Lattice structures that parameterize regulatory network dynamics

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ABSTRACT

We consider two types of models of regulatory network dynamics: Boolean maps and systems of switching ordinary differential equations. Our goal is to construct all models in each category that are compatible with the directed signed graph that describe the network interactions. This leads to consideration of lattice of monotone Boolean functions (MBF), poset of non-degenerate MBFs, and a lattice of chains in these sets. We describe explicit inductive construction of these posets where the induction is on the number of inputs in MBF.

Our results allow enumeration of potential dynamic behavior of the network for both model types, subject to practical limitation imposed by the size of the lattice of MBFs described by the Dedekind number.

1. Introduction

The goal of this paper is to construct lattice and poset structures that define proximity between regulatory network models and enable their enumeration.

Consider a regulatory network $N = (V, E, \delta)$ with a set of nodes V with $|V| = n$ and directed signed edges E with signs $\delta = \delta_{ji} \in \{\pm 1\}$ assigned to each edge (i, j) from node i to j. The directed edges of regulatory network represent monotone directed influence of one node on the other: $\delta = 1$ represents monotonically non-decreasing and $\delta = -1$ monotonically non-increasing influence.

The traditional Boolean network model assumes that each node has two states: the OFF state, usually represented by 0 and and the ON state, represented by 1. Let $\mathbb{B} := (\{0, 1\}; 0 \prec 1\})$ be a Boolean lattice with natural order $0 \prec 1$ and let $\mathbb{B}^n$ be a lattice of Boolean n vectors with order induced component-wise by $\prec$. The set $\mathbb{B}^n$ is a phase space of a Boolean network model, and its dynamics on $\mathbb{B}^n$ is given by a $F = (F_1, \dots, F_n)$, where function F_j updates state of node $j \in V$ based on the current states of the *source nodes* $\mathbf{S}(j)$. Function $F_j : \mathbb{B}^{|\mathbf{S}(j)|} \to \mathbb{B}$ is a *monotone* Boolean function with respect to each of its input respecting the sign δ_{ji} associated to each edge $i \multimap j$, $i \in \mathbf{S}(j)$.

Definition 1.1. *A function $f : \mathbb{B}^n \to \mathbb{B}$ is a positive (negative) monotone Boolean function if $b^1 \prec b^2$ implies $f(b^1) \preceq f(b^2)$ ($f(b^1) \succeq f(b^2)$).*

The collection $F = (F_1, \dots, F_n)$ is the Boolean model associated to the network N on the state space $\mathbb{B}^n$. Dynamics induced by F can be generated by various update rules from fully synchronous to asynchronous Curry, Monteiro and Chaouiya (2019); Paulevé, Kolcak and Chatain (2020); Chatain, Haar and Pauleve (June 2018).

An important problem in this context is the description of all monotone Boolean functions (MBF) that are consistent with the given network. The size of the set of monotone Boolean functions with k inputs, known as *Dedekind number*, is only known for $k \leq 9$ and grows faster that exponentially. This paper presents an explicit inductive construction of all MBFs with k inputs to help understand all potential dynamics of networks with less than $k \approx 6$ inputs, rather than to address the value of Dedekind number. The collection of all network-compatible MBFs can be used to study changes in network dynamics as a function of changing Boolean function F Abou-Jaoude and Monteiro (2019). Since this class of functions is large even for moderately sized networks, attention is often focused on *non-degenerate* MBFs where every input variable influences the output in at least one input state (see Definition 2.4). Characterization of non-degenerate MBFs with k inputs as anti-chains of the Boolean lattice $2^{[k]}$ of all subsets of a set $[k]$ with k elements is given in Curry et al. (2019). We describe an inductive algorithm that constructs the partially ordered set (poset) of non-degenerate MBFs as a subset of a lattice of all MBFs with k inputs from the poset of non-degenerate MBFs with $k - 1$ inputs and the lattice of all MBFs with $k - 1$ inputs.

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Monotone Boolean functions play a surprising role in some differential equation network models as a descriptor of their parameter regimes. Based on earlier work of Thomas Thomas (1973, 1991); Thomas, Thieffry and Kaufman (1995) and Snoussi Snoussi and Thomas (1993); Snoussi (1989), Glass Glass (1975); Glass and Kauffman (1973, 1972); Glass and Pasternack (1978) proposed to embed Boolean models into a set of ordinary differential equations. These systems became known as *Glass networks* or *switching networks*

$$\dot{x}_j = -x_j + \lambda_j F_j(b(x)) \tag{1}$$

where $x \in \mathbb{R}^{n+}$, F_j is a Boolean function, λ_j is a rate constant and $b = (b_1(x_1), \dots, b_n(x_n))$ is a function that assigns to a continuous variable x_i a Boolean variable $b_i(x_i)$ by using a *threshold* θ_i. We consider two possible functions

$$\sigma^+(x) = \begin{cases} 1 & x > \theta \\ 0 & x < \theta \end{cases} \qquad\qquad \sigma^-(x) = \begin{cases} 0 & x > \theta \\ 1 & x < \theta \end{cases}$$

Then $b_i(x) = \sigma^+(x)$, if the edge $i \to j$ is positive, and $b_i(x) = \sigma^-(x)$, if the edge $i \dashv j$ is negative.

Switching systems have relatively simple non-linearities and therefore considerable analysis can be done to understand their dynamics; however, serious challenges arise when one seeks to properly continue some solutions across threshold hyperplanes in $\mathbb{R}^{n+}$. A natural questions to what extend the switching systems predict dynamics of nearby smooth systems are delicate Ironi, Panzeri, Plahte and Simoncini (2011); Gedeon, Harker, Kokubu, Mischaikow and Oka (2017). Apart from piecewise constant nonlinearity, the assumption of simultaneous and identical effect of the node i onto all its *targets* $k \in \mathbf{T}(i)$ is biologically questionable. In particular, when the state i switches to 1 all downstream edges will get activated and thus all the target nodes either activated or repressed. A more realistic (and generic) assumption is that the activation of each of the targets of i happen at a different level of i; the single activation threshold θ_i should be replaced by a collection of thresholds $\{\theta_{ki}\}_{k\in\mathbf{T}(i)}$ where $\theta_{ki} \neq \theta_{si}$ for $s, k \in \mathbf{T}(i), k \neq s$. This assumptions also resolves some of the technical issues regarding continuation of solutions of switching systems across the thresholds. In addition, different orderings of these thresholds may lead to different dynamics.

We make another useful generalization. Rather than considering Boolean functions $\sigma^{\pm}$ multiplied by a single constant λ_j, we allow the values of σ^+ and σ^- to be arbitrary non-negative values $0 \leq L < U$ where L stands for "lower" and U for "upper" expression level:

$$\sigma^+(x) = \begin{cases} U & x > \theta \\ L & x < \theta \end{cases} \qquad\qquad \sigma^-(x) = \begin{cases} L & x > \theta \\ U & x < \theta \end{cases} \tag{2}$$

The three parameters L, U, θ associated to a function $\sigma^{\pm}(x_i)$ can have different values for each edge $i \to j$ of the network RN. Thus in general $L = L_{ji}, U = U_{ji}$ and $\theta = \theta_{ji}$ in expression (2). With these changes, a general switching system has the form Edwards (2001); Ironi et al. (2011); Cummins, Gedeon, Harker, Mischaikow and Mok (2016); Gedeon (2020)

$$\dot{x}_j = -\gamma_j x_j + F_j(\sigma(x)) \tag{3}$$

where $\sigma = (\sigma_{j1}^{\delta_{j1}}, \dots, \sigma_{jk}^{\delta_{jk}})$, $k = |\mathbf{S}(j)|$ is the number of sources of node j and the sign δ matches the sign of the corresponding edge (i, j) in the network N. The function F_j is often assumed to be a homogenous polynomial of its arguments Ironi and Panzeri (2009); Ironi et al. (2011), or have a specific form of product of sums Cummins et al. (2016); Kepley, Mishaikow and Zhnang (2021), but it can be more general Thieffry and Romero (1999); Bernot, Cassez, Comet, Delaplace, Muller and Roux (2007); Crawford-Kahrl, Cummins and Gedeon (2022).

