Homeomorphisms of one-dimensional hyperbolic attractors

by James Leo Jacklitch

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematical Sciences
Montana State University

Abstract:
Williams has shown that one-dimensional connected hyperbolic attractors can be realized as the inverse limits of bouquets of circles with single bonding maps that fix branch points. Suppose (formula not captured in OCR) are diffeomorphisms on manifolds with one-dimensional connected hyperbolic attractors Λ and Γ, respectively. Let A and B denote the transition matrices for the bonding maps on the bouquets of circles whose inverse limits correspond to X and Y with spectral radii μA and μB. We show that if λ and Γ are orientable and homeomorphic then A and B are weakly equivalent. We lso show that if λ and Gamma; are non-orientable and homeomorphic then there exist Perron numbers α and β and positive integers m and n such that αμA; = μ^mB and βμB = &mu^nA. These two results together imply, the weaker result, if A is homeomorphic to Γ and the topological entropies (formula not captured in OCR)then the algebraic extension fields Q(μand Q(μB are identical

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Doctor of Philosophy

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Mathematical Sciences

MONTANA STATE UNIVERSITY
Bozeman, Montana

April 2001

APPROVAL

of a dissertation submitted by

James Leo Jacklitch

This dissertation has been read by each member of the dissertation committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

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ACKNOWLEDGEMENTS

I would like to thank my committee members for their time, suggestions, and comments. I would also like to thank my friends and family for their support. Lastly, I would like to especially thank my advisor, Dr. Marcy Barge, for his guidance and patience, without which, this manuscript would not have been possible.

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ABSTRACT

Williams has shown that one-dimensional connected hyperbolic attractors can be realized as the inverse limits of bouquets of circles with single bonding maps that fix branch points. Suppose $f\colon X \to X$ and $g\colon Y \to Y$ are diffeomorphisms on manifolds with one-dimensional connected hyperbolic attractors Λ and Γ, respectively. Let A and B denote the transition matrices for the bonding maps on the bouquets of circles whose inverse limits correspond to X and Y with spectral radii μ_A and μ_B. We show that if Λ and Γ are orientable and homeomorphic then A and B are weakly equivalent. We also show that if Λ and Γ are non-orientable and homeomorphic then there exist Perron numbers α and β and positive integers m and n such that $\alpha\mu_A = \mu_B^m$ and $\beta\mu_B = \mu_A^n$. These two results together imply, the weaker result, if Λ is homeomorphic to Γ and the topological entropies $h_{\text{top}}(f|_\Lambda) = \log(\mu_A)$ and $h_{\text{top}}(g|_\Gamma) = \log(\mu_B)$ then the algebraic extension fields $\mathbb{Q}(\mu_A)$ and $\mathbb{Q}(\mu_B)$ are identical.

CHAPTER 1

INTRODUCTION

One important area in smooth dynamical systems is the study of structurally stable diffeomorphisms. In fact, one of the main problems introduced by Smale in his paper Differentiable Dynamical Systems [22] was the classification of structurally stable diffeomorphisms.

J. Palis and S. Smale made two fundamental conjectures in this area which were subsequently shown to be valid. The first was the Structural Stability Conjecture which states that a C^1 diffeomorphism is structurally stable if and only if it satisfies Axiom A and strong transversality. Robinson [20] showed the sufficiency and Mañé [14] showed the necessity of this conjecture. necessity of this conjecture. The second was the Ω-Stability Conjecture which states that a C^1 diffeomorphism is Ω-stable if and only if it satisfies Axiom A and has no cycles. Smale [22] showed the sufficiency and Palis [19] showed the necessity of this conjecture.

One theorem, due to Smale [21], concerning the non-wandering sets of C^1 Axiom A diffeomorphisms is the Spectral Decomposition Theorem which states that if f is a C^1 Axiom A diffeomorphism then there is a unique way of writing the non-wandering set $\Omega(f)$ as the finite union of disjoint, closed, invariant, indecomposable subsets $\Omega(f) = \Omega_1 \cup \ldots \cup \Omega_k$ in such a way that on each subset f is topologically transitive. The simplest subsets that appear in the spectral decomposition of an Ω-stable diffeomorphism, after isolated periodic sinks, are expanding attractors.

R. F. Williams realized an important tool for the study of expanding attractors was inverse limits. Williams [24] showed that if a diffeomorphism F has a one-dimensional hyperbolic attractor Λ then $F|_\Lambda$ is topologically conjugate to the shift homeomorphism $\hat{f}$ on an inverse limit of a branched one-manifold G with a single bonding map f satisfying certain properties. In the same paper, he proved a converse for diffeomorphisms on S^4. Williams [26] showed similar results for all diffeomorphisms with expanding attractors.

This relationship between expanding attractors and inverse limits sparked interest in the investigation of inverse limits of finite graphs in hopes that by studying the dynamical properties of the bonding maps that topological properties of the inverse limit spaces could be obtained, and vice versa.

One such result, relating the dynamics of a map on a finite graph and the topological properties of the associated inverse limit space, is the following due to Barge and Diamond [3]. Suppose $f : G \to G$ is a piecewise monotone map on a finite graph G. The following are equivalent: the topological entropy of f is positive; f has a horseshoe; there are positive integers r and M such that f has a periodic point of prime period rm for all $m \geq M$; and the inverse limit space $\varprojlim\{G, f\}$ contains an indecomposable subcontinuum.

Barge and Diamond [4] introduced the relation of weak equivalence for nonnegative square integral matrices to study the inverse limits of finite graphs. Barge, Jacklitch, and Vago [7] then introduced the stronger relation of weak equivalence for substitutions to further study the inverse limits of finite graphs.

The two main results of this dissertation are the following:

THEOREM 1. *Let G and G' be orientable bouquets of circles with branch points b and b', respectively, let $f : G \to G$ and $f' : G' \to G'$ be collapsing surjective immersions such that $f(b) = b$ and $f'(b') = b'$, let M_f be the transition matrix for f relative to an ordering of the components of $G\backslash\{b\}$, and let $M_{f'}$ be the transition matrix for f' relative to an ordering of the components of $G'\backslash\{b'\}$. If $\varprojlim\{G, f\}$ is homeomorphic to $\varprojlim\{G', f'\}$ then M_f is weakly equivalent to $M_{f'}$.*

THEOREM 2. *Let G and G' be finite connected graphs with branch points but without end points and let $f : G \to G$ and $f' : G' \to G'$ be aperiodic collapsing surjective immersions such that the branch sets of the graphs are invariant under the immersions. If $\varprojlim\{G, f\}$ is homeomorphic to $\varprojlim\{G', f'\}$ and the topological entropies $h_{\text{top}}(f) = \log(\lambda_f)$ and $h_{\text{top}}(f') = \log(\lambda_{f'})$ then there exist Perron numbers α and β and positive integers m and n such that $\alpha\lambda_f = \lambda_{f'}^m$ and $\beta\lambda_{f'} = \lambda_f^n$.*

The second theorem includes collapsing surjective immersions on bouquets of circles which fix branch points since they are automatically aperiodic. Both theorems give us corresponding results for one-dimensional connected hyperbolic attractors by results of Williams [24], [25].

CHAPTER 2

PRELIMINARIES

Finite Topological Graphs

A *compactum* is a nonempty compact metrizable space and a *continuum* is a connected compactum.

Suppose x is a point in a one-dimensional compactum K. If x is an end point of every arc in K containing x then x is called an *end point of* K. If K contains a simple n-od ($n \geq 3$) with vertex x then x is called a *branch point of* K. The set of all end points of K will be denoted by $\mathcal{E}(K)$ and the set of all branch points of K will be denoted by $\mathcal{B}(K)$.

A *finite graph* is a compactum which can be written as the union of finitely many arcs each pair of which intersect in at most a common end point. We will call a finite connected graph without endpoints, having only one branch point, a *bouquet of circles.* By definition, a finite graph is locally connected and locally path connected.

Suppose M is a metric space with metric d. The metric d is called *Menger-convex* if for every pair of distinct points $x, z \in M$ there exists a point $y \in M\backslash\{x, z\}$ such that $d(x, y) + d(y, z) = d(x, z)$. The following result is due to Bing [8] and Moise [17], independently.

THEOREM 2.1. *Every locally connected continuum admits a compatible Menger-convex metric.*

Since every finite connected graph is a locally connected continuum, it admits a compatible Menger-convex metric, by the above theorem.

If U is an arc or simple n-od in a compactum K such that $U\backslash\mathcal{E}(U)$ is open in K then we will call U a *star in K*. If U is a star in a finite graph without end points then $\text{Int}(U) = U\backslash\mathcal{E}(U)$.

PROPOSITION 2.2. *If V is an open neighborhood of x in a finite graph graph G without end points then there exists a star neighborhood U of x such that $U \subset V$.*

The following theorem is due to Menger [15].

THEOREM 2.3. *Given any two distinct points in a compact Menger-convex metric space, there exists an arc between the points such that the length of the arc in the metric is the same as the distance between the points.*

Suppose (M, d) is a compact Menger-convex metric space. By the above theorem, the distance between any two distinct points is the same as the length of the shortest arc between the points. So $B_d(x, \epsilon)$ is arc connected for each $x \in M$ and $\epsilon > 0$.

Suppose U is a star in a finite graph G and $\mathcal{A}(U)$ is a finite collection of distinct arcs in G whose union is U. If the end points of U can be partitioned into two subsets such that:

i) given an arc in $\mathcal{A}(U)$ the end points of that arc are not contained in the same element of the partition, *ii)* given a point from each element of the partition, the arc between the points that is contained in U is contained in $\mathcal{A}(U)$

then $\mathcal{A}(U)$ is called an *arc structure on U.* If U has n end points then there are $2^{n-1} - 1$ different arc structures that can be placed on U. An arc structure on U naturally induces an arc structure on any star contained in U.

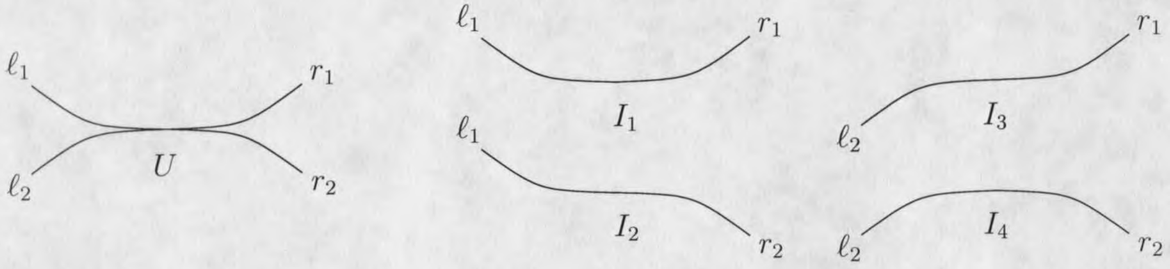

Figure 1. A star U with arc structure $\mathcal{A}(U) = \{I_1, I_2, I_3, I_4\}$.

Let G be a finite graph without end points. Since the collection of all stars in G with end points removed forms a basis for the topology on G, we only need to work with these kind of neighborhoods when studying G.

Suppose U and V are arc structured stars in G. Since a nondegenerate component of the intersection of two stars in G is a star, the arc structures on U and V induce arc structures on each nondegenerate component of $U \cap V$. If the arc structures on each nondegenerate path component of $U \cap V$, induced by the arc structure on U and the arc structure on V, are the same then U and V are said to have *coherent arc structures.* A choice of coherent arc structures on each star contained in G is called an *arc structure system for G.*

Suppose U is a arc structured star in G with arc structure $\mathcal{A}(U)$. If $\rho : U \to \mathbb{R}$ is a map, by which we always mean a continuous function, such that $\rho|_I : I \to \mathbb{R}$ is a topological embedding for each arc $I \in \mathcal{A}(U)$ then we will call ρ a *directing map on U.* Two directing maps, ρ_1 and ρ_2, on U are called *equivalent* if for each arc $I \in \mathcal{A}(U)$ the homeomorphism $(\rho_2|_I) \circ (\rho_1|_I)^{-1} : \rho_1(I) \to \rho_2(I)$ is increasing. This is an equivalence relation that partitions the collection of all directing maps on U into two equivalence classes, each of which we will call a *direction for U.* We will call an arc structured star together with a direction for it a *directed star.*

If U is a directed star then the direction for U naturally induces a direction for any star contained in U.

Suppose U and V are directed stars. If the directions for each nondegenerate path component of $U \cap V$, induced by the direction for U and the direction for V, are the same then we say U and V are *coherently directed.*

Suppose ρ_1 is a directing map on a star U_1 in G and ρ_2 is a directing map on a star U_2 in G. If $x \in \mathrm{Int}(U_1) \cap \mathrm{Int}(U_2)$ and there exists a star neighborhood V of x contained in $U_1 \cap U_2$ such that $\rho_1|_V$ and $\rho_2|_V$ are equivalent then ρ_1 and ρ_2 are said to be *equivalent near x.* This is an equivalence relation that partitions the collection of all directing maps on all star neighborhoods of x into two equivalence classes, each of which is called a *direction at x.* We will let $\mathcal{D}_x$ denote the collection of these two directions at x. The direction for a directed star naturally induces a direction at each point in the interior of that star.

Suppose (U, D_U) and (V, D_V) are directed stars in G. If $x \in \text{Int}(U) \cap \text{Int}(V)$ and there exists a star neighborhood W of x contained in $U \cap V$ such that the directions for W induced by D_U and D_V are equal then we say (U, D_U) and (V, D_V) are *coherently directed near* x. If (U, D_U) and (V, D_V) are coherently directed near x for each $x \in \text{Int}(U) \cap \text{Int}(V)$ then (U, D_U) and (V, D_V) are coherently directed.

Suppose G is endowed with an arc structure system. We say G is *orientable* if there exists a collection of coherently directed stars whose interiors cover G. If G is orientable then a choice of coherently directed stars whose interiors cover G induces a direction at each point of G. This continuous choice of directions at each point of G is called an *orientation of* G. If G is connected and orientable then there exist exactly two distinct orientations of G.

Suppose G' is finite graph without end points endowed with an arc structure system. We call a map $f : G \to G'$ an *immersion* if for each point x in G and each star neighborhood V of $f(x)$ in G' there exists a star neighborhood U of x in G such that the restriction of f to each arc in $\mathcal{A}(U)$ is a topological embedding into an arc in $\mathcal{A}(V)$. We call an immersion $f : G \to G'$ *collapsing* if for each point x in G and each star neighborhood V of $f(x)$ in G' there exists a star neighborhood U of x in G such that the image of U under f is contained in an arc in $\mathcal{A}(V)$.

If $f : G \to G$ is an immersion such that $f(\mathcal{B}(G)) = \mathcal{B}(G)$ then $\mathcal{B}(G) \subset f^{-1}(\mathcal{B}(G))$ and f is one-to-one on each component of $G \backslash f^{-1}(\mathcal{B}(G))$. Thus f is a Markov map and we can associate a matrix to f relative to an ordering of the components of

$G \backslash \mathcal{B}(G)$ in the following manner. The *transition matrix* for f relative to the ordered components, $\{J_1, \ldots, J_m\}$, of $G \backslash \mathcal{B}(G)$ is the $m \times m$ matrix whose ij-th entry is the number of times the component J_j covers the component J_i under f, that is, the number of components of $G \backslash f^{-1}(\mathcal{B}(G))$ in J_j that map onto J_i. If A and B are transition matrices for f relative to different orderings of the components of $G \backslash \mathcal{B}(G)$ then A is similar to B by a permutation matrix.

We define λ_f to be the spectral radius of the transition matrix for f relative to an ordering of the components of $G \backslash \mathcal{B}(G)$. This definition is independent of which ordering is chosen by the similarity remark made above.

It is a well known result, due to Block, Guckenheimer, Misiurewicz, and Young [9], that the topological entropy of a Markov map on an interval or circle is equal to the logarithm of the spectral radius of an associated transition matrix. We can easily extend this result to Markov maps on finite graphs using similar techniques. Thus $h_{\text{top}}(f) = \log(\lambda_f)$.

The following result about topological entropies is due to Bowen [10].

THEOREM 2.4. *Let $f : X \to X$ and $g : Y \to Y$ be maps where X and Y are compacta. If $k : X \to Y$ is a uniformly finite-to-one surjective map such that $k \circ f = g \circ k$ then $h_{\text{top}}(f) = h_{\text{top}}(g)$.*

Two maps $f : X \to X$ and $g : Y \to Y$ are called *shift equivalent* provided there are maps $r : X \to Y$ and $s : Y \to X$ such that $r \circ f = g \circ r$, $f \circ s = s \circ g$, $s \circ r = f^m$, and $r \circ s = g^m$ for some positive integer m.

LEMMA 2.5. *If G and G' are finite graphs and $f : G \to G$ and $f' : G' \to G'$ are surjective immersions such that f is shift equivalent to f' then $h_{\text{top}}(f) = h_{\text{top}}(f')$.*

PROOF. Since f is shift equivalent to f', there exist maps $r : G \to G'$ and $s : G' \to G$ such that $r \circ f = f' \circ r$, $f \circ s = s \circ f'$, $s \circ r = f^m$, and $r \circ s = (f')^m$ for some positive integer m. Since G and G' are compact and f^m and $(f')^m$ are surjective immersions, $f^m = s \circ r$ and $(f')^m = r \circ s$ are uniformly finite-to-one surjective maps. Then r and s are uniformly finite-to-one surjective maps. Thus $h_{\text{top}}(f) = h_{\text{top}}(f')$ by Theorem 2.4. □

A square nonnegative integer matrix A is called *aperiodic* if there exists a positive integer m such that A^m is a positive integer matrix. We will call f *aperiodic* if the transition matrix for f relative to an ordering of the components of $G \backslash \mathcal{B}(G)$ is aperiodic. This definition is again independent of which ordering is chosen by the earlier similarity remark.

A real number $\lambda \geq 1$ is called a *Perron number* if it is an algebraic integer that strictly dominates all of its other algebraic conjugates. The spectral radius of an aperiodic matrix is a Perron number. Thus, if f is aperiodic then λ_f is a Perron number.

We will say two square nonnegative integer matrices, A and B, are *weakly equivalent* if there exist sequences of positive integers, $\{n_i\}_{i=1}^{\infty}$ and $\{m_i\}_{i=1}^{\infty}$, and sequences of nonnegative integer matrices, $\{S_i\}_{i=1}^{\infty}$ and $\{T_i\}_{i=1}^{\infty}$, such that $S_iT_i = A^{n_i}$ and

$T_iS_{i+1} = B^{m_i}$ for all $i \in \mathbb{N}$. If A^j and B^k are weakly equivalent for positive integers j and k then A and B are weakly equivalent by definition. The following theorem is due to Barge and Diamond [4].

THEOREM 2.6. *Let A and B be aperiodic nonnegative integer matrices with spectral radii λ_A and λ_B, respectively. If A is weakly equivalent to B then there are Perron numbers, α and β, and positive integers, m and n, such that $\alpha\lambda_A = \lambda_B^m$ and $\beta\lambda_B = \lambda_A^n$.*

It should be noted that this Perron relationship between spectral radii is weaker than weak equivalence of the matrices. We provide an example of this in the next chapter.

Inverse Limit Spaces

An *inverse sequence of spaces* is a double sequence $\{X_i, f_i\}_{i=1}^{\infty}$ of spaces X_i, called *coordinate spaces*, and maps $f_i : X_{i+1} \to X_i$, called *bonding maps*. If $\{X_i, f_i\}_{i=1}^{\infty}$ is an inverse sequence of spaces then the *inverse limit* of $\{X_i, f_i\}_{i=1}^{\infty}$, denoted by $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$, is the subspace of the product space $\prod_{i=1}^{\infty} X_i$ defined by $\varprojlim\{X_i, f_i\}_{i=1}^{\infty} = \{(x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} X_i \mid f_i(x_{i+1}) = x_i \text{ for each } i \in \mathbb{N}\}$.

THEOREM 2.7. *If $\{X_i, f_i\}_{i=1}^{\infty}$ is an inverse sequence of compacta (continua) then $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ is a compactum (continuum).*

PROOF. See [18]. □

Let X be a space and $f : X \to X$ be a map. If $\pi_j : \prod_{i=1}^{\infty} X \to X$ is the projection map of the product space into the j-th coordinate space, defined by $\pi_j((x_i)_{i=1}^{\infty}) = x_j$, then $\pi_j|_{\varprojlim\{X,f\}} : \varprojlim\{X, f\} \to X$ is a map. It can be shown that if f is a surjective map then $\pi_j|_{\varprojlim\{X,f\}}$ is a surjective map. Although we should write $\pi_j|_{\varprojlim\{X,f\}}$ when referring to the j-th projection map acting on the inverse limit space $\varprojlim\{X, f\}$, we will write π_j without this restriction for brevity. It can be shown that the collection $\{\pi_i^{-1}(U) \mid U$ is an open set in X and $i \in \mathbb{N}\}$ is a basis for the subspace topology on $\varprojlim\{X, f\}$.

