

ANALYTICAL AND NUMERICAL MODELING OF CORONAL
SUPRA-ARCADE FAN STRUCTURES

by

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DEDICATION

For my father, Andrew Lowndes Scott, my mother, Marianne Benezet, and my sister, Ruthy Elizabeth Scott. I couldn't have asked for better. And for Beth Mendelsohn, whom I also hold dear.

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In my time at Montana State University I have benefited from the help and support of an incredible group of individuals. Without my good friends, Lucas Tarr, Paul T. Baker, and Adam Kobelski, who were always there with an encouraging word or a well timed trip to the Hauf, I would likely have caved under pressure. And I would certainly never have produced this collection of work without the guidance of my mentors, David McKenzie and Dana Longcope. In particular, I feel that I must thank David McKenzie for taking me in as a stray and giving me a chance to do science. I had done nothing to gain his confidence and I consider his faith in me at that time to be an act of incredible generosity. And finally, if not for the staff here in the physics department of MSU none of us would make it to the end of the day. Too often their efforts go unnoticed but they are the unsung heros of our community.

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ABSTRACT

Among the myriad of interesting phenomenon in the solar corona is the highly dynamic region above active region arcades, commonly referred to as the “supra-arcade” region. In the minutes and hours following the formation of an arcade of post-flare loops, we commonly observe the development of a curtain like structure, with spiny rays of enhanced emission in X-Ray and extreme ultra-violet. Additionally, these structures often exhibit dynamics over a variety of length scales, from large-amplitude coherent transverse oscillations, to the appearance of low-emission columns that seem to descend toward the solar limb. The wealth of dynamical aspects that are present in the supra-arcade seems to indicate that the plasma there is subject to a complex balance of influencing factors, which makes it difficult to develop a self-consistent hypothesis for describing all of the various features simultaneously. In this work we undertake to explain one such behavior as an isolated phenomenon. We argue that the descending low-emission voids, sometimes called Supra-Arcade Downflows (SADs) are consistent with the formation of a particular kind of shock in the vicinity of a retracting element of reconnected magnetic flux. We then use numerical simulations to expand this result to a broader parameter space, as well as investigating the details of a variety of other behavioral regimes. Finally, in an effort to understand the broader dynamics of the supra-arcade region, we undertake a study that incorporates imaging data into a numerical simulation, which can then be used to estimate the ambient plasma parameters in the supra-arcade region. In this way we show that the balance of influencing factors in the supra-arcade is indeed highly dynamic and that the simplifications offered in certain extremes of magnetohydrodynamics are ill-applied in this case.

1. INTRODUCTION

1.1. Studying the Solar Corona

Contrary to the norm, researchers in solar physics enjoy the benefits of an enormous international community with a policy of complete transparency in the sharing of information and data. Many ground based and space-borne instruments have been built as part of collaborative efforts between the European Space Agency (ESA), the Japanese Aerospace eXploration Agency (JAXA), and the National Aeronautics and Space Agency (NASA), as well as the many US research labs, including Lockheed Martin, Jet Propulsion Labs and the Naval Research Lab, to name a few. Without these, and the free sharing of the data that they collect, there would be nothing to study, and it is through their continued efforts that the solar physics community continues its progress toward a more complete understanding of the sun. And yet, for all the recent advancements in solar observation and data collection, the inherent difficulty in applying broad and well established principles to specific solar events and classes of events remains a problem that is ubiquitous in solar physics.

Nowhere is this more apparent than in solar active regions, where twisted and strained coronal magnetic fields undergo reconnection processes, thereby releasing their stored magnetic energy. The reconfiguration of the magnetic field through magnetic reconnection, which is critical to the abrupt changes in the magnetic field that

are observed in events like solar flares and Coronal Mass Ejections (CMEs), is no exception. Many models exist to describe the morphological changes that can occur through reconnection and yet the typically accepted value for the resistivity of coronal plasmas seems insufficient to allow for the reconfiguration of coronal magnetic field structures, which span hundreds of megameters, over times as short as a few minutes to several hours, as is commonly observed. And so, while the broader aspects of reconnection are well understood, and despite ongoing efforts to refine these models, the detailed applicability to specific events remains elusive.

Another difficulty that is intrinsic to solar physics is the simultaneous influence of the two fundamental forcing terms in magnetohydrodynamics (MHD): the pressure and the Lorentz force. These two contributions affect the evolution of magnetized plasmas in fundamentally different ways and so plasma systems are often characterized by the relative scaling of their contributions, with so called low- β and high- β regimes corresponding to magnetic and fluid dominated plasmas, respectively. For example, the solar wind is typically considered to be a high- β plasma, with the ion pressure providing the dominant term in the momentum equation, so that the magnetic field is simply swept along with the fluid, which is itself largely indifferent to the Lorentz force.

Active region cores are typified by the opposite extreme, with magnetic structures that seem to evolve in response to the Lorentz force with little to no consideration

for the ion pressure. These regions are typically considered to fall into the low- β regime, but that is not to say that the presence of the plasma is unimportant. Ironically, without the presence of coronal plasma in active regions there would be no emission there at all. And even when the pressure provides a vanishing contribution to the momentum equation, it is the plasma that provides the electrical conductivity that keeps the magnetic diffusivity small. Without this the field would quickly and smoothly evolve to the lowest possible energy state and would not exhibit the slow build-up of magnetic energy and subsequent loss of stability that ultimately leads to solar energetic events.

Yet, while many parts of the solar atmosphere conform to one limit of MHD or another, there are regions in which the balance of contributing effects is more complex. When the distribution of photospheric flux in an active region exhibits a predominantly linear polarity inversion line, the post-flare loops that form in the aftermath of magnetic reconnection often settle into a sort of elongated arch that straddles the polarity inversion line. This structure is referred to as a flare loop arcade and in the region above the arcade, the so-called supra-arcade, we often witness the formation of sheet-like structures (sometimes called Current Sheet / Thermal Halos or CSTHs) that extend radially outward along the spine of the arcade. One example, from the 2011 October 22 event, is shown in Figure 1.1. The flare loop arcade and supra-arcade are clearly visible in the 131 Å and 193 Å channels of the Atmospheric

Imaging Assembly (AIA) onboard the Solar Dynamics Observatory (SDO) (Lemen et al., 2012). And unlike active region cores, which can typically be modeled without consideration of plasma pressure, structures in the supra-arcade seem to evolve in response to a number of competing factors simultaneously.

1.2. Dynamics of the Supra-Arcade Region

Early observations of supra-arcade plasmas served as a contextual verification of the so-called standard flare model. In this model, also referred to as the CSHKP model (Carmichael, 1964; Sturrock, 1966; Hirayama, 1974; Kopp & Pneuman, 1976), the relaxation of a tortured magnetic field toward equilibrium results in the release of a flux rope (CME) while the lower lying loops form a flare-loop arcade, and between the arcade and the expanding CME there is a current sheet, which supports a rotational discontinuity in the magnetic field. Pneuman (1972) argued that coronal helmet streamers, which are morphologically analogous to supra-arcade structures, are fundamentally related to the structure of the coronal magnetic field and indicative of the close proximity of regions of opposing magnetic field, i.e. a current sheet. Then with the advent of coronal X-ray observations, the connection between coronal streamers and flaring active regions (flare-loop arcades and supra-arcade structures) was made rigorous (see, for example, Svestka et al., 1997; Švestka et al., 1998).

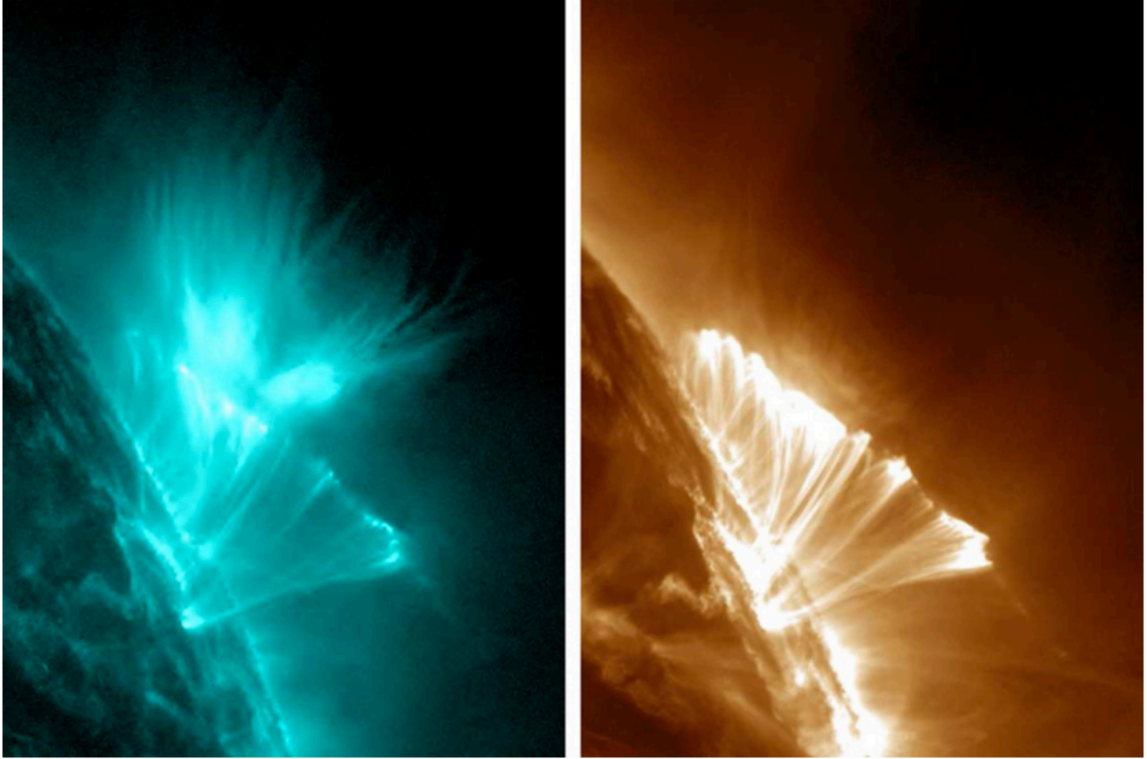


Figure 1.1. A flare loop arcade and supra-arcade current sheet / thermal halo were observed by AIA/SDO on 2011 October 22. These structures appear in 131 Å and 193 Å in the left and right panels, respectively.

The dynamics of the plasma within these structures were first characterized by McKenzie & Hudson (1999), who observed that, in addition to behaving as a tether between the low-lying arcade loops and the expanding CME, supra-arcade plasmas also exhibit complex internal flows and density variations that develop on length scales that are much shorter than the characteristic size of the underlying photospheric flux systems. Foremost among these observations was the early identification of descending streaks of reduced emission, which were later dubbed Supra-Arcade Downflows

(SADs) by McKenzie (2000). These can also be seen in Figure 1.1 as spiny, dark lanes in the 131 Å supra-arcade emission.

In recent years the supra-arcade region has been carefully scrutinized regarding the coincidence of the sheet like emission with the assumed presence of a current sheet, as well as the fundamental nature of the complex flows that are apparent in X-ray and extreme ultra-violet. Some of the more persistent questions are these: Does the supra-arcade emission originate from within the current sheet or is it associated with the surrounding plasma; and in either case, what mechanism is responsible for elevating the plasma temperature to allow for the observed emission spectra? What is the plasma β in the supra-arcade region and how is it structured? What causes the formation of dark lanes and why do they descend toward the limb? And if they are caused by some transient element, why do they not “fill in” after its passing? Is the apparent motion associated with variations in the plasma density or is it a thermal effect, exacerbated by instrumental effects? And if it is a density effect, is the apparent motion related to bulk displacements or wave-like amplitude perturbations?

In order to address these issues, several authors have undertaken investigations that attempt to disambiguate observations of supra-arcade structures and, in some cases, unify seemingly disparate behaviors under a single explanation. The initial explanation of SADs offered in McKenzie (2000) has been further expounded upon by Savage & McKenzie (2011) and Savage et al. (2012), with careful treatment of

the similarities in the dynamics of SADs and coincident retracting loop structures or Supra-Arcade Downflowing Loops (SADLs). In the current iteration, this model suggests that retracting post-reconnection loops, which originate high above the arcade (presumably as entrained flux behind the expanding CME) displace the supra-arcade plasma as they descend and thereby create a sort of wake, which ultimately leads to the low emission, dark lanes.

Other modeling efforts have been made by Cassak et al. (2013), who attempted to reproduce SADs using a steady reconnection outflow that continually displaces plasma within the arcade to maintain a lane of low-emission plasma, and Cécere et al. (2012), who invoke an initial over-pressurization of the plasma to stimulate a shock-driven interference effect. And, more recently, Guo et al. (2014) have recovered an effect that is qualitatively similar to SADs from simulations of magnetic reconnection within the supra-arcade current sheet, which they argue is susceptible to the Rayleigh-Taylor instability. Each of these seems plausible and yet, since each relies on a specific initial configuration, it is not yet obvious which, if any, is consistent with the conditions within the supra-arcade region.

Further work has gone into characterizing the apparent motion of supra-arcade plasma. It has been suggested that temperature fronts, propagating through the supra-arcade, could conspire with instrument response functions and thereby create the appearance of motion where there is none. Hanneman & Reeves (2014) performed

inversions of multi-spectral imaging observations and revealed that the temperature dependent Differential Emission Measure (DEM) in the supra-arcade region is highly dynamic above $\log_{10} T \sim 10^{6.5}$, but while their result could support the notion of thermal velocity artifacts, the specific temperature fluctuations that they observed are inconsistent with the observed changes in emission; it seems that changes in the overall density are necessary to explain the observations. McKenzie (2013) inferred velocity fields from Fourier Local Correlation Tracking (FLCT) that seem to be consistent with turbulent dynamics and suggest that the apparent motion is “real”, and not an instrumental effect. This view is further supported by the work of Innes et al. (2003), who found line of sight velocities that are consistent with the apparent flow velocities described in McKenzie (2013).

Perhaps the most controversial topic is that of the energy distribution. The magnetic dominance of the solar corona has often been invoked in the context of the supra-arcade region despite the enormous range of possible values of β as described, for example, in Gary (2001). Certainly β must be small in active region cores, and large in the region of the expanding CME, but in the supra-arcade, which is at an intermediate height, between these two regimes, the energy balance is not entirely obvious. McKenzie (2013) performed a Potential Field Source Surface (PFSS) extrapolation, combined with published values for the density and temperature, and

argued that, far from being small, β is likely of order unity or greater. This estimate is consistent with the apparent motion of density features in the supra-arcade, which seem inconsistent with the extreme limits of ideal MHD. In particular, SADs are suggestive of a complex interaction between the plasma and the magnetic field; a high β regime would be unlikely to support the transverse density stratification that is presumably responsible for the dark lanes, while a low β regime seems inconsistent with the slow, wavering motion of spiky fan features. Moreover, the transverse oscillations observed by Verwichte et al. (2005) appear to propagate at a speed that is smaller than but still of the same order as the Alfvén speed, which implies that either the slow magnetosonic speed is not significantly slower than the fast magnetosonic speed, or that their estimate of the Alfvén speed is too high, either of which point to a non-trivial value for β .

1.3. Research Described Herein

The research presented here consists of three self-contained manuscripts that, together, constitute an attempt to disambiguate certain aspects of the supra-arcade region and its characteristics using a magnetohydrodynamic framework under an assortment of simplifying assumptions. The organization and content of these papers is as follows. In Ch.2 we develop an analytical model for the flow of plasma around a magnetic element, which serves as proxy for a retracting flux tube as in the model

of SADs by Savage et al. (2012). This model is predicated on the assumption that β is vanishingly small so that the magnetic field is understood to be impervious to the response of the fluid. Through this study we show that the interpretation of Savage et al. (2012) has a viable foundation in magnetohydrodynamics, provided that the presence of a retracting magnetic element can be motivated.

Then, owing to the fact that the zero- β limit is likely to be inconsistent with the conditions of the supra-arcade region, we go on in Ch.3 to perform a numerical investigation that relaxes this limitation. We choose for this the same model setup as in Ch.2, which allows for the generalization of our previous findings to a broader parameter space of magnetohydrodynamics. This study, therefore, extends the results of Scott et al. (2013) and further solidifies the viability of Savage et al. (2012) by demonstrating the plausibility of that model for β up to values of order one. Moreover, the simulation technique that we employ demonstrates the possibility of rigorously mapping the interaction between the retracting flux tube and the surrounding plasma, which may be useful in characterizing the drag force exerted on the retracting flux tube, an effect that is of great importance to other aspects of coronal dynamics.

Finally, in Ch.4, we alter our approach and develop a different sort of numerical model in which extreme ultra-violet observational data is incorporated into an MHD simulation of plasma throughout the supra-arcade, not just in the near vicinity of a single element of reconnected flux. This model is then used to estimate the

approximate value of the magnetic field strength and plasma β in the supra-arcade region, which we find to be much higher than the assumed low β of the underlying arcade. We further show that this data-assimilation technique can be used to characterize the interaction between the supra-arcade plasma and the more diffuse plasma of the extended corona, which interaction may be of great importance to understanding and interpreting future in-situ observations from satellite missions into the inner heliosphere.

2. PERISTALTIC PUMPING NEAR POST-CME SUPRA-ARCADE CURRENT SHEETS

Contribution of Authors and Co-Authors

Manuscript in Chapter 2

Author: Roger B. Scott

Contributions: Conceived and developed analytical model. Wrote code for quasi-analytical evolution. Wrote manuscript content.

Co-Author: Dr. Dana W. Longcope

Contributions: Helped to conceive and develop analytical model. Provided feedback of analysis and comments on drafts of the manuscript.

Co-Author: Dr. David E. McKenzie

Contributions: Helped to conceive analytical model. Provided feedback of analysis and comments on drafts of the manuscript.

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ABSTRACT

Measurements of temperature and density near supra-arcade current sheets suggest that plasma on unreconnected field lines may experience some degree of “pre-heating” and “pre-densification” prior to their reconnection. Models of patchy reconnection allow for heating and acceleration of plasma along reconnected field lines but do not offer a mechanism for transport of thermal energy across field lines. Here we present a model in which a reconnected flux tube retracts, deforming the surrounding layer of unreconnected field. The deformation creates constrictions that act as peristaltic pumps, driving plasma flow along affected field lines. Under certain circumstances these flows lead to shocks that can extend far out into the unreconnected field, altering the plasma properties in the affected region. These findings have direct implications for observations in the solar corona, particularly in regard to such phenomena as high temperatures near current sheets in eruptive solar flares and wakes seen in the form of descending regions of density depletion or supra-arcade downflows.

2.1. Introduction

Since the development of X-ray and EUV solar imaging, observations of evolving arcade structures have become a ubiquitous signature of magnetic reconnection in solar flares. Many of these structures also exhibit vertical fans with highly emissive coronal plasma and what is presumed to be a nearly vertical magnetic field rising above the apex of the arcade (Švestka et al., 1998; McKenzie & Hudson, 1999; Webb et al., 2003). This picture is consistent with the standard flare model in which a current sheet separates antiparallel layers of magnetic field between an arcade of reconnected flux and a rising coronal mass ejection (Cliver & Hudson, 2002). But while

the general properties of these structures are well established, the mechanism responsible for increased emission from plasma in the supra-arcade fan remains unclear (Seaton & Forbes, 2009; Ko et al., 2010; Reeves et al., 2010).

One possibility is that the emitting plasma is within the current sheet itself and that its temperature has been increased as a result of ohmic heating. This explanation relies on the assumption that the line-of-sight depth of the current sheet is large enough to allow for a non-negligible emission measure. In cases where the current sheet is observed edge-on (or nearly so), the line-of-sight depth can easily exceed 10^5 km. These edge-on observations (e.g., Ciaravella & Raymond, 2008; Savage et al., 2010) also enable upper limits to be placed on the thickness of the current sheet: the measurements indicate thicknesses of no more than $5\text{--}50 \times 10^3$ km. Conversely, Tucker (1973) used theoretical arguments and estimated that post-CME current sheets should have a thickness of roughly 10^{-1} km. Such thickness estimates become crucial in cases where the current sheets are observed face-on, as in Švestka et al. (1998), McKenzie & Hudson (1999), Innes et al. (2003), Warren et al. (2011), and McKenzie (2013).

An alternative explanation is that the emission comes not from within the current sheet itself but rather from a *thermal halo* that surrounds the current sheet. The thermal halo could be orders of magnitude thicker than the current sheet and thus provide sufficient line-of-sight depth for observed emission. However, this scenario requires some mechanism for increasing the local plasma density above that of the

surrounding corona. Chromospheric evaporation is a likely candidate for this pre-densification, but one must still justify the timeliness of the evaporation, which is usually attributed to thermal conduction into the chromosphere (Cargill et al., 1995).

Reconnection within the current sheet is a likely source of energy both for the heating of plasma and for evaporation-driven pre-densification because it efficiently converts magnetic free energy into thermal and kinetic energy (Guidoni & Longcope, 2010; Priest, 1999). But while reconnection may provide sufficient energy to heat the surrounding plasma, thermal conductivity transverse to the magnetic field is very weak (Choudhuri, 1998). Even if we assume that there exists a well of thermal energy in the reconnected field, it remains unclear what mechanism could be responsible for transporting the energy across field lines. And while radiative transfer is not limited by thermal conduction it is also far too weak given the low optical depth that is typical of the corona. Nonlinear mode-coupling could play a role if the reconnection event somehow excited magnetosonic waves that then dissipated energy in the surrounding plasma.

Recent observations in EUV (Savage et al., 2012; Warren et al., 2011; Savage & McKenzie, 2011) have resolved what appear to be magnetic loops that descend through the supra-arcade fan. The loops seem to form wake-like structures that appear as density depletions or voids in the surrounding plasma. The nature of these voids was studied by Verwichte et al. (2005), who characterized the apparent wave

motion of their edges. They found that the boundary between the low density voids and the surrounding plasma exhibited transverse oscillating wavelets that propagated sunward at speeds in the range of 50 km s^{-1} to 500 km s^{-1} . Costa et al. (2009) simulated the formation of these dark lanes from an initial pressure perturbation. They found that the lanes could be interpreted as an interference pattern resulting from the reflection of magnetosonic shocks and rarefaction waves. More recently, Cassak et al. (2013) simulated the formation of dark lanes as “flow channels carved by sunward-directed outflow jets from reconnection.” The applicability of this last interpretation, which places the voids below the arcade itself, must be carefully considered when placed in the context of observations of supra-arcade features.

Another possibility is that these features are the result of patchy reconnection in which flux tubes retract toward the arcade under the influence of magnetic tension and are drawn through the surrounding, unreconnected field as depicted in Figure 2.1. Previous authors have modeled the dynamics of the retracting flux tube (Guidoni & Longcope, 2010; Longcope et al., 2009; Linton & Longcope, 2006), but have not yet considered its effect on the surrounding, unreconnected flux. Cargill et al. (1996) modeled the interaction of a magnetic cloud and the surrounding magnetic field, but their work focused on the high- β regime ($\beta = 8\pi p/B^2 \gg 1$). For our analysis we will consider the consequences of an extremely low- β scenario in which the magnetic field dominates all other energy contributions.