Series of papers Cummins et al. (2016); Cummins, Gedeon, Harker and Mischaikow (2018, 2017); Harker (2017); Harker and Cummins (2017); Gedeon (2020); Duncan, Gedeon, Kokubu, Oka and Mischaikow (2021); Duncan (2021) developed a modeling and computational approach DSGRN. DSGRN is based on observation that the parameter space of all $L_{ij}, U_{ij}, \theta_{ij}$ admits a finite decomposition into regions, where parameters in the same region support the same coarse dynamics, as described by a state transition graph (STG). Each such region is called a *DSGRN parameter* and is represented as a node in a parameter graph PG. The edges of PG capture spatial organization of DSGRN parameters by connecting nodes corresponding to parameter regions that share co-dimension one boundary. Analysis of STG dynamics across the entire PG have been used to examine dynamics of complex gene regulatory networks Xin, Cummins and Gedeon (2020); Gameiro, Gedeon, Kepley and Mischaikow (2021); Gedeon, Cummins, Harker and Mischaikow (2018); Diegmiller, Zhang, Gameiro, Barr, Alsous, Schedl, Shvartsman and Mischaikow (2021).

On the other hand, as was shown in Crawford-Kahrl et al. (2022), there is a one-to-one correspondence between nodes of PG and collections $f = (f_1, f_2, \ldots, f_n)$ where f_j is a $|\mathbf{T}(i)|$-chain of monotone Boolean functions (see section 2.1 for definition of a chain). This correspondence arises from an observation that DSGRN parameters correspond to all different ways how the input edges $j \to i, j \in \mathbf{S}(i)$ can activate collection of edges $i \to k, k \in \mathbf{T}(i)$. The input pattern can be represented as a Boolean vector $d \in \mathbb{B}^{\mathbf{S}(i)}$ that indicates which input edges are active and which are not. The activation pattern describes all the ways that nodes $k \in \mathbf{T}(i)$ respond to values d. This response, however, is transmitted through a varying level of a single variable x_i associated to node i. As the value of x_i passes through threshold θ_{ki}, associated with an edge $k \in \mathbf{T}(i)$, the edge k gets activated. Since the thresholds $\theta_{ki}, k \in \mathbf{T}(i)$ are linearly ordered, if an input d activates the s-th node, it will also has to activate all nodes $1, \ldots s-1$ associated with lower thresholds $\theta_{1i} < \theta_{2i} < \ldots < \theta_{s-1,i} < \theta_{si}$. If we represent the activation of a particular $s \in \mathbf{T}(i)$ at θ_{si} by a Boolean function f_s, by setting $f_s(d) = 0$ for all $d \in \mathbb{B}^{\mathbf{S}(i)}$ which do not activate s and $f_s(d) = 1$ for all $d \in \mathbb{B}^{\mathbf{S}(i)}$ activate s, then the linear order of the the thresholds $\theta_{ki}, k \in \mathbf{T}(i)$ imply that $f = (f_1, f_2, \ldots, f_n)$ is a $|\mathbf{T}(i)|$-chain of monotone Boolean functions.

As we will show later, the set of monotone Boolean functions is a graded lattice L and thus each node in PG corresponds to a collection $(f_1, f_2, \ldots, f_n)$, where f_i is a chain in a lattice L_i. The set of non-degenerate MBFs forms a partially ordered subset of lattice of all MBFs L, that inherits its graded structure.

The goal of this paper is to describe an inductive algorithm that builds lattice L^k_M of monotone Boolean functions with k inputs, as well as the poset of non-degenerate MBFs $L^k_N \subset L^k_M$, from L^{k-1}_M and L^{k-1}_N. To facilitate the initial step of this algorithm we explicitly describe $L^2_N \subset L^2_M$ and $L^3_N \subset L^3_M$, see Figure 3. As the last step, we describe an algorithm to construct parameter nodes in PG by constructing s-chains in a lattices L^k_M.

These results allow complete description of dynamical behavior of a network using Boolean models and differential equations switching models. The description is valid for networks of any size but there are clear practical limits imposed by quickly increasing Dedekind number.

2. Preliminaries

2.1. Posets and lattices

Given a set P, a partial order on P is a binary relation $\prec$ on P that satisfies the reflexivity, antisymmetry and transitivity properties. The pair $(P, \prec)$ defines a *partially ordered set* (poset). Elements $x, y \in P$ are *comparable* in $(P, \prec)$ if either $x \prec y$ or $y \prec x$.

A *chain* in a poset $(P, \prec)$ is a subset of P in which all the elements are pairwise comparable. An *antichain* in a poset $(P, \prec)$ is a subset of P in which no two elements are comparable. For any $x, y \in P$ we say that y *covers* x, if $x \prec y$ and there is no element $z \in P$ such that $x \prec z \prec y$.

A poset $(P, \prec)$ can be graphically represented by a *Hasse Diagram (HD)*, which is a transitive reduction of $(P, \prec)$.

Given a subset $X \subset P$, an element $u \in P$ is an upper bound of X in $(P, \prec)$ if $x \prec u$ for any $x \in X$. Similarly, $l \in P$ s a lower bound of X if $l \prec x$ for any $x \in X$.

Definition 2.1. *A poset $(P, \prec)$ is called a lattice if each two-element subset $\{x, y\} \subseteq P$ has a join (i.e. least upper bound) denoted by $x \vee y$ and a meet (i.e. greatest lower bound) denoted by $x \wedge y$. This definition makes $\wedge$ and $\vee$ binary operations. Both operations are monotone with respect to the given order: $x_1 \leq x_2$ and $y_1 \leq y_2$ implies that*

$$x_1 \vee y_1 \leq x_2 \vee y_2 \quad \text{and} \quad x_1 \wedge y_1 \leq x_2 \wedge y_2.$$

A bounded lattice is a lattice that additionally has a greatest element (also called maximum, or top element), and is often denoted by 1, as well a least element (also called minimum, or bottom element), often denoted by 0. Then $0 \leq x \leq 1$ for every $x \in P$.

A lattice $(L, \vee, \wedge)$ is a distributive lattice if for all $x, y, z \in L$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Definition 2.2. *Let $(L_1, \vee_1, \wedge_1)$ and $(L_2, \vee_2, \wedge_2)$ be two lattices. Then $(L, \vee, \wedge)$ is the direct product of lattices, with $L = L_1 \times L_2$ in which the binary operation $\vee$ and $\wedge$ on L satisfies for any $(x_1, y_1) \in L, (x_2, y_2) \in L$*

$$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee_1 x_2, y_1 \vee_2 y_2)$$
$$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge_1 x_2, y_1 \wedge_2 y_2)$$

We say that y covers x, written $x \lessdot y$, iff $x < y$ and there is no element z such that $x < z < y$.

Definition 2.3 (Stanley (1984); Butler (1994)). *A graded poset is a poset $(P, <)$ equipped with a rank function $\eta : P \to N$, where N the set of natural numbers. The rank function η must satisfy the following two properties:*

- *η is compatible with the ordering, meaning that for all $x, y \in P$, if $x < y$ then $\eta(x) < \eta(y)$; and*
- *η is consistent with the covering relation of the ordering, meaning that for all x and y, if y covers x then $\eta(y) = \eta(x) + 1$.*

The value of the rank function for an element of the poset is called its rank . A level of a graded poset is the subset of all the elements of the poset that have a given rank value. Fig. 1 shows a poset where nodes aligned horizontally have the same rank and thus are on the same level.

Following is a straightforward consequence of the definition of graded poset.

Lemma 2.1. *In graded poset P, if two nodes $x, y \in P$ satisfy $\eta(x) = \eta(y)$, then x and y are incomparable.*

2.2. Monotone Boolean functions

Every monotone Boolean function f can be represented in a Disjunctive Normal Form (DNF), a disjunction of clauses (*minterms*), defined by elementary conjunctions, where each variable appears either as an uncomplemented literal s_i (if f is positive in the input s_i), or as a complemented literal $\neg s_i$ (if f is negative in the input s_i). From such a representation, a (unique) canonical representation called the Complete DNF of the monotone Boolean function f can be obtained by deleting all redundant clauses, i.e., those that are absorbed by other clauses of the original DNF Crama and Hammer (2011).

Definition 2.4. *A variable b_k is an essential variable of a Boolean function f if there is at least one $\mathbf{b} \in B^n$ such that $f|_{b_k=0} \neq f|_{b_k=1}$. A Boolean function is said to be non-degenerate if all variables are essential Shmulevich, Dougherty, , Kim and Zhang (2002); Curry et al. (2019).*

Definition 2.5. *A truth set of a Boolean function $f : \mathbb{B}^k \to \mathbb{B}$ is the set*

$$\mathbb{T}(f) := \{x \in \mathbb{B}^k \mid f(x) = 1\}.$$

We are ready to define the main objects of our interest.