Let Y be a space, let $g : Y \to Y$ be a map, and let $\{p_i\}_{i=1}^{\infty}$ be a collection of maps $p_i : X \to Y$ such that $p_i \circ f = g \circ p_{i+1}$ for each $i \in \mathbb{N}$. If $(x_i)_{i=1}^{\infty} \in \varprojlim\{X, f\}$ then $(\prod_{i=1}^{\infty} p_i)((x_i)_{i=1}^{\infty}) = (p_i(x_i))_{i=1}^{\infty} \in \varprojlim\{Y, g\}$ since $g(p_{i+1}(x_{i+1})) = p_i(f(x_{i+1})) = p_i(x_i)$ for each $i \in \mathbb{N}$. So $(\prod_{i=1}^{\infty} p_i)|_{\varprojlim\{X,f\}}$ maps $\varprojlim\{X, f\}$ into $\varprojlim\{Y, g\}$. We will call $p_\infty \equiv (\prod_{i=1}^{\infty} p_i)|_{\varprojlim\{X,f\}} : \varprojlim\{X, f\} \to \varprojlim\{Y, g\}$ the *ladder map* induced by $\{p_i\}_{i=1}^{\infty}$.

If $p_\infty : \varprojlim\{X, f\} \to \varprojlim\{Y, g\}$ is the ladder map induced by $\{p_i\}_{i=1}^{\infty}$ and π_i and σ_i are the i-th projection maps acting on $\varprojlim\{X, f\}$ and $\varprojlim\{Y, g\}$, respectively, then $p_i \circ \pi_i = \sigma_i \circ p_\infty$ for each $i \in \mathbb{N}$. Note that p_∞ is the only map from $\varprojlim\{X, f\}$ into $\varprojlim\{Y, g\}$ such that $p_i \circ \pi_i = \sigma_i \circ p_\infty$ for each $i \in \mathbb{N}$.

THEOREM 2.8. *Let X and Y be compacta, let $f : X \to X$ and $g : Y \to Y$ be maps, and let $\{p_i\}_{i=1}^{\infty}$ be a collection of maps $p_i : X \to Y$ such that $p_i \circ f = g \circ p_{i+1}$ for each $i \in \mathbb{N}$. If each p_i is an open k-to-one surjective map such that $f|_{p_i^{-1}(\{y\})}$ is* *a one-to-one map for each $y \in p_i(X)$ then the ladder map induced by $\{p_i\}_{i=1}^{\infty}$ is an open k-to-one surjective map.*

PROOF. Assume $p_\infty : \varprojlim\{X,f\} \to \varprojlim\{Y,g\}$ is the ladder map induced by $\{p_i\}_{i=1}^{\infty}$ where each p_i is an open k-to-one surjective map such that $f|_{p_i^{-1}(\{y\})}$ is a one-to-one map for each $y \in p_i(X)$.

p_∞ *is surjective*: Let $(y_i)_{i=1}^{\infty} \in \varprojlim\{Y,g\}$. Since p_i is a surjective map, $\{y_i\}$ is closed in Y, and X is a compactum, the subspace $p_i^{-1}(\{y_i\})$ of X is a compactum. Then $\varprojlim\{p_i^{-1}(\{y_i\}), f_i|_{p_{i+1}^{-1}(\{y_{i+1}\})}\}_{i=1}^{\infty}$ is a compactum, by Theorem 2.7, which maps onto $(y_i)_{i=1}^{\infty}$.

p_∞ *is k-to-one*: Let $(y_i)_{i=1}^{\infty} \in p_\infty(\varprojlim\{X,f\}) = \varprojlim\{Y,g\}$. Since $y_i \in p_i(X) = Y$ and p_i is k-to-one, $p_i^{-1}(\{y_i\})$ has k elements. Suppose $\{x_{1,j}\}_{j=1}^{k} = p_1^{-1}(\{y_1\})$. Since $p_i \circ f = g \circ p_{i+1}$, $f(p_{i+1}^{-1}(\{y_{i+1}\})) \subset p_i^{-1}(\{y_i\})$. Since $p_i^{-1}(\{y_i\})$ has k elements for each $i \in \mathbb{N}$ and $f|_{p_{i+1}^{-1}(\{y_{i+1}\})}$ is a one-to-one map, $f(p_{i+1}^{-1}(\{y_{i+1}\})) = p_i^{-1}(\{y_i\})$. For each $x_{i,j} \in p_i^{-1}(\{y_i\})$, let $x_{i+1,j}$ be the unique element of $p_{i+1}^{-1}(\{y_{i+1}\})$ such that $f(x_{i+1,j}) = x_{i,j}$. Then $p_\infty^{-1}(\{(y_i)_{i=1}^{\infty}\}) = \{(x_{i,j})_{i=1}^{\infty}\}_{j=1}^{k}$.

p_∞ *is open*: For $i \in \mathbb{N}$, let π_i and σ_i be the i-th projection maps acting on $\varprojlim\{X,f\}$ and $\varprojlim\{Y,g\}$, respectively. Let $\pi_k^{-1}(U) \in \{\pi_i^{-1}(U) | U$ is an open set in X and $i \in \mathbb{N}\}$, which is a basis for $\varprojlim\{X,f\}$. Since p_k is open and U is an open set in X, $p_k(U)$ is an open set in Y. So $\sigma_k^{-1}(p_k(U))$ is open in $\varprojlim\{Y,g\}$. Let $(y_i)_{i=1}^{\infty} \in \sigma_k^{-1}(p_k(U))$. Let $x_k \in U \cap p_k^{-1}(\{y_k\})$; the latter set is nonempty since $y_k = \sigma_k((y_i)_{i=1}^{\infty}) \in p_k(U)$. Since $f^i(p_{k+i}^{-1}(\{y_{k+i}\})) = p_k^{-1}(\{y_k\})$ for $i \in \mathbb{N}$, let $x_{k+i} \in$

$p_{k+i}^{-1}(\{y_{k+i}\})$ such that $f^i(x_{k+i}) = x_k$. Then $\underline{x} = (f^{k-1}(x_k), \dots, f(x_k), x_k, x_{k+1}, \dots) \in \pi_k^{-1}(U)$ and $p_\infty(\underline{x}) = (y_i)_{i=1}^\infty$. So $(y_i)_{i=1}^\infty \in p_\infty(\pi_k^{-1}(U))$ and $\sigma_k^{-1}(p_k(U)) \subset p_\infty(\pi_k^{-1}(U))$. Since $\sigma_k \circ p_\infty = p_k \circ \pi_k$, $p_\infty(\pi_k^{-1}(U)) \subset \sigma_k^{-1}(p_k(U))$. So $p_\infty(\pi_k^{-1}(U)) = \sigma_k^{-1}(p_k(U))$ is an open set in $\varprojlim\{Y, g\}$. □

If $p : X \to Y$ is a map such that $p \circ f = g \circ p$ then we will call $(\prod_{i=1}^\infty p)|_{\varprojlim\{X,f\}} : \varprojlim\{X, f\} \to \varprojlim\{Y, g\}$ the *ladder map induced by* p. Let $\hat{f} : \varprojlim\{X, f\} \to \varprojlim\{X, f\}$ be the ladder map induced by f. By definition, $\pi_k = \pi_{k+1} \circ \hat{f}$ for each $k \in \mathbb{N}$. The map $\hat{f}$ is a homeomorphism called the *shift homeomorphism* induced by f.

If X is a compactum with metric d then $\underline{d}((x_i)_{i=1}^\infty, (z_i)_{i=1}^\infty) = \sum_{i=1}^{\infty} d(x_i, z_i)/2^i$ is compatible with the subspace topology on $\varprojlim\{X, f\}$.

Matchbox Continua

Let X be a continuum. Following closely notation introduced by Aarts and Martens [1], we make the following definitions. If C is a zero-dimensional compactum and $h : C \times [-1, 1] \to X$ is a topological embedding such that $h(C \times [-1, 1])$ is closed in X and $h(C \times (-1, 1))$ is open in X then the set $M \equiv h(C \times [-1, 1])$ is called a *matchbox in* X. For each $c \in C$, the set $h(\{c\} \times [-1, 1])$ is called a *match in* M. If every point in X has an open neighborhood homeomorphic to the product of a zero-dimensional compactum and an open arc then X is called a *matchbox continuum.*

Let G be a finite connected graph with branch points but without end points, let d be a Menger-convex metric compatible with G, and let $f : G \to G$ be a collapsing surjective immersion such that $f(b_i) = b_i$ for each $b_i \in \{b_1, \ldots, b_r\} \equiv \mathcal{B}(G)$. Let $\{J_1, \ldots, J_m\}$ be the path components of $G \backslash \mathcal{B}(G)$ and $\{b_1, \ldots, b_r, \ldots, b_s\} \equiv f^{-1}(\mathcal{B}(G))$.

PROPOSITION 2.9. *For each* $k \in \mathbb{N}$ *and* $J_i \in \{J_1, \ldots, J_m\}$, $\pi_k^{-1}(J_i) \subset \varprojlim\{G, f\}$ *is homeomorphic to the product of a zero-dimensional compactum and an open arc.*

PROOF. Let $\{I_1, \ldots, I_n\}$ be the path components of $G \backslash f^{-1}(\mathcal{B}(G))$ and σ be the discrete metric on the set $\{1, \ldots, n\}$. Let $C_i \equiv \{(s_j)_{j=1}^{\infty} \in \prod_{j=1}^{\infty}\{1, \ldots, n\} \mid J_i = f(I_{s_1})$ and $I_{s_j} \subset f(I_{s_{j+1}})$ for each $j \in \mathbb{N}\}$ which is a closed subset of the zero-dimensional compactum $\prod_{j=1}^{\infty}\{1, \ldots, n\}$ with metric $\underline{\sigma}$ defined by $\underline{\sigma}((s_j)_{j=1}^{\infty}, (t_j)_{j=1}^{\infty}) = \sum_{j=1}^{\infty} \sigma(s_j, t_j)/2^j$. Let $h_{i,k} : C_i \times J_i \to \pi_k^{-1}(J_i)$ be defined by $h_{i,k}((s_j)_{j=1}^{\infty}, x) = (f^{k-1}(x), \ldots, x, (f|_{I_{s_1}})^{-1}(x), (f|_{I_{s_2}})^{-1} \circ (f|_{I_{s_1}})^{-1}(x), \ldots)$.

$h_{i,k}$ *is surjective*: Let $\underline{x} \in \pi_k^{-1}(J_i)$. Then $\pi_k(\underline{x}) \in J_i$ and $f \circ \pi_{\ell+1}(\underline{x}) = \pi_\ell(\underline{x})$ for each $\ell \in \mathbb{N}$. Since $\pi_k(\underline{x}) \notin \mathcal{B}(G)$, $\pi_{k+j}(\underline{x}) \notin f^{-1}(\mathcal{B}(G))$ for each $j \in \mathbb{N}$. For each $j \in \mathbb{N}$, let s_j be the element of $\{1, \ldots, n\}$ such that $\pi_{k+j}(\underline{x}) \in I_{s_j}$. Since $\pi_{k+1}(\underline{x}) \in I_{s_1}$ and $f \circ \pi_{k+1}(\underline{x}) = \pi_k(\underline{x}) \in J_i$, $J_i = f(I_{s_1})$ and $(f|_{I_{s_1}})^{-1} \circ \pi_k(\underline{x}) = \pi_{k+1}(\underline{x})$. Since $\pi_{k+j+1}(\underline{x}) \in I_{s_{j+1}}$ and $f \circ \pi_{k+j+1}(\underline{x}) = \pi_{k+j}(\underline{x}) \in I_{s_j}$, $I_{s_j} \subset f(I_{s_{j+1}})$ and $(f|_{I_{s_{j+1}}})^{-1} \circ \pi_{k+j}(\underline{x}) = \pi_{k+j+1}(\underline{x})$. So $(s_j)_{j=1}^{\infty} \in C_i$ and $(f|_{I_{s_j}})^{-1} \circ \ldots \circ (f|_{I_{s_1}})^{-1} \circ \pi_k(\underline{x}) = \pi_{k+j}(\underline{x})$. Thus $((s_j)_{j=1}^{\infty}, \pi_k(\underline{x})) \in C_i \times J_i$ and $h_{i,k}((s_j)_{j=1}^{\infty}, \pi_k(\underline{x})) = (f^{k-1} \circ \pi_k(\underline{x}), \ldots, \pi_k(\underline{x}), (f|_{I_{s_1}})^{-1} \circ \pi_k(\underline{x}), (f|_{I_{s_2}})^{-1} \circ (f|_{I_{s_1}})^{-1} \circ \pi_k(\underline{x}), \ldots) = (\pi_1(\underline{x}), \ldots, \pi_k(\underline{x}), \pi_{k+1}(\underline{x}), \ldots) = \underline{x}$.

$h_{i,k}$ *is injective*: Suppose $h_{i,k}((s_j)_{j=1}^{\infty}, x) = h_{i,k}((t_j)_{j=1}^{\infty}, z)$. Then $x = \pi_k \circ h_{i,k}((s_j)_{j=1}^{\infty}, x) = \pi_k \circ h_{i,k}((t_j)_{j=1}^{\infty}, z) = z$. Also for $j \in \mathbb{N}$, $s_j = t_j$ since $\pi_{k+j} \circ h_{i,k}((s_j)_{j=1}^{\infty}, x) \in I_{s_j}$, $\pi_{k+j} \circ h_{i,k}((t_j)_{j=1}^{\infty}, z) \in I_{t_j}$, and $\pi_{k+j} \circ h_{i,k}((s_j)_{j=1}^{\infty}, x) = \pi_{k+j} \circ h_{i,k}((t_j)_{j=1}^{\infty}, z)$. Thus $((s_j)_{j=1}^{\infty}, x) = ((t_j)_{j=1}^{\infty}, z)$.

$h_{i,k}$ *is continuous*: Let $\epsilon > 0$ and $((s_j)_{j=1}^{\infty}, x) \in C_i \times J_i$. Choose a positive integer K such that $\text{diam}(G)/2^K < \epsilon/3$ and a positive integer ℓ such that $k + \ell \geq K$. Since f^{k-j} is continuous for each $j \in \{1, \ldots, k\}$, there exists a $\delta_j > 0$ such that if $d(x, z) < \delta_j$ then $d(f^{k-j}(x), f^{k-j}(z)) < \epsilon/3$. Since $(f|_{I_{s_{j-k}}})^{-1} \circ \ldots \circ (f|_{I_{s_1}})^{-1}$ is continuous on J_i for $j \in \{k+1, \ldots, k+\ell\}$, there exists a $\delta_j > 0$ such that if $z \in J_i$ and $d(x, z) < \delta_j$ then $d((f|_{I_{s_{j-k}}})^{-1} \circ \ldots \circ (f|_{I_{s_1}})^{-1}(x), (f|_{I_{s_{j-k}}})^{-1} \circ \ldots \circ (f|_{I_{s_1}})^{-1}(z)) < \epsilon/3$. Let $\delta = \min\{1/2^{k+\ell}, \delta_1, \ldots, \delta_{k+\ell}\}$ and $((t_j)_{j=1}^{\infty}, z) \in C_i \times J_i$ such that $\rho(((s_j)_{j=1}^{\infty}, x), ((t_j)_{j=1}^{\infty}, z)) < \delta$. Then $\underline{\sigma}((s_j)_{j=1}^{\infty}, (t_j)_{j=1}^{\infty}) < \delta$ and $d(x, z) < \delta$. So $\sigma(s_j, t_j)/2^j < \underline{\sigma}((s_j)_{j=1}^{\infty}, (t_j)_{j=1}^{\infty}) < 1/2^{k+\ell}$, that is $\sigma(s_j, t_j) = 0$, for $j \in \{1, \ldots, k+\ell\}$. Thus $\underline{d}(h_{i,k}((s_j)_{j=1}^{\infty}, x), h_{i,k}((t_j)_{j=1}^{\infty}, z)) = \sum_{j=1}^{\infty} d(\pi_j \circ h_{i,k}((s_j)_{j=1}^{\infty}, x), \pi_j \circ h_{i,k}((t_j)_{j=1}^{\infty}, z))/2^j = \sum_{j=1}^{k} d(f^{k-j}(x), f^{k-j}(z))/2^j + \sum_{j=k+1}^{k+\ell} d((f|_{I_{s_{j-k}}})^{-1} \circ \ldots \circ (f|_{I_{s_1}})^{-1}(x), (f|_{I_{t_{j-k}}})^{-1} \circ \ldots \circ (f|_{I_{t_1}})^{-1}(z))/2^j + \sum_{j=k+\ell+1}^{\infty} d(\pi_j \circ h_{i,k}((s_j)_{j=1}^{\infty}, x), \pi_j \circ h_{i,k}((t_j)_{j=1}^{\infty}, z))/2^j < \epsilon/3 + \epsilon/3 + \text{diam}(G)/2^{k+\ell} < \epsilon$.

$h_{i,k}^{-1}$ *is continuous*: Let $\epsilon > 0$ and $\underline{x} \in \pi_k^{-1}(J_i)$. Then there exists a positive integer ℓ such that $1/2^{\ell} < \epsilon$ and an $(s_j)_{j=1}^{\infty} \in C_i$ such that $((s_j)_{j=1}^{\infty}, \pi_k(\underline{x})) = h_{i,k}^{-1}(\underline{x})$. For $j \in \{1, \ldots, \ell\}$, since I_{s_j} is open in G, there exists a $\delta_j > 0$ such that $B_d(\pi_{k+j}(\underline{x}), \delta_j) \subset I_{s_j}$. Let $\delta = \min\{\epsilon/2^k, \delta_1/2^{k+1}, \ldots, \delta_\ell/2^{k+\ell}\}$ and $\underline{z} \in \pi_k^{-1}(J_i)$

such that $\underline{d}(\underline{x}, \underline{z}) < \delta$. Then $d(\pi_j(\underline{x}), \pi_j(\underline{z}))/2^j < \delta$ for each $j \in \mathbb{N}$ and there exists a $(t_j)_{j=1}^{\infty} \in C_i$ such that $((t_j)_{j=1}^{\infty}, \pi_k(\underline{z})) = h_{i,k}^{-1}(\underline{z})$. So $d(\pi_k(\underline{x}), \pi_k(\underline{z})) < \epsilon$ and $d(\pi_{k+j}(\underline{x}), \pi_{k+j}(\underline{z})) < \delta_j$ for each $j \in \{1, \ldots, n\}$. Then $\pi_{k+j}(\underline{z}) \in I_{s_j}$ for each $j \in \{1, \ldots, n\}$ and $\underline{\sigma}((s_j)_{j=1}^{\infty}, (t_j)_{j=1}^{\infty}) = \sum_{j=1}^{\infty} \sigma(s_j, t_j)/2^j = \sum_{j=\ell+1}^{\infty} \sigma(s_j, t_j)/2^j \leq 1/2^{\ell} < \epsilon$. Thus $\rho(h_{i,k}^{-1}(\underline{x}), h_{i,k}^{-1}(\underline{z})) = \rho(((s_j)_{j=1}^{\infty}, \pi_k(\underline{x})), ((t_j)_{j=1}^{\infty}, \pi_k(\underline{z}))) = \max\{\underline{\sigma}((s_j)_{j=1}^{\infty}, (t_j)_{j=1}^{\infty}), d(\pi_k(\underline{x}), \pi_k(\underline{z}))\} < \epsilon$. □

COROLLARY 2.10. *For each $k \in \mathbb{N}$ and $x \in G \backslash \mathcal{B}(G)$, $\pi_k^{-1}(U)$ is a matchbox in $\varprojlim\{G, f\}$ for every star neighborhood U of x contained in $G \backslash \mathcal{B}(G)$.*

PROOF. Let J_i be the component of $G \backslash \mathcal{B}(G)$ containing U and let C_i be the one-dimensional compactum and $h_{i,k} : C_i \times J_i \to \pi_k^{-1}(U)$ the homeomorphism defined in Proposition 2.9. Since U is an arc, let $\rho : U \to [-1, 1]$ be a homeomorphism. Then $(h_{i,k}|_{C_i \times U}) \circ (id_{C_i} \times \rho^{-1}) : C_i \times [-1, 1] \to \pi_k^{-1}(U)$ is a homeomorphism. □

PROPOSITION 2.11. *For each $k \in \mathbb{N}$ and $x_i \in \mathcal{B}(G)$, there exists a $\delta_{i,k} > 0$ such that $\pi_k^{-1}(U)$ is a matchbox in $\varprojlim\{G, f\}$ for every star neighborhood U of x_i contained in $B_d(x_i, \delta_{i,k})$.*