Our focus will be to consider how the plasma and unreconnected flux that surround the current sheet behave in response to a reconnection event. Toward this end we assume that a localized reconnection event has already occurred within a supra-arcade current sheet and has created a bundle of newly closed magnetic field lines, a flux tube, which retracts through the current sheet (Linton & Longcope, 2006; Longcope et al., 2009) as depicted in Figure 2.1. The retracting flux tube is a prescribed element whose radius and motion are parameters of the model. The primary effect we consider is the deformation it creates in the surrounding field. The deflection of a given field line is bounded by the radius of the retracting tube and is smaller farther away. Due to this smallness the deformation is typically dismissed as a minor effect, though Linton & Longcope (2006) did consider the possibility that the work required to deform the external field might contribute to a drag force on the retracting tube.

The retracting flux tube could have two possible orientations relative to the current sheet. The theoretical work of Linton & Longcope (2006) assumes that a section of the tube lies within the plane of the current sheet, as shown the right side of Figure 2.1. On the other hand imaging observations have been interpreted assuming the flux tube pierces the current sheet normally, as in the right panel of Figure 2.1 (McKenzie, 2000; Savage et al., 2012). Our modeling will be applicable to either scenario since both create identical deformations in the surrounding field. We hereafter focus on

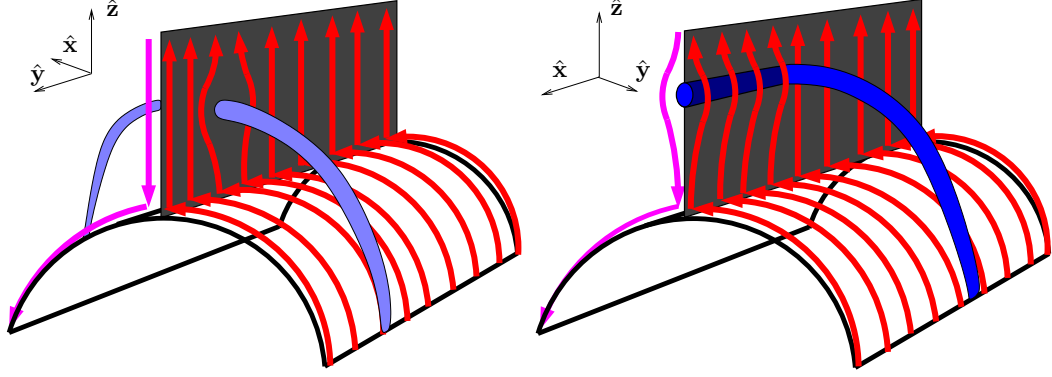


Figure 2.1. The left panel depicts a reconnected flux tube piercing normally through the current sheet as in Savage et al. (2012) while the right panel depicts a reconnected flux tube embedded within the current sheet as in Linton & Longcope (2006). Taking $\hat{\mathbf{z}}$ to point vertically away from the limb with $\hat{\mathbf{x}}$ pointing along the reconnected flux tube, $\hat{\mathbf{y}}$ is either normal to or in the plane of the current sheet, depending on the configuration.

the surrounding field which is roughly vertical, and refer to the retracting flux as an *intrusion*.

In the present work we show that the deformation takes the form of a constriction, which moves downward through the surrounding field at the same speed as the retracting flux tube. Observations clearly show this speed to be some fraction of the local Alfvén speed (Savage & McKenzie, 2011), and often in excess of the local sound speed. We observe that the moving constriction behaves as a peristaltic pump, resulting in field-aligned plasma flows, which we dub *peristaltic flows*. We show below that there are regimes in which these flows lead to slow magnetosonic shocks that develop in the surrounding field. These manifest in our model as hydrodynamic shocks and rarefaction waves, which travel along the field at speeds comparable to the sound

speed. The existence of such features leads to several dramatic effects, including significant heating and changes to the density and emission measure of plasma in the unreconnected field.

2.2. The Model

The model that we present here treats the unreconnected flux as current-free field along which plasma is constrained to move. We begin by determining the magnetic field subject to the influence of an intruding, reconnected flux tube. We assume that β is extremely small so that the magnetic field may be determined independent of the plasma. This dictates both the plasma flow trajectory and the cross section of parallel flow. Steady solutions are then found for plasma flow along each field line. Points where the flow is ill-defined are avoided through the introduction of rarefaction waves and acoustic shocks, which are a limiting form of slow magnetosonic shocks in very low β . The result is a piecewise continuous adiabatic series of solutions that evolve in time as the fluid jumps propagate. The 2D behavior is ultimately found through interpolating between solutions along representative field lines.

Our analysis will invoke two distinct reference frames. The *limb-frame* is stationary with respect to the solar surface and in this frame the undisturbed plasma is at rest. Alternatively, in the *comoving frame* it is the descending intrusion that is at rest and the plasma is taken to be rising uniformly at large distances from the intrusion.

It is in the comoving frame that the magnetic field is most easily determined because in this frame the boundary conditions are steady in time and therefore, so too is the field.

2.2.1. Deformed Potential Field

In the comoving frame the unreconnected magnetic field is a sum of the original magnetic field prior to distortion ($\mathbf{B}_0$) and a second field ($\mathbf{B}'$) representing the influence of the intruding flux tube

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}'. \quad (2.1)$$

Since the reconnected and unreconnected fields are topologically distinct we will impose the simplifying constraint that no unreconnected field lines may intersect the surface of the reconnected flux tube. Furthermore, as information about the reconnection event cannot have influenced the field at arbitrarily large distances, $\mathbf{B}$ must reduce to $\mathbf{B}_0$ far from the flux tube, so $\mathbf{B}'$ must vanish there.

We take the background field to be uniform and vertical ($\mathbf{B}_0 = \pm \hat{\mathbf{z}} B_0$) while the reconnected flux defines a uniform cylinder (S), centered at the origin, with radius R and symmetry axis pointing in the $\hat{\mathbf{x}}$ direction.¹ $\mathbf{B}'$ depends only on y and z . The total field is assumed to be potential with boundary conditions given by

$$\hat{\mathbf{z}} \times \mathbf{B}|_{r \rightarrow \infty} = \hat{\mathbf{r}} \cdot \mathbf{B}|_{r \in S} = 0. \quad (2.2)$$

¹This assumes that the radius of curvature of the reconnected flux is large compared to the embedded length within the fan.

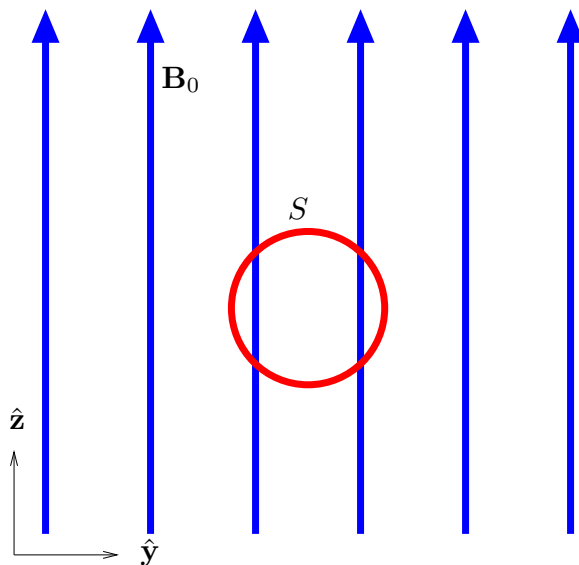


Figure 2.2. An initially uniform field ($\mathbf{B}_0$) is altered by the intruding surface (S) with $\mathbf{B}'$ introduced so that the net field ($\mathbf{B} = \mathbf{B}_0 + \mathbf{B}'$) satisfies the appropriate boundary conditions on S .

The first constraint ensures that the magnetic field is unaffected far from S while the latter ensures that no field lines intersect S .

The potential magnetic field, constrained by these boundary conditions, may be determined in terms of a flux function such that

$$\mathbf{B} = \hat{\mathbf{x}} \times B_0 \nabla f = -B_0 \nabla \times (\hat{\mathbf{x}} f), \quad (2.3)$$

where $\hat{\mathbf{x}}$, being the axis of symmetry of the intruding flux, is an ignorable direction.

In the far field, setting $f \rightarrow y$ ensures that $\mathbf{B} \rightarrow B_0 \hat{\mathbf{z}}$, while on the surface of the feature $\hat{\mathbf{r}} \cdot \mathbf{B} = 0$ so $\partial_\theta f|_S = 0$. Thus, in terms of f the boundary conditions become

$$\nabla f \times \hat{\mathbf{y}}|_{r \rightarrow \infty} = 0 \quad (2.4)$$

and

$$\nabla f \times \hat{\mathbf{r}}|_{r \rightarrow R} = 0. \quad (2.5)$$

And, since $\nabla \times \mathbf{B} = 0$, f satisfies Laplace's equation;

$$\nabla^2 f = 0. \quad (2.6)$$

With these conditions and the choice that f be symmetric in y , the flux function is uniquely specified as

$$f = \sin(\theta) \left(\frac{R^2}{r} - r \right) = y \left(1 - \frac{R^2}{y^2 + z^2} \right), \quad (2.7)$$

with θ measured from $-\hat{\mathbf{z}}$. Since the magnetic field is everywhere orthogonal to the gradient of f , contours of f are themselves field lines, denoted $\mathbf{X}_f$, which can be parameterized by solving for z in terms of f and y so that

$$z_f(y)^2 = y \frac{y^2 - fy - R^2}{f - y}, \quad (2.8)$$

where, for a given f , the y coordinate along the field line is bounded by

$$|f| < |y| < \sqrt{f^2/4 + 1} + |f|/2. \quad (2.9)$$

Figure 2.3 shows a contour plot of $f(y, z)$ that traces a representative set of field lines. For each field line, $y \rightarrow f$ as $|z| \rightarrow \infty$. Values of f are therefore the lateral positions of the field lines in the far field. The deflection of each field line is largest abreast of the intrusion, where $z = 0$. For $|f| \gg R$ this deflection goes to zero while for the most strongly deflected field line ($f = 0$) the deflection is $|\Delta y| = R$.

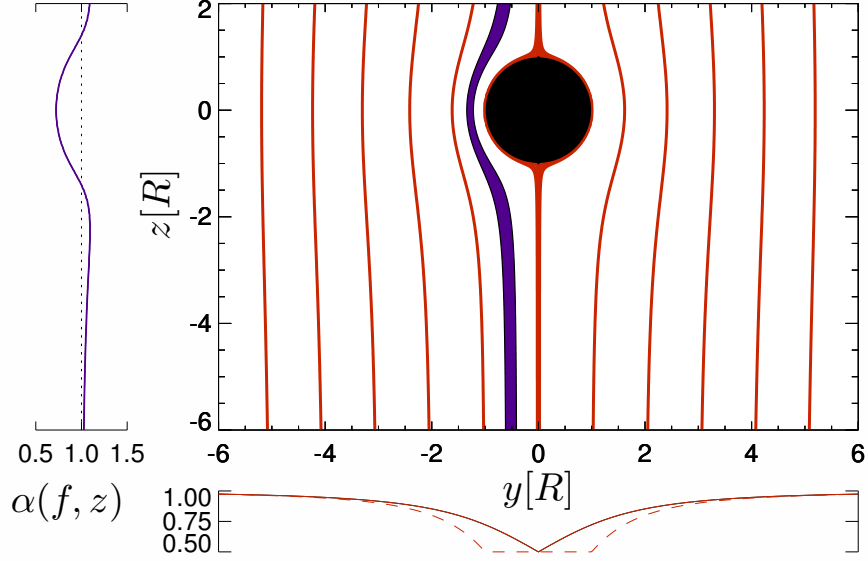


Figure 2.3. The potential magnetic field is deformed by the expansion of the origin onto a cylindrical surface of radius R . Field lines are deflected around the intrusion. The inverse normalized field strength (α) is shown on the left panel for the $f = -R/2$ field line, parameterized in z . The associated flux tube is traced in purple on the main panel. In the bottom panel the minimum inverse normalized field strength ($\alpha_{z=0}$) is plotted first as a function of f (solid red) and then as a function of y (dashed red).

Ultimately, we will be interested in the parameterized cross section of an arbitrary, unreconnected flux tube. From $\nabla \cdot \mathbf{B} = 0$ it follows that the cross section scales inversely with the field strength. Let $\alpha(\mathbf{X}_f)$ be the inverse of the dimensionless field strength of an infinitesimal flux tube, normalized to unity in the far field and parameterized along an arbitrary field line, $\mathbf{X}_f$, so that $1/\alpha \equiv |\mathbf{B}|/B_o$. In terms of f

$$\frac{1}{\alpha^2} = (\partial_y f)^2 + (\partial_z f)^2 \quad (2.10)$$

$$= \frac{2R^2(y^2 - z^2) + R^4}{(y^2 + z^2)^2} + 1, \quad (2.11)$$

which, after substituting in for $z_f(y)$, can be expressed as

$$\alpha^2 = \frac{y^2 R^2}{f^2 R^2 + 4y^2 (f - y)^2}. \quad (2.12)$$

Holding f fixed, we define $\alpha_f(y)$ to be the cross section of an infinitesimal flux tube centered on a field line $\mathbf{X}_f$, parameterized by the lateral deflection of the field line.

α_f achieves a minimum value at $z = 0$, where the field line passes abreast of the intrusion. This location is referred to as the throat of the flux tube and has a cross section of

$$\alpha_{min}(f) = \frac{1}{2} \frac{\sqrt{f^2 + 4R^2} + f}{\sqrt{f^2 + 4R^2}}, \quad (2.13)$$

which is necessarily less than one. Moving away from the throat along the field line the flux tube expands until it reaches a maximum value, which is necessarily greater than one, and then slowly contracts toward unity as $|z| \rightarrow \infty$. In general, field lines that pass close to the intrusion have the smallest minimum cross section and greatest overall variability while for large $|f|$ values α is nearly uniform along z . The $f = 0$ field line is both the most and least constricted with a cross section that diverges at $y = 0, z = \pm R$ and achieves the global minimum of $\alpha_{min}(f = 0) = \min[\alpha(y, z)] = 0.5$ at its throat.

2.2.2. Peristaltic Flow

Under the assumption of ideal magnetic induction, as a fluid element moves it must remain on the same field line and its cross section for flow parallel to the field

must be the same as that of the associated flux tube. Since the magnetic field is stationary with respect to the descending intrusion, the flow will be steady in the co-moving frame. The steady version of the continuity equation, $\nabla \cdot (\rho \mathbf{u}) = 0$, is satisfied by a constant mass flux

$$\dot{m} = \rho u \alpha, \quad (2.14)$$

where ρ is the density, u is the speed of the fluid, α is the cross section of the flux tube defined by f and $\dot{m}$ is a constant of integration that is conserved along $\mathbf{X}_f$. The steady flow must also satisfy the momentum equation

$$\rho(\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} + \nabla \cdot \underline{\underline{\sigma}} - \nabla p, \quad (2.15)$$

where gravity is omitted for simplicity. Here p is the plasma pressure and $\underline{\underline{\sigma}}$ is the viscous stress tensor. Since the flow must be parallel to the magnetic field, the Lorentz force makes no contribution to the parallel momentum equation

$$(\mathbf{u} \cdot \nabla) \frac{1}{2} u^2 + \frac{1}{\rho} (\mathbf{u} \cdot \nabla) p = \frac{1}{\rho} \mathbf{u} \cdot (\nabla \cdot \underline{\underline{\sigma}}). \quad (2.16)$$

This equation is the same as that of a neutral fluid passing through a nozzle. Together with an energy equation relating ρ and p , Eqs. (2.16) and (2.14) fully specify the spatial variation of the fluid along the length of the flux tube.

For simplicity we adopt the isothermal equation of state

$$p = C_s^2 \rho, \quad (2.17)$$

where C_s is the sound speed. This assumption is motivated by the very high thermal conduction along field lines. Combining Eq. (2.17) with Eq. (2.16) and integrating over the volume of a fluid element with parameterized length l leads to

$$\left[\frac{1}{2}u^2 + C_s^2 \ln \rho \right] \Big|_{l_1}^{l_2} = \int_{l_1}^{l_2} dl \, \hat{u} \cdot \left[\frac{1}{\rho} \nabla \cdot \underline{\underline{\sigma}} \right], \quad (2.18)$$

where l_1 and l_2 are two arbitrary locations along the flux tube. Under strong magnetization the viscous force is dominated by the parallel contribution (Guidoni & Longcope, 2010). Using a field-aligned coordinate system it can be shown that this contribution takes the form

$$\hat{u} \cdot \left[\nabla \cdot \underline{\underline{\sigma}} \right] = \frac{\mu^{(0)}}{\alpha^4} \left(\frac{4}{3} \partial_l^2 \left(\frac{u}{\alpha} \right) + 2 \partial_l \left(u \partial_l \frac{1}{\alpha} \right) \right), \quad (2.19)$$

where $\mu^{(0)}$ is the dominant coefficient of dynamic viscosity, which is proportional to $\rho \lambda C_s$, and λ is ion mean free path. When the flow is sufficiently smooth for the ion mean-free path to be negligible, the viscous contribution to the momentum equation may be neglected and the left hand side of Eq. (2.18) is conserved along the length of the flux tube. Only in the case of a shock, where fluid variation cascades to shorter length scales, is the viscous contribution significant.

After neglecting viscosity, Eqs. (2.18) and (2.14) lead to a relationship between the flux-tube cross section α and the Mach number ($M = u/C_s$) that is equivalent de Laval's equation for steady flow through a nozzle;

$$M^2 - \ln M^2 - \ln \alpha^2 = \mathcal{B}. \quad (2.20)$$

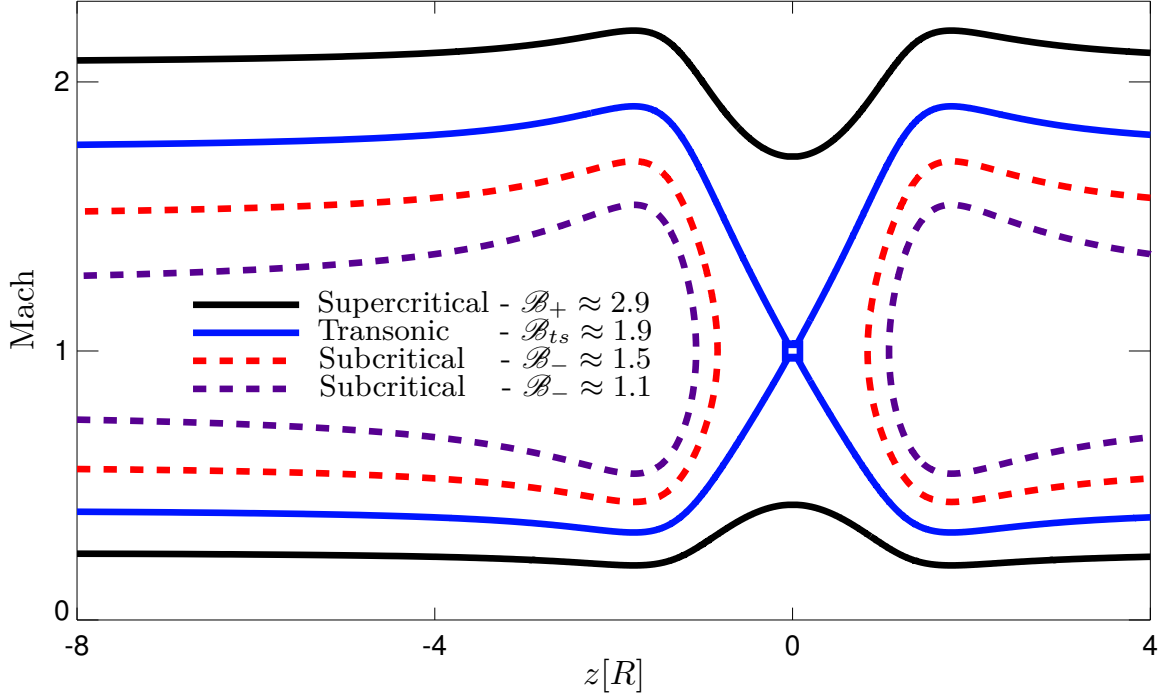


Figure 2.4. Contours of Eq. (2.20) are plotted for the $f = R/2$ field line. Each color represents a different value for $\mathcal{B}$. Supercritical solutions (black) are well-behaved. Subcritical solutions (purple, red) are ill-behaved where $M \rightarrow 1$ and ill-defined in the range $-z_{crit} < z < z_{crit}$. The two regimes are separated by the transonic contour (blue), which passes through $M = 1, z = 0$. Fluid flow is from left to right.

$\mathcal{B}$ is effectively Bernoulli's constant and is a conserved quantity along any time-independent, inviscid flow. $\mathcal{B}$ can, in principle, assume any real, positive value, and will generally be determined by evaluating M and α at a particular point of interest. For real values of M the quantity $M^2 - \ln M^2$ has a minimum value of unity at $M = 1$ and diverges monotonically as M goes to either zero or infinity. There are two solutions to Eq. (2.20) corresponding to any value of $\mathcal{B}$ – one subsonic and one supersonic.

The behavior of this system can be visualized by plotting contours of $\mathcal{B}$ in M - z phase space. Figure 2.4 shows a representative set of solutions for the $f = R/2$ field line with $\mathcal{B}$ ranging from approximately 1.1 for the dashed purple contour up to nearly 3 for the black contour. The qualitative behavior of these solutions is dictated by how $\mathcal{B}$ relates to the critical value of

$$\mathcal{B}_{ts}(f) = 1 - 2 \ln \alpha_{min}(f), \quad (2.21)$$

which defines the transonic flow contour for which $M \rightarrow 1$ exactly at the throat of the constriction where $\alpha \rightarrow \alpha_{min}$. The transonic contour separates the so called *supercritical* solutions, given by $\mathcal{B} \in \mathcal{B}_+ > \mathcal{B}_{ts}$, from the *subcritical* solutions, given by $\mathcal{B} \in \mathcal{B}_- < \mathcal{B}_{ts}$. As an example, consider the $f = R/2$ field line depicted in Figure 2.4. The minimum cross section is $\alpha_{min}(R/2) \approx 0.62$ and so $\mathcal{B}_{ts}(R/2) \approx 2$. Inverting Eq. (2.20) for $\alpha = 1$ (in the far field) we find the two transonic inflow values are $M_{ts}(R/2) \approx 1.75$ and 0.41 .