Definition 2.6.
- *Let L^k_M be a poset of monotone Boolean functions $f : \mathbb{B}^k \to \mathbb{B}$ with partial order $f_1 \preceq f_2$ if, and only if, $\mathbb{T}(f_1) \subset \mathbb{T}(f_2)$.*
- *Let $L^k_N \subset L^k_M$ be a poset of non-degenerate monotone Boolean functions.*

We immediately have the following Proposition.

Proposition 2.1. *Both L^k_N and L^k_M are graded posets for any $k \geq 1$, with grading $\eta : L^k_M \to N$ given by*

$$\eta(f) = |\mathbb{T}(f)|,$$

which is the size of the truth set of Boolean function $f : \mathbb{B}^k \to \mathbb{B}$.

For illustration of $L^3_N \subset L^3_M$, see Figure 1.

2.3. DSGRN parameters

Definitions in this section are taken from Duncan et al. (2021); Duncan (2021); Crawford-Kahrl et al. (2022) and interested reader is referred to these papers for more details.

We start by describing the set of *DSGRN parameters* , which we often shorten to *parameters,* associated to a regulatory network $N = (V, E, \delta)$. This set depends on the nodes V and the edges E, but not on the sign of the edges δ. We fix a network N and suppress dependencies on N in our notation.

Let $m_j := |\mathbf{T}(j)|$ denote the number of target nodes of j. A <u>j-order parameter</u> is a bijection $\theta_j : \mathbf{T}(j) \to \{1, \dots, m_j\}$ which defines an ordering of the out-edges of j. The set of j-order parameters is denoted by $\Theta(j)$. A collection $\theta := (\theta_j)_{j\in V}$ is an <u>order parameter</u>. The set of all order parameters is given by $\Theta := \prod_{j\in V} \Theta(j)$.

Definition 2.7. *A <u>j-logic parameter</u> is a collection of positive monotone Boolean functions*

$$f_j = (f_j^1, \dots, f_j^{m_j})$$

satisfying the <u>ordering condition</u>

$$f_j^1(b) \geq \cdots \geq f_j^{m_j}(b) \tag{4}$$

for all $b \in \mathbb{B}^{\mathbf{S}(j)}$. As a consequence, j-logic parameter is a $|\mathbf{T}(j)|$-chain of positive monotone Boolean functions $f_j^i : \mathbb{B}^{|\mathbf{S}(j)|} \to \mathbb{B}$.

The role of the function $f_j^{\theta_j(i)}$ is to describe when a given input b activates the edge $j \multimap i$. If $f_j^{\theta_j(i)}(b) = 1$ then b is activates the edge $j \multimap i$ and if $f_j^{\theta_j(i)}(b) = 0$ then b does not activate the edge $j \multimap i$. Note that the activation in this context is independent of the sign of the edge δ_{ij} as both positive and negative edges may be activated, or not activated. This means that we only need to describe the structure of all positive monotone Boolean functions and not all monotone Boolean functions. To show we can construct any monotone Boolean function from a positive monotone Boolean function we note that each MBF can be written in a disjunctive normal form (DNF) where each literal corresponding to positive input is un-complemented and each negative input is complemented Curry et al. (2019). As an example, consider a monotone Boolean function with three inputs A, B, C given by its DNF

$$f(A, B, C) = (A \wedge \neg C) \vee (B \wedge \neg C).$$

This function is monotone increasing with respect to A and B and monotone decreasing with respect to C. Importantly, there is a corresponding *positive* MBF which is monotone increasing with respect to all its inputs given by

$$(A \wedge C) \vee (B \wedge C).$$

Therefore we will only describe positive MBFs, since the others can be constructed by corresponding literal substitutions. In Fig. 1 we show a poset of all positive MBFs with three inputs.

A collection $f := (f_j)_{j\in V}$ is a <u>logic parameter</u>. The set of all j-logic parameters is denoted $\mathcal{L}(j)$, while the set of all logic parameters is

$$\mathcal{L} := \prod_{j\in V} \mathcal{L}(j). \tag{5}$$

The set of all parameters is the product of logic and order parameters $\mathcal{P} := \mathcal{L} \times \Theta$. We call

$$\mathcal{P}(j) := \mathcal{L}(j) \times \Theta(j) \tag{6}$$

the set of j-parameters; the set of all parameters are given by the product

$$\mathcal{P} = \prod_{j\in V} \mathcal{P}(j).$$

We endow the set $\mathcal{P}$ with structure of a graph by defining adjacency between elements of $\mathcal{P}$. Two j-parameter nodes, $(f_j, \theta_j), (h_j, \phi_j) \in \mathcal{P}(j)$ are <u>adjacent</u> if exactly one of the following conditions is satisfied.

- Order adjacency: $f_j = h_j$ and the values of the order parameters θ_j and ϕ_j are exchanged on a single pair of neighboring entries on which the logic parameters agree. Explicitly, there is an adjacent transposition π of $\{1,\dots,m_j\}$ such that $\theta_j = \pi \circ \phi_j$ and $f_j^i = h_j^{\pi(i)}$ for each i. Letting s be the index with $s+1 = \pi(s)$, we note that $f_j^s = h_j^{\pi(s)}$ and $f_j = h_j$ together imply $f_j^s = f_j^{s+1}$.
- Logical adjacency: $\theta_j = \phi_j$ and the j-logic parameters f_j and h_j differ in a single input. Explicitly, there is unique $i \in \{1,\dots,m_j\}$, and unique $b^0 \in \mathbb{B}^{\mathbf{S}(j)}$ such that $f_j^i(b^0) \neq h_j^i(b^0)$. For $s \neq i$, we require $f_j^s(b) = h_j^s(b)$ for all $b \in \mathbb{B}^n$ and for $b \neq b^0$ we require $f_j^s(b) = h_j^s(b)$.

The j-factor graph is the undirected graph

$$\mathsf{PG}(j) := (\mathcal{P}(j), \mathcal{E}(j)) \tag{7}$$

whose nodes are j-parameter nodes and whose edges connect adjacent nodes. The parameter graph $\mathsf{PG} := (\mathcal{P}, \mathcal{E})$ is the Cartesian product

$$\mathsf{PG} := \prod_{j\in V} \mathsf{PG}(j). \tag{8}$$

That is, there is an edge $(p^1, p^2) \in \mathcal{E}$ if and only if there is a unique $j \in V$ such that $(p_j^1, p_j^2) \in \mathcal{E}(u)$ and $p_i^1 = p_i^2$ for all $i \neq j$.

We close this section with few simplifications that allow us to reduce the number of parameter nodes that need to be described. First, we note that the second factor in (6) is simply a set of permutations of the set $\mathbf{T}(j)$. If we describe the poset of logical parameters $\mathcal{L}(j)$ then the set of nodes $\mathcal{P}(j)$ consists of $|\mathbf{T}(j)|$ copies of the poset $\mathcal{L}(j)$ with well-understood edges connecting these copies Andreas, Cummins and Gedeon (2024). Therefore it is sufficient to describe the poset of logical parameters $\mathcal{L}(j)$ with fixed, but arbitrary, order parameter $\theta_j \in \Theta(j)$.

Second, we introduce a set of *essential* logical parameters which is a strict subset of all logical parameters. This definition captures the notion that each edge of the network N plays a non-trivial role in at least one network configuration.

Definition 2.8 (Duncan et al. (2021)). *Let $p = (f, \theta) \in \mathcal{P}$ with $f_j = (f_j^1, \dots, f_j^{m_j})$ be a logical parameter.*

- *p is output essential if every function f_j^i is not constant.*
- *p is input essential if for every j there exists an $i \in \mathbf{S}(j)$ and input $b \in \mathbb{B}^{|\mathbf{S}(j)|}$ such that changing the i-th coordinate in b changes the value of f_j^k for at least one $k \in \mathbf{T}(j)$. In the language of Definition 2.4 for every $j \in V$ there is $i \in \mathbf{S}(j)$ such that the variable s_i is an essential variable for $f_j^k, k \in \mathbf{T}(j)$.*
- *p is essential parameter if p is both input and output essential.*

Observe that if the number of targets of node j is $m_j = 1$ this definition reduces to the definition of non-degenerate monotone Boolean function in Definition 2.4.