PROOF. For each $i \in \{1, \ldots, r\}$, let W_i be a star neighborhood of x_i such that the only element of $f^{-1}(\mathcal{B}(G))$ contained in W_i is x_i and let $\rho_i : W_i \to [-1, 1]$ be a map such that $\rho_i|_{I_j} : I_j \to [-1, 1]$ is a homeomorphism for each $I_j \in \mathcal{A}(W_i)$. We can choose each W_i so $\{W_1, \ldots, W_r\}$ is a pairwise disjoint collection. For $i \in \{1, \ldots, r\}$, let V_i be a star neighborhood of x_i contained in $f^{-1}(\mathrm{Int}(W_i)) \cap \mathrm{Int}(W_i)$. For $i \in \{r+1, \ldots, s\}$, let V_i be an arc neighborhood of x_i contained in $(G \backslash \bigcup_{j=1}^{r} W_j) \cap f^{-1}(\bigcup_{j=1}^{r} \mathrm{Int}(W_j))$. We can choose each V_i so $\{V_{r+1}, \dots, V_s\}$ is a pairwise disjoint collection. For $i \in \{1, \dots, r\}$, let $\mathcal{A}_i = \{I_\ell \in \bigcup_{j=1}^{s} \mathcal{A}(V_j) \mid I_\ell \cap f^{-1}(\{x_i\}) \neq \emptyset\}$. Then $\bigcap_{I_\ell \in \mathcal{A}_i} \rho_i^{-1} \circ \rho_i \circ f(I_\ell)$ is a star neighborhood of x_i contained in V_i. For $i \in \{1, \dots, s\}$, let U_i be the path component of $f^{-1}(\bigcup_{j=1}^{r} \bigcap_{I_\ell \in \mathcal{A}_i} \rho_i^{-1} \circ \rho_i \circ f(I_\ell)$ contained in V_i. For each $i \in \{1, \dots, r\}$, we choose a homeomorphism $h_i : W_i \to W_i$ such that the end points and branch point of W_i are fixed and the end points of $\bigcap_{I_\ell \in \mathcal{A}_i} \rho_i^{-1} \circ \rho_i \circ f(I_\ell)$ are mapped to the end points of U_i. Let $h : G \to G$ be the homeomorphism defined by $h(x) = h_i(x)$ for $x \in W_i$ and $h(x) = x$ otherwise. By definition, $h(J_i) = J_i$ and h preserves the canonical orientation of J_i. Let $f' : G \to G$ be the collapsing surjective immersion defined by $f' = h \circ f$. We now construct a homeomorphism $p_\infty : \varprojlim\{G, f\} \to \varprojlim\{G, \hat{f}\}$ in the following manner. Let $p_1 : G \to G$ be the identity map. Suppose that for $j = 1, \dots, N$, we have defined homeomorphisms $p_j : G \to G$ such that $p_j(x_i) = x_i$ for each $x_i \in f^{-1}(\mathcal{B}(G))$, $p_j(J_i) = J_i$, p_j preserves the canonical orientation of J_i for each $J_i \in \{J_1, \dots, J_m\}$, and the following diagram commutes.

$$
\begin{array}{ccccccc}
G & \xleftarrow{f} & G & \xleftarrow{f} & \cdots & \xleftarrow{f} & G \\
\downarrow p_1 & & \downarrow p_2 & & \cdots & & \downarrow p_N \\
G & \xleftarrow[f']{} & G & \xleftarrow[f']{} & \cdots & \xleftarrow[f']{} & G
\end{array}
$$

Let $p_{N+1}|_{J_{i,\ell}} : J_{i,\ell} \to J_{i,\ell}$ be the homeomorphism defined by $p_{N+1}|_{J_{i,\ell}} = (f'|_{J_{i,\ell}})^{-1} \circ p_N \circ f|_{J_{i,\ell}}$. Then p_{N+1} extends from $\bigcup_{i=1}^{m} \bigcup_{\ell=1}^{r_i} J_{i,\ell}$ to a homeomorphism on G such that

$p_N \circ f = f' \circ p_{N+1}$. Let $p_\infty : \varprojlim\{G, f\} \to \varprojlim\{G, \hat{f}\}$ be the ladder map induced by $\{p_i\}_{i=1}^\infty$. Since p_i is a homeomorphism for each $i \in \mathbb{N}$, p_∞ is a homeomorphism by Theorem 2.8. Let π'_k be the k-th projection map acting on $\varprojlim\{G, f'\}$.

Claim. For each $k \in \mathbb{N}$ and $U_i \in \{U_1, \dots, U_r\}$, $(\pi'_k)^{-1}(U)$ is a matchbox in $\varprojlim\{G, f'\}$ for every star neighborhood U of x_i contained in $\mathrm{Int}(U_i)$.

Proof of Claim. Let $\{U_{s+1}, \dots, U_t\}$ be the components of $G \setminus \bigcup_{i=1}^s \mathrm{Int}(U_i)$ and let $\{I_1, \dots, I_n\}$ be the collection of open arcs such that $\{\mathrm{Cl}(I_1), \dots, \mathrm{Cl}(I_n)\} = \bigcup\limits_{i=1}^{t} \mathcal{A}(U_i)$. For $i \in \{1, \dots, r\}$, we assume that $\mathrm{Cl}(I_i)$ is the arc in $\mathcal{A}(U_i)$ such that $f'(I_j) = I_i$ for every I_j with $\mathrm{Cl}(I_i) \in \mathcal{A}(U_i)$. Let $C_i = \{(s_j)_{j=0}^\infty \in \prod_{j=0}^\infty \{1, \dots, n\} \mid \mathrm{Cl}(I_{s_0}) \in \mathcal{A}(U_i)$ and $I_{s_j} \subset f'(I_{s_{j+1}})$ for $j = 0, 1, \dots\}$, which is a closed subset of the zero-dimensional compactum $\prod_{j=0}^\infty \{1, \dots, n\}$. Let U be a star neighborhood of x_i contained in $\mathrm{Int}(U_i)$ and $\rho : U \to [-1, 1]$ be a map such that $\rho|_{U \cap I_j} : U \cap I_j \to [-1, 1]$ is a homeomorphism for each I_j with $\mathrm{Cl}(I_j) \in \mathcal{A}(U_i)$. Let $h_{i,k} : C_i \times [-1, 1] \to (\pi'_k)^{-1}(U)$ be defined by $h_{i,k}((s_j)_{j=1}^\infty, t) = ((f')^{k-1} \circ (\rho|_{U \cap I_{s_0}})^{-1}(t), \dots, (\rho|_{U \cap I_{s_0}})^{-1}(t), (f'|_{I_{s_1}})^{-1} \circ (\rho|_{U \cap I_{s_0}})^{-1}(t), \dots)$.

$h_{i,k}$ is surjective: Let $\underline{x} \in (\pi'_k)^{-1}(U)$. Then $\pi'_k(\underline{x}) \in U \subset \mathrm{Int}(U_i)$, $\rho \circ \pi'_k(\underline{x}) \in [-1, 1]$, and $f' \circ \pi'_{j+1}(\underline{x}) = \pi'_j(\underline{x})$ for $j \in \mathbb{N}$. Since $\pi'_k(\underline{x}) \notin \mathcal{E}(U_i)$, $\pi'_{k+j}(\underline{x}) \notin (f')^{-1}(\bigcup_{i=1}^r \mathcal{E}(U_i))$ for $j \in \mathbb{N}$.

If $\pi'_{k+j}(\underline{x}) \in \mathrm{Int}(U_i)$ for each $j \in \{0, 1, \dots\}$ then $\pi'_{k+j}(\underline{x}) \in I_i$ for each $j \in \{0, 1, \dots\}$. So, let $s_j = i$ for each $j \in \{0, 1, \dots\}$. Then $\mathrm{Cl}(I_{s_0}) = \mathrm{Cl}(I_i) \in \mathcal{A}(U_i)$ and $I_{s_j} = I_i = f'(I_i) = f'(I_{s_{j+1}})$ for $j \in \{0, 1, \dots\}$. So $(s_j)_{j=1}^\infty \in C_i$. Since f' is one-to-one on I_i and $\pi'_{k+j}(\underline{x}) \in I_i$ for each $j \in \{0,1,\ldots\}$, $(f'|_{I_{s_j}})^{-1}(\pi'_{k+j}(\underline{x})) = \pi'_{k+j+1}(\underline{x})$ for each $j \in \{0,1,\ldots\}$.

If $\pi'_{k+j}(\underline{x}) \notin \text{Int}(U_i)$ for some $j \in \{0,1,\ldots\}$ then there exists an $\ell \in \{0,1,\ldots\}$ such that $\pi'_{k+\ell}(\underline{x}) \in \text{Int}(U_i)$ but $\pi'_{k+\ell+1}(\underline{x}) \notin \text{Int}(U_i)$. So $\pi'_{k+j}(\underline{x}) \notin \bigcup_{i=1}^{r} \text{Int}(U_i)$ and thus only one element of $\{I_1,\ldots,I_n\}$ contains $\pi'_{k+j}(\underline{x})$. For $j \geq \ell+1$, let s_j be the unique element of $\{1,\ldots,n\}$ such that $\pi'_{k+j}(\underline{x}) \in I_{s_j}$. Let $s_\ell \in \{1,\ldots,n\}$ such that $f'(I_{s_{\ell+1}}) = I_{s_\ell}$. For $0 \leq j < \ell$, let $s_j = i$. Then $\text{Cl}(I_{s_0}) \in \mathcal{A}(U_i)$ and $I_{s_j} \subset f'(I_{s_{j+1}})$ for $j \in \{0,1,\ldots\}$. So $(s_j)_{j=1}^{\infty} \in C_i$. Since f' is one-to-one on I_j and $\pi'_{k+j}(\underline{x}) \in I_j$ for each $j \in \{0,1,\ldots\}$, $(f'|_{I_{s_j}})^{-1}(\pi'_{k+j}(\underline{x})) = \pi'_{k+j+1}(\underline{x})$ for each $j \in \{0,1,\ldots\}$.

Then $((s_j)_{j=1}^{\infty}, \rho\circ\pi'_k(\underline{x})) \in C_i \times [-1,1]$ and $h_{i,k}((s_j)_{j=1}^{\infty}, \rho\circ\pi'_k(\underline{x})) = ((f')^{k-1}\circ (\rho|_{U\cap I_{s_0}})^{-1}(\rho\circ\pi'_k(\underline{x})),\ldots,(\rho|_{U\cap I_{s_0}})^{-1}(\rho\circ\pi'_k(\underline{x})), (f'|_{I_{s_1}})^{-1}\circ(\rho|_{U\cap I_{s_0}})^{-1}(\rho\circ\pi'_k(\underline{x})),\ldots) = ((f')^{k-1}\circ\pi'_k(\underline{x}),\ldots,\pi'_k(\underline{x}),(f'|_{I_{s_1}})^{-1}\circ\pi'_k(\underline{x}),\ldots) = (\pi'_1(\underline{x}),\ldots,\pi'_k(\underline{x}),\pi'_{k+1}(\underline{x}),\ldots) = \underline{x}$.

$h_{i,k}$ *is injective*: Suppose that $h_{i,k}((s_j)_{j=1}^{\infty},t) = h_{i,k}((s'_j)_{j=1}^{\infty},t')$. Then $t = \rho\circ(\rho|_{U\cap I_{s_0}})^{-1}(t) = \rho\circ(\rho|_{U\cap I_{s'_0}})^{-1}(t') = t'$. For purposes of contradiction, assume $(s_j)_{j=1}^{\infty} \neq (s'_j)_{j=1}^{\infty}$. Let ℓ be the smallest index such that $s_\ell \neq s'_\ell$. Without loss of generality, assume $s'_\ell \neq i$. Then $I_{s'_{\ell+1}} \cap \text{Int}(U_i) = \emptyset$. So $I_{s'_{\ell+1}} \cap I_{s_{\ell+1}} = \emptyset$ since $s_{\ell+1} \neq s'_{\ell+1}$. Then $(f'|_{I_{s'_{\ell+1}}})^{-1}\circ\ldots\circ(f'|_{I'_{s_1}})^{-1}\circ(\rho|_{U\cap I_{s'_0}})^{-1}(t') \neq (f'|_{I_{s_{\ell+1}}})^{-1}\circ\ldots\circ(f'|_{I_{s_1}})^{-1}\circ(\rho|_{U\cap I_{s_0}})^{-1}(t')$. But this contradicts the supposition that $h_{i,k}((s_j)_{j=1}^{\infty},t) = h_{i,k}((s'_j)_{j=1}^{\infty},t')$. Thus $((s_j)_{j=1}^{\infty},t) = ((s'_j)_{j=1}^{\infty},t')$.

$h_{i,k}$ *is continuous*: Similar to continuity argument in Proposition 2.9.

Since $C_i \times [-1,1]$ is compact and $(\pi'_k)^{-1}(U)$ is Hausdorff, $h_{i,k}$ is a homeomorphism and $(\pi'_k)^{-1}(U)$ is a matchbox in $\varprojlim\{G, f'\}$, which proves the claim.

To finish the proof of the proposition, let $x_i \in \mathcal{B}(G)$. Since p_k is a homeomorphism such that $p_k(x_i) = x_i$, there exists a $\delta_{i,k} > 0$ such that $p_k(B_d(x_i, \delta_{i,k})) \subset \mathrm{Int}(U_i)$. Let U be a star neighborhood of x_i contained in $B_d(x_i, \delta_{i,k})$. Then $p_k(U)$ is a star neighborhood of x_i contained in $\mathrm{Int}(U_i)$. So $(\pi'_k)^{-1}(p_k(U))$ is a matchbox in $\varprojlim\{G, f'\}$ by the above claim. Since p_∞ is a homeomorphism and $p_\infty(\pi_k^{-1}(U)) = (\pi'_k)^{-1}(p_k(U))$, $\pi_k^{-1}(U)$ is a matchbox in $\varprojlim\{G, f\}$ □

PROPOSITION 2.12. $\varprojlim\{G, f\}$ *is a matchbox continuum.*

PROOF. By Theorem 2.7, $\varprojlim\{G, f\}$ is a continuum. Let $\underline{x} \in \varprojlim\{G, f\}$. If $\pi_k(\underline{x}) \notin \mathcal{B}(G)$ for some $k \in \mathbb{N}$ then $\pi_k(\underline{x}) \in J_i$ for some $J_i \in \{J_1, \ldots, J_m\}$. Let I be an arc neighborhood of $\pi_k(\underline{x})$ contained in J_i. Then $\pi_k^{-1}(I)$ is a matchbox neighborhood of $\underline{x}$ by Corollary 2.10. If $\pi_k(\underline{x}) \in \mathcal{B}(G)$ then there exists a $\delta > 0$ such that $\pi_k^{-1}(U)$ is a matchbox in $\varprojlim\{G, f\}$ for every star neighborhood U of $\pi_k(\underline{x})$ contained in $B_d(\pi_k(\underline{x}), \delta)$ by Proposition 2.11. Let U be a star neighborhood of $\pi_k(\underline{x})$ contained in $B_d(\pi_k(\underline{x}), \delta)$. Then $\pi_k^{-1}(U)$ is a matchbox neighborhood of $\underline{x}$. □

Suppose U is a matchbox in a continuum. If $\rho : U \to \mathbb{R}$ is a map such that $\rho|_I : I \to \mathbb{R}$ is a topological embedding for each match I in U then we will call ρ a *directing map on U.* Two directing maps, ρ_1 and ρ_2, on U are called *equivalent* if for each match I in U the homeomorphism $(\rho_2|_I) \circ (\rho_1|_I)^{-1} : \rho_1(I) \to \rho_2(I)$ is increasing. This is an equivalence relation that partitions the collection of all directing maps on

U into equivalence classes, each of which we will call a *direction for U.* A matchbox together with a direction is called a *directed matchbox.*

PROPOSITION 2.13. *Let M be a matchbox in a continuum X and let ρ_1 and ρ_2 be directing maps on M. If M_+ is the union of all matches I in M such that $(\rho_2|_I) \circ (\rho_1|_I)^{-1} : \rho_1(I) \to \rho_2(I)$ is increasing and M_- is the union of all matches I in M such that $(\rho_2|_I) \circ (\rho_1|_I)^{-1} : \rho_1(I) \to \rho_2(I)$ is decreasing then M_+ and M_- are matchboxes or empty.*

PROOF. Let C be a zero-dimensional compactum and $h : C \times [-1,1] \to X$ be topological embedding such that $h(C \times [-1,1]) = M$ and $h(C \times (-1,1)) = \text{Int}(M)$. Assume σ is the metric on C and d is the metric on $C \times [-1,1]$ defined by $d((s_1,t_1),(s_2,t_2)) = \max\{\sigma(s_1,s_2), |t_1 - t_2|\}$. Let $C_+ = pr_1(M_+)$ and $C_- = pr_1(M_-)$. For each $s \in C$, let $\epsilon_s = \min\{|\rho_1 \circ h(s,-1) - \rho_1 \circ h(s,1)|/2, |\rho_2 \circ h(s,-1) - \rho_2 \circ h(s,1)|/2\}$. Since $\rho_1 \circ h$ and $\rho_2 \circ h$ are uniformly continuous, let $\delta_s > 0$ be such that if $d((s_1,t_1),(s_2,t_2)) < \delta_s$ then $|\rho_i \circ h(s_1,t_1) - \rho_i \circ h(s_2,t_2)| < \epsilon_s$. Let C_s be a clopen neighborhood of s such that $C_s \subset B_\sigma(s,\delta_s)$. For $i \in \{1,2\}$, if $\rho_i \circ h(s,-1) < \rho_i \circ h(s,1)$ and $s' \in C - s$ then $d((s,t),(s',t)) < \delta_s$ and $|\rho_i \circ h(s,t) - \rho_i \circ h(s',t)| < \epsilon_s$ for all $t \in [-1,1]$. Thus $\rho_i \circ h(s',-1) < \rho_i \circ h(s,-1) + \epsilon_s \leq (\rho_i \circ h(s,-1) + \rho_i \circ h(s,1))/2 \leq \rho_i \circ h(s,1) - \epsilon_s < \rho_i \circ h(s',1)$. Similarly, if $\rho_i \circ h(s,-1) > \rho_i \circ h(s,1)$ then $\rho_i \circ h(s',-1) > \rho_i \circ h(s',1)$ for each $s' \in C_s$. Thus $(\rho_2|_I) \circ (\rho_1|_I)^{-1} : \rho_1(I) \to \rho_2(I)$ is increasing for each match in the matchbox $h(C_s \times [-1,1])$ or $(\rho_2|_I) \circ (\rho_1|_I)^{-1} : \rho_1(I) \to \rho_2(I)$ is decreasing for each match in matchbox $h(C_s \times [-1,1])$. So $C_s \subset C_+$ or $C_s \subset C_-$ for each

$s \in C$. Since C is compact and $\{C_s\}_{s\in C}$ is an open covering of C there exists a finite subcollection $\{C_{s_1}, \ldots, C_{s_k}\}$ that covers C. Thus C_+ and C_- are a finite union of elements of $\{C_{s_1}, \ldots, C_{s_k}\}$ which means they are clopen subsets of C. If $C_+ \neq \emptyset$ then $M_+ = h(C_+ \times [-1,1])$ is a matchbox. If $C_+ = \emptyset$ then $M_+ = h(C_+ \times [-1,1]) = \emptyset$. Similarly M_- is either a matchbox or empty. □

If U is a directed matchbox in a continuum then the direction for U naturally induces a direction for any arc contained in U.

Suppose U and V are directed matchboxes in a continuum. If the directions for each nondegenerate path component of $U \cap V$ induced by the directions for U and V are the same then U and V are said to be *coherently directed.*

Suppose ρ_1 is a directing map on a matchbox U_1 and ρ_2 is a directing map on a matchbox U_2. If $x \in \text{Int}(U_1) \cap \text{Int}(U_2)$ and there exists a matchbox neighborhood of x contained in $U_1 \cap U_2$ such that $\rho_1|_V$ and $\rho_2|_V$ are equivalent then ρ_1 and ρ_2 are said to be *equivalent near x.* This is an equivalence relation that partitions the collection of all directing maps on all matchbox neighborhoods of x into two equivalence classes, each of which is called a *direction at x.* The direction for a directed matchbox naturally induces a direction at each point in the interior of that matchbox.

If $x \in \text{Int}(U) \cap \text{Int}(V)$ and there exists a matchbox neighborhood W of x contained in $U \cap V$ such that the directions for W induced by the direction for U and the direction for V are equal then we say U and V are *coherently directed near x.* If U and V are coherently directed near x for each $x \in \mathrm{Int}(U) \cap \mathrm{Int}(V)$ then U and V are coherently directed.

Suppose M is a matchbox continuum. We say M is *orientable* if there exists a collection of coherently directed matchboxes whose interiors cover M. If M is orientable then a choice of coherently directed matchboxes whose interiors cover M induces a direction at each point M. This continuous choice of directions at each point of M is called an *orientation of M.* If M is orientable then there exist exactly two distinct orientations of M.

