A dynamic programming model for range management decisions with suggestions for further research
by William Albert Bussing

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE in Agricultural Economics
Montana State University

Abstract:
This study is an attempt to formulate a dynamic programming decision model of the management process of a range-livestock operation and to establish the empirical measurements required for the application of the model.

The dynamic programming formulation is used for the model because the range management problem may be accurately represented by a multistage decision process.

The analysis is performed on a per-acre basis on the level of the individual ranch. The model determines an optimal policy for range improvement, through fertilization and artificial seeding, and for grazing practice, through the adjustment of the stocking rate. The optimal policy is stated in terms of decision rules that are dynamic extensions of the static equimarginal principle.

An example is presented to demonstrate the derivation of the decision rules from a model with specified functional forms. The properties of the decision rules are examined for changes in the system parameters and relevant prices.

The data requirements for the application of the model are presented. It is necessary to specify the response of livestock output to changes in moisture and nutrient levels, the stocking rate, and the specie composition of the range area. The effects of each decision on the future production of the range area must also be measured.

The measurement problems of the model may be classified as: (1) problems that are independent of animal effects; (2) problems that are dependent on animal effects, but not on animal performance; and (3) problems that are dependent on animal performance. It is suggested that problems of the first and second type may be resolved in small-plot experiments; problems of the third type must be resolved in grazing trials.

The importance of this study is the presentation of the empirical measurements required for future research. The problem of range management must be approached in a unified research effort by physical scientists as well as economists. This study expresses the economic viewpoint on the relevant information and appropriate data for such research.

STATEMENT OF PERMISSION TO COPY

In presenting this thesis in partial fulfillment of the requirements for an advanced degree at Montana State University, I agree that the Library shall make it freely available for inspection. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by my major professor, or, in his absence, by the Director of Libraries. It is understood that any copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Signature William Albert [illegible]

Date July 21, 1970

A DYNAMIC PROGRAMMING MODEL FOR RANGE MANAGEMENT DECISIONS WITH SUGGESTIONS FOR FURTHER RESEARCH

by

WILLIAM ALBERT BUSSING

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree

of

MASTER OF SCIENCE

in

Agricultural Economics

Approved:

R. J. McConnen

Head, Major Department

Oscar R Burt

Chairman, Examining Committee

H Goering

Graduate Dean

MONTANA STATE UNIVERSITY
Bozeman, Montana

August, 1970

ACKNOWLEDGMENTS

The author wishes to express his special thanks and appreciation to Dr. Oscar R. Burt, Dr. Richard O. Wheeler, and Mr. Maurice C. Taylor for their guidance and critical review of this thesis. The assistance and encouragement of the staff and fellow graduate students is likewise appreciated.

The major portion of this study was funded by the Economic Research Service of the United States Department of Agriculture, and the author gratefully acknowledges this aid.

Any error or omissions in this study, of course, are the responsibility of the author.

TABLE OF CONTENTS

LIST OF FIGURES

ABSTRACT

This study is an attempt to formulate a dynamic programming decision model of the management process of a range-livestock operation and to establish the empirical measurements required for the application of the model.

The dynamic programming formulation is used for the model because the range management problem may be accurately represented by a multi-stage decision process.

The analysis is performed on a per-acre basis on the level of the individual ranch. The model determines an optimal policy for range improvement, through fertilization and artificial seeding, and for grazing practice, through the adjustment of the stocking rate. The optimal policy is stated in terms of decision rules that are dynamic extensions of the static equimarginal principle.

An example is presented to demonstrate the derivation of the decision rules from a model with specified functional forms. The properties of the decision rules are examined for changes in the system parameters and relevant prices.

The data requirements for the application of the model are presented. It is necessary to specify the response of livestock output to changes in moisture and nutrient levels, the stocking rate, and the specie composition of the range area. The effects of each decision on the future production of the range area must also be measured.

The measurement problems of the model may be classified as: (1) problems that are independent of animal effects; (2) problems that are dependent on animal effects, but not on animal performance; and (3) problems that are dependent on animal performance. It is suggested that problems of the first and second type may be resolved in small-plot experiments; problems of the third type must be resolved in grazing trials.

The importance of this study is the presentation of the empirical measurements required for future research. The problem of range management must be approached in a unified research effort by physical scientists as well as economists. This study expresses the economic viewpoint on the relevant information and appropriate data for such research.

CHAPTER I

INTRODUCTION*

In 1969, Montana ranchers produced $319.4 million worth of livestock and livestock products, an increase of 11 percent over the previous year. Cattle and calves produced $292.2 million of the total, an increase of $27 million or 10 percent over the previous year. In 1968, livestock accounted for 62.1 percent of the total cash receipts for all farm commodities and cattle and calves accounted for 51.3 percent.

The marketings of cattle declined between 1968 and 1969, so the increase in cash receipts resulted from an increase in the average price. In 1967, the average prices per hundredweight for cattle and calves were $22.00 and $28.30, respectively. In 1968, the average prices rose to $22.80 and $29.20, and in 1969, to $25.70 and $32.70.

About 55 percent of Montana's total land area is used for grazing, and Montana ranchers are interested in methods of improving their range areas. One method is fertilization, and the cost of nitrogen fertilizer has dropped from 13 or 14 cents to about 10 cents a pound. It is hoped that the lower costs of fertilization and the higher prices of cattle make range improvement by fertilization a practical and profitable alternative. Even at 10 cents a pound, however, nitrogen fertilizer represents a considerable investment in range improvement.

---

*The figures reported are from the April 30, 1970 bulletin of the Montana Crop and Livestock Reporting Service.

Before making a decision on such a range improvement practice, the ranchers need to know the effect of the range improvement on their profits. This study is directed toward providing that information.

The Problem

The output of a range-livestock operation may be increased in several ways: by improving the forage yield of the range, by improving the livestock characteristics, and by improving the grazing management practices of the rancher. The forage yield of the range may be improved by a variety of methods. Those methods that have been used in Montana may be classified as mechanical and chemical treatments, artificial seeding, and fertilization.

In this paper, mechanical and chemical treatments of the range, and the characteristics of the livestock are considered to be determined by purely technical criteria. That is, the production function of the range-livestock operation is assumed to specify the maximum output, for a given factor mix, which results from the use of the best treatments and type of livestock which the existing state of technical knowledge provides. Then, the yield of the range-livestock operation, in pounds of marketable livestock products, is assumed to be determined by the nitrogen and moisture levels of the soil, the specie composition of the range, and the stocking rate of livestock. The site characteristics such as topography, available water, and native soil productivity are taken as given.

The nitrogen level, or the amount of available nitrogen per unit area of soil, is a measure of the controllable nutrient of the soil. It is expected that the range yield reacts positively, at least over some values, to an increase in the nitrogen level. Nitrogen fertilization also alters the specie composition of the range. It is expected that the change is toward more desirable specie with increased nitrogen levels.

The stocking rate, or the number of head of livestock grazed on a unit area or range, is a measure of range usage* The effect of an increased stocking rate on the range yield is complex.

> Grazing animals have harmful effects on plants because of over- and under-selective grazing; compaction of soils due to trampling, which retards water infiltration and root development; uneven fertility and its removal; and outright herbage wastage due to fouling, trampling, and selective grazing. [7:p.14]

Over-selective grazing is the case in which the animal is able to select the higher quality forage for its grazing. This case results from a stocking rate that is low relative to forage availability and produces more output per animal, generally at the expense of lower output per acre. Under-selective grazing is the case in which the animal is forced to graze low quality forage. This case results from a high

---

*When considering different ages and classes of livestock, the stocking rate will have to be adjusted by animal unit equivalents.

stocking rate and produces lower output per animal, although the output per acre may increase. The effect of increased stocking rates on specie composition is expected to be negative.

This is the central problem of grazing management : the compromise between gain per animal and gain per acre. Generalized curves have been suggested by Harlan [3:p.140] and Mott [5:p.606], which clearly illustrate this compromise. Figure 1 shows the curves presented by Mott. The vertical axis represents the output per animal (solid line) and the output per acre (dashed line), though the scale is not the same. The horizontal axis represents the stocking rate, running from no animals to heavy stocking rates. It is readily seen that at the maximum output per acre the output per animal is falling off rapidly, which results in animals with lower market value.

The net revenue function of the rancher is a function of both the output per acre and the output per animal, which are both functions of the stocking rate. Such a revenue function may be written as

$$R(s) = \Phi[x(s),y(s)]$$

where s is the stocking rate, x(s) is the output per acre, and y(s) is the output per animal. Figure 1 shows that over some range of values, the output per acre increases with increased stocking rates. The output per animal decreases for any increase in the stocking rate. An increase in either output per acre or output per animal results in an increase in revenue. These relationships may be expressed as

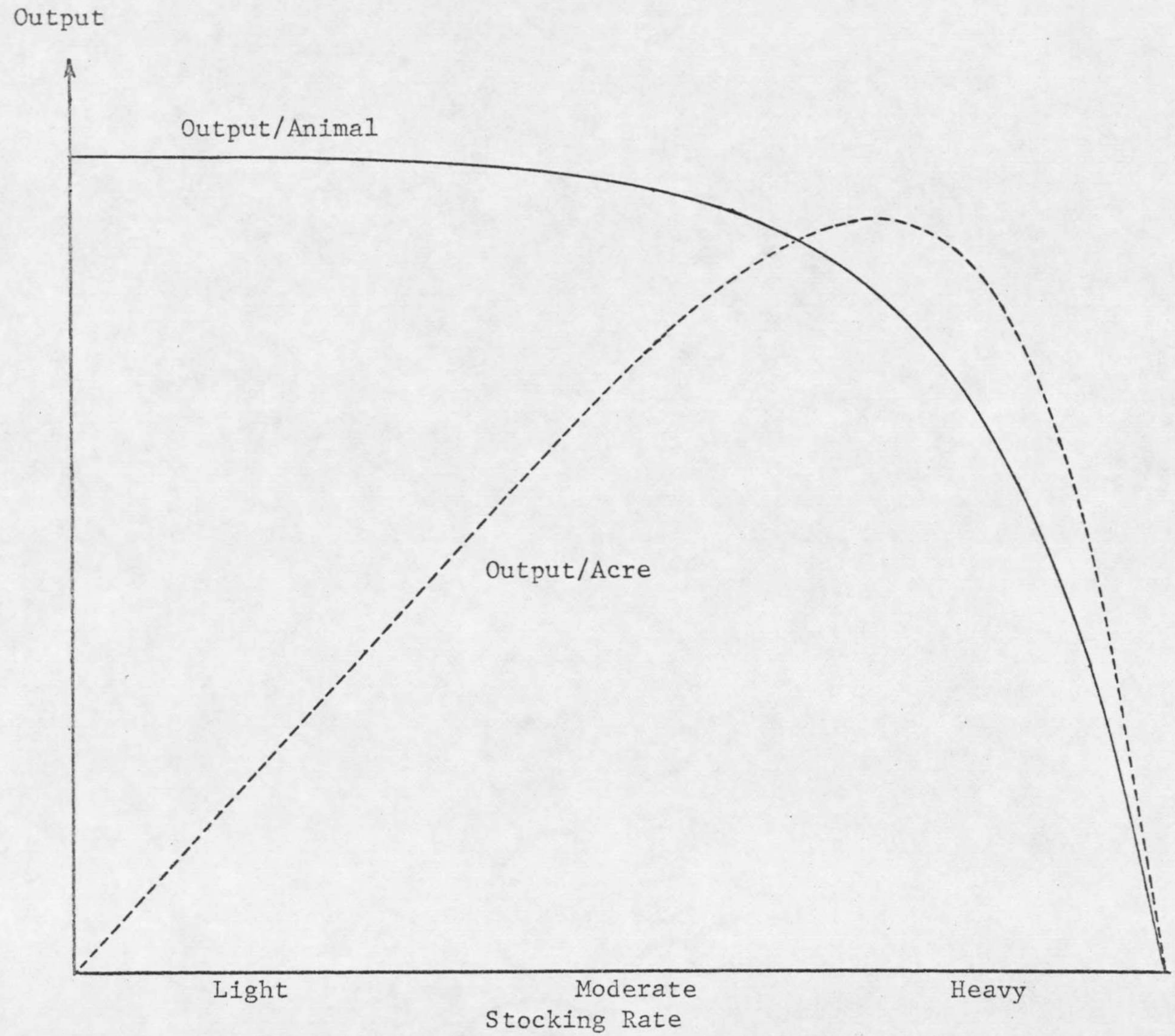

Figure 1. The influence of the stocking rate on output per animal and output per acre.

$$\frac{\partial y}{\partial s} < 0, \ \frac{\partial \Phi}{\partial x} > 0, \ \frac{\partial \Phi}{\partial y} > 0,$$

$$\frac{\partial^2 x}{\partial s^2} < 0 \text{ and } \frac{\partial x}{\partial s} > 0, \ s < s^*,$$

where s* is the stocking rate that results in the maximum output per acre ($\frac{\partial x}{\partial s} = 0$).