3. Structure of logic parameter graphs $\mathcal{L}$

3.1. Lattices and posets of monotone Boolean functions

In this section we describe the poset of logical parameters $\mathcal{L}$ defined in (5) and its subset of essential logical parameters. Since $\mathcal{L} = \prod_{j\in V} \mathcal{L}(j)$ is a product of its factor logic posets $\mathcal{L}(j)$, it is sufficient to describe the structure of $\mathcal{L}(j)$ for any j. This description will be in terms of chains within two posets:

- monotone Boolean functions L_M^k with k-inputs and
- non-degenerate monotone Boolean functions $L_N^k \subset L_M^k$.

We start our discussion by describing structure of the set L_M^k of positive monotone Boolean functions with k inputs.

Proposition 3.1. *For $f, g \in L_M^k$ let*

$$f \vee g = \max\{f, g\} \qquad \textit{and} \qquad f \wedge g = \min\{f, g\}.$$

Then $f \vee g \in L_M^k$ and $f \wedge g \in L_M^k$ and, consequently, $(L_M^k, \vee, \wedge)$ is a lattice for all integers $k \geq 1$.

Proof. Consider $f, g \in L_M^k$ and two inputs $x, y \in \mathbb{B}^k$ with $x \preceq y$. Since both f, g are monotone Boolean functions,

$$f(x) \leq f(y) \quad \text{and} \quad g(x) \leq g(y). \tag{9}$$

Set $h(z) := \min\{f, g\}$, Then by definition and monotonicity of f and g

$$h(x) \leq f(x) \leq f(y)$$
$$h(x) \leq g(x) \leq g(y),$$

and therefore $h(x) \leq \min\{f(y), g(y)\} = h(y)$. It follows that $h \in L_M^k$.

The argument for $\max\{f, g\}$ is analogous.

Let $2^{[k]}$ be the Boolean lattice of subsets of $[k] := \{1, \dots, k\}$ with join (meet) given by union (intersection) of sets. Consider an injective map $\tau : L_M^k \to 2^{[k]}$ given by

$$\tau(f) := \mathbb{T}(f), \tag{10}$$

which assigns to each $f \in L_M^k$ the truth set of f. Notice that

$$\mathbb{T}(f \vee g) = \mathbb{T}(f) \cup \mathbb{T}(g)$$

and

$$\mathbb{T}(f \wedge g) = \mathbb{T}(f) \cap \mathbb{T}(g),$$

and hence the map τ is a lattice homomorphism between $(L_M^k, \vee, \wedge)$ and $(2^{[k]}, \cup, \cap)$. It follows that L_M^k is a distributive lattice.

The following theorem is the main result of Curry et al. (2019):

Theorem 3.1 (Curry et al. (2019)). *The non-degenerate monotone Boolean functions L_N^k are in one-to-one correspondence with the set of anti-chains in Boolean lattice $2^{[k]}$.*

This theorem is based on a representation of each $f \in L_N^k$ as a set of anti-chains by using disjunctive normal form (DNF) representation of functions in $f \in L_N^k \subset L_M^k$. Each conjunction term can be represented as a set of indices in $\{1, 2, \dots, k\}$ of the corresponding variables and thus the DNF corresponds to a collection of subsets of $\{1, 2, \dots, k\}$. Because f is non-degenerate, these subsets form a partition of $\{1, 2, \dots, k\}$; the fact that they form anti-chain follows from the DNF representation of f.

Note that this is a different representation than the one implied by lattice homomorphism τ in (10) and so Theorem 3.1 does not imply that L_N^k is a sublattice of L_M^k, see Figure 1. Indeed, by direct inspection of Figure 1 the join of

$$A \vee (B \wedge C) \in L_N^3 \quad \text{and} \quad C \vee (A \wedge B) \in L_N^3 \quad \text{is} \quad A \vee C \notin L_N^3.$$

For a different example, Figure 3(a) shows $L_N^2 \subset L_M^2$. The poset L_N^2 contains only two functions AND and OR.

3.2. Chains of monotone Boolean functions

Recall from (5) that the logic parameter graph

$$\mathcal{L} = \prod_{j \in V} \mathcal{L}(j)$$

is a product of logic factor parameter graphs $\mathcal{L}(j)$ and each element $f_j \in \mathcal{L}(j)$, $f_u = (f_u^1, \dots, f_u^{|\mathbf{T}|})$ is a $|\mathbf{T}|$-chain in $L_M^{|\mathbf{S}(u)|}$, see Definition 2.7.

We now show that a collection of chains in a poset is itself a poset.

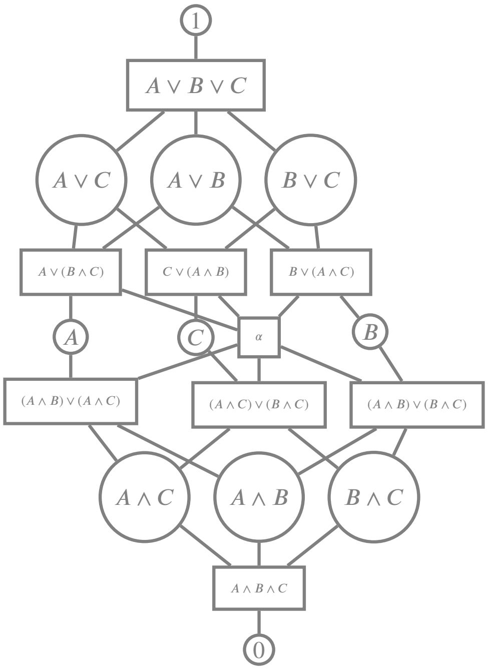


Figure 1: Lattice of positive monotone Boolean functions L^3_M. Nodes aligned horizontally have the same grade level $\mathbb{T}(f)$ which is the size of truth set, ranging from 0 at the bottom to $8 = 2^3$ on the top. Each node has DNF description of the function, where 0 and 1 are constant functions and $\alpha = (A \wedge C) \vee (A \wedge B) \vee (B \wedge C)$. Finally, the set of rectangular nodes is the poset of non-degenerate functions L^3_N.

Definition 3.1. *Consider poset $(A, \preceq)$ and two chains $p = (p_1 \preceq \dots \preceq p_k)$ and $q = (q_1 \preceq \dots \preceq q_k)$. We define poset $(CA, \leq)$ by setting p to be an immediate predecessor of q in new order $\leq$ where if there is index $s \in \{1, \dots, k\}$ such that*

- *p_s an immediate predecessor of q_s, $p_s \prec q_s$ in A;*
- *$p_j = q_j$ for all $j \neq s$.*

The order $\leq$ is transitive closure of the immediate predecessor relation.

We define a collection of posets of interest.

Definition 3.2. *Let $L^k_M(s)$ be a poset of s chains $(f^1, \dots, f^s)$ where $f^j \in L^k_M$ is a positive monotone Boolean function $B^k \to B$.*

Let $L^k_E(s) \subset L^k_M(s)$ be a poset of s chains $(f^1 \preceq \dots \preceq f^s)$ where $f^j \in L^k_M$ and at least one $f^i \in L^k_N$ is non-degenerate positive monotone Boolean function. The set $L^k_E(s)$ is the set of essential factor logic parameters.

Note that by construction, for any k and s we have a relationship of posets

$$L^k_E(s) \subset L^k_M(s). \tag{11}$$

Defining the product of lattices as in Definition 2.2 we have the following theorem on important posets of the logic parameter poset $\mathcal{L}$.

Proposition 3.2. *Consider a network $RN = (V, E)$ and a node $u \in V$. Then*

1. *The set*

$$\mathcal{L}_E := \prod_{j \in V} L_E^{|\mathbf{S}(j)|}(|\mathbf{T}(j)|)$$

is poset of essential logic parameters.

2. *The set*

$$\mathcal{L} = \prod_{j \in V} L_M^{|\mathbf{S}(j)|}(|\mathbf{T}(j)|)$$

is lattice of all logic parameters.

Proof. (1) follows directly from Definition 3.1 and Proposition 2.1. The statement **(2)** follows from Proposition 2.1 and Definition 2.2.

