number of the students seemed to use symbolic representation as place holders to which they could later return and which they could refine in successive returns. These representations seemed not to be intended to represent the actual mathematical ideas the students were entertaining. Instead, these representations apparently reminded the students that, although they had not yet represented the exact relationships, they knew the magnitude and position of what they intended and would later revisit to refine the representations.

Research also demonstrates that to successfully teach subject matter understanding, instructors must possess conceptual and flexible representations of the material they are teaching (McDiarmid, Ball, and Anderson, 1989). However, recent research indicates that preservice teachers may have difficulty learning and using multiple representations in problem solving. In a study of involving understandings of the arithmetic mean, Gfeller, Niess, and Lederman (1999) found that many of the subjects (pre-service teachers in a Masters of Arts in Teaching program) had trouble viewing a mean in multiple ways. Many of the subjects were able to solve problems involving the mean using a computational algorithm, but were unable to provide solution using a graphical method. These researchers assert that professional development courses geared at building pre-service teachers' repertoire of representations should pay careful attention to representations in different settings. Pre-service teachers may feel confident in knowing various views, yet may be limited in their understanding in alternative settings.

Bishop (1989) remarks "whilst it is tempting to believe that all visualizations [representations] must necessarily play a useful role in mathematical activity, we clearly need research which helps us understand more about which of their features contribute significantly to the role in a given mathematical problem situation." (p. 9) It was with this broad objective that the present research was initiated.

Visualization and Analytic Strategies during Problem Solving

Any researcher or teacher would find it difficult to argue with Piaget's (1977) contention that some people are particularly visual; whereas, others prefer motor, auditory, or other modes. However, whether or not we believe there are visual, motor, and/or auditory individuals, identifying and classifying individuals of groups as such has been very difficult for researchers to accomplish.

Various researchers have attempted to define and analyze visualization and visual thinking, included among these are definitions rooted in intuition, pictorial representations of objects, geometrical and graphical representations, mixed Visualizer/Analyzer model, and as questions of internal versus external representations (Simmermann & Cunningham, 1991; Lean & Clements, 1981; Pylyshyn, 1973; Zazkis, Dubinsky, & Dautermann, 1996; Webb, 1979).

Recent research by Zazkis, Dubinsky, and Dautermann (1996) suggests the relationships between visualization, visual thinking, and analytical thinking are very complex and directly affect one another. Further, their research may indicate that a visual or analytic classification of an individual may represent only the individual's mental process, understandings, and preferences at a given moment. That is, as a problem solver matures in his or her understanding, the visual and analytic aspects of his or her thinking become more closely related and integrated.

To arrive at their conclusions, Zazkis and colleagues (1996) studied students' ($n = 32$) use of visualization and analysis in mathematical thinking. The subjects, all of whom were enrolled in their first abstract algebra course, were given the task of listing the elements of the dihedral group and finding the product of elements. The subjects were familiar with the dihedral group elements and structure from class discussions and group work. The subjects modeled the group in one of two ways. The model that relates to visual thinking expresses the group in terms of the symmetries of a square. In the visual-thinking model, the elements of the group are the four rotations of the square together with the four reflections about the diagonals or line segments connecting the midpoints of the opposite sides. The product of two symmetries is modeled by performing two of the operations on the square.

The model associated with analytic thinking represented the group in terms of permutations of four objects. The product of two of these objects is obtained by applying a specific multiplication algorithm to the elements.

On the basis of their research, Zazkis, Dubinsky, and Dautermann (1996) constructed a model of mathematical thinking that synthesized visualization and analytic thinking. In this model, which the researchers refer to as the Visualizer/Analytic model, the thinker begins with an act of visualization or a mental picture. Then, using the visualization, an analytic process takes occurs that either solves the problem or refines the previous visualization. The thinker then, through the use of the refined visualization, re-analyzes the problem and either solves the problem or once again refines the visualization. Zazkis and colleagues state that; as the analysis and visualization cycle continues

the acts of visualization and analysis become, for the individual (but not necessarily for the observer), successively closer. This is meant in at least two senses. At first, acts of visualization and analysis may be seen by the individual as being very separate and quite different. The passage from one to the other may represent a major mental effort, which can even be resisted in some cases. Gradually, the two kinds of thought become more interrelated and the movement between them becomes less of a concern. Part of the reason for this development is the second sense in which the two kinds of acts approach each other. In the beginning, the individual may interpret visualization as relating to events that are mainly external to her or himself (as in picturing a physical square or drawing a picture on the chalkboard), whereas analysis may be seen more as originating from the individual (as in inventing and using a symbol system). As the acts continue, both kinds of thinking become more and more deeply interiorized within the mind of the individual. After some time...analysis and visual understandings are synthesized so that it may be very hard for the individual or the observer to distinguish between them. (p. 447)

Clements (1982a, 1982b) argues that, while there may be “visualizers” and “verbalizers,” there also appears to be a group of learners he identifies as “mixers.” The “mixers” are individuals who “do not appear to have a tendency one way or another” (Clements, 1982b, p. 34). If, as Clements suggests, there do exist individuals that have traits of both “visualizers” and “verbalizers,” then there should be some identifiable gradation of “visualization” or “verbalization” that can be used to categorize the learner. However,

Anderson (1978) contends that it is not possible to make this type of differentiation among learners.

If there do exist classifiable groups of "visualizers" and "verbalizers" then a mathematics curriculum and pedagogy could be designed to cater to the individual needs of the each group. That is, the "visualizer" would receive mathematical instruction via a curriculum that would best use his or her visual talents. Likewise, a student identified as "verbal" would be instructed in a more verbal manner and be expected to produce his or her school work in a verbal made. However, would this curriculum then relieve the "verbalizer" from learning to read and analyze information using graphs, tables, and drawings? Likewise, would the "visualizer" not be trained to use symbolic algebra and mathematical notation to analyze problems or to verbalized descriptions to his or her mathematical solutions. Zazkis et al. (1996) suggest that if such a learning trait categorization could be designed then "defining some people as a 'visualizers' may imply that certain individuals are closed out of understanding certain mathematical concepts that have no concrete antecedents" (p. 437).

So, it is seen that the literature demonstrates the following:

1. Students do not naturally make connections between the various representations. And in fact, students may not see any connection between physical representations and mathematical representations. This indicates that research is needed that will demonstrate how best to teach so that students can begin to understand the relationship of the various representations. It may take considerable practice on the students part and very directed teaching for the student to begin to make connections between the representations.
2. Lesh and colleagues (1987) studied how good problem solvers use representations. The research is much less complete in examining how less competent problems solvers use translations. In particular, there is little research that demonstrates the difficulties that the

less competent problem solvers have using representations and how their use of representations differs from that of the competent problem solvers.

3. The research indicates that there may be "visualizers," "verbalizers," and there may be problem solvers that mix these two traits or have no tendency toward either of the two traits. However, the research does not clarify if the ability to mix the two traits is learned or inherent. Research is needed that demonstrates if the mixing of the two traits makes for more successful problem solving and if students can learn to strengthen and mix their verbal and visual skills to enhance their problem solving skills.
4. The research demonstrates that both translations and representations are important aspects of problem solving (e.g., Lesh, Post, and Behr, 1987). The research is much less comprehensive in explaining how the problem solver uses representations, and in particular, why a problem solver chooses to use any particular representation. If the choice of representations is "instinctive" as is asserted by Lesh and his colleagues, then the natural question is; why does this "instinct" develop? Further, if a problem solver has an instinct to use a particular representation, and the representation is not appropriate for the problem, how and when does the problem solver chose to use a none instinctive representation.

CHAPTER 3

METHODOLOGICAL DETAIL OF THE STUDY

Introduction

The data analysis of this study took place at Montana State University-Bozeman and Oregon Institute of Technology. The data were collected during a three-week period at Indiana University in Bloomington, Indiana. For the study, five probability problems were administered to a set of twenty-one participants. Each of the probability problems is solvable using set algebra and/or Venn diagrams. The study participants were student volunteers drawn from four sections of a finite mathematics course. Each subject was (a) interviewed; (b) videotaped while attempting to solve the five probability problems, and (c) interviewed after the problem-solving session. The first interview was intended to build rapport between the subject and the interviewer, make the subject comfortable with the videotaping equipment, and explain how the problem-solving session would be conducted. Also, each participant was asked a set of question about the his or her mathematical skills and educational background. An outline of these questions is included in Appendix D. However, because many of the subjects volunteered information about themselves, not all the questions were asked of each interviewee.

The post-session interview was used to have the subject explain any aspect of the problem-solving session not understood by the researcher. In general, the post-session was used to ask questions about any remarks made during the problem-solving session that were either not heard clearly or not understood by the researcher.

The pre-session interview and post-session interview were audiotaped. The problem-solving session was videotaped and the students' written work was collected.

This chapter includes descriptions of the pilot study, subjects, method of data collection, data sources, and method of data analysis.

Pilot Study

A pilot study undertaken at Montana State University--Bozeman (MSU) was used to develop and refine a set of probability problems to be used in the study. Prior to the pilot study, seven probability problems were developed. Each of the seven questions was designed to be similar to problems found in the finite mathematics (Math 118) course textbook (Maki and Thompson, 1996) used at Indiana University. This was done so that the study subjects, each of whom would be a student taking Math 118, would be familiar with the mathematical tools needed to solve the problems.

Five students from the Department of Mathematical Sciences at MSU volunteered to take part in the pilot study. Four of the volunteers were masters-level graduate students and one a senior undergraduate. Each of the pilot volunteers was asked to work the questions in the problem set while being videotaped, just as would be the research subjects. After each volunteer completed the problem set, he or she was questioned about the problems to (a) make sure that the problems were unambiguous and (b) ensure that each problem was worded as intended. As each volunteer reacted to the questions, the questions were refined as needed. The refined questions were then used with the next pilot study volunteer. It was found that the videotape recording was clearer if each of the five questions was printed on a separate sheet of paper with the participants identification included on the sheet. Also, this ensured each student's work was attached to the proper problem.

During the course of the pilot study, it became clear that the problem set contained too many questions. Consequently, two of the seven questions were eliminated. Also, the order of the questions was changed so that the questions the volunteers perceived to be easier would be earliest in the set.

Throughout the pilot study the researcher positioned the video camera at various locations and angles attempting to find a position that would give the clearest picture of the student's writing and give the clearest sound reproduction while not interfering with the student's work. It was discovered that the camera angle and position must be changed for left-handed and right-handed subjects. In general, the best position for the camera was behind the subject while viewing the subject's work over the subject's shoulder on the side of the writing hand. Because of the video recordings limited resolution, the font size of the problem set was increased from twelve to fourteen. Also, it was found that the participants were much more verbal and explain their think more thoroughly if they read each probability problem aloud before attempting to solve the problem.

It became apparent during the pilot that a four function calculator and pencil and scratch paper should be supplied to the participants.

After the five pilot volunteers had completed the probability problems, the set was again reviewed by Dr. James Robison-Cox and Dr. Theodore Hodgson. Several of the problems in the set were revised on their counsel. The problem set as given to the students can be found in Appendix E. Notice that only one problem is on each sheet of paper. This was done so that each student's written solution would be attached to the related problem when it was collected.

Participant Selection

Subjects participating in the study were drawn from students enrolled in Indiana University's introductory finite mathematics course (Math 118), during fall semester, 1996. In Math 118 the daily curriculum is prescribed by the Department of Mathematics and parallels the sequence of topics presented in Maki and Thompson (1996). During the initial weeks of the course, the topics covered in lecture included: sets, set operations, Venn diagrams, partitions, size of sets, set outcomes, and tree diagrams. Thus, it was expected that

the participants would be exposed, both by direct instruction and via the homework, to examples and problems much like those presented during the problem-solving sessions.

An announcement of the study was mailed to all Math 118 instructors, and several of the instructors read the announcement to his or her section. The researcher was permitted to personally make an announcement about the study and lobby for participants in three Math 118 sections. The prospective volunteers were told that each would receive ten dollars for participating. Twenty-one students volunteered for the study. The researcher did not question the participants as to which section of Math 118 they were enrolled; however, several of the participants mentioned their Math 118 teacher's name or the section number so it is known that there were participants from at least three sections of Math 118 and therefore, from at least three distinct instructors.

Participant Description

The subjects included in this study were recruited from Indiana University's introductory finite mathematics course, Math 118. The organization and selection of topics in this course are guided by Maki and Thompson's (1996) text, *Finite Mathematics*. The text and classroom instruction included many examples of worked probability problems. The text problems require the use of college-level algebra and (in many cases) Venn diagrams. The use of the Venn diagrams and algebra was demonstrated frequently during classroom instruction. Many problems were assigned that required the use of Venn diagrams to find the solutions. Also, classroom quizzes were given, collected, and graded, covering the probability sections of the text.

Twenty-one students, eleven men and ten women, from three or more sections of Math 118 volunteered to participate in this study. Nine of the students were majoring in business-related subjects (business, finance, accounting, or marketing), two were elementary education majors, and one major from each of German studies, public administration,

computer science, telecommunications, environmental affairs, English, speech, and music. Two participants were still undecided about their majors. Of the twenty-one, seven said they disliked mathematics, six said they liked mathematics, and eight were undecided as to whether they liked or disliked mathematics. Fifteen of the participants were Freshman, three were Seniors, one a Junior and one a graduate student. One of the participants, a foreign student, did not state an academic standing.

Data Collection

The data gathering includes material from each of the following: (a) A pre-session interview used to acquaint each volunteer with the research process, audio and video equipment, and establish a small biographical base, (b) a problem-solving session in which each participant attempted to solve five probability questions, followed by (c) a post-session interview, in which each participant explained (if asked) his or her solution technique, (d) any written material produced by the participant during the problem-solving session, and (e) a field journal which record the researchers observations and comments. The pre-session and the post-session were audiotaped. The problem-solving session was vidiotaped. The field journal was used to provide written reminders to the researcher of questions to ask of the participant for the post-session and to record the general impressions of the researcher. The journal had no formal order and was not written in sentence structure.

The following methods were used to collect and triangulate the research data:

Pre-Session Interview

Each participant, prior to attempting the problem set, was audiotape recorded answering a number of questions about their mathematics skills, academic endeavors, previous mathematics course, and their opinions and feelings about mathematics. The purpose of this preliminary interview was to gather the necessary data to identify and

classify the participants, to help each of the participants become comfortable with the recording equipment (the videotaping and audiotaping equipment was quite obtrusive), to encourage them to speak loudly, and to familiarize themselves with the data collection process and the researcher. To accomplish these objectives each student was asked about his or her major, high school and college mathematics course work, favorite and least favorite subjects, if the liked or disliked mathematics and Math 118, to identify their academic year, and any other relevant questions that might come up during the conversation with the researcher. An outline of the questions asked in the pre problem-solving session is included in Appendix C.

Videotaped Problem-Solving Sessions

Each of the selected research participants was videotaped attempting to solve a set of five probability problems. Participants were asked to read each probability problem aloud before beginning the problem and to verbalize their thinking while attempting to solve the problems.³ The video recorder was focused on the student's written work to detail the physical manifestation of the problem-solving process. Also, each student's written work was collected to document the process and to identify when translations occurred. As each participant worked, the researcher operated the video camera and (when possible) made notes about his own observation of the student's work. These notes were then used to question the participant during the post problem-solving interview. In particular, the researcher attempted to identify and note any patterns that were occurring in the student's work, record any sections of the student's work that were not clear to the researcher, and note any uniqueness or originality in the student's work.

³ "Ericsson and Simon's (1980) 'Verbal Reports as Data' concludes that certain kinds of talking aloud instructions--those that ask for verbalization as one solves a problem, without calling for explanation (elaboration or retrospection of what one is doing)--do not seem to affect people's performance while solving problems. The issue is still very much unresolved." (Schoenfeld, 1985, p. 275)

Although the problem set had been refined by the pilot study, it became apparent during the first three interviews that the first problem was too easy--the participants were doing the problem in their heads without using diagrams or symbolic algebra. Hence, problem one was refined and, because it then became much the same as problem three, problem three was also changed. Consequently, the data collected on problems one and three from the first three participants have not been included in the analysis.

Post-Session Interview

The post problem-solving interview followed immediately after the participant's problem-solving session. This interview was audio recorded. The post session interview was used to clarify the researcher's understanding of the mathematical techniques used during his or her problem-solving session. In particular, each participant was asked to clarify or explain any mathematical steps she or he had not explained sufficiently during the problem-solving session. If Venn diagrams were used the, participant was asked why he or she used a Venn and asked when in his or her mathematical schooling he or she had first been exposed to Venn diagrams and similar graphs. If the student used only set algebra to solve a problem, he or she was asked why he or she had not used Venn diagrams and asked what cues the student used in deciding when to use a Venn or when to use an algebraic method. Also, because there were variations among the participants as to how they drew their respective Venn diagrams, several were asked why they had drawn their Venn diagrams in a particular manner. For example, some students drew a box representing the universal set around their Venn circles while other students did not draw the box around the circles on some or all of their Venn pictures. The researcher asked each of the participants who had not included the box on some of his or her diagrams why he or she had not included the Venn box. If the participant could not recall what steps she or he had taken on a problem, the researcher and the participant would watch the problem session video together and

consequently the participant was often able to explain what steps he or she followed and why.

The notes taken during the problem-solving session were not formalized--that is, the notes were (generally) (a) reminders of questions to ask of the subject at the end of the problem-solving session, (b) impressions the researcher had concerning some aspect of a problem or session, or (c) a reminder to "look-at" some aspect of the session at a future time. The notes are not included with the data--rather, they were used as a memory aid while the researcher was reviewing and analyzing the problem-solving session video data.

Fieldnotes

The researcher's fieldnotes include the audiotape and videotape of participants' verbal and nonverbal activities, and a written record of those individual activities that might not be (fully) recorded by the videotape or audiotape. These notes also record the researcher's thoughts, impressions, and reflections of the participants' activities during the course of the problem-solving sessions and follow-up interviews. The researcher's notes include his thoughts and questions that were written during the problem-solving session and any impressions and thoughts that were noted after each participant had concluded her or his session.

Bogdan and Biklen (1992) advocate fieldnotes containing descriptive and reflective aspects. They recommend that the descriptive aspects of the fieldnotes should encompass the following areas:

1. *Portraits of the subject.* The subject portraits should include descriptions of the subjects demeanor, interview behavior, and mannerisms. The descriptions should also include any aspects of the subject that might set them apart from others or reveal thinking and behavior patterns.

2. *Reconstruction of Dialogue.* The reconstruction of dialogue should contain a record of all non-taped conversations and private dialog between the participants and the researcher. The written record is to supplement and clarify the audiotape and video records and to record any dialog that may have been missed by the electronic recordings. The researcher only included a record of the dialogue in his note if he believed that the recorder had failed to record something because of outside noise or equipment failure.
3. *Accounts of particular events.* The notes of a particular event are a record of who was involved in an event, in what manner they were involved, and a description of the characteristics or nature of the event.
4. *Depiction of activities.* This is to record detailed descriptions of the behavior of the research participants to reproduce the sequence of behaviors, reactions, and the nature of the actions.
5. *The observer's behavior.* Bogdan and Biklen (1992) recommend the following

In qualitative research, the subjects are the people interviewed and found in the research setting, but you should treat yourself as an object of scrutiny as well. Because you are the instrument of data collection, it is very important to take stock of your own behavior, assumptions, and whatever else might affect the data that are gathered and analyzed. ... the descriptive part of the notes also should contain materials on such things as your dress, actions, and conversations with subjects. Although you attempt to minimize your effect on the setting, always expect some impact. Keeping a careful record of your behavior can help assess untoward influences (p. 121).

The field journal also includes a reflective account of the observations. According to Bogdan and Biklen (1992), the "emphasis is on speculation, feelings, problems, ideas, hunches, impressions, and prejudices. Also included is material in which you lay out plans for future research as well as clarify and correct mistakes or misunderstandings in your fieldnotes" (p. 121). They recommend that the reflective account of the journal include the following categories:

1. *Reflections on analysis.* The reflections on analysis contain initial speculation about the patterns and themes that may be emerging from the observations and recorded data. This includes speculation about what the researcher is learning and how various parts of the data and observations may be connected.
2. *Reflections on method.* The reflection on method section contains comments about the (a) procedures and strategies employed in the study's data collection process (b) decisions made about the studies design, and (c) necessary changes made to the design.
3. *Reflections on ethical dilemmas and conflicts.*
4. *Reflections on the researcher's frame of mind.* This portion of the field journal contains comments concerning the researcher's personal opinions and assumptions about the subjects and setting of the study. These reflections were used to control researcher bias and personal assumptions during data analysis.
5. *Points of clarification.* This portion of the notes includes any post problem-solving session information provided by the participants and any corrections that need to be inserted in the researcher's notes or transcripts of the sessions or interviews.

For the purpose of this research, the field notes contained (a) audiotapes of the pre problem-solving and post problem-solving sessions, (b) videotape of the problem-solving session, (c) written notes made by the researcher during the sessions, (d) transcripts of the various sessions (e) written notes of participant's behavior that were made while the researcher viewed the videotapes of the problem-solving sessions, and (f) the researcher's notes concerning the study's methodology.

Data Analysis

The process of constant comparison of data was used during and after the period of data collection. Glasser (1978) enumerates the steps in the constant comparison method of data analysis and theory formulation as follows:

1. Begin collecting data.
2. Look for key issues, recurrent events, or activities in the data that become categories of focus.
3. Collect data that provide many incidents of the categories of focus, with an eye to seeing the diversity of the dimensions under the categories.
4. Write about the categories you are exploring, attempting to describe and account for all the incidents you have in your data while continually searching for new incidents.
5. Work with the data and emerging model to discover basic social processes and relationships.
6. Engage in sampling, coding, and writing as the analysis focuses on the core categories.

Glasser (1978) notes that although the constant comparison method is enumerated above as a series of steps, the method is not linear. That is, the researcher could be involved with several of the steps at any given moment and any step may reoccur as research and analysis proceed.

For this research the database includes the participants' verbal remarks, written algebraic and graphical material, and observable motor actions collected via videotape during problem-solving sessions and during pre and post problem-solving session interviews. The videorecording of each problem-solving session was transcribed and correlated with the subjects' written work. The transcripts were also made of any comments, explanations, and translation information that came from the post session interview. The transcripts of the pre and post problem-solving session were used (when needed) to ensure the accuracy of the transcript of the problem-solving session.

In the transcripts of the students' work (protocols) presented in the Analysis of the Data (chapter four), the left column is a transcript of the students' spoken remarks as they worked the problem. The right column is a transcript of any written work, including any graphs and algebra. Each drawing is correlated with the verbal transcript found immediately

to the left of the drawing. That is, (as closely as possible) each drawing in the right column corresponds with the transcript of the spoken words appearing to the drawings immediate left. Breaks in the subjects' verbalization (quiet times) are distinguished by line spaces, although there is no correlation between the number of line spaces and the length of the quiet time.

All marks (with three exceptions) included on the transcript of the written work are also found on the subjects' written work. The three exceptions are (a) a printed hand is used to indicate that a student pointed to some part of his or her drawing, (b) a stylized calculator indicates that the subject used a calculator, and (c) "underlining" indicates the student made an addition to an existing equation or drawing. When the subject pointed to an item on his or her drawing, a hand is included on the transcript of the written work with a finger indicating the position on the drawing to which the subject was referring. The words spoken by the subject as he or she pointed are found in the transcript immediately to the left of the hand.

All data, numbers, or graphics that the subjects added to their drawings during the problem-solving sessions are included in the transcript. These additions are underlined the first time they appear on the drawing. The figures in the right column of the protocols appear (as closely as possible) identical to the students' written work. In the transcript of the spoken words, a segment number is included in brackets (e.g., [Segment 1]). Insertions are included so that the reader can locate in the protocol any words or phrases referred to in the discussion of the protocol.

Protocol 3.1 is a transcript of Miseon doing problem number two. The numbers in parenthesis inserted in the spoken transcript record the elapsed time. The timing started the moment she finished reading the problem and began working on a solution. (For the sake of readability, these timings are not included in the protocols in chapter four).

Protocol 3.1

Name: Miseon Problem: Two

At Giant State University a survey of students taking college mathematics found that 40% took college algebra, 30% took statistics, and 42% took neither course. If one of the students from the survey is chosen at random, what would be the probability that she or he took either algebra or statistics but not both courses?

[Segment 1]

Okay. (5 seconds)

$$A1 \Rightarrow 40\%$$

I'll go--okay, forty percent took college math.

Thirty percent statistics--thirty percent.

$$St \Rightarrow 30\%$$

Forty-two percent took neither.

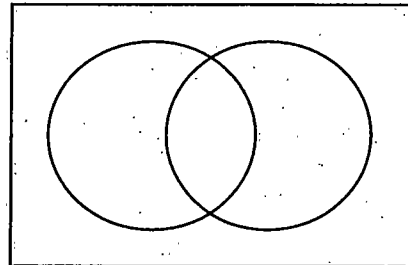
That means like A--S is--.

$$(A \cup S)' \Rightarrow 42\%$$

(35 seconds)

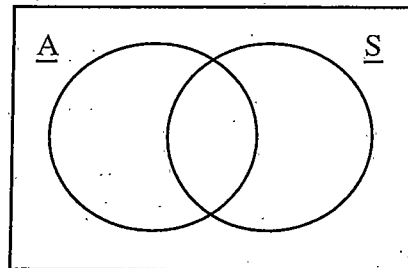
Okay, so the question is--asks.

(40 seconds)

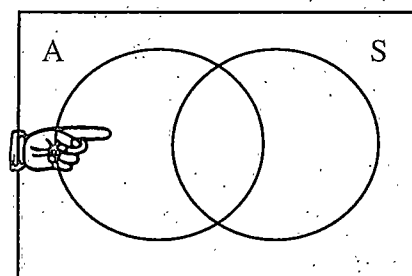


[Segment 2]

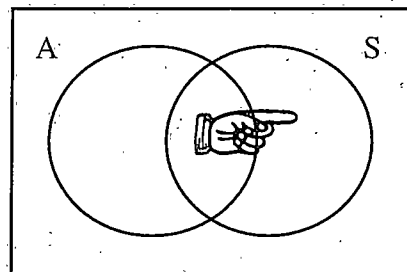
This is algebra and this is statistics.



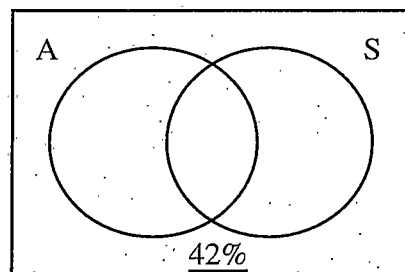
They want to know either Algebra or the Statistics but not both courses, meaning that you want to have--this side



or this side. (60 seconds)



So this is forty-two percent.