PROPOSITION 2.14. *G is orientable if and only if $\varprojlim\{G, f\}$ is orientable.*

PROOF. By Proposition 2.11, we can choose a pairwise disjoint collection of sets in G, $\{U_1, \ldots, U_r\}$, such that U_i is a star neighborhood of the branch point b_i and $\pi_k^{-1}(U_i)$ is a matchbox in $\varprojlim\{G, f\}$ for each $i \in \{1, \ldots, r\}$. We can then choose a pairwise disjoint collection of sets in G, $\{U_{r+1}, \ldots, U_{r+m}\}$, such that U_{r+i} is an arc contained in J_i for each $i \in \{1, \ldots, m\}$ and $G \backslash \bigcup_{i=1}^{r} \mathrm{Int}(U_i) \subset \bigcup_{i=1}^{m} \mathrm{Int}(U_{r+i})$.

Assume G is orientable and choose an orientation for G. Since $G = \bigcup_{i=1}^{r+m} \mathrm{Int}(U_i)$, we can choose a directing map ρ_i on U_i for each $i \in \{1, \ldots, r+m\}$ such that if U is a nondegenerate path component in $U_i \cap U_j$ then $\rho_j|_I \circ \rho_i|_I^{-1} : \rho_i(I) \to \rho_j(I)$ is an increasing map for each $I \in \mathcal{A}(U)$. Let $\underline{\rho}_i$ be the directing map on $\pi_k^{-1}(I_i)$ defined by $\underline{\rho}_i = \rho_i \circ \pi_k$. For each match I of $\pi_k^{-1}(U_i)$, $\pi_k|_I$ is a homeomorphism onto $\pi_k(I) \in \mathcal{A}(U_i)$ and $\underline{\rho}_i|_I = \rho_i|_{\pi_k(I)} \circ \pi_k|_I$ is a topological embedding. If I is a nondegenerate path component of $\pi_k^{-1}(U_i) \cap \pi_k^{-1}(U_j) = \pi_k^{-1}(U_i \cap U_j)$ and U is the nondegenerate path component of $U_i \cap U_j$ containing $\pi_k(I)$ then $\pi_k(I) \in \mathcal{A}(U)$. Since $\underline{\rho}_j|_I \circ \underline{\rho}_i|_I^{-1} = \rho_j|_{\pi_k(I)} \circ \pi_k|_I \circ \pi_k|_I^{-1} \circ \rho_i|_{\pi_k(I)}^{-1} = \rho_j|_{\pi_k(I)} \circ \rho_i|_{\pi_k(I)}^{-1}$, $\underline{\rho}_j|_I \circ \underline{\rho}_i|_I^{-1} : \underline{\rho}_i(I) \to \underline{\rho}_j(I)$ is an increasing map. So the matchboxes $\pi_k^{-1}(U_i)$ and $\pi_k^{-1}(U_j)$ with directions $[\underline{\rho}_i]$ and $[\underline{\rho}_j]$, respectively, are coherently directed. Since $\varprojlim\{G, f\} = \pi_k^{-1}(\bigcup_{i=1}^{r+m} (\text{Int}(U_i))) = \bigcup_{i=1}^{r+m} \text{Int}(\pi_k^{-1}(U_i))$, $\varprojlim\{G, f\}$ is orientable.

Assume $\varprojlim\{G, f\}$ is orientable and choose an orientation of $\varprojlim\{G, f\}$. For $i \in \{1, \ldots, r\}$, let $\underline{\rho}_i$ be a directing map representing the direction for $\pi_k^{-1}(U_i)$ and I_i be the match in $\pi_k^{-1}(U_i)$ containing $\underline{b}_i \equiv (b_i, b_i, \ldots)$. Choose a directing map ρ_i on U_i such that $(\rho_i \circ \pi_k)|_{I_i} \circ \underline{\rho}_i|_{I_i}^{-1} : \underline{\rho}_i(I_i) \to \rho_i \circ \pi_k(I_i)$ is an increasing map. Let $\underline{U}_{i,+}$ be all matches I in $\pi_k^{-1}(U_i)$ such that $(\rho_i \circ \pi_k)|_I \circ \underline{\rho}_i|_I^{-1} : \underline{\rho}_i(I) \to \rho_i \circ \pi_k(I)$ is an increasing map. Then $\underline{b}_i \in I_i \subset \underline{U}_{i,+}$. Since $\rho_i \circ \pi_k$ and $\underline{\rho}_i$ are directing maps on $\pi_k^{-1}(U_i)$, $\underline{U}_{i,+}$ is a matchbox in $\varprojlim\{G, f\}$ contained in $\pi_k^{-1}(U_i)$ by Proposition 2.13. Choose $\epsilon > 0$ such that $B_d(\underline{b}_i, \epsilon) \subset \underline{U}_{i,+}$ and positive integer ℓ such that $\text{diam}(G)/2^{k+2\ell} < \epsilon$. Then for $\underline{x} \in \pi_{k+2\ell}^{-1}(\{b_i\})$, $\underline{d}(\underline{b}_i, \underline{x}) = \sum_{j=k+2\ell+1}^{\infty} d(b_i, \pi_j(\underline{x}))/2^j < \epsilon$. Choose a star neighborhood V_i of b_i contained in U_i such that $f^{2\ell}(V_i) \subset U_i$. Since f is an immersion, $\rho_i|_{f^{2\ell} \circ \pi_{k+2\ell}(I)} \circ f^{2\ell} \circ \rho_i|_{\pi_{k+2\ell}(I)}^{-1} : \pi_{k+2\ell}(I) \to \pi_k(I)$ is an increasing map for each match I in $\pi_{k+2\ell}^{-1}(V_i)$. Since $\pi_{k+2\ell}^{-1}(\{b_i\}) \subset B_d(\underline{b}_i, \epsilon)$ and $f^{2\ell}(V_i) \subset U_i$, $\pi_{k+2\ell}^{-1}(V_i) \subset \underline{U}_{i,+}$. Thus $(\rho_i \circ \pi_k)|_I \circ \underline{\rho}_i|_I^{-1} : \underline{\rho}_i(I) \to \rho_i \circ \pi_k(I)$ is an increasing map for each match I in $\pi_{k+2\ell}^{-1}(V_i)$ and $[(\rho_i \circ \pi_k)|_{\pi_{k+2\ell}^{-1}(V_i)}] = [\underline{\rho}_i|_{\pi_{k+2\ell}^{-1}(V_i)}]$. For each match I in $\pi_{k+2\ell}^{-1}(V_i)$, consider the map $(\rho_i \circ \pi_k)|_I \circ (\rho_i \circ \pi_{k+2\ell})|_I^{-1} : \pi_{k+2\ell}(I) \to \pi_k(I)$, which is an increasing map since $(\rho_i \circ \pi_k)|_I \circ (\rho_i \circ \pi_{k+2\ell})|_I^{-1} = \rho_i|_{\pi_k(I)} \circ \pi_k|_I \circ \pi_{k+2\ell}|_I^{-1} \circ \rho_i|_{\pi_{k+2\ell}(I)} = \rho_i|_{f^{2\ell} \circ \pi_{k+2\ell}(I)} \circ f^{2\ell} \circ \rho_i|_{\pi_{k+2\ell}(I)}^{-1}$.

So $[\rho_i|_{V_i} \circ \pi_{k+2\ell}] = [(\rho_i \circ \pi_k)|_{\pi^{-1}_{k+2\ell}(V_i)}] = [\underline{\rho}_i|_{\pi^{-1}_{k+2\ell}(V_i)}]$. Since $\hat{f}$ is a homeomorphism, $\underline{\rho}_i|_{\hat{f}^{-2\ell}(I)} \circ \hat{f}^{-2\ell} \circ \underline{\rho}_i|^{-1}_I : \underline{\rho}_i(I) \to \underline{\rho}_i \circ \hat{f}^{-2\ell}(I)$ is an increasing map for each match I in $\pi_k^{-1}(V_i)$. For each match I in $\pi_k^{-1}(V_i)$, consider the map $(\rho_i \circ \pi_k)|_I \circ \underline{\rho}_I^{-1} : \underline{\rho}_i(I) \to \rho_i \circ \pi_k(I)$, which is increasing since $(\rho_i \circ \pi_k)|_I \circ \underline{\rho}_i|^{-1}_I = (\rho_i \circ \pi_k)|_I \circ \hat{f}^{2\ell} \circ \underline{\rho}_i|^{-1}_{\hat{f}^{-2\ell}(I)} \circ \underline{\rho}_i|_{\hat{f}^{-2\ell}(I)} \circ \hat{f}^{-2\ell} \circ \underline{\rho}_i|^{-1}_I = ((\rho_i \circ \pi_{k+2\ell})|_{\hat{f}^{-2\ell}(I)} \circ \underline{\rho}_i|^{-1}_{\hat{f}^{-2\ell}(I)}) \circ (\underline{\rho}_i|_{\hat{f}^{-2\ell}(I)} \circ \hat{f}^{-2\ell} \circ \underline{\rho}_i|^{-1}_I)$. So $\pi_k^{-1}(V_i) \subset \underline{U}_{i,+}$. Since $\pi_k^{-1}(\{b_i\})$ is a cross-section of both $\pi_k^{-1}(U_i)$ and $\pi_k^{-1}(V_i)$, $\pi_k^{-1}(U_i) = \underline{U}_{i,+}$. Thus $[\rho_i \circ \pi_k] = [\underline{\rho}_i]$. For $j \in \{r+1, \ldots, r+m\}$, let $\underline{\rho}_j$ be a representative of the directing map for $\pi_k^{-1}(U_j)$ and I_j be a match in $\pi_k^{-1}(U_j)$. Choose a directing map ρ_j on U_j such that $(\rho_j \circ \pi_k)|_{I_j} \circ \underline{\rho}_j|^{-1}_{I_j} : \underline{\rho}_j(I_j) \to \rho_j \circ \pi_k(I_j)$ is an increasing map. If $U_i \in \{U_1 \ldots, U_r\}$ and U is a path component of $U_i \cap U_j$ then each match in $\pi_k^{-1}(U_i \cap U_j)$ maps onto the arc U. Let I be the match in $\pi_k^{-1}(U_i \cap U_j)$ that is contained in I_i. The map $\rho_i|_{\pi_k(I)} \circ \rho_j|^{-1}_{\pi_k(I)} : \rho_j \circ \pi_k(I) \to \rho_i \circ \pi_k(I)$ is increasing since $\rho_i|_{\pi_k(I)} \circ \rho_j|^{-1}_{\pi_k(I)} = (\rho_i \circ \pi_k)|_I \circ (\rho_j \circ \pi_k)|^{-1}_I = ((\rho_i \circ \pi_k)|_I \circ \underline{\rho}_i|^{-1}_I) \circ (\underline{\rho}_i|_I \circ \underline{\rho}_j|^{-1}_I) \circ (\underline{\rho}_j|_I \circ (\rho_j \circ \pi_k)|^{-1}_I)$. So the directing maps $[\rho_i]$ and $[\rho_j]$ coherently direct the stars U_i and U_j. Thus G is orientable. □

CHAPTER 3

MAIN RESULTS

Orientable Bouquets of Circles

Our goal in this section is to prove the following theorem utilizing techniques similar to those used by Mioduszewski [16] and by Barge and Diamond [4] to construct "nearly" commuting infinite diagrams from homeomorphic inverse limit spaces.

THEOREM 3.1. *Let G and G' be orientable bouquets of circles with branch points b and b', respectively, let $f : G \to G$ and $f' : G' \to G'$ be collapsing surjective immersions such that $f(b) = b$ and $f'(b') = b'$, let M_f be the transition matrix for f relative to an ordering of the components of $G\backslash\{b\}$, and let $M_{f'}$ be the transition matrix for f' relative to an ordering of the components of $G'\backslash\{b'\}$. If $\varprojlim\{G, f\}$ is homeomorphic to $\varprojlim\{G', f'\}$ then M_f is weakly equivalent to $M_{f'}$.*

We know begin the proof of the above theorem. Let $\{J_1, \ldots, J_m\}$ be an ordering of the components of $G\backslash\{b\}$ and let $\{J'_1, \ldots, J'_n\}$ be an ordering of the components of $G'\backslash\{b'\}$. Since two matrices that are similar by a permutation matrix are weakly equivalent, without loss of generality, we assume M_f is the transition matrix for f relative to $\{J_1, \ldots, J_m\}$ and $M_{f'}$ is the transition matrix for f' relative to $\{J'_1, \ldots, J'_n\}$.

Assume $\varprojlim\{G, f\}$ is homeomorphic to $\varprojlim\{G', f'\}$. Let ϕ be a homeomorphism from $\varprojlim\{G, f\}$ to $\varprojlim\{G', f'\}$. We will spend the rest of this section showing that M_f is weakly equivalent to $M_{f'}$.

Since G and G' are connected, $f^2 : G \to G$ and $(f')^2 : G' \to G'$ are orientation preserving collapsing surjective immersions. Since $\varprojlim\{G, f\}$ is homeomorphic to $\varprojlim\{G', f'\}$, $\varprojlim\{G, f^2\}$ is homeomorphic to $\varprojlim\{G', (f')^2\}$. If we can show that the transition matrix M_{f^2} of f^2 relative to $\{J_1, \ldots, J_m\}$ is weakly equivalent to the transition matrix $M_{(f')^2}$ of $(f')^2$ relative to $\{J'_1, \ldots, J'_n\}$ then M_f is weakly equivalent to $M_{f'}$ since $M_{f^2} = (M_f)^2$ and $M_{(f')^2} = (M_{f'})^2$. Thus, without loss of generality, we assume that f and f' are orientation preserving.

Choose Menger-convex metrics, d and d', compatible with the topologies on G and G', respectively, and let $\underline{d}$ and $\underline{d}'$ be the induced metrics on $\varprojlim\{G, f\}$ and $\varprojlim\{G', f'\}$, respectively. Let π_k be the k-th projection map acting on $\varprojlim\{G, f\}$ and let π'_k be the k-th projection map acting on $\varprojlim\{G', f'\}$.

Choose an orientation of G. Since $\varprojlim\{G, f\}$ is orientable by Proposition 2.14, choose the orientation of $\varprojlim\{G, f\}$ induced by the orientation of G using the map π_k. Since f is orientation preserving this orientation is independent of which π_k is used. Since $\varprojlim\{G', f'\}$ is orientable by Proposition 2.14, choose the orientation of $\varprojlim\{G', f'\}$ induced by the orientation of $\varprojlim\{G, f\}$ using the map ϕ. Then choose the orientation of G' induced by the orientation of $\varprojlim\{G', f'\}$ using the map π'_k. Since f' is orientation preserving this orientation is independent of which π'_k is used.

For each $J_i \in \{J_1, \ldots, J_m\}$, let C_i be the zero-dimensional compactum and $h_{i,k} : C_i \times J_i \to \pi_k^{-1}(J_i)$ the homeomorphism defined in Proposition 2.9. For each $i \in \{1, \ldots, m\}$, choose an $\underline{s}_i \in C_i$. For $k \geq 1$, let $e_k : G \to \varprojlim\{G, f\}$ be the function defined by $e_k(x) = h_{i,k}(\underline{s}_i, x)$ for $x \in J_i$ and $e_k(b) = \underline{b}$. For each $J_i' \in \{J_1', \ldots, J_n'\}$, let C_i' be the zero-dimensional compactum and $h_{i,k}' : C_i' \times J_i' \to (\pi_k')^{-1}(J_i')$ the homeomorphism defined in Proposition 2.9. For each $i \in \{1, \ldots, n\}$, choose an $\underline{s}_i' \in C_i'$. For $k \geq 1$, let $e_k' : G' \to \varprojlim\{G', f'\}$ be the function defined by $e_k'(x') = h_{i,k}'(\underline{s}_i', x')$ for $x' \in J_i'$ and $e_k'(b') = \underline{b}'$. By definition $e_k|_{G\setminus\{b\}}$ and $e_k'|_{G'\setminus\{b'\}}$ are orientation preserving topological embeddings. Let $t_{\ell,k} : G \to G'$ and $s_{\ell,k} : G' \to G$ be defined by $t_{\ell,k}(x) = \pi_\ell' \circ \phi \circ e_k(x)$ and $s_{\ell,k}(x') = \pi_\ell \circ \phi^{-1} \circ e_k'(x')$.

Given $\epsilon > 0$, a metric space (Y, d), and two functions $f : X \to Y$ and $g : X \to Y$, we will use the notation $f \underset{\epsilon}{=} g$ when $d(f(x), g(x)) < \epsilon$ for all $x \in X$.

Lemma 3.2. *Given $\epsilon > 0$ and a positive integer n, there exists a positive integer M such that $s_{n,m} \circ t_{m,\ell} \underset{\epsilon}{=} f^{\ell-n}$ for all integers $m \geq M$ and $\ell > n$.*

Proof. Let $\epsilon > 0$ and let n be a positive integer. Since $\pi_n \circ \phi^{-1}$ is uniformly continuous on $\varprojlim\{G', f'\}$, there exists a $\delta > 0$ such that if $\underline{x}', \underline{z}' \in \varprojlim\{G', f'\}$ with $\underline{d}'(\underline{x}', \underline{z}') < \delta$ then $d(\pi_n \circ \phi^{-1}(\underline{x}'), \pi_n \circ \phi^{-1}(\underline{z}')) < \epsilon$. Choose a positive integer M such that $\operatorname{diam}(G')/2^M < \delta$. Let m and ℓ be integers such that $m \geq M$ and $\ell > n$. If $x \in G$ then $\underline{d}'(e_m' \circ \pi_m' \circ \phi \circ e_\ell(x), \phi \circ e_\ell(x)) = \sum_{j=1}^{\infty} d'(\pi_j' \circ e_m' \circ \pi_m' \circ \phi \circ e_\ell(x), \pi_j' \circ \phi \circ e_\ell(x))/2^j = \sum_{j=1}^{m} d'((f')^{m-j} \circ \pi_m' \circ \phi \circ e_\ell(x), \pi_j' \circ \phi \circ e_\ell(x))/2^j + \sum_{j=m+1}^{\infty} d'(\pi_j' \circ e_m' \circ \pi_m' \circ \phi \circ e_\ell(x), \pi_j' \circ \phi \circ e_\ell(x))/2^j \leq \sum_{j=m+1}^{\infty} \operatorname{diam}(G')/2^j = \operatorname{diam}(G')/2^m < \delta$. By definition of

e_ℓ, $f^{\ell-n} = \pi_n \circ e_\ell$. Thus $d(s_{n,m} \circ t_{m,\ell}(x), f^{\ell-n}(x)) = d(\pi_n \circ \phi^{-1} \circ e'_m \circ \pi'_m \circ \phi \circ e_\ell(x), \pi_n \circ \phi^{-1} \circ \phi \circ e_\ell(x)) < \epsilon$. □

LEMMA 3.3. *Given $\epsilon' > 0$ and a positive integer m, there exists a positive integer N such that $t_{m,n} \circ s_{n,\ell} \underset{\epsilon'}{=} (f')^{\ell-m}$ for all integers $n \geq N$ and $\ell > m$.*

PROOF. As above. □

LEMMA 3.4. *Given $\epsilon' > 0$ and a positive integer m, there exists a positive integer L such that for each integer $\ell \geq L$ there exists a $\Delta_\ell > 0$ such that if $x, z \in G$ with $d(x,z) < \Delta_\ell$ then $d'(t_{m,\ell}(x), t_{m,\ell}(z)) < \epsilon'$.*

PROOF. Let $\epsilon' > 0$ and let m be a positive integer. Since $\pi'_m \circ \phi$ is uniformly continuous on $\varprojlim\{G, f\}$ there exists a $\sigma > 0$ such that if $\underline{x}, \underline{z} \in \varprojlim\{G, f\}$ with $\underline{d}(\underline{x}, \underline{z}) < \sigma$ then $d'(\pi'_m \circ \phi(\underline{x}), \pi'_m \circ \phi(\underline{z})) < \epsilon$. Choose a positive integer L such that $\text{diam}(G)/2^L < \sigma/2$. Let ℓ be a integer such that $\ell \geq L$. Since $f^{\ell-j}$ is uniformly continuous on G for $1 \leq j \leq \ell$, there exists a $\delta_j > 0$ such that if $x, z \in G$ with $d(x,z) < \delta_j$ then $d(f^{\ell-j}(x), f^{\ell-j}(z)) < \sigma/2$. Let $\Delta_\ell = \min\{\delta_1, \ldots, \delta_\ell\}$. If $x, z \in G$ with $d(x,z) < \Delta_\ell$ then $\underline{d}(e_\ell(x), e_\ell(z)) = \sum_{j=1}^{\infty} d(\pi_j \circ e_\ell(x), \pi_j \circ e_\ell(z))/2^j = \sum_{j=1}^{\ell} d(f^{\ell-j}(x), f^{\ell-j}(z))/2^j + \sum_{j=\ell+1}^{\infty} d(\pi_j \circ e_\ell(x), \pi_j \circ e_\ell(z))/2^j < \sum_{j=1}^{\ell} \sigma/2^{j+1} + \sum_{j=\ell+1}^{\infty} \text{diam}(G)/2^j < \sigma/2 + \text{diam}(G)/2^\ell < \sigma$. Thus $d'(t_{m,\ell}(x), t_{m,\ell}(z)) = d'(\pi'_m \circ \phi \circ e_\ell(x), \pi'_m \circ \phi \circ e_\ell(z)) < \epsilon$. □