Supercritical solutions have the property that $M_+(z)^2 - \ln M_+(z)^2 > 1$ for all values of $\alpha_f(z)$ so that $M(\mathcal{B}_+, f, z)$ is well defined along the entire flux tube. The black contours in Figure 2.4 represent the supersonic and subsonic solutions for a particular value of $\mathcal{B}_+$. Note that these contours are everywhere either supersonic or subsonic and are well defined as $z \rightarrow 0$. Subcritical solutions do not have this property and are ill-defined at any location where the cross section is smaller than

the so called *critical cross section*, given by

$$\alpha_{crit} = e^{(1-\mathcal{B}_-)/2}. \quad (2.22)$$

Subcritical solutions are defined by the existence of a set of *critical points*, given by the two locations, $z = \pm z_{crit}(f)$, that satisfy Eq. (2.22). At these critical points $M(\mathcal{B}_-, f, z_{crit}) = 1$, while over the interval $-z_{crit} < z < z_{crit}$ the Mach number is ill-defined. The dashed red and purple contours of Figure 2.4 represent subcritical solutions for two different values of $\mathcal{B}_-$. In both cases the Mach number goes to unity at $z = \pm z_{crit}(f)$ and is ill-defined over the interval between the two critical points. A third solution is visible as the blue line in Figure 2.4 and corresponds to the *transonic* contour with $\mathcal{B} = \mathcal{B}_{ts}$. This solution has the unique property that $\alpha_{crit} = \alpha_{min}(f) = \alpha_f(z = 0)$ so that the two critical points occur exactly at the throat of the flux tube.

The two branches of the transonic contour separate the M - z phase space into the subcritical region, which is located between the transonic contours, and the supercritical region, which is located above and below the supersonic and subsonic branches of the transonic solution, respectively. As with subcritical solutions, the supersonic and subsonic branches of the transonic solution eventually intersect as the Mach number goes to unity. But, unlike the subcritical contours, the transonic contour is well defined over the entire length of the flux tube. And since $\partial_z \alpha_f(z) = 0$ at $z = 0$, the fluid is well-behaved at this point even as the Mach number goes to unity. This contour

is therefore the only viable solution that allows for a fluid to smoothly pass between supersonic and subsonic flows while conserving the value of $\mathcal{B}$.

2.2.3. Transitional Flows

If we were free to choose the value of $\mathcal{B}$ to be always equal to or greater than $\mathcal{B}_{ts}$ the steady solutions described in 2.2.2 would be sufficient. Since the fluid is at rest in the limb frame, in the intrusion frame the fluid velocity in the far field is given by $\mathbf{u}_{far} = u_{in}\hat{\mathbf{z}} = \hat{\mathbf{z}}C_sM_{in}$, where u_{in} is the speed at which the intrusion descends and M_{in} is its Mach number. Since M_{in} must be allowed to assume any real value we are forced to consider that the far field boundary condition might correspond to a subcritical flow. The inadmissibility, between $-z_{crit}$ and z_{crit} , of a solution with the value of $\mathcal{B}$ fixed by the boundary condition demands that the overall solution be one in which $\mathcal{B}$ is not conserved. This solution will take the form of several regions of constant $\mathcal{B}$, each connected by a transition in which Bernoulli's equation does not hold. The transitions are either shocks or rarefaction waves whose locations change with time. The complete flow combines two shocks, both propagating upstream, enclosing a transonic flow on which $\mathcal{B} = \mathcal{B}_{ts}$, and then a rarefaction wave propagating downstream away from the intrusion. For a careful discussion of shocks and rarefaction waves see Chapters IX and X of Landau & Lifshitz (1959). The following is a more specific discussion, aimed at our particular problem.

Shocks – In cases where the length scale of the fluid becomes comparable to the ion mean free path the viscosity has a non-negligible contribution to the momentum equation and cannot be ignored as it was leading to Eq. (2.20). The resulting behavior is referred to as a shock, which is a thin transition from one value of $\mathcal{B}$ to another. In a reference frame co-moving with the shock the flow must be steady and conserve mass and momentum. In the isothermal case these conditions lead to a version of the Rankine-Hugoniot condition (see Appendix A, §A.5.1), amounting to conservation of

$$M' + \frac{1}{M'} \quad (2.23)$$

across the jump, where $M' = (u - u_s)/C_s$ is the Mach number viewed from a frame moving at the shock speed u_s . This conservation law differs from Eq. (2.20) because the jump is assumed so thin that viscosity cannot be ignored and α is approximately constant across it. Discounting the trivial case where the Mach number is unchanged, leads to the relation

$$M'_2 = \frac{1}{M'_1} \quad (2.24)$$

between upstream and downstream Mach numbers, M'_1 and M'_2 . It is evident that one of these will be subsonic while the other is supersonic.

In terms of Mach numbers M_j in the frame of the intrusion, Eq. (2.24) takes the form

$$M_2 - M_s = \frac{1}{M_1 - M_s} \quad , \quad (2.25)$$

where $M_s = u_s/C_s$ is the Mach number of the shock. Knowing the upstream and downstream Mach numbers then gives the shock Mach number as

$$M_s = \frac{M_1 + M_2}{2} - \sqrt{\left(\frac{M_1 - M_2}{2}\right)^2 + 1}, \quad (2.26)$$

assuming $M_1 > M_2 > 0$. The shock will move leftward ($M_s < 0$) if $M_1 < 1/M_2$, and rightward if $M_1 > 1/M_2$. Mass conservation, in the shock reference frame, then leads to the relation

$$\rho_2 = \rho_1 \frac{M_1 - M_s}{M_2 - M_s} = \rho_1 \frac{\sqrt{(M_1 - M_2)^2 + 4} + (M_1 - M_2)}{\sqrt{(M_1 - M_2)^2 + 4} - (M_1 - M_2)}, \quad (2.27)$$

between pre-shock and post-shock density. A shock must have $M'_1 > 1 > M'_2$, and therefore $\rho_2 > \rho_1$: it is compressive.

Rarefaction Waves – A jump to lower density, not possible in a shock, must occur in a rarefaction wave. In cases without externally defined length scale the rarefaction wave will be *self-similar* (Landau & Lifshitz, 1959, §92), depending on space and time through a single similarity variable $(z - z_0)/t$. A rarefaction wave is inherently time-dependent and so Bernoulli's equation is again invalid. In our solution, a shock and rarefaction wave will be generated simultaneously at $t = 0$ from the single point $z = z_{crit}$. This initial state lacks a length scale and we may take the downstream rarefaction wave to be of the self-similar form. It will be bounded by weak discontinuities at its edges. The leading edge, at $z = z_2$, propagates into the

(unperturbed) downstream plasma at $u_2 + C_s$. Upstream of this the velocity, and thus Mach number, is linear (Landau & Lifshitz, 1959)

$$M = M_2 - \frac{z_2 - z}{C_s t}. \quad (2.28)$$

Upstream of the trailing edge, at $z = z_1$, the flow is again constant with $M = M_1 < M_2$. Thus the extent of the rarefaction wave grows in time as $\Delta z = (M_2 - M_1)C_s t$, beginning as a discontinuity at $t = 0$. The initial discontinuity at $z = +z_{crit}$ decomposes into this rarefaction wave and a shock, in the manner of a Riemann problem.

Within the rarefaction wave the density is an explicit function of velocity (see Appendix A, §A.5.2).

$$\rho = \rho_2 e^{M-M_2} = \rho_2 \exp\left(\frac{z - z_2}{C_s t}\right). \quad (2.29)$$

The upstream and downstream densities are therefore related by

$$\rho_2 = \rho_1 e^{M_2 - M_1}, \quad (2.30)$$

across the ever-expanding rarefaction wave. In order for this solution to apply the interior size of the rarefaction wave must be much smaller than the length scale of variation of the fluid cross-section, α . Fortunately, the rarefaction wave, while growing in time, propagates very quickly into the far field so that no matter how large it becomes, the scale over which α varies is always larger still.

2.2.4. Composite Flow

The complete solution, defined over the entire length of the affected flux tube, will be piecewise continuous using shocks, rarefaction waves and regions of pertistaltic flow so that the fluid velocity and density are treated in an internally consistent manner. The locations of the transition flows will travel along the length of the flux tubes in order to satisfy their governing equations and will therefore introduce time variations into the system despite our previous assumption of time independence. In letting this system evolve we are assuming that it can be treated as an adiabatic series of time-independent solutions. This assumption will be valid so long as the timescale over which a given feature evolves is long compared to its fluid crossing time.

In order to form a solution we use physical consideration to motivate the choice of initial conditions in the region between the critical points. Far above and below the intrusion we demand that the plasma density and velocity be unchanged and continuity demands that every jump in velocity have a corresponding jump in density. We therefore require at least two jumps with at least one unspecified intermediate value of $\mathcal{B}$ in order to have sufficient degrees of freedom to satisfy the boundary conditions on both velocity and pressure, which is equivalent to density in the isothermal limit.

If the fluid velocity in the far field is supersonic and lies below the transonic contour, then $\mathcal{B}_{in}$ is subcritical and M will be ill-defined between the two critical

points.² In order for the fluid to avoid an infinite acceleration at the upstream critical point there must be a transition away from $\mathcal{B}_{in}$ and onto some well-behaved flow, $\mathcal{B} \in \mathcal{B}_+$. In order for solution to remain well defined the transition must propagate upstream, away from the critical point. This is only possible in the case of a shock. A rarefaction wave would not propagate upstream with sufficient speed to escape the critical region. The downstream flow must therefore be subsonic in order for the system to be well defined at the upstream critical point.

At the downstream critical point there must again be a transition to connect the flow that resulted from the upstream shock back to the original contour $\mathcal{B}_{in}$. If the interior flow were everywhere subsonic the jump would again have to be a rarefaction wave and would propagate at Mach 1 into the higher density, subsonic fluid, ultimately making its way into the critical region and leaving the downstream critical point again ill-behaved. Thus, in the downstream region, the flow that resulted from the upstream shock must be supersonic at the critical point. Only the transonic contour $\mathcal{B}_{ts}$ can satisfy this condition without introducing yet another jump within the critical region. We therefore reach the conclusion, well known in nozzle problems, that the flow must cross from subsonic to supersonic at the throat.

The density change across the leading shock, which connects the subcritical inflow to the transonic interior flow, is fixed by the relative speeds of the fluid on either

² M_{in} could also be subsonic and above the lower transonic contour. For now we consider only supersonic inflows.

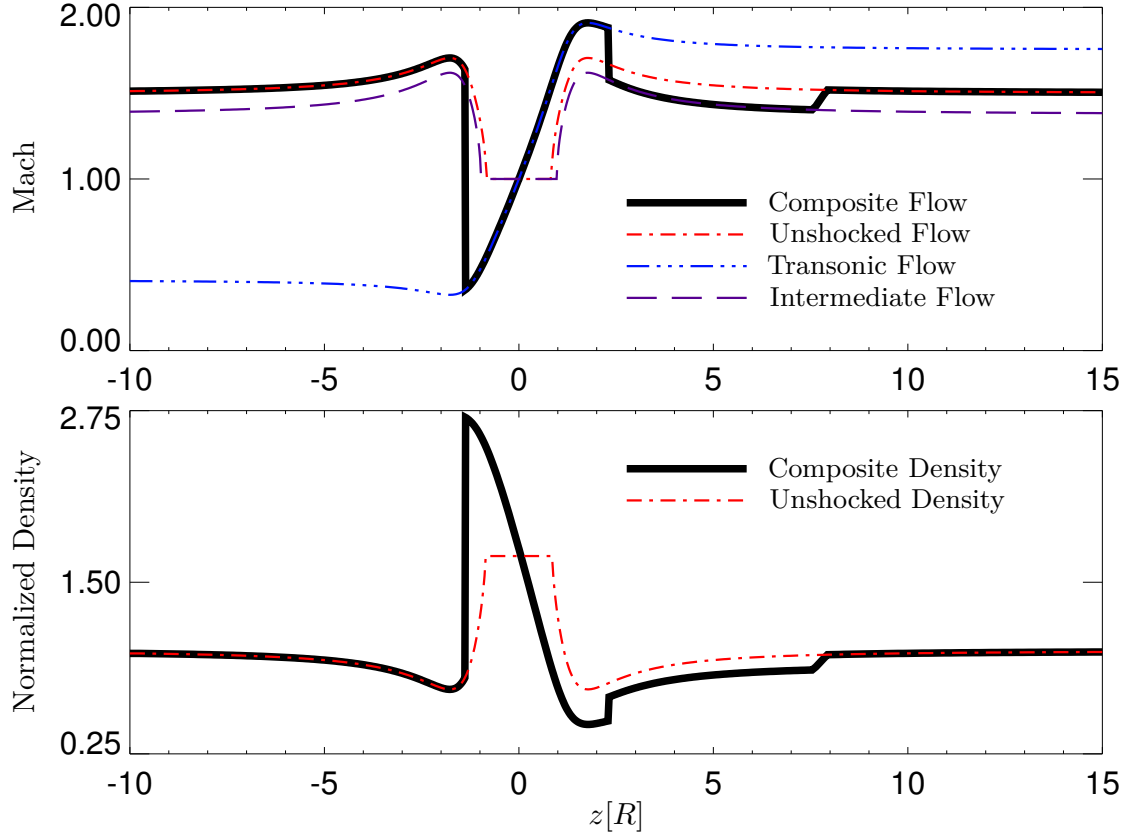


Figure 2.5. The piecewise continuous, composite flow is formed by connecting the subcritical flow from the far field with the transonic solution in the interior and an unspecified subcritical solution in the intermediate downstream region. The unshocked density and Mach number indicate the far field subcritical solution in the absence of shocks. As in Figure 2.4 fluid flows from left to right.

side of the shock. At the downstream critical point the speed of the fluid is given by the transonic solution and the density is fixed through the continuity equation. Any transition from the transonic flow back to the original flow must therefore satisfy the disparity in both speed and density at this point, a feat not achievable for either a

shock or a rarefaction wave. The jump at the downstream critical point must therefore decompose into both a shock and a rarefaction wave, just as in an asymmetric Riemann problem (Landau & Lifshitz, 1959). The rarefaction wave then propagates at Mach 1 into the downstream fluid and therefore moves away from the intrusion at speeds in excess of Mach 2. The shock propagates subsonically upstream into the transonic flow, which is itself supersonic, and therefore moves more slowly away from the intrusion.

Between the downstream shock and the rarefaction wave there is an initially infinitesimal intermediate region in which the fluid lies on an unspecified contour of $\mathcal{B}$, which will be determined such that the net effect of the two downstream transitions exactly compensates for the upstream shock and transonic interior. The intermediate flow is supersonic but also slower than the initial, subcritical flow, so it too has critical points and these must be accounted for when the initial locations of the three transitions are chosen. The whole system is evolved by using the current velocity of each feature to determine its location at some future time and then constructing the new velocity and density profiles for the whole system at that time. This construct is shown in Figure 2.5 for the $f = 0.5$ field line with an inflow condition of $M_{in} = 1.5$. The system is shown a short time after launch so that the transitions are spatially separated and can be easily distinguished. The original, subcritical flow is unphysical

at $z \approx -R$ but the composite flow is transonic at this point and thus exhibits no critical phenomena.

Relative to the intrusion, the upstream transonic flow and downstream intermediate flow are both slower than the subcritical flow in the far field. In the limb frame the plasma in these regions is actually descending toward the limb along with the intrusion. The two shocks similarly descend toward the limb with the leading shock pushing ahead and the trailing shock lagging ever farther behind while the rarefaction wave is sufficiently fast that it is not entrained with the intrusion and escapes rapidly upward. Note however that these shocks are not standoff shocks. They evolve in time and move steadily away from the intrusion, ultimately finding their way into the far field where their evolution slows and the assumption of time-independence becomes increasingly exact.

2.3. Results

In order to gain insight into solar dynamics from this model we must construct synthetic observables which can be compared to actual observations. To this end we begin by constructing 2D maps of density and velocity, made by interpolating between a representative sample of field lines, each determined with the same boundary conditions. Features of the 1D fluid solution manifest in the 2D maps as broad fronts, and regions of high or low density. Figure 2.6 shows one such map of plasma density

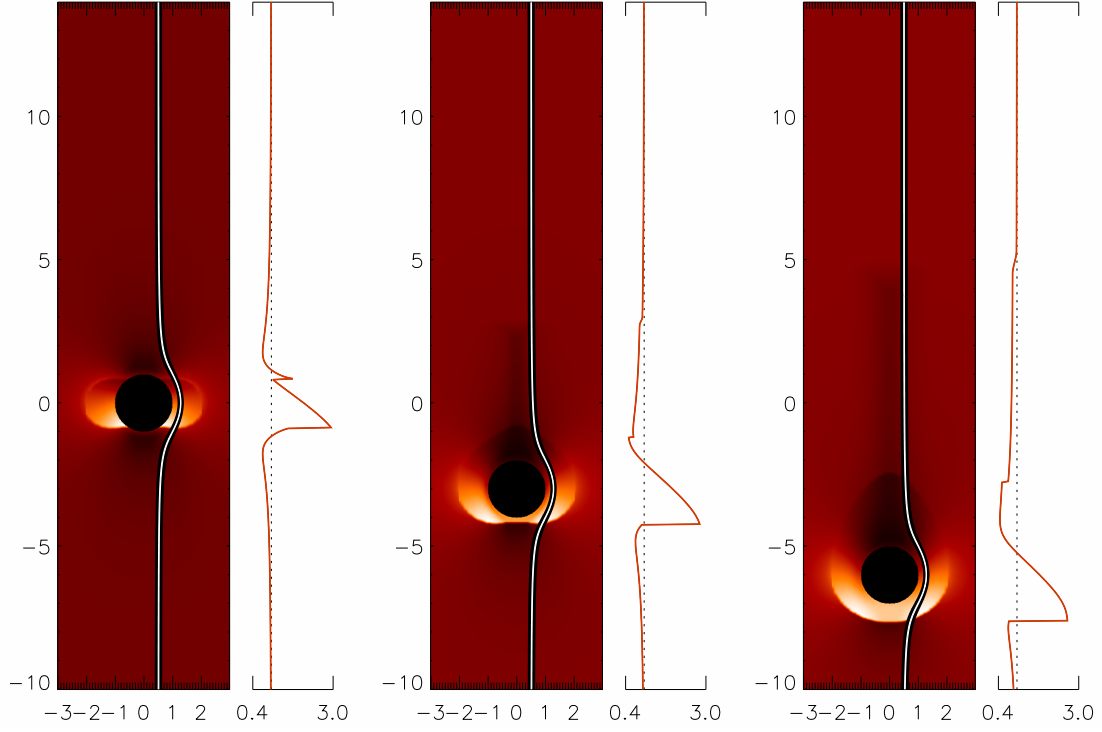


Figure 2.6. A $M_{in} = 1.5$ descending intrusion is shown in the limb frame at times $t = \{0, 3, 6\}R/C_s$. The high and low density regions are seen in red scale and the plots to the right of each panel show the exact density profile for the field line traced in white. The dashed lines indicate a normalized density of 1, as in the far field.

given at three successive times. From this example several features are visible. The leading shock, trailing shock and rarefaction wave are all distinguishable as abrupt changes in the plasma density. The high and low density “head” and “tail” grow in time and descend toward the limb while a slightly less rarefied region between the trailing shock and the rarefaction wave grows quickly upward as the rarefaction wave escapes away from the limb.

2.3.1. Emission Measure

As a proxy for synthetic images of the optically thin corona we choose the emission measure density ($\epsilon \propto \rho^2$). The emission measure profile will depend on the viewing angle. To begin with we consider a line of sight that is normal to the current sheet, consistent with many imaging observations of sheet-like structures above post-CME solar arcades (Švestka et al., 1998; Gallagher et al., 2002; Innes et al., 2003; Savage & McKenzie, 2011; Savage et al., 2012; McKenzie, 2013). If the intrusion pierces normally through the current sheet as in the left panel of Figure 2.1 then ϵ can be constructed simply by squaring the 2D density maps such as in Figure 2.6. This viewing angle also applies to cases where the intrusion is imbedded within the current sheet (as in the right panel of Figure 2.1), which is itself viewed edge on. The resulting emission measure maps will exhibit the same features as Figure 2.6. A collection of four such emission measure maps is shown in Figure 2.7 for four different descending intrusions, each depicted at the instant of launch and then again after the shocks have propagated into the far field. At $t = 0$ the shocks trace out the loci of critical points for each field line. Then, as $t \rightarrow \infty$ the shock fronts move into the far field so that the shock column has infinite vertical extent both above and below the intrusion.

Relative to the diameter of the intrusion, the width of the shocked column depends only on the speed of the intrusion M_{in} , which dictates the fluid velocity in the far field. For intermediate speeds (between Mach 1.4 and 1.7) the column width is

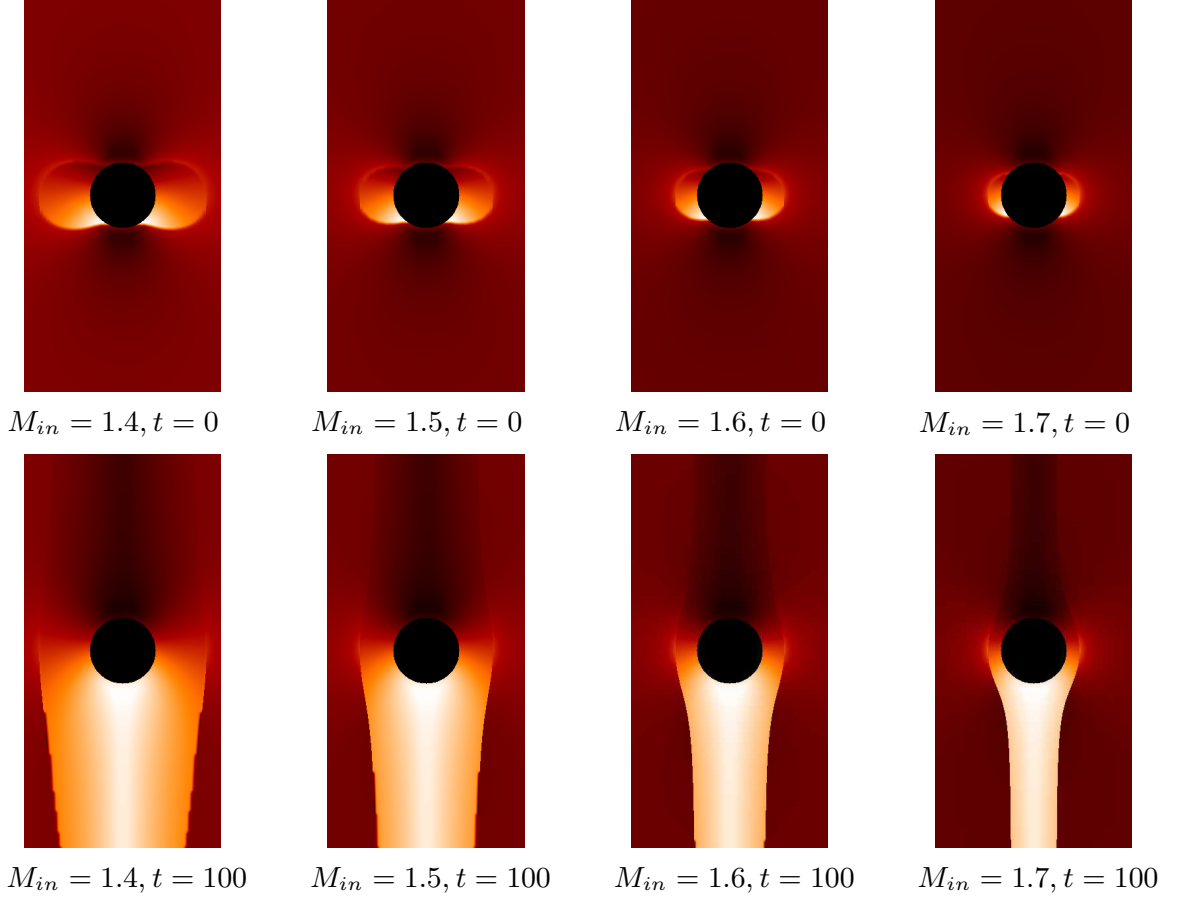


Figure 2.7. Four peristaltic shocks are shown for $M_{in} = \{1.4, 1.5, 1.6, 1.7\}$ (left to right) and for times $t = \{0, 100\}R/C_s$ (top to bottom). The edges of the shocked column in the lower row trace out the field lines that are transonic for each value of M_{in} , which separate the shocked and unshocked regions.

of the same order as the intrusion diameter. The upper limit occurs as $M_{in} \rightarrow 1.92$ at which point all field lines become non-critical so the shocked column vanishes. In the opposite limit, as $M_{in} \rightarrow 1$ all field lines exhibit critical behavior and the shocked column becomes infinite in width but with vanishing amplitude in the far field.