(1 minute 5 seconds)

[Segment 3]

So uhh all this total should be one hundred minus forty-two. I think fifty-eight percent, right. (1 minute 25 seconds)

$$100 - 42 = 58\%$$

Like total A S is A plus S minus--uhh S.

$$A \cup S = A + S - A \cap S$$

So fifty-eight equals forty plus thirty minus--.

$$58 = 40 + 30 - X$$

Okay, X will be--

$$X =$$

seventy minus fifty-eight.

$$\begin{array}{r} 70 \\ - 58 \\ \hline 12 \end{array}$$

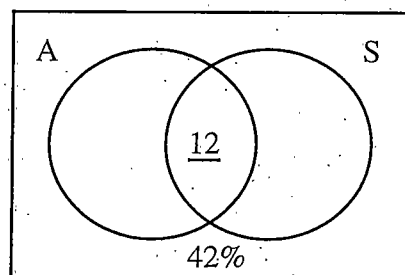
Umm--Okay.

Twelve.
(1 minute 50 seconds)

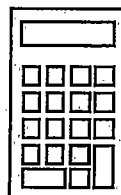
$$X = \underline{12}$$

[Segment 4]

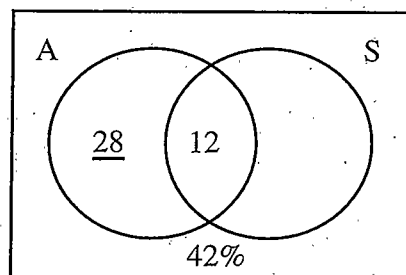
We know that's--this is twelve.



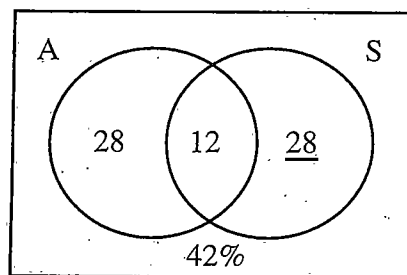
Like, so what we do is subtract forty
[laugh] minus twelve is like--



twenty-eight.



Thirty minus twelve is eighteen.



(2 minutes 5 seconds)

[Segment 5]

So like--either Algebra or Statistics--so you just have to add it.

Twenty-eight plus twenty-eight--fifty-six!

$$28 + 28 = 56$$

(2 minutes 20 seconds)

(End of Protocol 3.1)

The videotapes, transcripts, and students' written work were analyzed to determine (a) what representations the students used in each solution attempt, (b) the beginning and ending times of each translation task, (c) whether or not the students could successfully translate between representations, and (d) whether or not the students were competent using the representations. Each protocol's representation and timing information [(a) and (b) above] was coded onto a time-line. An example of a time-line is given in Time-line 3.1, included below, which relates to protocol 3.1. An explanation of the procedures used to develop the time-lines is included in Appendix A, and an example of the parsing of Protocol 3.1 is included in Appendix B. The time-lines (and the method of parsing the protocols) are an adaptation of those developed by Schoenfeld (1985) in his research identifying Polya's problem-solving steps. The protocol tables identify each student's activities in five second intervals. Any change recorded in a student's behavior is rounded to the nearest five second timing break.

The time-lines are used to identify: (a) any representations the student used, (b) patterns in the use of the representations, and (c) time patterns in the student's translations. The tables from the various participants are compared in chapter four, with a particular focus on identifying, describing, and comparing the behavior patterns of successful and unsuccessful problem solvers.

Time-Line 3.1

Student: Miseon

Problem: Two

Comments: Competent, Incorrect Answer (subtraction error on final step of the problem)

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	0:	:05	:10	:15	:20	:25	:30	:35	:40	:45	:50	:55	1:00	1:05	1:10	1:15	1:20	1:25	1:30

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	1:30	1:35	1:40	1:45	1:50	1:55	2:00	2:05	2:10	2:15	2:20	2:25	2:30	2:35	2:40	2:45	2:50	2:55	3:00

Note. Context refers to the contextual (English) representation, Venn refers to the Venn diagram representation, and Symbolic refers to a symbolic representation. $C \leftrightarrow V$ indicates a translation between the contextual mode and a Venn diagram. $V \leftrightarrow S$ indicates a translation between a Venn diagram and a symbolic mode, and $C \leftrightarrow S$ indicates a translation between the contextual mode and a symbolic mode. The times are in five second intervals.

Time-line 3.1 indicates that Miseon started her attempt by using 5 seconds analyzing the problem in context, that is, she first studied the problem as an English statement. Miseon then spent 30 seconds translating the problem into symbolic notation, as is seen in the first portion of Protocol 3.1. She then returned to the contextual problem statement (5 seconds) and translated information from the problem into a Venn diagram, taking her 25 seconds. Miseon then spent 20 seconds translating information from the Venn diagram into symbolic algebra, building an algebraic equation. She then spent 25 seconds solving these equations for the unknown amounts and 15 seconds more transferring (translating) this information back into her Venn diagrams. Miseon saw she had the information necessary to answer the problem and then spent 15 seconds translating the information from the Venn diagram into the mode in which the problem was stated.

Research Questions:

The researcher entered the study with four research questions, two that were considered "Primary" and two "Secondary" questions. Primary research questions focus on the mechanics of students representation use, whereas secondary questions focus on the role of representation use in problem solving.

Primary Research Questions:

1. Is there a difference in successful and unsuccessful students' use of multiple representations to solve probability problems?
2. Is there a difference in the patterns exhibited by successful and unsuccessful students with regard to their translation between representations?

Secondary Research Questions:

3. With regard to the problem-solving process as a whole, what benefits (if any) do students gain through the use of multiple representation?
4. What difficulties (if any) do students encounter in their use of multiple representations?

CHAPTER FOUR

REVIEW AND ANALYSIS OF THE DATA

Overview

This chapter begins with a review of data collected from twenty-one students attempting to solve five probability problems. The data are used to develop definitions describing the students' problem-solving skills and translation activities. Two groups of problem solvers are identified ("competent" and "error-prone") and described. Tables enumerate the number of correct and incorrect solutions (itemized by both student and problem number) and include related details about the students' solutions. The chapter continues by describing the translation and problem-solving activities of the competent and error-prone problem solvers and contrasting the work of these two groups. In particular, the were are examined for patterns of behavior that characterize students' use of translations and multiple representations.

Introduction to the Data

The following summarizes the types of answers given by twenty-one students attempting five probability word problems. The students, data gathering methodology, and student selection process are described in chapter three. The data for problems one and three were provided for eighteen students instead of for twenty-one. (Questions one and three were changed after the data from the first three students were collected.)

For the purpose of this research, being unable to answer a question is distinct from answering a question incorrectly. An incorrect answer is one in which the student wrote or verbalized an answer but the numeric portion of the answer was incorrect. However, if a student did not finish a problem or did not give an answer that included a number, either

written or spoken, then he or she was recorded as having a DNF (did-not-finish). Further, the term not solved the problem refers either to an answer that is incorrect or to a problem that was not finished.

Table 4.1 categorizes by correct or incorrect the answers given by each student. The right column of the table lists (from left to right) the number of correct answers, incorrect answers, and DNFs provided by each student. For example, an entry of (3,2,1) indicates the student answered three questions correctly, two questions incorrectly, and did not finish one problem. The bottom row lists the total number of correct, incorrect, and incomplete solutions given by the students for each problem.

Table 4.1 indicates that no student correctly answered all five questions. Notice, however, Kelli is recorded as having no incorrect answers and no problems she failed to finish (i.e., DNF). It is recorded as such because the data for problems one and three were not included for Kelli, George, and Miseon (as explained in chapter three). Although Kelli's answers for questions one and three are not recorded, she did correctly answer a different version of the two problems. Likewise, both George and Miseon correctly answered a different version of problems one and three. Thus (if the unrecorded versions were included) of the twenty-one students, three had no correct answers, four had one correct, four had two correct, four had three correct, five had four correct, and one had all five questions correct.

There were seven students who missed four or more of the problems, that is, had either none or only one problem correct. Of these, Don and Mindi each had four DNFs. Bob, with four incorrect, and David, with three incorrect, each had only one problem he failed to finish. Thus, for the students missing four or more of the problems, there is a large variation between the number of incorrect solutions and the number of problems not completed.

Problem three appears to be the most difficult with fifteen of the twenty-one

students missing the problem. Five of the students did not finish problem three, and ten students gave an incorrect solution. However, the students also had difficulty with both problems two and four. Problem two had the highest number of DNFs (7) with a total of eleven students not solving the problem. Thirteen students were not able to solve problem fourteen.

Table 4.1. Answers Given by Problem and Student

Student:	Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	C,I,DNF
Arnie	DNF	DNF	Incorrect	Incorrect	Incorrect	(0,3,2)
Bob	Incorrect	DNF	Incorrect	Incorrect	Incorrect	(0,4,1)
Chad	Correct	Correct	Correct	Incorrect	Correct	(4,1,0)
David	Correct	Incorrect	Incorrect	DNF	Incorrect	(1,3,1)
Deb	Incorrect	Incorrect	Incorrect	Correct	Correct	(2,3,0)
Don	DNF	DNF	Correct	DNF	DNF	(1,0,4)
Erin	Correct	Correct	DNF	Incorrect	Correct	(3,1,1)
George	—	Correct	—	Incorrect	Correct	(2,1,0)
Greg	Correct	DNF	Correct	Incorrect	Correct	(3,1,1)
Jack	Incorrect	Correct	Incorrect	Incorrect	Correct	(2,3,0)
Karen	Correct	DNF	Incorrect	Correct	Correct	(3,1,1)
Kelli	—	Correct	—	Correct	Correct	(3,0,0)
Kyle	Incorrect	Correct	DNF	DNF	DNF	(1,1,3)
Liz	Correct	Incorrect	DNF	Incorrect	DNF	(1,2,2)
Lori	Correct	DNF	Incorrect	Correct	DNF	(2,1,2)
Marty	Correct	Correct	DNF	Correct	Correct	(4,0,1)
Meredith	Correct	Correct	Incorrect	Correct	Correct	(4,1,0)
Mike	Incorrect	Correct	Incorrect	Incorrect	Correct	(2,3,0)
Mindi	DNF	DNF	DNF	DNF	Incorrect	(0,1,4)
Miseon	—	Incorrect	—	Correct	Correct	(2,1,0)
Sue	Incorrect	Correct	Incorrect	Correct	Correct	(3,2,0)
TOTALS	(9,6,3)	(10,4,7)	(3,10,5)	(8,9,4)	(13,4,4)	(43,33,23)

Note. (n = 21) DNF indicates the student did not finish the problem. Dashes indicate data were not recorded. Amounts in parenthesis record numbers of problems correct, incorrect, and not finished, respectively.

Performance-Based Definitions

Based upon an analysis of the students' solutions and the data in Table 4.1, two classes of problem solvers were defined.

1. *Competent* problem solvers are able to analyze a problem, develop an appropriate solution strategy, translate between representations (modes) accurately, and operate within the representations competently. As a result, competent problem solvers are able to analyze a problem, find a method to solve the problem, choose and use appropriate translations, and, but for minor errors, find a correct solution. The competent students use translations and representations in a manner appropriate to the problem and can quickly and accurately translate between the various representations.

It should be noted that competent problem solvers are so labeled because of their ability to use the mathematical tools and concepts available to them in a competent manner, not because of their ability to always find a correct solutions to the problems. Indeed, the fact that competent problem solvers can translate between representations and operate within the various representations enables each of them to find correct solutions. However, the competent and the error-prone students both made "slips" (Norman, 1981). That is, individuals from either group made errors in arithmetic, copying, or reading the problem.

2. *Error-prone* problem solvers have less success (than the competent students) at solving problems because they are unable to (a) develop or implement an appropriate problem-solving strategy, (b) complete the necessary translations, or (c) perform the necessary arithmetic or algebraic operations while working within the various modes. That is, error-prone students often have difficulty translating the English statement of the problem into a Venn diagram and often have difficulty translating information from a Venn or English representation into an algebraic statement. Also, the error-prone students may be unable to operate successfully within the various representations. For instance, several of the

students had difficulty performing algebraic operations (e.g., Karen's Protocol 4.3) and, as a result, had difficulty finding numeric values representing regions in the Venn diagrams. Further, there is a distinct subgroup of the error-prone students that appears unable to successfully analyze the English representation of the problem(s). Consequently, the members of this subgroup are unable to translate the problem from English into any other representation. For the purpose of clarity, the error-prone students are not strictly the same group as the set labeled unsuccessful. Several of the unsuccessful students were not error-prone, rather, they were careless in their work or unconcerned about getting a correct answer. To summarize, the error-prone students are classified as such because each is prone to making errors in translation, in mathematical manipulations, or in problem analysis, not because their answers are in error. In fact, incorrect answers are caused by the student's inability to complete a translation, by mistakes made during a translations, and, to a lesser degree, by mistakes made while operating in some non-English mode.

Solutions Data by Subject Class

This section looks again at the solution attempts of the twenty-one students; however, the data are classified into sets of error-prone and competent problem solvers. These two classes were formed through analysis of each student's work.

As before, being unable to answer a question correctly is distinct from answering a question incorrectly. An incorrect answer is one in which the student wrote or verbalized an answer, but the numeric portion of the answer was incorrect. However, if a student either did not finish a problem, did not produce an answer that included a number (the number could be either written or spoken), or expressed that he or she did not understand some portion of a problem, then he or she was recorded as having a DNF (did-not-finish). Being unable to finish a question does not imply that the student did not understand how to approach the problem or that her or his method was incorrect. Cases were recorded wherein a student was

unable to complete some particular translation or manipulation, which prohibited her or him from finishing. However, it was clear that she or he understood how to approach the problem and would have found a correct solution, but for an inability to finish steps in the problem (e.g., Protocol 4.6).

The following two tables display the type of answer (correct, incorrect, or DNF) given on each problem for each student in the two classes of problem solvers.

Six of the twenty-one students are classified as competent. Only one of the competent students did all five of the problems correctly but none of the six missed more than one question. Also, only one of these students was unable to give an answer and that was only on one problem.

Table 4.2: Types of Answers Given by Competent Students

Student	Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	C,I,DNF
Chad	Correct	Correct	Correct	Incorrect	Correct	(4,1,0)
George	—	Correct	—	Incorrect	Correct	(2,1,0)
Kelli	—	Correct	—	Correct	Correct	(3,0,0)
Marty	Correct	Correct	DNF	Correct	Correct	(4,0,1)
Meredith	Correct	Correct	Incorrect	Correct	Correct	(4,1,0)
Miseon	—	Incorrect	—	Correct	Correct	(2,1,0)
TOTALS						(19,4,1)

Note. (n = 6) DNF indicates the student did not finish the problem. Dashes indicate the problem was not recorded. Amounts in parenthesis record numbers of problems correct, incorrect, and not finished, respectively.

From Table 4.2 it is seen that the competent students answered 79% of the problems correctly (19 of 24 questions), 17% incorrectly (4 of 24), and were unable to answer a question only 4% of the time (one question of the twenty-four). So, 21% percent of the time competent students either did not give a solution or gave an incorrect answer.

As shown in Table 4.3, fifteen of the twenty-one research students are classified as error-prone. Three of the fifteen were unable to answer any of the five problems correctly,

four were able to answer one problem correctly, four answered two problems correctly, and four students answered three of the problems correctly. None of the error-prone students were able to correctly answer more than three of the five questions.

Table 4.3 shows that the error-prone students correctly answered 32% of the questions (24 of 75 questions), 39% incorrectly (29 of 75), and were unable to answer 29% of the questions (22 of 75). We see then that the error-prone students either did not finish a problem or gave an incorrect answer 68% of the time (29 + 22 of 75 questions).

Table 4.3: Types of Answers Given by Error-Prone Students

Student	Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	C,I,DNF
Arnie	DNF	DNF	Incorrect	Incorrect	Incorrect	(0,3,2)
Bob	Incorrect	DNF	Incorrect	Incorrect	Incorrect	(0,4,1)
David	Correct	Incorrect	Incorrect	DNF	Incorrect	(1,3,1)
Deb	Incorrect	Incorrect	Incorrect	Correct	Correct	(2,3,0)
Don	DNF	DNF	Correct	DNF	DNF	(1,0,4)
Erin	Correct	Correct	DNF	Incorrect	Correct	(3,1,1)
Greg	Correct	DNF	Correct	Incorrect	Correct	(3,1,1)
Jack	Incorrect	Correct	Incorrect	Incorrect	Correct	(2,3,0)
Karen	Correct	DNF	Incorrect	Correct	Correct	(3,1,1)
Kyle	Incorrect	Correct	DNF	DNF	DNF	(1,1,3)
Liz	Correct	Incorrect	DNF	Incorrect	DNF	(1,2,2)
Lori	Correct	DNF	Incorrect	Correct	DNF	(2,1,2)
Mike	Incorrect	Correct	Incorrect	Incorrect	Correct	(2,3,0)
Mindi	DNF	DNF	DNF	DNF	Incorrect	(0,1,4)
Sue	Incorrect	Correct	Incorrect	Correct	Correct	(3,2,0)
TOTALS						(24,29,22)

Note. (n = 15) DNF indicates the student did not finish the problem. Amounts in parenthesis record numbers of problems correct, incorrect, and not finished, respectively.

Tables 4.4 and 4.5, respectively, show the number of correct, incorrect, and DNFs for each problem by the competent and error-prone problem-solvers. Again, we see that problem three appears to be the most difficult for the students. Of the two competent students who missed the problem, one gave a wrong answer and the other gave no answer.

However, thirteen of the fifteen error-prone students missed this problem. Nine of the fifteen answered incorrectly and four were unable to finish the problem.

Table 4.4: Answers Given by Competent Students
(n = 6)

Question #	Correct	Incorrect	DNF
Question 1	3	0	0
Question 2	5	1	0
Question 3	1	1	1
Question 4	4	2	0
Question 5	6	0	0
TOTALS	19	4	1

Four of the six competent students had one incorrect solutions, and one student had a DNF. None of the six missed more than one, and one student missed none of the problems.

It should be noted here that although the number of incorrect or DNF solutions given by the competent students may appear high (5 of 24 problems not solved), their wrong answers are very different in nature from those of the error-prone students. How and why these answers are different will be discussed in later sections.

Table 4.5: Answers Given by Error-Prone Students
(n = 15)

Question #	Correct	Incorrect	DNF
Question 1	6	6	3
Question 2	5	3	7
Question 3	2	9	4
Question 4	4	7	4
Question 5	7	4	4
TOTALS	24	29	22

Notice that eleven of the fifteen error-prone students were unable to find an answer to one or more of the five problems, and fourteen answered one or more question(s) incorrectly. Typically, error-prone students giving incorrect answers either (1) have some

misunderstanding about the facts and relationships expressed in the problem statement, (2) make a translation error, or (3) make an arithmetic or algebraic mistake; however, each of these students did give a numerical solution, albeit incorrect. Twelve of the error-prone students correctly answered one or more question(s) but were unable to find an answer to other problems.

Characteristics of Competent Problem Solvers

The examples and analysis that follows purposes to (a) identify and characterize successful and unsuccessful students' use of (a) multiple representations and (b) translations between representations during problem solving. This is done in order to answer the following questions:

1. Is there a difference in successful and unsuccessful students' use of multiple representations to solve probability problems?
2. Is there a difference in the patterns exhibited by successful and unsuccessful students with regard to their translation between representations?
3. With regard to the problem-solving process as a whole, what benefits (if any) do students gain through the use of multiple representation?
4. What difficulties (if any) do students encounter in their use of multiple representations?

This section seeks to identify and describe various characteristics demonstrated by the competent problem solvers, specifically, the successful students' use of multiple representations and translations. Included are several detailed examples demonstrating these characteristics and the students' problem-solving techniques.

In particular, this research focuses on the following attributes demonstrated by competent problem solvers:

1. An ability to make translations between the various representations accurately and efficiently.

2. A clear understanding of the type of information that will be revealed by a translation and consequently, a realization of the need for a translation and new representation.
3. The realization of the nature of the connection and relationships between the various representations. In particular, that a translation does not change data and relationships, but rather, that translations and representations are used to reveal relationships and categorize data.
4. The ability to refocus their attention from the "big picture" onto "smaller" aspects of a problem until they gained from the smaller piece some insight or information that is needed to solve the larger problem.
5. A realization of the relevance and non-relevance of data, relationships, and facts expressed in the problem statement or revealed by a translation or representation.

Examples of Competent Characteristics:

Competent problem solvers display an ability to make translations between the various representation modes efficiently (and generally, quickly) with few errors. They view the Venn diagram as a tool to assist them during problem solving and see algebra and symbolic notation as tools to use while completing the Venn diagrams. It should be noted here that, in both the competent and error-prone groups, algebra and symbolic notation are not one tool, but rather, two distinct but (often) inseparable tools. As will be seen in the examples that follow, symbolic notation is used when translating between the contextual and Venn diagrams or used while solving algebraic equations. However, algebra is a tool using both symbols and algebraic steps to solve for an unknown quantity. In the examples, the unknown quantity is most often an unknown value represented by a region of a Venn diagram. The two tools are generally indistinguishable (to an outside observer) because, in most instances, the act of translating to the symbolic mode includes developing a symbolic algebraic relationship. However, in the error-prone students' protocols, a different kind

translation act was observed: a contextual to symbolic translation in which the students "symbolized" the nouns referred to in the problem statement. During this "symbolization step," symbols were assigned to each of the objects mentioned in the problem statement. This symbolizing process often took place while the student was translating the problem statement into a Venn diagram. The symbols were assigned to the regions of the Venn diagram and also used in the algebra equations that were used to find the values represented by the Venn diagram regions.

Regarding attribute three, the competent students made very strong connections between the representations. The strong connections allowed them to realize that a logical contradiction on a Venn diagram or in an algebraic equation implies that they had made an error either during a translation or while operating in an earlier mode.

Notice that the fifth characteristic is closely related to the second and forth characteristic. In particular, the competent students realized how a translation would help in solving a problem, what data would be revealed by the translation, and how the data would be categorized by a translation. They were able to take their focus off the larger problem to refocus on some particular aspect(s) of the problem (such as a particular piece of a Venn diagram) and, while doing so, were able to see what facts and data were relevant to that piece of the problem.

Instances of the ease of translation and the competent use of the various modes are found in nearly every one of the competent students' work. As an example, consider Kelli's use of the various modes as she solves problem five. In this protocol, Kelli wrote out almost every algebraic steps needed to solve the problem. However, many of the competent students were so comfortable with algebra and solving Venn diagrams that they were able to minimize their written algebra by (a) skipping steps or (b) doing much of the work in their head. Notice Kelli's efficiency as she translated between the various modes (characteristic one), how she appeared to understand why and when she should use a translation

(characteristic two and five), and how easily she change her focus from the larger problem to the algebra when needed (characteristic four).

Notice also Kelli's remark as she begins the problem [segment one] "Okay, again a little Venn diagram." Kelli knew that she needed a Venn diagram, and, she knew why she needed a Venn diagram. She was aware that the Venn diagram would aid her in categorizing and describing the data in a manner that would lead to an answer, demonstrating characteristics two and five.

As a reminder, the protocols are a transcripts of the students work and words, only. None of the words recorded were spoken by the researcher. The left column is a transcript of the student's spoken words and the right column is of their written work. Chapter three provides a detailed explanation of the protocols.

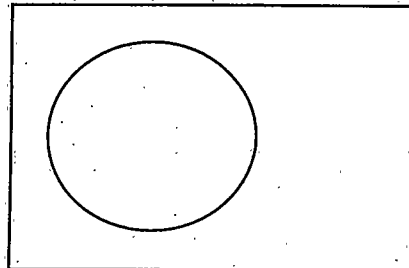
Protocol 4.1

Name: Kelli Problem: Five

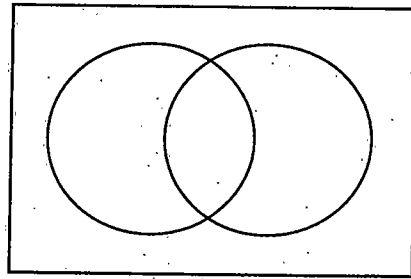
A securities analyst is reviewing the performance of a group of computer manufacturers. She finds that 75 percent have increased sales and 30 percent have increased earnings. She also finds the 15 percent have increased neither sales nor earnings. One manufacturer is selected at random from the group. What is the probability that it has increased both sales and earnings?

[Segment 1] Okay.

Okay, again a little Venn diagram.
So this one will be sales.



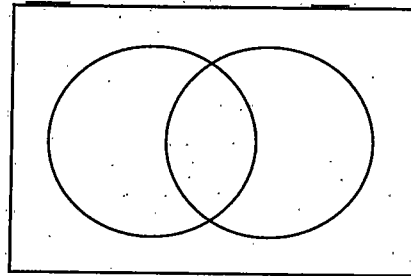
And this one will be earnings.



So sales, earnings.

$S =$

$E =$

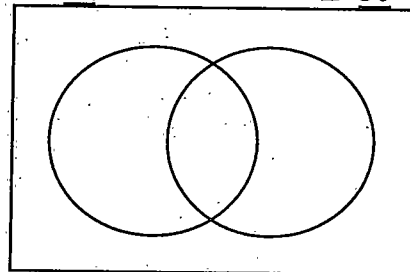


[Segment 2]

Seventy-five percent have increased on their sales and thirty percent have increased on their earnings.

$S=75$

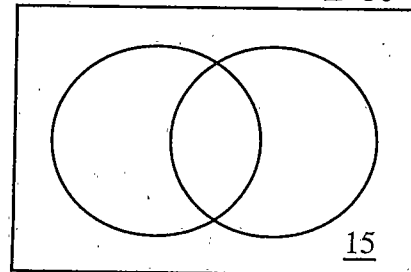
$E=30$



She finds that fifteen percent have increased for neither of them so that goes right there.

$S=75$

$E=30$

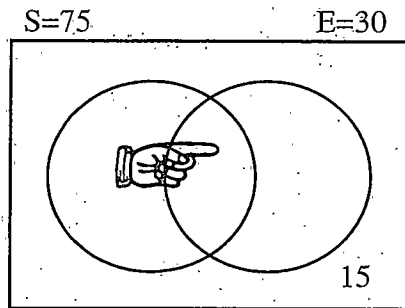


[Segment 3]

One manufacture is selected at random. What is the probability that they have increase both sales and earnings?

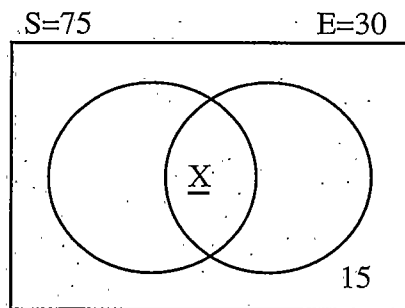
[Segment 4]

So you want how--what's the probability of its falling? uhh in this category right here.

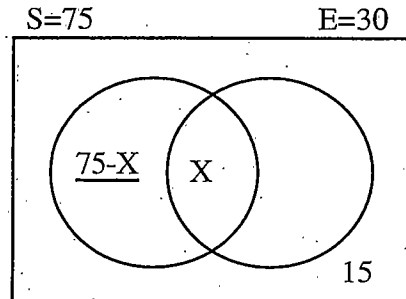


[Segment 5]

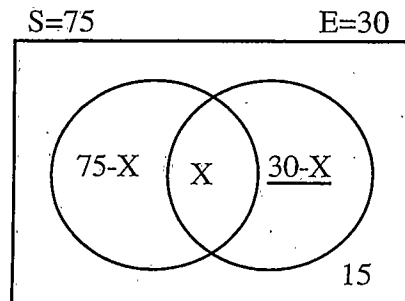
So again, you just have to make that X.