LEMMA 3.5. *Given $\epsilon > 0$ and a positive integer n, there exists a positive integer M such that for each integer $m \geq M$ there exists a $\Delta'_m > 0$ such that if $x', z' \in G'$ with $d'(x', z') < \Delta'_m$ then $d(s_{n,m}(x'), s_{n,m}(z')) < \epsilon$.*

PROOF. As above. □

Since d is a Menger-convex metric, there exists an $\mathrm{E} > 0$ such that $B_d(x, \mathrm{E})$ is an open star neighborhood of x for each $x \in G$ and the shortest arc between any two distinct points in $\mathrm{Cl}(B_d(x, \mathrm{E}))$ is contained in $\mathrm{Cl}(B_d(x, \mathrm{E}))$. Similarly, there exists an $\mathrm{E}' > 0$ such that $B_{d'}(x', \mathrm{E}')$ is an open star neighborhood of x' for all $x' \in G'$ and the shortest arc between any two distinct points in $\mathrm{Cl}(B_{d'}(x', \mathrm{E}'))$ is contained in $\mathrm{Cl}(B_{d'}(x', \mathrm{E}'))$. Given two distinct points in a star there exists a unique arc, contained in the star, between the points. By Lemma 3.4, given $0 < \epsilon' < \mathrm{E}'$ and a positive integer m, there exists a positive integer L such that for each integer $\ell \geq L$ there exists a $\Delta_\ell > 0$ such that if $d(x, z) < \Delta_\ell$ then $d'(t_{m,\ell}(x), t_{m,\ell}(z)) < \epsilon'$. For $0 < \delta < \min\{\Delta_\ell, \mathrm{E}\}$, we define a map $t_{m,\ell,\delta} : G \to G'$ such that $t_{m,\ell,\delta}(x) = t_{m,\ell}(x)$ for all $x \notin B_d(b, \delta)\backslash\{b\}$. If $x \in B_d(b, \delta)\backslash\{b\}$ then we define $t_{m,\ell,\delta}(x)$ in the following manner. Let z be the end point of the star $\mathrm{Cl}(B_d(b, \delta))$ such that x is on the arc between b and z contained in $B_d(b, \mathrm{E})$. If $t_{m,\ell}(b) = t_{m,\ell}(z)$ then define $t_{m,\ell,\delta}(x)$ to be $t_{m,\ell}(b)$. If $t_{m,\ell}(b) \neq t_{m,\ell}(z)$ then define $t_{m,\ell,\delta}(x)$ to be the point x' on the arc between $t_{m,\ell}(b)$ and $t_{m,\ell}(z)$ contained in $B_{d'}(t_{m,\ell}(b), \mathrm{E}')$ such that $d'(t_{m,\ell}(b), x')/d'(t_{m,\ell}(b), t_{m,\ell}(z)) = d(b, x)/d(b, z)$. By definition, $d'(t_{m,\ell,\delta}(b), t_{m,\ell,\delta}(x)) < \epsilon'$ for all $x \in B_d(b, \delta)$. Define the map $s_{n,m,\delta'} : G' \to G$ similarly.

LEMMA 3.6. *Given $0 < \epsilon' < \mathrm{E}'$ and a positive integer m, there exists a positive integer L such that for each integer $\ell \geq L$ there exists a $0 < \Delta_\ell < \mathrm{E}$ such that*

$t_{m,\ell,\delta} \underset{\epsilon'}{=} t_{m,\ell}$ *and if* $x, z \in G$ *with* $d(x,z) < \Delta_\ell$ *then* $d'(t_{m,\ell,\delta}(x), t_{m,\ell,\delta}(z)) < \epsilon'$ *for* $0 < \delta < \Delta_\ell$.

PROOF. Let $0 < \epsilon' < \mathrm{E}'$ and let m be a positive integer. By Lemma 3.4, there exists a positive integer L such that for each integer $\ell \geq L$ there exists a $0 < \Delta_\ell < \mathrm{E}$ such that if $x, z \in G$ with $d(x,z) < \Delta_\ell$ then $d'(t_{m,\ell}(x), t_{m,\ell}(z)) < \epsilon'/3$.

Let $x \in G$ and $0 < \delta < \Delta_\ell$. If $x \notin B_d(b,\delta)\backslash\{b\}$ then $d'(t_{m,\ell,\delta}(x), t_{m,\ell}(x)) = d'(t_{m,\ell}(x), t_{m,\ell}(x)) = 0 < \epsilon'$. If $x \in B_d(b,\delta)\backslash\{b\}$ then $d'(t_{m,\ell,\delta}(x), t_{m,\ell}(x)) \leq d'(t_{m,\ell,\delta}(x), t_{m,\ell,\delta}(b)) + d'(t_{m,\ell}(b), t_{m,\ell}(x)) < \epsilon'/3 + \epsilon'/3 < \epsilon'$.

Let $x, z \in G$ with $d(x,z) < \Delta_\ell$ and $0 < \delta < \Delta_\ell$. If $x, z \notin B_d(b,\delta)\backslash\{b\}$ then $d'(t_{m,\ell,\delta}(x), t_{m,\ell,\delta}(z)) = d'(t_{m,\ell}(x), t_{m,\ell}(z)) < \epsilon'/3 < \epsilon'$. If $x, z \in B_d(b,\delta)\backslash\{b\}$ then $d'(t_{m,\ell,\delta}(x), t_{m,\ell,\delta}(z)) \leq d'(t_{m,\ell,\delta}(x), t_{m,\ell,\delta}(b)) + d'(t_{m,\ell,\delta}(b), t_{m,\ell,\delta}(z)) < \epsilon'/3 + \epsilon'/3 < \epsilon'$. If $x \in B_d(b,\delta)\backslash\{b\}$ and $z = b$ then $d'(t_{m,\ell,\delta}(x), t_{m,\ell,\delta}(z)) < \epsilon'/3 < \epsilon'$. If $x \in B_d(b,\delta)\backslash\{b\}$ and $z \notin B_d(b,\delta)$ then $x \in B_d(z,\Delta_\ell)$. Let w be the end point of $\mathrm{Cl}(B_d(b,\delta))$ on the arc between x and z contained in $B_d(z,\Delta_\ell)$. So $d'(t_{m,\ell,\delta}(x), t_{m,\ell,\delta}(z))$ $< d'(t_{m,\ell,\delta}(x), t_{m,\ell,\delta}(b)) + d'(t_{m,\ell,\delta}(b), t_{m,\ell,\delta}(w)) + d'(t_{m,\ell,\delta}(w), t_{m,\ell,\delta}(z)) =$ $d'(t_{m,\ell,\delta}(x), t_{m,\ell,\delta}(b)) + d'(t_{m,\ell}(b), t_{m,\ell}(w)) + d'(t_{m,\ell}(w), t_{m,\ell}(z)) < \epsilon'/3 + \epsilon'/3 + \epsilon'/3 =$ ϵ'. □

LEMMA 3.7. *Given* $0 < \epsilon < \mathrm{E}$ *and a positive integer* n, *there exists a positive integer* M *such that for each integer* $m \geq M$ *there exists a* $0 < \Delta'_m < \mathrm{E}'$ *such that* $s_{n,m,\delta'} \underset{\epsilon}{=} s_{n,m}$ *and if* $x', z' \in G'$ *with* $d'(x',z') < \Delta'_m$ *then* $d(s_{n,m,\delta'}(x'), s_{n,m,\delta'}(z')) < \epsilon$ *for* $0 < \delta' < \Delta'_m$.

PROOF. As above. □

LEMMA 3.8. *Given* $0 < \epsilon < \mathrm{E}$ *and positive integers* n *and* M*, there exists an integer* $m \geq M$*, a* $0 < \Delta' < \mathrm{E}'$*, and an integer* $L > n$ *such that for each integer* $\ell \geq L$ *there exists a* $0 < \Delta_\ell < \mathrm{E}$ *such that* $s_{n,m,\delta'} \circ t_{m,\ell,\delta} \underset{\epsilon}{=} f^{\ell-n}$ *for* $0 < \delta' < \Delta'$ *and* $0 < \delta < \Delta_\ell$.

PROOF. Let $0 < \epsilon < \mathrm{E}$ and let n and M be positive integers. By Lemma 3.2, there exists a positive integer M_1 such that for integers $m \geq M_1$ and $\ell > n$, $d(s_{n,m} \circ t_{m,\ell}(x), f^{\ell-n}(x)) < \epsilon/3$ for all $x \in G$. By Lemma 3.7, there exists a positive integer M_2 such that for each integer $m \geq M_2$ there exists a $0 < \Delta'_m < \mathrm{E}'$ such that if $d'(x', z') < \Delta'_m$ then $d(s_{n,m,\delta'}(x'), s_{n,m,\delta'}(z')) < \epsilon/3$ and $d(s_{n,m,\delta'}(x'), s_{n,m}(x')) < \epsilon/3$ for $0 < \delta' < \Delta'_m$ and $x', z' \in G'$. Choose an integer $m \geq \max\{M, M_1, M_2\}$. By Lemma 3.6, there exists a positive integer L_1 such that for each integer $\ell \geq L_1$ there exists a $0 < \Delta_\ell < \mathrm{E}$ such that $d'(t_{m,\ell,\delta}(x), t_{m,\ell}(x)) < \Delta'_m$ for $0 < \delta < \Delta_\ell$ and $x \in G$. Let $L = \max\{L_1, n+1\}$. If ℓ is an integer such that $\ell \geq L$ then $d(s_{n,m,\delta'} \circ t_{m,\ell,\delta}(x), f^{\ell-n}(x)) \leq d(s_{n,m,\delta'} \circ t_{m,\ell,\delta}(x), s_{n,m,\delta'} \circ t_{m,\ell}(x)) + d(s_{n,m,\delta'} \circ t_{m,\ell}(x), s_{n,m} \circ t_{m,\ell}(x)) + d(s_{n,m} \circ t_{m,\ell}(x), f^{\ell-n}(x)) < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$ for $0 < \delta' < \Delta'_m$ and $0 < \delta < \Delta_\ell$. □

LEMMA 3.9. *Given* $0 < \epsilon' < \mathrm{E}'$ *and positive integers* m *and* N*, there exists an integer* $n \geq N$*,* $0 < \Delta < \mathrm{E}$*, and an integer* $L > m$ *such that for each integer* $\ell \geq L$ *there exists a* $0 < \Delta'_\ell < \mathrm{E}'$ *such that* $t_{m,n,\delta} \circ s_{n,\ell,\delta'} \underset{\epsilon'}{=} (f')^{\ell-m}$ *for* $0 < \delta < \Delta$ *and* $0 < \delta' < \Delta'_\ell$.

PROOF. As above. □

LEMMA 3.10. *Given* $0 < \epsilon < \min\{E, E'\}$ *there exist two strictly increasing sequences* $\{n_i\}_{i=1}^{\infty}$ *and* $\{m_i\}_{i=1}^{\infty}$ *of positive integers and two sequences* $\{\delta_i'\}_{i=1}^{\infty}$ *and* $\{\delta_i\}_{i=2}^{\infty}$ *of positive numbers such that* $s_{n_i,m_i,\delta_i'} \circ t_{m_i,n_{i+1},\delta_{i+1}} \underset{\epsilon}{=} f^{n_{i+1}-n_i}$ *and* $t_{m_i,n_{i+1},\delta_{i+1}} \circ s_{n_{i+1},m_{i+1},\delta_{i+1}'} \underset{\epsilon}{=} (f')^{m_{i+1}-m_i}$ *for each* $i \in \mathbb{N}$.

PROOF. Let $0 < \epsilon < \min\{E, E'\}$. Choose positive integers n_1 and M. By Lemma 3.8, there exist an integer $m_1 \geq M$, $0 < \Delta' < E'$, and an integer $N > n_1$ such that for each integer $n \geq N$ there exists a $0 < \Delta_n < E$ such that $s_{n_1,m_1,\delta'} \circ t_{m_1,n,\delta} \underset{\epsilon}{=} f^{n-n_1}$ for $0 < \delta' < \Delta'$ and $0 < \delta < \Delta_n$. Choose $0 < \delta_1' < \Delta'$. Assume n_i, m_i, δ_i', and $N > n_i$ are given such that for each integer $n \geq N$ there exists a $0 < \Delta_n < E$ such that $s_{n_i,m_i,\delta_i'} \circ t_{m_i,n,\delta} \underset{\epsilon}{=} f^{n-n_i}$ for $0 < \delta < \Delta_n$. By Lemma 3.9, there exists an integer $n_{i+1} \geq N$, $0 < \Delta < E$, and an integer $M > m_i$ such that $t_{m_i,n_{i+1},\delta} \circ s_{n_{i+1},m,\delta'} \underset{\epsilon}{=} (f')^{m-m_i}$ for $0 < \delta < \Delta$ and $0 < \delta' < \Delta_m'$. Choose $0 < \delta_{i+1} < \min\{\Delta_{n_{i+1}}, \Delta\}$. Then $s_{n_i,m_i,\delta_i'} \circ t_{m_i,n_{i+1},\delta_{i+1}} \underset{\epsilon}{=} f^{n_{i+1}-n_i}$. Assume m_i, n_{i+1}, δ_{i+1} and $M > m_i$ are given such that for each integer $n \geq M$ there exists a $0 < \Delta_m' < E'$ such that $t_{m_i,n_{i+1},\delta_{i+1}} \circ s_{n_{i+1},m_{i+1},\delta'} \underset{\epsilon}{=} (f')^{m-m_i}$ for $0 < \delta' < \Delta_m'$. By Lemma 3.8, there exists an integer $m_{i+1} \geq M$, $0 < \Delta' < E'$, and an integer $N > n_{i+1}$ such that $s_{n_{i+1},m_{i+1},\delta'} \circ t_{m_{i+1},n,\delta} \underset{\epsilon}{=} f^{n-n_{i+1}}$ for $0 < \delta' < \Delta'$ and $0 < \delta < \Delta_n$. Choose $0 < \delta_{i+1}' < \min\{\Delta_{m_{i+1}}', \Delta'\}$. Then $t_{m_i,n_{i+1},\delta_{i+1}} \circ s_{n_{i+1},m_{i+1},\delta_{i+1}'} \underset{\epsilon}{=} (f')^{m_{i+1}-m_i}$. □

The collections of 1-cycles $\{\mathrm{Cl}(J_1), \ldots, \mathrm{Cl}(J_m)\}$ and $\{\mathrm{Cl}(J_1'), \ldots, \mathrm{Cl}(J_n')\}$ form bases for the free abelian groups $H_1(G;\mathbb{Z})$ and $H_1(G';\mathbb{Z})$, respectively. Let F and F'

be the matrices of f_* and f'_*, respectively, relative to the above bases for $H_1(G;\mathbb{Z})$ and $H_1(G';\mathbb{Z})$. Since f and f' are orientation preserving, F and F' are square nonnegative integer matrices.

LEMMA 3.11. *F and F' are weakly equivalent.*

PROOF. Let $0 < \epsilon < \min\{\mathrm{E}, \mathrm{E}'\}$. By Lemma 3.10, there exist two strictly increasing sequences $\{n_i\}_{i=1}^{\infty}$ and $\{m_i\}_{i=1}^{\infty}$ of positive integers and two sequences $\{\delta'_i\}_{i=1}^{\infty}$ and $\{\delta_i\}_{i=2}^{\infty}$ of positive numbers such that $s_{n_i,m_i,\delta'_i} \circ t_{m_i,n_{i+1},\delta_{i+1}} \underset{\epsilon}{=} f^{n_{i+1}-n_i}$ and $t_{m_i,n_{i+1},\delta_{i+1}} \circ s_{n_{i+1},m_{i+1},\delta'_{i+1}} \underset{\epsilon}{=} (f')^{m_{i+1}-m_i}$ for each $i \in \mathbb{N}$. Let $H : G \times [0,1] \to G$ be the map defined in the following manner. Given $x \in G$, there exists a unique arc I_x, possibly degenerate, contained in $B_d(f^{n_{i+1}-n_i}(x), \mathrm{E})$ between the points $f^{n_{i+1}-n_i}(x)$ and $s_{n_i,m_i,\delta_i} \circ t_{m_i,n_{i+1},\delta'_{i+1}}(x)$. Let $H(x,t)$ be the unique point on I_x such that $d(f^{n_{i+1}-n_i}(x), H(x,t)) = t \cdot d(f^{n_{i+1}-n_i}(x), s_{n_i,m_i,\delta_i} \circ t_{m_i,n_{i+1},\delta'_{i+1}}(x))$. Since $H(x,0) = f^{n_{i+1}-n_i}(x)$ and $H(x,1) = s_{n_i,m_i,\delta_i} \circ t_{m_i,n_{i+1},\delta'_{i+1}}(x)$ for all $x \in G$, $f^{n_{i+1}-n_i}$ is homotopic to $s_{n_i,m_i,\delta_i} \circ t_{m_i,n_{i+1},\delta'_{i+1}}(x)$. Thus $(f_*)^{n_{i+1}-n_i} = (f^{n_{i+1}-n_i})_* = (s_{n_i,m_i,\delta_i} \circ t_{m_i,n_{i+1},\delta'_{i+1}})_* = (s_{n_i,m_i,\delta_i})_* \circ (t_{m_i,n_{i+1},\delta'_{i+1}})_*$. Similarly, $(f'_*)^{m_{i+1}-m_i} = (t_{m_i,n_{i+1},\delta'_{i+1}})_* \circ (s_{n_{i+1},m_{i+1},\delta_{i+1}})_*$. For $i \in \mathbb{N}$, let S_i and T_i be the matrices of $(s_{n_i,m_i,\delta_i})_*$ and $(t_{m_i,n_{i+1},\delta'_{i+1}})_*$, respectively, relative to the bases for $H_1(G;\mathbb{Z})$ and $H_1(G';\mathbb{Z})$. Then $S_iT_i = F^{n_{i+1}-n_i}$ and $T_iS_{i+1} = (F')^{m_{i+1}-m_i}$. Since s_{n_i,m_i,δ_i} is orientation preserving on $G \backslash B_d(b,\delta_i)$ and $s_{n_i,m_i,\delta_i}(B_d(b,\delta_i))$ is contained in $B_{d'}(s_{n_i,m_i,\delta_i}(x), \mathrm{E}')$, S_i is a nonnegative integer matrix. Similarly, T_i is a nonnegative integer matrix. Thus F and F' are weakly equivalent. □

Since $M_f = F$ and $M_{f'} = F'$, M_f is weakly equivalent to $M_{f'}$, which concludes the proof of the Theorem 3.1.

Orientable Finite Connected Graphs without Endpoints

We first prove the following lemma utilizing techniques developed by Williams [25].

LEMMA 3.12. *If G is an orientable finite connected graph with branch points but without end points and $f : G \to G$ is an aperiodic collapsing surjective immersion such that $f(\mathcal{B}(G)) = \mathcal{B}(G)$ then there exist a positive integer n, an orientable bouquet of circles G_* with branch point b_*, and a collapsing surjective immersion $f_* : G_* \to G_*$ such that $f_*(b_*) = b_*$ and f_* is shift equivalent to f^n.*

PROOF. If $\#\mathcal{B}(G) = 1$ then G is an orientable bouquet of circles whose branch point is fixed by f. Thus by letting $n = 1$, $G_* = G$ and $f_* = f$, f_* is trivially shift equivalent to f^n.

If $\#\mathcal{B}(G) \geq 2$ then we first construct a positive integer n, an orientable finite connected graph G_1 without end points where $\#\mathcal{B}(G_1) = \#\mathcal{B}(G)$ such that there exists a branch point of G_1 that intersects the closure of every component of $G_1 \backslash \mathcal{B}(G_1)$, and a collapsing surjective immersion $f_1 : G_1 \to G_1$ such that $f_1(b) = b$ for all $b \in \mathcal{B}(G_1)$ and f_1 is shift equivalent to f^n.

If there exists a branch point of G that intersects the closure of every component of $G\backslash\mathcal{B}(G)$ then by letting n be a positive integer such that $f^n(b) = b$ for each $b \in \mathcal{B}(G)$, $G_1 = G$, and $f_1 = f^n$, f_1 is trivially shift equivalent to f^n.

If there does not exist a branch point of G that intersects the closure of every component of $G\backslash\mathcal{B}(G)$ then we construct n, G_1, and f_1 as follows.