In order to construct these posets we need to be able to perform three basic constructions:

1. Construct lattices $L_M^k, k = 2, 3, \ldots$. We describe direct construction using Karnaugh maps in section 3.3 for $k = 3, 4$ and then provide an inductive construction of L_M^k from L_M^{k-1} in section 3.4.
2. Construct the lattices $L_N^k \subset L_M^k$. We describe this is section 3.4.
3. Since $\mathcal{L}_E \subset \mathcal{L}$ are product of chains in L_N^k or L_M^k respectively, we describe a construction of a graded lattice $L(s)$ of all s chains in L for arbitrary graded lattice L in section 3.5.

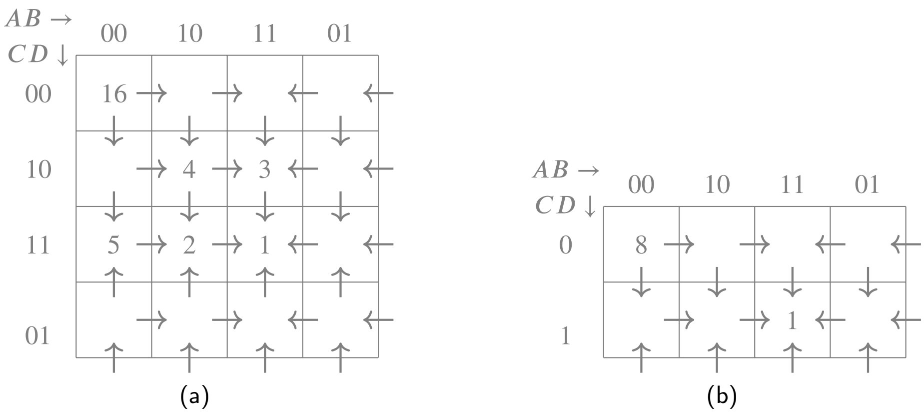


Figure 2: (a) Karnaugh grid cells for Boolean function with four inputs A, B, C, D form a 2-D torus. The directed field is indicated with the top cell denoted by 1 and bottom cell by 16. Each collection of cells is a truth set of a Boolean function, but not all collections correspond to a truth set of a monotone Boolean function. The single cell 1 corresponds to function $A \wedge B \wedge C \wedge D$ and is an MBF; so are functions whose truth values correspond to cells $1 \cup 2$ or $1 \cup 3$, but a function with truth cells $1 \cup 4$ is not an MBF. Function with truth values $1 \cup 2 \cup 3 \cup 3 \cup 4$ is MBF which can be simplified to $A \wedge C$. (b) Karnaugh grid cells for Boolean function with three inputs with top grid marked 1 and bottom grid marked 8.

3.3. Explicit construction of L_M^k and L_N^k for $k \leq 3$.

We present an algorithm using Karnaugh maps Karnaugh (1953), widely used tool for bringing a Boolean function to its DNF. The Karnaugh map transfers Boolean function truth table into a grid where cells are ordered in Gray code Brown (2012 [2003, 1990]). Each cell is assigned the corresponding output value of the Boolean function. Let S be the collection of grid elements. Each $s \in S$ is assigned two values: its input sequence $\sigma(s) \in \mathbb{B}^k$ and Boolean output $o(s) = f(\sigma(s)) \in \mathbb{B}$. Neighboring groups of neighboring 2-cells, 4-cells and, in general 2^n cells can be used to construct a minimal disjunctive normal form of the Boolean function. Karnaugh maps for Boolean functions with up to 4 inputs can be represented as planar grids which are projections of a grid on a 2-torus. We use Karnaugh maps to generate a lattice of monotone Boolean functions for $k = 3$, see Figure 2. To do so, we first define directed graph on the Karnaugh grid. Since in Gray code neighboring grid elements $s_1, s_2 \in S$ differ in exactly one input value, $\sigma(s_1)$ and $\sigma(s_2)$ are ordered in lattice $\mathbb{B}^k$. We define directed edge $s_1 \to s_2$ between such neighboring s_1, s_2 if, and only if, $\sigma(s_1) \prec \sigma(s_2)$ in $\mathbb{B}^k$. Because of Gray encoding, there is a directed edge between any two neighboring grid elements that border through a codimension 1 hyperplane.

A Boolean function $f : \mathbb{B}^k \to \mathbb{B}$ is monotone, if its Karnaugh map respects the directed edges:

$$f(s_2) > f(s_1) \quad \text{at neighboring grid elements} \quad s_1, s_2 \quad \text{if and only if } s_1 \to s_2.$$

This characterization of monotone Boolean functions allows us to construct the graded lattice of MBFs for $k = 2$ and $k = 3$. We start with the unique function f with rank 0 i.e $|\mathbb{T}(f)| = 0$, which is the constant function $f \equiv 0$. Karnaugh directed graph has a single cell where all edges are directed inwards that corresponds to the highest value of the input $(1, 1) \in \mathbb{B}^2$ (or $(111) \in \mathbb{B}^3$). Therefore there is unique function that assumes value 1 on this input and value 0 for any other input and this function is the AND function $x_1 \wedge x_2 \wedge \ldots \wedge x_n$ for $n = 2$ ($n = 3$). This is the unique rank one function f. Proceeding in this way we construct the lattice L^2_M in Figure 3 and L^3_M in Figure 1. The subset $L^3_N \subset L^3_M$ is also depicted in Figure 1.

Note that usefulness of Karnaugh maps is limited to low dimensions $k \leq 4$ of the input set $\mathbb{B}^k$, since the Gray code requires that Boolean functions with $2n - 1$ and $2n$ inputs are represented by cubical grids in dimension n. Gray code is impractical when $n \geq 3$.

Having constructed L^3_N and L^3_M, we now describe an inductive construction of L^k_N and L^k_M from L^{k-1}_M and L^{k-1}_N.

3.4. Inductive construction of L^k_N and L^k_M

We first describe an inductive construction of lattice L^k_M from lattice L^{k-1}_M and then the construction of poset L^k_N from L^{k-1}_M and L^{k-1}_N. Both of these constructions are based on the construction that was introduced in Crawford-Kahrl et al. (2022), which is a special case of a more general result Fidytek, Mostowski, Somla and Szepietowski (2001) used recently in Jakel (2023) to compute the 9-th Dedekind number.

Let $D^k = (D^k, \leq)$ be free distributive lattice with k generators. Then

Theorem 3.2 (Fidytek et al. (2001)). *There is a one-to-one correspondence between elements of D^k and monotone mappings from 2^s into D^{k-s}.*

As a corollary, there is a one-to-one correspondence of elements from D^k and pairs (x, y) of elements x, y from D^{k-1} with $x \leq y$. The explicit construction in the next definition is consistent with this corollary.

Definition 3.3 (Crawford-Kahrl et al. (2022)). *Given a Boolean lattice $\mathbb{B}^k$ and arbitrary component $i \in \{1, \ldots, k\}$, we define two subspaces of $\mathbb{B}^k$.*

The ceiling in the i-th direction is the set

$$C_i := \{(y_1, y_2, \ldots, y_k) \in \mathbb{B}^k \mid y_i = 1\}$$

and the floor in the i-th direction is the set

$$F_i := \{(y_1, y_2, \ldots, y_k) \in \mathbb{B}^k \mid y_i = 0\}.$$

Notice that C_i and F_i are both isomorphic to $\mathbb{B}^{k-1}$.

This leads to following construction.

Definition 3.4. *Consider graded posets (K^{k-1}, η_K) and (Q^{k-1}, η_Q) where both $K^{k-1}, Q^{k-1} \subset L^{k-1}_M$. Let $K^{k-1} \oplus Q^{k-1} \subset L^k_M$ be the set of Boolean functions $h := f \oplus g$ where*

1. *$f \prec g$ and*
2. *restriction of h to ceiling C_{k-1} of $\mathbb{B}^k$ is $g \in Q^{k-1}$ and restriction to floor F_{k-1} of $\mathbb{B}^k$ is $f \in K^{k-1}$.*
3. *Define rank $\eta(h) := \eta_K(f) + \eta_Q(g)$.*

In other words, elements of $K^{k-1} \oplus Q^{k-1} \subset L^k_M$ are 2-chains in L^{k-1}_M, where each element is required to be in K^{k-1} and Q^{k-1}, respectively.

Theorem 3.3 (Crawford-Kahrl et al. (2022)). *The graded lattice of monotone Boolean functions (L^k_M, η_k) satisfies*

$$(L^k_M, \eta_k) = (L^{k-1}_M, \eta_{k-1}) \oplus (L^{k-1}_M, \eta_{k-1}).$$

The range of η_s is $[0, 2^s]$ for all s and $\eta_k = \eta_{k-1} + \eta_{k-1}$.