In a static framework, the rancher seeks to maximize the revenue function, R(s), which implies that $\Phi[x(s),y(s)]$ is a maximum. The necessary condition for the maximum is

$$\frac{\partial \Phi}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial \Phi}{\partial y} \cdot \frac{\partial y}{\partial s} = 0.$$

As the stocking rate increases toward s*, the output per acre increases at a decreasing rate and the output per animal falls at an increasing rate. The optimum stocking rate is that at which the incremental addition to revenue due to an incremental increase in output per acre is equal to the loss in revenue due to an incremental decrease in output per animal. The optimum stocking rate will not be s* unless an increase in the stocking rate does not result in a decrease in the output per animal or in revenue. The management problem is to determine the optimum stocking rate.

The rancher approaches this problem with his ability to adjust the stocking rate through the retention of young stock as yearlings.* In the

*The term young stock will be used to refer to the offspring of the mature livestock from birth to one year of age, such as calves. The term yearling will be used to refer to the offspring from age one to age two, such as yearling heifers and steers.

face of a good year, when moisture is expected to be high and forage yield larger than normal, the rancher may hold yearlings in order to take advantage of the higher yield. Before an expected bad year, the rancher may sell all of his young stock in order to reduce the grazing pressure on the expected low forage yield.

The specie composition of the range might be expressed as an index of the relative abundance of the various specie that comprise the forage. The index must reflect the desirability of the forage, including such things as energy, protein and fat content, plant characteristics such as root structure, and so on. It is then expected that the range yield reacts positively to an improvement in the index of specie composition. If a suitable index cannot be constructed, the specie composition may have to be expressed as a vector of forage quality measurements.

The moisture level is a measure of the available water in a unit area of the soil. It is expected that the range yield reacts positively to an increase in the moisture level. The moisture level, measured, for example, as soil moisture in the spring, is assumed to be determined by the precipitation during the year. The precipitation is assumed to be determined exogenously in a stochastic manner.

The yield of the range-livestock operation is to be expressed as the output of animal products per acre, which may be measured directly or based upon measurements of output per animal and the number of animals per acre, and is to be measured at the end of the production period. The revenue function is the market value of the output that is sold, measured on a per acre basis.

The assumption of a stochastic moisture level variable imposes a probability distribution on the return function*, but in order to achieve analytical results it is desirable to reduce the stochastic return function to a meaningful deterministic value. This is accomplished by assuming some specific behavior on the part of the rancher. For simplicity, it is assumed here that the rancher is concerned with his expected returns, the mean value of the return function over the probability distribution associated with it. The expected return function does not depend on the moisture level.

The managerial control of the rancher takes the form of alternatives through which he may influence his expected returns: he may adjust the nitrogen level of the soil by applying fertilizer, he may adjust the stocking rate by adding or removing livestock from the range area, and he may adjust the specie composition by reseeding the range.

The application of fertilizer entails some fixed cost per acre for application plus an increasing cost function of the amount of fertilizer applied per acre.

*The fact that the return function itself is stochastic allows the imposition of subjective probabilities on the returns. For example, it might be desirable, on a behavioral basis, to weight higher moisture (and thereby, higher returns) by a greater probability than climatic data specifies. It is also possible to attach probabilities to the predicted future prices of livestock. The analysis remains the same.

The option of reseeding the range, if carried out, entails some fixed cost per acre for preparing and seeding the range.

It is assumed that the rancher has an initial breeding herd which may be represented by a specific stocking rate. Further, it is assumed that the rancher maintains the breeding herd at a constant level. In other words, the offspring of the breeding stock are fed for one production period and then sold, or fed for one additional production period, as yearlings, and then sold. Any young livestock held for longer than two production periods is considered to be held for replacement of older breeding stock, the older stock then being sold in place of the young. Thus, the adjustment in the stocking rate available to the rancher consists of the number of young livestock he holds over as yearlings, and entails a marketing cost for selling young stock and yearlings plus a wintering cost for holding young stock over to yearlings.*

The total cost function of the range-livestock operation is the sum of the three cost functions, and therefore depends on nitrogen application, changes in the stocking rate, and the option of reseeding.**

---

*It is conceivable that the rancher may even wish to purchase additional stock to feed during an exceptionally good year. In that case, the cost of adjusting grazing intensity also entails the purchasing costs of the additional livestock.

**The cost function, like the return function, may be stochastic, in which case the rancher is assumed to consider his <u>expected</u> costs.

It is assumed that the rancher utilizes the alternatives available to him in such a way as to increase the output of his operation, so that the addition to his expected revenue exceeds his costs, and maximizes discounted expected profit over his planning horizon. The increased output may take the form of heavier animals or more animals or both. The objective of this paper is to establish a conceptual framework within which a rational policy of utilizing the alternatives may be found.

Dynamic Programming

The dynamic programming formulation was selected for the problem structure mainly because its characteristic multistage decision process most nearly approximates the decision process of the rancher.

> A multistage decision process is characterized by the task of finding a sequence of decisions which maximizes (or minimizes) an appropriately defined objective function. The <u>stage</u> is the interval into which the process is divided, a decision being made at each stage in the sequence of stages comprising the decision process. The <u>state</u> of the process, at a particular stage, describes the condition of the process, and is defined by the magnitudes of state variables and/or qualitative characteristics. Decision making at a given stage controls the state in which the process will be found in the following stage, the control being either deterministic (with certainty) or stochastic (with a probability distribution). [2:p.121]

These concepts will become clearer as the present model is formulated in the next chapter.

Several other characteristics of the dynamic programming formulation are also desirable. The dynamic nature of the formulation is important, as the continued operation of a range area is a dynamic problem. As noted above, the decision process may be stochastic, and the formulation serves to determine an optimum decision under uncertainty. The specific decision problem of any particular rancher is imbedded in a general formulation which specifically embodies any length of planning horizon and any initial state of the range operation, and thus contains two dimensions of sensitivity analysis.

The method of solution will be "value iteration," in which the maximum value of the objective function is computed at each stage. The principal advantage of this method is that the general solution displays, in its development, the optimal decision for each stage of the process.

The investigation will begin with the statement of the range management problem, the general assumptions and the relevant variables. The problem will then be stated in terms of a multistage decision process and formulated as a dynamic programming problem. The solution of the problem will be found, in the form of the necessary conditions for optimization. An example will be presented to illustrate the use of the model with specified functional forms. The properties of the solution of the example will be considered. Finally, the problems of empirical measurements will be outlined and suggestions for their resolution will be made.

CHAPTER II

THE MODEL

Formulation

In order to utilize the dynamic programming algorithm, it is necessary to formulate the problem stated above in terms of a multistage decision process.

Define a stage of the process to be one year, ending in late fall when the livestock is sold. The state of the process, at any given stage, is determined by the nitrogen level, the stocking rate, and the index of specie composition.

The functional form of the decision process for dynamic programming is restricted only by the Markov requirement:

> After any number of decisions, say k, the effect of the remaining N-k stages of the decision process upon the total return depends only upon the state of the system at the end of the k-th decision and subsequent decisions. [1:p.54]

In other words, at any time during the decision process, the rancher's future decisions depend only on the value of the state variables at the end of the last production period. It is assumed that the formulation of this model meets the Markov requirement, although problems with accomplishing the Markov requirement can usually be avoided by introducing additional state variables.

As a matter of notation, a superscript always refers to the time period or stage number. Exponents will always be written with parentheses.

The following variables are defined on a per acre basis to allow for any ranch size:

y^t = expected range yield at the end of period t, in animal products per acre,

x_1^t = nitrogen level of the range at the end of period t, in pounds per acre,

x_2^t = additional stocking rate in period t+1, due to yearlings held, in head per acre,

x_3^t = index of specie composition at the end of period t,

θ_1^t = nitrogen application at the beginning of period t, in pounds per acre,

θ_2^t = adjustment in stocking rate at the end of period t, due to the sale of young livestock, in head per acre,

θ_3^t = reseeding option, carried out at the beginning of period t.

To clarify the relationships of the state and decision variables to the stages, Figure 2 is presented. The horizontal line represents time, future time to the right. The occurrences of the state and decision variables are indicated by the variables' symbols.

A very important aspect of the multistage decision process is the effect of the decision variables at a given stage upon the state variables in the next stage. These effects, expressed as transformations of the state variables, provide the dynamic link between stages. The transformations are indicated in Figure 2 by the dashed arrows.

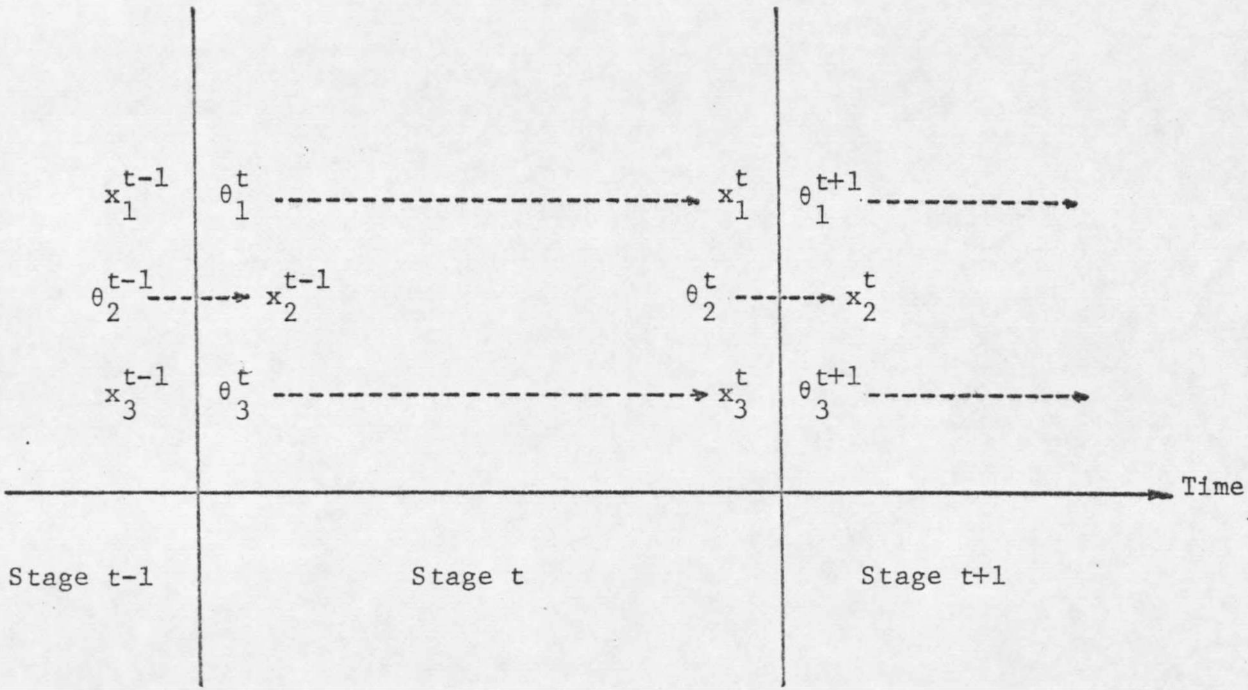

Figure 2. The relationship of the state and decision variables to the stages.

It is expected that the nitrogen level at the end of each stage is given by some percentage of the nitrogen level at the beginning of the stage, or, some percentage of carryover of the sum of the nitrogen level at the end of the previous stage and the nitrogen applied at the beginning of the stage, that is,

$$(1)\quad x_1^t = \alpha(x_1^{t-1} + \theta_1^t),\quad 0 \leq \alpha \leq 1,\quad (t=1,2,\ldots,T),$$

where α is the percentage of carryover.*

The stocking rate in each stage is given by the sum of breeding stock, offspring of that stock, and yearlings held over from the previous stage, each on a head per acre basis. Breeding stock per acre, is assumed to be constant at K. Then, assuming a proportional number of weaned young stock, say γ, young stock per acre is γK. The additional stocking rate from yearlings in each stage is simply the difference between the young crop of the previous stage and the young stock sold at the end of the previous stage, or

$$(2)\quad x_2^t = (\gamma K - \theta_2^t),\quad \theta_2^t \leq \gamma K,\quad (t=1,2,\ldots,T),$$

The specie composition at the end of each stage is expected to depend on the nitrogen level, the stocking rate, and the specie composition of the range at the end of the previous stage, and also the nitrogen

---

*The percentage of carryover may depend on the level of moisture, the level of nitrogen and the production of forage, but for simplicity is assumed to be independent of such effects.

application and reseeding operation at the beginning of the stage. There is not sufficient information on the changes in specie composition to make more than a general transformation statement, such as

$$(3)\quad x_3^t = \tau_3^t(x_1^{t-1}, x_2^{t-1}, x_3^{t-1}; \theta_1^t, \theta_3^t | K, \gamma K), \quad (t=1,2,\ldots,T).$$

According to the assumptions, the expected returns of the range-livestock operation at the end of period t may be written as

$$(4)\quad R^t(X^{t-1}; \theta^t | K, \gamma K) = p^t y^t(X^{t-1}; \theta^t | K, \gamma K), \quad (t=1,2,\ldots,T),$$

where X^{t-1} represents the vector $(x_1^{t-1}, x_2^{t-1}, x_3^{t-1})$, θ^t represents the vector $(\theta_1^t, \theta_2^t, \theta_3^t)$, and p^t is the market price of livestock.* It should be noted that the stocking rate of the breeder stock and the young stock, K and γK, are assumed to be constant and become parameters. Also, the stocking rate of the yearling stock, x_2^{t-1}, appears explicitly as a result of the assumption requiring their sale at the end of the period.