Let m be a positive integer such that $f^m(b) = b$ for all $b \in \mathcal{B}(G)$. Since there does not exist a branch point of G that intersects the closure of every component of $G\backslash\mathcal{B}(G)$, we choose a $b \in \mathcal{B}(G)$ and let $\{V_1, \ldots, V_j\}$ be the components of $G\backslash\mathcal{B}(G)$ such that $b \notin \mathrm{Cl}(V_i)$. If U is a star neighborhood of b then there exists an $I \in \mathcal{A}(U)$ such that $f^m(V) \subset I$ for all sufficiently small star neighborhoods V of b since f^m is a collapsing immersion. For each $V_i \in \{V_1, \ldots, V_j\}$, there exists a positive integer k_i and an $x_i \in V_i$ such that $f^{k_i m}(x_i) = b$ and $f^{k_i m}(V) \subset I$ for sufficiently small star neighborhoods V of x_i since f^m is an aperiodic collapsing immersion. Let $n = km$ where $k = \max\{k_1, \ldots, k_j\}$.

Let G_1 be the partition of G consisting of the finite set $b_1 \equiv \{b, x_1, \ldots, x_j\}$ and the one-point sets $\{x\}$ for $x \notin \{b, x_1, \ldots, x_j\}$ and let $r : G \to G_1$ be the surjective function that takes each point of G to the element of G_1 containing it. Assume G_1 is given the quotient topology determined by r. Since only a finite number of points of G are identified to obtain G_1, G_1 is a finite connected graph without end points and $\#\mathcal{B}(G_1) \geq 2$. By the construction of G_1, b_1 intersects the closure of every component of $G_1\backslash\mathcal{B}(G_1)$.

We now proceed to define an arc structure system for G_1. Each star in G_1 which is an arc will have the natural arc structure an arc possesses. If $x \in \mathcal{B}(G)\backslash\{b\}$ then there exists a star neighborhood U of x in G such that $U \cap b_1 = \emptyset$. Since r is one-to-one on U, $r(U)$ is a star neighborhood of the branch point $\{x\}$ in G_1. Then $\mathcal{A}(r(U)) = \{r(I) \mid I \in \mathcal{A}(U)\}$ is the arc structure on $r(U)$ induced by the arc structure on U. This arc structure on $r(U)$ then induces a coherent arc structure on every star neighborhood of $\{x\}$. To construct an arc structure on every star neighborhood of b_1 in G_1, we choose a star neighborhood U_0 of b and a star neighborhood U_i of each $x_i \in \{x_1, \ldots, x_j\}$ such that U_i is an arc and $\{U_0, \ldots, U_j\}$ are pairwise disjoint. Then $r(\bigcup_{i=0}^{j} U_i)$ is a star neighborhood of b_1 in G_1. For each $U_i \in \{U_0, \ldots, U_j\}$, let $\mathcal{E}_\ell(U_i)$ be the left end points of U_i and $\mathcal{E}_r(U_i)$ be the right end points of U_i determined by an orientation of G. Then $\mathcal{E}_1(r(\bigcup_{i=0}^{j} U_i)) = \bigcup_{i=0}^{j} r(\mathcal{E}_\ell(U_i))$ and $\mathcal{E}_2(r(\bigcup_{i=0}^{j} U_i)) = \bigcup_{i=0}^{j} r(\mathcal{E}_r(U_i))$ is a partition of $\mathcal{E}(r(\bigcup_{i=0}^{j} U_i))$ into two nonempty subsets. Let $\mathcal{A}(r(\bigcup_{i=0}^{j} U_i))$ be the arc structure on $r(\bigcup_{i=0}^{j} U_i)$ induced by this partition. This arc structure on $r(\bigcup_{i=0}^{j} U_i)$ then induces a coherent arc structure on every star neighborhood of b_1 in G_1. Thus we have defined an arc structure system for G_1. By definition of the arc structure system for G_1, r is a surjective immersion which is not collapsing and an orientation of G induces an orientation of G_1.

Let $s : G_1 \to G$ be the function defined by $s(x) = f^n \circ r|_{G\backslash b_1}^{-1}(x)$ for $x \in G_1\backslash\{b_1\}$ and $s(b_1) = b$. Since $r|_{G\backslash b_1}$ is a topological embedding, $f^n \circ r|_{G\backslash b_1}^{-1} : G_1\backslash\{b_1\} \to G$ is continuous. So s is continuous at each $x \in G_1\backslash\{b_1\}$. The image under s of every sequence of points converging to b_1 converges to $b = s(b_1)$. So s is continuous at b_1. By definition, s is a collapsing surjective immersion such that $s(b_1) = b$.

Let $f_1 : G_1 \to G_1$ be the map defined by $f_1 = r \circ s$. Since s is a collapsing surjective immersion such that $s(b_1) = b$ and r is a surjective immersion such that $r(b) = b_1$, f_1 is a collapsing surjective immersion such that $f_1(b_1) = b_1$. Since $f^n = s \circ r$, $f_1 = r \circ s$, $r \circ f^n = f_1 \circ r$, and $s \circ f_1 = f^n \circ s$, f_1 is shift equivalent to f^n.

Now we construct an orientable bouquet of circles with branch point b_* and a collapsing surjective immersion $f_* : G_* \to G_*$ such that $f_*(b_*) = b_*$ and f_* is shift equivalent to f_1 in the following manner.

Let $\{U_1, \ldots, U_m\}$ be the components of $G_1 \backslash \{b_1\}$. Then each U_i is an open arc or simple open n-od ($n \geq 3$) in G_1. For each U_i, there exists a unique collection of distinct open arcs $\{I_{i,1}, \ldots, I_{i,\ell_i}\}$ such that there exists a component of $G_1 \backslash f_1^{-1}(\{b_1\})$ whose image under f_1 is $I_{i,j}$. Since f_1 is surjective, $\bigcup_{j=1}^{\ell_i} I_{i,j} = U_i$.

For each $1 \leq i \leq m$ and $1 \leq j \leq \ell_i$, let $O_{i,j}$ be the set of ordered triples (x, i, j) such that $x \in \mathrm{Cl}(I_{i,j})$ and let $p_{i,j} : O_{i,j} \to G_1$ be the function defined by $p_{i,j}(x, i, j) = x$. Since $p_{i,j}$ is one-to-one, the topology on G_1 naturally induces a topology on $O_{i,j}$ such that $p_{i,j}$ is a topological embedding. Since $\mathrm{Cl}(I_{i,j})$ is a circle in G_1 and $p_{i,j}(O_{i,j}) = \mathrm{Cl}(I_{i,j})$, $O_{i,j}$ is a finite connected graph without end points or branch points, that is, a circle.

Assume $G_2 = \bigcup_{\substack{1 \leq i \leq m \\ 1 \leq j \leq \ell_i}} O_{i,j}$ is given the topology generated by the union of the topologies on each space $O_{i,j}$. Then G_2 is a finite graph without end points or branch points, that is, a finite union of disjoint circles. Let $p : G_2 \to G_1$ be the map defined by $p(x, i, j) = p_{i,j}(x, i, j)$.

Let G_* be the partition of G_2 consisting of the one-point sets $\{(x, i, j)\}$ for $x \neq b_1$ and the finite set $b_* = \{(b_1, i, j) \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq \ell_i\}$ and $q : G_2 \to G_*$ be the surjective function that takes each point of G_2 to the element of G_* containing it. Assume G_* is given the quotient topology determined by q. Since only a finite number of points in G_2 are identified to obtain G_*, G_* is a bouquet of circles with branch point b_*.

Let $r : G_* \to G_1$ be the function defined by $r(x_*) = p \circ q|_{G_2 \backslash b_*}^{-1}(x_*)$ for $x_* \in G_* \backslash \{b_*\}$ and $r(b_*) = b_1$. Since q is one-to-one on $G_2 \backslash b_*$, $p \circ q|_{G_2 \backslash b_*}^{-1}$ is continuous. So r is continuous at each $x_* \in G_* \backslash \{b_*\}$. The image under r of every sequence of points converging to b_* converges to $b_1 = r(b_*)$. So r is continuous at b_*.

We now proceed to define an arc structure system for G_* using the arc structure system for G_1. Each star in G_* which is an arc will be given the natural arc structure an arc possesses. Let U be a star neighborhood of b_1 in G_1. Then $r^{-1}(U)$ is a star neighborhood of b_* in G_*. The arc structure on U naturally induces an arc structure on $r^{-1}(U)$ in the following manner. The end points of U, $\mathcal{E}(U)$, can be partitioned into two subsets, $\mathcal{E}_1(U)$ and $\mathcal{E}_2(U)$, such that the end points of each arc of $\mathcal{A}(U)$ are not contained in the same partition and given a point from each partition the arc between the points contained in U is an element of $\mathcal{A}(U)$. Then $\mathcal{E}_1(r^{-1}(U)) = r^{-1}(\mathcal{E}_1(U))$ and $\mathcal{E}_2(r^{-1}(U)) = r^{-1}(\mathcal{E}_2(U))$ is a partition of $\mathcal{E}(r^{-1}(U))$ into two nonempty disjoint subsets. Let $\mathcal{A}(r^{-1}(U))$ be the arc structure on $r^{-1}(U)$ induced by this partition. This arc structure on $r^{-1}(U)$ then induces a coherent arc structure on every star neighborhood of b_*. Thus we have defined an arc structure system for G_*.

By definition of the arc structure system, r is a surjective immersion such that $r(b_*) = b_1$ which is not collapsing and an orientation of G_1 induces an orientation of G_*.

Let $\{V_1, \dots, V_n\}$ be the components of $G_1 \backslash f_1^{-1}(\{b_1\})$. For each V_i, $f_1(V_i)$ is an open arc such that $f_1(\mathrm{Cl}(V_i))$ is a circle in G_1. By the construction of G_*, there exists one and only one open arc I_i of $G_* \backslash \{b_*\}$ such that $r(I_i) = f_1(V_i)$ and for each open arc I of $G_* \backslash \{b_*\}$ there exists a $V \in \{V_1, \dots, V_n\}$ such that $r(I) = f_1(V)$. Let $s : G_1 \to G_*$ be the function defined by $s(x) = (r|_{I_i})^{-1} \circ f_1(x)$ for $x \in V_i$ and $s(x) = b_*$ for $x \in f_1^{-1}(\{b_1\})$. Since $r|_{I_i}$ is a topological embedding, $(r|_{I_i})^{-1} \circ f_1|_{V_i} : V_i \to I_i$ is continuous. So s is continuous at each $x \in G_1 \backslash f_1^{-1}(\{b_1\})$. If $x \in f_1^{-1}(\{b_1\})$ then the image under s of every sequence of points converging to x converges to $b_* = s(x)$. So s is continuous at each $x \in f_1^{-1}(\{b_1\})$. By definition, s is a collapsing surjective immersion such that $s(b_1) = b_*$ and $f_1 = r \circ s$.

Let $f_* : G_* \to G_*$ be the map defined by $f_* = s \circ r$. Since r is a surjective immersion such that $r(b_*) = b_1$ and s is a collapsing surjective immersion such that $s(b_1) = b_*$, f_* is a collapsing surjective immersion such that $f_*(b_*) = b_*$. Since $s \circ f_1 = s \circ r \circ s = f_* \circ s$ and $r \circ f_* = r \circ s \circ r = f \circ r$, f_* is shift equivalent to f_1.

Thus f_* is shift equivalent to f^n. □

THEOREM 3.13. *Let G and G' be orientable finite connected graphs with branch points but without end points and let $f : G \to G$ and $f' : G' \to G'$ be aperiodic collapsing surjective immersions such that $f(\mathcal{B}(G)) = \mathcal{B}(G)$ and $f'(\mathcal{B}(G')) = \mathcal{B}(G')$. If $\varprojlim\{G, f\}$ is homeomorphic to $\varprojlim\{G', f'\}$ then there exist Perron numbers α and β and positive integers m and n such that $\alpha\lambda_f = \lambda_{f'}^m$ and $\beta\lambda_{f'} = \lambda_f^n$.*

PROOF. By Lemma 3.12, we can choose positive integers, ℓ and ℓ', orientable bouquets of circles, G_* and G'_*, with branch points b_* and b'_*, respectively, and collapsing surjective immersions, $f_* : G_* \to G_*$ and $f'_* : G'_* \to G'_*$, such that $f_*(b_*) = b_*$, $f'_*(b'_*) = b'_*$, f_* is shift equivalent to f^ℓ, and f'_* is shift equivalent to $(f')^{\ell'}$. Then $\log(\lambda_{f_*}) = h_{\text{top}}(f_*) = h_{\text{top}}(f^\ell) = \ell h_{\text{top}}(f) = \log(\lambda_f^\ell)$ and $\log(\lambda_{f'_*}) = h_{\text{top}}(f'_*) = h_{\text{top}}((f')^{\ell'}) = \ell' h_{\text{top}}(f') = \log(\lambda_{f'}^{\ell'})$ by Lemma 2.5. So $\lambda_{f_*} = \lambda_f^\ell$ and $\lambda_{f'_*} = \lambda_{f'}^{\ell'}$.

Since f_* is shift equivalent to f^ℓ, $\varprojlim\{G_*, f_*\}$ is homeomorphic to $\varprojlim\{G, f^\ell\}$ which in turn is homeomorphic to $\varprojlim\{G, f\}$. Similarly, $\varprojlim\{G'_*, f'_*\}$ is homeomorphic to $\varprojlim\{G', f'\}$.

Let M_{f_*} be the aperiodic transition matrix for f_* relative to some ordering of the components of $G_* \backslash \mathcal{B}(G_*)$ and $M_{f'_*}$ be the aperiodic transition matrix for f'_* relative to some ordering of the components of $G'_* \backslash \mathcal{B}(G'_*)$. Suppose $\varprojlim\{G, f\}$ is homeomorphic to $\varprojlim\{G', f'\}$. Then $\varprojlim\{G_*, f_*\}$ is homeomorphic to $\varprojlim\{G'_*, f'_*\}$ and M_{f_*} is weakly equivalent to $M_{f'_*}$ by Theorem 3.1. By Theorem 2.6, there exists a Perron number a and a positive integer η such that $a\lambda_{f_*} = \lambda_{f'_*}^\eta$. Let $\alpha = a\lambda_f^{\ell-1}$, which is a Perron number since λ_f is a Perron number and Perron numbers are closed under multiplication [13], and let $m = \ell'\eta$, which is a positive integer. Then $\alpha\lambda_f = a\lambda_f^\ell = a\lambda_{f_*} = \lambda_{f'_*}^\eta = (\lambda_{f'}^{\ell'})^\eta = \lambda_{f'}^m$. Similarly, there exists a Perron number β and a positive integer n such that $\beta\lambda_{f'} = \lambda_f^n$. □

Non-Orientable Finite Connected Graphs without Endpoints

Our goal in this section is to prove the following theorem utilizing techniques similar to those used by Fokkink [12] to analyze matchbox manifolds.

THEOREM 3.14. *Let G and G' be non-orientable finite connected graphs with branch points but without end points and let $f : G \to G$ and $f' : G' \to G'$ be aperiodic collapsing surjective immersions such that $f(\mathcal{B}(G)) = \mathcal{B}(G)$ and $f'(\mathcal{B}(G')) = \mathcal{B}(G')$. If $\varprojlim\{G, f\}$ is homeomorphic to $\varprojlim\{G', f'\}$ then there exist Perron numbers α and β and positive integers m and n such that $\alpha\lambda_f = \lambda_{f'}^m$ and $\beta\lambda_{f'} = \lambda_f^n$.*

We begin the proof of the above theorem by constructing an orientable double cover for each finite graph. Let $\widetilde{G}$ be the set of all ordered pairs (x, D_x) such that $x \in G$ and $D_x \in \mathcal{D}_x$. Given a directed star (U, D_U) in G, let $\widetilde{U}_{D_U}$ be the set of all $(x, D_x) \in \widetilde{G}$ such that $x \in \text{Int}(U)$ and D_x is the direction at x induced by D_U.

The set $\widetilde{G}$ naturally inherits a topology from the topology on G in the following manner. Let $\widetilde{\mathcal{S}}_G$ be the collection of all subsets of $\widetilde{G}$ induced by all directed stars in G. If $(x, D_x) \in \widetilde{G}$ then there exists a directed star (U, D_U) such that $x \in \text{Int}(U)$ and D_x is the direction at x induced by D_U. So $(x, D_x) \in \widetilde{U}_{D_U} \in \widetilde{\mathcal{S}}_G$. If $\widetilde{U}$ and $\widetilde{V}$ are elements of $\widetilde{\mathcal{S}}_G$ then there exist directed stars (U, D_U) and (V, D_V) in G that induce

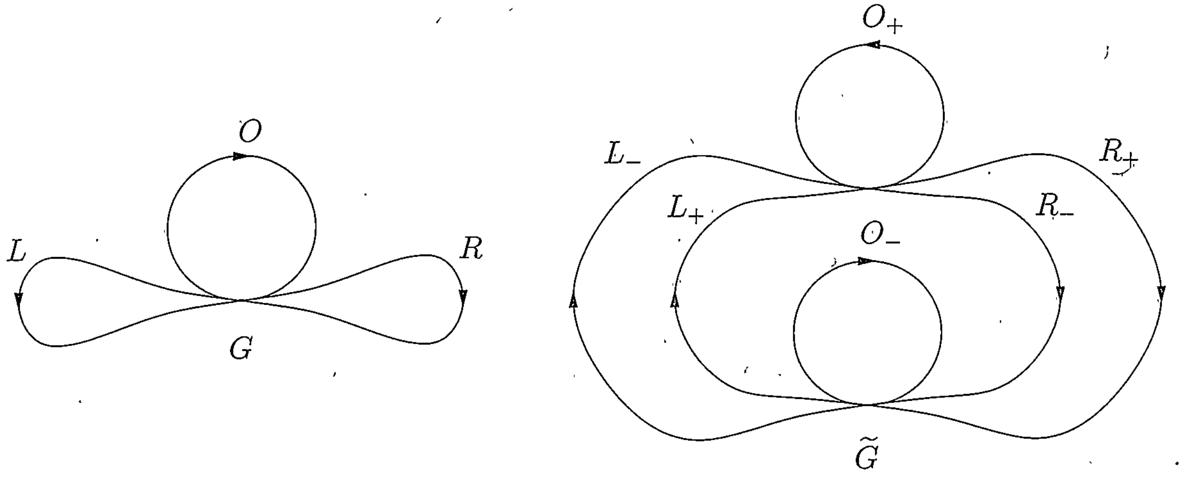

Figure 2. A non-orientable graph G with corresponding orientable covering space $\widetilde{G}$.

$\widetilde{U}$ and $\widetilde{V}$, respectively. If $(x, D_x) \in \widetilde{U} \cap \widetilde{V}$ then (U, D_U) and (V, D_V) are coherently directed near x. So there exists a directed star neighborhood (W, D_W) of x such that $W \subset U \cap V$, (W, D_W) and (U, D_U) are coherently directed, and (W, D_W) and (V, D_V) are coherently directed. Thus $(x, D_x) \in \widetilde{W}_{D_W} \subset \widetilde{U} \cap \widetilde{V}$. Therefore $\widetilde{S}_G$ is a basis for a topology on $\widetilde{G}$. We will assume the set $\widetilde{G}$ is endowed with the topology generated by $\widetilde{S}_G$.

Let $pr_1 : \widetilde{G} \to G$ be the open surjective two-to-one map defined by $pr_1(x, D_x) = x$. Let $x \in G$ and (U, D_U) be a directed star neighborhood of x in G. Then $\widetilde{U}_{D_U}$ and $\widetilde{U}_{-D_U}$ are two disjoint open sets in $\widetilde{G}$ whose union is $pr_1^{-1}(\text{Int}(U))$ and $pr_1|_{\widetilde{U}_{D_U}}$ and $pr_1|_{\widetilde{U}_{-D_U}}$ are homeomorphisms onto $\text{Int}(U)$. Thus $\widetilde{G}$ is a two-fold covering of G with covering map pr_1.

LEMMA 3.15. *$\widetilde{G}$ is a finite graph with branch points but without end points.*

Proof. Since G is a compact second-countable metric space and $pr_1 : \widetilde{G} \to G$ is a two-to-one covering map, $\widetilde{G}$ is a compact second-countable metrizable space. Since pr_1 is a two-to-one local homeomorphism, $\widetilde{G}$ is a finite graph with branch points but without end points. □

Let $\widetilde{U}$ be an element of $\widetilde{\mathcal{S}}_G$ and (U, D_U) be the directed star in G that induces it. Then $\mathrm{Cl}(\widetilde{U})$ is a star in $\widetilde{G}$ and $pr_1|_{\mathrm{Cl}(\widetilde{U})}$ is a topological embedding. Let $\mathcal{A}(U)$ be the arc structure on U and $\mathcal{A}(\mathrm{Cl}(\widetilde{U})) = \{pr_1|^{-1}_{\mathrm{Cl}(\widetilde{U})}(I) | I \in \mathcal{A}(U)\}$. Since $pr_1|_{\mathrm{Cl}(\widetilde{U})}$ is a homeomorphism onto U, $\mathcal{A}(\mathrm{Cl}(\widetilde{U}))$ is an arc structure on the star $\mathrm{Cl}(\widetilde{U})$. Thus an arc structure system for G naturally induces an arc structure system for $\widetilde{G}$. When considering $\widetilde{G}$, we will assume it is endowed with the arc structure system induced by the arc structure system for G.