We illustrate this Theorem in Figure 3. In panel (a) we present the graded lattice of monotone Boolean functions L^2_M where the inputs are denoted by A and B. On the bottom (top) we have constant zero (one) functions; the other four functions are $A \wedge B$, $A \vee B$, A and B. The latter two functions are true whenever A (or B) is true, respectively. Note that the rows of the lattice represent the grading of the lattice in terms of size of the truth set starting from 0 on the bottom to 4 on the top. The nodes $A \wedge B$ and $A \vee B$, enclosed in rectangles, represent the non-degenerate MBF that belong to L^2_N.

In panel (b) we illustrate the Theorem 3.3 by taking the lattice L^3_M of MBFs in Figure 1 and identifying each node as a chain $(f \prec g) \in L^2_M$ shown in panel (a). Therefore the entire poset is

$$L^3_M = L^2_M \oplus L^2_M.$$

We point out another important subsets of L^3_M that play a key role in the inductive argument that follows.

- All rectangular nodes belong to the set $(L^2_N \oplus L^2_M) \cup (L^2_M \oplus L^2_N)$ which consists of 2-chains in L^2_M where at least one function belongs to $L^2_N = \{AND, OR\}$.
- The set

  $$diag(L^2_N \oplus L^2_N) := \{(AND, AND), (OR, OR)\},$$

  marked as dashed rectangles, contains constant 2-chains in L^2_N i.e. where both functions are the same non-degenerate Boolean function.
- Importantly, the non-dashed rectangle nodes comprise the set $L^3_N \subset L^3_M$ (compare Figure 2), Note that the set L^3_N satisfies

  $$L^3_N = (L^2_N \oplus L^2_M) \cup (L^2_M \oplus L^2_N) \setminus diag(L^2_N \oplus L^2_N).$$

This observation leads to the main result of the paper.

Theorem 3.4. *Let $diag(L^{k-1}_N \oplus L^{k-1}_N)$ be the set of functions $h = (f, f)$ were $f \in L^{k-1}_N$. Then*

1. $$L^k_N = \big((L^{k-1}_N \oplus L^{k-1}_M) \cup (L^{k-1}_M \oplus L^{k-1}_N)\big) \setminus diag(L^{k-1}_N \oplus L^{k-1}_N).$$

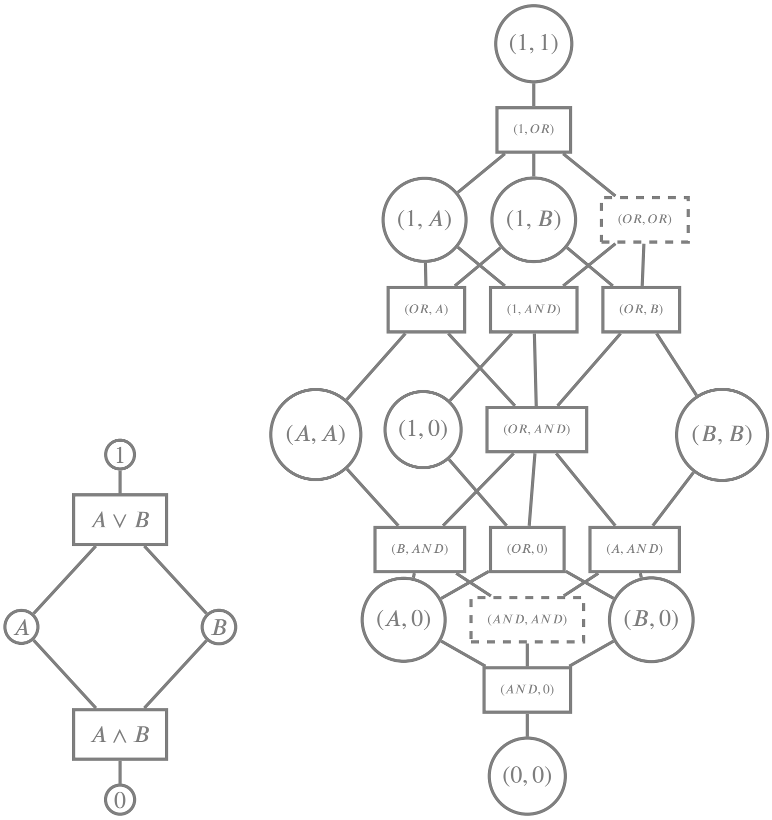
   Here $\big((L^{k-1}_N \oplus L^{k-1}_M) \cup (L^{k-1}_M \oplus L^{k-1}_N)\big)$ is the set of all $f \preceq g$ where either f or g is non-degenerate element of L^{k-1}_N.
2. *The layer $\eta(\cdot) = 1$ in L^k_M contains unique element f_b (bottom) and layer $\eta(\cdot) = 2^k - 1$ in L^k_M contains unique element f_t (top). Both $f_b, f_t \in L^k_N$;*
3. *The range of the grading function η evaluated on non-degenerate functions L^k_N is $[0, 2^k] \setminus \{0, 2, 2^k - 2, 2^k\}$.*

Proof.
(1) Consider 2-chains $f \preceq g$ with $f, g \in L^{k-1}_M$, where at least one of the functions f, g belongs to L^{k-1}_N i.e.

$$f, g \in ((L^{k-1}_N \oplus L^{k-1}_M) \cup (L^{k-1}_M \oplus L^{k-1}_N).$$

Since $L^{k-1}_N \subset L^{k-1}_M$ then $h = f \oplus g \in L^k_M$ by Theorem 3.3. Since either f or g belongs to L^{k-1}_N the variables $y_1, \ldots, y_{k-1}$ in $\mathbb{B}^{k-1}$ are essential for either f or g and hence essential variables of the Boolean function h. For h to belong to L^k_N the variable y_k perpendicular to both floor F_k and the ceiling C_k also must be essential. For this to happen, the necessary and sufficient condition is that $f \neq g$. Hence all unequal pairs $f \prec g$, where at least one function is in L^{k-1}_N give rise to $h \in L^k_N$.

On the other hand the pairs $f \preceq f, f \in L^{k-1}_N$ belong to $diag(L^{k-1}_N \oplus L^{k-1}_N)$. Therefore every pair $(f, g) \in \big((L^{k-1}_N \oplus L^{k-1}_M) \cup (L^{k-1}_M \oplus L^{k-1}_N)\big)$ either gives rise to $h \in L^k_n$ when $f \neq g$, or, when $f = g$, then f belongs to $diag(L^{k-1}_N \oplus L^{k-1}_N)$. The result follows.



(a) (b)

Figure 3: (a) The set L^2_M for inputs marked A and B. The set of L^2_N consists of two nodes $A \wedge B(= AND)$ and $A \vee B(= OR)$. (b) 2-chains of L^2_M mapped onto the structure of L^3_M. The nodes are marked by pair of chains from (a) and directly correspond to nodes in Figure 1. Rectangles correspond to 2-chains where at least one element belongs to the set $L^2_N = \{AND, OR\}$. The dashed rectangles correspond to constant 2-chains where both elements are in L^2_N. The set L^3_N is a difference of these sets and consists of functions in full rectangles.

(2) The only monotone Boolean function with the size of truth set $|\mathbb{T}| = 1$ has to achieve this value on the highest input vector in $\mathbb{B}^k$ which is the vector of all ones $\mathbf{1} := (1, \dots, 1)$. This is the function

$$f_b = x_1 \wedge x_2 \wedge \dots \wedge x_k.$$

Similarly, the only monotone Boolean function with the size of truth set $|\mathbb{T}| = 2^k - 1$ is the function which is 0 only at the lowest input vector in $\mathbb{B}^k$ which is the vector of all zeros $\mathbf{0} := (0, \dots, 0)$. This is the function

$$f_t = x_1 \vee x_2 \vee \dots \vee x_k.$$

The formulas for f_b and f_t show that both are non-degenerate.

(3) The statement is true for $j = 2$ by inspection of Figure 3 (a). Assume, by induction, that the statement is true for $j = k - 1$.