As an alternative we consider the system viewed from the side, such as if the line of sight were along $\hat{\mathbf{y}}$ in the right side of Figure 2.1. In this case the emission measure is constructed by integrating transversely across the 2D domain. The resulting profile will resemble that of an individual field line but will be somewhat smoother, having effectively averaged over all shocked field lines. We define the background-subtracted, normalized column emission measure as

$$\epsilon(z) = \int_{-L}^L dy \left(\frac{\rho(y, z)^2}{\rho_0^2} - 1 \right), \quad (2.31)$$

so that $\epsilon(z)$ goes to zero for $\rho(y, z) = \rho_0$ and is negative or positive where ρ is depleted or enhanced with respect to the ambient plasma. Figure 2.8 shows a stack-plot of successive time-steps of $\epsilon(z, t)$ for $M_{in} = 1.3$. The propagating shocks and rarefaction wave can be seen as abrupt jumps in the red scale emission.

To visualize how this kind of structure might manifest in the current sheet consider the case of a slightly sheared supra-arcade magnetic field. According to Guidoni & Longcope (2010), a local reconnection event will result in a growing, descending trapezoidal plateau that leads to something like the right side of Figure 2.1 and Figure 2.9. The horizontal segment of the reconnected field, i.e. the intrusion, is embedded in the current sheet and drives peristaltic flows in the surrounding layers of field. As the plateau descends the bends move outward so that more of the reconnected flux is embedded in the current sheet.

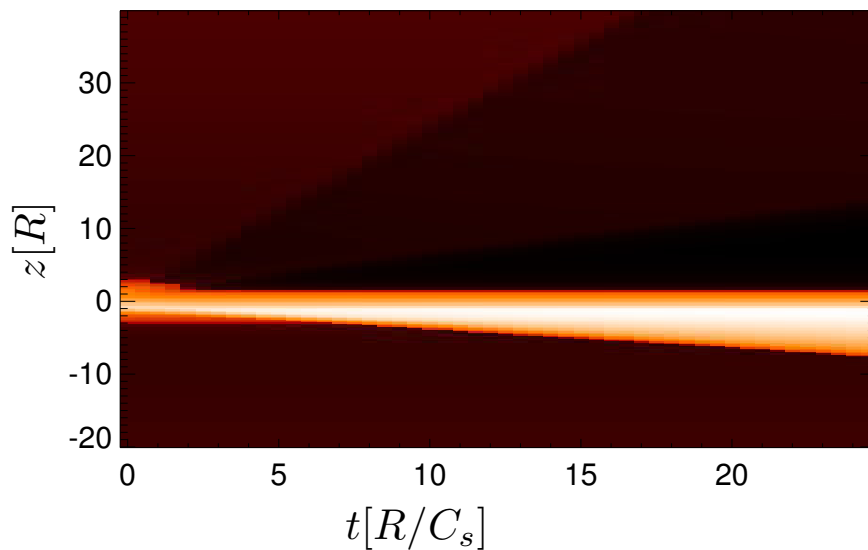


Figure 2.8. The line-of-sight integrated emission measure depicted here as a stack plot. Higher emission is indicated in yellow. Time increases to the right with the vertical profile at any given time corresponding to $\epsilon(z, t)$.

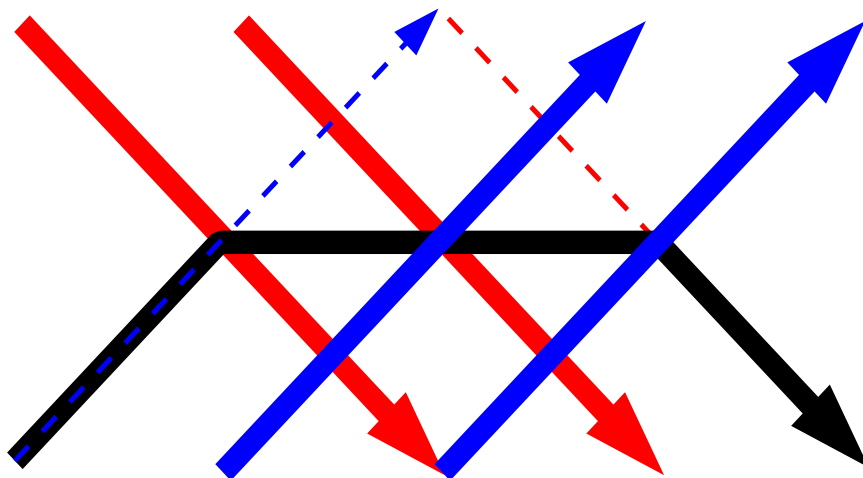


Figure 2.9. A sheared magnetic field results in a reconnected, horizontal field line which is dragged downward through the adjacent layers of unreconnected field, forming a plateau.

Because the field is sheared, we treat the two respective layers of magnetic field independently. They both exhibit peristaltic flows which result in an emission such as in Figure 2.8, but in one field the flow is slanted slightly to the right while the other is slanted to the left. The composite flows launch first on the field lines closest to the initial reconnection point. Then, as the plateau descends, those same field lines continue to evolve while field lines that are newly exposed to the growing plateau are initiated each in turn. The net result is a locus of shocked flows that grows as the plateau grows.

Such a system is depicted in Figure 2.10. The unreconnected field is angled up and to the right in the foreground and down and to the right in the background. The layers of field that pass close to the horizontal segment of reconnected field exhibit peristaltic pumping. The emission measure on each field line is given by $\epsilon(\tilde{z}, t - t_f)$, where $\tilde{z}$ is the distance along the angled field line and t_f is the time at which that field line was initiated. The collection of lower shocks leads to a vaguely arch shaped high density region while the upper shock leads to a similar rarefied region. The antisunward rarefaction waves lead to a nearly vertical column of low density owing to the fact that these waves propagate supersonically outward along field lines at a rate comparable to the growth of the plateau.

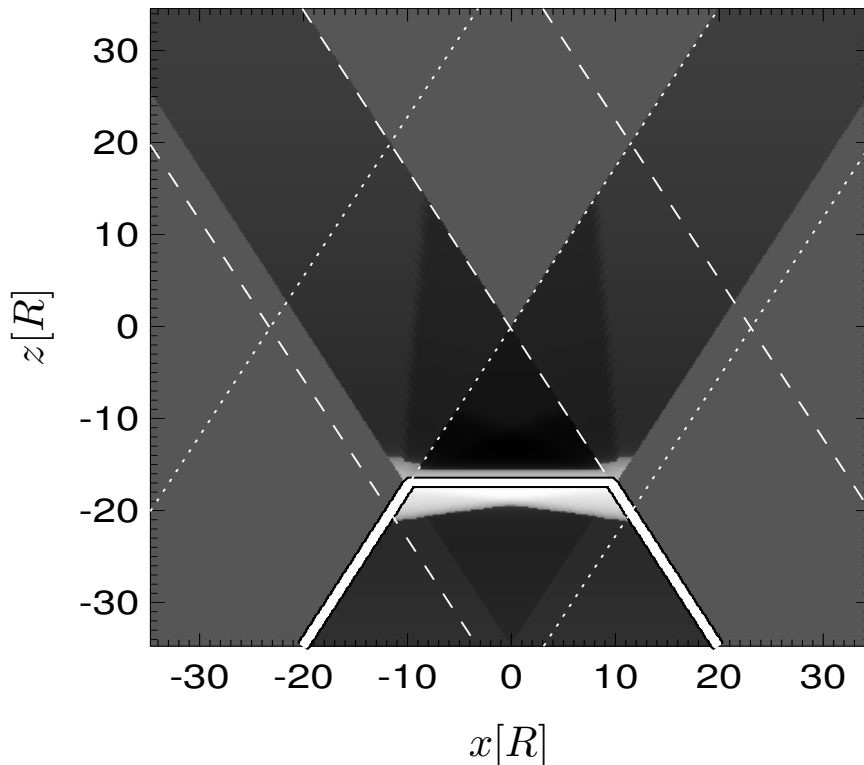


Figure 2.10. A high contrast plot of emission measure for a sheared peristaltic event with two sets of field lines, dashed and dotted. The locus of shock features are visible above and below the plateau created by the embedded segment of reconnected flux, depicted as a solid white line.

2.3.2. Momentum in the Fluid

Since the descending intrusion generates plasma motion in the surrounding field, there should be associated energy and momentum transfer into that fluid. For any finite domain we can find the momentum in the plasma through numerical integration of the plasma density and velocity. Figure 2.11 displays a representative plot of momentum density.

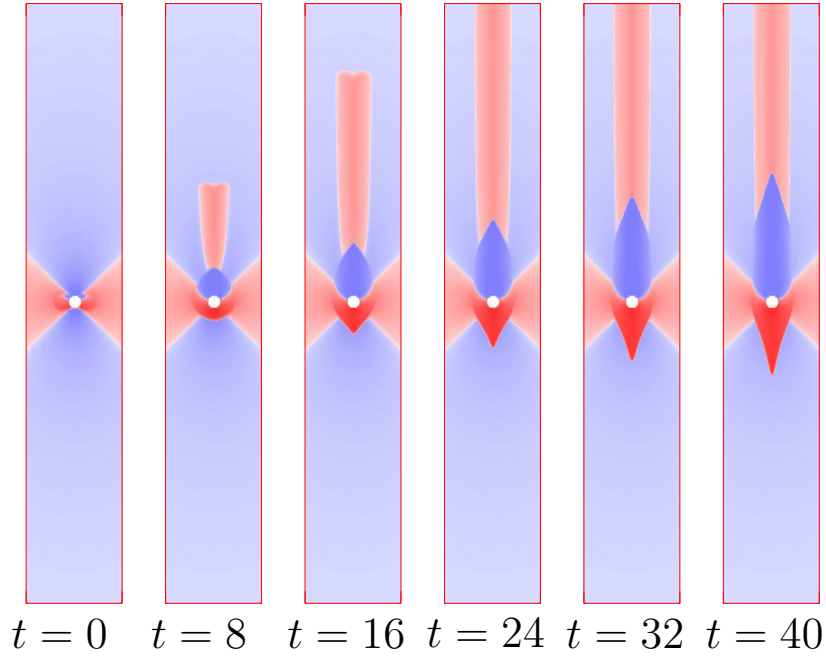


Figure 2.11. A time series of fluid momentum density. Red indicates sunward momentum while blue is antisunward.

Due to the fact that the normalized cross section in the far field is asymptotic to but always greater than unity, the supersonic fluid far from the intrusion always propagates slightly faster than M_{in} in the rest frame of the intrusion. In the limb frame this fluid is slowly rising so the momentum density far from the intrusion is always anti-sunward. Closer to the intrusion it is directed sunward as the cross section becomes constricted and the fluid is slowed below M_{in} . Immediately above the lower shock the momentum density is strongly sunward and then becomes anti-sunward as the transonic flow passes abreast of the intrusion and again exceeds M_{in} . It then

becomes sunward again across the second shock before finally returning to the far-field limit across the rarefaction wave.

To explore this more carefully we observe that the force per unit fluid cross section on a shocked flux tube may be found explicitly through

$$f = \partial_t \int_{-\infty}^{\infty} d\mathbf{z} \cdot (\rho \mathbf{u} \alpha) \quad (2.32)$$

$$= \partial_t \left(\int_{z_1-\delta}^{z_1+\delta} + \int_{z_2-\delta}^{z_2+\delta} + \int_{z_3-\delta}^{z_3+\delta} \right) d\mathbf{z} \cdot (\rho \mathbf{u} \alpha) \quad (2.33)$$

$$= v_{z_1} [\rho u \alpha]_{z_1}^{z_1+\delta_1} + v_{z_2} [\rho u \alpha]_{z_2}^{z_2+\delta_2} + v_{z_3} [\rho u \alpha]_{z_3}^{z_3+\delta_3} , \quad (2.34)$$

where δ_i represents the width of each jump. This can be calculated numerically for every field line within a finite domain under the assumption that all three jumps have propagated into the far field where the shock speeds become steady and $\alpha \rightarrow 1$. In Figure 2.12 we see that the net force on the fluid appears to be finite even in the limit of $M \rightarrow 1$, where the shock becomes infinitely wide. In order to confirm this the contribution from the far field may be approximated analytically. This calculation is not included in the present work but it can be shown that the force contributed by the shocks in the far field vanishes with an inverse power of distance greater than unity so that, indeed, the net force on the fluid remains finite even as the shocks fill all of space.

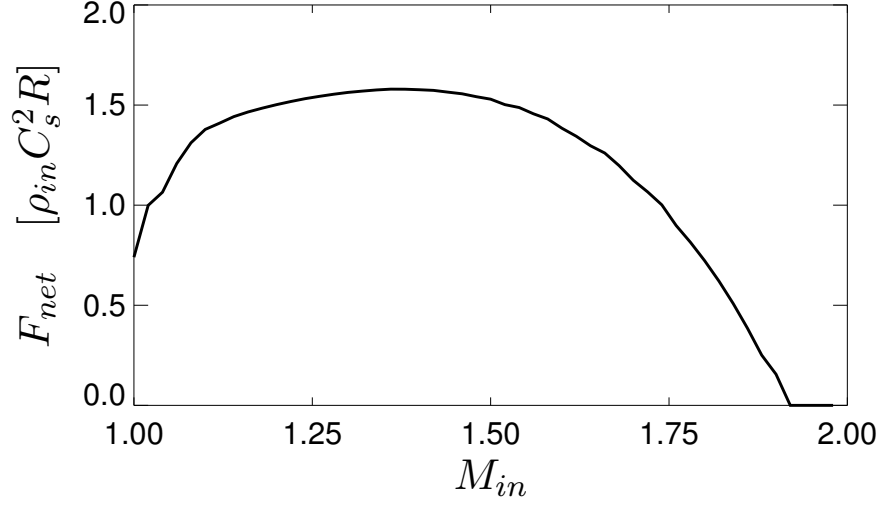


Figure 2.12. The time rate of change of momentum in the fluid yields a net force on the fluid that vanishes when the width of the shocked column goes to zero as $M_{in} \rightarrow 1.92$ and remains finite as the width expands and $M_{in} \rightarrow 1$.

2.3.3. Drag Force

If the total momentum in our model contained only two contributing terms we could use Figure 2.12 as a proxy for the drag force on the intrusion. In actuality the far-field boundary conditions also contribute momentum to the system and the drag force must be calculated explicitly by integrating the plasma pressure over the surface of the intrusion. The pressure is related to the density, which can be found explicitly by calculating the behavior of fluid on the $f = 0$ streamline. The net vertical force due to the pressure $p_s(\theta)$ on the surface S is then

$$F_z = 2 \int_0^\pi r d\theta p_s \hat{\mathbf{n}} \cdot \hat{\mathbf{z}} = 2 \int_0^\pi p_s(\theta) r \cos(\theta) d\theta . \quad (2.35)$$

The plasma on the $f = 0$ field line is always on the transonic contour. The $f = 0$ streamline intersects the surface of the intrusion at the two magnetic null points $\theta = \{0, \pi\}$, with θ measured here from $-\hat{\mathbf{z}}$. At these points the fluid cross section diverges as the magnetic field strength vanishes. For $\theta = \pi$ the fluid Mach number also diverges so ρ must be zero by continuity. For $\theta = 0$ the plasma is subsonic so M goes to zero as $\alpha \rightarrow \infty$.

The actual computation must be done numerically but to ensure that the pressure is well behaved, let us first explore the behavior at $\theta \rightarrow \{0, \pi\}$. At the upper pole the behavior is simple. Both the cross section and the flow velocity are divergent and so the density and pressure must vanish in order to satisfy the continuity equation. The lower pole is harder to intuit. Here the plasma velocity goes to zero but the cross section becomes infinite so it is not clear how the pressure will behave. Evaluating Eq. (2.12) with $f = 0$ and substituting $y \rightarrow R \sin(\theta)$ the flow along the transonic contour is stated explicitly as

$$M^2 - \ln(M^2) + \ln(4\theta^2) = 1 + \ln(4) \quad (2.36)$$

where, for small θ we have use the small angle approximation. For small M and small θ this expression becomes

$$\ln(M^2) \approx \ln(4\theta^2) \quad \Rightarrow \quad M \propto \theta \quad (2.37)$$

and thus, at the lower pole, the pressure goes as

$$p(\theta \rightarrow 0) = \frac{p_0 M_0}{M\alpha} \propto \frac{p_0 M_0}{\theta/\theta} = p_0 M_0. \quad (2.38)$$

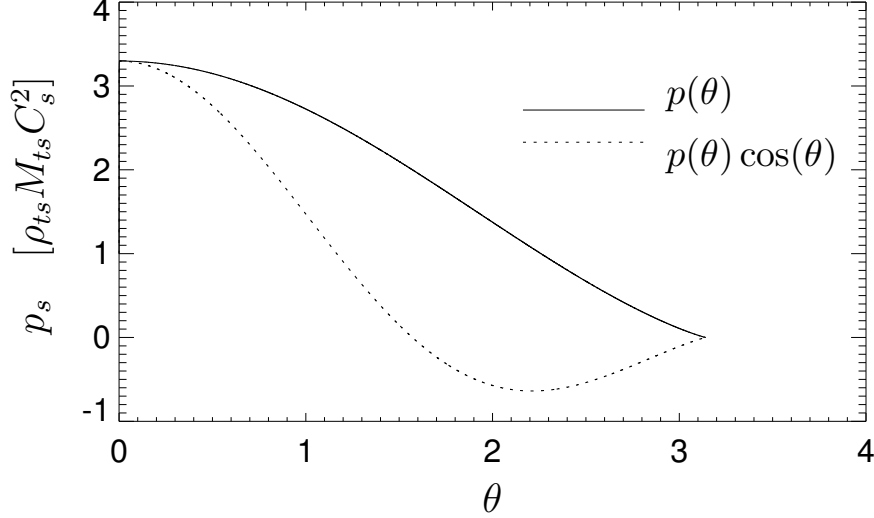


Figure 2.13. Pressure along the surface of the intrusion, parameterized by polar angle. ρ_{ts} and M_{ts} are the transonic density and Mach number far below the intrusion. The integrated pressure gives the force on the intrusion (per unit inserted length) as $F \approx 4.8 \times \rho_0 M_0 R C_s^2$.

Having demonstrated that the behavior at the lower pole is non-divergent we may extrapolate from the numerical limit as $\alpha \rightarrow \infty$. In this way we find that $p(\theta \rightarrow 0) \approx 3.3\rho_{ts}$, where ρ_{ts} is the plasma density on the transonic contour far from the intrusion. $p(\theta)$ is therefore well-behaved for all θ and can be calculated numerically as in Figure 2.13. The pressure decreases monotonically with θ , indicating an upward net force on the intrusion. The z -component of this force is given explicitly by the area under the dashed curve.

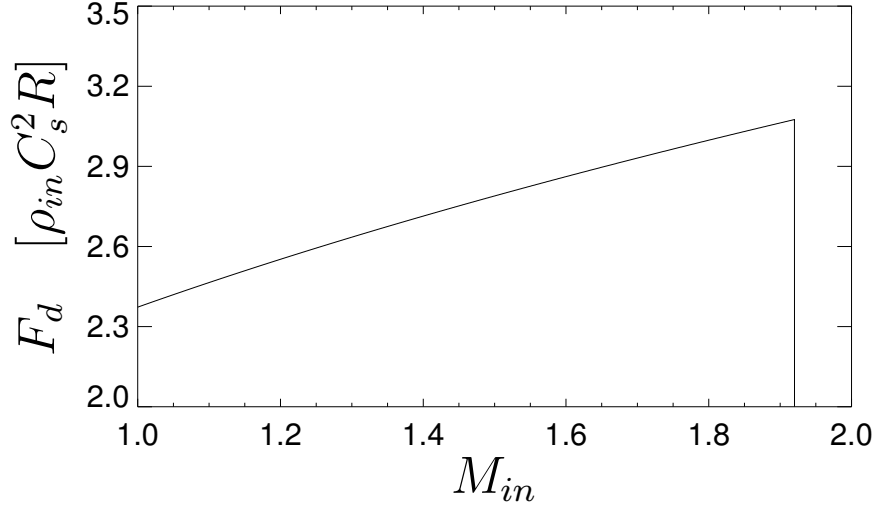


Figure 2.14. The drag force is found by evaluating the density jump across the lower shock for a given intrusion Mach number and then combining the result with the integration factor from Figure 2.13.

In order to find how the drag force depends on M_{in} we must find the pressure jump across the leading shock, which will dictate ρ_{ts} and hence p_{ts} . The shock velocity M_s is determined by M_{in} and M_{ts} . But, in the far field $M_{ts} \rightarrow 0.319$ for $f = 0$, $\alpha = 1$. Thus, ρ_{ts} depends only on M_{in} and, when multiplied by the integration factor from Figure 2.13, the resulting drag force can be found as depicted in Figure 2.14. The drag is lowest for the Mach 1 limit and increases almost linearly up to $M_{in} \approx 1.92$, at which point the column disappears. At and above Mach 1.92 the drag is zero due to the symmetry of the de Laval flow solutions, just as in D'Alembert's paradox. Below Mach 1 we have not calculated the drag profile but we expect, given the extent of

the subsonic critical regime, that the drag will remain finite down to the minimum critical value of $M_{min} \approx 0.32$, at which point the shocked column again vanishes.