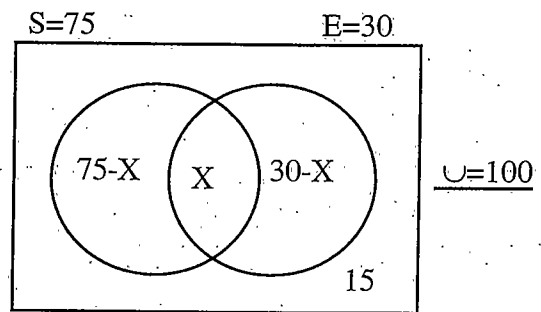
And then seventy-five minus X



thirty minus X,



and since this union will equal one hundred you have to have--umm.



[Segment 5]

So one-hundred will have to equal this fifteen plus seventy-five minus X plus X plus thirty minus X.

$$100 = 15 + 75 - X + X + 30 - X$$

And then those cancel out each other.

$$100 = 15 + 75 - \cancel{X} + \cancel{X} + 30 - X$$

One-hundred equals fifteen plus seventy-five plus thirty.

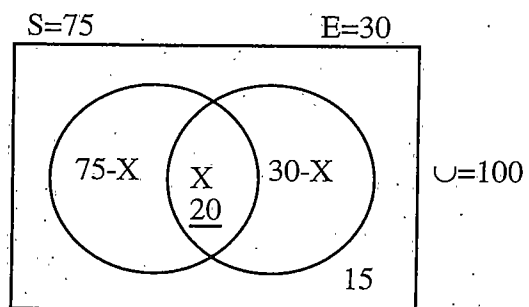
A hundred--and twenty minus X.

$$100 = 120 - X$$

And then add X here, add X here and subtract a hundred, subtract a hundred.

$$\begin{array}{r} 100 = 120 - X \\ +X - 100 \quad -100 + X \\ \hline \end{array}$$

And X equal to twenty.



[Segment 6]

So it's just twenty.

So X equals twenty.

$$X = 20$$

(End of Protocol 4.1)

Kelli completed this problem quickly and with little hesitation. She translated the English statement into a Venn diagram accurately and efficiently [Segments 1 and 2] and she was able to immediately see what region of the Venn was needed [Segment 3] to solve

the problem (demonstrating characteristic one). She used symbolic algebra accurately, competently, and efficiently [Segment 5] as needed to solve the Venn diagram regions. Notice also that once Kelli started her algebra solution she focused entirely on the algebra until she had a solution, that is, a value for her X variable. She then transferred the algebraic information back to the Venn representation and only then did she refocus her attention [Segment 5 and 6] to solving the larger problem, demonstrating both characteristics one and four.

In Protocol 4.1, and in other competent students' problems, it is seen that the students understood the connections between and among the English, Venn, and symbolic representations. More than just seeing that a connection exists between the various modes, these students understood the relationships between the various modes. These students understood that a Venn representation allows the data to be organized or categorized in a manner that helps one to "see" data relationships in the information presented in the contextual mode. Further, these students were confident that their algebraic steps were correct and were comfortable using symbolic notation. Consequently, the competent students translated competently and confidently between the various modes.

Because competent problem solvers have such a good understanding of the connections between and among the representations, they realize that (if the translation is performed correctly) the modes represent the same information but one representation may present data or information in a clearer manner than another. For this reason, if a competent problem solver (1) completes a translation and begins operating in the new representation but (2) finds a statement that appears illogical or incompatible with other information he or she (3) will then conclude that either a mistake had been made in translation or that there is an error in her or his analysis of the problem. The competent problem solver will then work to find the error and not continue with attempts to solve the problem until he or she has resolved the apparent contradiction.

Chad's solution to problem one (Protocol 4.2) demonstrates a competent problem solver's use of translation (characteristics one and two), error analysis and correction procedure (characteristic 3), and again demonstrates how the competent student is able to focus attention away from the larger problem and onto a portion of the problem (characteristic 4). Notice also that both Kelli and Chad have a clear understanding of the relationships given in the problem and consequently are both able to quickly analyze the problem and see a solution method.

Protocol 4.2:

Name: Chad Problem: One

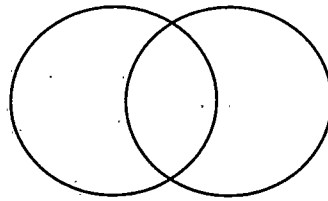
In the evening, between 7 and 9 PM, Ann's phone is not busy 75% of the time, John's phone is busy 35% of the time, and both of their phones are busy 10% of the time. What is the probability that neither of their phones will be busy at some moment during this part of the evening?

[Segment 1]

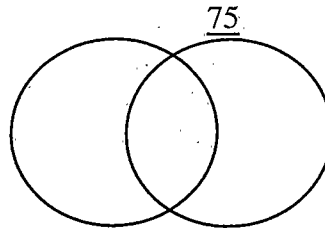
All right. So, when I do this I'm looking for—that neither of the phones will be busy. So I know that Ann's phone is not busy seventy-five percent of the time.

[Segment 2]

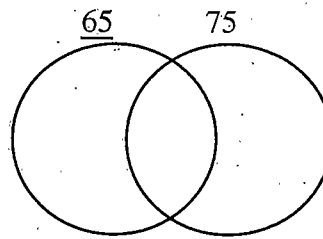
I'm going to do the circle diagram—the Venn thing—and I'm just going to put—



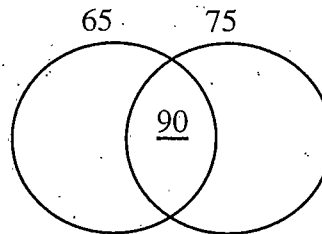
I know seventy-five's in this circle.



And I know his, John's, is busy at thirty-five percent so I know its not busy at sixty-five.



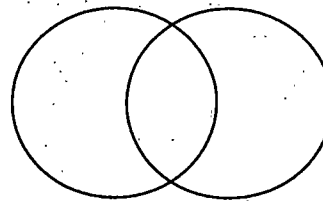
And then, both of their phones are busy at ten percent of the time so their not busy at ninety percent. That will go in the middle.



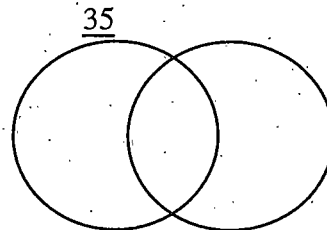
That doesn't work. Hold on.

[Segment 3]

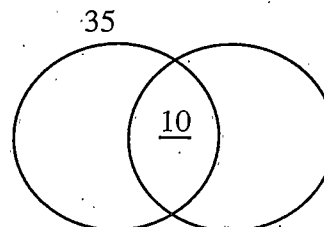
Okay, I'm going to switch this around.



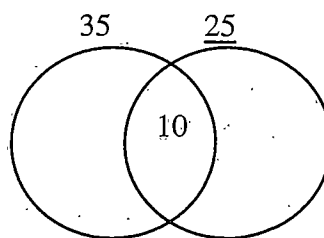
Okay, I'm going to switch it around and make this thirty-five percent over here.



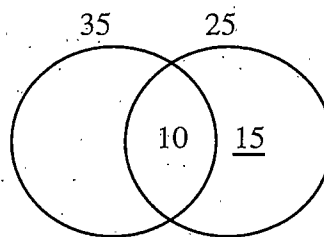
Both are busy at ten percent—that's in the middle.



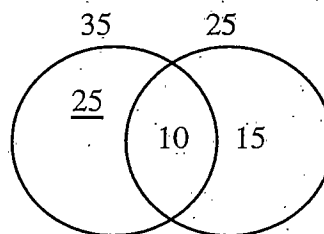
And, hers, Ann's, is not—is busy twenty-five percent.



So, I know—I know that Ann's is busy fifteen percent of the time



and John is not—John's isn't at all. And his is busy twenty-five percent of not all so—.



[Segment 4]

Okay, so you add all that together—. I want to find the probability that neither of the phones were busy at some moment during this part of the evening so to find that I'm just going to find out what's the probability—what percent will be and just add all this together which is twenty-five, ten, fifteen which is fifty.

$$\begin{array}{r} 1 \\ 25 \\ 10 \\ 15 \\ \hline 50 \end{array}$$

So, it's one minus fifty percent. The answer is fifty percent.

$$1 - 50\% = 50\%$$

[Segment 5]

I'll just write it out. Probability not busy—.

$$\Pr(\text{Not busy}) = 50\%$$

That's figured out.

(End of Protocol 4.2)

Notice that Chad was very positive about his need to draw a diagram. He realized immediately that the Venn would allow him to categorize the information needed for the solution. However, he did not draw a box around his Venn circles. This failure may have mislead him on his first attempt at solving the problem in that he did not realize the universe of possible outcomes includes the percentages for "busy" and "not busy." Other students that didn't place the Venn circles in a box representing the outcome universe also had difficulty solving those problems.

Chad's initial approach to solving the problem [Segment 1 and 2] was essentially correct. However, as he filled out the Venn diagram with the "not busy" percentage, he realized that the numbers he had placed in the various regions were not compatible with the "90%" that he thought should go into the Venn diagram's region of intersection. He saw that the 90% he had placed in the Venn intersection was not possible if the two circles had the values seventy-five and sixty-five. He then stated "That doesn't work. Hold on." [Segment 2]. He then redrew the Venn diagram [Segment 3] and changed the Venn circles to represent the percentages for "busy phones" instead of "not busy phones." Chad continued working until he solved the Venn diagram, found a correct answer, and translated his answer back into English.

Chad is very efficient and competent with the translations, in fact, so much so that when he saw the contradiction and then realized there was a more appropriate Venn representation for the problem he didn't hesitate to stop, change the representations in his Venn diagrams, and re-translate the English information into the new Venn. Chad saw the contradiction because he had a clear understanding of the type of information that would be revealed by the Venn diagram. On noticing the contradiction, he almost immediately saw that a different representation of the Venn diagram would more appropriately represent the information needed for his solution method. Chad is so comfortable with the process of

solving the Venn diagrams and the related algebra that he saw no need to write out the equations. Instead, he did the algebra in his head while verbalizing the needed numbers.

Notice that Chad realizes the values given in the problem (35 and 25) do not represent the values inside the Venn regions, which is the reason he writes these values [Segment 3] on top of the circles instead of inside the circles. Like was seen in Protocol 4.1, once Chad begins to perform the algebraic operations [Segment 4] he focuses his complete attention on only that aspect of the problem and continues to do so until he finds values for all of the regions of the Venn. However, unlike with Kelli, it is not clear that Chad saw initially what regions of the Venn he needed for the solution. This may be the reason he solved for all the regions of the Venn. Or, he may have realized that he needed the outside regions of the circles and that he would have to solve the entire Venn to find those values.

Many competent problem solvers become so focused on finding the mathematical solution to a problem that they do not finish the problem with their answer stated in the contextual mode, that is, as an English statement. Rather, they quit working, believing they are done with the problem, as soon as they find a number that symbolically represents the solution without translating the number into the contextual mode. As an example of this, notice how Kelli (Protocol 4.1) presents her solution. Kelli stated "So X equals twenty" [Segment 6] rather than stating "The probability of the manufacturer has increased sales and earnings is twenty percent." Also, when Chad first stated his solution (Protocol 4.2) he said [Segment 4], "The answer is fifty percent," but he didn't write anything. Then, after a pause [Segment 5], he stated, "I'll just write it out. Probability not busy." However, he didn't write the word "probability." Rather, he uses the mathematical symbol "Pr" for the probability. These examples may indicate that competent students construct such strong connections between representations that they believe they have answered the question appropriately using symbolic notation. That is, they believe there is no need to translate back

into the English representation because, to them, the symbolic or algebraic representation portrays the information just as clearly as does a contextual statement of the solution.

During problem solving, competent problem solvers often refocus their attention for a period of time on some selected aspect or portion of the problem in an attempt to understand that particular piece of the problem. In general, to gain an understanding of the problem segment, the competent problem solver translates some portion of the problem into a different representation, such as symbolic algebra, in an attempt to clarify, classify, or solve (i.e., find a value) for some part of the problem.

The previous examples serve to demonstrate two aspects of the use of representations that characterize the competent students: (a) The competent problem solver translates all or some appropriate portion of the problem from one mode into another mode, performs some operations or analysis in the new mode, and after obtaining the needed understanding of the portion of the problem that was translated, translates the new information back into the original mode and attempts to integrate this new knowledge into the larger problem. However (b), if the competent student performs a translation but then concludes that the new representation is not useful in analyzing or solving the problem, he or she will then return attention to some earlier representation, make a new translation, and use a different representation to help solve the problem. This second characteristic was demonstrated by Chad in Protocol 4.2.

These two characterizations support research by Lesh, Post, and Behr (1987) in which they reported (during problem solving sessions) "students seldom worked through solutions in a single representational mode. Instead students frequently use several representational systems, in series and/or in parallel, with each depicting only a portion of the given situation" (p. 37). Lesh and his colleagues report that while students were solving word problems their solution paths often "weave back and forth among several representational systems, each of which typically is well suited for representing some parts

of the situation but is ill suited for representing others" (p. 38). This is clearly seen in the various students' use of Venn diagrams to categorize the numerical data from the English statement of the problem and with their use of algebra to solve for the values in Venn diagrams. However, the students did not try to represent the problem statement algebraically or with a Venn, rather, they used the representations to analyze and categorize the data given in the problem statement.

It should be noted that in many cases the competent problem solvers were able to translate and analyze a portion of a problem without writing out any of the translated information, that is, without displaying any written work in the new mode. In these cases, the students verbalized why the translation and new representation was needed, arrived at a mathematical decision or obtained a numeric value with a calculator (or other means), and then translated back to the original mode. These steps were all accomplished without the students writing down an algebraic equation or making a diagram in the new mode. In fact, competent problem solvers are so comfortable doing algebra and solving the Venn circles that often they will only verbalized their work and not write out their algebraic manipulations. This is demonstrated in Protocol 4.2 [Segment 4]; Chad is able to analyze the Venn diagram without writing any symbolic notation and arrives at the needed values by using addition rather than by solving an algebraic equation.

Competent problem solvers are able to separate the relevant facts from the irrelevant facts within the problem statement as they analyzed the problem. These students generally appear to understand the details given in the problem statement and are able to see the logical consequences and relationships demanded by the facts stated in the problem statement and consequently are able to discover or visualize a possible solution method before translating the problem or some portion of the problem into a non-contextual representation. That is, competent problem solvers appear to be cognizant of what information will be revealed by any given translation. The competent problem solver might

not know the details or the exact numbers that will be revealed by the translation, but he or she does know the type of information or character of the data that will be revealed by the translation. Consequently, the competent problem solver chooses a translation because he or she knows that a particular mode will classify or categorize the information from the problem in a manner useful for finding a solution.

Characteristics of Error-Prone Problem Solvers

This section describes various characteristics demonstrated by the error-prone problem solvers. Included are several detailed examples demonstrating these characteristics and the students' problem-solving techniques.

In particular, this research focuses on the following attributes demonstrated by the error-prone problem solvers:

1. Error-prone students often (initially) use an English to Venn translation in an attempt to discover a method of finding a solution rather than as a method of finding a solution.
2. The error-prone students often appear to have no understanding of the purpose of the translations or the various representations. Rather, they seem to use the translation blindly in the hope of "getting somewhere" on a problem, or because it is a technique demonstrated by the class instructor.
3. Error-prone students are very apt to make errors during translations. In particular, many error-prone students displayed difficulty translating the contextual problem statement into a Venn diagram.
4. Error-prone students display significant difficulties understanding and interpreting the English statement of the problem. In particular, these students often demonstrate an inability to understand the problem statement well enough to determine what initial course should be taken to solve the problem and appear to have poor understanding of the relationships expressed in the contextual statement of the problem.

Error-prone students often use the English to Venn translation in an attempt to discover a solution method rather than as method to find a solution. These students use the translation as a method of restating the problem in the hope that the "restatement" will somehow reveal a technique or method for solving the problem. Note however, these students do not appear to regard the Venn diagram specifically as a restatement of the problem. Rather, the Venn diagram is used in conjunction with the English statement of the problem to analyze the question while they are attempting to find a method or technique that can be used to solve the problem.

The error-prone students often express something to the effect that "they are not sure why a translation is need but that they [the translations] always seem to work for solving the problems." These students often appear to have no understanding of the purpose of the translations--or, at least, some of the translations. Rather, they seem to use the translations blindly in the hope of "getting somewhere" on the problem, or because it was a technique demonstrated by the class instructor.

Further, the error-prone students are very likely to make certain types of translation errors. In particular, many of the students demonstrated difficulty translating from the contextual to the Venn diagram. However, they display much less difficulty translating information from the Venn diagrams into English statements than they do translating English into a Venn diagram. Also, many of these students appear to have little difficulty understanding and analyzing Venn diagrams.

Characteristic one implies that the error-prone students used translations in a much different manner and for a much different purpose than the use typically displayed by competent problem solvers. As demonstrated in the previous section, the competent problem solver analyzes the problem in context, determines a solution method, and then uses the Venn diagram and other mathematical tools to analyze and categorize the data with the intent of finding a number or set of numbers that will answer the given question. As will be seen

in the protocols that follow, the error-prone students often are unable to successfully analyze the problem in context and thus are unable to readily determine a solution method. Consequently, they then attempt to translate problem statement into a Venn diagram in the hope that this will then lead them to some understanding of how to solve the problem--or to the solution to the problem without their having gained any understanding. However, it was noted that many of the error-prone students, after they had translated the problem statement to a Venn diagram and "discovered" a solution method, then used the Venn diagrams and other representations as tools to re-represent and categorize data. That is, after the error-prone student discovered a solution method via a translation or Venn diagram, he or she would use the Venn and other representations in much the same manner as did the competent student.

Characteristic two will be demonstrated both explicitly and implicitly in the protocols that follow. In particular, it is demonstrated by the error-prone students use of algebra in solving for the regions of the Venn diagrams. Because the competent students knew exactly what they were looking for and how the Venn regions were related with one another, many of them were able to use short cuts, skip algebra steps, or solve in their heads for the numbers to put in the regions. However, it is seen that the error-prone students were much more likely to follow a pattern or algorithm when solving a Venn, that is, a method that was demonstrated by an instructor. Few of the error-prone students (successfully) skipped steps or solved the Venn in their head. Also, two of the error-prone students attempted to solve some of the problems using a tree diagram; this happened immediately after the technique was demonstrated in class to solve a conditional probability problem.

Note that characteristic three and four are somewhat related. It is natural that if a student does not understand the statement of a problem or what the problem is asking for, then he or she may well have difficulty translating the contextual information into a Venn diagram. However, there were cases where the student was unclear about what the problem

was asking (in context) but as (or after) she or he translated the problem into a Venn diagram came to enough of an understand of the question so as to able to find an answer. Also, there were cases where the student (seemingly) understood the problem in context but was unable to translate the contextual information into a Venn diagram.

Examples of Error-prone Characteristics:

Again, several detailed examples are given to demonstrate characteristics of error-prone students' problem solving skills. However, it is very important to note that some of the characteristics that most clearly demonstrate why students are error prone are found in their failure to display certain explicit attributes, attributes that are displayed as beneficial to problem solving by the competent students. Therefore, throughout this section the explicit characteristics are demonstrated through the use of examples. The non-explicit characteristics are demonstrated by comparing various problem-solving sessions. These comparisons are used to identify the techniques, skills, or knowledge needed to solve the problem(s) that were not displayed (or used) by error-prone students.

Each of the fifteen error-prone problem solvers appeared to know that it is appropriate to use Venn diagrams on the problems. That is, every error-prone student voiced that he or she should use a Venn diagram on these problems or attempted on at least some of the problems to translate English or algebraic information into a Venn diagram. This may be due to their having seen the Venn diagrams used both in class and in the text to solve similar problems. Eight of the fifteen used Venn diagrams on each of the problems. Five other students attempted to use a Venn diagram on one or more of the problems, and further, these students didn't try any other methods on the problems that were not attempted with a Venn diagram. Two of the students (Mike and Arnie) attempted the problems with a mix of Venn diagrams and tree diagrams. Mike tried a tree diagram with problem three but

used a Venn diagram with the other problems. Arnie used a tree diagram on numbers three and five but used Venn diagrams on numbers two and four.

Furthermore, nine of the fifteen error-prone students knew how to perform the needed algebra. That is, each of the nine demonstrated during the problem solving session that she or he possessed the algebraic skills necessary for solving the problems. Six of the error-prone students did not show enough algebra to afford an assessment of their algebraic skills. This is not to say that the error-prone students didn't have trouble with at least some of the algebra and algebraic concepts. However, it is clear that many of the error-prone students were able to perform the necessary algebraic manipulations, even if they (might) not have understood why the manipulations were needed or not understood what the algebra symbols and equations represented after they had completed a series of algebraic manipulations.

Karen's problem number two (Protocol 4.3) provides an example of an error-prone student's misunderstandings of the relationships expressed in an algebra equation. Notice that Karen did all her algebra in a formal or "classroom demonstration" manner, just as would an instructor when demonstrating a problem to a class.

Protocol 4.3

Name: Karen Problem: Two

At Giant State University a survey of students taking college mathematics found that 40% took college algebra, 30% took statistics, and 42% took neither course. If one of the students from the survey is chosen at random, what would be the probability that she or he took either algebra or statistics but not both courses?

[Segment 1]
Okay. So, let's see.

Algebra is point four.

$$A = .4$$

Statistics point three.

$$S = .3$$

And, A intersect S complement is--and forty-two percent took neither course--

$$(A \cap S)' =$$

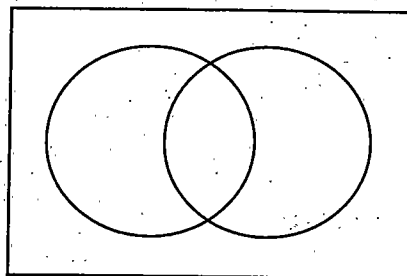
point four-two.

$$(A \cap S)' = \underline{.42}$$

One of the students from the survey is chosen at random, what would be the probability that she or he took either algebra or statistics but not both? Hmm--Okay.

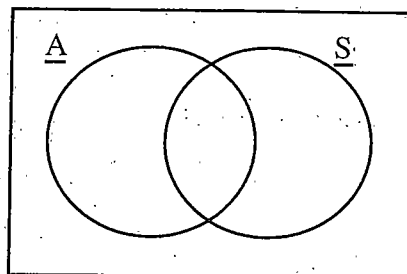
[Segment 2]

Well, the first thing--my instinct is always to draw a Venn diagram and fill in what I know. And then, try and put that together with what's being asked.



.1

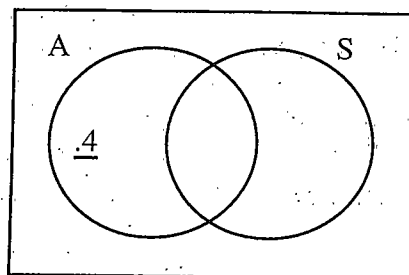
So, we have A, which is algebra, and S is statistics.



.1

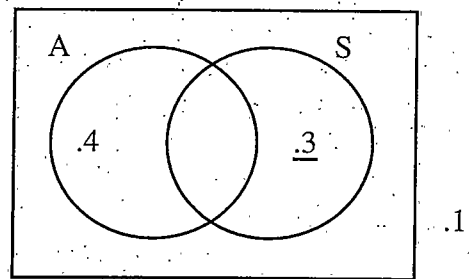
We know that forty percent took algebra, thirty-two took statistics, and forty-two took neither.

Okay, so that is algebra--point four,

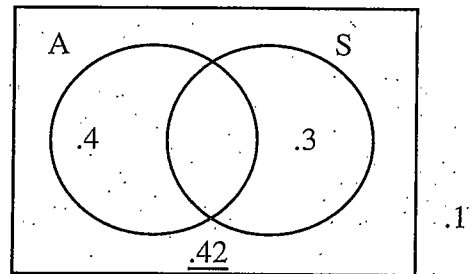


.1

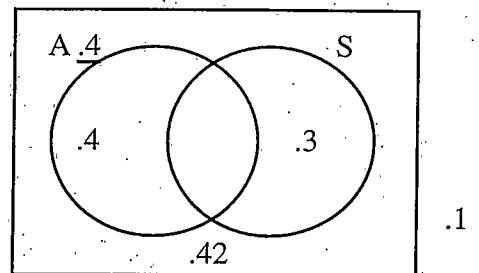
statistics point three,



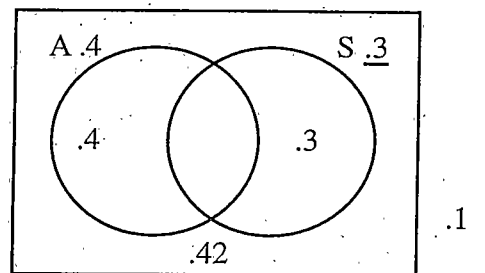
and outside of them both is point four-two.



Umm--Actually, point four

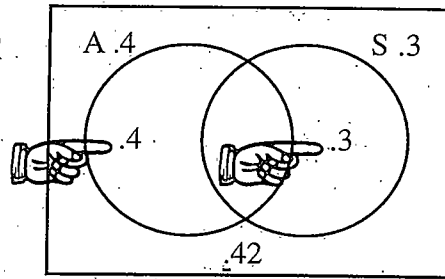


and point three.



[Segment 3]

And, if you have an intersection--either algebra or statistics--okay, so you have to find the intersection first before you actually know what this part is out here.



Okay--.

[Segment 4]

One is the total.

One equals

1

1 =

point four-two plus point four minus X plus point three minus X plus X.

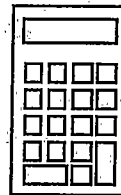
$$1 = .42 + (.4 - X) + (.3 - X) + X$$

Okay, so we have point four-two plus point seven minus X equal one.

$$.42 + .7 - X = 1$$

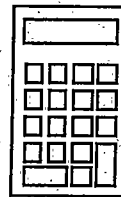
One minus point four-two minus point seven.
One minus point four-two equals point five-eight minus point seven

hmm.



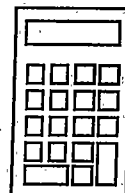
Don't think so. Oh, I know.

Point four--and point seven equals--.



OOPS, that's not right.
I'm stumped.

Try this, minus point four two--.



Oh, I do know.

No, I'm getting negatives.

[Segment 5]

Okay. All right, this is not a difficult problem.
Do not panic.

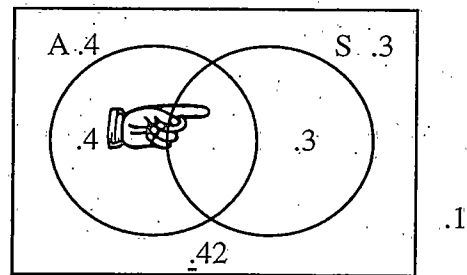
If one of the students from the survey is
chosen at random, what would be the
probability that she or he took either algebra
or statistics but not both?

I understand the problem. I'm just having a
hard time with the math, which is not
surprising. That's what happens to them all.