Clearly if $h \in L^k_N$ and $\eta(h) = 0$, then h must have the form $h = f \oplus g$ where $\eta(f) = \eta(g) = 0$. However, such a pair (f, g) does not belong to $L^{k-1}_N \oplus L^{k-1}_M$ by inductive hypothesis. Similar argument applies to h in L^k_N with $\eta(h) = 2^k$. Such an h must have the form $h = f \oplus g$ where $\eta(f) = \eta(g) = 2^{k-1}$.

Now we discuss the value 2 of the grading function $\eta(\cdot)$. By statement (1) every element h in L^k_N with $\eta(h) = 2$ has the form $h = f \oplus g$ where $\eta(f) = \eta(g) = 1$ (i.e. type $(1, 1)$) or $\eta(f) = 0$ and $\eta(g) = 2$ (i.e type $(0, 2)$) with both $f, g \in L^{k-1}_M$ and at least one of $f, g \in L^{k-1}_N$. By induction hypothesis, a pair (f, g) of type $(0, 2)$ does not belong to $L^{k-1}_N \oplus L^{k-1}_M$. On the other hand, by statements (2) and (3), there is unique function h with grading type $(1, 1)$ must be $h = f_m \oplus f_m$ which belongs to $diag(L^{k-1}_N \oplus L^{k-1}_N)$.

In a similar way, every element h in L_N^k with $\eta(h) = 2^k - 2$ has the form $h = f \oplus g$ where $\eta(f) = \eta(g) = 2^k - 1$, or $\eta(f) = 2^k, \eta(g) = 2^k - 2$, with $f, g \in L_M^{k-1}$ and at least one of $f, g \in L_N^{k-1}$. By induction hypothesis, any node whose layers satisfy the latter sum does not belong to $L_N^{k-1} \otimes L_M^{k-1}$. On the other hand, by statements (2) and (3), the function h with the type $(2^k - 1, 2^k - 1)$ must be the combination $h = f_M \oplus f_M$ which belongs to $diag(L_N^{k-1} \oplus L_N^{k-1})$. The result follows.

Theorem 3.4 leads to the following inductive algorithm constructing L_N^k and L_M^k for any k.

Algorithm

1. $L_N^2 \subset L_M^2$ are in Figure 3(a). This is the base step of the induction.
2. Given $L_N^{k-1} \subset L_M^{k-1}$ we construct $L_N^k \subset L_M^k$ as follows:
   (i) Since by Theorem 3.3 $L_M^k = L_M^{k-1} \oplus L_M^{k-1}$, every $h \in L_M^k$ has the form $h := f \oplus g$ where $f \preceq g$ is a 2-chain in L_M^{k-1} .
   (ii) Using Theorem 3.4 we construct L_N^k by

$$L_N^k := \big((L_N^{k-1} \oplus L_M^{k-1}) \cup (L_M^{k-1} \oplus L_N^{k-1})\big) \setminus diag(L_N^{k-1} \oplus L_N^{k-1}).$$

3.4.1. An example: Construction of L_M^4

Figure 1 depicts the lattice L_M^3 of MBFs with 3 inputs. We now illustrate construction of the lattice L_M^4 as 2-chains $f \preceq g$ where both $f, g \in L_M^3$. We only construct functions $f \in L_M^4$ with $\eta(f) \in \{0, 1, 2\}$.

Let $U(k) \subset L_M^3(2)$ be the set of 2-chains $f \prec g$ with $\eta(f) + \eta(g) = k$. Using Theorem 3.3 we abuse notation and use $U(k)$ to also denote the set of all functions $h \in L_M^4$ of the form $h = f \oplus g, f \prec g \in L_M^3$.

To construct $U(0)$, we note that there is always unique bottom node $\ell := \mathbf{0} \oplus \mathbf{0} \in L_M^3 \oplus L_M^3$ with $\eta(\ell) = 0$, where $\mathbf{0}$ is the unique node in L_M^3 with $\eta(\mathbf{0}) = 0$ (Figure 1).

To construct the set of nodes $U(1) \subset L_M^3(2)$ we first write integer $1 = 0 + 1$ as a sum of smaller integers. There is unique node in the set $U(1)$

$$U(1) := \{(\mathbf{0}, f_b)\},$$

where $\eta(f_b) = 1$ is the unique node in layer 1 of L_M^3. In Figure 1 the node f_b has the form $A \wedge B \wedge C$.

The set of nodes $U(2) \subset L_M^3 \oplus L_M^3$ is obtained by first writing rank 2 as all a sum of smaller integers $2 = 2 + 0$ or $2 = 1 + 1$. Then

$$U(2) := \{(\mathbf{0}, A \wedge C), (\mathbf{0}, A \wedge B), (\mathbf{0}, B \wedge C), (f_b, f_b)\},$$

where

$$\eta(A \wedge C) = \eta(A \wedge B) = \eta(B \wedge C) = 2$$

are the three nodes with $\eta(\cdot) = 2$ of L_M^3 in Figure 1.

The pairs of nodes that are connected in $L_M^3 \oplus L_M^3$ are

- between layers $U(0)$ and $U(1)$: $(\mathbf{0}, \mathbf{0}) \in U(0)$ connects to $(\mathbf{0}, f_b) \in U(1)$;
- between layers $U(1)$ and $U(2)$: $(\mathbf{0}, f_b)$ connects to all four nodes in $U(2)$.

Continuing in this way, one can construct all layers $U(k)$ with $k = 0, 1, 2, 3 \ldots 16$. Table 1 summarizes numbers of elements in each layer of L_M^4, number of elements of essential pairs $(L_N^3 \oplus L_M^3) \oplus (L_M^3 \oplus L_N^3)$ (which we denote due to lack of space by $L_N^3 \oplus L_M^3$) and the number of elements of $L_N^4 \subset L_M^4$. The three bottom rows provide the sum of the top rows (which differ by different integer decomposition of the rank in the top row) along each column.

Layer	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Type	0+0	1+0	2+0	3+0	4+0	5+0	6+0	7+0	8+0	8+1	8+2	8+3	8+4	8+5	8+6	8+7	8+8
L^4_M	1	1	3	3	4	3	3	1	1	1	3	3	4	3	3	1	1
$L^3_N \oplus L^3_M$	0	1	0	3	1	3	0	1	0	1	0	3	1	3	0	1	0
L^4_N	0	1	0	3	1	3	0	1	0	1	0	3	1	3	0	1	0
Type			1+1	2+1	3+1	4+1	5+1	6+1	7+1	7+2	7+3	7+4	7+5	7+6	7+ 7		
L^4_M			1	3	3	4	3	3	1	3	3	4	3	3	1		
$L^3_N \oplus L^3_M$			1	3	3	4	3	3	1	3	3	4	3	3	1		
L^4_N			0	3	3	4	3	3	1	3	3	4	3	3	0		
Type					2+2	3+2	4+2	5+2	6+2	6+3	6+4	6+5	6+6				
L^4_M					3	6	9	9	9	9	9	6	3				
$L^3_N \oplus L^3_M$					0	6	3	9	0	9	3	6	0				
L^4_N					0	6	3	9	0	9	3	6	0				
Type							3+3	4+3	5+3	5+4	5+5						
L^4_M							3	6	9	6	3						
$L^3_N \oplus L^3_M$							3	6	9	6	3						
L^4_N							0	6	9	6	0						
Type									4+4								
L^4_M									4								
$L^3_N \oplus L^3_M$									1								
L^4_N									0								
L^4_M	1	1	4	6	10	13	18	19	24	19	18	13	10	6	4	1	1
$L^3_N \oplus L^3_M$	0	1	1	6	4	13	9	19	11	19	9	13	4	6	1	1	0
L^4_N	0	1	0	6	4	13	6	19	10	19	6	13	4	6	0	1	0

Table 1
First row is the layer number of L^4_M. Each triple row contains the type of combination $a+b$ of layer number of pairs of nodes from L^3_M, total number of nodes of this type that belong to L^4_M, the number of those that belong to the set $(L^3_N \oplus L^3_M) \cup (L^3_M \oplus L^3_N)$, and the number of those that belong to L^4_N. The three rows on the bottom list is the sum of the each category in the column. The difference between last two rows are nodes in $diag(L^3_N \oplus L^3_N)$ which can only happen for the type $k+k$. Note that in types $3+2, 4+2, 4+3, 5+4, 6+4$ and $6+5$ the assumption that the pair $f \prec g$ be ordered in L^3_M restricts the number of admissible pairs.