As indicated, the drag curve in Figure 2.14 does not match the net force on the fluid from Figure 2.12. This is not surprising since the transonic flow is asymmetric along the vertical direction and thus the centripetal force on each fluid element is unbalanced. Thus the magnetic field must deform asymmetrically in order to balance the fluid pressure. It follows that there must be some infinitesimal asymmetry between the far field magnetic field above and below the intrusion which in turn leads to a net force exerted on the system by the far field boundary conditions.

2.4. Discussion

In this work we have shown how field line retraction following a local reconnection event can manifest as a descending constriction in the nearby unreconnected field. This constriction behaves in many respects as a peristaltic pump, which leads to peristaltic flows and ultimately to shocks and rarefaction waves that alter the velocity and density of plasma on affected field lines. These shocks are not to be confused with standoff shocks, which form at a fixed distance in front of a traveling obstacle and are thereafter stationary in time. The fluid jumps that we have described cannot exist as time-independent solutions and must necessarily propagate away from the

intrusion. The region between these jumps therefore grows in time and is, in and of itself, a dynamic feature.

The composite flows that form in this model are restricted to a column whose width is defined by the field-line that exhibits transonic flow for a particular boundary condition, M_{in} . This width increases monotonically as the speed of the intrusion decreases toward the sound speed. The minimum width of zero occurs when the speed of the descending intrusion reaches $M_{max} = 1.92$ while the maximum width is arbitrarily large as $M_{in} \rightarrow 1$. In this analysis we considered only supersonic values for M_{in} but we acknowledge that the peristaltic flows will continue to exhibit critical behavior even for subsonic values of M_{in} down to the point where $\mathcal{B}$ is again larger than $\mathcal{B}_{ts}$ on the $f = 0$ field line.

Our profiles for the 2D density and emission measure maps bear striking resemblance to observations of voids and Supra-Arcade Downflows (SADs) in post-CME flares (McKenzie, 2000; Savage et al., 2012). In our analysis we considered an isothermal plasma in order to make the development more tractable. We offer, without proof, that an adiabatic plasma would exhibit the same qualitative behavior with the addition that plasma in the region between the lower shock and the rarefaction wave would exhibit an increase in temperature. The rarefied tail and high-emission leading edge may even be useful as thermal diagnostics since a temperature increase

in the rarefied column could move the emission outside of a particular observation band-pass, thereby increasing the contrast in these features.

We also considered an alternate geometry in which the particular shape of the emission profile is exchanged for a column integrated emission measure which occurs everywhere along the length of an embedded flux tube. This geometry also offers an interpretation for down-flowing features but may be more accurately used to describe how reconnection events and the contraction of reconnected flux can lead to heating of plasma along a broadly distributed volume of unreconnected field.

Our model also helps to explain how a thermal halo (Seaton & Forbes, 2009) might form around the current sheet. Here we have described only the creation of shocks along constricted field lines. But these shocks could very well travel down the unreconnected field all the way to the chromosphere where they would then drive evaporation exactly as in conduction dominated flare loops (Cargill et al., 1995). This evaporation might then increase the density on “post-peristaltic” field lines, which could then undergo their own reconnection or even experience another “peristaltic process” in the event of a second nearby reconnection event.

We further describe how the alterations to velocity and density relate to the momentum density in the fluid and the subsequent net force (per unit embedded length) on the fluid. This force is related to but not equal to the net force on the retracting flux tube since a third contribution comes from the boundary conditions

which maintain the field profile in the far field. The net force on the intrusion is found from a direct calculation of pressure integrated over its surface. This force points in the direction opposite the motion of the intrusion and is of order $RC_s^2 \rho_{in}$. It increases almost linearly by nearly a factor of two over the range $1 < M_{in} < 2$. Larger descent speeds correspond to a larger drag force so that, if this force is sufficient to influence the kinematics of the descending intrusion, the drag will decrease as the intrusion slows.

This drag force offers a possible explanation to the fact that reconnection outflows appear to move sub-Alfvénically despite the predictions of reconnection models such as described by Seaton & Forbes (2009). If reconnection outflows originate in locations where peristaltic shocks can form then this could lead to a drag force that would keep their velocities below the Alfvén speed. However, if the outflow velocity ever exceeds the maximum shock velocity then the drag force should vanish according to our model. These loops would descend much more rapidly in its absence. This line of reasoning suggests there may be a bimodal distribution in the velocity of retracting magnetic loops. Loops that move fast enough to avoid launching peristaltic shocks would remain fast-moving while slower moving loops would be damped by the momentum transferred into the plasma.

In order for the aforementioned drag force to have a non-negligible influence, the plasma pressure must be comparable to the magnetic energy density in the retracting

flux tube. But pressure balance between the intrusion and its surroundings demands that the field strength must be comparable between the retracting flux and the un-reconnected field. Thus, the drag force will only be significant if the plasma pressure is comparable to the magnetic energy density in the unreconnected field. While this is not unlikely in reality it is in conflict with the zero- β assumption and therefore cannot be reconciled with our model in its current form.

Our model assumes an extremely low β value in order to invoke the rigid magnetic field. However, observations suggest that this may not be an accurate assumption in the supra-arcade region (McKenzie, 2013). It may be that by the time supra-arcade downflows become visible in observations the local plasma β has already been increased due to previous instances of peristaltic pumping and that our model only applies to the early stages of flare activity, when the plasma density and temperature are still relatively low. Future work will therefore require the relaxation of the low β approximation, which will necessitate a numerical simulation. Another issue with the model is that we have been forced to stitch together time-independent solutions in an adiabatic fashion. The validity of this approximation, as well as those used in deriving the 1D MHD simplifications, will likewise need to be tested through simulations.

When comparing to observations, some key differences are also apparent. Our model cannot reproduce the oscillatory behavior on the edges of voids as seen in Verwichte et al. (2005), although it does predict a discontinuity in plasma density

transverse to the field, which could support surface modes if the zero- β assumption were relaxed. Also, while we predict that these features should occur for $M_{in} \lesssim 2$, Savage & McKenzie (2011) measured a typical downflow speed of on the order of 10^2 km s $^{-1}$ with some instances of much higher values. Depending on the local sound speed these velocities may fall above our Mach 2 prediction. A more careful study of SAD speeds and the associated local sound speed will need to be conducted in order to refine this estimate. Moreover, the upper limit yielded by our simple model can be relaxed by generalizations to non-circular intrusions. As a first attempt we calculated that for elliptical intrusions the value of M_{max} could be increased by a factor of nearly two before the aspect ratio of the ellipse became unrealistic. It may be that an appropriate choice of intrusion cross section could reconcile any lingering disparities between the model and observations of flare loops. Ultimately we intend to further the investigation with a regimen of numerical simulations. The ultimate success of this model will be in providing a theoretical framework for interpreting features seen in more complex numerical simulations.

2.5. Acknowledgements

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3. NUMERICAL SIMULATIONS OF PLASMA DYNAMICS IN THE VICINITY OF A RETRACTING FLUX TUBE

Contribution of Authors and Co-Authors

Manuscript in Chapter 3

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Contributions: Conceived and developed numerical model. Wrote manuscript content.

Co-Author: Dr. Dana W. Longcope

Contributions: Helped to conceive of numerical model and aided with subsequent analysis. Provided feedback of analysis and comments on drafts of the manuscript.

Co-Author: Dr. David E. McKenzie

Contributions: Helped to conceive of numerical model. Provided feedback of analysis and comments on drafts of the manuscript.

Manuscript Information Page

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ABSTRACT

In a previous paper we presented an analytical, zero- β model for supra-arcade downflows in which a retracting flux tube deforms the surrounding magnetic field, constricting the flow of plasma along affected field lines and, in some cases, forcing the plasma to exhibit shocks. Here we present a numerical simulation based on the same model construction – a retracting flux tube is treated as a rigid boundary around which the plasma is forced to flow and the magnetic field and plasma evolve according to the governing equations of magnetohydrodynamics. We find that the collimated shocks described in our previous study are recovered for plasma β in the range of $0 \leq \beta \lesssim 1$ while for $1 \lesssim \beta$ the behavior is similar to the simpler hydrodynamic case, with conical bow shocks forming when the acoustic Mach number approaches or exceeds unity. Further, we find that while the plasma β is important in identifying the various types of behaviors, more important still is the Alfvén Mach number, which, if large, implies that the bulk kinetic energy of the fluid exceeds the internal energy of the magnetic field, thereby leading to the formation of unconfined, fast-mode magnetohydrodynamic shocks, even in the limit of small β .

3.1. Introduction

The flow of plasma around a circular boundary is a topic of great importance for solar physics, in which individual flux tubes are often treated as discrete elements that interact with their environment through the forces exerted across their surface. For instance, Linton & Longcope (2006) developed a model for the evolution of a post-reconnection flux tube that retracts under the influence of its own internal magnetic tension. They attempted to include a drag-like force by considering the mass that is continually swept up by the retracting flux tube, but had no means by which to treat actual drag, which is a result of pressure forces exerted on the surface of

the tube and must depend on its speed relative to the surrounding medium. Subsequent investigations (Longcope et al., 2009; Guidoni & Longcope, 2010) ignored these external forces entirely.

McKenzie & Hudson (1999) suggested that a retracting flux tube could be the cause of low-emission voids above solar flare arcades, which they went on to name Supra-Arcade Downflows (SADs) (McKenzie, 2000). This view has been expanded upon (Savage et al., 2012) but largely without a detailed understanding of the mechanism by which the flux tube might impart energy and momentum to the surrounding medium. In Scott et al. (2013) we took this model a step farther and presented an analytical solution in which the flux tube serves as a boundary condition for the surrounding magnetic field, which then constrains the flow of plasma and causes the subsequent formation of shocks. We also made preliminary estimates of the drag force that the retracting flux tube would experience; however, the assumption of an arbitrarily strong magnetic field, on which Scott et al. (2013) is built, is seemingly inconsistent with recent estimates of the energy balance in the supra-arcade region (McKenzie, 2013; Scott et al., submitted: 2015).

Here we present a numerical magnetohydrodynamic (MHD) study of the flow of magnetized plasma around a flux tube, which we treat as a rigid, circular boundary. This study is an extension of Scott et al. (2013), though its implications are farther reaching than the interpretation of SADs. The magnetic field and fluid are treated

two-dimensionally, with the flux tube extending along the line of symmetry, out of the plane. Treatment of the flux tube as a boundary is accomplished through the implementation of a conformal coordinate system in which the circular cross section of the intrusion is reduced to a branch-cut. The fluid is treated isothermally, without viscous or adiabatic heating and with no thermal conduction. An isotropic viscosity is used to maintain stability and to ensure that any shocks that develop are fully resolved.

As our model setup allows for arbitrary ratios of thermal to magnetic energy ($\beta = 8\pi p/B^2$, where p is the pressure and B the magnetic field strength), with the magnetic field strength and acoustic Mach number specified as initial conditions, we are able to explore a variety of different regimes simultaneously. The extreme limits of low and high β , which can be compared to analytical results in order to check the integrity of our model, are immediately applicable to retracting flare loops in active region cores and expanding magnetic clouds in the extended corona, respectively. And, yet, it is the behavior for intermediate values of β , which seem increasingly likely to correspond to the conditions in the supra-arcade region, that is most immediately of interest. Specifically, we hope to see how the analytical model of Scott et al. (2013) is altered as the zero- β assumption is relaxed and whether the predictions of that model can be applied in instances of non-trivial energy balance.

The organization of this paper is as follows: In §3.2 we develop the specific details and assumptions of the model, including a unique coordinate system for treating the interior boundary condition and a description of our numerical techniques. In §3.3 we describe the various behavioral regimes that are accessible to the system and in §3.4 we survey the parameter space, identifying where each type of behavior has been observed. Then in §3.5 we summarize our results and expound upon the possible implications of this work.

Code for this project was written in the Interactive Data Language (IDL) and depends on internal functionality as well as the usual user-contributed routines from *SolarSoft* (Freeland & Handy, 1998). IDL is a proprietary language that is maintained by Exelis Visual Information Systems (commonly known as Exelis). Licenses for using IDL must be leased from Exelis but our code can be made publicly available upon request.

3.2. The Model

To investigate the behavior of a magnetized fluid in the near vicinity of a retracting magnetic element we will employ the same model configuration as in Scott et al. (2013) but with the inclusion of an evolving magnetic field. We assume a two-dimensional (2D) solution, in which the plasma and the magnetic field are fully described by an $x - y$ coordinate plane with the Line-Of-Sight (LOS) coordinate ($\mathbf{z} = z\hat{\mathbf{x}} \times \hat{\mathbf{y}}$)

taken to be an ignorable direction of symmetry. We then consider the following MHD equations:

$$\partial_t \rho = -\nabla \cdot \rho \mathbf{u} \quad (3.1)$$

$$\partial_t p = -\nabla \cdot p \mathbf{u} + (1 - \gamma)p \nabla \cdot \mathbf{u} + (\gamma - 1)\dot{Q} \quad (3.2)$$

$$\partial_t \mathbf{u} = -(\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{1}{\rho} \left(\frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} + \nabla \cdot \underline{\underline{\sigma}} - \nabla p \right) \quad (3.3)$$

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}), \quad (3.4)$$

where ρ , $\mathbf{u}$, $\mathbf{B}$ and p are the plasma density and velocity, the magnetic field, and the plasma pressure, respectively, $\underline{\underline{\sigma}}$ is the viscous stress tensor and $\dot{Q}$ is the volumetric heating due to thermal conduction and viscous heating. Here we have used the ideal form of the induction equation so that the magnetic field does not reconnect.

In order to simplify this system, we choose to represent the magnetic field in terms of a flux function, φ , so that

$$\mathbf{B} = B_0 \hat{\mathbf{z}} \times \nabla \varphi, \quad (3.5)$$

where B_0 is the overall magnetic field strength. In terms of φ we may then restate the induction equation in its uncurred form:

$$\partial_t \varphi = -\mathbf{u} \cdot \nabla \varphi. \quad (3.6)$$

This representation, which is only afforded to us in 2D, incorporates the requirement that $\nabla \cdot \mathbf{B} = 0$ and has the benefit of reducing the number of independent variables

in the system. Additionally, since the magnetic field is everywhere perpendicular to the gradient of φ , contours of φ are themselves field lines, making visualization of the magnetic field a simpler matter.

In the reference frame that is co-moving with the retracting flux tube we may treat the surface of the flux tube, which we refer to as *the intrusion*, as a stationary circle of radius R , centered at the origin. In this reference frame, the surrounding plasma, which is taken to be initially stationary in the rest frame of the solar limb, is perceived to flow upward with a velocity that is uniform in the asymptotic region, far from the intrusion. This far-field plasma velocity is understood to represent the speed at which the flux tube retracts toward the limb. The intrusion is then treated as a rigid, stationary boundary in the interior of the simulation domain and we demand that the normal component of both $\mathbf{B}$ and $\mathbf{u}$ vanish along its surface, i.e.

$$\mathbf{u} \cdot \mathbf{r}|_{r=R} = 0; \quad (3.7)$$

$$\mathbf{B} \cdot \mathbf{r}|_{r=R} = 0, \quad (3.8)$$

where $r = \sqrt{x^2 + y^2}$.

3.2.1. Conformal Coordinate System

Because the interior boundary is defined along a curved surface, implementation of this boundary condition in cartesian coordinates is problematic. If the spatial resolution of the numerical grid is not sufficiently fine then the boundary surface will

be coarse, which will tend to lead to numerical artifacts. We therefore adopt an alternative grid, based on coordinates that are defined with respect to the potential field solution for the magnetic field around a conducting sphere. This is equivalent to the coordinate system used in our previous analytical model and offers an additional benefit in terms of transport coefficients.

Since the thermal and viscous transport coefficients are generally anisotropic with respect to the magnetic field direction, any attempt to model these effects on a cartesian grid is always limited by a geometric effect relating to the angle between the magnetic field and the coordinate system. While the magnetic field in our model is dynamic in general, we expect it to more closely resemble a potential magnetic field, which naturally satisfies the boundary condition along the surface of the intrusion, than a cartesian grid, which forms a non-trivial angle to the surface of the intrusion and, therefore, must also form a non-trivial angle to the magnetic field along the same surface. By aligning the coordinate system to the potential field we reduce the typical angle between the magnetic field and the grid, thereby minimizing the numerical error associated with these anisotropic transport coefficients. In the current study these benefits offer little gain as we will employ isotropic viscosity and omit thermal conduction; however, we intend to treat these terms more carefully in future refinements of our investigation.

The potential field that we choose for our grid must be tangent to the surface of the intrusion and vertical in the far field. We also require a second coordinate that is everywhere orthogonal to the first so that the two together define a basis.¹ This is accomplished by introducing the complex function

$$\hat{F} = \zeta - \frac{R^2}{\zeta}, \quad (3.9)$$

which is a conformal transform that defines a mapping between two complex variables, $\zeta = x + iy$ and $\hat{F} = f + ig$, such that if ζ lies on a circle of radius R , i.e. $|\zeta| = R$, then the real and imaginary parts of $\hat{F}$ will satisfy $\Re(\hat{F}) = 0$ and $|\Im(\hat{F})| \leq 2R$. That is, the mapping equates the circle given by $x^2 + y^2 = R^2$ to a vertical cut given by $f = 0, |g| \leq 2R$. As such, the treatment of the interior boundary is simply satisfied in terms of f and g by demanding that both $\mathbf{u}$ and $\mathbf{B}$ have no component perpendicular to f for $f = 0$ and $|g| \leq 2R$. Far from the intrusion ($|\zeta| \gg R$) we see that $\hat{F} \rightarrow \zeta$, so f and g reduce simply to x and y in the asymptotic region. Thus, the condition that the velocity and the magnetic field be uniform and vertical in the asymptotic region is equivalent in the two coordinate systems.

A visualization of the transformation is depicted in Figure 3.1, and a detailed treatment is as follows. By taking the real and imaginary parts of $\hat{F}$, we find that f and g are

$$f = x\left(1 - \frac{R^2}{r^2}\right), \quad (3.10)$$

¹In principle orthogonality is not required but it is generally preferred.

$$g = y(1 + \frac{R^2}{r^2}). \quad (3.11)$$

Since $\hat{F}$ preserves angles locally, f and g are orthogonal so we can then use the gradients of f and g to define our basis vectors for the simulation grid with the metric, m , given by

$$m = |\nabla f| = |\nabla g|. \quad (3.12)$$

The gradients of f and g also satisfy

$$\nabla f \times \nabla g = m^2 \hat{\mathbf{z}} \quad (3.13)$$

and

$$\nabla f \cdot \nabla g = \nabla^2 f = \nabla^2 g = 0, \quad (3.14)$$

the latter owing to the fact that $\hat{F}$ is analytic. The metric, m , is most easily written

$$m^2 = 1 + \frac{R^4 + 2R^2(x^2 - y^2)}{(x^2 + y^2)^2} \quad (3.15)$$

and the cartesian coordinates are themselves found by inverting Equation (3.9), i.e.,

$$x + iy = \zeta = \frac{1}{2} \left(\hat{F} \pm \sqrt{\hat{F}^2 + 4R^2} \right), \quad (3.16)$$

which leads directly to

$$x = \frac{f}{2} \pm \frac{1}{2} \Re \left(\sqrt{f^2 + 2ifg - g^2 + 4R^2} \right) \quad (3.17)$$

and

$$y = \frac{g}{2} \pm \frac{1}{2} \Im \left(\sqrt{f^2 + 2ifg - g^2 + 4R^2} \right). \quad (3.18)$$

Here the sign of each root is taken to correspond to the sign of f , which then maps

$\hat{F} \in \mathbb{C}$ to $\hat{\zeta} \in |\mathbb{C}| \geq R$, where $\mathbb{C}$ is the set of all complex numbers.²

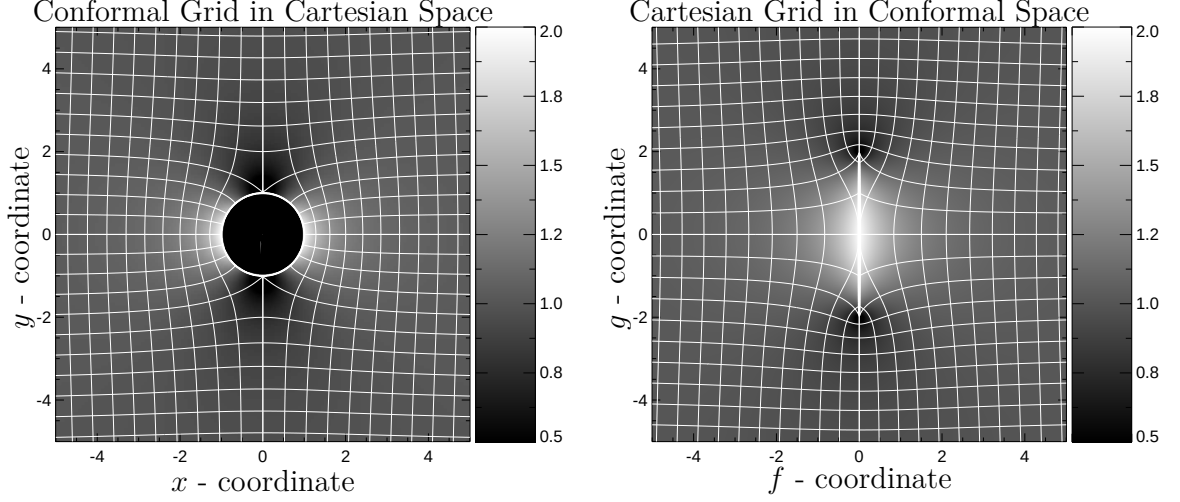


Figure 3.1. A conformal grid, composed of coordinates f and g , is depicted in a cartesian domain in the left panel for $R = 1$. The gray-scale image represents the value of the f - g coordinate metric, m . In the right panel, the cartesian grid of coordinates x and y is depicted in the conformal domain, also for $R = 1$. The intrusion is not within the conformal domain and thus the coordinate values of x corresponding to $x^2 < 1 - y^2$ are also not within the domain, provided that $|y| \leq 1$.

We can now express the MHD state variables in the new coordinate system. Scalar variables need not be altered as their representation is coordinate-independent; and since we have elected to represent the magnetic field in terms of a flux function, only the fluid velocity is given in vector form. As with cartesian coordinates, we express this vector in terms of components:

$$\mathbf{u} = u_f \nabla f + u_g \nabla g \quad (3.19)$$

²The opposite choice corresponds to a mapping from $\hat{F} \in \mathbb{C}$ to $\hat{\zeta} \in |\mathbb{C}| \leq R$, i.e. the *interior* of the retracting flux tube.

where u_g is given by

$$u_g = \frac{1}{m^2} \nabla g \cdot \mathbf{u} \quad (3.20)$$

and similarly for f . In terms of f and g the governing equations are found by projecting onto these bases and then simplifying as needed.