The rancher incurs his costs, except the marketing costs, as much as a year before he realizes the revenue that they produce, so the marketing cost and the revenue function should be discounted to their present values. Assuming that they are, the cost function may be written as

*The revenue function expressed in equation (4) should be considered to be completely general. The price, p^t, may be a vector of the prices of livestock of different classes and ages and the production function, y^t, would then be a vector of the output produced by the various livestock. The revenue function would then be the simple sum of the revenue due to young stock, yearlings, and mature stock of each class.

(5) $C^{t}(x_2^{t-1};\Theta^{t})$, $(t=1,2,\ldots,T)$.*

Again, the stocking rate of the yearlings enters the cost function as a result of the assumption that they must be sold and thus entail a marketing cost.

The rancher is assumed to act so as to maximize his expected profits

(6) $\pi^{t}(X^{t-1};\Theta^{t}) = R^{t}(X^{t-1};\Theta^{t}) - C^{t}(x_2^{t-1};\Theta^{t})$, $(t=1,2,\ldots,T)$.

In order to move to the dynamic programming formulation, the problem must be restated to allow recursion "backwards", from the last stage of the planning horizon to the current year. This is accomplished by letting, n, the number of stages remaining in the rancher's planning horizon, become a variable.** Define $f_n(X^{n+1})$ to be the discounted expected value of the future profits from the n-stage decision process under an optimal policy when the initial state is given by the vector $X^{n+1} = (x_1^{n+1}, x_2^{n+1}, x_3^{n+1})$. The future profits will be discounted by the discount factor, β, which is the reciprocal of one plus the relevant interest rate. The problem may be restated as follows:

---

*The cost function, like the revenue function, is very general and may comprise different costs of different classes of livestock.

**If the rancher's planning horizon is T years, from year 1 to year T, then at the end of year T-1, there is one production period left in the planning horizon, so n is equal to one. The transformation between n and t is:

$$n=T-1 \quad , \quad \begin{array}{l}(n=N,N-1,\ldots,1,0),\\ (t=1,2,\ldots,T-1,T),\end{array}$$

$$N=T-1.$$

(7) Maximize $f_N(X^{N+1}) = \sum_{i=0}^{N} (\beta)^{N-i}[R^i(X^{i+1};\theta^i) - C^i(X_2^{i+1};\theta^i)]$,

(8) subject to $x_1^n, x_2^n, x_3^n \geq 0$,

(1a) $x_1^n = \alpha(x_1^{n+1} + \theta_1^n)$,

(2a) $x_2^n = \delta(\gamma K - \theta_2^n)$,

(3a) $x_3^n = \tau_3^n(X^{n+1};\theta^n)$, $(n=0,1,2,\ldots,N)$.*

The general recursion relations in $f_n(X^{n+1})$ may be found by the application of Richard Bellman's principle of optimality:

> An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. [1;p.56]

Application of the principle to the present problem yields the recursion relation

(9) $f_n(X^{n+1}) = \underset{\theta^n}{\text{Max}} [R^n(X^{n+1};\theta^n) - C^n(x_2^{n+1};\theta^n) + \beta f_{n-1}(X^n)]$.

To clarify the derivation of equation (9), start at the end of the planning horizon, n equal to zero, and work back to the present, n, utilizing the principle of optimality at each stage. At the end of the planning horizon, the end of stage one, it is assumed that the salvage land value of the range area depends on its state and is given by

*The constraints may be written concisely in vector form as: $X^n = T(X^{n+1};\theta^n)$, $(n=0,1,2,\ldots,N)$. This implies equations (1a), (2a), and (3a).

$$R^0(x_1^1,x_2^1,x_3^1) = L(x_1^1,x_2^1,x_3^1).$$

There are no decisions possible, so the expected immediate profit is

$$f_0(X^1) = L(X^1).$$

Iterating once for the optimal policy for a one-year planning horizon, the expected profit is

$$f_1(X^2) = \underset{\theta^1}{\text{Max}}\ [R^1(X^2;\theta^1) - C^1(x_2^2;\theta^1) + \beta f_0(X^1)]$$

where X^1 is the state resulting from the transformation of the state X^2 by the decision θ^1, according to equations (1a), (2a), and (3a). The optimal policy for a one-year planning horizon is to maximize the expected immediate profit plus the discounted salvage value of the land. Continuing for planning horizons of two, three, etc. years, the iterations are:

$$f_2(X^3) = \underset{\Theta^2}{\text{Max}}\ [R^2(X^3;\Theta^2) - C^2(x_2^3;\Theta^2) + \beta f_1(X^2)],$$

$$f_3(X^4) = \underset{\Theta^3}{\text{Max}}\ [R^3(X^4;\Theta^3) - C^3(x_2^4;\Theta^3) + \beta f_2(X^3)],$$

$$(10)\qquad \vdots$$

$$f_{n-1}(X^n) = \underset{\Theta^{n-1}}{\text{Max}}\ [R^{n-1}(X^n;\Theta^{n-1}) - C^{n-1}(x_2^n;\Theta^{n-1}) + \beta f_{n-2}(X^{n-1})].$$

The preceding iterations and their solutions are those used in the value-iteration algorithm to solve the general problem. The iterations continue until n equals N, the number of years in the rancher's planning horizon.

The maximization problem indicated in each iteration may be solved by any adequate maximization technique. If the variables and functions are continuous, the method of Lagrangian multipliers may suffice. If the variables are discrete, an exhaustive search procedure is required. Any technique will lead to certain necessary conditions for maximization, and within these conditions will be found the optimal policy to be followed by the rancher.

The Necessary Conditions

The general form of the necessary conditions for the maximum of equation (9) to exist may be found by applying the discrete version of Pontryagin's maximum principle (see Appendix A). In general, the maximum, subject to the constraints expressed in equations (8), (1a), (2a), and (3a), is given by the decision policy $*\theta^n$, where the component decisions, $*\theta_1^n$, $*\theta_2^n$, $*\theta_3^n$, are the solutions of

$$\text{(A-20)} \quad \frac{\partial R^n}{\partial \theta_1^n} + \sum_{i=0}^{n-1} (\alpha\beta)^{n-i} \frac{\partial R^i}{\partial x_1^{i+1}} + \sum_{i=0}^{n-1} (\beta)^{n-i} \frac{\partial \tau_3^n}{\partial \theta_1^n} \frac{\partial R^i}{\partial x_3^n}$$

$$+ \sum_{i=0}^{n-2} \sum_{j=0}^{i} (\alpha)^{n-i-1} (\beta)^{n-j} \frac{\partial \tau_3^{i+1}}{\partial x_1^{i+2}} \frac{\partial R^j}{\partial x_3^{i+1}} = \frac{\partial C^n}{\partial \theta_1^n}$$

and

$$\text{(A-21)}\quad \frac{\partial R^n}{\partial \theta_2^n} - \beta\frac{\partial R^{n-1}}{\partial x_2^n} - \sum_{i=0}^{n-2}(\beta)^{n-i}\,\frac{\partial \tau_3^{n-1}}{\partial x_2^n}\,\frac{\partial R^i}{\partial x_3^{n-1}}$$
$$= \frac{\partial C^n}{\partial \theta_2^n} - \beta\frac{\partial C^{n-1}}{\partial x_2^n}.$$

These two decision rules are conditional upon θ_3^n, the decision to reseed. Given θ_3^n equal to zero, indicating that the reseeding is not carried out, equations (A-20) and (A-21) simultaneously determine values of θ_1^n and θ_2^n. Similarly, the equations determine some other values for θ_1^n and θ_2^n, given the decision that θ_3^n is equal to one, indicating that the reseeding is carried out. The two values of θ^n, $(\theta_1^n,\theta_2^n\theta|_3^n=0)$ and $(\theta_1^n,\theta_2^n\theta|_3^n=1)$, must be compared by substitution into the right-hand side of equation (9). One of the two values should produce a greater value of $f_n(X^{n+1})$ and the maximum operator requires the selection of the greater result. The associated vector is $*\theta^n$, the optimal decision policy.

Consider equation (A-20). Except for the summation terms on the left-hand side, of the equation, the decision rule represents the classical static equimarginal relationship between revenue and cost. The summation terms have been introduced by the dynamic formulation of the problem, through the state transformation functions.

The application of nitrogen to the soil represents a resource flow to an asset stock, namely, soil nitrogen. Because nitrogen has a two-fold effect on range yield, a direct change in quantity and an indirect change in quality through specie composition effects, the change in the value of the asset results from the two separate effects. The first summation term is the discounted expected future returns due to the percentage of an increment of nitrogen retained in the soil, through its direct effect on forage yield. The second summation term is the discounted expected future returns due to an increment of nitrogen applied, through its effect upon the specie composition of the range in all future periods. The third summation term is the discounted expected future returns due to the percentage of an increment of nitrogen retained in the soil, through its effect upon the specie composition of the range in each subsequent period. All of these effects have been assumed to be positive, so the three summation terms are positive. The sum of the terms is the present value of an incremental investment in soil nitrogen.

The first two terms on the left-hand side of equation (A-20) represent the response of the revenue function to a change in forage yield due to nitrogen. The last two terms represent the response of the revenue function to a change in specie composition due to nitrogen. The first and third terms represent the total response of the revenue function to applied nitrogen; the second and fourth terms represent the total response to soil nitrogen retained.

The timing of the effects is indicated by indices on the summation signs. The application of nitrogen directly increases the revenue in stage n and changes the specie composition in the future, beginning at stage n-1. The percentage of nitrogen retained directly increases the revenue in stage n-1 and changes the specie composition in the future, beginning at stage n-2.

Equation (A-21) exhibits some similarity. Like the first equation, it is a modified form of the static equimarginal principle. By assumption, however, the stocking rate may not be maintained at an increased level by indefinitely retaining the same livestock: yearlings have to be sold. As a result, the stocking rate decision spans only two production periods: the rancher may sell young stock in the present period, or yearlings in the next period. Thus, the rancher considers any difference in revenue, and the associated costs, that might result from the retention of young stock to be sold as yearlings. The first two terms on the left-hand side of equation (A-20) are this difference in revenue, and the right-hand side is the difference in costs.

The decision to increase the stocking rate also influences the value of a potential asset, namely, forage yield, through the effect of increased stocking rates on the specie composition of the range. The summation term in equation (A-21) is the discounted expected future returns due to the stocking rate in the next period through its effect upon the specie composition of the range in the future. The effect of livestock on specie composition has been assumed to be negative, so the third term represents a loss in the future revenue due to yearlings.

Utilizing the one-to-one (inverse) correspondence of young stock sold and yearlings held,

$$\frac{\partial x_2^n}{\partial \theta_2^n} = -1,$$

equation (A-21) may be written

$$\text{(A-21a)} \quad \frac{\partial R^n}{\partial \theta_2^n} + \beta\frac{\partial R^{n-1}}{\partial \theta_2^n} + \sum_{i=0}^{n-2} (\beta)^{n-i} \frac{\partial \tau_3^{n-1}}{\partial \theta_2^n} \frac{\partial R^i}{\partial x_3^{n-1}} = \frac{\partial C^n}{\partial \theta_2^n} + \beta\frac{\partial C^{n-1}}{\partial \theta_2^n}.$$

Now the left-hand side of equation (A-21a) is clearly the total change in revenue associated with an increment in the decision to sell young stock, and the right-hand side is the total change in cost.

Keeping the foregoing discussion in mind, the decision rules may be stated as:

(1) The present value of the marginal revenue produced by fertilizer in all future periods must equal the marginal cost of fertilizer.

and

(2) The present value of the marginal revenue produced by the sale of young stock must equal the marginal cost of selling young stock.

Again, these decision rules are conditional upon the decision to reseed the range area.

The present value of the expected profits under an optimal policy, given by equation (9), is net of all costs except interest on the investment in land and a return to the rancher. The derivation of an optimal policy of fertilization and stocking, however, also yields an estimate of the value of the range land in terms of the capitalized future income that can be derived from the land. Thus, the maximum price per acre that can be paid for land can be found by using the optimum solution to equation (9) and specifying the desired return on investment and management of the rancher.

Sensitivity Analysis

In any optimization problem, it is desirable to be able to investigate the response of the optimal decisions and returns to changes in the parameters of the system. In economic analysis, this capability, called sensitivity analysis, is particularly important because the nature of the economic system presents significant measurement problems. The dynamic programming formulation embodies sensitivity analysis in two dimensions: the values of the state variables and the number of stages. [6:p.66]

When the dynamic programming problem has been solved in its general form, as in equations (A-20) and (A-21), the optimal decision policy is stated in terms of the initial values of the state variables, the vector X^{n+1}, in the present case. Given any initial values of the state variables, the solution determines a decision policy at each stage that will optimize the objective function.