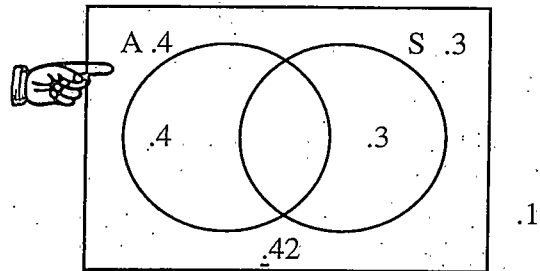
[Segment 6]

The question is: What is the probability that
they took either algebra or statistics but not
both courses?

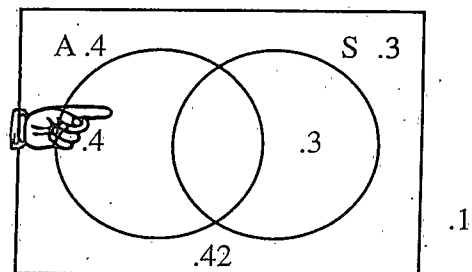
So, you have to first find the intersection



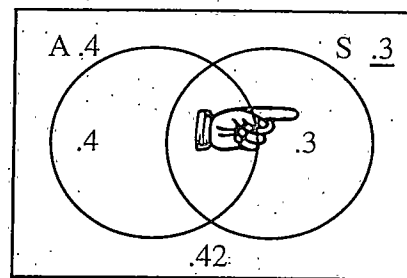
and subtract it from here.



You want to find this area of the diagram.



plus this area of the diagram.



Now, I know the formula for finding the intersection.

Why am I having such trouble?

[Segment 7]

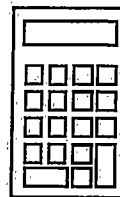
Okay. Point four minus X plus point three minus X plus X equals A intersect S.

$$(.4 - X) + (.3 - X) + X = A \cap S$$

The first thing that I'll do here, given this formula, is I cancel plus X and a minus X.

$$(.4 - X) + (.3 - X) + X = A \cap S$$

So then I have point four plus point three



is point seven minus X equals

$$.7 - X =$$

what I had before.

[Segment 8]

Laughs. Okay now, one more way to find that, given this information, is:

I know one's the total and know point four-two is out here so one minus point four-two minus point seven--.

$$1 - .42 - .7$$

I'm frustrated. I'm ready to give up and go on to the next one. Can I do that?

{She returned to this problem after completing problem four}

[Segment 9]

I'll redo this one.

Algebra equals point four.

Statistics equals point three.

And, A union S complement equals point four-two.

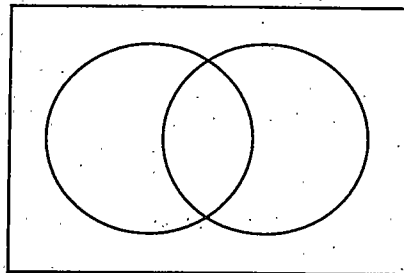
[Segment 10]

[Draws Venn diagram]

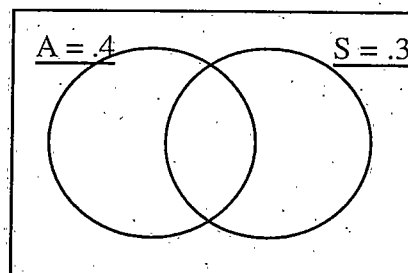
$$A = .4$$

$$S = .3$$

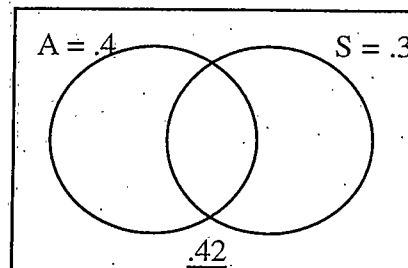
$$(A \cap S)' = .42$$



A point four. S point three.



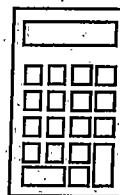
Neither point four-two.



So, point four minus X plus point three minus X plus X plus point four-two equals one.

$$(.4 - X) + (.3 - X) + X + .42 = 1$$

Point four minus ummm plus um--



Point four

.4

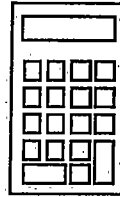
$$(.4 - X) + (.3 - X) + X =$$

$$+ .42 = 1$$

Point four minus X plus point four-two equals one.

$$.4 - X + .42 = 1$$

Point four plus point four-two equals--.



{Laugh.} I guess I won't work these--.

Point eight-two minus X equals one.

That doesn't compute.

So, I give. The end.

(End of Protocol 4.3)

Karen tried to solve the problem four times. On each of her attempts she correctly represented the Venn relationships algebraically. However, on each attempt when she reached the point [as in Segment 9] of solving for X in the equation:

$$(.4 - X) + (.3 - X) + X + .42 = 1$$

(X represents the value of intersection of the two Venn circles) she became confused. She correctly arrived at the equation;

$$.4 - X + .3 + .42 = 1.$$

and found;

$$-X = 1 - .4 - .3 - .42 = -.12$$

At this point she saw the value “-.12” and concluded that a negative number could not be the correct numerical value for the Venn intersection. But, she didn't realize that the negative value was equal to (-X) and therefore the value for X was actually positive. On each attempt

she arrived at this point in her algebra and became stymied by the equation. Karen finally gave up completely after her fourth attempt when she was not able to resolve the perceived contradiction.

The error-prone students are characterized by their non-competence, (i. e., prone to making errors) while translating between modes and while working in the various representations, and lack of confidence when translating from the English to the Venn diagram mode. (This does not mean the error-prone were not competent translating from the Venn mode to the English mode. In fact, the error-prone students appeared to be quite competent translating Venn diagrams into English statements.) Notice Karen's remark "Okay. All right, this is not a difficult problem. Do not panic." [Segment 5]. Remarks like these were not uncommon among the error-prone students. Several other error-prone students also expressed a lack of confidence about either their solutions or work.

Greg's attempt to solve problem two (Protocol 4.4) demonstrates an error-prone students inability to translate a problem from context into a Venn diagram.

Protocol 4.4

Name: Greg Problem: Two

At Giant State University a survey of students taking college mathematics found that 40% took college algebra, 30% took statistics, and 42% took neither course. If one of the students from the survey is chosen at random, what would be the probability that she or he took either algebra or statistics but not both courses?

[Segment 1]

At Giant State University a survey of students taking college mathematics found that forty percent took college algebra.

A equals forty.

$$A = 40$$

Thirty percent took statistics. S equal thirty.

$$S = 30$$

Forty two percent took neither. Uhh--all right so, $N \setminus (A \cup S)$ --no, that's wrong.

$$n(A \cup S)'$$

I'll do that later. I'll just leave that there.

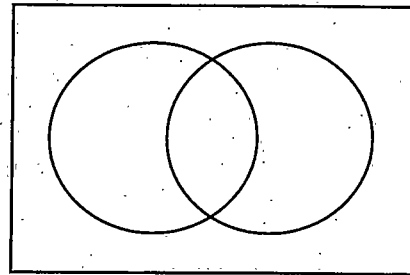
[Segment 2]

If one of the students from the survey is chosen at random, what would be the probability that she or he took either algebra or statistics but not both courses?

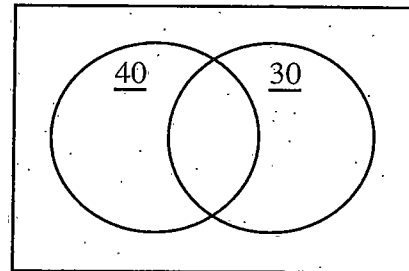
Okay, I've got--I really don't know what I'm doing.

[Segment 3]

Umm--I suppose I need to do this.



Algebra. Statistics.



[Segment 4]

And forty-two percent took neither. So, if neither is the complement of both {laugh} then fifty-eight is one-hundred minus forty-two.

$$\begin{array}{r} 100 \\ - 42 \\ \hline 58 \end{array}$$

Fifty-eight took--.

[Segment 5]

No, that's not right at all.

Boy, I--.

If one--if one of the students from the survey is chosen at random, what would be the probability that she or he took either algebra or statistics but not both courses?

Umm--Or is union. So we need to find the union of algebra and statistics.

And forty-two is N A intersect--A prime intersect S prime.

$$n(A' \cap S')$$

So, uhh I guess that would be N A intersect S prime.

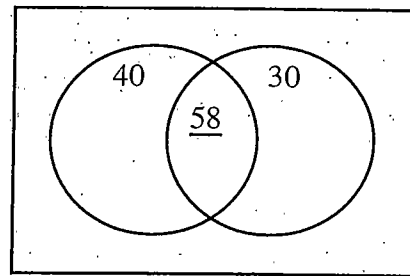
$$n[(A \cap S)']$$

So, we will--. It doesn't make sense to me because--.

[Segment 6]

Okay, well uhh we're going to have to do-- find the complement of that because that's all I know how to do. And that's fifty-eight once again.

So this would go here.



[Segment 7]

Now the union of A and S equals forty plus thirty minus fifty-eight.

$$n(A \cup S) = 40 + 30 - 58$$

Which equals twelve.

$$n(A \cup S) = 40 + 30 - 58 = \underline{12}$$

Okay, so that's the union of the--. One of the students from the survey is chosen at random--but not both.

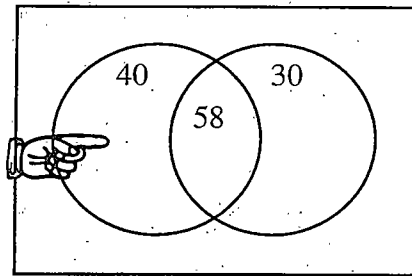
[Segment 8]

I'm very confused right now because I always have problems thinking these out.

Umm--It's not the words really.

It's just the--the fact that I can't decide what to use to get to where I am--where I'm going.

Umm--So we need to find that.

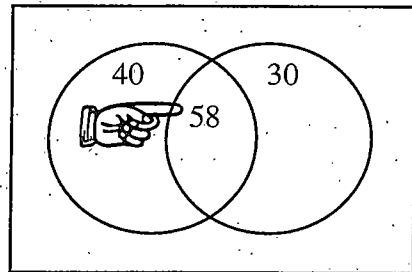


We need to find--.

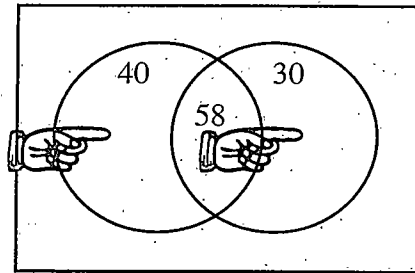
Okay, I know how to do it. I just don't know--.

[Segment 9]

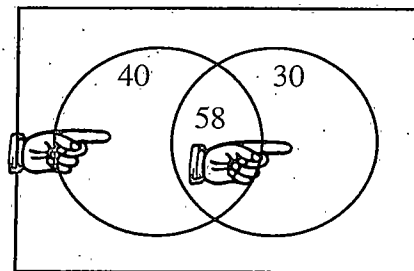
We need to find what this is for real, because it can't be fifty-eight. Because that would make these negative numbers if I'm partitioning uhh--.



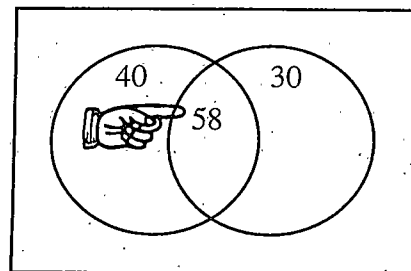
And we need to figure out how many did take both so that I can add those two together to get the answer.



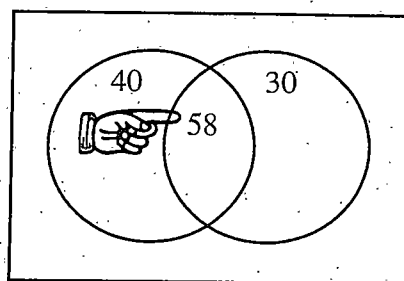
Only, I don't know how to do that because once you partition these two with--



with this.

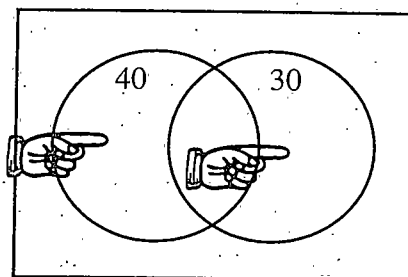


Once--once this is subtracted from



both of these--

then you add whatever you get from these subtractions--these two together--and you get the answer.



Because that would be either algebra or statistics but not both.

[Segment 10]

I just don't know how to get there. Umm-- So-- Umm--I don't know what to get. My-- my mind is blank because of trying to uhh-- show what would be the probability-- Took algebra or statistics but not both courses. So forty-two percent took neither. That does-- that doesn't mean that forty-two--that fifty-eight took both though. So that's totally wrong. Umm--Forty took algebra, thirty took statistics, and forty-two took neither. So forty-two is-- Maybe fifty--umm.

Forty-two is $N(A \cup S)$ prime.

$$n[(A \cup S)']$$

[Segment 11]

I think umm--. That's a problem because I don't know how--I don't know what to use these numbers with. That's the furthest I think can go with this problem because I'm totally--totally blanked right now. Umm--I--. Point-four. Point-three. I'm not sure. I--I don't know where to go. I know what should be done but I don't know how to do it.

(End of Protocol 4.4)

In the protocol of Greg's attempt to solve problem two, it is seen that he did not understand that the forty-two percent that took "neither course" should fall outside the two Venn circles. Because of his inability to translate facts given in English into a Venn diagram he was unable to find the number of individuals in the intersection (those taking both algebra and statistics) of the Venn circles.

Greg was correct when he subtracting 42 from 100 [Segment 4], thus obtaining the percent of students taking "either or both courses." And Greg was correct [Segment 5] when he stated, "So we need to find the union of algebra and statistics." However, he did not understand that the number (58) he found [Segment 4] represents the area covered by both Venn circles rather than just the intersection of the circles. Due to this error in his understanding, he incorrectly placed [Segment 6] the number 58 in the intersection of the Venn circles. However, Greg was able to translate from the Venn diagram back into English, so as soon as he tried to interpret (i.e., translate into the contextual mode) his Venn diagram [Segments 8 and 9] he realized that the numbers in the regions of the Venn had to be incorrect. After concluding that he had made an error, he was able to analyze the Venn and [Segment 9] discern that he had to find the sum of the numbers representing the areas of the two Venn circles outside there intersection. But, even at this point he remained unable to translate the given English statement into his Venn. After struggling to find his error, he finally ended the attempt [Segment 11] stating, "I know what should be done but don't know how to do it."

Greg was able to look at the Venn and understand what each part represented. Also, he was able to identify the portion of the Venn diagram for which he needed to find representative values. But, Greg was not able to translate the English statement of the problem into the Venn diagram and consequently was also unable to find the numbers need to describe each region of the Venn diagram. This made it impossible for him to finish the problem.

Interestingly, although Greg had difficulty translating from English to the Venn representations, he was much more competent translating Venn diagram information back into context. Because of this competence, he was able to correctly answer problem one (Protocol 4.5). That is, on question one Greg was unable to translate the English statement of the problem into the Venn diagram. But, after drawing a Venn diagram, he was able to describe what each Venn region represented. Consequently, by the process of elimination, he was able to arrive at a description of a region on his Venn diagram that matched the sets given in the problem statement.

Protocol 4.4 also demonstrates another attribute of the error prone students; Greg states several times that he is unsure of what to do and suggests [Segment 8] that he doesn't really understand the relationships expressed in the problem statement. This insecurity (actually, a more accurate definition may be the word unsure rather than insecure) caused his inability to translate from the English to the Venn, and further, his inability to translate caused him to remain unsure of what the problem statement was asking. Even after correctly answering the problem, he remained unsure that his answer was really an answer for the question asked in the contextual problem statement.

Protocol 4.5.

Name: Greg Problem: One

In the evening, between 7 and 9 PM, Ann's phone is not busy 75% of the time, John's phone is busy 35% of the time, and both of their phones are busy 10% of the time. What is the probability that neither of their phones will be busy at some moment during this part of the evening.

[Segment 1]

In the evening, between seven and nine PM,
Ann's phone is not busy seventy-five percent
of the time.

Ann not busy seventy-five,

Ann - not busy 75%

so is busy twenty-five.

Ann - not busy 75%
is busy 25%

Ann's phone is not busy seventy-five percent of the time. John's phone is busy thirty-five percent of the time.

John not busy one hundred minus thirty-five is sixty-five--sixty-five.

John - not busy 65%

Is busy thirty-five.

John - not busy 65%
is busy 35%

And both of their phones are busy ten percent of the time.

Both not busy ninety percent.

Both - not busy 90%

Are busy ten percent.

Both - not busy 90%
are busy 10%

[Segment 2]

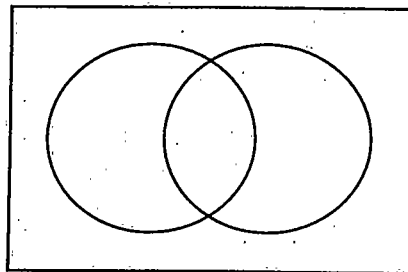
What is the probability that neither of their phones will be busy at the same moment during this part of the evening?

Okay, let's see what I need to do. Well, if neither of their phones will be busy at the same moment during this part of the evening--so if--.

[Segment 3]

I think what we need to do is draw the Venn diagram.

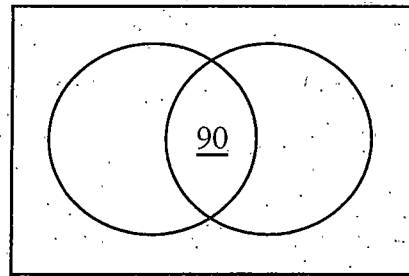
So we'll draw one and box around it.



And--now let's see, if they're both--if the phones are not busy--if both are not busy ninety percent of the time.

Let's try that. That's weird.

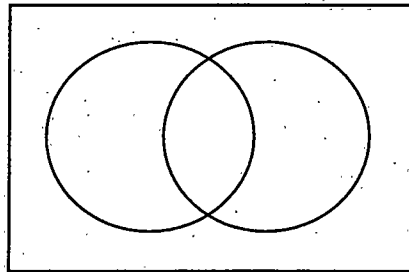
Okay.



Oh--what is it that both will be. I'll leave that there.

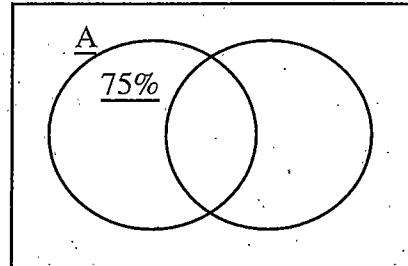
[Segment 4]

Umm. Draw another one.

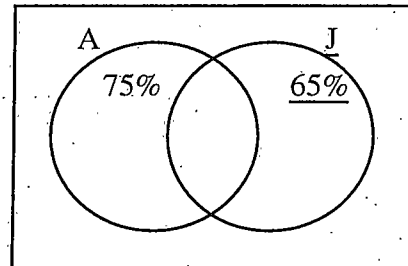


Okay--both. That neither of their phones will be--will be busy during this part of the evening. Okay.

Ann's is not busy seventy-five--put that there.



John's is not busy sixty-five.



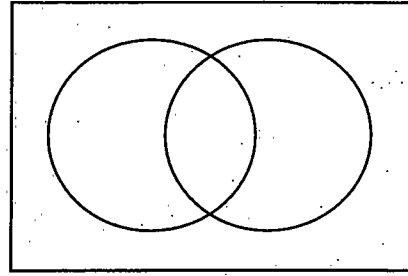
[Segment 5]

And umm let's see--how do we do this? How do you define when their phones will--will not be busy at the same moment. So, we probably need umm--.

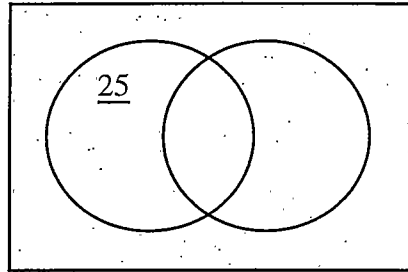
[Segment 6]

If it's busy--let's try it this way.

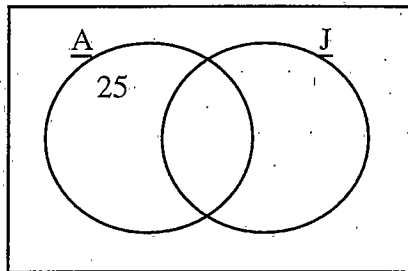
I'm very unsure what to do right now.



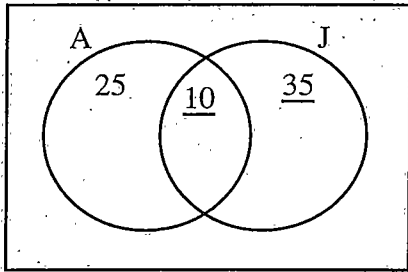
Okay, if it is busy--twenty-five.



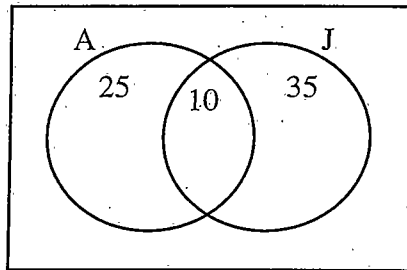
This is Ann's--John's .



And thirty-five and ten.

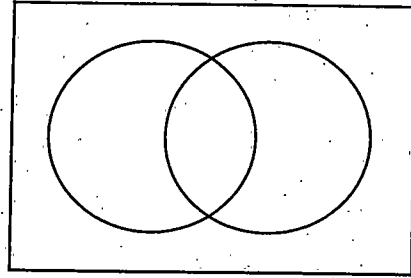


That makes a partition--that.

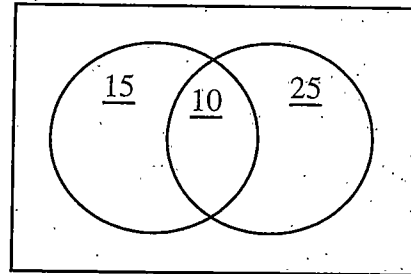


[Segment 7]

So we'll try it this way.



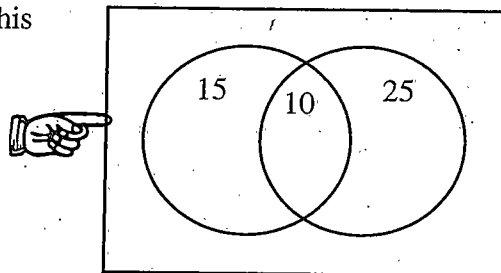
I'll draw one more, so that would be fifteen, ten and twenty-five.



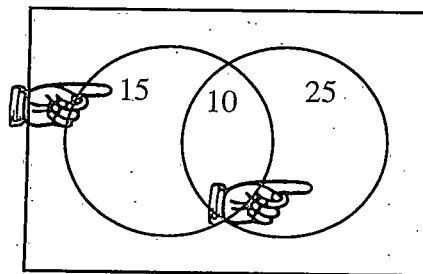
And this is when they are busy. So if we--if we can find--.

[Segment 8]

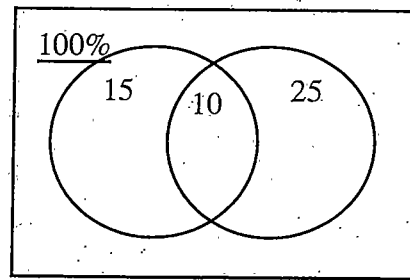
We need to find the--the compliment of this or something.



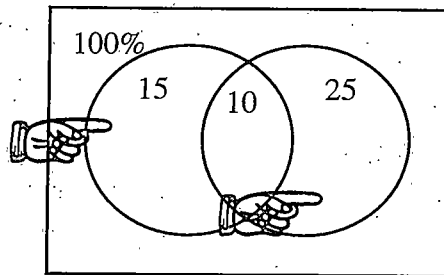
So let's see--the probability that neither of their phones will be busy at the same moment--so this is when--this is phone being busy.



So I guess it's one hundred percent.



Okay umm So that would be-- Okay, these are busy.



[Segment 9]

And that-- Okay, if she--if her phone is busy. Let's see. Probably need to add, from the partition, Ann's and John's since-- Maybe we don't.

Maybe we need to find the union of the Ann and John phone being busy. Because then that would equal one-hundred percent. Let's see.

Seventy-five percent plus thirty-five.

$$\begin{array}{r} 1 \\ 75 \\ + 35 \\ \hline 110 \end{array}$$

No, that's not right. Okay.

Umm--twenty-five percent plus thirty-five equals sixty

$$\begin{array}{r} 1 \\ 25 \\ + 35 \\ \hline 60 \end{array}$$

then minus ten.

$$\begin{array}{r} 1 \\ 25 \\ + 35 \\ \hline 60 \\ - 10 \\ \hline 50 \end{array}$$

[Segment 10]

So N Ann union John equals fifty. So altogether their phone's busy fifty percent.

$$n(A \cup J) = 50$$

So, if I'm doing this correctly, then the complement of that should equal--I'm not--I don't know if I'm doing that correctly but I think I am--umm fifty percent also.

$$n(A \cup J)' = 50\%$$

So, the probability that neither of their phones will be busy at the same moment during this part of the evening is fifty percent.

$$50\%$$

Or one over two.

$$\frac{1}{2}$$

I'm not sure how it's suppose to be answered. I think I--I guess we're suppose to break this down to point umm-- Well, we'll just try it that way. I think that's it.

(End of Protocol 4.5)

Notice Greg's remark [Segment 3], "I think what we need to do is draw the Venn diagram." This remark parallels [Protocol 4.4, Segment 3] the remark "Umm--I suppose I need to do this" made in problem two just as he begins to draw a Venn diagram. In both of these problems Greg was using the Venn as a tool to find a solution method rather than as a tool to describe and categorize data. (However, the Venn diagram does categorize data whether or not he uses it for that purpose--that is the reason the Venn diagram helps the error-prone student find a solution method). Greg's two remarks about using a Venn diagram indicates that he does not understand how the Venn diagram will categorize data or what it will reveal about data. Rather, he is using the Venn "blindly" in the hopes the Venn will lead him into a method to solve the problem. He hopes the Venn diagram will help him understand the problem and help him see what he must do to solve it.

The Venn diagram categorized the data for both the error-prone and competent students, no matter the intention of the student. However, the competent students knew before translating from the contextual to Venn mode what manner of information would be

revealed by the Venn diagram and how this would help them to solve the problem. The error-prone students appeared to not understand how the Venn would help solve the problem, only that it might somehow help solve the problem.

Protocol 4.6 is Bob's problem two solution attempt. This example again demonstrates an error-prone student's inability to translate the English statement into a Venn. However, Bob does appear to understand what each of the Venn regions represent. Bob gave his answer as "A intersect S," [Segment 3] which is correct symbolically. But, he was unable to find the numeric value for this region because, as he stated [Segment 2], he did not know "what's not both." That is, Bob was not able to see that "not both" means the region of the Venn that includes the two Venn circles except for their zone of intersection.