3.2.2. Fluid Equations

Within the framework of these coordinates we must restate the governing equations. As an example, consider the continuity constraint. Substituting our definition for the velocity from Equation (3.19) into the Right-Hand Side (RHS) of Equation (3.1) we have

$$\begin{aligned} \nabla \cdot \rho \mathbf{u} &= \nabla \cdot (\rho u_f \nabla f) + \nabla \cdot (\rho u_g \nabla g) \\ &= \nabla f \cdot \nabla \rho u_f + \nabla g \cdot \nabla \rho u_g + \rho u_f \nabla^2 f + \rho u_g \nabla^2 g \\ &= (\nabla f \cdot \nabla f) \partial_f \rho u_f + (\nabla g \cdot \nabla g) \partial_g \rho u_g. \end{aligned} \quad (3.21)$$

From here we see that the time rate of change of ρ is

$$\partial_t \rho = -m^2 (\partial_f \rho u_f + \partial_g \rho u_g), \quad (3.22)$$

where we have made use of the fact that f and g are harmonic and orthogonal and employed the chain rule in vector form; $\nabla \equiv \nabla f \partial_f + \nabla g \partial_g$.

The remaining expressions are addressed in similar fashion so that the five scalar equations governing the system are:

$$\partial_t \rho = -m^2(\partial_f \rho u_f + \partial_g \rho u_g), \quad (3.23)$$

$$\partial_t p = -m^2(\partial_f p u_f + \partial_g p u_g), \quad (3.24)$$

$$\partial_t \varphi = -m^2(u_f \partial_f + u_g \partial_g) \varphi, \quad (3.25)$$

$$\begin{aligned} \partial_t u_f = & -m^2(u_f \partial_f + u_g \partial_g) u_f - m(u_f^2 + u_g^2) \partial_f m - \frac{1}{\rho} \partial_f p \\ & + \frac{1}{\rho} m^2(\partial_f \varphi)(\partial_f^2 + \partial_g^2) \varphi + \frac{m}{\rho} \mu(\partial_f^2 + \partial_g^2) m u_f, \end{aligned} \quad (3.26)$$

and

$$\begin{aligned} \partial_t u_g = & -m^2(u_f \partial_f + u_g \partial_g) u_g - m(u_f^2 + u_g^2) \partial_g m - \frac{1}{\rho} \partial_g p \\ & + \frac{1}{\rho} m^2(\partial_g \varphi)(\partial_f^2 + \partial_g^2) \varphi + \frac{m}{\rho} \mu(\partial_f^2 + \partial_g^2) m u_g. \end{aligned} \quad (3.27)$$

In the latter two expressions we have foregone detailed consideration of the viscous stress in favor of the simpler isotropic form with μ representing the coefficient of dynamic viscosity. The energy equation has been similarly simplified, with $\gamma \rightarrow 1$ to suppress both adiabatic and volumetric heating. This is essentially an isothermal condition but with the benefit that the temperature can be set differently within the domain and, once set, each fluid element will carry its own temperature with

it. Careful derivations of these expressions are given in Appendix A, §A.2, along with detailed consideration of thermal conduction, viscous heating, and the viscous contribution to the momentum equations.

3.2.3. Numerical Techniques

For numerical simplicity the f and g coordinates were normalized to the radius of the intrusion. The density and sound-speed (C_s) were both normalized to their asymptotic values, far from the intrusion, and the time variable was normalized to the acoustic crossing time of the intrusion, referenced to the asymptotic sound-speed, i.e.

$$\tilde{\rho} \rightarrow \rho / \bar{\rho}, \quad (3.28)$$

$$\tilde{C}_s \rightarrow C_s / \bar{C}_s, \quad (3.29)$$

$$\tilde{t} \rightarrow t \bar{C}_s / R, \quad (3.30)$$

$$(3.31)$$

where the tilde and over-line indicate the normalized and asymptotic value of each, respectively. The velocity, pressure and magnetic field strength were then normalized using the asymptotic sound-speed, and density;

$$\tilde{p} \rightarrow p / \left(\bar{\rho} \bar{C}_s^2 \right), \quad (3.32)$$

$$\tilde{u}^2 \rightarrow u^2 / \bar{C}_s^2, \quad (3.33)$$

$$\tilde{B}_0^2 \rightarrow B_0^2 / \bar{\rho}, \quad (3.34)$$

$$(3.35)$$

The magnetic flux function was then normalized to the coordinate system according to

$$\tilde{\varphi} = \tilde{B}_0(f + \varphi_1), \quad (3.36)$$

where φ_1 tracks the deviation from the potential field, i.e.

$$\tilde{\mathbf{B}} = \tilde{B}_0 \hat{z} \times \nabla(f + \varphi_1) \quad (3.37)$$

$$= \tilde{B}_0 \nabla g + \tilde{B}_0 (\nabla g \partial_f - \nabla f \partial_g) \varphi_1. \quad (3.38)$$

For future discussion it will be understood that all variables are given in their normalized form and we will omit the use of the tilde.

The initial magnetic field is always in a potential state and aligned to the g coordinate lines, which is accomplished by setting $\varphi_1 = 0$ at $t = 0$. The initial plasma velocity is dictated in such a way that the *speed* of the fluid is uniform and equal to the asymptotic value with velocity streamlines aligned to the magnetic field. This is accomplished by initially specifying the f and g components of the velocity to be

$$u_f(t = 0) = 0, \quad u_g(t = 0) = \frac{1}{m} \bar{u}_g. \quad (3.39)$$

The initial density is set to unity in the asymptotic region while within the region of interest it varies linearly with m so that the mass flux along each streamline is

initially uniform, i.e.

$$\partial_t \rho = \nabla \cdot \rho \mathbf{u} = m^2 \partial_g \bar{\rho} \bar{u}_g = 0. \quad (3.40)$$

Of all the initial configurations that we tried, this seemed the least susceptible to transient effects. The pressure was initialized to be equal to the density so that the isothermal sound speed would be uniform with a value of $C_s = \bar{C}_s = 1$. Our model does, however, allow for the possibility of a non-uniform initial temperature.

With this configuration, only the asymptotic velocity and magnetic field strength remain unspecified. These were chosen at the beginning of each simulation run in order to explore the behavior of the system in various regimes. In every case the asymptotic velocity was one of

$$\bar{u}_g = \{0.29, 0.44, 0.67, 1.00, 1.50, 2.25, 3.38\}, \quad (3.41)$$

which correspond to increasing powers of $3/2$, and the plasma- β was chosen from

$$\bar{\beta} = \{0.0, 0.1, 0.5, 1.0, 2.0, 10.0, \infty\}. \quad (3.42)$$

For non-zero values of $\bar{\beta}$ this was done by specifying the magnetic field strength. For the case of $\beta = 0$ we fixed the transverse component of the velocity to be forever zero, which prevents the initial magnetic field from evolving and therefore imitates the system in the case of an impervious potential magnetic field, i.e. zero- β .³

³Setting the sound speed to zero would also give a zero- β scenario, but then the acoustic Mach number would necessarily be infinite and we could not explore the range of effects from varying M_s .

We chose to solve Equations (3.23), (3.24), (3.25), (3.26) and (3.27) on a uniform grid of 200 x 800 pixels spanning the domain $f \in [0, 10]$, $g \in [-20, 20]$. The surface of the right hemisphere of the intrusion sits at $f = 0$ between $g = \pm 2$ but is not within the numerical domain. The $f = 0$ boundary is understood to be a line of reflectional symmetry so this numerical domain captures the evolution of the entire system. The anti-sunward direction is taken to be up, in the direction of increasing g .

We used upwind differencing for the continuity, induction, and energy equations and centered differencing for the momentum equation. For the viscosity and the components of the Lorentz force we used a second order, centered differencing scheme for high order derivatives. At the bottom, right, and top boundaries, the normal components of the gradient were calculated using off-center differencing and the value of the laplacian was inferred from the adjacent cell. At the left boundary the normal component of the gradient was found by assuming that ρ, p and u_g were mirror symmetric while u_f and φ_1 were assumed to be mirror antisymmetric. Along the surface of the intrusion, however, we used exclusively off-center differencing, lest information about the mirror domain be somehow communicated through the body of the intrusion.

On the lower (sunward-most) boundary we fixed the value of φ in order to anchor the field lines without dictating the direction of the magnetic field. The fluid velocity at the lower boundary was held fixed at the initial value and the velocity component

perpendicular to the field was held at zero along this boundary. The density and pressure along this boundary were also held fixed. Together, these conditions guaranteed that the far-upstream conditions never varied from the initial condition. Along the right and left boundary the value of φ_1 and u_f were specified to be zero so that the magnetic field and fluid velocity are always parallel to these boundaries. The other state variables, ρ, p and u_g were allowed to vary freely along these boundaries.

At the upper boundary, where we desired for the fluid to flow out of the domain with minimal reflections, we specified that the magnetic field be orthogonal to the boundary. That is, the value of φ at the boundary was set equal to the value just inside the boundary, guaranteeing that $\nabla\varphi_1$ point only in f , consistent with a vertical magnetic field. For the remaining state variables we used an open boundary condition at the top of the domain with $\partial_g\{\rho, p, u_i\} = 0$. We considered that a non-reflecting boundary condition might better serve to suppress reflections; however, since the fluid velocity is always upward and often supersonic, it was observed that any artifacts that were created were quickly swept out of the domain.

Each simulation run was allowed to evolve until a maximum time of $t_{max} = 20 t_s$, which was found to be sufficient for the system to settle into a steady state in every case. Forward integration was accomplished by first dividing the simulation time into 400 time steps and then evolving the system from one step to the next through a series of unreported sub-steps. The integration time for each sub-step was found

from the Courant condition and given by

$$t_c = R_c \min \left(\frac{\Delta}{|\mathbf{u}| + \max(C_s + V_A + V_\sigma)} \right), \quad (3.43)$$

where $V_A = |\mathbf{B}|/\sqrt{4\pi\rho}$ is the Alfvén speed, $V_\sigma \equiv \Delta/(\mu m)$ is the characteristic speed of viscous diffusion, μ is the viscosity, Δ is the grid resolution, $R_c = 0.25$ is a safety factor. The viscosity coefficient was set to $\mu = 0.1$ giving a Reynolds number in the range of $Re = \bar{u}/\mu \sim \{3, 30\}$, depending on the initial velocity. As such, we did not anticipate the formation of boundary layers or vortex sheets, which typically do not occur for $Re < 100$. Each sub-step was further dividing into two micro-steps according to a second order Runge-Kutta scheme:

$$S(t_{i+1}) = S(t_i) + \delta t S'_R(t_i), \quad (3.44)$$

$$S_R(t_i) = S(t_i) + R_R \delta t S'(t_i). \quad (3.45)$$

Here $S(t_i)$ is representative of the state of the system at time $t = t_i$, S' is the generalized time derivative of the system, and S_R is the state of the system incremented forward by a time step $\delta t_R = R_R \delta t$. The value $R_R = 0.51$ was set to control the size of the intermediate sub-step, though any value over 0.5 is considered to be stabilizing.

The computational time to complete a simulation to a run-time of $t \sim 20 t_s$ varied depending on the strength of the magnetic field with high β runs completing in a few minutes and low β runs taking several days to return. A truly zero- β run would take infinite time to converge since it requires an infinite magnetic field strength, hence the

need to impose the zero- β limit by suppressing u_g . The opposite limit ($\beta \rightarrow \infty$) poses no problem since it can be achieved by setting $B_0 = 0$, which reduces the simulation to a hydrodynamic one with only a little computational overhead.

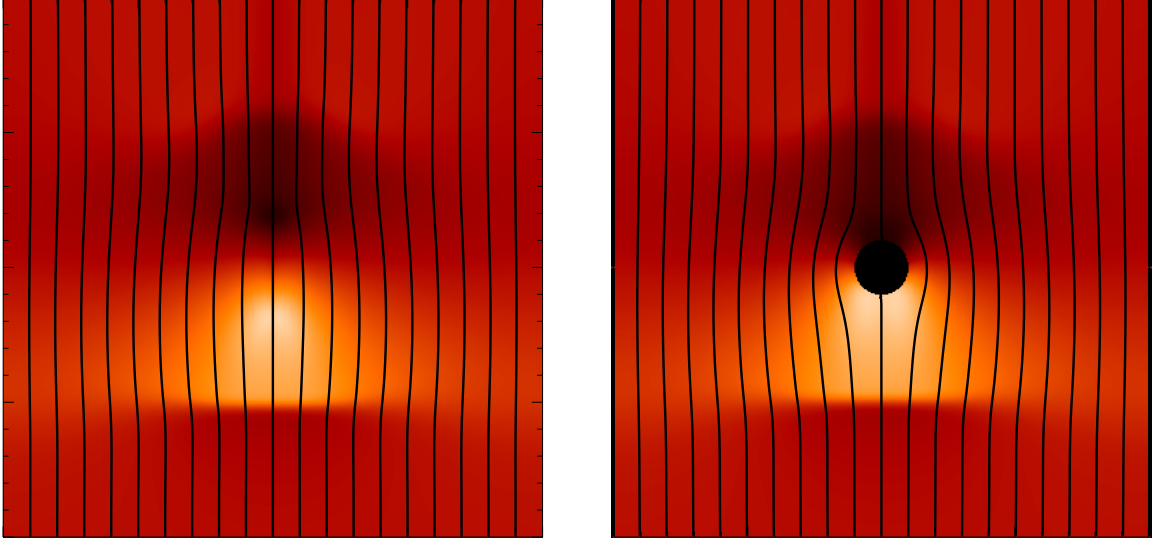


Figure 3.2. The plasma density is shown in the native coordinate system (left) and remapped to the cartesian coordinate system (right) in a red-yellow color scale. The magnetic field lines are indicated in black. This example is taken at a time of $t = 20 t_s$ with initial conditions $\bar{\beta} = 1.0$ and $\bar{M}_s = 1.0$.

Because each simulation is performed in the f, g coordinate domain, the results of each run were then remapped to the cartesian domain for interpretation. The cartesian grid allows for a more intuitive understanding of the dynamics as it explicitly shows the location of the intrusion. However, the deformation of the field is more easily seen in the native coordinates where the potential field is represented by straight lines and any curvature is indicative of a non-potential configuration. The results of

the remapping (which was accomplished using the `trigrid.pro` and `triangles.pro` routines) can be seen in Figure 3.2. While the information content is identical between the two panels, the curvature of the field is obvious in the native coordinates (left) and nearly indistinguishable from the potential field in cartesian coordinates (right). Once remapped, the velocity must be reconstructed from its component representation with the cartesian x component found from the conformal f and g components according to

$$\begin{aligned}
 u_x &= \hat{\mathbf{x}} \cdot \mathbf{u} \\
 &= \hat{\mathbf{x}} \cdot (u_f \nabla f + u_g \nabla g) \\
 &= u_f \partial_x f + u_g \partial_x g
 \end{aligned} \tag{3.46}$$

and the y component given similarly as

$$u_y = u_f \partial_y f + u_g \partial_y g. \tag{3.47}$$

By contrast, the remapped scalar parameters are immediately native to the cartesian grid.

3.3. Behavioral Modes

As we have mentioned, the setup of this model can be made broadly applicable and so we have chosen to explore a rather broad parameter space, with both $\bar{\beta}$ and $\bar{M}_s$ varying from very low to very high. In the following discussion we will consider

all possible initial configurations, and it will be understood that the applicability to specific solar features will vary depending on the assumed ambient conditions in different solar regimes.

In exploring these myriad regimes, several distinct modes of behavior have become apparent. In order to understand these it is useful to define two pressure-like quantities associated with the system, the magnetic pressure (also known as the magnetic energy density),

$$p_B = \frac{B^2}{8\pi}, \quad (3.48)$$

and the ram pressure (twice the bulk kinetic energy density)

$$p_r = \rho u^2. \quad (3.49)$$

Whereas the plasma pressure represents the internal energy (per degree of freedom) that is available to do work within the system, the magnetic and ram pressures represent the available energy in the magnetic field and bulk kinetic motion that contribute meaningfully to the momentum equation. In terms of these pressures the momentum equation becomes

$$\partial_t \mathbf{u} \approx -\frac{1}{\rho} \nabla (p + p_B + p_r) + \frac{1}{\rho} \nabla \cdot \underline{\underline{\sigma}}, \quad (3.50)$$

where we have omitted terms relating to the rotation and compressibility of the fluid or the slow, field aligned variation of the magnetic field.

The above expression is insufficient to recover the detailed dynamics of an evolving plasma, but does give insight into the various regimes that develop. The ratios of these pressures are easily related to three well known fluid parameters. Namely,

$$\beta \equiv \frac{8\pi p}{B^2} = \frac{p}{p_B}; \quad (3.51)$$

$$M_s^2 \equiv \frac{u^2}{C_s^2} = \frac{p_r}{p}; \quad (3.52)$$

$$M_A^2 \equiv \frac{u^2}{V_A^2} = \frac{p_r}{2p_B}, \quad (3.53)$$

where M_s is the acoustic Mach number (with $\gamma = 1$), V_A is the Alfvén speed, and M_A is the associated Alfvén Mach number. From here it is clear that only two of these fluid parameters can be independently set, with the third subsequently dictated according to

$$\beta = 2 \frac{M_A^2}{M_s^2}. \quad (3.54)$$

The various modes of behavior that we observe are each related to these fundamental ratios, and are described accordingly.

3.3.1. Continuous Nozzle Flow

One of the goals of this investigation was to confirm our previous analytical solution in which we considered an arbitrarily strong magnetic field, which suppressed all flows perpendicular to the field and therefore forced the plasma streamlines to be uniform in time, despite the temporal variance of the density and flow speed. In that work, we argued that this behavior should be manifest in cases of extremely low β , so

that the Lorentz force was dominant in the direction perpendicular to the field and the interesting fluid dynamics were all aligned to the magnetic field.

In Scott et al. (2013) we showed this case lends itself not to a single two-dimensional (2D) treatment, but rather to a class of 1D treatments, with the flow along each field line decoupled from adjacent flows. We parameterized each flow according to the expansion ratio (α) of the associated infinitesimal flux tube, which must go as $\alpha = 1/m$ in order that $\nabla \cdot \mathbf{B} = 0$, and applied the solution famously developed by Gustaf de Laval to describe the behavior of a compressible gas flowing through a nozzle with variable cross section. The acoustic Mach number of the plasma is then governed by a form of Bernoulli's equation;

$$M_s^2 - \ln M_s^2 + \ln m^2 = \mathcal{B}, \quad (3.55)$$

where $\mathcal{B}$ is a constant of the motion.

This solution has a number of interesting aspects. First, it is multivalued – for every solution with $M_s > 0$ there is an equivalent solution with $M_s < 0$. Second, it can be shown that if $M_s = 1$ then the value of $\alpha = 1/m$ must be a minimum and therefore m must be a maximum. From this, it follows that along each streamline there is a unique value for $\mathcal{B}$, which we call $\mathcal{B}_{crit}$, such that if $\mathcal{B} < \mathcal{B}_{crit}$ then there will exist a location where $M_s^2 - \ln M_s^2 < 1$ and therefore M_s cannot be real-valued at that location. And, since the maximum value of m along each streamline increases monotonically toward the intrusion, for any streamline that fails this criterion every

streamline closer to the intrusion will also fail the criterion. Conversely, if the central streamline, which passes along the surface of the intrusion, satisfies the critical criterion, then all streamlines will likewise satisfy the criterion.

We can then combine these conditions for individual streamlines into a global condition by simultaneously requiring that the solution along *every* streamline be globally real-valued, for which the requirement that the central streamline be everywhere real-valued is both necessary and sufficient. Since the narrowest point along the central streamline occurs abreast of the intrusion at $f = g = 0$, we can define the global critical value of $\mathcal{B}$ by setting $M_s = 1$ with $m = 2$ so that

$$\mathcal{B}_{crit}^{(max)} = 1 + \ln 4. \quad (3.56)$$

Then, observing that in the asymptotic region, where $M_s \rightarrow \overline{M}_s$, the metric goes to one, the asymptotic value of M_s is related to $\mathcal{B}$ according to

$$\overline{M}_s - \ln \overline{M}_s^2 \geq \mathcal{B}_{crit}^{(max)}, \quad (3.57)$$

which is now the global condition for the 2D de Laval nozzle solution. These observations together imply that for magnetically dominated systems there are, in fact, two sub-classes of behavior that each result in continuous, unshocked flow.

Subsonic Nozzle Flow – Because Equation (3.57) typically has two roots, there are two regimes that permit a globally continuous solution. Taking the subsonic root, we find that a smooth solution is permissible so long as $\overline{M}_s \lesssim 0.31$. We refer

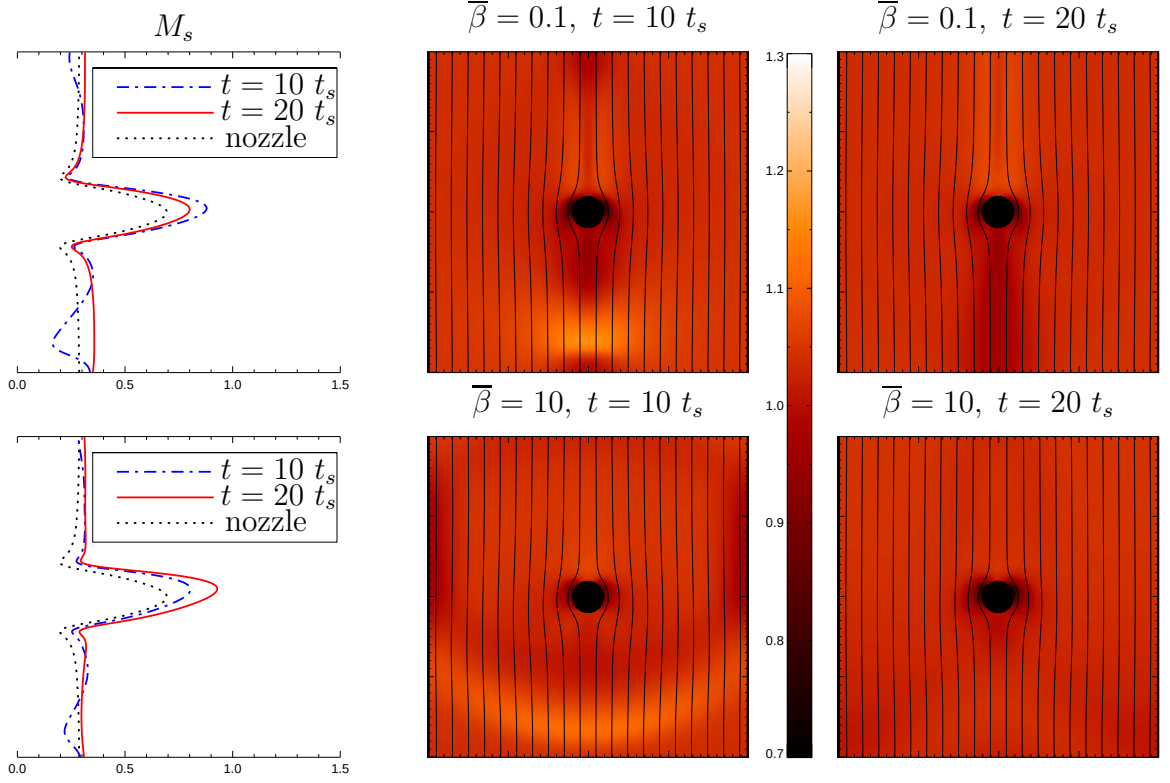


Figure 3.3. The subsonic nozzle regime (L^-) is depicted along the upper row for $\bar{\beta} = 0.1$ while the traditional, hydrodynamic solution is shown in the lower row for $\bar{\beta} = 10$. The asymptotic Mach number is $\bar{M}_s = 0.29$, below the critical threshold for L^- . The curves on the far left indicate the acoustic Mach number parameterized along a near-central streamline. The black dashed curves correspond to the zero- β , analytical solution. In the center and right-most panels the red-yellow color scale indicates density while the black field contours indicate magnetic field lines.

to behavior along the subsonic branch of the magnetically guided nozzle regime as

Subsonic Nozzle Flow (L^-), which behavior can be seen in the upper three panels of

Figure 3.3. The use of the letter L is in acknowledgement of Gustaf de Laval. In the

upper-left panel we show the parameterized solution along a near-central field line.⁴

⁴The central field line exhibits a magnetic null at $g = \pm 2$ and therefore can never satisfy the low- β criteria. It must rely on the surrounding field to maintain its shape, which it apparently does, but the details of the solution there are less ideal.