This is actually a double advantage. First, the model is generalized to livestock operations with any initial nitrogen level, grazing intensity, and specie composition. Second, for a particular operation the response of the optimal decisions and returns may be determined for <u>any</u> variation in the values of the state variables.

In the other dimension, the solution of the dynamic programming problem in its general form is stated in terms of a general planning horizon, N. Again, this generalizes the problem to livestock operations with <u>any</u> length of planning horizon, and for a particular operation the response of the optimal decisions and returns may be determined for <u>any</u> variation in the length of the planning horizon.

Although not embodied in the formulation of the dynamic programming problem, there is another dimension of sensitivity analysis which is useful in the present problem. There are several prices and costs which must be predicted for each stage of the planning horizon, and there will be error and uncertainty involved in these predictions. The response of the optimal solution to variations in such prices may be determined in order to examine the stability of the solution.

Convergence

The property of convergence is closely related, in computation, to the dimension of sensitivity analysis on the planning horizon. If the profit function defined in equation (7) is bounded and does not depend on the stage, the presence of the discount factor insures that, for large values of n, the policy vector converges to a single policy vector that is independent of the stage. Such a convergent policy is very useful for economic interpretation of changes in the parameters of the system. A sufficient, but not necessary, condition for the convergence of the model presented is that the prices and costs remain constant over time.

Another type of convergence results if the values of the state variables approach some steady-state values. The conditions for convergence in this sense are not stated, but may be derived from the necessary conditions for maximization. The convergence of the state variables has some intuitive appeal for this problem for these reasons: (1) if there is a well-defined optimum value of soil nitrogen, it is reasonable to expect that, in the long run, the nitrogen state variable will approach that value, perhaps corrected by the dynamics of the situation; (2) with a steady-state value of soil nitrogen, it is to be expected that the specie composition of the range will approach an equilibrium state; and (3) with steady-state values of both nitrogen and specie composition, and thereby a steady-state forage yield, there must be some optimum steady-state grazing practice. So, it is probably not unreasonable to expect convergence in this sense and the existence of steady-state values will be considered in an example to follow.

Uncertainty

The dynamic programming formulation allows for uncertainty in the form of stochastic transformation functions. In this paper, the transformation functions were assumed to be deterministic for the ease of analytical manipulation. However, in an application of the model, such things as the carryover of nitrogen or the change in specie composition might be considered to be stochastic processes. Another state variable might also be included, in the form of a moisture level measure, which by assumption, is stochastic.

The addition of a moisture level variable is considered to be an unambiguous improvement in the practical value of the model, and will serve as an example of a stochastic process. In order to include such a variable, the production function would have to be restated as a function of moisture level.

Let x_4^n be the soil moisture of the range area, measured at the end of period n. Assume that x_4^n is stochastic and distributed according to the conditional probability density function, $Pr(y|x_4^{n+1})$; where y is a dummy variable. Then the production function and the revenue function become stochastic, distributed with the conditional probability $Pr(y|x_4^{n+1})$:

$$R^n(X^{n+1};\Theta^n) \sim Pr(y|x_4^{n+2}), \quad \text{where } X^{n+1} = (x_1^{n+1}, x_2^{n+1}, x_3^{n+1}, x_4^{n+1}).$$

Assuming that the rancher seeks to maximize his discounted future <u>expected</u> profits, the problem may be restated as

$$\text{Maximize } f_n(X^{n+1}) = E \sum_{i=0}^{N} (\beta)^{N-i}[R^i(X^{i+1};\Theta^i) - C^i(x_2^{i+1};\Theta^i)] \quad ,$$

$$\text{subject to } x_1^n, x_2^n, x_3^n, x_4^n \geq 0,$$

$$x_1^n = \alpha(x_1^{n+1} + \theta_1^n),$$

$$x_2^n = (\gamma K - \theta_2^n),$$

$$x_3^n = \tau_3^n(X^{n+1};\theta_2^n),$$

$$x_4^n \sim \Pr(y|x_4^{n+1}), \quad (n=0,1,2,\ldots,N),$$

where E is the mathematical expectation operator. Assuming that the moisture level variable, x_4^n, is discrete*, equation (9) may be rewritten as

$$f_n(X^{n+1}) = \underset{\Theta^n}{\text{Max}}\, E[R^n(X^{n+1};\Theta^n) - C^n(x_2^{n+1};\Theta^n) + \beta f_{n-1}(X^n)],$$

or,

$$f_n(X^{n+1}) = \underset{\Theta^n}{\text{Max}}\, [\bar{\bar{R}}^n(X^{n+1};\Theta^n) - C^n(x_2^{n+1};\Theta^n) + \sum_{y} {}_n\beta f_{n-1}(X^n)\Pr(y\ x_4^{n+1})],$$

where $\bar{\bar{R}}^n(X^{n+1};\Theta^n)$ is the mathematical expected value of $R^n(X^{n+1};\Theta^n)$. The solution of this problem may be found in the same manner as in the deterministic case, but the generality of the probability distribution and the expectation operator prevent the simple exposition of analytical results.

---

*When the stochastic variable is discrete, the mathematical expectation of any function Z is defined by:

$$E[Z] = \sum_{y} Z\cdot\Pr(y|x^{n+1}).$$

If the stochastic variable is continuous, the summation becomes and integral, or,

$$E(Z) = \int_y Z \circ Pr(y|X_4^{n+1}) \circ dy.$$

The two cases are completely analogous, and the analysis remains the same.

CHAPTER III

AN EXAMPLE

In order to illustrate the use of the model, a simplified example is presented. The example is not intended to be definitive, although the functional forms are thought to be reasonably accurate. Because there is so little known about the causes and effects of specie composition and its changes, the state and decision variables associated with specie composition are not included. The terminal value of land is taken to be constant with respect to the state variables and, since then its magnitude is not important for decision-making, is set equal to zero. (For a sufficiently long planning horizon, the terminal value of land has no effect because of discounting.) The statistical reduction of the production function with respect to moisture level is assumed to have been accomplished, so that the functions presented may be considered to be the mean-value functions.

The production surface for livestock is derived from the generalized curves suggested by Harlan [3:p.140] and Mott [5:p.606]. For a given level of forage yield, the output per animal unit (AU) grazed is assumed to be a function of the stocking rate of the form

(11) $y_2 = k - ab^{z_2}$,

where y_2 is the output per AU in pounds, z_2 is the stocking rate in AU per acre, and k, a, and b are parameters. Such a curve is illustrated in Figure 3.

For a given stocking rate, the response of the output per animal unit grazed to soil nitrogen as reflected through forage yield is assumed to be quadratic:

$$(12) \quad y_1 = c + dx_1 - e(x_1)^2,$$

where y_1 is the output per AU in pounds, x_1 is the soil nitrogen level in pounds per acre, and c, d and e are constants. This curve is illustrated in Figure 4.

An analysis of equation (11) suggests that the paramter k represents the output per AU due to the quantity of forage yield. An increase in k results in an increase in output at all rates of stocking, an increase in the stocking rate at the point of no gain, but no change in the slope or the curvature of the output function. An autonomous increase in the quantity of forage yield should produce the same effects. The increase in the quantity of forage per AU results in greater animal gains at the same rate of stocking or in a greater stocking rate at the same level of animal gain. Since the plant-animal relationship is qualitatively unchanged, the increase in the quantity of forage does not affect the rate of gain versus the stocking rate.

A change in a or b results in a change in the slope and curvature of the output curve, indicating a changing plant-animal relationship. It is felt that this effect is similar to a change in specie composition or in livestock characteristics.

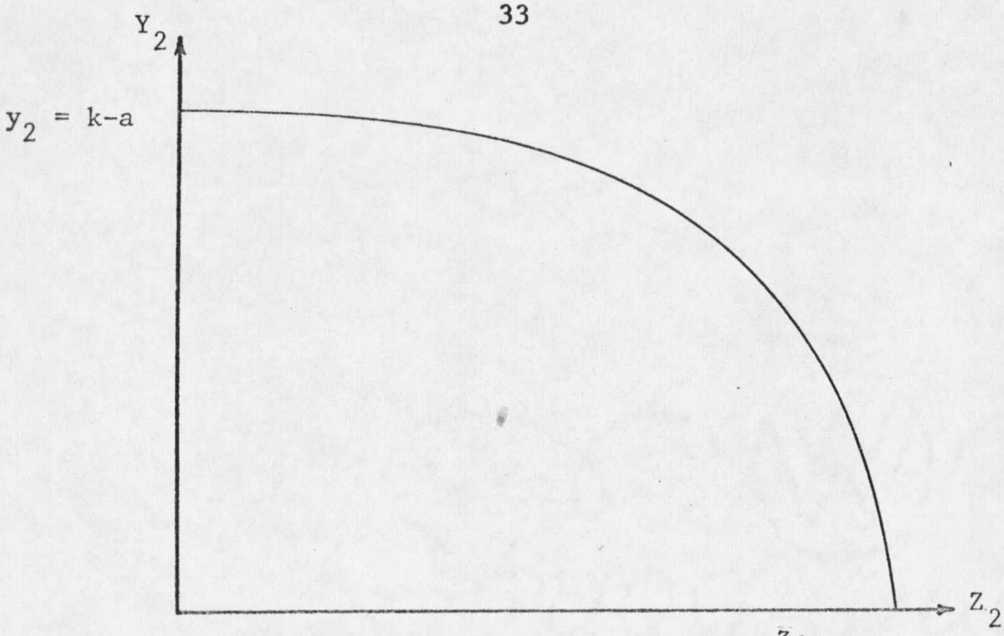

Figure 3. General form of the curve: $y_2 = k-ab^{z_2}$.

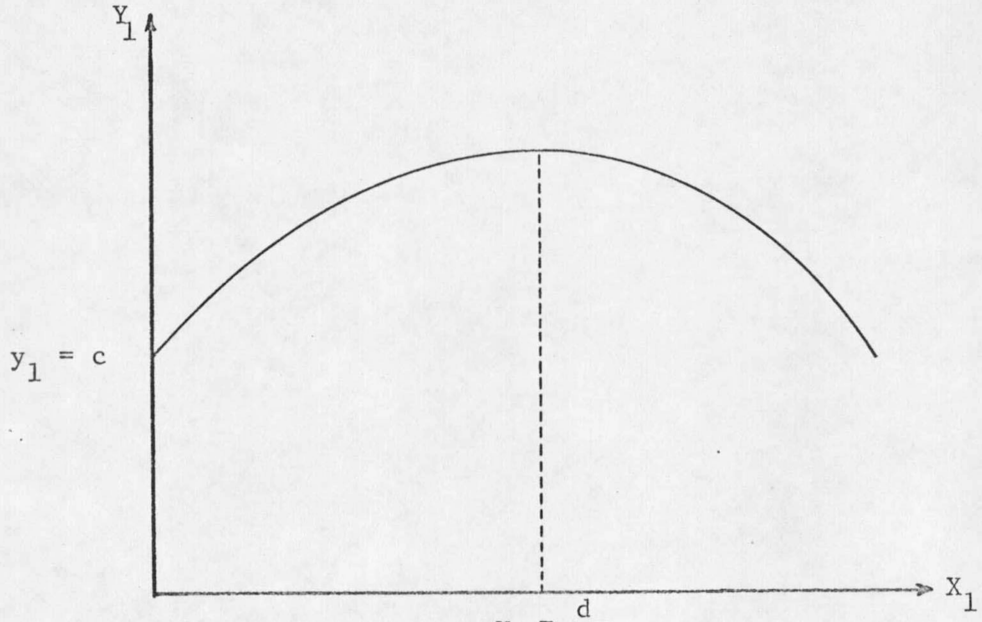

Figure 4. General form of the curve: $y_1 = c+dx_1-e(x_1)^2$.

On the basis of that interpretation of the parameters of equation (11), the response of the output per animal unit to the level of nitrogen may be included by allowing the parameter k to vary with nitrogen, according to equation (12). The result is a family of gain curves as shown in Figure 5, where the functional form is

$$(13) \quad y = [c + dx_1 - e(x_1)^2] - ab^{z_2},$$

where y is the output per AU, x_1 is the nitrogen level, and z_2 is the stocking rate. Parameters a through e represent the genetic characteristics of the livestock, the quality of soil and forage, and so on.

The output of animal products per acre is simply the product of the output per animal unit times the number of animal units grazed per The family of curves shown in Figure 6 are the output per acre curves. The revenue function is the product of the price times the number of animal units per acre sold times the output per animal unit.