3.4.2. Structure of L^4_M and L^4_N.

We make several observation about the structure of L^4_M and L^4_N. First, we have

$$|L^4_M| = 168 \qquad \text{and} \qquad |L^4_N| = 108.$$

Second, note that the layers 2 and 14 do not contain any functions in L^4_N; this illustrates Theorem 3.4.4.

Third, the difference

$$|(L^3_N \oplus L^3_M) \cup (L^3_M \oplus L^3_N)| - |L^4_N| = 9$$

is small. The functions $h \in (L^3_N \oplus L^3_M) \cup (L^3_M \oplus L^3_N) \setminus L^4_N$ have $\eta(h)$ divisible by 2 since such functions have the form $h = f \oplus f$ with $f \in L^3_N$. Therefore $(L^3_N \oplus L^3_M) \cup (L^3_M \oplus L^3_N)$ seems to be a good approximation of the set L^4_N.

Finally, note that sets L^4_M, L^4_N but also $(L^3_N \oplus L^3_M) \cup (L^3_M \oplus L^3_N)$ have the same number of nodes in layer k and layer $16-k$. The fact that L^4_M has this symmetry follows directly from Boolean duality in Boolean algebra Eiter, Makino and Gottlob (2008), where the dual function $g : \mathbb{B}^k \to \mathbb{B}$ to Boolean function $f : \mathbb{B}^k \to \mathbb{B}$ is a function where $\wedge$ and $\vee$ are interchanged and values 0 and 1 of the function are also interchanged. The symmetry of the set $(L^3_N \oplus L^3_M) \cup (L^3_M \oplus L^3_N)$ follows from the fact that if $p_j \in L^3_N \oplus L^3_M$ then at least one of the two function in the chain f_j is in L^3_N. Then this is also true for the dual parameter and therefore the dual parameter q_j also belongs to $L^3_N \oplus L^3_M$. The symmetry of L^4_N follows from the symmetry of $L^3_N \oplus L^3_M$ and symmetry of $diag(L^3_N \oplus L^3_N)$.

The posets L^4_M, L^4_N and $(L^3_N \oplus L^3_M) \cup (L^3_M \oplus L^3_N)$ have approximately the "double cone structure" where the middle layers with $\eta \approx 8$ are larger than the extremal layers close to 0 or close to 16, which is reflected in a triangular shape of the filled rows of the table. This comes from two sources. First, to construct a function $h \in L^4_M$ with $\eta(h) = k$ we write the number $k \in [0, 16]$ as a sum $k = r + s$ of two non-negative integers $r, s \in [0, 8]$. There are more such sums when $k \approx 8$. The second source is the inductive construction of these posets: since $h = (f, g)$ with $f, g \in L^3_M$

and the set L^3_M has the same double cone structure with most functions $f \in L^3_M$ with $\eta(f) = 3, 4, 5$. However, the double cone structure is not perfect for $(L^3_N \oplus L^3_M) \cup (L^3_M \oplus L^3_N)$ and L^4_N since the number of nodes are neither monotonically increasing between layers 0 and 8, nor monotonically decreasing between layers 8 and 16.

3.5. Construction of lattice of all graded lattice chains

In this section we provide the last step in construction of DSGRN logic parameters f_j as chains in L^k_M. Recall that DSGRN logic parameters $\mathcal{L}$ have the form $f = (f_1, \dots, f_n)$ where f_j is the j-logic parameter. If $|\mathbf{T}(j)| = m_j$ and $|\mathbf{S}(j)| = k_j$, then

$$f_j \text{ is an } m_j \text{ chain in } L^{k_j}_M.$$

Similarly, essential logic parameters $\mathcal{L}_E$ are m_j chains where at least one $f^i_j \in L^{k_j}_N$. Therefore this section focuses on construction of graded poset of k-chains $L(k)$ of a graded poset L.

Theorem 3.5. *Given graded poset (L, η) with grading $\eta \in [0, \ell]$, the set $L(k) := (L(k), \omega)$ of all k-chains in L is a graded poset with grading $\omega : L(k) \to [0, k\ell]$ given by*

$$\omega(p) = \sum_{i=1}^{k} \eta(p_i) \quad \textit{where} \quad p = (p_1, \dots, p_k) \;\; \textit{with} \;\; p_i \in L. \tag{12}$$

Proof. Given a graded poset L with grading η we first construct the set of nodes p with grading $\omega(p) = s$:

$$Q(s) := \{p \in L(k) \mid p = (p_1 \preceq p_2 \preceq \dots \preceq p_k), p_j \in L(k), \omega(p) = s\}.$$

Let $U(a) \subset L$ be the set of nodes p with $\eta(p) = a$ i.e. the a-layer in the base lattice L.

We start our construction by listing all decomposition of non-negative integer s as a sum of k non-negative integers a_j

$$s = a_1 + \dots + a_k, \qquad \text{with} \quad a_1 \le a_2 \le \dots \le a_k.$$

For every such decomposition, consider sets $U_1(a_1), \dots U_k(a_k) \subset L$. Note that when $a_i = a_{i+1} = \dots = a_{i+s_i}$ the set $U(a_i)$ appears s_i times in this collection. We call s_i the multiplicity of a_i. Let $\hat{U}_1(b_1) \dots \hat{U}(b_j)$ be a collection of distinct sets in the collection of sets $U_1(a_1), \dots U_k(a_k)$ for $b_1 \lneq b_2 \lneq \dots \lneq b_j$. Then the multiplicity of $\hat{U}_i(b_i)$ is s_j for some j. By renumbering we can assume that the multiplicity of $\hat{U}_i(b_i)$ is s_i.

To construct $p \in L(k)$ we pick $p = (p_1, \dots, p_k)$ with $p_j \in U_j$ in the following way:

Initial step Pick $p_1 \in \hat{U}(b_1)$. Since the multiplicity of b_1 is s_1 we pick $p_1 = p_2 = \dots = p_{s_1}$. Note that since $\hat{U}(b_1)$ is not ordered by Lemma 2.1, and we are constructing a chain p, we must pick the same element of the set $\hat{U}(b_1)$ s_1 times.

Induction step Assume $p_{i-1} \in \hat{U}(b_{i-1})$ has been selected. Let $R(p_{i-1}) \in L$ be the upper set of p_{i-1} in poset L. Pick $p_i \in R(p_{i-1}) \cap \hat{U}(b_i)$ s_i times.

This construction creates a set of nodes $Q(s)$ for $s = 0, \dots, k\ell$ where ℓ is the maximal value of the grading η. To construct the poset $L(k)$ we set

$$p \prec q \quad \text{for } p = (p_1, \dots, p_k) \in Q(s) \text{ and } q = (q_1, \dots, q_k) \in Q(s+1)$$

if there is $s \in \{1, \dots, k\}$ such that

- $p_s \prec q_s$ in L;
- $p_j = q_j$ for all $j \neq s$.

Clearly ω, as defined in (12), is a rank function.

4. Conclusions

The problem of understanding potential dynamics of regulatory network dynamics arises in study of gene regulation. Models of gene regulation are very difficult to parameterize, the precise nonlinearities for a differential equation models are uncertain, yet there is great desire to understand how the structure of the network constraints possible dynamics. We propose that for two widely used models of gene regulation, Boolean and switching systems of ODEs, one can bypass the need for parameterization by considering all models in each class that are compatible with the network. Once this collection is identified, one can analyze qualitative combinatorial dynamics generated by these models. Since it is possible to enumerate all models that are compatible with the network structure within classes of models, their analysis provides user with a dictionary of possible network dynamics.

In order to facilitate enumeration collection of all models compatible with network structure we identify and provide algorithmic construction of posets and lattices that generate these collections.

In particular, given lattice L_M^{k-1} of monotone Boolean functions with $k-1$ inputs and its poset L_N^{k-1} of non-degenerate MBFs with $k-1$ inputs, we construct

- lattice of monotone Boolean functions L_M^k as a $f \oplus g$ with $f, g \in L_M^{k-1}$;
- poset of non-degenerate monotone Boolean functions L_N^k ;

The collections of L_M^k describe all possible Boolean dynamics compatible with network dynamics.

To describe all potential dynamics of switching ODE systems we construct

- all logical parameters $\mathcal{L}$ of parameter graph PG as chains in L_M^j of varying length $j = |\mathbf{T}(i)|$;
- all essential parameters $\mathcal{L}_E$ as subsets of $\mathcal{L}$.

We hope that this work will facilitate computational examination of range of behavior associated to regulatory network dynamics.

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