In addition to the analytical solution, a transient solution is visible in both the density and acoustic Mach number up to a time of $t \sim 10 t_s$. By $t = 20 t_s$ the transient is gone and the behavior is nearly an exact match to the analytical solution, indicated by the black-dotted curve. The upper-central and upper-right panels indicate the plasma density, also at $t = 10 t_s$ and $t = 20 t_s$. For the first the transient is visible as a narrow front of enhanced density while for the second this feature is no longer in evidence.

Figure 3.3 also shows the behavior in the subsonic, large- β limit. There the restoring force of the magnetic field cannot play a role, but the variations from the L^- regime are small. This is due to the fact that the dipole magnetic field is itself a solution to the streamlines of an incompressible flow around a cylinder, which is a decent approximation for this system provided that $\overline{M}_s$ is small. As the acoustic Mach number approaches unity, however, the fluid streamlines are no longer held fixed and the system relaxes to the compressible, subsonic solution, which remains smooth up to flow speeds approaching the sound speed, provided that the Reynolds number is not too large. We will revisit the transition from subsonic to supersonic flow in the large- β limit at a later time.

Supersonic Nozzle Flow – Taking the upper root of Equation (3.57) we find that the magnetically guided flow should remain continuous provided that $\overline{M}_s \gtrsim 1.92$. Tacit in this statement, however, is the assumption that the ram pressure does not

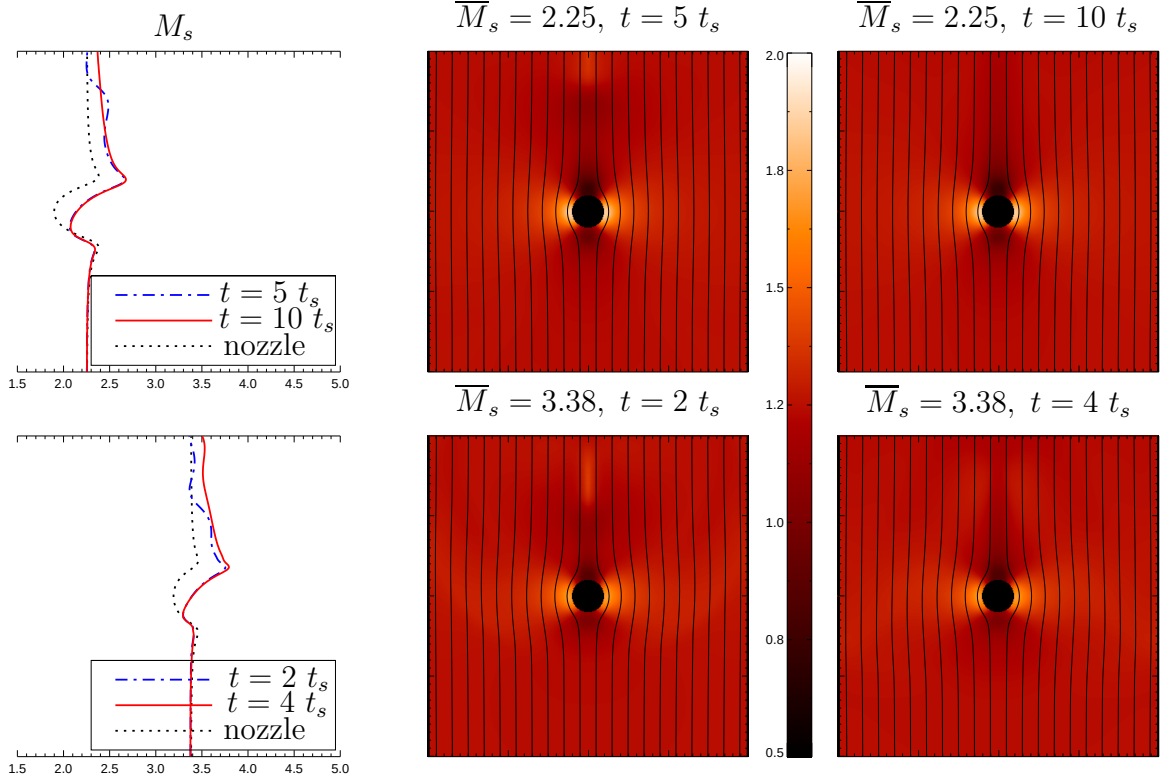


Figure 3.4. Plasma density is depicted in red-yellow color scale with overlaid field lines for two simulation examples in the supersonic nozzle regime (L^+). In both cases $\overline{\beta} = 0.1$, with $\overline{M}_s$ varying from 2.25 in the upper panel to 3.38 in the lower panel. The far left panel indicates the acoustic Mach number as parameterized along a near-central field line with the center and right panels showing two representative times for each case. In the upper and lower left panels the dotted line again shows the analytical solution, which is qualitatively similar to the numerical result.

dominate the magnetic field. This was understood to be the case for subsonic flows, however, for supersonic flows there exists the possibility that $\overline{M}_A \geq 1$, regardless of the value of $\overline{\beta}$. The conditions for *Supersonic Nozzle Flow* (L^+) are therefore more strongly constrained than are subsonic nozzle flows. Specifically,

$$1.92^2 < \overline{M}_s^2 \quad \text{and} \quad \overline{M}_A^2 < 1, \quad (3.58)$$

which, together, ensure that $\beta < 1$ and, therefore, $\{p, p_r\} \ll p_B$.

This mode of behavior is shown in Figure 3.4, which depicts two separate simulations, one with $\overline{M}_s = 2.25$ and the other with $\overline{M}_s = 3.38$. In both cases $\overline{\beta} = 0.1$ so that the Alfvén Mach number for these two runs is $\overline{M}_A = 0.50$ and 0.76 , ensuring that both runs are magnetically dominated. As we predicted in our previous paper the flow is smooth and largely resembles the analytical solution, once the transients have radiated away. Note that this mode of behavior, unlike the L^- regime, which is largely indistinguishable from the subsonic hydrodynamic case, is a significant departure from simple hydrodynamics. The existence of a supersonic, shock-free solution in the presence of a localized, rigid boundary, may be a unique characteristic of magnetically guided flows; it is certainly a departure from classical fluid dynamics.

3.3.2. Magnetically Confined Shocks

The existence of magnetically confined shocks is seen by reversing the logic in the preceding discussion of supersonic and sub-sonic nozzle flows: namely, by specifying, as a boundary condition, that the asymptotic value of M_s lie within the critical range. And, as with the L^+ regime, we require that $\overline{\beta} \ll 1$ and $\overline{M}_A \ll 1$, in order to maintain the condition of magnetic dominance. Then, since no globally real-valued solution can exist, the system is forced to develop shocks, which allow for local departures from the de Laval nozzle solution, thereby circumventing the unphysical aspects of that description.

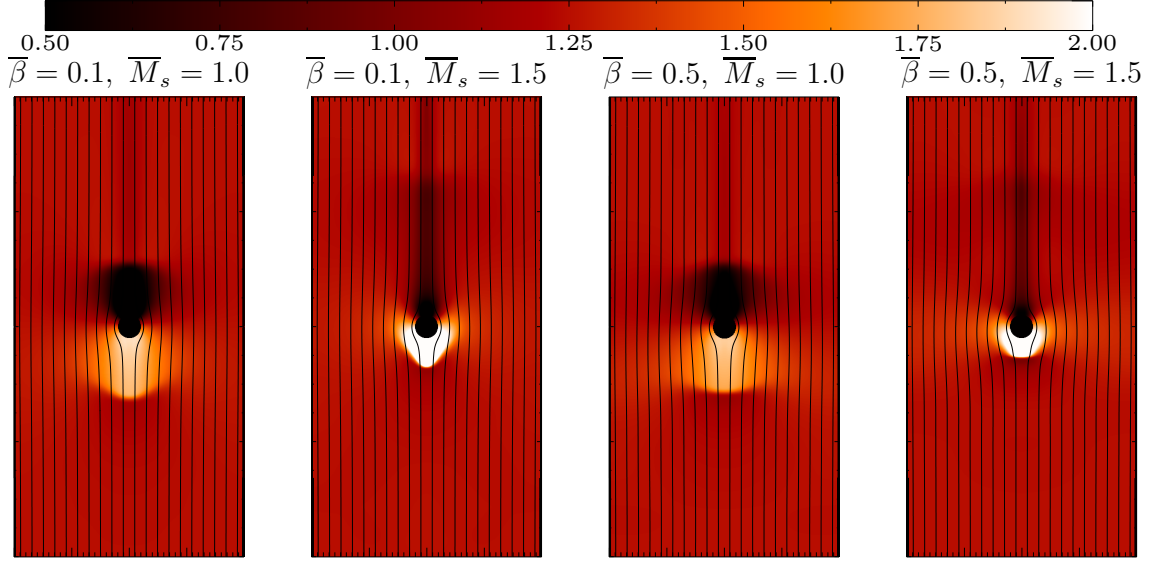


Figure 3.5. The plasma density is shown in red-yellow color scale for four examples of magnetically confined shocks, each at a time of $t = 20 t_s$. In each of the four examples $\bar{\beta}$ and $\bar{M}_s$ have been selected from one of $\bar{\beta} = \{0.1, 0.5\}$ and $\bar{M}_s = \{1.0, 1.5\}$.

The width of the shocked column depends directly on $\bar{M}_s$ and can be estimated by specifying M_s in Equation (3.55) with $m = 1$, thereby dictating the value of $\mathcal{B}$. This value is then related to f through the coordinate dependence of m and Equation (3.55). Setting the local Mach number to $M_s = 1$ then dictates the value of m corresponding to the transonic solution. For each streamline, the maximum value of m along that streamline is related to the critical value for transonic flow. Thus, for every value of M_s there is a critical streamline that defines the width of the shocked column – all streamlines outside of this column remain free of shocks while every streamline inside the column must undergo a shock – and the width decreases monotonically with $\mathcal{B}$, and therefore increases monotonically as $\bar{M}_s \rightarrow 1$.

While the need for shocks and the width of the shocked column are easily motivated, the details of the shock formation and the interior solution are more complex. As we discussed in Scott et al. (2013), the solution requires a leading shock that precedes the intrusion and then a transonic interior solution that ultimately connects to a trailing shock and rarefaction wave, which trail behind. Each of these features then propagates away from the intrusion at a speed dictated by the Rankine-Hugoniot conditions (or equivalent requirement in the case of the rarefaction wave). This behavior, which we predicted in our previous paper has never before been confirmed in a fully two-dimensional model with an evolving magnetic field.⁵ Moreover, in our analytical work, we only demonstrated the existence of a solution in the case of supersonic flows, while speculating that a corresponding subsonic, shocked solution should exist and exhibit similar properties. Numerical confirmation of these behaviors is presented here for the first time.

The ensemble behavior of the system in this regime, which was previously dubbed “peristaltic pumping”, is here referred to as a *Magnetically Confined Slow Shock* (S_c). The identification of these shocks as slow magnetosonic shocks relies on a number of observations. First, because the notion of magnetic confinement is predicated on a sub-Alfvénic regime of MHD, the Alfvén Mach number is necessarily small; therefore they cannot be fast shocks. If they are slow shocks then the relevant wave

⁵In Scott et al. (2013) we created a 2D composite solution from individual 1D solutions, each with a rigid magnetic field.

velocity against which to compare them is the slow magnetosonic speed in the case of propagation parallel to the field; this is the lesser of the sound speed (C_s) and the Alfvén speed (V_A). But of course the Alfvén speed is large in this regime, so the sound speed *is* the slow magnetosonic speed. The only remaining question is this: why does a slow magnetosonic shock form a column and not a cone? We will return to this at a later time but for now, consider that in the absence of a stationary shock cone, the fluid is forced to find a traveling solution, as we presented in the preceding analysis.

Four examples of this behavior are shown in Figure 3.5, with variations of $\bar{\beta}$ and $\bar{M}_s$. While this behavior was predicted through an analytical model that assumed a vanishingly small value of β , we anticipated that the behavior would persist for increasingly large values of β , perhaps as large as $\bar{\beta} = 0.1$. We did not anticipate that this behavior would persist all the way up to $\bar{\beta} = 0.5$, as seen in the right two panels of Figure 3.5. One explanation might be the local magnetic field strength of the dipole solution, which is generally enhanced relative to the ambient value, though other factors may also contribute.

As we have discussed, the different flow speeds have a strong effect on the width of the shocked column, which can be clearly seen when comparing either the left two panels or the right two panels in Figure 3.5. The formation of shock in a localized column, with the shock amplitude falling off abruptly outside of that column, is

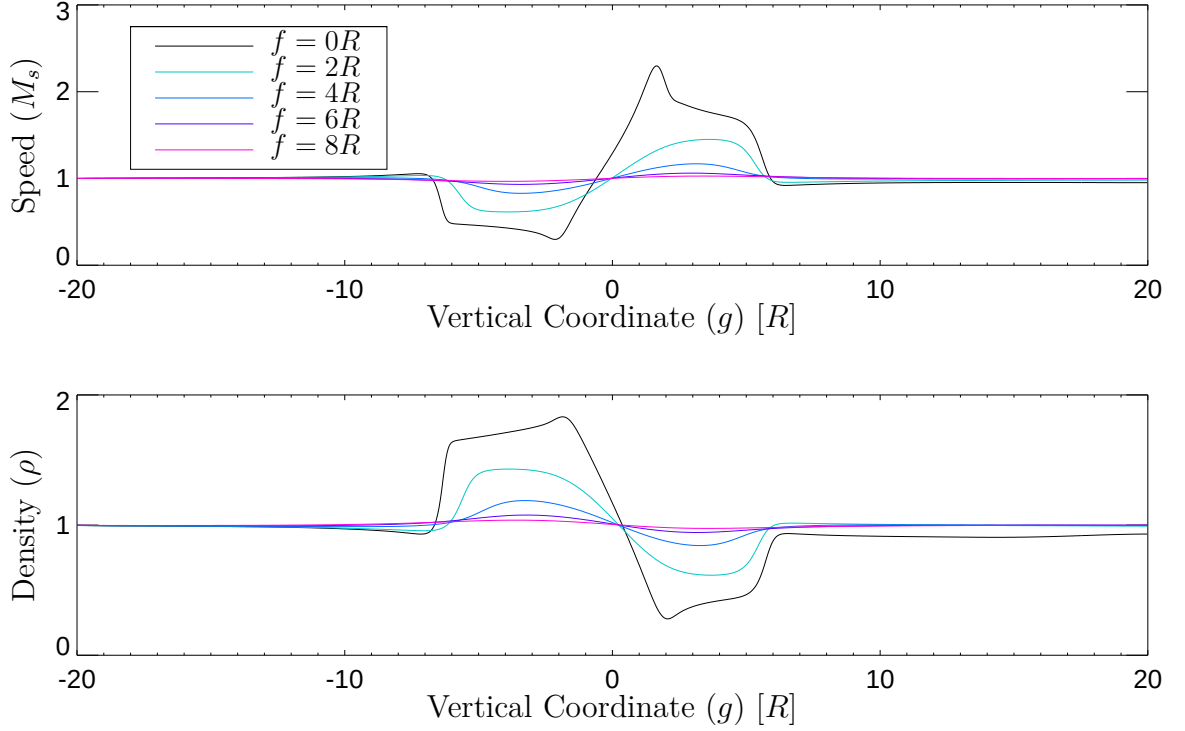


Figure 3.6. The fluid Mach number and plasma density are depicted in the upper and lower panels for a magnetically confined slow shock with $\bar{\beta} = 0.5$ and $\bar{M}_s = 1.0$. Each curve indicates the vertical profile at a given lateral distance from the intrusion. The leading shock, trailing shock, and transonic interior flow are all clearly visible.

unique to this solution. This effect is demonstrated in Figure 3.6, which indicates the parameterized solution as a function of transverse distance from the intrusion. Adjacent to the intrusion, a leading shock, transonic flow, and trailing shock are clearly visible, with the rarefaction wave already passed outside of the domain. But at a distance of $f = 4R$ the shock amplitude has diminished dramatically and the flow appears to be free of shocks.

Further demonstrating the uniqueness of this solution is the time evolution of the shocks. Unlike traditional bow shocks, which set up a fixed distance from the object that they precede, magnetically confined slow shocks continue to propagate away from the intrusion, even after the system has settled into a steady state. This is a result of the limited number of free parameters in the case of one dimensional flow, where the speed of the shock is dictated by the demand that it connect the far-field solution to the transonic solution. The fact that the system can be viewed as in a steady state is owing to the translational invariance of the magnetic field far from the intrusion – the leading and trailing shocks propagate away from the intrusion but the upstream, downstream, and interior flows are unchanged so the details of the shocks are steady, with only their position changing in time.

This behavior is demonstrated in Figure 3.7. Here a sequence of parameterized solutions are shown successively for the same conditions as in Figure 3.6, with $\bar{\beta} = 0.5$ and $\bar{M}_s = 1.0$. Initially the speed of the fluid is uniform and the density mimics the field strength as dictated by continuity. A shock solution quickly forms with leading and trailing shocks and a transonic interior, as in the $f = 0R$ profile of Figure 3.6. The interior solution remains largely unchanged after this point, while the leading and trailing shocks propagate steadily away from the intrusion. Not shown, but described in Scott et al. (2013), is the same solution in the rest-frame of the solar-limb, in which case the trailing shock propagates away from the intrusion but *not as quickly as the*

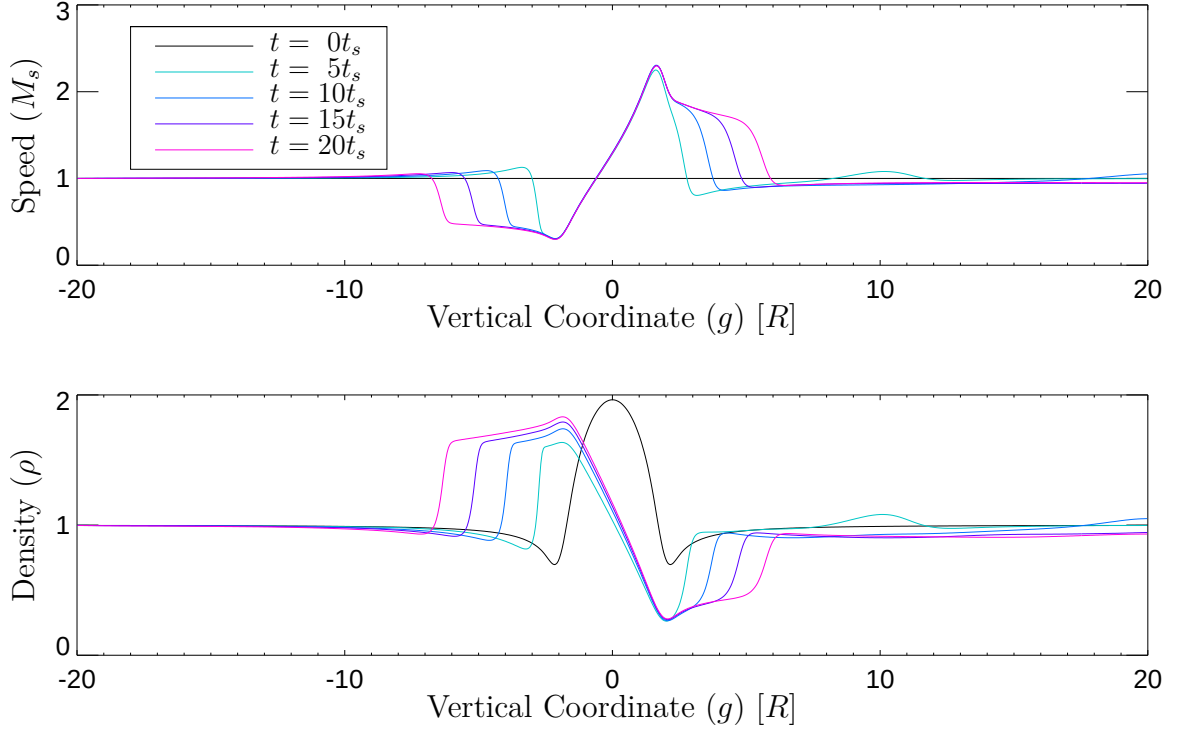


Figure 3.7. The vertical profiles for the evolution the Mach number and plasma density of a magnetically confined shock are depicted in the upper and lower panel. Each curve indicates the same transverse position but at successively later times. Again, the leading and trailing shock are in evidence, as is the interior, transonic solution.

intrusion descends so that the trailing shock is, in fact, entrained with the descending intrusion.

3.3.3. Ballistic Shocks

In each of the cases that we have described, with the exception of the subsonic, high- β scenario, the Alfvén Mach number has been small. The relaxation of this condition has two effects. The first is that the ram pressure is no longer small compared to the magnetic pressure and so the fluid is dominant over the magnetic field,

regardless of the value of β . The second effect is that the fast magnetosonic speed, which is generally larger than either the Alfvén speed or the sound speed, ceases to be significantly greater than the speed of the flow, and that in turn makes the system susceptible to the formation of fast magnetosonic shocks.⁶

As it happens, the formation of fast shocks has little to do with $\bar{\beta}$ and everything to do with $\bar{M}_s$ and $\bar{M}_A$, both of which depend critically on the ram pressure. Owing to this, we refer to the formation of fast magnetosonic shocks as the *Ballistic Shock* regime (S_b). The fact that the fast magnetosonic wave is independent of the plasma- β is a feature of the propagation speeds of fast and slow magnetosonic waves, which are given by the upper and lower roots of

$$v_{f,s}^2(\theta) = \frac{1}{2}(V_A^2 + C_s^2) \pm \frac{1}{2}\sqrt{(V_A^2 + C_s^2)^2 - 4V_s^2V_A^2\cos^2\theta}. \quad (3.59)$$

These depend critically on the angle of propagation with respect to the magnetic field (θ), the so-called *fast speed* ($v_{fast}^2 = V_A^2 + C_s^2$), and a sort of eccentricity factor given by $\chi = V_A C_s / (V_A^2 + C_s^2)$, all of which are invariant under interchange of V_A with C_s . That is, for any two values of V_A and C_s , with $\beta = 2C_s^2/V_A^2$, the values of V_A and C_s can be interchanged, so that $\beta \rightarrow \beta' = 4/\beta$, and the fast and slow magnetosonic speeds will be totally unaffected.