Notice also Bob's remark, "Okay, this looks like one of those good diagram things." [Segment 1]. He made this remark just as he began to draw the Venn diagram. As was seen in the earlier error-prone examples, Bob is using the Venn as a tool to discover a solution, or because he saw in class that problems like this were done with Venn diagrams. In any case, at this point in his solution attempt, he is not using the Venn diagram to order and classify data given in the problem statement.

Protocol 4.6

Name: Bob Problem: Two

At Giant State University a survey of students taking college mathematics found that 40% took college algebra, 30% took statistics, and 42% took neither course. If one of the students from the survey is chosen at random, what would be the probability that she or he took either algebra or statistics but not both courses?

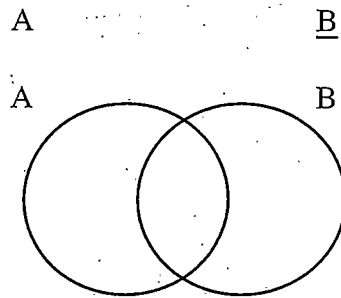
[Segment 1]

Okay, this looks like one of those good diagram things.

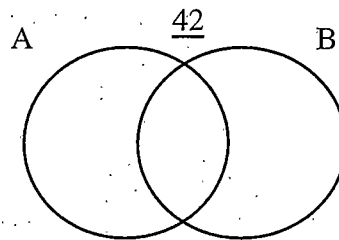
I'll say algebra is over here

A

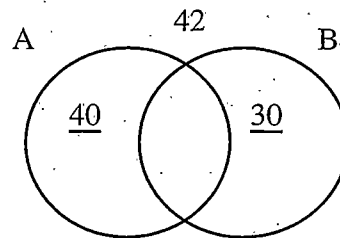
and statistics over here.



And forty-two took neither so they're both going to be outside.

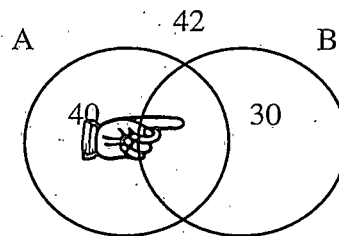


So forty's going to go over here and thirty's going to go over here.



[Segment 2]

Find the probability that she or he took either algebra or statistics but not both. We don't know what's not both. This.



Don't know how many--what percentage takes both.

So it should be seventy percent minus umm--.

70 -

[Segment 3]

You want the student that are going to take both, so you take off A intersect S.

70 - $(A \cap S)$

That's my answer. {Laugh}

(End of Protocol 4.6)

The error-prone students were often unable to determine or discover a solution method while working with the English representation of the problem. These students frequently translated (or attempted to translate) the problem into a non-contextual representation with the intent of discovering a solution method. However, in many cases the translation was performed incorrectly, and consequently, the new mode incorrectly represented the original information. Further, unlike the competent problem solvers, the error-prone students generally did not realize that a logical contradiction in a mode indicated that an error had been made during the translation. In several cases, the error-prone students were so lacking in confidence in their mathematical skills that they were unable to determine whether or not there was a logical error in the new representation. Instead of making a logical analysis of whether or not they had an error, these students just assumed that their mathematics had to be wrong. In many cases their belief was correct, although, in several instances they were incorrect in the assessment of their work and their work really was correct.

However, the error-prone students, once they accomplished an initial translation (whether or not the representation accurately represented the facts given in the problem statement) and discovered a solution method, then used the representations and further translations in a manner or process similar to that of the competent problem solvers. That is, the error-prone students (a) often had a difficult time determining how to solve a problem and (b) often made their initial translation without any knowledge of how or why the translation would aid with solving the problem. However, after they determined an approach for solving the problem, they used subsequent translations and representations in much the same manner as competent students. Upon finding a possible solution method (that is, after their initial translation) error-prone students will use subsequent translations (a) to clarify particular aspects of the problem, (b) to find the values of the regions in a Venn diagram, (c) to represent aspects of the problem symbolically, and (d) to solve algebraic equations. It

appeared as if many of the error-prone student possessed the mathematical tools to solve the probability problems but did not know how or when to apply the tools. That is, they often did not know how to translate the problem into a mode in which they could effectively apply their mathematical tools.

The error-prone students frequently displayed interpretation errors in their understanding of the problem statement. They often did not appear to understand the relationships and conditions expressed in the English statement of the problem. These students appear to have problems with high-level reading analysis. They seemed to have understood the words and sentences in the problems statement, but appeared unable to decipher the problem-statement paragraph. That is, these students understood the words, understood the sentences, but did not understand how to interconnect the ideas and relationships expressed by the sentences.

Sadly, the students that did not understand the Venn diagrams, or were not competent using the Venn diagrams, or were not successful making translations between Venn diagrams and other modes had no other tool or method that enabled them to categorize and organize the numerical elements of the problems. So then, if a student was unable to use the Venn diagrams in the manner described by the text (or in class) then he or she was also unable to devise any other method of solving the problems. This would indicate that those students who have learning styles that are not conducive to using Venn diagrams and graphical methods may have to spend extra time and effort mastering these problem-solving tools.

Some students understood the function of the Venn diagrams and were competent using Algebra, but were not competent and not confident with making the translation from the English into the Venn diagram. These students either had logic errors in there understanding of the contextual statement of the problem or they didn't understand how the question related to the set of elements as given in the problem.

Several of the error-prone students, when using the Venn diagrams, were not clear about when the Venn circles should or should not overlap. These individuals appeared to have no clear understanding that the overlapped region could be empty. This misunderstanding may indicate that these error-prone individuals believe that Venn diagrams are spatial or geometric representations and thereby contain some measure aspect rather than there being set representations.

Those students that could not translate (or could not translate accurately) were unable to solve the problems. Indeed, only one student was able to do any of the problems without translating--and he only solved one problem. Further, if a student could successfully perform the initial translation then she/he was likely to solve the problem, but if that same student, when he/she attempted a different problem, was unable to complete the initial translate then he or she was unlikely to solve the problem.

Protocol 4.7 characterizes the error-prone students that were unable to finish a problem. This final protocol demonstrates Lori's attempt to use the contextual to Venn translation as a method of discovering "how" to solve the problem (characteristic 1) and her lack of understanding of what will be revealed by the Venn diagram (characteristic 2).

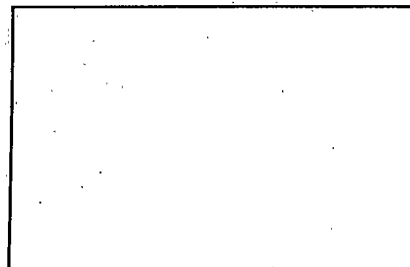
Protocol 4.7

Name: Lori Problem: Three

Each Monday a student attends finite mathematics class with probability 0.8, skips accounting class with probability 0.4, and skips both with probability 0.1. What is the probability that he attends at least one of these two classes on Monday?

[Segment 1]

I'd like to draw another Venn diagram--they seem to work for about everything.



Well, if he attends finite with point eight and
he attends accounting class with point six and
attends both with point nine--.

.8
.6
.9

What is the probability that he attends at least
one of these classes?

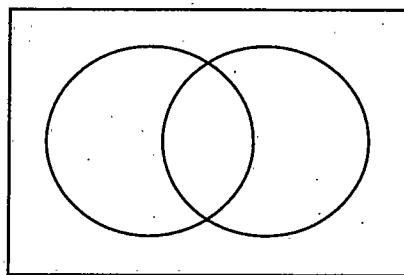
I'll come back to that one.

{She returned to this problem after attempting
problem five}

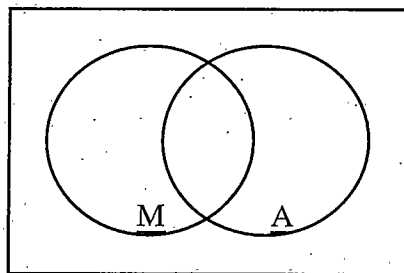
[Segment 2]

I'll try this one again.

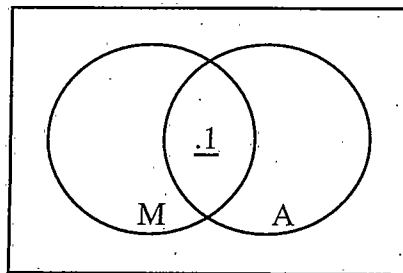
Okay, I'll draw a Venn diagram.



This one is for math class and this one is for
accounting.

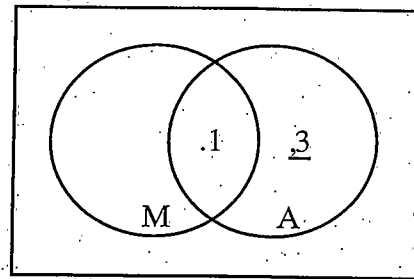


Attends both with a probability of point one.

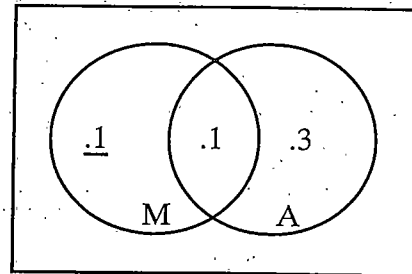


Uhh--That's not going to help.

He skips both--skips accounting class with a probability of point four.



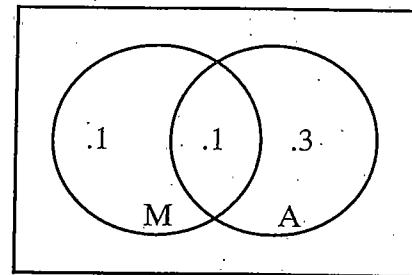
He skips this one with a probability of point two.



And he attends at least on of these two classes?

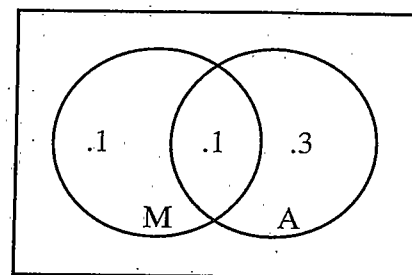
[Segment 3]

Well, if he's skipping one of these with point four then he'd have to be attending at least one of these--point six.



.6

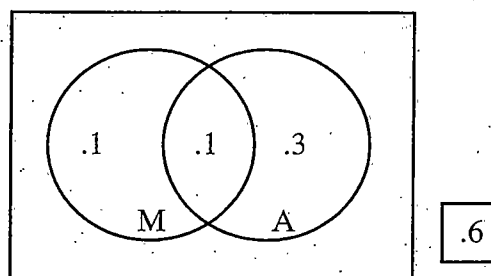
Point one, point three, point four when he skips--



.6

$$\underline{.1 + .3 = .4}$$

So it would have to be point six in terms of when he's going.



$$.1 + .3 = .4$$

[Segment 4]

Or point nine.

I vote point six because I don't know how to do it.

I have the point six because if he goes to math he attends with a probability of point eight--that means he'll skip with a probability of point two.

[Segment 5]

See, I messed up on my diagram. I need to find where he attends and I found where he skips. I don't know how to do this one.

[Segment 6]

If he attends class with a probability of point eight, then he skips with a point two.

Then if he attends accounting--if he skips accounting class with probability point four and skips both with point one--and so to find the probability that he attends at least one of these classes--.

The probability that he skips at least one of these classes is point four so my only guess would be to subtract that from one and get point six.

But that just doesn't sound right.
I don't know how to do it.

(End of Protocol 4.7)

Lori's remark in segment 1 of protocol 4.7, "I'd like to draw a Venn diagram--they seem to work for everything," again demonstrates how the error-prone students used Venn

diagrams. To many of these students, the Venn was just a picture used to find a method of solving the problem. They did not see it as a device used to sort and categorize data--even though that's why the Venn is effective at helping them find a solution method.

Students became very single minded while performing translations. Many of the students, from both the competent and the error-prone groups, became so involved in the translation process that they lost sight of the original problem. These students would continue with a translation until every piece the problem was translated into the new mode when in fact the piece of information they needed for answering the question was found early on in the translation. That is, they could have stopped before finishing the full translation. There is no indication that the students did this so that they could check their answers (seeing if the Venn or algebra made sense to them), but rather, it appears that they were so involved in the translation, and the related mathematical success, that they lost sight of the original reason for performing the translation. This is particularly true among the error-prone students, possibly because they are not as confident of what the question is asking and how to find the answer. However, almost none of the error-prone students reviewed their work to see if the answers made sense with the problem statement. In fact, those that did check their answers were generally the more competent of the error-prone students. The competent students were able to perform error checking as they performed the various translations and solved the problems.

The most competent students appeared to be able to visualize the problem pieces, data, and relationships in their minds. The Venn diagrams were used as memory or visual aids for these internal visualizations. The error-prone students appeared unable to generate these internal visualizations of the relationships. The error-prone students used the Venn diagrams to discover the relationships expressed in the problem statement. That is, the error-prone students used the Venn diagrams to discover and categorize the relationships between

and among the ideas expressed in the problem statement--whether or not the ideas were internalized is unclear.

Time-Protocol Comparison of Problem Solving Behavior

The following section contains tables that compare the times-of-usage of translations and representations by competent and error-prone problem solvers. The comparison is accomplished through the analysis of a representative group of time-line tables. The procedures and method of analysis of the time-line tables are explained in chapter three. However, it is worth noting the following about using the tables to analyze student problem solving behaviors: First, for each problem solving task, the table records (a) how much time was spent translating between representations and (b) how much time was spent using each representation. Secondly, and possibly more importantly, the time-line tables record what translations and representations were used and the sequence of use. Notice also, the time-line tables are analyzed with reference to the information gleaned from the previous qualitative analysis of the students' problem solving.

The qualitative analysis of the previous sections is useful in examining (a) how the problem solvers used representations, (b) what difficulties they had with the representations and translations, and (c) how the representations and translations aided the students in solving problems. The time-line tables are used to (a) record which representations and translations are used by the successful and error-prone problem solvers, (b) record the order or sequence of use of the tools, (c) record the time spent using the various representations and translations, and (d) compare the patterns of behaviors of the successful and error-prone problem solvers as they transitioned through the various translations and representations.

The seventeen time-line tables that follow were all developed via a timed task analysis of problem number two. The students of the first six time-lines (Time-line 4.1

through 4.6) are representative of the group of competent problem solvers. The students of the remaining time-lines (Time-line 4.7 through 4.17) are samples from the set of error-prone students. The first five of the error-prone time-lines (Time-line 4.7 through 4.11) represent those error-prone students that were able to find an answer, whether correct or incorrect, to problem two. Time-lines 4.12 through 4.17 represent the error-prone students that were unable to find a solution (DNF) to problem two.

In particular, in comparing the competent and error-prone groups of problem solvers, this research focuses on the following:

1. Competent and error-prone students display quite different methods of analysis and problem-solving strategies from one another during the initial seconds of attempting a problem.
2. Competent problem solvers appear to use symbolic algebra and algebraic manipulation much more frequently than do the error-prone students. In fact, the time-line tables demonstrate that many error-prone students don't even reach the point of needing to use symbolic algebra during the problem-solving sessions. The error-prone students however, do spend much more time "symbolizing" the problem statement (i.e., translating the problem statement from context into symbolic notation) than do competent problem solvers.
3. Competent students appear to be much more successful translating information between modes than do the error-prone students. In particular, the time-line tables illustrate how difficult it is for many of the error-prone students to express with a Venn diagram the relationships given in the contextual problem statement.

With reference to characteristic one, the time-line tables demonstrate that when presented with a problem the competent problem solver will spend a period of time analyzing the problem in context before proceeding with a solution method. On the other hand, the time-line tables demonstrate that many error-prone student will either spend no

perceptible time or a very long period of time on the initial contextual analysis. When an error-prone student spends a lengthy period of time on the initial analysis, it appears as if he or she has a poor understanding of the problem statement and consequently will have difficulty conceptualizing a method of solving the problem.

Characteristic two is demonstrated by the time-line tables in that (a) the competent problem solvers were much more likely (than were the error-prone students) to attempt a translation between a Venn diagram and symbolic algebra, and (b) the competent students' Venn diagram to symbolic algebra translations were more frequently completed (correctly) than those of the error-prone students. Item (b) is discernible by comparing the time-line tables of the competent and error-prone. The time-line tables demonstrate patterns in the use of the various representations and translations and these patterns were compared. Notice however, because the competent students more often used correct algebraic techniques they are much more likely (than are the error-prone students) to find the correct values represented by the regions in the Venn diagrams.

Characteristic three is demonstrated by noting that the error-prone students using much more time translating (or attempting to translate) the problems into the Venn mode than did the competent students. The qualitative analysis demonstrated that the contextual to Venn translation is very difficult, and was the cause of many errors, for the error-prone students. This difficulty with the Contextual-Venn translation was particularly noticeable with the error-prone students who were unable to finish the problem(s) (DNF). The time-lines demonstrate that these individuals spent much more time attempting the contextual to Venn translation than did the competent or error-prone finishers.

Before beginning the analysis of the time-line tables, it is worth noting the following about the coding of a translation event into a time-line table: The time-line tables demonstrate only the initial and final modes of any translation attempted during a particular time period. That is, if a time-line table records a student attempting a Contextual to Venn

translation, (i.e., $C \leftrightarrow V$), this indicates only that the student started the translation in the contextual mode and attempted to finish the translation in the Venn mode. However, in many instances a more accurate representation of the process may well be $C \leftrightarrow S \leftrightarrow V$ because the student was also translating the "nouns" given in the contextual material into symbols used on the Venn diagram. In particular, many of the competent students symbolized the contextual information during their translation from the contextual to the Venn diagram. The time-line tables demonstrate that the competent students spent little time translating from context to symbolic as compared with the time spent on this translation by the error-prone students.

Note also that the symbol " $C \leftrightarrow S$ " (indicating that the student is translating between the contextual and symbolic modes) is, in many cases, not indicating that the student is making an algebraic equation. In many cases students used this translation to identify each "noun" found in the contextual mode with a representative symbol. For example, in Protocol 4.4 (the qualitative analysis associated with Time-line 4.12), Greg used the first several seconds [segment 1] symbolizing the words "Algebra," "Statistics," and "Neither," but he did not make an equation using the symbols until much later.

Time-line Tables: Competent Problem Solvers

The first time-line table is for Marty, a competent student that successfully solved problems one, two, four, and five but was not able to finish number three.

Time-Line 4.1

Student: Marty

Problem: Two

Comments: Competent, Correct Solution

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	0:	:05	:10	:15	:20	:25	:30	:35	:40	:45	:50	:55	1:00	1:05	1:10	1:15	1:20	1:25	1:30

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	1:30-	1:35	1:40	1:45	1:50	1:55	2:00	2:05	2:10	2:15	2:20	2:25	2:30	2:35	2:40	2:45	2:50	2:55	3:00

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	3:00-	3:05	3:10	3:15	3:20	3:25	3:30	3:35	3:40	3:45	3:50	3:55	4:00	4:05	4:10	4:15	4:20	4:25	4:30

After his initial reading of the problem, Marty spent fifteen seconds studying the problem and formulating an approach to solving the problem. Marty then spent 55 seconds translating the problem from context into a Venn diagram and another 15 seconds analyzing his Venn diagram and attempting to solve each of the regions of the Venn. At this point Marty realized that he needed more information from the problem statement and so spent 20 seconds translating information from the contextual mode into his Venn diagram. Marty then spent 50 seconds analyzing his Venn, attempting to identify what regions he would need for a solution and figuring out how to solve for each of the regions. Next he spent 5 seconds translating information from the Venn into symbolic notation and making a

Time-Line 4.2
Student: George
Problem: Two
Comments: Competent, Correct Solution

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	1:30-	1:35	1:40	1:45	1:50	1:55	2:00	2:05	2:10	2:15	2:20	2:25	2:30	2:35	2:40	2:45	2:50	2:55	3:00

Time-line 4.2 indicates that George did not appear to spend any time at the onset analyzing the problem. Instead, he began immediately to translate the problem into a Venn diagram. However, an examination of the video tape and transcript of his work demonstrates

that George analyzed the problem while he was reading it aloud and realized then that he would have to draw a Venn diagram. However, he did not immediately know which regions of the Venn diagram were needed for the solution. In fact, it wasn't until 1:40, after he had finished solving for the values of all the Venn regions, that he looked back to the problem in context to find what region of the Venn was needed for the solution.

Also notice that George did not spend any perceptible time translating information from his Venn into the symbolic mode. Instead, he solved the Venn using arithmetic and a symbol free notional system of his own invention, as is described in the analysis of Protocol 4.4.

George finished the problem in two minutes including checking his answer. His is one of the fastest and most efficient problem-solving tasks demonstrated by the students.

Time-line 4.3 is of Kelli's work. She answer all five of the questions correctly; however, as explained earlier, the data from her questions one and three were not recorded.

Time-Line 4.3

Student: Kelli

Problem: Two

Comments: Competent, Correct Solution

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	04	:05	:10	:15	:20	:25	:30	:35	:40	:45	:50	:55	1:00	1:05	1:10	1:15	1:20	1:25	1:30

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	1:30	1:35	1:40	1:45	1:50	1:55	2:00	2:05	2:10	2:15	2:20	2:25	2:30	2:35	2:40	2:45	2:50	2:55	3:00

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	3:00-	3:05	3:10	3:15	3:20	3:25	3:30	3:35	3:40	3:45	3:50	3:55	4:00	4:05	4:10	4:15	4:20	4:25	4:30

Notice that both Kelli and Marty used several seconds at the onset of the problem-solving session analyzing the problem determining what exactly the problem was asking and how to approach the problem. Kelli's time-line table clearly demonstrates how a competent student uses the translations and modes in a logical and progressive manner to solve a problem. At the onset Kelli analyzed the problem, next she spent 45 seconds translating the problem into a Venn diagram, she spent 35 seconds analyzing the Venn diagram. She then spent a few seconds clarifying her symbols and another ten seconds translation the Venn into an algebraic equation. She used 55 seconds solving her equations and another 40 seconds translating the solutions back into the Venn diagrams. She used 5 seconds analyzing the Venn diagram for the answer and another five seconds stating her answer in English. In particular, notice how Kelli (like other competent students) has a very logical progression through the various modes and translations.

Miseon (Time-line 4.4) was one of the most mathematically competent students. She missed only one problem (number two) and that was due to a subtraction error as she completed solving her Venn diagram. Like is seen in the earlier time-line tables, she used several seconds at the onset of the problem analyzing the problem and determining what approach she to use.

Notice that Miseon has much the same time-line pattern as Kelli. Miseon however, unlike Kelli, used 30 seconds "symbolizing" the problem before proceeding with translating to the Venn diagram. From that point of time onward, her pattern is very much like Kelli's in that both of these students had a very logical and ordered progression

Comments: Competent, Incorrect Answer (subtraction error on final step of the problem)

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	1:30-	1:35	1:40	1:45	1:50	1:55	2:00	2:05	2:10	2:15	2:20	2:25	2:30	2:35	2:40	2:45	2:50	2:55	3:00

Marty's, Kelli's, and Miseon's time-lines illustrate how the prototypical successful student uses the various modes and translations. Each of the three analyzed the problem before starting a translation, translated from contextual into a Venn diagram, translated from the Venn to the symbolic, and used the symbolic mode to solve the Venn diagram. Each then proceeded to finish the problem with a time-line pattern (loosely) the reverse of this first pattern. That is, each translated information from the symbolic back to the Venn

diagram and then (from the Venn diagram) stated their answer. Kelli and Miseon translated from the Venn diagram to the contextual to state their answer; Marty translated the Venn into the symbolic mode as he stated his answer.

Chad (Time-line 4.5) successfully solved four of the problems but gave a incorrect answer to problem four. Chad demonstrated a quite different approach to solving problem two than was seen in Time-lines 4.1 through 4.4. Many competent students translated from the contextual mode to the Venn diagram, then solved for the values in the Venn diagram using symbolic algebra. That is, the focus of their problem solving method was the Venn diagram. Chad however translated from the contextual mode to the symbolic mode and then from the symbolic to the Venn diagram. By time 2:25 Chad had finished analyzing the Venn and knew which regions of the Venn diagram he needed values for to solve the problem. He then returned to the English statement of the problem (contextual mode), translated information from context into symbolic algebra, and solved equations for the variables representing the regions of the Venn diagram. However, Chad did not use the Venn diagram as the focus of his problem solving. Instead, he used symbols to represent each region in the Venn; that is, he solved $\Pr(A \cup S)' = X$ without identifying this with a particular Venn region. He finished solving the symbolic algebra at 2:55. He used from time 2:55 to 3:15 symbolizing and explaining how to solve for the other regions. From 3:15 to 3:25, he translated the Venn diagram into English and explained that values for one of the regions did not need to be found to solve the problem. From time 3:25 to 3:40, he read and explained the problem statement. From 3:40 to 4:00, he concluded the problem solving by adding together the values represented by the regions in the Venn diagram and identified each with a particular phrase from the contextual (English) problem statement. For example, during this period he explained, "So you add the what's in A not in S, which is the twenty-eight percent, plus the eighteen percent which is in S not in A and you get forty-six percent

Comments: Competent, Correct Solution

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	3:00-	3:05	3:10	3:15	3:20	3:25	3:30	3:35	3:40	3:45	3:50	3:55	4:00	4:05	4:10	4:15	4:20	4:25	4:30

The next time-line table student, Meredith, correctly answered all the problems except number three. On problem number two Meredith appeared initially to not understand what the problem statement is asking. In fact, Meredith made three attempts before she was

Comments: Competent, DNF

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	1:30-	1:35	1:40	1:45	1:50	1:55	2:00	2:05	2:10	2:15	2:20	2:25	2:30	2:35	2:40	2:45	2:50	2:55	3:00

Comments: Competent, DNF

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	0:	:05	:10	:15	:20	:25	:30	:35	:40	:45	:50	:55	1:00	1:05	1:10	1:15	1:20	1:25	1:30

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	1:30-	1:35	1:40	1:45	1:50	1:55	2:00	2:05	2:10	2:15	2:20	2:25	2:30	2:35	2:40	2:45	2:50	2:55	3:00

Student: Meredith
Problem: Two, Third of Three Attempts
Comments: Competent, Correct Solution

EPISODE																		
Context																		
$C \leftrightarrow V$																		
Venn																		
$V \leftrightarrow S$																		
Symbolic																		
$C \leftrightarrow S$																		
Time	0-:05	:10	:15	:20	:25	:30	:35	:40	:45	:50	:55	1:00	1:05	1:10	1:15	1:20	1:25	1:30

EPISODE																				
Context																				
$C \leftrightarrow V$																				
Venn																				
$V \leftrightarrow S$																				
Symbolic																				
$C \leftrightarrow S$																				
Time	1:30-	1:35	1:40	1:45	1:50	1:55	2:00	2:05	2:10	2:15	2:20	2:25	2:30	2:35	2:40	2:45	2:50	2:55	3:00	

On her first attempt Meredith, began the problem in much the same manner as did the other competent students. She first used several seconds analyzing the problem. She drew a Venn diagram upon realizing that one would be needed, but she didn't attempt to translate information from context to the Venn until later. Meredith then spent approximately 25 seconds symbolizing the problem, that is, translating the contextual material into symbolic notation and algebraic relationships. She then used 15 seconds attempting to translate her symbolized problem into a Venn diagram. At this point she realized that she didn't fully understand the problem and ended her attempt at translation between the symbolic and Venn modes. She then re-read the problem (five seconds in context) and attempted to translate the problem statement directly from context to the Venn

diagram. After 30 seconds she realized she was not able to complete the translation. She spent about one and a half minutes attempting once again to translate the contextual statement into symbols. On realizing she was not going to be able to finish this translation, she went on to the next problem.