In terms of the notation of Chapter II, the nitrogen level at the beginning of stage n is the sum of the nitrogen level at the end of stage n+1, x_1^{n+1}, and the nitrogen applied at the beginning of stage n, r_1^n. The stocking rate in stage n is the sum of breeding stock, K, young stock, γK, and yearlings held, x_2^{n+1}, each on a per acre basis. The stocking rate may be expressed in terms of animal units per acre as

$$Z_2^n = m_1(K) + m_2(\gamma K) + m_3(x^{n+1})$$

where Z_2^n is the stocking rate in AU's per acre, and m_1, m_2 and m_3 are the

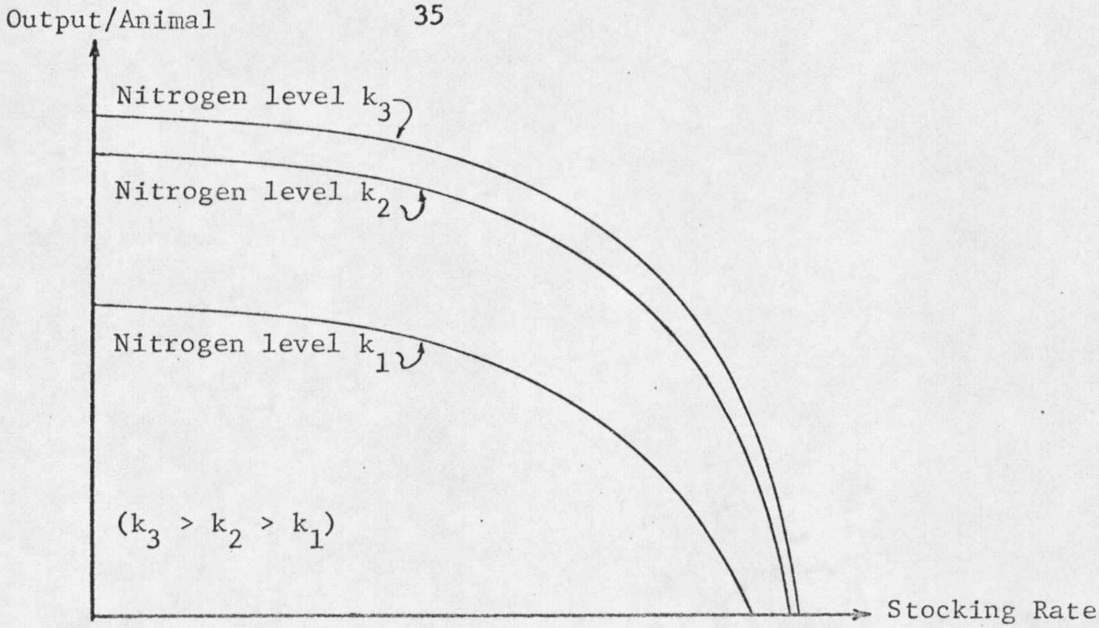

Figure 5. Representative members of the family of output per animal curves.

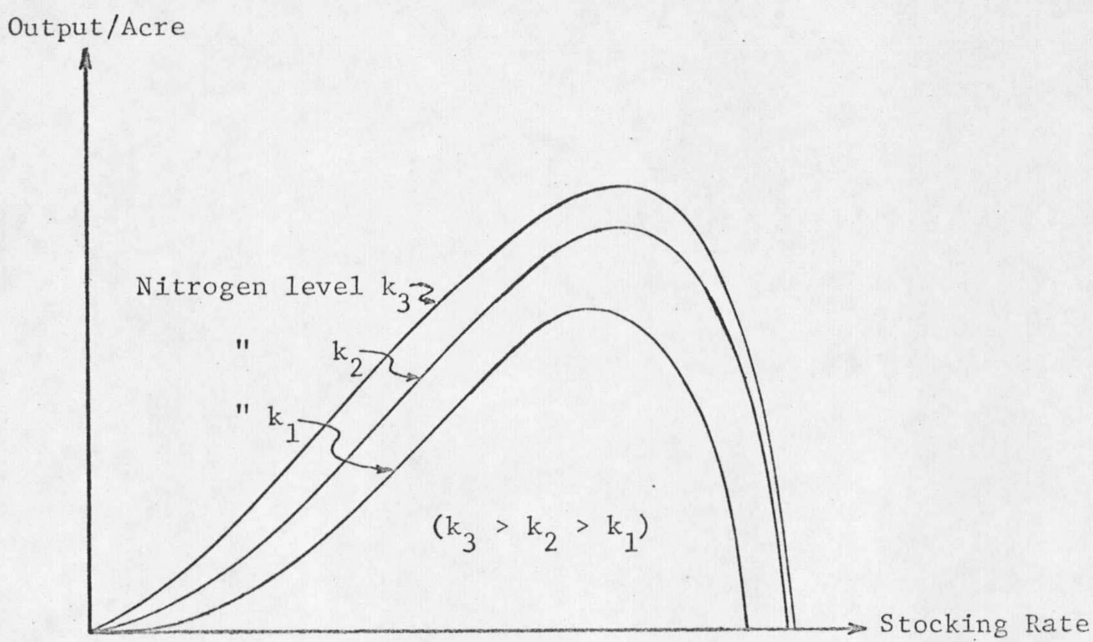

Figure 6. Representative members of the family of output per acre curves.

correction factors, in AU per head, for mature stock, young stock and yearlings, respectively.

At the time of sale, the weight of young stock and yearlings is some function of the output per animal unit, equation (13). For simplicity, assume that the weight functions are proportional, say

$$w_1^n = g_1 \cdot y^n$$

where w_1^n is the average weight per head of young stock at the end of stage n, y^n is the output per AU at the end of stage n, and g_1 is a constant, and

$$w_2 = g_2^n \cdot y^n$$

where w_2^n is the average weight per head of yearlings at the end of stage n, and g_2 is a constant. Then, the revenue produced at the end of stage n is given by

$$(14)\quad R^n = p_\theta^n(w_1^n \cdot \theta_2^n) + p_x^n(w_2^n \cdot x_2^{n+1}), \qquad (n=1,2,\ldots,N)$$

$$= p_\theta^n \cdot g_1 \cdot y^n \cdot \theta_2^n + p_x^n \cdot g_1 \cdot y^n \cdot x_2^{n+1}$$

$$= [p_\theta^n \cdot q_1 \cdot \theta_2^n + p_x^n \cdot q_2 \cdot x_2^{n+1}][c+d(x_1^{n+1}+\theta_1^n)-e(x_1^{n+1}+\theta_1^n)^2-a(b)^{z_2^n}],$$

where p_r^n is the price per pound for young stock and p_x^n is the price per pound for yearlings at the end of stage n.

The cost function in stage n is assumed to be a linear combination of the costs of fertilizer applied, θ_1^n, young stock wintered, $\gamma K-\theta_2^n$, and the marketing costs of the animal products sold, $w_2^n x_2^{n+1} + w_2^n x_2^{n+1}$.

cost function may be written

$$(15)\quad C^n = p_1^n\ \theta_1^n + C_1^n(\gamma K - \theta_2^n) + C_2^n(w_1^n\theta_2^n + w_2^n x_2^{n+1}),\quad (n=1,2,\ldots,N),$$

where p_1^n is the price per pound of fertilizer at the beginning of stage n, C_1^n is the cost per head of wintering young stock, and C_2^n is the marketing cost per pound, both taken at the end of stage n. The marketing and wintering costs are assumed to be discounted to the time of decision, and the revenue is assumed to be received one year after the time of decision. Thus, the profit at stage n is given by

$$(16)\qquad \pi^n = \beta R^n - C^n,\quad (n=1,2,\ldots,N).$$

where β is the discount factor.

Necessary Conditions

Substituting equations (14), (15), and (16) into the model and assuming an interior solution at each stage n,* the decision rule for nitrogen, equation (A-20), may be solved for θ_1^n: (see Appendix B)

$$(B\text{-}6)\quad \theta_1^n = \frac{d}{2e} - x_1^{n+1} - \frac{p_1^n - \alpha\beta p_1^{n-1}}{\beta(p_\theta^n g_1 \theta_2^n + p_x^n g_2 x_2^{n+1})(2e)}$$

---

*The assumption of an interior solution requires θ_1^n to be greater than zero at each stage n. This allows the consideration of a continuous solution by avoiding the discontinuity imposed by the constraint $\theta_1^n \geq 0$, $(n=0,1,2,\ldots,N)$.

The first term on the right-hand side of equation (B-6) is the physically optimum level of nitrogen, i.e., the level of nitrogen at which the marginal productivity of nitrogen is zero (see Figure 4), hereafter written x_1^*. The second term is the amount of nitrogen presently in the soil. The third term is an adjustment from the physically optimum level of nitrogen that reflects the asset value of soil nitrogen in terms of its cost and its productivity. The form of this term is due to the the simplicity of the functional forms assumed, primarily, the linearity of the cost function.

Since the percentage of carryover, α, and the discount factor, β, are both less than one, the third term is negative for a constant price of fertilizer. This implies that the cost of fertilization diminishes the profits so that something less than the physically optimum level of fertilizer should be applied. The term is more negative if the price of fertilizer in the next stage, p_1^{n-1}, is less than the current price, p_1^n. This implies that the reduced cost of fertilization in the next stage results in the decision to apply less nitrogen in the current stage, presumably in favor of applying more in the subsequent stages. The term becomes positive if the price in the next stage is sufficiently greater than the current price. This implies that the increased cost of fertilization in the next stage results in the decision to apply more fertilizer in the current stage, so that a greater nitrogen level will be retained through carryover.

The denominator of the third term is the marginal value product of an increment of nitrogen. An increase in the discount factor, the prices of livestock, or in the number of livestock sold, represents an increase in the value of soil nitrogen as an asset. Under such circumstances, the third term decreases, resulting in the decision rule moving closer to the physically optimum level of nitrogen.

The decision rule for the stocking rate, equation (A-21), becomes*

$$(17)\quad \beta p_\theta^n w_1^n - \beta^2[p_x^{n-1} w_2^{n-1} + (p_\theta^{n-1}\theta_2^{n-1} \frac{\partial w_1^{n-1}}{\partial x_2^n} + p^{n-1} x^n \frac{\partial w_2^{n-1}}{\partial x_2^n})]$$

$$= C_2^n w_1^n - \beta C_2^{n-1}[w_2^{n-1} + (\theta_2^{n-1} \frac{\partial w_1^{n-1}}{\partial x_2^n} + x_2^n \frac{\partial w_2^{n-1}}{\partial x_2^n})] - C$$

The left-hand side of equation (17) is the discounted difference in marginal revenue between young stock and yearlings. The first term is the discounted marginal revenue of young stock sold in the present stage, and the second term is the discounted marginal revenue of yearlings sold in the next stage. It should be noted that the second term includes an additional, differential expression. This expression represents the loss in output per AU which results from the increased stocking rate that the holding of yearlings in the next stage entails. The reason for the

*The expression for θ_2^n which results from equation (17) is in the form of a quadratic function and an exponential function. Such a form cannot be analytically solved for θ_2^n at this point.

expression is that the stocking rate of the young stock, even those sold, is already included as a parameter in the revenue function, but the presence of yearlings in any stage entails a net increase in the rate of stocking.

The right-hand side of equation (17) is the difference in marginal cost between young stock and yearlings. The first term is the marketing cost per head of young stock and the second term is the discounted marketing cost per head of yearlings. Their sum is an increasing cost per head of young stock sold, or a decreasing cost of young stock held over. The third term is the wintering cost per head of young stock that is a decreasing cost of selling young stock.

The decision rule spans only two production periods, a result of the assumption that yearlings must be sold.

The relationship between the two decisions, stocking and fertilizing, is represented more clearly in equation (B-6). An increased number of animals sold represents increased returns to the range-livestock operation. This allows the rancher to cover the costs of fertilization and results in the decision to apply more fertilizer. From the other viewpoint, an increased level of nitrogen represents an increased revenue per head of livestock, and results in the decision to sell more stock.

Convergence

It is instructive to consider a particular type of convergence at this point. The properties of the solution will be considered under the assumption that, for values of n greater than some value n_0, the state variables, x_1^n and x_2^n, converge to some steady-state value. The assumption may be written as

(18) $x_1^n = x_1$,

(19) $x_2^n = x_2$, $n > n_0$,

where x_1 and x_2 are constants. The transformation functions then become

$$x_1 = \alpha(\theta_1^n + x_1),$$

$$x_2 = \gamma K - \theta_2^n, \quad n > n_0,$$

and may be solved for θ_1^n and θ_2^n:

(20) $\theta_1^n = \frac{(1-\alpha)}{\alpha} x_1$,

(21) $\theta_2^n = \gamma K - x_2$, $n > n_0$.

Since x_1 and x_2 are constants, the decision variables must also converge to steady-state values, say θ_1 and θ_2, which are given by the right-hand sides of equations (20) and (21).