Lemma 3.16. *$\widetilde{G}$ is orientable.*

Proof. Let $\{\widetilde{U}_1, \dots, \widetilde{U}_n\}$ be a finite collection of elements of $\widetilde{\mathcal{S}}_G$ that cover $\widetilde{G}$. For $i = 1, \dots, n$, let (U_i, D_{U_i}) be the directed star in G that induces $\widetilde{U}_i$. Let ρ_i be a directing map for U_i representing D_{U_i} and $\tilde{\rho}_i : \mathrm{Cl}(\widetilde{U}_i) \to \mathbb{R}$ be the map defined by $\tilde{\rho}_i = \rho_i \circ pr_1|_{\mathrm{Cl}(\widetilde{U}_i)}$. If $\tilde{I} \in \mathcal{A}(\mathrm{Cl}(\widetilde{U}_i))$ then $pr_1(\tilde{I}) \in \mathcal{A}(U_i)$ and $\tilde{\rho}_i|_{\tilde{I}} = \rho_i|_{pr_1(\tilde{I})} \circ pr_1|_{\tilde{I}}$ is a topological embedding. So $\tilde{\rho}_i$ is a directing map for $\mathrm{Cl}(\widetilde{U}_i)$. Let $(\mathrm{Cl}(\widetilde{U}_i), D_{\mathrm{Cl}(\widetilde{U}_i)})$ be the directed star with direction $D_{\mathrm{Cl}(\widetilde{U}_i)} = [\tilde{\rho}_i]$. If $(x, D_x) \in \widetilde{U}_i \cap \widetilde{U}_j$ then (U_i, D_{U_i}) and (U_j, D_{U_j}) are coherently directed near x. So there exists a star neighborhood V of x in G contained in $U_i \cap U_j$ such that $\rho_i|_V$ is equivalent to $\rho_j|_V$. Let (V, D_V) be the directed star with direction $D_V = [\rho_i|_V] = [\rho_j|_V]$. Then $(x, D_x) \in \widetilde{V}_{D_V} \subset \widetilde{U}_i \cap \widetilde{U}_j$.

If $\tilde{I} \in \mathcal{A}(\mathrm{Cl}(\widetilde{V}_{D_V}))$ then $pr_1(\tilde{I}) \in \mathcal{A}(V)$. Since $\tilde{\rho}_j|_{\tilde{I}} \circ \tilde{\rho}_i|_{\tilde{I}}^{-1} = \rho_j|_{pr_1(\tilde{I})} \circ pr_1|_{\tilde{I}} \circ pr_1|_{\tilde{I}}^{-1} \circ \rho_i|_{pr_1(\tilde{I})}^{-1} = \rho_j|_{pr_1(\tilde{I})} \circ \rho_i|_{pr_1(\tilde{I})}^{-1}$, $\tilde{\rho}_j|_{\tilde{I}} \circ \tilde{\rho}_i|_{\tilde{I}}^{-1} : \tilde{\rho}_i(\tilde{I}) \to \tilde{\rho}_j(\tilde{I})$ is an increasing map. So $\tilde{\rho}_i|_{\mathrm{Cl}(\widetilde{V}_{D_V})}$ is equivalent to $\tilde{\rho}_j|_{\mathrm{Cl}(\widetilde{V}_{D_V})}$ and $(\mathrm{Cl}(\widetilde{U}_i), D_{\mathrm{Cl}(\widetilde{U}_i)})$ and $(\mathrm{Cl}(\widetilde{U}_j), D_{\mathrm{Cl}(\widetilde{U}_j)})$ are coherently directed near (x, D_x). Thus $\{(\mathrm{Cl}(\widetilde{U}_1), D_{\mathrm{Cl}(\widetilde{U}_1)}), \dots, (\mathrm{Cl}(\widetilde{U}_n), D_{\mathrm{Cl}(\widetilde{U}_n)})\}$ is a collection of coherently directed stars in $\widetilde{G}$ whose interior covers $\widetilde{G}$. Therefore $\widetilde{G}$ is orientable. □

LEMMA 3.17. *$\widetilde{G}$ is connected.*

PROOF. For purposes of contradiction, assume $\widetilde{G}$ is not connected. Since pr_1 is a two-to-one clopen map and G is connected, there exist two disjoint clopen subsets, $\widetilde{G}_1$ and $\widetilde{G}_2$, of $\widetilde{G}$ such that $\widetilde{G} = \widetilde{G}_1 \cup \widetilde{G}_2$ and $pr_1(\widetilde{G}_1) = pr_1(\widetilde{G}_2) = G$. Since $\widetilde{G}$ is orientable and $\widetilde{G}_1$ is homeomorphic to G, G is orientable. But this contradicts the fact that G is non-orientable. Thus $\widetilde{G}$ is connected. □

Let $pr_2 : \widetilde{G} \to \bigcup_{x \in G} \mathcal{D}_x$ be the surjective function defined by $pr_2(x, D_x) = D_x$. Assume $\bigcup_{x \in G} \mathcal{D}_x$ is endowed with the quotient topology induced by pr_2. Then $\{pr_2(\widetilde{U}) | \widetilde{U} \in \widetilde{\mathcal{S}}_G\}$ is a basis for the topology on $\bigcup_{x \in G} \mathcal{D}_x$. Let $(x, D_x) \in \widetilde{G}$ and V be a star neighborhood of $f(x)$ in G. Since f is a collapsing immersion, there exists a star neighborhood U of x in G such that the restriction of f to each arc in $\mathcal{A}(U)$ is a topological embedding into an arc of $\mathcal{A}(V)$ and $f(U)$ is contained in an arc of $\mathcal{A}(V)$. Let D_U be the direction for U that induces D_x and let ρ_V be a directing map on V. Then $\rho_V \circ f|_U : U \to \mathbb{R}$ is a directing map on U. Let $f_+ : \bigcup_{x \in G} \mathcal{D}_x \to \bigcup_{x \in G} \mathcal{D}_x$ be the function defined in the following manner. If $[\rho_V \circ f|_U] = D_U$, let $f_+(D_x)$ be the direction at $f(x)$ induced by $[\rho_V]$. If $[\rho_V \circ f|_U] = -D_U$, let $f_+(D_x)$ be the direction at $f(x)$ induced by $-[\rho_V]$.

Let $D_x \in \bigcup_{x\in G} \mathcal{D}_x$ and $pr_2(\widetilde{V}) \in \{pr_2(\widetilde{U}) | \widetilde{U} \in \widetilde{\mathcal{S}}_G\}$ such that $f_+(D_x) \in pr_2(\widetilde{V})$. Let (V, D_V) be the directed star neighborhood of $f(x)$ that induces $\widetilde{V}$. Then $f_+(D_x)$ is the direction at $f(x)$ induced by D_V. Since f is continuous at x there exists a star neighborhood U of x such that $f(U) \subset V$. Let D_U be the direction for U that induces D_x. Then $pr_2(\widetilde{U}_{D_U})$ is a neighborhood of D_x such that $f_+(pr_2(\widetilde{U}_{D_U})) \subset pr_2(\widetilde{V})$. Thus f_+ is continuous at D_x. Hence f_+ is continuous.

Let $\tilde{f} : \widetilde{G} \to \widetilde{G}$ be the function defined by $\tilde{f}(x, D_x) = (f \circ pr_1(x, D_x), f_+ \circ pr_2(x, D_x))$. Then $pr_1 \circ \tilde{f} = f \circ pr_1$ by definition.

LEMMA 3.18. *$\tilde{f}$ is an aperiodic collapsing surjective immersion such that $\tilde{f}(\mathcal{B}(\widetilde{G})) = \mathcal{B}(\widetilde{G})$.*

PROOF. *$\tilde{f}$ is surjective*: Let $(z, D_z) \in \widetilde{G}$ and (V, D_V) be a directed star neighborhood of z such that D_V is coherent with D_z. Since f is a surjective map, there exists an $x \in G$ such that $f(x) = z$ and a star neighborhood U of x such that $f(U) \subset V$. Let $D_U = [\rho_V \circ f|_U]$ where ρ_V is a representative of D_V, and D_x be the direction at x induced by D_U. Then $f_+(D_x) = D_z$. Thus $\tilde{f}(x, D_x) = (f \circ pr_1(x, D_x), f_+ \circ pr_2(x, D_x)) = (z, D_z)$.

$\tilde{f}$ is a collapsing immersion: Since $f \circ pr_1$ and $f_+ \circ pr_2$ are continuous, $\tilde{f}$ is continuous. Let $(x, D_x) \in \widetilde{G}$ and $\widetilde{V} \in \widetilde{\mathcal{S}}_G$ containing $\tilde{f}(x, D_x)$. Then $\mathrm{Cl}(\widetilde{V})$ is a star neighborhood of $\tilde{f}(x, D_x) = (f(x), f_+(D_x))$ and $V \equiv pr_1(\mathrm{Cl}(\widetilde{V}))$ is a star

neighborhood of $f(x)$. Let D_V be the direction for V that induces $f_+(D_x)$. Since f is a collapsing immersion at x there exists a star neighborhood U of x and a $J \in \mathcal{A}(V)$ such that the restriction of f to each arc in $\mathcal{A}(U)$ is a topological embedding into J. Then $pr_1|^{-1}_{\mathrm{Cl}(\widetilde{V})} \circ f|_I$ is a topological embedding into $pr_1|^{-1}_{\mathrm{Cl}(\widetilde{V})}(J) \in \mathcal{A}(\mathrm{Cl}(\widetilde{V}))$ for each $I \in \mathcal{A}(U)$. Let D_U be the direction for U that induces D_x. Then $\mathrm{Cl}(\widetilde{U}_{D_U})$ is a star neighborhood of (x, D_x). For $\tilde{I} \in \mathcal{A}(\mathrm{Cl}(\widetilde{U}_{D_U}))$, $pr_1(\tilde{I}) \in \mathcal{A}(U)$ and $\tilde{f}|_{\tilde{I}} = pr_1|^{-1}_{\mathrm{Cl}(\widetilde{V})} \circ f|_{pr_1(\tilde{I})} \circ pr_1|_{\tilde{I}}$ is a topological embedding into $pr_1|^{-1}_{\mathrm{Cl}(\widetilde{V})}(J)$. Thus $\tilde{f}$ is a collapsing immersion at (x, D_x).

$\tilde{f}$ is aperiodic: Let $\{J_1, \dots, J_n\}$ be the components of $G \backslash \mathcal{B}(G)$. Since f is aperiodic, there exists a positive integer M such that $f^m(J_i) = G$ for each J_i and $m \geq M$. Let $\{J_{i,1}, J_{i,2}\}$ be the components of $\widetilde{G} \backslash \mathcal{B}(\widetilde{G})$ such that $J_{i,1} \cup J_{i,2} = pr_1^{-1}(J_i)$. So $pr_1 \circ \tilde{f}^m(J_{i,\ell}) = f^m \circ pr_1(J_{i,\ell}) = f^m(J_i) = G$ for each component $J_{i,\ell}$ of $\widetilde{G} \backslash \mathcal{B}(\widetilde{G})$ and $m \geq M$. By definition of $\tilde{f}$, if $J_{i,1} \subset \tilde{f}(J_{i,\ell})$ then $J_{i,2} \subset \tilde{f}(J_{i,3-\ell})$. So either $J_{i,1} \subset \tilde{f}^m(J_{i,\ell})$ or $J_{i,2} \subset \tilde{f}^m(J_{i,\ell})$. Choose $m \geq M$ such that $\tilde{f}^m(b) = b$ for each $b \in \mathcal{B}(\widetilde{G})$ and $J_{i,\ell} \subset \tilde{f}^m(J_{i,\ell})$ for each $J_{i,\ell}$. Since the number of branch points and components of $\widetilde{G} \backslash \mathcal{B}(\widetilde{G})$ are finite there exists a k such that $\tilde{f}^{km}(J_{i,\ell}) = \tilde{f}^{(k+1)m}(J_{i,\ell})$ for each $J_{i,\ell}$. Let $A_1 = \tilde{f}^{km}(J_{1,1})$ and $A_2 = \tilde{f}^{km}(J_{1,2})$. For purposes of contradiction, assume $A_1 \cap A_2 = \emptyset$. If $x \in A_1 \backslash \mathcal{B}(\widetilde{G})$ then there exists a $J_{i,\ell}$ containing x that is contained in A_1. So $J_{i,\ell}$ is an open neighborhood of x contained in A_1. If $x \in A_1 \cap \mathcal{B}(\widetilde{G})$ then given a star neighborhood W of x there exists a star neighborhood V of x and an $I \in \mathcal{A}(W)$ such that $\tilde{f}^m(V) \subset I$. Since $x \in A_1$, there exists a $z \in J_{1,1}$ and a star

neighborhood U of z such that $\tilde{f}^{(k+1)m}(U) \subset I$. So $I \subset A_1$. Since $\tilde{f}^m(V) \subset I$ and $A_1 \cap A_2 = \emptyset$, $V \subset A_1$. So Int(V) is an open neighborhood of x contained in A_1. Thus A_1 is an open set. Similarly, A_2 is an open set. Since A_1 and A_2 are two disjoint open sets whose union is $\widetilde{G}$, $\widetilde{G}$ is not connected. But this contradicts the fact that $\widetilde{G}$ is connected. Thus $A_1 \cap A_2 \neq \emptyset$. If $x \in A_1 \cap A_2 \backslash \mathcal{B}(\widetilde{G})$ then there exists a $J_{i\ell}$ containing x that is contained in $A_1 \cap A_2$. If $J_{1,1} \subset \tilde{f}^{km}(J_{i,\ell})$ then $A_1 = \tilde{f}^{km}(J_{1,1}) \subset A_1 \cap A_2$. Thus $A_1 = A_2 = \widetilde{G}$. Similarly, if $J_{1,2} \subset \tilde{f}^{km}(J_{i,\ell})$ then $A_1 = A_2 = \widetilde{G}$. If $x \in A_1 \cap A_2 \cap \mathcal{B}(\widetilde{G})$ then given a star neighborhood W of x there exists a star neighborhood V of x and an $I \in \mathcal{A}(W)$ such that $\tilde{f}^m(V) \subset I$. Since $x \in A_1$, there exists a $z_1 \in I_{1,1}$ and a star neighborhood U_1 of x contained in $I_{1,1}$ such that $\tilde{f}^{(k+1)m}(I_{1,1}) \subset I$. Since $x \in A_2$, there exists a $z_2 \in I_{1,2}$ and a star neighborhood U_2 of x contained in $I_{1,2}$ such that $\tilde{f}^{(k+1)m}(I_{1,2}) \subset I$. So $I \subset A_1 \cap A_2$. If $J_{i,\ell}$ is a component of $\widetilde{G} \backslash \mathcal{B}(\widetilde{G})$ such that $J_{i,\ell} \cap I \neq \emptyset$ then $J_{i,\ell} \subset A_1 \cap A_2$. So $A_1 = A_2 = \widetilde{G}$ by earlier argument. If $J_{i,\ell}$ is a component of $\widetilde{G} \backslash \mathcal{B}(\widetilde{G})$ then $J_{1,1}$ or $J_{1,2}$ is a subset of $\tilde{f}^m(J_{i,\ell})$. If $J_{1,1} \subset \tilde{f}^m(J_{i,\ell})$ then $G = \tilde{f}^{km}(J_{1,1}) \subset \tilde{f}^{(k+1)m}(J_{i,\ell})$. Similarly, if $J_{1,2} \subset \tilde{f}^m(J_{i,\ell})$ then $G = \tilde{f}^{(k+1)m}(J_{i,\ell})$. Thus $\tilde{f}$ is aperiodic. □

Let $P : \varprojlim\{\widetilde{G}, \tilde{f}\} \to \varprojlim\{G, f\}$ be the ladder map induced by pr_1.

LEMMA 3.19. *P is a two-to-one covering map.*

PROOF. Since pr_1 is two-to-one covering map and $\tilde{f}|_{pr_1^{-1}(\{x\})}$ is a one-to-one map for each $x \in pr_1(\widetilde{G}) = G$, P is an open two-to-one surjective map by Theorem 2.8. Let π_i and $\widetilde{\pi}_i$ be the i-th projection maps acting on $\varprojlim\{G, f\}$ and $\varprojlim\{\widetilde{G}, \tilde{f}\}$, respectively.

Let $\underline{x} \in \varprojlim\{G, f\}$ and (U, D_U) be a directed star neighborhood of $\pi_1(\underline{x})$ in G. Since $\text{Int}(U)$ is an open neighborhood of $\pi_1(\underline{x})$ in G, $\pi_1^{-1}(\text{Int}(U))$ is an open neighborhood of $\underline{x}$ in $\varprojlim\{G, f\}$. By definition, $\widetilde{U}_{D_U}$ and $\widetilde{U}_{-D_U}$ are two disjoint open sets in $\widetilde{G}$ such that $pr_1^{-1}(\text{Int}(U)) = \widetilde{U}_{D_U} \cup \widetilde{U}_{-D_U}$ and both $pr_1|_{\widetilde{U}_{D_U}}$ and $pr_1|_{\widetilde{U}_{-D_U}}$ are homeomorphisms onto $\text{Int}(U)$. So $\widetilde{\pi}_1^{-1}(\widetilde{U}_{D_U})$ and $\widetilde{\pi}_1^{-1}(\widetilde{U}_{-D_U})$ are two disjoint open sets in $\varprojlim\{\widetilde{G}, \tilde{f}\}$ such that $P^{-1}(\pi_1^{-1}(\text{Int}(U))) = \widetilde{\pi}_1^{-1}(pr_1^{-1}(\text{Int}(U))) = \widetilde{\pi}_1^{-1}(\widetilde{U}_{D_U}) \cup \widetilde{\pi}_1^{-1}(\widetilde{U}_{-D_U})$. Since P is an open map, $P|_{\widetilde{\pi}_1^{-1}(\widetilde{U}_{D_U})}$ and $P|_{\widetilde{\pi}_1^{-1}(\widetilde{U}_{-D_U})}$ are open maps. Let $\underline{z} \in \pi_1^{-1}(\text{Int}(U))$. As a result of the construction of $P^{-1}(\{\underline{z}\})$ given an element $\underline{z}$ of $\varprojlim\{G, f\}$, in Theorem 2.8, if $\{\underline{\tilde{z}}_1, \underline{\tilde{z}}_2\} = P^{-1}(\{\underline{z}\})$ then $\{\widetilde{\pi}_1(\underline{\tilde{z}}_1), \widetilde{\pi}_1(\underline{\tilde{z}}_2)\} = pr_1^{-1}(\{\pi_1(\underline{z})\})$. Since $pr_1|_{\widetilde{U}_{D_U}}$ and $pr_1|_{\widetilde{U}_{-D_U}}$ are homeomorphisms onto $\text{Int}(U)$, one element of $pr_1^{-1}(\{\pi_1(\underline{z})\})$ is contained in $\widetilde{U}_{D_U}$ while the other element is contained in $\widetilde{U}_{-D_U}$. Thus one element of $P^{-1}(\{\underline{z}\})$ is contained in $\widetilde{\pi}_1(\widetilde{U}_{D_U})$ while the other element is contained in $\widetilde{\pi}_1(\widetilde{U}_{-D_U})$. So $P|_{\widetilde{\pi}_1^{-1}(\widetilde{U}_{D_U})}$ and $P|_{\widetilde{\pi}_1^{-1}(\widetilde{U}_{-D_U})}$ are homeomorphisms onto $\pi_1^{-1}(\text{Int}(U))$. Then $\pi_1^{-1}(\text{Int}(U))$ is evenly covered by P. Thus P is a two-to-one covering map. □

Since P is a local homeomorphism the direction at each $\tilde{x} \in \varprojlim\{\widetilde{G}, \tilde{f}\}$ induces a direction at $P(\tilde{x})$.