On her second attempt to solve problem two, Meredith attempted to translate directly between the contextual and Venn modes. Meredith continued with her second attempt for three minutes before once again going on to another problem. The time-line shows she focused for the entire time on an attempt to translate between the contextual and Venn modes.

On starting a third attempt, Meredith realized she was not going to be able to translate the problem from context into the Venn diagram. She instead used a technique much like the method described in Protocol 4.5. Meredith first re-read the problem then went directly to the Venn diagram and examined the various Venn regions. Using a process of elimination of the various regions of the Venn diagram, she was able to conclude that only one region of the Venn diagram could possibly represent the number for which the problem is asking. She then was able to solve the Venn diagram for the unknown number and answer the question. Notice that once she was able to determine what the problem statement asked and found a solution method, Meredith's time-line table pattern is very much like the patterns of the other competent students. However, unlike many of the competent students, Meredith did restate her answer in the contextual mode, thereby reassuring herself that she had really answered the problem.

Time-Line Tables: Error-Prone with Solutions

The next five time-line tables are examples of error-prone students' work. Each of the students was able to find a numeric solution on problem two. Two students found a correct answer; three found an incorrect answer.

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	3:00	3:05	3:10	3:15	3:20	3:25	3:30	3:35	3:40	3:45	3:50	3:55	4:00	4:05	4:10	4:15	4:20	4:25	4:30

David (Time-line 4.8) incorrectly answered problem two due to a subtraction error. He answered problem one correctly, answered three and five incorrectly, and did not finish number four.

David approached problem two in much the same manner as did Mike in Time-line 4.7. David used no discernible time analyzing the problem in context. Immediately after reading the problem, he attempted to translated the problem into symbolic notation. He used 1:55 minutes translating from the contextual mode into the symbolic mode and then used 45 seconds more translating the information from the symbolic mode into the Venn diagram. David, like Mike, did not use symbolic algebra to solve for the values in the Venn diagram.

Time-Line 4.8

Student: David

Problem: Two

Comments: Error-prone, Incorrect Answer (Subtraction Error)

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	0:	:05	:10	:15	:20	:25	:30	:35	:40	:45	:50	:55	1:00	1:05	1:10	1:15	1:20	1:25	1:30

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	1:30	1:35	1:40	1:45	1:50	1:55	2:00	2:05	2:10	2:15	2:20	2:25	2:30	2:35	2:40	2:45	2:50	2:55	3:00

EPISODE																				
Context																				
$C \leftrightarrow V$																				
Venn																				
$V \leftrightarrow S$																				
Symbolic																				
$C \leftrightarrow S$																				
Time	3:00	3:05	3:10	3:15	3:20	3:25	3:30	3:35	3:40	3:45	3:50	3:55	4:00	4:05	4:10	4:15	4:20	4:25	4:30	

EPISODE																				
Context																				
$C \leftrightarrow V$																				
Venn																				
$V \leftrightarrow S$																				
Symbolic																				
$C \leftrightarrow S$																				
Time	4:30	4:35	4:40	4:45	4:50	4:55	5:00	5:05	5:10	5:15	5:20	5:25	5:30	5:35	5:40	5:45	5:50	5:55	6:00	

Kyle correctly answered question number two (Time-line 4.9) on his second attempt,

answered number one incorrectly, and did not finish either of numbers three, four, or five:

Time-Line 4.9

Student: Kyle

Problem: Two, First of Two Attempts

Comments: Error-prone, DNF

EPISODE																				
Context																				
$C \leftrightarrow V$																				
Venn																				
$V \leftrightarrow S$																				
Symbolic																				
$C \leftrightarrow S$																				
Time	0:	:05	:10	:15	:20	:25	:30	:35	:40	:45	:50	:55	1:00	1:05	1:10	1:15	1:20	1:25	1:30	

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	1:30-	1:35	1:40	1:45	1:50	1:55	2:00	2:05	2:10	2:15	2:20	2:25	2:30	2:35	2:40	2:45	2:50	2:55	3:00

EPISODE																		
Context																		
$C \leftrightarrow V$																		
Venn																		
$V \leftrightarrow S$																		
Symbolic																		
$C \leftrightarrow S$																		
Time	3:00	3:05	3:10	3:15	3:20	3:25	3:30	3:35	3:40	3:45	3:50	3:55	4:00	4:05	4:10	4:15	4:20	4:30

EPISODE																				
Context																				
$C \leftrightarrow V$																				
Venn																				
$V \leftrightarrow S$																				
Symbolic																				
$C \leftrightarrow S$																				
Time	4:30	4:35	4:40	4:45	4:50	4:55	5:00	5:05	5:10	5:15	5:20	5:25	5:30	5:35	5:40	5:45	5:50	5:55	6:00	

Kyle was unable to translate the problem data from the contextual mode into a Venn diagram. Initially he examined the problem in context. Then he attempted a contextual to Venn translation. This translation attempt lasted for over two minutes before he realized he was not able to complete the translation. He then focused his attention to the problem in the contextual mode and attempted to determine what exactly the problem was asking. However, at time 4:05 he decided he was not able to understand the problem statement and so quit. He then went on to try problem three before coming back to retry problem two.

Student: Kyle

Problem: Two, Second of Two Attempts

Comments: Error-prone, Correct Solution

EPISODE																																				
Context																																				
$C \leftrightarrow V$																																				
Venn																																				
$V \leftrightarrow S$																																				
Symbolic																																				
$C \leftrightarrow S$																																				
Time	0:	05	:	10	:	15	:	20	:	25	:	30	:	35	:	40	:	45	:	50	:	55	:	1:00	:	1:05	:	1:10	:	1:15	:	1:20	:	1:25	:	1:30

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	1:30	1:35	1:40	1:45	1:50	1:55	2:00	2:05	2:10	2:15	2:20	2:25	2:30	2:35	2:40	2:45	2:50	2:55	3:00

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	3:00	3:05	3:10	3:15	3:20	3:25	3:30	3:35	3:40	3:45	3:50	3:55	4:00	4:05	4:10	4:15	4:20	4:25	4:30

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	4:30	4:35	4:40	4:45	4:50	4:55	5:00	5:05	5:10	5:15	5:20	5:25	5:30	5:35	5:40	5:45	5:50	5:55	6:00

On Kyle's second attempt, he focused on the problem in context for 1:15 minutes. At this point he tried to translate information into a Venn diagram but was unable to complete the translation. At that time he made the remark, "I know a Venn diagram is a good way to look at it, but it's not the correct way to solve the problem." He then began to analyze the problem using symbolic notation and what he called "common sense" instead of "formulas or anything." Although Kyle was able to find a correct solution to this problem, his last statement before quitting was; "I'm not sure whether that's correct or whether that would be a correct probability answer but right now--in my eyes--I guess that's the way I'd answer it." Kyle's statement illustrates one of the most noticeable non-mathematical differences between the competent and error-prone students--that of confidence in their solutions to the problems. Competent students are so able at performing translations between the various modes that, after having completed a series of translations,

they appear to be able to "see" across the translations into the various modes and thereby detect and correct their translation errors. Consequently, the competent students express much more confidence, both during the problem-solving session and with their answers, than do the error-prone students.

In particular, Kyle had no means available to him to check his solution for errors. His common sense had not failed him on this problem--but he had no way of comparing his number with the other values in the problem to determine if his answer made sense. Kyle would have been able to check his solution if he had made a correct translation from the symbolic mode to the Venn diagram and then found numeric values for each of the various regions of the Venn diagram representing relations or data given in the problem statement.

Liz (Time-line 4.10) gave an incorrect answer on problem two. She was able to answer number one correctly, did not finish numbers three and five, and gave an incorrect answer to problem four.

Liz thought this was a probability problem and consequently attempted to find the probability of each event taking place (e.g., probability of a student taking statistics). She did not try to draw a Venn diagram or attempt to translate the contextual information into a picture or graph.

As stated earlier, many of the error-prone students used a Venn diagram to help discover a solution method. While this is an appropriate use for the Venn, Liz's attempt illustrates why it is best that students understand what a Venn will describe and in what situations the Venn and other graphs are useful problem-solving tools. If Liz had known that a Venn could be used to categorize the contextual data, she would have attempted a contextual to Venn translation. If she had correctly performed the translation, she would have immediately known the values of the probabilities she was attempting to find. The point is, Liz failed to use this mathematical tool because she didn't know what the tool

“did.” Aside from the issue of knowing how to make and use a Venn, she didn’t know when to use the tool.

Time-Line 4.10

Student: Liz

Problem: Two

Comments: Error-prone, Incorrect Answer (Failed to account for probability of taking both classes)

EPISODE

Context	0-:05	:05-:10	:10-:15	:15-:20	:20-:25	:25-:30	:30-:35	:35-:40	:40-:45	:45-:50	:50-:55	:55-1:00	1:00-1:05	1:05-1:10	1:10-1:15	1:15-1:20	1:20-1:25	1:25-1:30
$C \leftrightarrow V$																		
Venn																		
$V \leftrightarrow S$																		
Symbolic																		
$C \leftrightarrow S$																		
Time	0-:05	:05-:10	:10-:15	:15-:20	:20-:25	:25-:30	:30-:35	:35-:40	:40-:45	:45-:50	:50-:55	:55-1:00	1:00-1:05	1:05-1:10	1:10-1:15	1:15-1:20	1:20-1:25	1:25-1:30

EPISODE

EPISODE																				
Context																				
$C \leftrightarrow V$																				
Venn																				
$V \leftrightarrow S$																				
Symbolic																				
$C \leftrightarrow S$																				
Time	1:30	1:35	1:40	1:45	1:50	1:55	2:00	2:05	2:10	2:15	2:20	2:25	2:30	2:35	2:40	2:45	2:50	2:55	3:00	

EPISODE

Context																		
C ↔ V																		
Venn																		
V ↔ S																		
Symbolic																		
C ↔ S																		
Time	3:00-3:05	3:05-3:10	3:10-3:15	3:15-3:20	3:20-3:25	3:25-3:30	3:30-3:35	3:35-3:40	3:40-3:45	3:45-3:50	3:50-3:55	3:55-4:00	4:00-4:05	4:05-4:10	4:10-4:15	4:15-4:20	4:20-4:25	4:25-4:30

EPISODE

[illegible]

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	6:00-	6:05	6:10	6:15	6:20	6:25	6:30	6:35	6:40	6:45	6:50	6:55	7:00	7:05	7:10	7:15	7:20	7:25	7:30

Time-Line 4.10 Continued

Student: Liz

Problem: Two

Comments: Error-prone, Incorrect Answer (Failed to account for probability of taking both classes)

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	7:30	7:35	7:40	7:45	7:50	7:55	8:00	8:05	8:10	8:15	8:20	8:25	8:30	8:35	8:40	8:45	8:50	8:55	9:00

Deb (Time-line 4.11) answered problems one, two, and three incorrectly but did find correct solutions to problems four and five. She attempted to use a Venn diagram on problem two but did not draw the region of intersection for the two Venn circles; hence, she was unable to find correct numerical values for the various regions.

Time-Line 4.11

Student: Deb

Problem: Two

Comments: Error-prone, Incorrect Answer (Did not intersect Venn regions, did not correctly analyze problem statement)

EPISODE																		
Context																		
$C \leftrightarrow V$																		
Venn																		
$V \leftrightarrow S$																		
Symbolic																		
$C \leftrightarrow S$																		
Time	0:	:05	:10	:15	:20	:25	:30	:35	:40	:45	:50	:55	1:00	1:05	1:10	1:15	1:20	1:30

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	1:30-	1:35	1:40	1:45	1:50	1:55	2:00	2:05	2:10	2:15	2:20	2:25	2:30	2:35	2:40	2:45	2:50	2:55	3:00

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	3:00-	3:05	3:10	3:15	3:20	3:25	3:30	3:35	3:40	3:45	3:50	3:55	4:00	4:05	4:10	4:15	4:20	4:25	4:30

It is seen that the error-prone students who were able to find a solution were less likely to spend time analyzing the problem in context at the onset, and those that did spend time in context spent more time in this mode than did the competent students. Further, none of the error-prone students with solutions used symbolic algebra to solve the Venn diagram. Each of the students that used a Venn diagram either solved it by some means of non-symbolic arithmetic. Notice that if Deb had used the standard algebraic method (as was demonstrated in the text) she would have caught and corrected her error of not intersecting the two Venn circles.

Notice that there does not appear to be any significant difference between the competent and the error-prone with answer groups in the amount of time spent attempting to make the translation into the Venn diagram. Nor does there appear to be a lot of difference in the total time spent solving the problem from onset to stating an answer.

Time-Line Tables: Error-Prone "Did Not Finish"

Time-line Tables 4.12 through 4.17 are examples of students who did not finish problem two. As was typical with the error-prone DNF students, none of these students successfully translated the contextual information into a Venn diagram. Each of these

students demonstrated that they recognized that a Venn diagram was needed for a solution; they showed this by trying to translation the contextual information into a Venn diagram. However, not one of these students was able to successfully translate the information stated in the contextual mode into a Venn diagram. Some of these students tried to translate directly between the contextual and Venn diagram modes. Others attempted the translation using the symbolic mode as an intermediate step in the translation. In any case, these students were not able to successfully complete the translation from the contextual mode to the Venn diagram.

Greg answered problems one, three, and five correctly. He answered number four incorrectly and did not finish number two.

Time-Line 4.12

Student: Greg

Problem: Two

Comments: Error-prone, DNF

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	0:	:05	:10	:15	:20	:25	:30	:35	:40	:45	:50	:55	1:00	1:05	1:10	1:15	1:20	1:25	1:30

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	1:30-	1:35	1:40	1:45	1:50	1:55	2:00	2:05	2:10	2:15	2:20	2:25	2:30	2:35	2:40	2:45	2:50	2:55	3:00

EPISODE																
Context																
$C \leftrightarrow V$																
Venn																
$V \leftrightarrow S$																
Symbolic																
$C \leftrightarrow S$																
Time	3:00-3:05	3:10-3:15	3:20-3:25	3:30-3:35	3:40-3:45	3:50-3:55	4:00-4:05	4:10-4:15	4:20-4:25	4:30-4:35	4:40-4:45	4:50-4:55	5:00-5:05	5:10-5:15	5:20-5:25	5:30-5:35

EPISODE																
Context																
$C \leftrightarrow V$																
Venn																
$V \leftrightarrow S$																
Symbolic																
$C \leftrightarrow S$																
Time	4:30-4:35	4:40-4:45	4:50-4:55	5:00-5:05	5:10-5:15	5:20-5:25	5:30-5:35	5:40-5:45	5:50-5:55	6:00						

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	6:00	6:05	6:10	6:15	6:20	6:25	6:30	6:35	6:40	6:45	6:50	6:55	7:00	7:05	7:10	7:15	7:20	7:25	7:30

Greg first performed a translation from the contextual mode to the symbolic mode, symbolizing the contextual terms. He then returned to the context and attempted a contextual-to-Venn diagram translation. When he discovered he couldn't successfully perform this translation, he tried to find the values in the Venn diagram using a contextual-to-symbolic-to-Venn diagram translation. This attempt was also unsuccessful. He then used 2:00 minutes analyzing the Venn diagram hoping he could figure out what each region should represent and thereby complete one of the translations. When this was also unsuccessful, he tried again to complete a translation from the contextual to the Venn diagram and then a context-to-symbolic-to-Venn diagram translation. These attempts were also unsuccessful. He quit at time 4:45.

Mindi (Time-line 4.13) was unable to finish problem two. In fact, she was unable to find an answer to any of the first four problems and found an incorrect answer to problem five.

Notice that Mindi, on both attempts at problem two, began by using a few seconds analyzing the problem. On her first attempt, Mindi tried to translate from the contextual mode to a Venn diagram. At time 1:35 she realized she was not going to be able to finish

this translation. She then tried to build the Venn using a translation from the symbolic mode to a Venn diagram. She realized at time 4:40 that this was also unsuccessful:

On her second attempt she again tried to translate the problem from context into a Venn diagram. She realized at 1:00 that she was not going to be able to finish the translation and so ended her attempt. As she finished her second attempt, she made the revealing comment, “I don’t know where to start.”

Time-Line 4.13

Student: Mindi

Problem: Two, First of Two Attempts

Comments: Error-prone, DNF

EPISODE																		
Context																		
$C \leftrightarrow V$																		
Venn																		
$V \leftrightarrow S$																		
Symbolic																		
$C \leftrightarrow S$																		
Time	0:	05	:10	:15	:20	:25	:30	:35	:40	:45	:50	:55	1:00	1:05	1:10	1:15	1:20	1:30

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	1:30-	1:35	1:40	1:45	1:50	1:55	2:00	2:05	2:10	2:15	2:20	2:25	2:30	2:35	2:40	2:45	2:50	2:55	3:00

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	3:00	3:05	3:10	3:15	3:20	3:25	3:30	3:35	3:40	3:45	3:50	3:55	4:00	4:05	4:10	4:15	4:20	4:25	4:30

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	4:30	4:35	4:40	4:45	4:50	4:55	5:00	5:05	5:10	5:15	5:20	5:25	5:30	5:35	5:40	5:45	5:50	5:55	6:00

Student: Mindi

Problem: Two, Second of Two Attempts

Comments: Error-prone, DNF

EPISODE																																				
Context	■																																			
$C \leftrightarrow V$		■	■	■	■	■	■	■	■	■	■	■	■	■																						
Venn																																				
$V \leftrightarrow S$																																				
Symbolic																																				
$C \leftrightarrow S$																																				
Time	0:	05	:	10	:	15	:	20	:	25	:	30	:	35	:	40	:	45	:	50	:	55	:	1:00	:	1:05	:	1:10	:	1:15	:	1:20	:	1:25	:	1:30

Lori correctly answered numbers one and four, incorrectly answered number three, and did not finish number two. On both of her attempts at problem two, she focused on making a translation from the contextual mode to a Venn diagram. She was not able to complete the translation and ended her first attempt at time 2:35 and her second at time 1:40.

Time-Line 4.14

Student: Lori

Problem: Two; First of Two Attempts

Comments: Error-prone, DNF

EPISODE																		
Context																		
$C \leftrightarrow V$																		
Venn																		
$V \leftrightarrow S$																		
Symbolic																		
$C \leftrightarrow S$																		
Time	0:	05	:10	:15	:20	:25	:30	:35	:40	:45	:50	:55	1:00	1:05	1:10	1:15	1:20	1:30

EPISODE																				
Context																				
$C \leftrightarrow V$																				
Venn																				
$V \leftrightarrow S$																				
Symbolic																				
$C \leftrightarrow S$																				
Time	1:30-	1:35	1:40	1:45	1:50	1:55	2:00	2:05	2:10	2:15	2:20	2:25	2:30	2:35	2:40	2:45	2:50	2:55	3:00	

Student: Lori

Problem: Two; Second of Two Attempts

Comments: Error-prone, DNF

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	0:	05	:10	:15	:20	:25	:30	:35	:40	:45	:50	:55	1:00	1:05	1:10	1:15	1:20	1:25	1:30

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	1:30-	1:35	1:40	1:45	1:50	1:55	2:00	2:05	2:10	2:15	2:20	2:25	2:30	2:35	2:40	2:45	2:50	2:55	3:00

Don (Time-line 4.15) answered problem three correctly but was unable to finish the other four problems. He attempted problem two three times. On each attempt he was unable to successfully translate information from the contextual mode into a Venn diagram. On his first attempt, Don first tried to make a translation from the Context into a Venn diagram. At time 0:20 he realized he didn't understand the problem. He then used from time 0:20 until 1:35 in the contextual mode analyzing the problem. He quit at time 1:35 and went on to another problem. On his second attempt, he again tried to translate from the context to a Venn diagram. When he realized he was unable to finish the translation, he changed his attention to the Venn diagram and attempted to analyze what each of the regions of the Venn diagram represented in the context of the problem statement. This reverse translation attempt

is somewhat like that of Greg on problem one as seen in Protocol 4.5. Don ended his second attempt at time 1:50. Don's third try was again an attempt to translate directly from context to the Venn. He realized after 0:45 that he still did not understand the problem statement in context and so ended his attempt.

Time-Line 4:15

Student: Don

Problem: Two; First of Three Attempts

Comments: Error-prone, DNF

EPISODE																		
Context																		
$C \leftrightarrow V$																		
Venn																		
$V \leftrightarrow S$																		
Symbolic																		
$C \leftrightarrow S$																		
Time	0:	05	:10	:15	:20	:25	:30	:35	:40	:45	:50	:55	1:00	1:05	1:10	1:15	1:20	1:30

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	1:30-	1:35	1:40	1:45	1:50	1:55	2:00	2:05	2:10	2:15	2:20	2:25	2:30	2:35	2:40	2:45	2:50	2:55	3:00

Student: Don

Problem: Two; Second of Three Attempts

Comments: Error-prone; DNF

EPISODE																		
Context																		
$C \leftrightarrow V$																		
Venn																		
$V \leftrightarrow S$																		
Symbolic																		
$C \leftrightarrow S$																		
Time	0-:05	:10	:15	:20	:25	:30	:35	:40	:45	:50	:55	1:00	1:05	1:10	1:15	1:20	1:25	1:30

EPISODE																				
Context																				
$C \leftrightarrow V$																				
Venn																				
$V \leftrightarrow S$																				
Symbolic																				
$C \leftrightarrow S$																				
Time	1:30	1:35	1:40	1:45	1:50	1:55	2:00	2:05	2:10	2:15	2:20	2:25	2:30	2:35	2:40	2:45	2:50	2:55	3:00	

Student: Don

Problem: Two; Third of Three Attempts

Comments: Error-prone, DNF

EPISODE																				
Context																				
$C \leftrightarrow V$																				
Venn																				
$V \leftrightarrow S$																				
Symbolic																				
$C \leftrightarrow S$																				
Time	0:	:05	:10	:15	:20	:25	:30	:35	:40	:45	:50	:55	1:00	1:05	1:10	1:15	1:20	1:25	1:30	

Bob (Time-line 4.16) incorrectly answered problem one, three, four, and five. He did not finish problem two. Bob first attempted a translation from Context to a Venn diagram. And, as was observed with several other students, when he was unable to finish the translation, he focused his attention to the Venn diagram attempting to analyze the various regions and thereby “back” translate the problem. That is, at 1:15, it became obvious to him that he was not going to be able to translate the information from the contextual mode to the Venn diagram. At 1:25 he tried another technique. He attempted to figure out what each of the regions of the Venn diagram represented (in context to the information given in the problem statement.) He was not able to decide what each of the regions represented and so ended this attempt at time 2:00. He then tried a translation between the symbolic and Venn diagram modes. This was also unsuccessful, and he ended this attempt at time 2:25. He then re-read the problem and quit at time 2:30.

Time-Line 4.16

Student: Bob

Problem: Two

Comments: Error-prone, DNF

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	0:	:05	:10	:15	:20	:25	:30	:35	:40	:45	:50	:55	1:00	1:05	1:10	1:15	1:20	1:25	1:30

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	1:30	1:35	1:40	1:45	1:50	1:55	2:00	2:05	2:10	2:15	2:20	2:25	2:30	2:35	2:40	2:45	2:50	2:55	3:00

Arnie incorrectly answered problems three, four, and five. He was unable to finish problems one and two.

On his first attempt at problem two, Arnie spend 0:30 analyzing the problem (in context) and then attempted a translation between the contextual and symbolic modes. He realized he did not know how to finish this translation at 1:25 and, after reading the problem again, ended his attempt at time 1:30.

Arnie then tried a context to Venn translation. This attempt was also unsuccessful and he quit at time 2:15. He then remarked "I'm just going to keep re-reading the problem to look for a hint of something else I can do." After another ten seconds he decides to go on to another problem.

Time-Line 4.17

Student: Arnie

Problem: Two, First of Two Attempts:

Comments: Error-prone, DNF

EPISODE																			
Context																			
$C \leftrightarrow V$																			
Venn																			
$V \leftrightarrow S$																			
Symbolic																			
$C \leftrightarrow S$																			
Time	0-:05	:10	:15	:20	:25	:30	:35	:40	:45	:50	:55	1:00	1:05	1:10	1:15	1:20	1:25	1:30	

Student: Arnie

Problem: Two, Second of Two Attempts

Comments: Error-prone, DNF

EPISODE																		
Context																		
$C \leftrightarrow V$																		
Venn																		
$V \leftrightarrow S$																		
Symbolic																		
$C \leftrightarrow S$																		
Time	0-:05	:10	:15	:20	:25	:30	:35	:40	:45	:50	:55	1:00	1:05	1:10	1:15	1:20	1:25	1:30

EPISODE																				
Context																				
$C \leftrightarrow V$																				
Venn																				
$V \leftrightarrow S$																				
Symbolic																				
$C \leftrightarrow S$																				
Time	1:30-	1:35	1:40	1:45	1:50	1:55	2:00	2:05	2:10	2:15	2:20	2:25	2:30	2:35	2:40	2:45	2:50	2:55	3:00	

Finally, the data analysis and time-lines demonstrate an important point about the relationship of translations to representations in problem solving: The students used the representations to (a) clarify or organize data and (b) restate the problem (or part of the problem) in a solvable form. However, although the representations may clarify the data and problem, the translation is the tool that allows the solver to achieve the clarification. In most cases, the problems were solved by the translation, that is, via the translation. An appropriate translation is the process whereby the problem information is arranged or categorized in

such a manner that an "answer" is either available upon inspection or after a set of transpositions (e.g., algebra steps). Indeed, that is the purpose of translating. The time-line demonstrate that after a translation is completed (if the proper representation was chosen) little time is actually spent examining the new representation. Rather, if the proper translation and representation was chosen, the needed information was either available upon inspection (i.e., obvious) or it was clear what steps should be taken to get the needed information (e.g., algebraic steps). And, most likely, if it is not obvious how to get the answer the student will make another translation. The time-lines demonstrate that the students spent very little time in problem analysis in any mode other than contextual. But, the time-lines do demonstrate that the students did spend a significant portion of the time making translations.

So then, a translation (if done correctly) will represent the problem so that either: (a) the answer is available upon inspection, (b) the answer will be available after a series of transpositions, (c) it is clear that another (or more) translation(s) is needed, or (d) it is clear that the representation is not appropriate and a different translation should have been chosen.

CHAPTER FIVE

IMPLICATIONS, DISCUSSION, AND RECOMMENDATIONS

Overview of the Study

The following chapter provides an overview of this research. Included are the research questions, a review of the methods of data acquisition, and a summary of the techniques used to analyze the data. Also included is a review of the major findings of the study and a number of remarks regarding the findings. The chapter concludes with recommendations and questions for further research.