To illustrate the convergence of the decision rules, consider the nitrogen decision rule, equation (B-6). Substituting equations (20) and (21) into the decision rule for nitrogen results in the expression

$$\theta_1 = x_1^* - \left(\frac{\alpha}{1-\alpha}\right)\theta_1 - \frac{p_1^n - \alpha\beta p_1^{n-1}}{\beta[p_\theta^n q_1 \theta_2 + p_x^n q_2(\gamma K - \theta_2)](2e)}.$$

The assumption of steady-state values requires the prices to retain the same relationship for all n greater than n_0, which may be accomplished by assuming constant prices, p_1, p_x and p_2. The steady-state decision rule for nitrogen is

$$(24) \quad \theta_1 = (1-\alpha)[x_1^* - \frac{(1-\alpha\beta)p_1}{\beta[(p_\theta g_1 - p_x g_2)\theta_2 + p\ g_2\gamma K](2e)}].$$

The second term on the right-hand side of equation (24) is the steady-state optimum level of nitrogen. In the steady-state situation, the rancher applies nitrogen until total soil nitrogen is up to the optimum level, and maintains that level by applying the percentage of nitrogen lost in each stage, $(1-\alpha)$, as equation (24) states.

A decrease in the price of nitrogen, p_1, results in the convergence of the nitrogen level to a higher steady-state value; an increase in the prices of livestock has the same result. An increase in either the carryover percentage or the discount factor increases the discounted future returns due to nitrogen and results in the convergence to a higher steady-state value of the nitrogen level. An increase in the carryover percentage increases the future returns because more nitrogen is available in future periods. An increase in the discount factor increases the present value of the future returns because the opportunity cost of capital is less.

As the product, $\alpha\beta$, approaches one, the optimum level of nitrogen approaches the point of zero marginal productivity, x_1^*, regardless of any finite cost of fertilizer. If the product approaches zero, then either no nitrogen is carried over or future returns are not considered at all so that the decision is reduced to a static consideration, and the optimum level of fertilizer converges to the level at which marginal cost and marginal revenue are equal in a static sense.

Sensitivity Analysis

For the purposes of sensitivity analysis, the response of the nitrogen decision rule to changes in the prices of livestock and fertilizer may be found. For example, if the price of fertilizer increases by Δp_1^v, then the new decision rule for nitrogen is

$$\theta_1 + \Delta\theta_1 = x_1^* - x_1 - \frac{(1-\alpha\beta)(p_1+\Delta p_1)}{[(p_\theta g_1 - p_x g_2)\theta_2 + p_x g_2 \gamma K](2e)}$$

or subtracting θ_1 from both sides, the change in the decision rule is

$$(25) \quad \Delta\theta_1 = -\frac{(1-\alpha\beta)\Delta p_1}{[(p_\theta g_1 - p_x g_2)\theta_2 + p_x g_2 \gamma K](2e)}$$

An increase of Δp_1 in the price of fertilizer decreases the amount of nitrogen to be applied by the amount on the right-hand side of equation (25). The relationship stated in equation (25) permits the researcher to examine the effects of price changes on the optimum solution. Such effects are particularly important when the variables are discrete valued.

Notice that equation (B-6) states the decision to apply nitrogen, θ_1^n, in terms of any level of existing soil nitrogen, x_1^{n+1}, any rate of stocking, $x_2^{n+1} - \theta_2^n$, and for any stage n. These are the two dimensions of sensitivity analysis that are embodied in the dynamic programming formulation. The same dimensions of analysis are available on the decision variable for stocking, θ_2^n, in equation (17).

CHAPTER IV

DIRECTIONS FOR RESEARCH

Data Requirements

The examination of the necessary conditions for maximization, equations (A-20) and (A-21), points out the vital information demanded for the application of the model presented. The nature of the equations indicates that research must be directed toward the specification of the revenue function and the state transformation functions.

The importance of the revenue function is emphasized by its repeated use in the necessary conditions, over the entire planning horizon. The direction of this paper has stressed animal output per acre as the measure of production and a means of evaluating range forage. If the model in its present form is to be applied, the specification of the revenue function must be the first and most important aim of further range research. As a second aim, the transformation functions may be specified by the data gathered in the process of determining the revenue function.

There are two requirements for the specification of the revenue function: the prices of livestock must be estimated and predicted over the planning horizon; and the response of the production function to fertilizer applications, stocking rates, moisture and specie composition must be determined. Specification of the transformation functions entails at least three information requirements: the carryover percentage of nitrogen, the response of specie composition to nitrogen, and the response of specie composition to stocking.

One point should be kept in mind in the development of experimental design. The model presented, in its full generality, allows for much more general functional forms than those assumed. The researcher must not feel bound to the assumption, for example, of a constant carryover percentage. The limiting requirement on the revenue, cost and transformation functions is the Markov requirement [p.10]: it must be the case that, given the state of the system at any stage, the revenue, cost and transformation functions depend only on that state and the subsequent decisions.

Another point that is most important to the researcher is that the economic viewpoint of the model presupposes the technical conditions on the production function. Though these conditions are not required for the operation of the model, they are necessary for its specification, hence, its application. The technical conditions include: the characteristics of the range land and the livestock, the use of mechanical and chemical treatments on the range land, and the techniques of fertilization and seeding.

With these two points well in mind, consider the problems of data demands.

The Prices

The decision rules depend on the marginal revenue produced by the decision in all future periods, so the livestock price must be known, either with certainty or with some established probability, for each stage in the planning horizon. A prediction model of such scope is not to be considered lightly, but there is hope in one fact: the future returns are discounted to present value and corrected by carryover percentages. If the returns are relatively constant, then with a discounted carryover percentage, the product $\alpha\beta$, of 0.8, the error associated with ignoring all stages more than 10 years in the future is on the order of only 10 percent; if the product is 0.7, the error is less than 10 percent after only 7 years and less than 3 percent after 10 years. The declining importance of future returns also implies the declining importance of measurement error on future prices. At the end of 10 years, the error is reduced to 3 percent and 1 percent of its value, when the product, $\alpha\beta$, is 0.8 and 0.7, respectively. It is not unreasonable to expect the product to be less than 0.8, so, if some level of error can be called acceptable, the demands on a prediction model for prices may be relaxed in length and accuracy.

The Production and Transformation Functions

The specification of the production and transformation functions cannot be so lightly dismissed. This model has set extremely high objectives for range research, attempting to explain forage quantity and quality, and animal performance by the variability of fertilizer, specie composition, and grazing pressure in order to form a relation between the latter variables and the output of animal products from range land. The task of conducting a single large-scale experiment to derive all of the necessary data is surely either impractical or impossible. The task must be divided into units that may be handled. Chamblee suggests that the problems of forage evaluation may be divided into three categories: (1) problems that are largely independent of animal effects; (2) problems that are interrelated with animal effects, but that may be resolved without animal measurements; and (3) problems that cannot be resolved without animal measurement. [7:p.150]

Problems Independent of Animal Effects

To the extent that the effects of animals on forage yield are independent of the effects of nitrogen, the response of forage yield to nitrogen and moisture can be included in the first category.*

Then, it is suggested that small-plot experiments with no animal trials would be adequate for resolving the problems of nitrogen and moisture response.

---

*This does not imply that the animal has no effect on forage yield; the fact that it does is undeniable. The implication is that the effects of nitrogen are entirely separate effects, and may be studied by themselves.

The specific items to be considered in this category include: the response of forage quantity to nitrogen application, the response of specie composition to nitrogen application, and the percentage of nitrogen retained in the soil, all measured at various levels of moisture. The results of the measurements are to be stated as the quantity of forage and the value of the index of specie composition produced by each experimental level of nitrogen at the various moisture levels.

The percentage of nitrogen carryover should be causually related to the yield and may also depend on the moisture level and specie composition. Small-plot experiments should be useful for obtaining the information necessary for determining such relationships.

These results should specify, at least partially, to the extent of the independence assumption, the nitrogen transformation function, equation (1a), the specie composition transformation function, equation (3a), and the forage quantity and quality due to nitrogen, specie composition and moisture.

If the reseeding option is to be tested, it must also fall in this category. It is expected that a separate transformation function for specie composition would result after artificial seeding.

It is suggested that the measurement of soil nitrogen and moisture be a physical one, not an estimate based on residual effects. Then, the measurement of nitrogen applied must be in consistent units, such as the actual amount of the applied nitrogen that results in available soil nitrogen. The forage quantity may be measured by clipping and stated in pounds. There are several survey methods available for the measurement of the index of specie composition but, as indicated previously, the structure of the index must first be decided.

The general design of the experiments in this category should be rather simple. An area of small plots is laid out and measured immediately for soil characteristics, moisture and nitrogen level and specie composition. Treatments are then applied to the plots, principally experimental levels of nitrogen but also mechanical, chemical and seeding treatments, if the latter are to be measured. Samples are taken yearly of moisture and nitrogen levels, specie composition and yield, and treatments reapplied in such a way as to produce an adequate cross-section of results. The precipitation each year should also be recorded in order to correlate precipitation with the moisture level. The times associated with green-up and green feed season might be recorded for use in future improvements of the model by timing effects.

Problems Interrelated to Animal Effects: No Performance Measurement

The second and third categories stated above are largely complementary, and considerable overlapping will be evident. The value of the experiments not involving animal measurements depends mainly on the practicality of conducting grazing trials and animal measurements. If grazing trials are not practical, then the results of the second category of experiments must be used to approximate animal effects. When grazing trials are carried out, the results of the second category of experiments are useful as fundamental knowledge on which grazing trials may be based.

Problems that might be resolved without the measurements of animal performance include the effects of the animal on range yield. Again, it is suggested that small-plot experiments would be adequate for resolving the problems of animal effects.

The specific items to be considered in this category include: the response of yield to the animal due to trampling, nutrient return by excreta and fouling, and the response of specie composition to grazing. The results of the measurements might be stated as the quantity of yield lost per head of livestock due to animal effects, and the change in specie composition, including animal preference and plant recovery, under grazing. If the animal effects appear to contribute significantly to the carryover percentage of nitrogen, this response should also be noted.

An important problem that must be resolved is the influence of yield on the percentage of offspring produced by the breeding stock. This influence is clearly a factor in the decision between selling young stock and selling yearlings and may also affect the decision to fertilizer. Small-plot experiments would be useful in resolving this problem.

These results should complete the specification of the nitrogen transformation function, equation (1a), and should contribute to the specification of the specie composition transformation function, equation (3a), and of the response of range yield to stocking rates.

The measurements, most likely on a per head basis, must be made in units consistent with those of the first category in terms of yield, specie composition and nitrogen level. The plot size must be recorded, of course, as the results may have to be stated in terms of acreage and used to approximate the total animal effects.

The experimental design should be similar to that of the first category and, in fact, should probably represent a simple expansion of the small-plot design. If it is reasonable to expect a homogeneous effect of an animal on an area of several small plots, the experiments may even approach actual grazing trials. The yield, specie composition, nitrogen and moisture level are measured before and after the animal is placed on the range area. The necessity of an animal depends on how closely artificial methods, such as clipping, approximate the animal effects. If an animal must be used, it seems that there is no reason why the animal performance cannot be measured, thereby making these experiments actual grazing trials.

Problems Interrelated to Animal Effects: Performance Measurements

The response of the animal to forage quantity and quality and the response of forage and animal yield to the stocking rate must be measured by animal trials. As noted above, if animal effects can be considered to be homogeneous over several small plot areas, then the same experimental design may be expanded and utilized for grazing trials.

The specific items to be considered in the third category include: the response of the livestock output to forage yield and to stocking rates, and the change in specie composition due to stocking rates. Since the output of livestock products is the desired end of the range operation, the results of the experiments under this category are by far the most important. Unfortunately, the necessity of grazing trials also makes the results the most difficult to achieve. The results of the measurements are to be stated as the output of animal products, either per head or per acre, produced by each level of forage yield, as measured in the first category, and by each experimental rate of stocking, and the value of the index of specie composition produced by each rate of stocking. It might also be desirable to record the response of forage yield to stocking rates.

These results should complete the specification of the specie composition transformation function, equation (3a), and the production function for livestock, by inference from the forage yield experiments.

It is suggested that the measurement of output be made in pounds of marketable livestock products per acre at each level of forage yield and rate of stocking. All other units must be consistent with the measurements in the other two categories.

If experiments of a scale sufficient to produce these results are not feasible, then the results must be inferred from extensive experiments of the second class. It is possible that time and cost factors may necessitate just such a direction of research.

Reid [7:p.45] suggests several indicator methods for forage evaluation which could possibly be used in conjunction with limited grazing trials to produce output results more quickly and more cheaply. Nitrogen fecal-indicator techniques have some appeal for this model because the emphasis here is on nitrogen responses.

The nitrogen indicator technique appears to be a valid measure of the digestibility and intake of forage. Grazing trials could be used to establish the relationships between these variables and animal output, and these variables and stocking rates. Small plot experiments could be used to establish the relationship of these variables to soil nitrogen and specie composition. Then, it does not seem unreasonable to expect that a direct measurement of the production function could be accomplished through the link provided by the nitrogen indicator in fewer grazing trials. In addition, the indicator technique provides a direct measure of fecal nitrogen return. For these reasons, it is strongly suggested that the methods of the nitrogen indicator be investigated as a useful measurement technique.

The Cost Function

The cost function does not seem to present nearly as many problems as the production function. The estimation and prediction of costs over the planning horizon presents the same problem as in the case of livestock prices, and the previous discussion applies equally well to both. The measurement of the cost function entails the specification of the costs associated with applying fertilizer, wintering calves, and marketing livestock. No significant difficulties seem to be apparent.