LEMMA 3.20. *The directions at each $x \in \varprojlim\{G, f\}$ induced by the directions at the two elements of $P^{-1}(\{x\})$ are different.*

PROOF. Let $\varprojlim\{G, f\}_+$ be the set of all $x \in \varprojlim\{G, f\}$ such that the directions at x induced by the two elements of $P^{-1}(\{x\})$ are the same and $\varprojlim\{G, f\}_-$ be the set of all $x \in \varprojlim\{G, f\}$ such that the directions at x induced by the both elements of $P^{-1}(\{x\})$ are different. If $\varprojlim\{G,f\}_- = \emptyset$ then the orientation of $\varprojlim\{\widetilde{G},\tilde{f}\}$ would induce an orientation of $\varprojlim\{G,f\}$. Since $\varprojlim\{G,f\}$ is non-orientable, $\varprojlim\{G,f\}_- \neq \emptyset$. For purposes of contradiction, assume $\varprojlim\{G,f\}_+ \neq \emptyset$. Let $x \in \varprojlim\{G,f\}_+$. Since P is a two-to-one covering map, there exists a matchbox neighborhood M of x such that $P^{-1}(M)$ is the union of two disjoint matchboxes $\tilde{M}_1$ and $\tilde{M}_2$ such that the restriction of P to each is a homeomorphism onto M. Let ρ_1 be a directing map on $\tilde{M}_1$ and ρ_2 be a directing map on $\tilde{M}_2$ coherent with the orientation of $\varprojlim\{\widetilde{G},\tilde{f}\}$. Then $\rho_1 \circ P|_{\tilde{M}_1}^{-1}$ and $\rho_2 \circ P|_{\tilde{M}_2}^{-1}$ are two directing maps on M. Let M_+ be the union of all matches I in M such that $(\rho_2 \circ P|_{\tilde{M}_2}^{-1})|_I \circ (\rho_1 \circ P|_{\tilde{M}_1}^{-1})|_I^{-1} : \rho_1 \circ P|_{\tilde{M}_1}^{-1}(I) \to \rho_2 \circ P|_{\tilde{M}_2}^{-1}(I)$ is increasing. Then M_+ is a matchbox neighborhood of x contained in $\varprojlim\{G,f\}_+$ by Proposition 2.13. So $\varprojlim\{G,f\}_+$ is open. Similarly, $\varprojlim\{G,f\}_-$ is open. Since $\varprojlim\{G,f\}_+$ and $\varprojlim\{G,f\}_-$ are two disjoint open sets whose union is $\varprojlim\{G,f\}$, $\varprojlim\{G,f\}$ is not connected. But this contradicts the fact that $\varprojlim\{G,f\}$ is a continuum. Thus $\varprojlim\{G,f\}_+ = \emptyset$. □

We construct $\widetilde{G}'$, $pr_1' : \widetilde{G}' \to G'$, $\tilde{f}' : \widetilde{G}' \to \widetilde{G}'$, and $P' : \varprojlim\{\widetilde{G}',\tilde{f}'\} \to \varprojlim\{G',f'\}$ similarly.

Assume $\varprojlim\{G,f\}$ and $\varprojlim\{G',f'\}$ are homeomorphic. Let $\phi : \varprojlim\{G,f\} \to \varprojlim\{G',f'\}$ be a homeomorphism. For each element x in $\varprojlim\{\widetilde{G},\tilde{f}\}$, let $\tilde{\phi}(x)$ be the element of $\varprojlim\{\widetilde{G}',\tilde{f}'\}$ such that the direction at $\phi \circ P(x)$ induced by the direction at x is the same as the direction at $\phi \circ P(x)$ induced by the direction at $\tilde{\phi}(x)$.

LEMMA 3.21. *$\tilde{\phi}$ is a homeomorphism.*

PROOF. $\tilde{\phi}$ *is surjective*: Let $x' \in \varprojlim\{\widetilde{G}', \tilde{f}'\}$. By Lemma 3.20 and the fact that ϕ is a homeomorphism, there exists one and only one $x \in (\phi \circ P)^{-1}(\{P'(x')\})$ such that the direction at $\phi \circ P(x)$ induced by the direction at x is the same as the direction at $P'(x')$ induced by the direction at x'. Then $\tilde{\phi}(x) = x'$.

$\tilde{\phi}$ *is injective*: Suppose that $\tilde{\phi}(x) = \tilde{\phi}(z)$. Then $\phi \circ P(x) = \phi \circ P(z)$ and the direction at $\phi \circ P(x)$ induced by the direction at x is the same as the direction at $\phi \circ P(z)$ induced by the direction at z. Then $x = z$ by Lemma 3.20.

$\tilde{\phi}$ *is continuous*: Let $x \in \varprojlim\{\widetilde{G}, \tilde{f}\}$ and V be an open neighborhood of $\tilde{\phi}(x)$. Since $\phi \circ P$ and P' are two-to-one covering maps, there exists a matchbox neighborhood M of $\phi \circ P(x)$ contained in $P'(V)$ such that $(\phi \circ P)^{-1}(M)$ is the union of two disjoint matchboxes such that $\phi \circ P$ restricted to each is a homeomorphism onto M and $(P')^{-1}(M)$ is the union of two disjoint matchboxes such that P' restricted to each is a homeomorphism onto M. Let $\tilde{M}$ be a matchbox neighborhood of x such that $\phi \circ P|_{\tilde{M}}$ is a homeomorphism onto M and $\tilde{M}'$ be a matchbox neighborhood of $\tilde{\phi}(x)$ such that $\tilde{M}' \subset V$ and $P'|_{\tilde{M}'}$ is a homeomorphism onto M. Let ρ be a directing map on $\tilde{M}$ coherent with the orientation of $\varprojlim\{\widetilde{G}, \tilde{f}\}$ and ρ' be a directing map on $\tilde{M}'$ coherent with the orientation of $\varprojlim\{\widetilde{G}', \tilde{f}'\}$. Then $\rho \circ (\phi \circ P|_{\tilde{M}})^{-1}$ and $\rho' \circ P|_{\tilde{M}'}^{-1}$ are the induced directing maps on M. Let M_+ be the union of the matches I in M such that $(\rho \circ (\phi \circ P|_{\tilde{M}})^{-1})|_I \circ (\rho' \circ P|_{\tilde{M}'}^{-1})|_I^{-1} : \rho' \circ P|_{\tilde{M}'}^{-1}(I) \to \rho \circ (\phi \circ P|_{\tilde{M}})^{-1}(I)$ is increasing. By definition of $\tilde{\phi}(x)$, the match in M containing $\phi \circ P(x)$ is contained in M_+. Let $\tilde{M}_+ = (\phi \circ P|_{\tilde{M}})^{-1}(M_+)$ and $\tilde{M}'_+ = P'|_{\tilde{M}'}^{-1}(M_+)$. Then $\tilde{M}_+$ is a matchbox neighborhood of x such that $\tilde{\phi}(\tilde{M}_+) = \tilde{M}'_+$. Since $\tilde{M}_+ \subset V$, $\tilde{\phi}$ is continuous at x. Thus $\tilde{\phi}$ is continuous at each $x \in \varprojlim\{\tilde{G}, \tilde{f}\}$. Since $\varprojlim\{\tilde{G}, \tilde{f}\}$ is compact and $\varprojlim\{\tilde{G}', \tilde{f}'\}$ is Hausdorff, $\tilde{\phi}$ is a homeomorphism. □

Since $\widetilde{G}$ and $\widetilde{G}'$ are orientable finite connected graphs with branch points but without end points and $\tilde{f} : \widetilde{G} \to \widetilde{G}$, $\tilde{f}' : \widetilde{G}' \to \widetilde{G}'$ are aperiodic collapsing surjective immersions such that $\tilde{f}(\mathcal{B}(\widetilde{G})) = \mathcal{B}(\widetilde{G})$ and $\tilde{f}'(\mathcal{B}(\widetilde{G}')) = \mathcal{B}(\widetilde{G}')$, and $\varprojlim\{\widetilde{G}, \tilde{f}\}$ is homeomorphic to $\varprojlim\{\widetilde{G}', \tilde{f}'\}$, there exist Perron numbers α and β and positive integers m and n such that $\alpha\lambda_{\tilde{f}} = \lambda^m_{\tilde{f}'}$ and $\beta\lambda_{\tilde{f}'} = \lambda^n_{\tilde{f}}$ by Theorem 3.13. Since $pr_1 : \widetilde{G} \to G$ and $pr'_1 : \widetilde{G}' \to G'$ are two-to-one covering maps such that $pr_1 \circ \tilde{f} = f \circ pr_1$ and $pr'_1 \circ \tilde{f}' = f' \circ pr'_1$, $h_{\text{top}}(\tilde{f}) = h_{\text{top}}(f)$ and $h_{\text{top}}(\tilde{f}') = h_{\text{top}}(f')$ by Theorem 2.4. So $\lambda_{\tilde{f}} = \lambda_f$ and $\lambda_{\tilde{f}'} = \lambda_{f'}$. Thus $\alpha\lambda_f = \lambda^m_{f'}$ and $\beta\lambda_{f'} = \lambda^n_f$, which concludes the proof of the theorem.

Finite Connected Graphs without Endpoints

THEOREM 3.22. *Let G and G' be finite connected graphs with branch points but without end points and let $f : G \to G$ and $f' : G' \to G'$ be aperiodic collapsing surjective immersions such that $f(\mathcal{B}(G)) = \mathcal{B}(G)$ and $f'(\mathcal{B}(G')) = \mathcal{B}(G')$. If $\varprojlim\{G, f\}$ is homeomorphic to $\varprojlim\{G', f'\}$ then there exist Perron numbers α and β and positive integers m and n such that $\alpha\lambda_f = \lambda^m_{f'}$ and $\beta\lambda_{f'} = \lambda^n_f$.*

PROOF. Assume $\varprojlim\{G, f\}$ is homeomorphic to $\varprojlim\{G', f'\}$. Then either $\varprojlim\{G, f\}$ and $\varprojlim\{G', f'\}$ are both orientable or both non-orientable. So either G and G' are both orientable or both non-orientable. If G and G' are both orientable then then there exist Perron numbers α and β and positive integers m and n such that $\alpha\lambda_f = \lambda_{f'}^m$ and $\beta\lambda_{f'} = \lambda_f^n$ by Theorem 3.13. If G and G' are both non-orientable then then there exist Perron numbers α and β and positive integers m and n such that $\alpha\lambda_f = \lambda_{f'}^m$ and $\beta\lambda_{f'} = \lambda_f^n$ by Theorem 3.14. □

Barge and Diamond [4] have shown that given two Perron numbers λ and γ, if there are Perron numbers α and β and positive integers m and n such that $\alpha\lambda = \gamma^m$ and $\beta\gamma = \lambda^n$ then $\mathbb{Q}(\lambda) = \mathbb{Q}(\gamma)$. This results in the following corollary to the above theorem.

COROLLARY 3.23. *Let G and G' be finite connected graphs with branch points but without end points and let $f : G \to G$ and $f' : G' \to G'$ be aperiodic collapsing surjective immersions such that $f(\mathcal{B}(G)) = \mathcal{B}(G)$ and $f'(\mathcal{B}(G')) = \mathcal{B}(G')$. If $\varprojlim\{G, f\}$ is homeomorphic to $\varprojlim\{G', f'\}$ then $\mathbb{Q}(\lambda_f) = \mathbb{Q}(\lambda_{f'})$.*

It should be noted that the algebraic extension relationship between spectral radii is weaker than the Perron relationship between the spectral radii. To see this, let $\lambda_f = 2$ and $\lambda_{f'} = 3$. Then $\mathbb{Q}(\lambda_f) = \mathbb{Q}(\lambda_{f'})$ but there do not exist Perron numbers, α and β, and positive integers, m and n such that $\alpha\lambda_f = \lambda_{f'}^m$ and $\beta\lambda_{f'} = \lambda_f^n$.

Two Examples

Our first example shows that we can use weak equivalence of matrices to determine when certain inverse limits of collapsing surjective immersions on orientable bouquets of circles which fix branch points are not homeomorphic. Our second example shows that weak equivalence of transition matrices is not a complete topological invariant for inverse limits of collapsing surjective immersions on orientable bouquets of circles which fix branch points. The second example requires a slight understanding of substitution tilings. For a discussion of this topic, consult Anderson and Putnam [2].

Example 1

Let G be an oriented bouquet of two circles S_1 and S_2. Let $f : G \to G$ be an immersion which fixes the branch point such that $S_1 \to S_1S_1S_2S_1S_1$ and $S_2 \to S_1$. Let $f' : G \to G$ be an immersion which fixes the branch point such that $S_1 \to S_1S_2S_2S_1S_1$ and $S_2 \to S_1S_2S_1$. Then f and f' are collapsing surjective immersions. The transition matrices for f and f' relative to the natural ordering of the components of $G\backslash\mathcal{B}(G)$ are $M_f = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}$ and $M_{f'} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$. Swanson and Volkmer [23] show that these primitive matrices are not weakly equivalent by an ideal argument. Thus $\varprojlim\{G, f\}$ and $\varprojlim\{G, f'\}$ are not homeomorphic by Theorem 3.1.

Since $\lambda_f = 2 + \sqrt{5} = \lambda_{f'}$, there trivially exists Perron numbers α and β and positive integers m and n such that $\alpha\lambda_f = \lambda_{f'}^m$ and $\beta\lambda_{f'} = \lambda_f^n$. Thus we can not use Theorem 3.22 or Corollary 3.23 to determine whether $\varprojlim\{G, f\}$ and $\varprojlim\{G, f'\}$ are homeomorphic or not.

Example 2

Let φ be the substitution on two letters given by $\varphi(1) = 11221$, $\varphi(2) = 1$ and $\mathcal{T}_\varphi$ be the associated tiling space. Let χ be the substitution on two letters given by $\chi(1) = 12121$, $\chi(2) = 1$ and $\mathcal{T}_\chi$ be the associated tiling space. Using techniques developed by Anderson and Putnam [2], there exist orientable bouquets of two circles G_φ and G_χ and collapsing surjective immersions $f_\varphi : G_\varphi \to G_\varphi$ and $f_\chi : G_\chi \to G_\chi$, which fix branch points, such that $\mathcal{T}_\varphi$ is homeomorphic to $\varprojlim\{G_\varphi, f_\varphi\}$ and $\mathcal{T}_\chi$ is homeomorphic to $\varprojlim\{G_\chi, f_\chi\}$. We can order the components of $G_\varphi\backslash\mathcal{B}(G_\varphi)$ and $G_\chi\backslash\mathcal{B}(G_\chi)$ such that transition matrices for f_φ and f_χ relative to these orderings are $M_\varphi = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} = M_\chi$. These transition matrices are, trivially, weakly equivalent. But, Barge and Diamond [6] show that $\mathcal{T}_\varphi$ and $\mathcal{T}_\chi$ are not homeomorphic using weak equivalence of associated substitutions. Thus $\varprojlim\{G_\varphi, f_\varphi\}$ is not homeomorphic to $\varprojlim\{G_\chi, f_\chi\}$.

One-Dimensional Hyperbolic Attractors

We first state the following result due to Williams [24], [25].

THEOREM 3.24. *If $g : X \to X$ is a diffeomorphism of a manifold having a connected one-dimensional hyperbolic attractor Ω then there exist two positive integers k and ℓ, a compact branched one-manifold G which is topologically a bouquet of circles, and an expansive immersion $f : G \to G$ where the branch point of G is fixed under f, every point of G is non-wandering under f, and every point of G has a neighborhood* *whose image under f^ℓ is an arc such that $g|_\Omega^k$ is topologically conjugate to the shift homeomorphism induced by f.*

Let $g : X \to X$ be a diffeomorphism of a manifold with one-dimensional connected hyperbolic attractor Ω. By Theorem 3.24, there exist two positive integers k and ℓ, a compact branched one-manifold G which is topologically a bouquet of circles, and an expansive immersion $f : G \to G$ where the branch point of G is fixed under f, every point of G is non-wandering under f, and every point of G has a neighborhood whose image under f^ℓ is an arc such that the shift homeomorphism induced by f is topologically conjugate to $g|_\Omega^k$.

Assume G is endowed with the arc structure system that the differentiable structure on G naturally induces. Williams [25] showed that if I is an arc in G then there exists a positive integer j such that $G \subset f^j(I)$, f as above. So f^ℓ is a collapsing surjective immersion that fixes the branch point of G.

Let $g' : X' \to X'$ be a diffeomorphism of a manifold with one-dimensional connected hyperbolic attractor Ω'. By Theorem 3.24, there exist two positive integers k' and ℓ', a compact branched one-manifold G' which is topologically a bouquet of circles, and an expansive immersion $f' : G' \to G'$ where the branch point of G' is fixed under f', every point of G' is non-wandering under f', and every point of G' has a neighborhood whose image under $(f')^{\ell'}$ is an arc such that the shift homeomorphism induced by f' is topologically conjugate to $g'|_{\Omega'}^{k'}$.

Assume G' is endowed with the arc structure system that the differentiable structure on G' naturally induces. Then $(f')^{\ell'}$ is a surjective collapsing immersion that fixes the branch point of G'.

Assume Ω is homeomorphic to Ω'. Since $g|_{\Omega}^{k}$ is topologically conjugate to the shift map $\hat{f}$ induced by f and $g'|_{\Omega'}^{k'}$ is topologically conjugate to the shift map $\hat{f}'$ induced by f', $\varprojlim\{G, f\}$ is homeomorphic to $\varprojlim\{G', f'\}$. Thus $\varprojlim\{G, f^{\ell}\}$ is homeomorphic to $\varprojlim\{G', (f')^{\ell'}\}$.

Let M_f be the transition matrix for f relative to an ordering of the components of $G\backslash\mathcal{B}(G)$ and let $M_{f'}$ be the transition matrix for f' relative to an ordering of the components of $G'\backslash\mathcal{B}(G')$.

PROPOSITION 3.25. *If Ω and Ω' are orientable then M_f is weakly equivalent to $M_{f'}$.*

PROOF. Since Ω and Ω' are orientable, $\varprojlim\{G, f^{\ell}\}$ and $\varprojlim\{G', (f')^{\ell'}\}$ are orientable. So $M_{f^{\ell}}$ is weakly equivalent to $M_{(f')^{\ell'}}$ by Theorem 3.1. Since $M_{f^{\ell}} = (M_f)^{\ell}$ and $M_{(f')^{\ell'}} = (M_{f'})^{\ell'}$, M_f is weakly equivalent to $M_{f'}$. □

Assume $h_{\text{top}}(g|_{\Omega}) = \log(\omega)$ and $h_{\text{top}}(g'|_{\Omega'}) = \log(\omega')$. Using results of Williams [24], there exists a connected finite graph K and an aperiodic collapsing surjective immersion $h : K \to K$ where $h(\mathcal{B}(K)) = \mathcal{B}(K)$ such that the shift homeomorphism $\hat{h}$ induced by h is topologically conjugate to $g|_{\Omega}$. Bowen [10] showed that the the topological entropy of the induced shift homeomorphism on an inverse limit of a compactum with single surjective bonding map is equal to the topological entropy of the bonding map. By Theorem 2.4 and above result, $h_{\text{top}}(h) = h_{\text{top}}(\hat{h}) = h_{\text{top}}(g|_\Omega) = \log(\omega)$. If M_h is the transition matrix for h relative to an ordering of the components of $K\backslash\mathcal{B}(K)$ then the spectral radius of M_h is ω. Since M_f is aperiodic, ω is a Perron number. Similarly, ω' is a Perron number.

PROPOSITION 3.26. *There exist Perron numbers, α and β, and positive integers, m and n, such that $\alpha\omega = (\omega')^m$ and $\beta\omega' = \omega^n$.*

PROOF. By Theorem 2.4, $h_{\text{top}}(\hat{f}) = h_{\text{top}}(g|_\Omega^k) = k\log(\omega) = \log(\omega^k)$ and $h_{\text{top}}(\hat{f}') = h_{\text{top}}(g'|_{\Omega'}^{k'}) = k'\log(\omega') = \log((\omega')^{k'})$. Since the topological entropy of the induced shift homeomorphism on an inverse limit of a compactum with single surjective bonding map is equal to the topological entropy of the bonding map, $h_{\text{top}}(f) = h_{\text{top}}(\hat{f}) = \log(\omega^k)$ and $h_{\text{top}}(f') = h_{\text{top}}(\hat{f}') = \log((\omega')^{k'})$. Then $h_{\text{top}}(f^\ell) = \ell\log(\omega^k) = \log(\omega^{k\ell})$ and $h_{\text{top}}((f')^{\ell'}) = \ell'\log((\omega')^{k'}) = \log((\omega')^{k'\ell'})$. So $\lambda_{f^\ell} = \omega^{k\ell}$ and $\lambda_{(f')^{\ell'}} = (\omega')^{k'\ell'}$. By Theorem 3.22, there exists a Perron number a and a positive integer η such that $a\lambda_{f^\ell} = \lambda_{(f')^{\ell'}}^\eta$. Let $\alpha = a\omega^{k\ell-1}$, which is a Perron number since ω is a Perron number, and let $m = k'\ell'\eta$, which is a positive integer. Then $\alpha\omega = a\lambda_{f^\ell} = \lambda_{(f')^{\ell'}}^\eta = ((\omega')^{k'\ell'})^\eta = (\omega')^m$. Similarly, there exists a Perron number β and positive integer n such that $\beta\omega' = \omega^n$. □

The above Perron relationship implies the weaker result that $\mathbb{Q}(\omega) = \mathbb{Q}(\omega')$.

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