Through an investigation of processes, methods, and procedures displayed during problem solving tasks, this study examines the use of representations and translations during mathematical problem solving. Specifically, the study focused on how problem solvers used translations and representations with reference to the following questions:

1. Is there a difference in successful and unsuccessful students' use of multiple representations to solve probability problems?
2. Is there a difference in the patterns exhibited by successful and unsuccessful students with regard to their translation between representations?
3. With regard to the problem-solving process as a whole, what benefits (if any) do students gain through the use of multiple representation?
4. What difficulties (if any) do students encounter in their use of multiple representations?

The problem solving data were collected during interviews with individuals selected from students enrolled in a college finite-mathematics course. The identification as to whether a student was a competent or error-prone problem solver was made post hoc, on the basis of whether the student was able to successfully use mathematical tools, accurately

translate between the various representation modes, and use the modes to solve probability problems.

For the purpose of this research:

1. *Competent* problem solvers are able to analyze a problem, develop an appropriate solution strategy, translate between representations (modes) accurately, and operate within the representations competently. As a result, competent problem-solvers are able to analyze a problem, find a method to solve the problem, choose and use appropriate translations, and, but for minor errors, find a correct solution. The competent students use translations and representations in a manner appropriate to the problem and can quickly and accurately translate between the various representations.

2. *Error-prone* problem solvers have less success (than the competent students) at solving problems because they are unable to (a) develop or implement an appropriate problem solving strategy, (b) complete the necessary translations, or (c) perform the necessary arithmetic or algebraic operations while working within the various modes. Error-prone students often have difficulty translating the English statement of the problem into a Venn diagram and often have difficulty translating information from a Venn or English representation into an algebraic statement. Also, the error-prone students may be unable to operate successfully within the various representations. To summarize, the error-prone students are classified as such because each is prone to making errors in translation, in mathematical manipulations, or in problem analysis, not because their answers are in error. Competent problem solvers are so labeled because of their ability to use the mathematical tools and concepts available to them in an competent manner, not because of their ability to always find a correct solutions to the problems.

Students selected from several sections ($n=21$) of a finite mathematics course demonstrated their mathematical and translation skills as they attempted to solve a set of five

probability problems. Each student was asked to verbalize his or her thinking process as he or she attempted to solve the problems. The problem solving sessions were video recorded.

The videotapes and written work were used to make a transcript of the students' problem sessions. The transcripts correlate the students' spoken words with any written material, symbolic algebra, and graphical data made during the sessions.

The videotaped problem-solving sessions and transcripts were analyzed with focus on students' use of multiple representations and translations. Also, the video recordings were used to identify the times (from the beginning of the session) when the subjects made a translation, the length of time taken to complete the translation, and how long the subject operated within each representation. These data were transferred to a time-line. The time-lines were used to look for patterns in the time of use of the representations and translations.

In particular, this research focused on (a) the students' use of representations and (b) the links the students made between the representation modes, that is, the students' use of translations.

The purpose of the representation research was to identify (a) which representations the competent and error-prone students use, (b) when they choose to use any particular representation, (d) how frequently they use each representation, and (e) how successful they are at using the representations. This research looked for patterns of behavior that characterize the competent and error-prone students use of representations and translations.

As illustrated in Figure 4.2, for the purpose of this research, (a) the contextual representation was probability problems written in English; (b) the algebraic mode was Symbolic Algebra (college introductory level); and (c) the visual or graphical mode was Venn diagrams. The translations were between these three representations

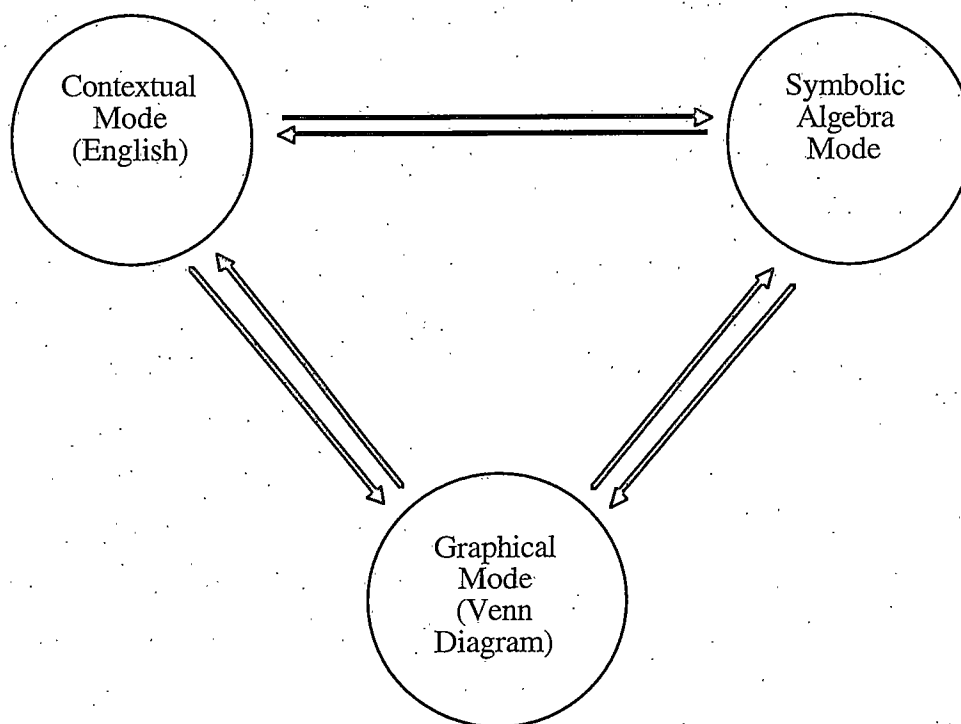


Figure 4.1: Translations and Representations

The purpose of the translation research was (a) to identify patterns of movement between the representations and (b) to identify patterns in the time spent translating and operating in each of the representations.

Summary and Implications

A Venn diagram is a graph that uses circles to represent sets. The circles overlapping, and non-overlapping zones represent set relations. The relations are demonstrated via the inclusion, exclusion, or intersection of the zones of the circles drawn on the Venn diagram. That is, for the purpose of this research, Venn diagrams allowed students to categorize and enumerate set data.

The primary purpose of this research was to answer the following two questions:

1. Is there a difference in successful and unsuccessful students' use of multiple representations to solve probability problems?

2. Is there a difference in the patterns exhibited by successful and unsuccessful students with regard to their translation between representations?

The data suggest that there is a difference in the use of multiple representations by the two groups. Further, the time lines demonstrate a difference in patterns of behavior of the two groups.

This research demonstrates that the competent problem solvers used Venn diagrams to categorize and enumerate the problem data. The data suggest that competent students are cognizant that the purpose of the Venn diagrams is to categorize and enumerate data. However, the research data indicate that many error-prone problem solvers used the Venn diagram as a vehicle of discovery, that is, as a tool used to "figure out" how to solve a problem. Many of the error-prone students were unable to discern how to solve the problems while studying the English statement of the problem and consequently, often translated to a Venn diagram while attempting to "discover" a solution method. They used a translation and a new representation to "see" if they could figure out how to do the problem. This is not to say that error-prone students didn't also use the Venn to categorize and enumerate problem data. In fact, that is what a Venn diagram does to data, no matter the intention of the translator. Indeed, it is because Venn diagrams so clearly categorize logical set relations that the error-prone students were able to solve any of the problems. The error-prone students generally translated the problem (often incorrectly) into a Venn representation to see if they could discover a solution method, and once the translation was complete they often did see how to complete the problem. The fact that the error-prone students were enabled, via the Venn representation, to discover a solution path serves to demonstrate the appropriateness and power of translations and multiple representations.

Competent students appeared (generally) to understand the written problem statements and were able to see or discover a solution method before translating the problem into a non-English representation. Moreover, the competent students appeared very

confident and competent doing translations and hence, often translated portions of a problem to discover the relevance of a particular piece of information or to clarify some aspect of the problem or their solution method.

The error-prone students, on the other hand, were very prone to make errors in translations, and often expressed a lack of confidence in their translations. In particular, the error-prone students had difficulty translating English statements into Venn diagrams. Interestingly, many of the error-prone subjects appeared to have little difficulty translating information from a Venn diagram into an English statement. The fact that the error-prone students were able to translate information from the Venn into an English statement, but were unable to successfully translate from the English to the Venn, demonstrates both the purpose and usefulness of Venn diagrams and graphical representations. By design, the Venn diagram is meant to clarify and classify information so that the problem solver can easily understand and interpret data relationships. Hence, the error-prone students were able to see the relationships (expressed via the Venn diagram) in the data and express these relationships as an English statement. In several instances, the error-prone students arrived at a correct solution without really realizing the purpose of the Venn representation or really knowing why they were using a Venn diagram.

The time-lines indicate that error-prone students jumped between using the various translations and representations much more frequently than did the competent students. The error-prone students often would try making a translation to some mode, would not be successful making the translation, and then would immediately try a different translation. They would continue this process of trying different representations until they finally gave up in frustration. Notice though, the error-prone students were not successful using the representations because they were unable to successfully complete the translations necessary to use the representation.

The time lines show that the competent students tended to stay within a representation until reaching some conclusion as to how the representation would benefit them in finding a solution, or until they concluded that a different representation was needed. They would then translate (if they needed to) and use another representation to clarify data relationships or find a needed numeric value. The competent students continued this process of translating and clarifying the problem and finding the needed values until they had found a solution.

It is seen then that the competent subjects used translations to arrive at conclusions about aspects of the problem. The translations were seen as a tool to use in the solution process. However, the error-prone students used translations (a) as a method of discovering a solution method or (b) because they "just don't know what to do." The group that used a Venn diagram to discover a solution method demonstrate that the error-prone students realized that translating helps with finding a solution. (Even though they did not know why or how it helps.) The students that "don't know what to do" illustrate the error-prone students that do not know how to do the problem and do not realize that a translation might help them to solve the problem. This second group used the translation because "that's what was done in class and that's all I know to do."

The following remarks are made regarding the subjects that chose a solution method on the basis of "that was how I saw it done in class." One hopes that through sufficient practice, and with clear classroom demonstrations, the error-prone subjects will master the skills displayed by the competent problem solvers, and hence, become competent problem solvers. If error-prone students, through study and practice, are able to master the problem-solving techniques, then this research is quasi-longitudinal in that the interviews are a "before" and "after" view of students problem-solving skills. It appears that the error-prone subjects, because they were unable to develop their own problem solving strategies--or because they had insufficient out-of-class practice--used classroom instruction as their

problem solving model. These students mimicked the successful behaviors they observed in the classroom and attempted to apply it to any and all problems that they believed to be somewhat similar to the demonstrated problems. Interestingly, those error-prone subjects with insufficient practice often did not realize why they could not do the problems and did not understand why their solutions were not working out as smoothly as those demonstrated in the classroom.

A secondary purpose for this research was to answer the following two questions:

1. With regard to the problem-solving process as a whole, what benefits (if any) do students gain through the use of multiple representation?
2. What difficulties (if any) do students encounter in their use of multiple representations?

The following comments are made in regard to these questions.

Lesh, Post, and Behr (1987) reported "good problem solvers tend to be sufficiently flexible in their use of a variety of relevant representational systems that they *instinctively* (italics added) switch to the most convenient representational system at any given point in the solution process" (p.38). This research indicates that the choice of representational systems is not instinctive. Rather, it appears as if subjects learn from practice and demonstration which representational choices are most appropriate for the various types of probability problems. This learning cycle becomes evident as the students struggle to interpret and analyze the probability problems. The researcher recorded several instances that indicate that if a student understands, or appears more comfortable with, one particular translation and/or representation rather than another, he or she will use the "comfortable" translation and representation rather than analyzing the problem to see if the particular representation chosen is appropriate for solving the problem. This "comfortable choice" is particularly true for those students who do not have a clear understand of what a particular translation and representation will reveal about the data, that is, the error-prone students. This indicates that if students are to become competent problem solvers they need to be able

to (a) operate in many different representations, (b) translate accurately and confidently between the representations, and (c) be able to discern, before starting the translation, how the new representation will present the data and relationships expressed in the problem. Further, the research suggests that students need early exposure to the various representations and lots of practice making translations between the representations. Moreover, it should be made clear to students, both by text and instruction, which representations are appropriate and which are not appropriate for the various types of problems presented in the curriculum. For example, some of the subjects attempted to solve the research problems with a tree diagram, not realizing that tree diagrams were not an appropriate graphical method for representing the problems.

Therefore, for the error-prone problem-solvers to become competent problem solvers, they need to (a) have some understanding of the problem statement, (b) understand why a translation is needed, (c) discern which translation and representation is appropriate, and (d) be able to make the translation. Indeed, no individual will use a tool if he or she doesn't possess a knowledge of the tool's existence, know why the tool is useful, where the tool is usable, and how to use the tool.

According to Janvier's (1987) discussion of "the Source and Target Paradigm," to achieve a translation, the translator has to transform the source "target wise" or, in other words, to look at the source from a "target point of view." Therefore, in a direct translation, the translator must examine the source (the representation to be translated) in the context of the target. This research reveals that, as subjects attempt to translate between representations, they often switched back and forth between the modes; that is, they translate information in one direction (in relation to the representations) for a time; then they appear to be translating in the other direction for a time. This "back and forth" aspect of the translation process reveals the students' efforts to transform the source "target wise" into a form compatible with the target representation. Note, however, it also points out the

limitations of using the source-target analogy in that, while an arrow only goes one direction, as does a translation, many of the students went back and forth between representations (that is, between the source and the target) many times before finishing the translation and building a correct "target" representation.

The time lines together with the analysis of the students' protocols demonstrate that the competent students are able to "hit the target" (i.e., translate competently) while the error-prone students commonly either (a) are unable to understand the relationships expressed in the problem statement well enough to be able to translate from context into relational symbols or, (2) do not understand how to represent the source information in the new representation mode. Many of the error-prone subjects were able to translate from the Venn diagram to the English mode but not vice-versa. That is, many error-prone subjects understood the problem when the data was presented via a Venn diagram, but did not understand how to solve the problem when it was presented as a written English statement. This demonstrates how powerful the Venn diagram is as a tool for categorizing and presenting data, and hence, how important translation and representation skills are in problem solving.

It is important that students, particularly those that are the least confident and competent mathematically, begin to see the connections between the representations. And they need to become cognizant of the "translation direction" during a translation attempt. If a student has an understanding of the connections and translation direction but is unable to translate forward from one mode to another then they may be able to decode the translation by looking at the reverse translation. This translation technique was observed when one of the students, although he did not understand what is being asked for in the problem statement, knew what each region of the Venn diagram represented in English. Then, by using a process of elimination of the various regions of the Venn diagram, he was able to decode the problem statement and figure out what it was asking. He related each zone of the

Venn diagram with a relation expressed in the problem statement, rather than relating each statement to a zone of the Venn diagram. This then, in the context of Janvier's "Source and Target Paradigm," would have the student use the target to decode the source. The research herein suggests that for the error-prone subjects to attain problem-solving and translation competence they must (a) have much more practice translating between representations, (b) have the source-target decoding techniques demonstrated and explained by text and teacher, and (c) somehow come to an understanding of why to translate, that is, what the new representation will reveal about the data or relationships expressed in the problem statement. This research does not indicate how these mathematical skills can best be accomplished or at what educational level this knowledge and skill can best be attained.

Many of the error-prone subjects seemed unable to come to a good understanding of the relationships expressed in the problem statement. These students had no chance of translating the problem into a Venn diagram because either (a) they did not understand the English expression of the relationships or (b) they were unable make a cognitive connection between the two modes. That is, they did not apprehend that the new representation revealed for them anything useful about the contextual statement of the problem. This finding indicates that mathematics teachers need to spend time teaching "reading mathematics for understanding." Teachers should assign and demonstrate logic problems (i.e., that give the students opportunity to read problems and clarify problem statements). And, the students need to practice the algorithms (i.e., standard methods demonstrated by text and class lecture) to solve the problems. The NCTM (2000) appears to acknowledge these needs with following comment:

An important part of learning mathematics is learning to use the language, conventions, and representations of mathematics. Teachers should introduce students to conventional mathematical representations and help them use those representations effectively by building on the students' personal and idiosyncratic representations when necessary. *It is important for teachers to highlight ways in which different representations of the same objects can convey different information.*

and to emphasize the importance of selecting representations suited to the particular mathematical tasks at hand. [Italics added] (p.361-362)

Many of the error-prone students appeared to be unable to make a cognitive connection between the representations. Regarding this, on several occasions error-prone students expressed that they knew they needed to make a Venn diagram but that they were not sure why they needed the Venn diagram. That is, students knew they needed a Venn diagram (that was the only way they had seen the problem solved) but were unsure what the Venn diagram revealed about the problem and data. Also, they were unclear why the Venn diagram was the method used to analyze and solve the problem. In general, teachers act and teach like competent problem solvers, thereby never giving students the opportunity to explore (in a class setting with teacher participation) various methods and approaches to solving the problems. It may be that teachers should teach (sometimes) as if they were not competent problem solvers. That is, teachers need to explore with the students. They need to present and explore a variety of problems in greater detail. And teachers need to look at the various translation and representations and explore how the representations may, or may not, apply to problems and thereby demonstrate which translations and representations are appropriate for solving the various types of problems. Further, the students would then understand (hopefully) why certain representations are appropriate for some types of problems and why other representations are not appropriate, and maybe understand why a representation will produce wrong answers for one problem type and produce right answers for a different type of problem.

Recommendations for Further Research

This research indicates that further research is justified. The comments and questions that follow address some of these topics.

Is it possible to teach translation skills to students and how best can this be accomplished? On the basis of this research, it appears as though students have a much easier time translating information from Venn diagrams into native language than from native language to the Venn diagram. This then may indicate that students should first learn to translate Venn diagrams into native language and then use this translation skill to “deconstruct” English statements, via a Venn diagram, and thereby learn to translate information from the native language (context) into the Venn diagram. However, it may be that students best learn by exposure and practice in translating both directions at the same time. Could the student, by practicing the translations in both directions during the same period of time learn to use the “target” to decode the “source” information? Further research may reveal whether students learn translation skills best by repeated practice of one translation, or if students should be exposed to and practice two or even several types and directions of translations during the learning session.

This research indicates that there may be a correlation between reading ability and translation skill. Many of the error-prone students were competent at reading the words of the probability problems, but appeared “functionally illiterate” at decoding the relationships implied by the problem statement. It would be of interest to discover whether or not there exists a statistical correlation between higher level reading skills and mathematical translation skills. Further, if this correlation does exist, it would appear to indicate that training in reading and decoding would improve mathematical understanding and translation skills.

If problem-solving skills are developed via the student "mimicking" observed translation and problems solving behaviors, then one has to wonder how, and when, does a successful student use these mimicked skills to develop her or his own set of problem solving skills? That is, by what process does the competent student "internalize" the translation skills observed in the classroom? Further, how does the competent student integrate all the observed methods and decide which to use during a problem-solving tasks? If translation and representation skills can be taught, then it would seem that a study modeled on Schoenfeld's (1985) study of heuristic instruction and problem-solving abilities would demonstrate and clarify how, and when, students integrate translation and representation skills into their problem-solving repertoire. Such research could provide direction as to when and how teachers should integrate translation and representation skills into problem-solving sessions. In particular, the question arises: do children learn translation skills more efficiently at a younger age (as they appear to do with language skills) or should the teaching of translations and representations be reserved (for the major part) until the student is algebra ready?

REFERENCES CITED

- Anderson, J. R. (1978). Arguments concerning representations for mental imagery. Psychological Review, 85 249-277.
- Behr, M., Lesh, R., Post, T., & Silver, E. (1983). Rational-Number concepts. In R. Lesh, & M. Landau (Eds.), Acquisition of mathematics concepts and processes (pp. 91-126). New York: Academic Press.
- Bell, A. W. (1976). The learning of general mathematical strategies. Doctoral dissertation. Shell Centre for Mathematics Education, University of Nottingham.
- Bishop, A. J. (1989). Review of research on visualization in mathematics education. Focus on Learning Problems in Mathematics, 11 (1), 7-16.
- Bogdan, R. C., & Biklen, S. K. (1992). Qualitative research: An introduction to theory and methods (2nd ed.). Boston: Allyn & Bacon.
- Bradshaw, J. (1925). Drawings for teachers of solid geometry. The Mathematics Teacher, 17 (1), 37-45.
- Clements, K. (1982a). Visual imagery and school mathematics (part 1). For the learning of mathematics, 2 (2), 2-39.
- Clements, K. (1982b). Visual imagery and school mathematics (concluded). For the learning of mathematics, 2 (3), 33-39.
- Gfeller, M., Niess, M., & Lederman, N. (1999). Preservice teachers' use of multiple representation in solving arithmetic mean problems. School Science and Mathematics 99 (5), 250-257.
- Glasser, B. (1978). Theoretical sensitivity: Advances in the methodology of grounded theory. Mill Valley, CA: Sociology Press.
- Haertter, L. (1931). An effective method of teaching pupils how to solve verbal problems. The Mathematics Teacher, 24 (3), 166-175.
- Heid, K. M., Kerr, K., Blume, G. W., Marshall, J., Flanagan, K., & Iseri L. (1997). Conjecture and representational style in CAS-assisted mathematical problem solving. In J. Dossey, J. Swafford, M. Parmantie, & A. Dossey (Eds.), Psychology of Mathematics Education: Vol. 2. Discussion Groups (pp.585-592). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.
- Heid, K. M., Kerr, K., Blume, G. W., Marshall, J., Flanagan, K., & Iseri L. (1997). Roles of symbolic representation in CAS-assisted mathematical problem solving. In J. Dossey, J. Swafford, M. Parmantie, & A. Dossey (Eds.), Psychology of Mathematics Education: Vol. 2. Discussion Groups (pp.593-600). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

- Hodgson, T. (1992). The construction of Venn diagrams representations of set expressions: An analysis of a procedural task. Doctoral dissertation. School of Education, Indiana University.
- Hodgson, T. (1996). Students' ability to visualize set expressions: An initial investigation. Educational Studies in Mathematics, 30, 159-178.
- Hughes-Hallet, D., Gleason, A. (1994). Calculus. New York: John Wiley & Sons.
- Janvier, C. (1987). Translation Processes in Mathematics Education. In C. Janvier (Ed.) Problems of Representation in the Teaching and Learning of Mathematics (pp. 27-32). Hillsdale, NJ: Lawrence Erlbaum.
- Janvier, C., Girardon, C., & Morand, J. (1993). Mathematical symbols and representations. In P. Wilson (Ed.) Research Ideas for the Classroom: High School Mathematics (pp. 79-102). New York: MacMillan.
- Lean, G., & Clements, M. A. (1981). Spatial ability, visual imagery, and mathematical performance. Educational Studies in Mathematics, 12, 267-299.
- Lesh, R. (1981). Applied mathematical problem solving. Educational Studies in Mathematics, 12, 235-264.
- Lesh, R., & Landau, M. (Eds.). (1983). Acquisition of mathematics concepts and processes. New York: Academic Press.
- Lesh, R., Landau, M., & Hamilton, E. (1983). Conceptual models and applied mathematical problem-solving research. In R. Lesh, & M. Landau (Eds.), Acquisition of mathematics concepts and processes (pp. 263-307). New York: Academic Press.
- Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.) Problems of representation in the teaching and learning of mathematics (pp. 33-40). Hillsdale, NJ: Lawrence Erlbaum.
- Maki, D., & Thompson, M. (1996). Finite mathematics (4th ed.). New York: McGraw-Hill
- McDiarmid, G., Ball, D., & Anderson C. (1989). Why staying one chapter ahead doesn't really work: Subject specific pedagogy. In M. C. Reynolds, (ED.), Knowledge base for beginning teachers (pp. 193-206). New York: Pergammon Press.
- Mathematics Association of America (1923). National committee of mathematical requirements of the mathematics association of America, the reorganization of mathematics in secondary education. Washington D.C.
- National Council of Teachers of Mathematics. (1989). Curriculum and evaluation standards for school mathematics. Reston, VA: Author.
- National Council of Teachers of Mathematics. (1991). Professional standards for teaching mathematics. Reston, VA: Author.

- National Council of Teachers of Mathematics. (1995). Assessment standards for school mathematics. Reston, VA: Author.
- National Council of Teachers of Mathematics. (2000). Principles and Standards for School Mathematics. Reston, VA: Author.
- Norman, D. A. (1981). Categorization of Action Slips. Psychological Review 88, 1-15.
- Nyberg, J. A. (1925). Recent changes in the teaching of algebra. The Mathematics Teacher, XVII (1), 10-21.
- Patton, M. Q. (1990). Qualitative evaluation and research methods. Newbury Park, CA: Sage Publications.
- Piaget, J. (1977). Mental images. In H. E. Gruber & J. J. Voneche (Eds.) The essential Piaget. (pp.652-684). New York: Basic Books.
- Presmeg, N. C. (1992). Prototypes, metaphors, metonymies and imaginative rationality in high school mathematics, Educational Studies in Mathematics, 23, 595-610.
- Pylyshyn, Z. W. (1973). What the mind's eye tells the mind's brain: A critique of mental imagery, Psychological Bulletin, 80, 1-24.
- Schoenfeld, A. H. (1985). Mathematical problem solving. Orlando FL: Academic Press.
- Skemp, R. (1987). The psychology of learning mathematics: Expanded american edition. Hillsdale, NJ: Lawrence Erlbaum.
- Vinner, Shlomo (1989). The avoidance of visual considerations in calculus students. Focus on Learning Problems in Mathematics, 11, 149-156.
- Webb, N. L. (1979). Process, conceptual knowledge, and mathematical problem-solving ability. Journal for Research in Mathematics Education, 10, 83-93.
- Zazkis, R., Dubinsky, E. & Dautermann, J. (1996). Coordinating Visual and analytical strategies: A study of students' understanding of the group D4. Journal for Research in Mathematics Education, 27 (4) 435-457.
- Zimmermann, W. & Cunningham, S (Eds.). (1991). Visualization in teaching and learning mathematics. MAA Notes #19.

APPENDICES

APPENDIX A

ANALYSIS OF PROBLEM-SOLVING PROTOCOLS

A Framework for the Macroscopic Analysis of Problem-Solving Protocols

The method of protocol analysis employed in this study is a direct adaptation of Schoenfeld's (1985) problem-solving research. In his study, Schoenfeld reviewed protocols generated by university students during problem-solving sessions and categorized students behavior into one of four categories: reading, planning, implementing, and checking. Likewise, this research examined problem-solving protocols generated by university students. Rather than focusing on students' problem solving behaviors, however, this research examined students' (1) use of representations, and (2) translation between the representations. The overall intent of the method was to identify a set of potentially important problem-solving tools and the manner in which students use these tools to solve probability problems. Although specific to the problems administered as part of this study, the method of analysis is generalizable. That is, the method could (with some adaptation) serve as a framework for investigating students' use of multiple representations in other problem settings.

In general, translations are activities (cognitive and overt) that result in a representation of problem information. That is, the act of "translating" allows the problem-solver to depict all or part of a problem in a new and different manner. Thus, it is appropriate to look for translation activities at points in the protocol data where the problem-solver significantly changes the problem representation. A representation is identified by the manner in which the problem is (1) depicted on paper or (2) spoken of by the problem-solver. Using this definition, the method of protocol analysis employed in this study sought to distinguish between the act of translating between representations from the *use* of a particular representation.