Extensions

The extension of the model by the addition of a moisture level variable has already been discussed. The model might be extended by the addition of other state variables. One reported effect of fertilization on range land is an earlier green-up and longer green feed season. A state variable that expresses such timing effects provides more precision in the returns function and may provide another decision variable for the control of the operation. A decision to place livestock on the range area earlier or for longer periods of time, for example, might significantly reduce wintering costs.

To a certain degree, there are already several extensions available in the formulation of the model. The revenue, cost and transformation functions may be considered to be vector functions of a vector of state variables.

For example, nutrients other than nitrogen might be considered. This would entail a transformation function for each nutrient, not necessarily independent, and the inclusion of the nutrient variables in the revenue and cost functions. If the variables x_1^n and θ_1^n are considered as vectors of nutrients, the analysis presented is generalized, in the form of vector equations, to the case of several nutrients. Similarly the variables x_2^n and θ_2^n can be generalized to include several types of livestock or even game animals.

In line with the vector state variables, the specification of separate revenue and cost functions for different ages, grades or types of livestock should result in a more precise model. The separate functions could take the form of a vector of revenue and of cost functions, and if R^n and C^n are considered as vectors, the analysis above is generalized to this case. This change would entail production functions for each class of livestock, which may be a prohibitive consideration for research.*

---

*Note, however, the treatment of two classes of livestock in the example on page 34, where the vector of prices is (p_θ^n, p_x^n) and the production function vector is $(w_1^n \theta_2^n, w_2^n x_2^{n+1})$.

Possibly the most useful extension of the model is the expansion to two stages per year. One stage might be defined to begin in the spring and end in the fall; the second stage might include the remainder of the year. The consideration of two stages allows for decision-making at two points in the year, so that the wintering of livestock may also be included in the analysis of the range operation. The change to two stages creates more flexibility in the decision process and permits the rancher to examine the climatic and range conditions for changes that might influence his decisions during the year.

The formulation of a stochastic model has been discussed, and may be more realistic than a deterministic model. The carryover of nitrogen and the change in specie composition, for example, are very likely to be stochastic processes. The treatment of the stochastic model has been discussed above.

An extremely important extension of the model is the development of a macro-model to include an entire region of individual ranches. A region following similar range policies might profoundly influence the cost and revenue functions. It is conceivable, for example, that the construction of a fertilizer plant or a packing plant might become feasible in such a region.

Finally, in speaking of improvements, the objective function must not be ignored. The definition of an objective is completely general, allowing for a combination of several objectives, such as maximum profits at minimum risk, or a combination of conditions, such as minimizing the maximum losses. The generality of the objective function permits the application of the model to different types of operations and operators.

Conclusions

The data requirements and extensions must stand as the results of this paper. The problem of range improvement is a complex one which must be approached in a unified effort by soil and animal scientists as well as economists. This is not to imply that all range research should be carried out by an organized task force, but suggests the need for better understanding and coordination of research activities. The information on the various phases of range technology must come from the research work of agronomists and animal scientists on production and utilization, agricultural engineers on technical probabilities, and production economists and management specialists on profit maximization. Available data is all too often inadequate, conflicting, confusing or inapplicable, largely because of duplicated efforts. The research problems are especially acute in range research, where limitations are imposed by the number of treatments possible, given livestock, land and physical resource levels. [7:p.204]

It is hoped that the direction of this paper has expressed the economist's viewpoint on two major problems of research: (1) what kind of information is relevant to the analysis; and (2) how the appropriate data may be obtained. The relevant information is outlined above, and some suggestions have been made on data gathering. It has been pointed out that the research task can be broken down into segments that may be more manageable, and a weak statement on the priority of various research work has been made. The demand is now made on the joint effort of economists and technical scientists to develop a research program that will yield the answers to the problem at hand.

APPENDICES

APPENDIX A

THE DERIVATION OF THE NECESSARY CONDITIONS FOR MAXIMIZATION

The general solution of the dynamic programming problem stated in the text in equations (7), (8), (1a), (2a), and (3a) may be found by solving the recursion relation, equation (9), at each stage. However, application of the discrete version of the Pontryagin maximum principle [4:p.139] yields the general necessary conditions without iteration in the function space (under suitable assumptions).

The discrete version of Pontryagin's maximum principle may be stated as follows:

> The state of the process denoted by an s-dimensional vector $X=(x_1,x_2,\ldots,x_s)$, is transformed at each stage according to an r-dimensional decision vector, $\Theta=(\theta_1,\theta_2,\ldots,\theta_r)$, which represents the decisions made at that stage. The transformation of the process at the n-th stage is described by a set of performance equations in vector form.
>
> (A-1) $X^n = T^n(X^{n+1};\Theta^n)$, $(n=0,1,2,\ldots,N)$
>
> $X^{N+1} = A.$
>
> A typical optimization problem associated with such a process is to find a sequence of Θ^n, $n=0,1,2,\ldots,N$, subject to the constraints
>
> (A-2) $\psi_i^n[\theta_1^n,\theta_2^n,\ldots,\theta_r^n] \leq 0$, $(n=0,1,2,\ldots,N;\quad i=1,2,\ldots,r)$
>
> which makes a function of the state variable of the final stage
>
> (A-3) $S = \sum_{i=1}^{s} h_i x_i^0$, $\qquad (h_i = \text{constant})$
>
> an extremum when the initial condition $X^{N+1} = A$ is given.

The function, S, which is to be maximized (or minimized), is the objective function of the process.

The procedure for solving such an optimization problem by a discrete version of the maximum principle is to introduce an s-dimensional adjoint vector Z^n and a Hamiltonian function H^n, which satisfy the following relations:

$$\text{(A-4)} \quad H^n = \sum_{i=1}^{s} Z_i^n T_i^n(X^{n+1};\Theta^n),$$

$$\text{(A-5)} \quad Z_i^{n+1} = \partial H^n / \partial x_i^{n+1},$$

and

$$\text{(A-6)} \quad Z_i^0 = h_i. \quad (n=0,1,2,\ldots,N; \quad i=1,2,\ldots,s)$$

If the optimal decision vector function $^*\Theta^n$, which makes the objective function S an extremum is interior to the set of admissible decisions Θ^n, the set given by equation (A-2), a necessary condition for S to be (local) extremum with respect to Θ^n is

$$\text{(A-7)} \quad \partial H^n / \partial \Theta^n = 0. \quad (n=0,1,2,\ldots,N)$$

If $^*\Theta^n$ is at a boundary of the set, it can be determined from the condition that H^n is (locally) extremum.

In order to apply the above principle to the present problem it is necessary to state the objective function as a state variable. Let x_4^n be defined so that

$$\text{(A-8)}\quad x_4^n = (1+r)x_4^{n+1} + R^n(X^{n+1};\theta^n) - C^n(x^{n+1};\theta^n), \quad (n=0,1,2,\ldots,N)$$

$$x_4^{N+1} = 0.$$

where $R^0(X^1;\theta^0)$ is $L(X^1)$, $C^0(x^1;\theta^0)$ is zero and r is the interest rate used for discounting. The solution of the difference equation (A-8) with the given boundary condition yields, for n equals zero:

$$x_4^0 = \sum_{i=0}^{N}(1+r)^i[R^i(X^{i+1};\theta^i) - C^i(x_2^{i+1};\theta^i)].$$

By equation (7), the relation between x_4^0 and $f_n(X^{n+1})$ is

$$x_4^0 = (1+r)^N f_n(X^{n+1}),$$

or, since β is equal to the reciprocal of (1+r), the objective function may be stated as:

$$\text{(A-9)}\quad \text{Maximize } (\beta)^N x_4^0.$$

Equations (1a), (2a), and (3a) satisfy equation (A-1), equation (8) satisfies equation (A-2), and equation (7) satisfies equation (A-3), where h_1, h_2, and h_3 are zero and h_4 is $(\beta)^N$. So introduce the vector Z^n and the Hamiltonian function H^n, defined by equations (A-4), (A-5), and (A-6):

$$\text{(A-10)}\quad H_1^n = Z_1^n[\alpha(x_1^{n+1}+\theta_1^n)] + Z_2^n[\gamma K-\theta_2^n] + Z_3^n[\tau_3^n(X^{n+1};\theta^n)]$$
$$+ Z_4^n[(1+r)x_4^{n+1}+R^n(X^{n+1};\theta^n) - C^n(x_2^{n+1};\theta^n)],$$

$$Z_1^{n+1} = \alpha Z_1^n + \frac{\partial\tau_3^n}{\partial x_1^{n+1}} Z^n + \frac{\partial R^n}{\partial x_1^{n+1}} Z^n,$$

$$(\text{A-11}) \quad Z_2^{n+1} = \frac{\partial\tau_3^n}{\partial x_2^{n+1}} Z_3^n + [\frac{\partial R^n}{\partial x_2^{n+1}} - \frac{\partial C^n}{\partial x_2^{n+1}}] Z_4^n,$$

$$Z_3^{n+1} = \frac{\partial\tau_3^n}{\partial x_3^{n+1}} Z^n + \frac{\partial R^n}{\partial x_3^{n+1}} Z^n,$$

$$Z_4^{n+1} = (1+r) Z_4^n,$$

and

$$Z_1^0 = 0,$$

$$Z_2^0 = 0,$$

$$(\text{A-12}) \quad Z_3^0 = 0,$$

$$Z_4^0 = (\beta)^N.$$

If the optimal decision vector $*\theta^n$, is interior to the set of admissible decisions, i.e.,

$$-*\theta_1^n < 0,$$

$$(\text{A-13}) \quad *\theta_2^n < \gamma K,$$

$$*\theta_3^n = 0 \text{ or } 1,$$

Then the necessary condition for (A-9) to be maximized is given by equation (A-7), evaluated at θ_3^n equal to zero or one, whichever yields the larger value of (A-9). Differentiating equation (A-10), according to equation (A-7), the necessary conditions are:

$$\frac{\partial H^n}{\partial \theta_1^n} = \alpha Z_1^n + \frac{\partial \tau_3^n}{\partial \theta_1^n} Z_3^n + [\frac{\partial R^n}{\partial \theta_1^n} - \frac{\partial C^n}{\partial \theta_1^n}] Z_4^n = 0,$$

and

$$\frac{\partial H^n}{\partial \theta_2^n} = -Z_2^n + [\frac{\partial R^n}{\partial \theta_2^n} - \frac{\partial C^n}{\partial \theta_2^n}] Z_4^n = 0, \quad \text{since } \frac{\partial \tau_3^n}{\partial \theta_2^n} = 0,$$

which may be written

$$\text{(A-14)} \quad [\frac{\partial R^n}{\partial \theta_1^n} - \frac{\partial C^n}{\partial \theta_1^n}] + \frac{Z_1^n}{Z_4^n} + \frac{\partial \tau_3^n}{\partial \theta_1^n} \cdot \frac{Z_3^n}{Z_4^n} = 0,$$

and

$$\text{(A-15)} \quad [\frac{\partial R^n}{\partial \theta_2^n} - \frac{\partial C^n}{\partial \theta_2^n}] - \frac{Z_2^n}{Z_4^n} = 0.$$

The values of Z_i^n may be found by solving the difference equations (A-11) with the boundary conditions (A-12). The iterations on Z_4, for $n=0,1,2,\ldots,$ are:

$$\text{(A-12)} \quad Z_4^0 = (\beta)^N,$$

$$Z_4^1 = (1+r)(\beta)^N = (\beta)^{N-1},$$

$$Z_4^2 = (1+r)(\beta)^{N-1} = (\beta)^{N-2},$$

$$\vdots$$

(A-16) $$Z_4^n = (1+r)(\beta)^{N-n-1} = (\beta)^{N-n}.$$

It can readily be seen that Z_4^n represents the discount factor associated with the returns at stage n, with respect to the beginning of the planning horizon, stage N.