To assist in the categorization of student behavior, the protocols were partitioned into macroscopic (time) segments of uniform behavior called "episodes." An episode is a

period during which a problem-solver is engaged in only one task. For the purpose of this research, the tasks identified are either (1) making a translation or (2) performing some piece of the problem-solving task using a single representation (non-translation task). Each task was categorized into one of the following six classes of episodes:

- I. Translation Behavior (between representations):
 - a) Context and Venn diagram
 - b) Venn diagram and Symbolic Algebra
 - c) Context and Symbolic Algebra
- II. Non-translation Behavior (Inner-representation activity):
 - a) Context
 - b) Venn Diagram
 - c) Symbolic Algebra

It is important to note that student behaviors must be identifiable to be coded. Toward this end, the method employed by this study is consistent with that of Lucas and his colleagues (1980). All behavior is required to be explicit. Otherwise, it is not coded.

Once a protocol was parsed into episodes, the moment at which the transition between episodes took place was identified. In other words, the researcher sought to identify episodic junctures--points at which the problem-solver changed from "using a representation" to "constructing a representation," or vice-versa.

The following is a list of the criteria used for parsing the protocols. Appendix B presents an example of how the criteria were used to parse a protocol.

Criteria 1. An explicit action indicating a translation.

Criteria 2. An explicit action indicating no translation taking place.

Criteria 3. Verbalization indicating a translation.

Criteria 4. Verbalization indicating no translation taking place.

As a technical issue, it should be noted that the presence of the verbal criteria in a protocol does not necessarily verify that a translation did or did not take place. In part, this is due to the nature of the data: talk-aloud protocols. In the process of solving the problem, an explicit action (criteria one or two) was generally accompanied by a verbal explanation of the activity (criteria three or four). Together, the two served as clear indicators of cognitive activities in which the problem-solver was engaged. However, a verbalization indicating a translation (criteria three) was not always accompanied by an observable translation. Rather, the subject may only have been thinking (aloud) of taking the action. The verbal criteria, therefore, are indicators of actions, but do not guarantee that the verbalized activity occurred.

The following explicit actions (criteria one) were viewed as indicative of translation behavior:

- (1) Placing values (numbers) from the problem statement into regions of the Venn diagram [contextual to Venn diagram translation],
- (2) Constructing symbolic expressions and equations for the purpose of the numerical value of a region of a diagram [Venn diagram to symbolic algebra translation], or
- (3) Transferring the solutions obtained through symbolic (algebraic) manipulations into the regions of a Venn diagram [symbolic algebra to Venn diagram translation].

Explicit actions indicative of no translation (criteria two) include the following:

- (1) Solving an algebraic equation for an unknown [symbolic algebra mode],
- (2) Making sense (through pointing or other outward action) of the information contained in the Venn diagram (after having placed the values given in the problem statement into regions to the Venn diagram) and voicing something to the effect, "I think I need to find the number that goes in this intersection." [Venn diagram mode]

Note that for category (2), the explicit action indicating no translation was identified by a combination of the subjects' body language, verbal activity, and the fact that the "natural"

next activity (natural to someone that has done, or knows how to do the problem) hasn't taken place. In this case, the "natural" next step is to use algebra to find the unknown value represented by a region of the Venn. Thus, subjects operated within the Venn diagram mode in order to decide what course of action to pursue.

The following types of verbalizations (criteria three) were viewed as indicative of translation behavior:

- (1) Stating that a given value should be placed in a particular Venn region [Context to Venn diagram translation], or
- (2) Verbalizing an equation that relates the values of the regions of a Venn diagram that can be solved to find an unknown value represented by one region of the Venn diagram [Venn diagram to symbolic algebra translation].

Verbalizations indicative of no translation (criteria four) include the following:

- (1) Explanation of an algebraic step while solving for an unknown value [symbolic algebra mode],
- (2) Making sense of (i.e. re-reading aloud) the information stated in the problem statement [Contextual mode].

The parsed protocol in Appendix B (and in the body of the text) was constructed by considering the problem-solving behaviors displayed on the video record of the problem sessions and identifying distinct episodes. Therefore, it is indicative of the precise point in the problem-solving session where translation activities (or non-translation activities) changed. Once the episodes were identified, the time at which the change of behavior took place was identified by a re-checked through the use of video recordings and transcripts.

Reliability in the parsing of protocols appears to be quite high. After several of the protocols were parsed (by the researcher), the transcripts were reviewed by committee co-chair Ted Hodgson. He independently reviewed the protocol data and developed his parsed version of the data. The parsed transcripts were then compared. Differences in the

researchers' interpretation of the data were then noted and discussed. A consensus was then reached by reviewing the video-recordings of the problem-solving session. Although no data were collected regarding inter-rater reliability (this is certainly one area in which the study can be improved and will need to be improved prior to consideration by the broader scientific community), no major (and very few minor) differences in the researchers' parsings were detected. This finding is consistent with the Schoenfeld's (1985) assessment of his parsing process. In particular, Schoenfeld remarked that "with a small amount of training, partitioning protocols into episodes and delineating new-information points are both rather straightforward processes." To parse the protocols in his study, Schoenfeld used a team of three undergraduates (that he had trained). As with the present study, he found the reliability to be quite high, albeit Schoenfeld presents no formal data to support his claim.

APPENDIX B

AN EXAMPLE OF PROTOCOL ANALYSIS AND PARSING

An Example of Protocol Analysis and Parsing

Protocol B, below, is a transcript of one student's written and spoken work. To make the protocol, the audio tapes and the audio portion of the video tapes of the subjects' interviews were transcribed. Then, the video tape of the problem-solving session, the students written work, and the transcript of the students' verbal comments were used to make a full transcript of the problem-solving session. The full transcript correlates the subjects spoken words with his or her written work.

Protocol B is a transcript of Miseon solving problem number two. In this protocol, and those included in the Analysis of the Data (chapter four), the left column is a transcript of the students spoken remarks as she or he worked the problem. The numbers in parenthesis inserted in the spoken transcript are the elapsed times from when Miseon started working on her solution. (For the sake of readability, the timing marks are not included in the chapter four examples.) The timing started when the student finished reading the statement of the problem aloud. The left column is a transcript of all of the problem solver's spoken words. The right column is a transcript of the written work, including all graphs and algebraic symbols. Each drawing or figure is correlated with the verbal remarks immediately to it's left. That is, each drawing in the right column corresponds (as closely as possible) with the transcript of the spoken words appearing to the drawings immediate left. Breaks in the subject's verbalization (quiet times) are indicated by line spaces in the transcript, although there is no correlation between the number of line spaces and the length of quiet time. All marks (with three general exceptions) included on the transcript of the written work are also found on the subjects written work. The three exceptions are (a) a printed hand indicates that the student pointed to some part of his or her drawing, (b) a stylized calculator indicates that the subject used a calculator for a period of time, and (c) an underline indicates that a number or some other mark has been added to an existing graph,

drawing, or equation. The underline is included only at the first occurrence of the new mark. The underline will not be shown on any subsequent appearance of the drawing or equation. If a student pointed to something on his or her drawing a hand is included on the transcript of the written work with a finger indicating the place where the student pointed. The words spoken by the subject as he or she pointed are found in the transcript immediately to the left of the hand. All data, numbers, or graphics that the student used during the problem solving session are included in the transcript. The figures in the right column of the transcript appear (as closely as possible) identical to the written work turned in by the student. In Protocol B, as in the protocols included in chapter four, a segment number is included in brackets (e.g., [Segment 1].) These are included so that the reader can easily locate in the transcript any of the subjects words or phrases that are referred to in the data analysis discussion.

Protocol B

Name: Miseon Problem: Two

At Giant State University a survey of students taking college mathematics found that 40% took college algebra, 30% took statistics, and 42% took neither course. If one of the students from the survey is chosen at random, what would be the probability that she or he took either algebra or statistics but not both courses?

[Segment 1]

Okay. (5 seconds)

$A1 \Rightarrow 40\%$

I'll go--okay, forty percent took college math.

Thirty percent statistics--thirty percent.

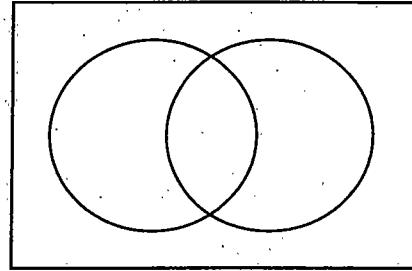
$St \Rightarrow 30\%$

Forty-two percent took neither.

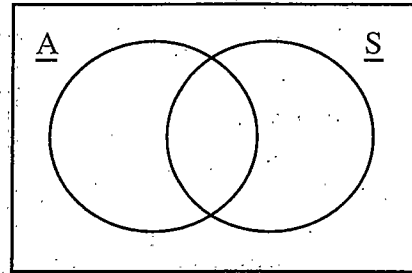
That means like A--S is--.
(35 seconds)

$(A \cup S)' \Rightarrow 42\%$

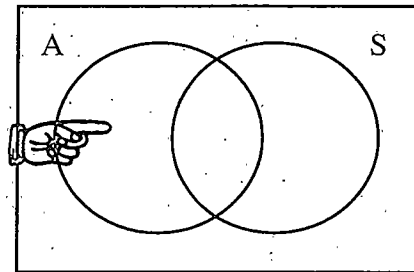
Okay, so the question is--asks.
(40 seconds)



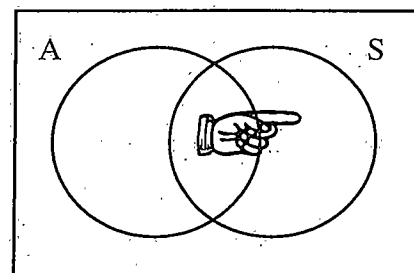
[Segment 2]
This is algebra and this is statistics.



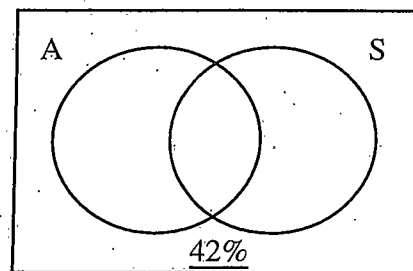
They want to know either Algebra or the
Statistics but not both courses, meaning that
you want to have--this side



or this side. (60 seconds)



So this is forty-two percent.



(1 minute 5 seconds)

[Segment 3]

So uhh all this total should be one hundred minus forty-two. I think fifty-eight percent, right.

$$100 - 42 = 58\%$$

(1 minute 25 seconds)

Like total A S is A plus S minus--uhh S.

$$A \cup S = A + S - A \cap S$$

So fifty-eight equals forty plus thirty minus--.

$$58 = 40 + 30 - X$$

Okay, X will be--

$$X =$$

seventy minus fifty-eight.

$$\begin{array}{r} 70 \\ - 58 \\ \hline 12 \end{array}$$

Umm--Okay.

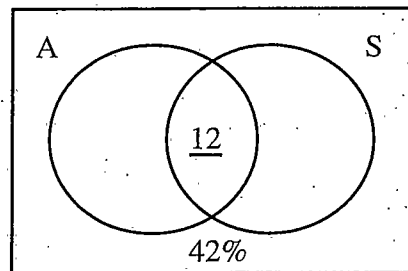
Twelve.

$$X = \underline{12}$$

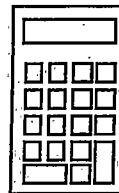
(1 minute 50 seconds)

[Segment 4]

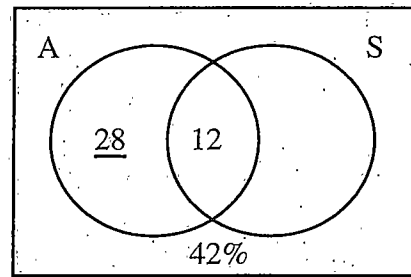
We know that's--this is twelve.



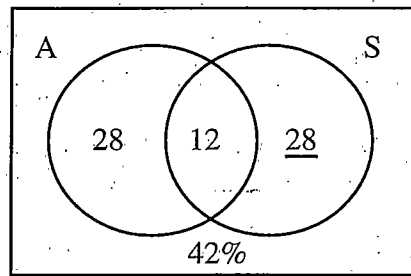
Like, so what we do is subtract forty
[laugh] minus twelve is like--



twenty-eight.



Thirty minus twelve is eighteen.



(2 minutes 5 seconds)

[Segment 5]

So like--either Algebra or Statistics--so you just have to add it.

Twenty-eight plus twenty-eight--fifty-six!

$$28 + 28 = 56$$

(2 minutes 20 seconds)

(End of Protocol B)

The video-tapes, transcripts, and student's written work were analyzed to determine (a) what representations the student used, (b) the beginning and ending times of each translation task, (c) whether or not the student could successfully translate between representations, (d) and if the subject was competent using the representations. How the various protocols was parsed is described in Appendix A. The timing information was transferred to a time-line. An example is shown in Time-line B, included below. The time-lines are an adaptation of Schoenfeld's (1985) time-line tables. The time-lines identify the student's activities in five second intervals. Any change in the student's behavior is

recorded and the time of the change is rounded and recorded to the nearest five second interval.

The following is a list of the criteria used for parsing the protocols.

Criteria 1. An explicit action indicating a translation.

Criteria 2. An explicit action indicating no translation taking place.

Criteria 3. Verbalization indicating a translation.

Criteria 4. Verbalization indicating no translation taking place.

Protocol A is parsed as follows;

Time 0:00 until 0:05; Miseon studied the problem in context (criteria 2). No words were spoken. Because she is studying the problem in context this is a contextual episode.

Time 0:05 until 0:35; Miseon is associating values from the problem statement with symbols (criteria 1) and verbalizing the values assigned (criteria 3). This is a translation between Context and Venn diagram.

Time 0:35 until 0:40; Miseon reads problem statement (criteria 2 and 4). She is operating in Context representation.

Time 0:40 until 1:05; She drew a Venn diagram, described the regions of the Venn diagram (associating them with names given in the problem statement), and placed a value found in the problem statement in a region of the Venn diagram (criteria 1 and 3). She is translating between Context and Venn diagram.

Time 1:05 until 1:25; Miseon wrote down but did not solve a series of equations (criteria 1). The equations used symbols representing regions of the Venn diagram and values she had previously placed in the Venn diagram. She explained the process as she wrote the equations (criteria 3). She was translating between the Venn diagram mode and the symbolic mode.

Time 1:25 until 1:50; Miseon developed and solved a set of equations, but the values and symbols were not taken from the Venn diagram or earlier work (criteria 2). She

described her work as it developed (criteria 4) without referring to any other mode. The solution to her equation was assigned to a symbol but not assigned (until later) to a region of the Venn diagram (criteria 2). She was operating in the symbolic mode.

Time 1:50 until 2:05; First, she wrote a value on the Venn diagram (criteria 1). This was the number she had found using algebra during time 1:25 to 1:50. She then used her calculator and algebra to find other values that were written on the Venn diagram (criteria 1). As she found the values, she explained what the values were and where they were to be placed on the Venn (criteria 3). This was a translation between the Symbolic mode and the Venn diagram mode. Notice, she made an error on the Venn diagram. She spoke the word "eighteen" but wrote 28 on the Venn diagram.

Time 2:05 until 2:20; Miseon looks at the problem statement (Contextual mode) and reads "Algebra or Statistics--so you just add it." (Criteria 3) The "it" being the regions of the Venn diagram representing algebra and statistics but not the intersection of the two regions. She wrote down the values from the Venn diagram (criteria 1) and added them. This is a translation between the Venn diagram mode and the contextual mode.

The parsing and episode information is coded into Time-line B included below.

Time-line B

Student: Miseon

Problem: Two

Comments: Competent, Incorrect Answer (subtraction error on final step of the problem)

EPISODE	
Context	
$C \leftrightarrow V$	    
Venn	
$V \leftrightarrow S$	   
Symbolic	
$C \leftrightarrow S$	     
Time	0:00:05:10:15:20:25:30:35:40:45:50:55:1:00:1:05:1:10:1:15:1:20:1:25:1:30

EPISODE	
Context	
$C \leftrightarrow V$	  
Venn	
$V \leftrightarrow S$	   
Symbolic	   
$C \leftrightarrow S$	
Time	1:30:1:35:1:40:1:45:1:50:1:55:2:00:2:05:2:10:2:15:2:20:2:25:2:30:2:35:2:40:2:45:2:50:2:55:3:00

Note. Context refers to the contextual (English) representation, Venn refers to the Venn diagram representation, and Symbolic refers to a symbolic representation. $C \leftrightarrow V$ indicates a translation between the contextual mode and a Venn diagram. $V \leftrightarrow S$ indicates a translation between a Venn diagram and a symbolic mode, and $C \leftrightarrow S$ indicates a translation between the contextual mode and a symbolic mode. The times are in five second intervals.

Time-line B indicates that Miseon started her attempt at solving problem two by spending five seconds analyzing the problem in context, that is, she first thing she did was study the problem as an English statement. Miseon then used thirty seconds translating the problem into symbolic notation (as is seen in the first portion of Protocol B.) She then returned to the contextual problem statement (five seconds) and translated information from the problem into a Venn diagram, using 25 seconds. Miseon then used 20 seconds translating information from the Venn diagram into symbolic algebra, forming an algebraic equation. She next spent 25 seconds solving these equations for the unknown amounts and 15 seconds more transferring (translating) this information back into her Venn diagrams. Miseon realized she had the information necessary to answer the problem and used 15 seconds translating the information from the Venn diagram into the representation in which

the problem was originally stated, that is, translation between the Venn diagram and the contextual mode..

The protocols and time-lines are used to identify: (a) any representations the student used, (b) patterns in the use of the representations, and (c) time patterns in the student's translations.

APPENDIX C

SAMPLE QUESTIONS FOR PRE-SESSION INTERVIEW
AND SAMPLE INTERVIEW

Pre Problem-Solving Interview Questions

1. What year of college are you in?
2. What is your major?
3. What High School Mathematics classes did you take?
4. What grades did you get in your high school mathematics classes?
5. Have you had any prior college math classes?
6. Are you ever afraid of Math?
7. What is your favorite Subject?
8. And your least favorite subject?
9. Do you feel confident about Math?
10. Do you feel confident about Math 118?
11. In what subjects are you most competent?
12. In what subjects are you least competent?
13. Do you feel confident about doing Math 118?
14. What grade are you getting in Math 118?
15. What grade did you get on the first test in Math 118?
16. Are there any questions you would like to ask me?

Sample Interview Transcription

Interviewer: What year are you in college?

Kyle: I'm a freshman.

Interviewer: Is this your first semester?

Kyle: Yes, it is.

Interviewer: What math did you take in high school?

Kyle: I took algebra and College Algebra my senior year--and then geometry.

Interviewer: The geometry was your senior year?

Bran: I believe the geometry was my sophomore year or my junior year--my sophomore year.

Interviewer: So you had freshman algebra then geometry and then--what did you have your junior year?

Kyle: It was algebra again then College Algebra. It was just the advanced--I mean it touched on some calculus but it was basically college algebra.

Interviewer: What grades did you make in high school math?

Kyle: On average in the area of a B--about.

I: What grade are you making in Math 118?

Kyle: I believe probably in the area of a high D or low C.

Interviewer: What did you get on the first test?

Kyle: A fifty-five percent.

Interviewer: A sixty was about average?

Kyle: Yeah, I think a fifty-five came to a C.

Interviewer: Do you like math?

Kyle: Yes.

Interviewer: Do you like Math 118?

Kyle: Umm--I do. I mean, I can appreciate what it is and understand it.

Interviewer: What is your major in college?

Kyle: Umm--Undecided at this point in time.

Interviewer: What are you leaning towards?

Kyle: Umm--It may be some form of business or on the opposite end, some form of journalism.

Interviewer: Why are you taking Math 118?

Kyle: As a basic class. It applies to a lot of majors and being undecided I decided to take uhh classes that would sort of fill some prerequisites for other classes, higher levels. Just to get me started.

Interviewer: Have you ever been afraid of math or been afraid when you do math?

Kyle: No.

Interviewer: For your tests, do you ever get anxious?

Kyle: No, I really don't get anxious. I'm pretty confident in math. Umm--It's always been one of my stronger subjects. I just think sometimes whether or not I'm anxious is affected by whether or not I've studied or know the material.

Interviewer: What is your favorite subject?

Kyle: My favorite subjects would umm basically a form of English. I do enjoy history and all the other classes but uhh I enjoy English and music.

Interviewer: What is your least favorite.

Kyle: Probably any form of science. Biology or probably chemistry.

Interviewer: On the other hand, not favorite and least favorite, what's your strongest subject?

Kyle: My strongest subject would have to be uhh that would be math. Because of the way that I--I guess I learn it. I learn it in a practical sense, common sense.

Interviewer: And what's your weakest?

Kyle: My weakest would probably be chemistry and biology. But that--because I'm--I'm lazy to it. I don't appreciate it or enjoy it so I don't exactly put the effort forth and therefore I suffer for it.

Interviewer: Do you consider yourself pretty competent in mathematics?

Kyle: Yes, I do.

Interviewer: And in Math 118?

Kyle: Umm--I do. But again, I think a lot of the times, my effort shows that umm I end up not understanding it because I don't put the effort forth.

Interviewer: Are you pretty confident when you do math? Not competent but confident.

Kyle: Yes, I am. I'm confident. Umm--I'm confident that I can understand the concepts and it--it will really register in my head, and I--I do. I do understand it a lot of the time, most of the time.

Interviewer: What about in Math 118? Are you confident when you do the problems and when you do the tests?

Kyle: I am confident in the material that I know. The test was a bit confusing. Umm--I was a little bit caught off guard being my first year. Umm, I am confident when I take quizzes and test. I--I just umm sometimes my lack of knowledge on the material ends up affecting my grade.

Interviewer: What grades have you been getting on the quizzes?

Kyle: The first few quizzes I was receiving uhh I think it was ten out of ten, eight out of ten umm, but the last few quizzes I've received four out of ten's and I know I had a quiz today that I got zero out of ten on.

Interviewer: Why is that?

Kyle: Uhh--I haven't been understanding the material lately. The last few classes I haven't--it hasn't registered. He has been going over it too fast for me in class and I think I'm trying to learn it a different way on my own and it's just not working.

Interviewer: Do you study a lot on your own?

Kyle: No, I do not.

Interviewer: Should you be?

Kyle: Yeah, I should.

Interviewer: Are there help sessions and stuff?

Kyle: Yes, there are. My schedule at night conflicts with a lot of the help sessions so I tend to be out on my own on it and I just work beyond in class.

APPENDIX D

CONSENT FORM

Subject Consent Form For Participation In Human Research**Indiana University****Students Use of Multiple Representations in Mathematical Problem Solving**

You are being asked to participate in a study of how students use mathematical representations while probability problem solving. The term "mathematical representation" means any of the written or spoken forms which can be used to represent any part of a mathematical problem. These forms could include written or spoken English, graphs or diagrams, or algebraic equations.

This research will help us better understand how students use the various representations while problem solving. This research will help to identify how various tasks are completed using representations and where students have trouble using the representations. With this information it may be possible to give teachers information about better ways to teach mathematics.

You are being asked to participate in this research because you are enrolled in Math 118. That is, because the subject material you are studying in Math 118 uses both algebraic and graphical data and the probability problems in Math 118 require you to use these representations to find solutions. For this reason the representations being observed in this research will be familiar to you. There will be subjects used from several of the Math 118 sections. How you solved the problems will be examined, but there is no grading of the problems or comparisons made between you and other students.

If you agree to participate you will be asked to solve between four (4) and six (6) probability problems while being video taped. Your face will not appear in the video, only your hands will show as you work the various problems on a sheet of paper. You will also be asked to explain what steps you are using to solve the problem as you attempt to work

each of them. Some of the problems you might solve with little difficulty. Others are designed to be difficult.

It may take you one-half to one hour to work through the problem set. After you have completed the problems I will ask you to view the video with me so that you can explain why you used various procedures or methods to solve the problems. That is, I will ask you to explain your thinking. This last part of the interview will be recorded with a tape recorder.

The entire session should require between one and one and one-half hours of your time.

There is no risk to you as far as your grade in Math 118 and no one else besides myself and Dr. Ted Hodgson will view the video recording. I will make a written transcript of the video and tape recordings for my research, but you will not be identified in any way on the written transcript. That is, no one will be able to identify you through the transcript. Further, the video tapes will be erased after they have been transcribed. The transcriptions will be coded so that there will be no way to identify any individual's work.

After the interview you will be given ten dollars.

If you choose not to participate there will be no repercussions and there will be no effect on your Math 118 grade.

Do you have any questions now? Be sure to ask if you have questions at any time during the research.

You can reach me, Jim Ballard, at Ted Hodgson's office 201 Swain East (855-1960) through October 10, 1996. After that time I can be reached at: Math Dept, MSU, Bozeman, MT 59715.

If you have further questions at any time about the rights of Human Subjects they can be answered by the Human Subjects Committee Chairperson here at Indiana University, phone 855-3067.

Authorization: I have read the above and understand the discomforts, inconvenience and risk of this study. I,

_____, agree to participate in this research. I understand that I may later refuse to participate, and that I may withdraw from the study at any time. I have received a copy of this consent form for my own records.

Signed _____

Investigator _____

Date _____

APPENDIX E

PROBLEM SET

Problem 1**Problem 1** Name: _____

In the evening, between 7 and 9 PM, Ann's phone is not busy 75% of the time, John's phone is busy 35% of the time, and both of their phones are busy 10% of the time. What is the probability that neither of their phones will be busy at some moment during this part of the evening?

Problem 2

Problem 2 Name: _____

At Giant State University a survey of students taking college mathematics found that 40% took college algebra, 30% took statistics, and 42% took neither course. If one of the students from the survey is chosen at random, what would be the probability that she or he took either algebra or statistics but not both courses?

Problem 3

Problem 3 Name: _____

Each Monday a student attends finite mathematics class with probability 0.8, skips accounting class with probability 0.4, and skips both with probability 0.1. What is the probability that he attends at least one of these two classes on Monday?

Problem 4**Problem 4** Name: _____

A group of television viewers were questioned about whether they enjoyed Star Trek DSNine, Star Trek Voyager, or Star Trek Next Generation. The following data were obtained.

- 38 percent commented favorably about Next Generation.
- 15 percent commented favorably about Voyager.
- 17 percent commented favorably about DSNine.
- 7 percent commented favorably about both Next Generation and Voyager.
- 8 percent commented favorably about both Next Generation and DSNine.
- 5 percent commented favorably about DSNine and Voyager.
- 3 percent commented favorably about Star Trek Voyager, Next Generation, and DSNine.

A questionnaire is selected at random. Find the probability that the viewer commented favorably on exactly one of the three Star Trek shows?

Problem 5**Problem 5** Name: _____

A securities analyst is reviewing the performance of a group of computer manufacturers. She finds that 75% have increased sales and 30% have increased earnings. She also finds that 15% have increased neither sales nor earnings. One manufacturer is selected at random from the group. What is the probability that it has increased both sales and earnings?

MONTANA STATE UNIVERSITY - BOZEMAN



3 1762 10344168 7