The iterations of Z_3, according to (A-11), (A-12), and (A-16), are:

$$Z_3^1 = (\beta)^N \frac{\partial R^0}{\partial x_3^1},$$

$$Z_3^2 = (\beta)^N \frac{\partial \tau_3^1}{\partial x_3^2} \frac{\partial R^0}{\partial x_3^1} + (\beta)^{N-1} \frac{\partial R^1}{\partial x_3^2},$$

$$Z_3^3 = (\beta)^N \frac{\partial \tau_3^2}{\partial x_3^3} \frac{\partial \tau_3^1}{\partial x_3^2} \frac{\partial R^0}{\partial x_3^1} + (\beta)^{N-1} \frac{\partial \tau_3^2}{\partial x_3^3} \frac{\partial R^1}{\partial x_3^2} + (\beta)^{N-2} \frac{\partial R^2}{\partial x_3^3},$$

$$\vdots$$

$$Z_3^n = (\beta)^N \frac{\partial \tau_3^{n-1}}{\partial x_3^n} \frac{\partial \tau_3^{n-2}}{\partial x_3^{n-1}} \cdots \frac{\partial \tau_3^1}{\partial x_3^2} \frac{\partial R^0}{\partial x_3^1}$$

$$+ (\beta)^{N-1} \frac{\partial \tau_3^{n-1}}{\partial x_3^n} \cdot \cdot \cdot \frac{\partial \tau_3^2}{\partial x_3^3} \frac{\partial R^1}{\partial x_3^2}$$

$$\vdots$$

$$+ (\beta)^{N-n+1} \frac{\partial R^{n-1}}{\partial x_3^n}.$$

Note that x_3^n and τ_3^n are the same, by

(3a) $x_3^n = \tau_3^n(X^{n+1};\Theta^n)$, $(n=0,1,2,\ldots,N)$,

so that

$$\frac{\partial \tau_3^{n-1}}{\partial x_3^n} = \frac{\partial x_3^{n-1}}{\partial x_3^n}, \quad (n=0,1,2,\ldots,N),$$

and the chain rule may be applied to Z_3^n to get

$$Z_3^n = (\beta)^N \frac{\partial R^0}{\partial x_3^n} + (\beta)^{N-1} \frac{\partial R^1}{\partial x_3^n} + \ldots + (\beta)^{N-n+1} \frac{\partial R^{n-1}}{\partial x_3^n},$$

or,

$$\text{(A-17)} \quad Z_3^n = \sum_{i=0}^{n-1} (\beta)^{N-i} \frac{\partial R^i}{\partial x_3^n}.$$

The multiplier Z_3^n represents the discounted future marginal revenue due to an incremental change in the specie composition in stage n.

The value of Z_2^n may be calculated directly from equations (A-11), (A-16), and (A-17):

$$\text{(A-18)} \quad Z_2^n = \frac{\partial \tau_3^{n-1}}{\partial x_2^n} \left[\sum_{i=0}^{n-2} (\beta)^{N-i} \frac{\partial R^i}{\partial x_3^{n-1}} \right] + (\beta)^{N-n+1} \left[\frac{\partial R^{n-1}}{\partial x_2^n} - \frac{\partial C^{n-1}}{\partial x_2^n} \right].$$

The first term on the right represents the discounted future marginal revenue due to the specie composition change caused by holding yearlings into stage n-1. The second term represents the discounted net marginal revenue received from yearlings in stage n-1. Thus, Z_2^n represents the total discounted future net marginal revenue due to yearlings held in stage n-1.

Finally, the iterations of Z_1, by (A-11), (A-12), and (A-16) are:

$$Z_1^1 = (\beta)^N \frac{\partial R^0}{\partial x_1^1},$$

$$Z_1^2 = \alpha(\beta)^N \frac{\partial R^0}{\partial x_1^1} + \frac{\partial \tau_3^1}{\partial x_1^2} Z^1 + (\beta)^{N-1} \frac{\partial R^1}{\partial x_1^2},$$

$$Z_1^3 = (\alpha)^2(\beta)^N \frac{\partial R^0}{\partial x_1^1} + \alpha \frac{\partial \tau_3^1}{\partial x_1^2} Z_3^1 + \alpha(\beta)^{N-1} \frac{\partial R^1}{\partial x_1^2} + \frac{\partial \tau_3^2}{\partial x_3^3} Z_3^2 + (\beta)^{N-2} \frac{\partial R^2}{\partial x_1^3},$$

$\cdot$
$\cdot$
$\cdot$

$$Z_1^n = (\alpha)^{n-1}(\beta)^N \frac{\partial R^0}{\partial x_1^1} + (\alpha)^{n-2}(\beta)^{n-1} \frac{\partial R^1}{\partial x_3^2} + \ldots + (\beta)^{N-n+1} \frac{\partial R^{n-1}}{\partial x_1^n}$$

$$+ (\alpha)^{n-2} \frac{\partial \tau_3^1}{\partial x_1^2} Z_2^1 + (\alpha)^{n-3} \frac{\partial \tau_3^2}{\partial x_1^3} Z_3^2 + \ldots + \frac{\partial \tau_3^{n-1}}{\partial x_1^n} Z_3^{n-1}.$$

From equation (A-17), the value of Z_3^{i+1} is

$$Z_3^{i+1} = \sum_{j=0}^{i} (\beta)^{N-j} \frac{\partial R^j}{\partial x_3^{i+1}}.$$

Substituting these values for Z_3^{i+1} into the expression for Z_1^n yields

$$\text{(A-19)} \quad Z_1^n = \sum_{i=0}^{n-1} (\alpha)^{n-i-1}(\beta)^{N-i} \frac{\partial R^i}{\partial x_1^{i+1}} + \sum_{i=0}^{n-2} \sum_{j=0}^{i} (\alpha)^{n-i-2}(\beta)^{n-j} \frac{\partial \tau_3^{i+1}}{\partial x_1^{i+2}} \frac{\partial R^j}{\partial x_3^{i+1}}.$$

The first term on the right represents the discounted future marginal revenue produced by the percentage of nitrogen carried over into stage n-1. The second term represents the discounted future marginal revenue produced by the specie composition change due to the nitrogen carried over into all future periods. Thus, Z_1^n, is the total discounted future marginal revenue produced by nitrogen carried over into stage n-1.

Dividing by Z_4^n is the same as appreciating the future returns to their present value at stage n. Substituting the values of the Z_i^n into equations (A-14) and (A-15), the necessary conditions for the maximization of (A-9) are:

$$\text{(A-20)}\quad \frac{\partial R^n}{\partial \theta_1^n} + \sum_{i=0}^{n-1} (\alpha\beta)^{n-i} \frac{\partial R^i}{\partial x_1^{i+1}} + \sum_{i=0}^{n-1} (\beta)^{n-i} \frac{\partial \tau_3^n}{\partial \theta_1^n} \frac{\partial R^i}{\partial x_3^n}$$

$$+ \sum_{i=0}^{n-2} \sum_{j=0}^{i} (\alpha)^{n-i-1} (\beta)^{n-j} \frac{\partial \tau_3^{i+1}}{\partial x_1^{i+2}} \frac{\partial R^j}{\partial x_3^{i+1}} = \frac{\partial C^n}{\partial \theta_1^n}$$

and

$$\text{(A-21)}\quad \frac{\partial R^n}{\partial \theta_2^n} = \beta \frac{\partial R^{n-1}}{\partial x_2^n} - \sum_{i=0}^{n-2} (\beta)^{n-i} \frac{\partial \tau_3^{n-1}}{\partial x_2^n} \frac{\partial R^i}{\partial x_3^{n-1}}$$

$$= \frac{\partial C^n}{\partial \theta_2^n} - \beta \frac{\partial C^{n-1}}{\partial x_2^n}.$$

APPENDIX B

AN EXAMPLE OF THE SOLUTION OF THE DYNAMIC PROGRAMMING PROBLEM

The decision rule for nitrogen may be derived from the first necessary condition for maximization, given by

$$\text{(B-1)}\quad \frac{\partial R^n}{\partial \theta_1^n} + \sum_{i=1}^{n-1} (\alpha\beta)^{n-i} \frac{\partial R^i}{\partial x_1^{i+1}} = \frac{\partial C^n}{\partial \theta_1^n}, \quad (n=1,2,\ldots,N),$$

where τ_3^n is assumed to be constant in all stages.

The revenue function is given by equation (14) and the cost function by equation (15). Consider the last stage, n equal to one. The decision rule becomes

$$\beta[p_\theta^1 g_1 \theta_2^1 + p_x g_2 x_2^2][d-2e(x_1^2+\theta_1^1)] = p_1^1$$

or, solving for θ_1^1,

$$\text{(B-2)}\quad \theta_1^1 = \frac{d}{2e} - x_1^2 - \frac{p_1^1}{\beta[p_\theta^1 g_1 \theta_2^1 + p_x^1 g_2 x_2^2](2e)}.$$

Assuming that the solution is interior, i.e., θ_1^1 is strictly greater than zero, consider the stage when n is equal to two. The decision rule becomes

$$\beta[p_\theta^2 g_1 \theta_2^2 + p_x g_2 x_2^3][d-2e(x_1^3+\theta_1^2)] + \alpha\beta.\beta[p_\theta^1 g_1 \theta_2^1 + p_x^1 g_2 x_2^2][d-2e(x_1^2+\theta_1^1)] = p_1^2$$

or, by equation (B-2)

$$\beta[p_\theta^2 g_1 \theta_2^2 + p_x^2 g_2 x_2^3][d-2e(x_1^3+\theta_1^2)] + \alpha\beta p_1^1 = p_1^2$$

or, by solving for θ_1^2,

$$\text{(B-3)} \quad \theta_1^2 = \frac{d}{2e} - x_1^3 - \frac{p_1^2 - \alpha\beta p_1^1}{\beta[p_\theta^2 g_1 \theta_2^2 + p_x^2 g_2 x_2^3]}.$$

This is the value-iteration algorithm: Equation (B-2) is a partial necessary condition for the maximum to exist in stage one, and equation (B-3) is a partial necessary condition for the maximum to exist in stage two. Coupled with the second necessary condition, equations (B-2) and (B-3) represent the solution of the value iterations at stages one and two. The algorithm continues iteration until stage n is reached, but by the use of the principlé of mathematical induction the general result may be established without further iteration.

Assume that the solution for the stage where n is equal to k is given by

$$\text{(B-4)} \quad \theta_1^k = \frac{d}{2e} - x_1^{k+1} - \frac{p_1^k - \alpha\beta p_1^{k-1}}{\beta[p_\theta^k g_1 \theta_2^k + p_x^k g_2 x_2^{k+1}](2e)}, \; p_1^0 = 0.$$

When n is equal to k+1, the decision rule becomes

$$\beta[p_\theta^{k+1} g_1 \theta_2^{k+1} + p_x^{k+1} g_2 x_2^{k+1}][d-2e(x_1^{k+2}+\theta_1^{k+1})] +$$

$$\sum_{i=1}^{k} (\alpha\beta)^{k-i+1} \beta[p_\theta^i g_1 \theta_2^i + p_x^i g_2 x_2^{i+1}][d-2e(x_1^{i+1}+\theta_1^i)] = p_1^{k+1}, \quad p_1^0 = 0.$$

but by (B-4), the summation term is

$$\sum_{i=1}^{k}(\alpha\beta)^{k-i+1}\beta[p^{i}g_1\theta_2^{i}+p_x^{i}g_2x_2^{i}][d-2e(\frac{d}{2e}-\frac{p_1^{i}-\alpha\beta p_1^{i-1}}{\beta[p_\theta^{i}g_1\theta_2^{i}+p_x^{i}g_2x_2^{i}](2e)})]$$

or,

$$\sum_{i=1}^{k}(\alpha\beta)^{n-i+1}(p_1^{i}-\alpha\beta p_1^{i-1})$$

which reduces to $\alpha\beta p_1^{k}$. Then, solving the decision rule for θ_1^{k+1},

$$\text{(B-5)} \quad \theta_1^{k+1}=\frac{d}{2e}-x_1^{k+2}-\frac{p_1^{k+1}-\alpha\beta p_1^{k}}{\beta[p_\theta^{k+1}g_1\theta_2^{k+1}+p_x^{k+1}g_2x_2^{k+2}](2e)},\ p_1^{0}=0,$$

which, by the principle of mathematical induction, establishes equation (B-4) for any n, particularly,

$$\text{(B-6)} \quad \theta_1^{n}=\frac{d}{2e}-x_1^{n+1}-\frac{p_1^{n}-\alpha\beta p_1^{n-1}}{\beta[p_\theta^{n}g_1\theta_2^{n}+p_x^{n}g_2x_2^{n+1}](2e)},\ p_1^{0}=0.$$

LITERATURE CITED

1 Bellman, Richard. Adaptive Control Processes, Princeton University Press, Princeton, New Jersey, 1961.

2 Burt, Oscar R. and Allison, John R. "Farm Management Decisions with Dynamic Programming," Journal of Farm Economics, Vol. XLV, No. 1, February 1963.

3 Harlan, Jack R. "Generalized Curves for Gain per Head and Gain per Acre in Rates of Grazing Studies," Journal of Range Management, Vol. 11, No. 3, May 1958.

4 Hwang, C.L. and Fan, L.T. "A Discrete Version of Pontryagin's Maximum Principle," Operations Research, Vol. 15, No. 1, January-February 1967.

5 Mott, G.O. "Grazing Pressure and the Measurement of Pasture Production," Proceedings of the Eighth International Grassland Congress, 1960.

6 Neuhauser, George L. Introduction to Dynamic Programming, John Wiley and Sons, Inc., New York, 1966.

7 Pasture and Range Research Techniques, Comstock Publishing Associates, Ithaca, New York, 1962.

Bussing, William A
A dynamic programming model for range management decisions with suggestions for further research

	NAME AND ADDRESS
5-23-72	Ag Econ (F) JS McConnen by Berbert
4-11-74	1106 Hedges N (2) Susan Goodey SS
JUL 5 1974	Ellwood Soderlind AG ECON Dep T (S)
NOV 2 8 1978	Craig Simons 587584 1328 W. Babcock