⁶Flow speeds in excess of the fast magnetosonic speed are a necessary condition of fast magnetosonic shocks, but not a sufficient condition. There must still be a trigger for the formation of strong gradients or no shock will form.

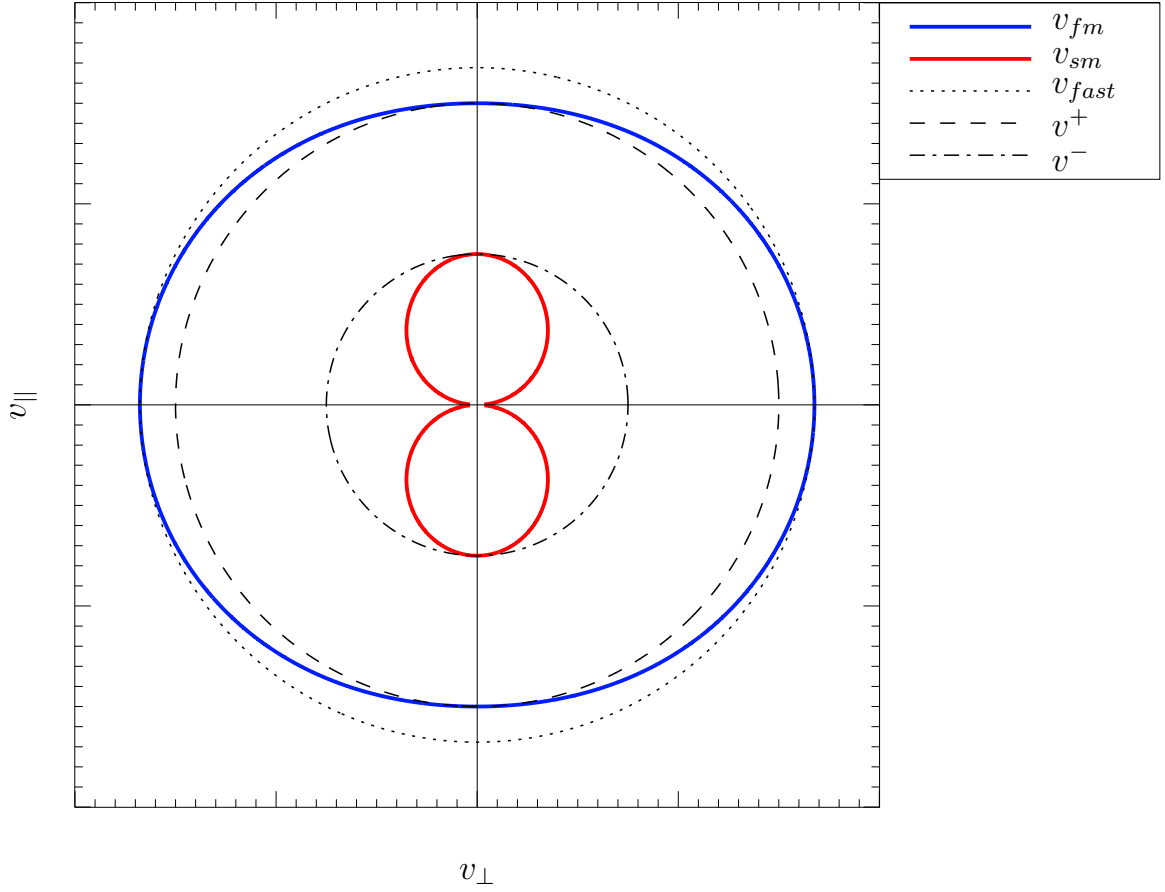


Figure 3.8. The Friedrich's diagram above indicates the propagation speeds of the various magnetosonic modes in terms of their components relative to the direction of the magnetic field. The relative size of the two components indicates the angle with respect to the field, i.e. $v_{\parallel} = \hat{\mathbf{b}} \cdot \mathbf{v}$. The concentric circles indicate the magnitude of the fast speed and the larger and lesser of the Alfvén speed and the sound speed, respectively. The speed of fast and slow magnetosonic waves (v_{fm} , v_{sm}) are represented by the blue and red curves. For each, the speed at a given propagation direction is indicated by the distance from the origin to the velocity surface at the corresponding polar angle.

A detailed representation of the angular dependence of magnetosonic wave speeds is offered in Figure 3.8. Note that while the fast magnetosonic mode is non-zero at all propagation angles, the slow magnetosonic mode has a phase speed of zero

for perpendicular propagation. From here we can begin to see how a shock wave might form in a moving fluid and why it is that only the fast magnetosonic wave permits the formation of shock cones. Shock waves occur when steep gradients create non-negligible contributions from the viscous stress, which corresponds to a non-conservative force. The formation of steady bow shocks can be understood in terms of the linearized wave theory as the interference of expanding radial wave fronts, each of which originates from a different place within the fluid, but with a sort of coherent phase velocity so that they all stagnate and mutually interfere along a single surface.

The shock cone angle can be derived by considering a shock front whose surface normal forms an as-yet unspecified angle to the upstream fluid flow, which is taken to be uniform and vertical. If the shock front is flat then the system will be translationally invariant with respect to its tangent and we may transform into a reference frame where the fluid velocity is normal to the shock front without loss of generality. The speed of the fluid in this frame is related to the speed of the fluid in the rest frame by

$$u_{\perp} = u_0 \cos \theta, \tag{3.60}$$

where θ is the angle between the shock normal and the fluid velocity in the lab frame and u_0 is the speed of the fluid in the lab frame. If the asymptotic strength of the shock (far from whatever obstruction led to its formation) is weak, then it will propagate through the fluid *at the corresponding wave speed*. That is, in the case of a

weak shock, the fast magnetosonic shock propagates at the fast magnetosonic speed. Therefore, a stationary shock is found by setting the flow speed into the shock equal to the magnetosonic speed at the same angle, as given by Equation (3.59). And *the angle of the shock front with respect to the fluid velocity* is the angle that makes these two speeds equal; the opening half-angle of the shock cone is the complement of this angle.

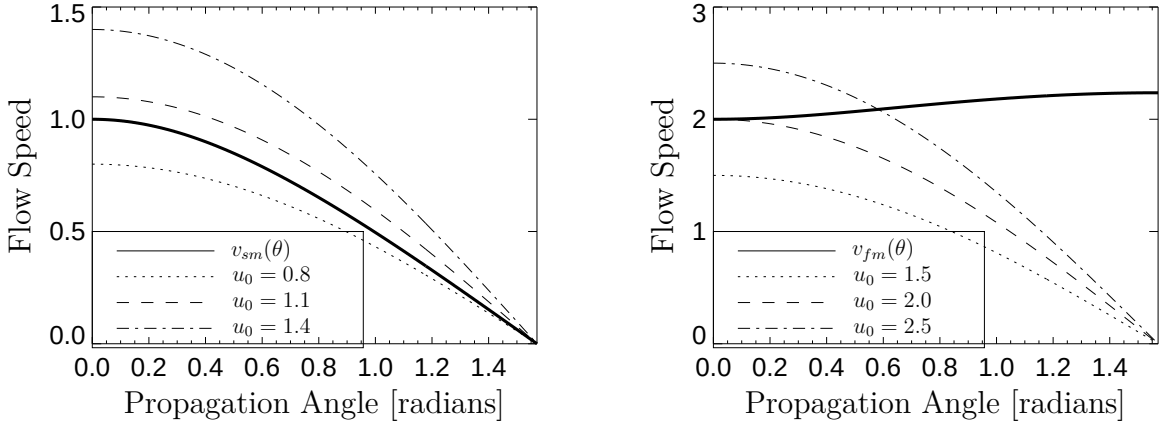


Figure 3.9. The angular dependence of wave propagation speed is plotted as a function of the angle between an assumed shock front and the (vertical) magnetic field. The slow magnetosonic mode is shown on the left while the fast mode is shown on the right. The normal component of the fluid velocity is also shown for three characteristic flow speeds in each case. Viable shock cone angles correspond to intersections of these curves.

Let us consider a few limiting cases. Suppose that the Alfvén speed is very small compared to the sound speed and β is very large. In this case the slow mode collapses and the speed of fast mode reduces to the isotropic sound speed. The shock front

angle is then given by

$$C_s = u_0 \cos \theta \implies \cos^{-1} \theta = \frac{1}{M_s} = \sin^{-1} \phi, \quad (3.61)$$

which is identical to the expected hydrodynamic limit, with the shock cone half-angle, $\phi = \pi/2 - \theta$.

Applying the same analysis with the magnetosonic wave speed demonstrates why it is that slow magnetosonic shocks do not form shock cones. The lower root of Equation (3.59) falls off as approximately $\cos \theta$, as does the normal component of the flow velocity. There is no guarantee that an angle exists at which these can be made to match and, in general, there is no solution. This can be seen in the left panel of Figure 3.9. The wave speed is plotted in black for $V_A = 2C_s = 2$. The normal component of the velocity is plotted for $u_0 = \{0.8, 1.1, 1.4\}C_s$. Only in the limit that $u_0 \rightarrow C_s$ do these curves intersect, and there the angle of intersection is ambiguous.

This helps, in part, to explain why the collimated shocks of Scott et al. (2013) are predicted to be infinitely wide at $\overline{M}_s = 1$; it is the only speed that supports slow-mode shock cones and the predicted cone angle at that speed is highly unstable to variations in $\overline{M}_s$. By contrast, the fast magnetosonic wave speed increases monotonically with propagation angle relative to the field. It follows that if the ambient flow speed is larger than the speed of parallel propagation (equal to the larger of C_s and V_A), then a solution for the shock cone angle is guaranteed.

We now return to the case where both $\overline{M}_s$ and $\overline{M}_A$ are greater than one. This condition is implied any time $M_s > 1$ and $\beta > 2$; however, it is possible for a low- β system to be simultaneously supersonic and super-Alfvénic, provided that $M_s^2 > 2M_A^2/\beta$. In such cases the formation of a fast magnetosonic shock is natural any time the fluid density forms strong gradients, as in the case of supersonic flow past a circular boundary where the stagnation point at the lower apex of the intrusion necessarily collects mass until a pressure gradient forms that is sufficiently strong to alter the fluid velocity in that region.

The fast magnetosonic Mach cone that forms in this case can be predicted using the same line of reasoning as described above; the angle-dependent wave speed must be matched to the projected velocity. In the case of fast magnetosonic waves the angular wave speed increases monotonically with the angle of propagation while the projected fluid velocity decreases monotonically. A solution for the Mach angle is guaranteed, therefore, anytime the upstream velocity is greater the parallel propagation speed of fast magnetosonic waves. This is evidenced in the right panel of Figure 3.9, which depicts the angular dependence of the fast magnetosonic speed corresponding to the same conditions as in the left panel. The minimum value for the fast magnetosonic speed is 2 in this case, and a shock cone solution is guaranteed any time the flow speed exceeds this value.

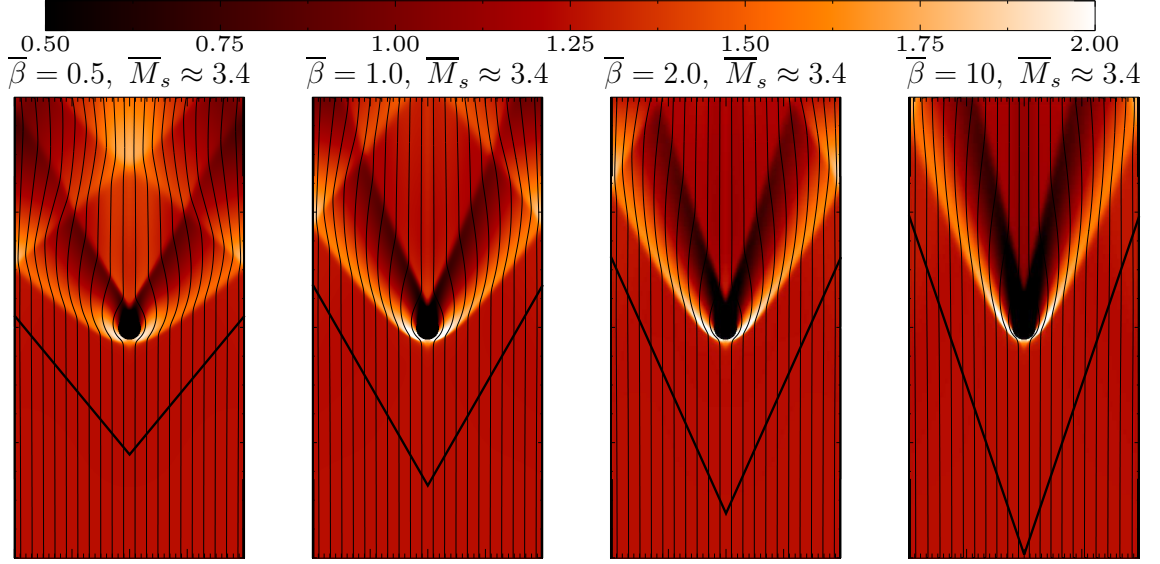


Figure 3.10. Ballistic shocks are depicted for a strongly supersonic flow $\overline{M}_s = 3.38$ with $\overline{\beta}$ ranging from 0.5 to 10. The associated Alfvén Mach number therefore ranges from $\overline{M}_A = 1.7$ on the far left to $\overline{M}_A = 7.6$ on the far right. The predicted Mach cone shape is depicted by the bold black wedge.

The formation of fast magnetosonic shock cones can be seen in Figure 3.10, which shows four separate instances, each with a different magnetic field strength. In each of these cases the acoustic Mach number is set to $\overline{M}_s = 3.38$ while the Alfvén Mach number ranges from $\overline{M}_A = 1.7$ in the case of $\overline{\beta} = 0.5$ up to $\overline{M}_A = 7.6$ in the case of $\overline{\beta} = 10$. For each of these the Mach cone angle was calculated as previously described. Note that the correspondence in the high- β regime is quite good while for decreasing β values the predicted cone shape deviates from the simulation result. We attribute this discrepancy to the weak shock approximation, which does not seem to be valid within the simulation domain for the low- β scenario. However, upon careful

inspection, the demonstrated shock front does appear to be curved, even toward the boundaries of the simulation, and we anticipate that for a larger simulation box the predicted cone angle may become increasingly accurate.

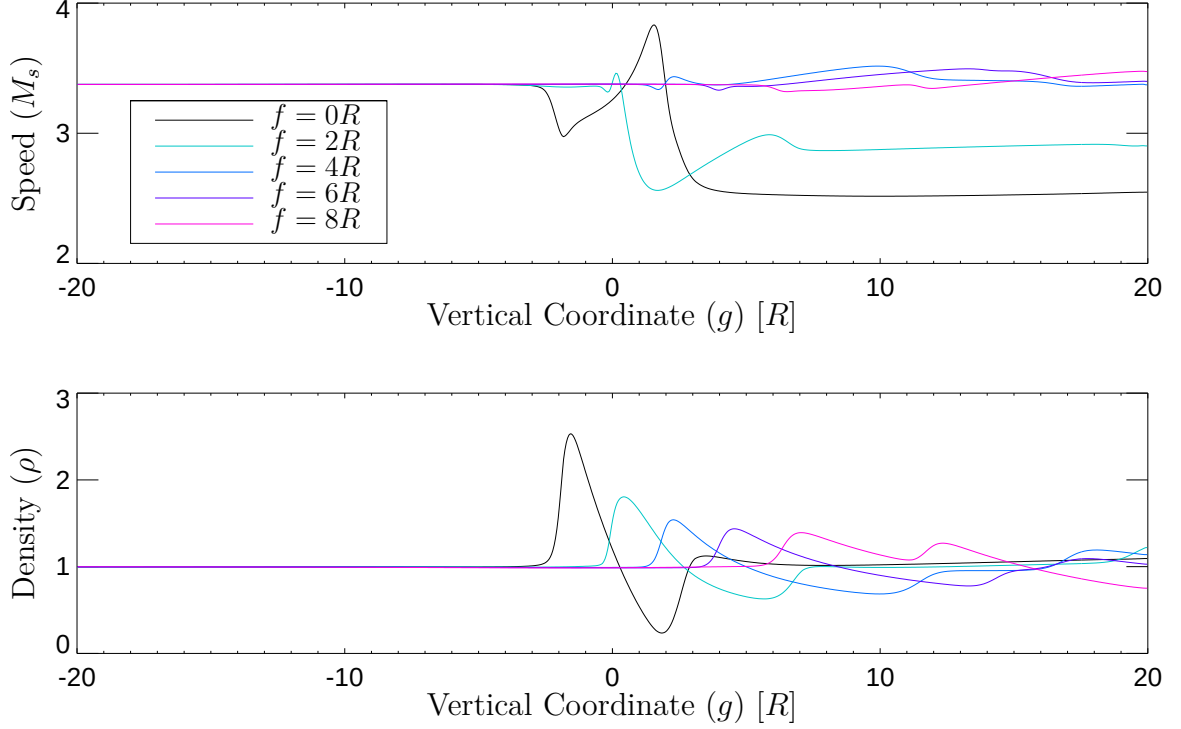


Figure 3.11. The vertical profile of acoustic Mach number and density are shown for a ballistic shock as a function of transverse distance from the intrusion. This run was performed with $\overline{M}_s = 3.38$ and $\overline{\beta} = 1.0$ giving an Alfvén Mach number of $\overline{M}_A = 2.4$.

Comparing the vertical profile of fast shock cones to slow confined shocks it is immediately apparent that while the confined shock amplitude falls off abruptly at some critical lateral distance from the intrusion, the amplitude of the fast magnetosonic shock decays much more slowly. This is evidenced in Figure 3.11, which depicts the

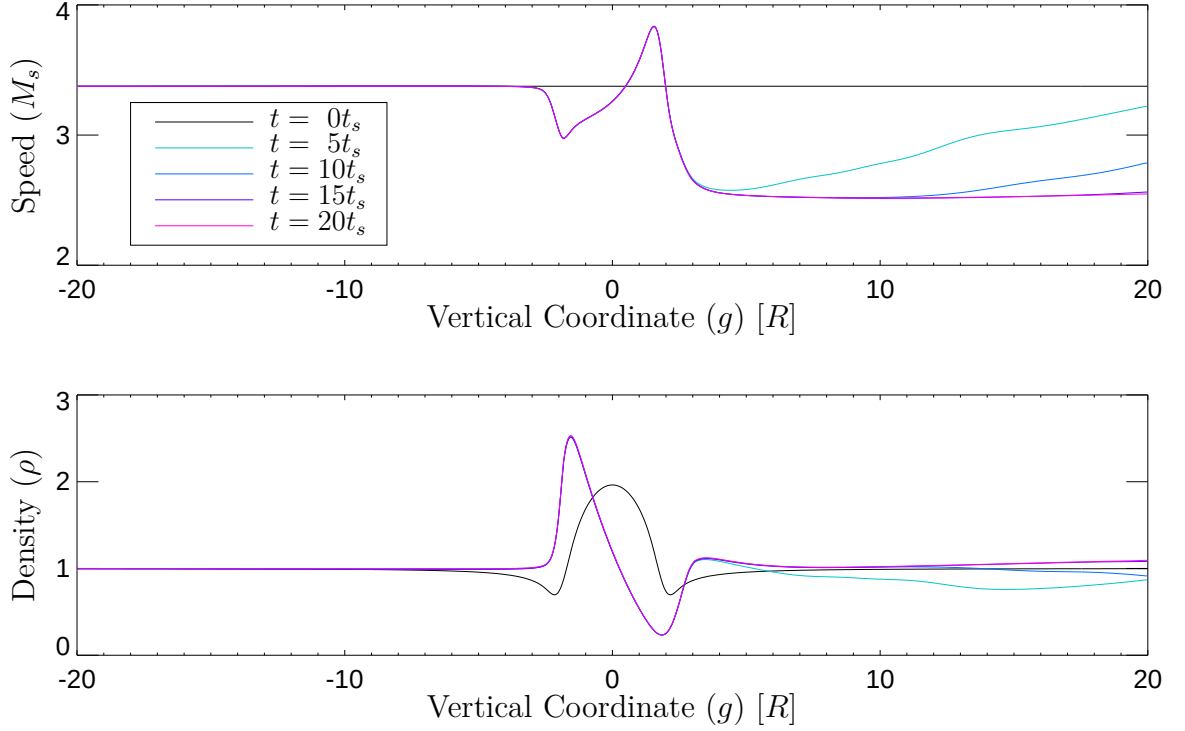


Figure 3.12. The vertical profile of plasma velocity and density are indicated for a ballistic shock as a function of time. As in Figure 3.11, $\overline{M}_s = 3.38$, $\overline{\beta} = 1.0$, and $\overline{M}_A = 2.4$.

acoustic Mach number and density along vertical cuts that are successively farther from the intrusion. Just as in the case of confined shocks there is a leading and a trailing shock. However, unlike the slow confined shocks, the total density change from upstream of the leading shock to downstream of the trailing shock is very small. And from the bottom panel of Figure 3.11 it is apparent that the strength of the shocks falls off geometrically (or perhaps logarithmically in this 2D simulation).

The temporal evolution of ballistic shock cones is also very different from the slow confined shocks. As seen in Figure 3.12, the initial shock formation happens very

quickly, after which the only evolution is the slow decay of the transient downstream solution. The final, steady state of the system is characterized by a net density change across both shocks of zero and a net decrease in the fluid velocity inside the shock cone, despite the intermediate increase while the fluid velocity is transonic between the two shocks. Critically, the position of the leading and trailing shocks relative to the intrusion is steady. These are standoff shocks, that form at a given position and are stationary thereafter.

3.4. Parameter Survey

In order to refine our understanding of the transitions between the various regimes we conducted a parameter survey of all possible combinations of values in Equations (3.41) and (3.42). A summary of these simulations is depicted in Figure 3.13, which shows the results of a subset of these runs, each taken at a time of $t = 20 t_s$, well after the steady solution has developed. In the figure the acoustic Mach number increases from left to right while the plasma β increases downward. The Alfvén Mach number increases from upper left to lower right, with the diagonal entries from lower left to upper right corresponding to $\overline{M}_A \sim 1$.

The behavior in the extreme limits of $\overline{M}_s = \{0.29, 3.38\}$ and $\overline{\beta} = \{0, \infty\}$ differs little from their nearest adjacent runs and has been omitted from Figure 3.13. The transition from subsonic nozzle flow, through magnetically confined slow shocks, to

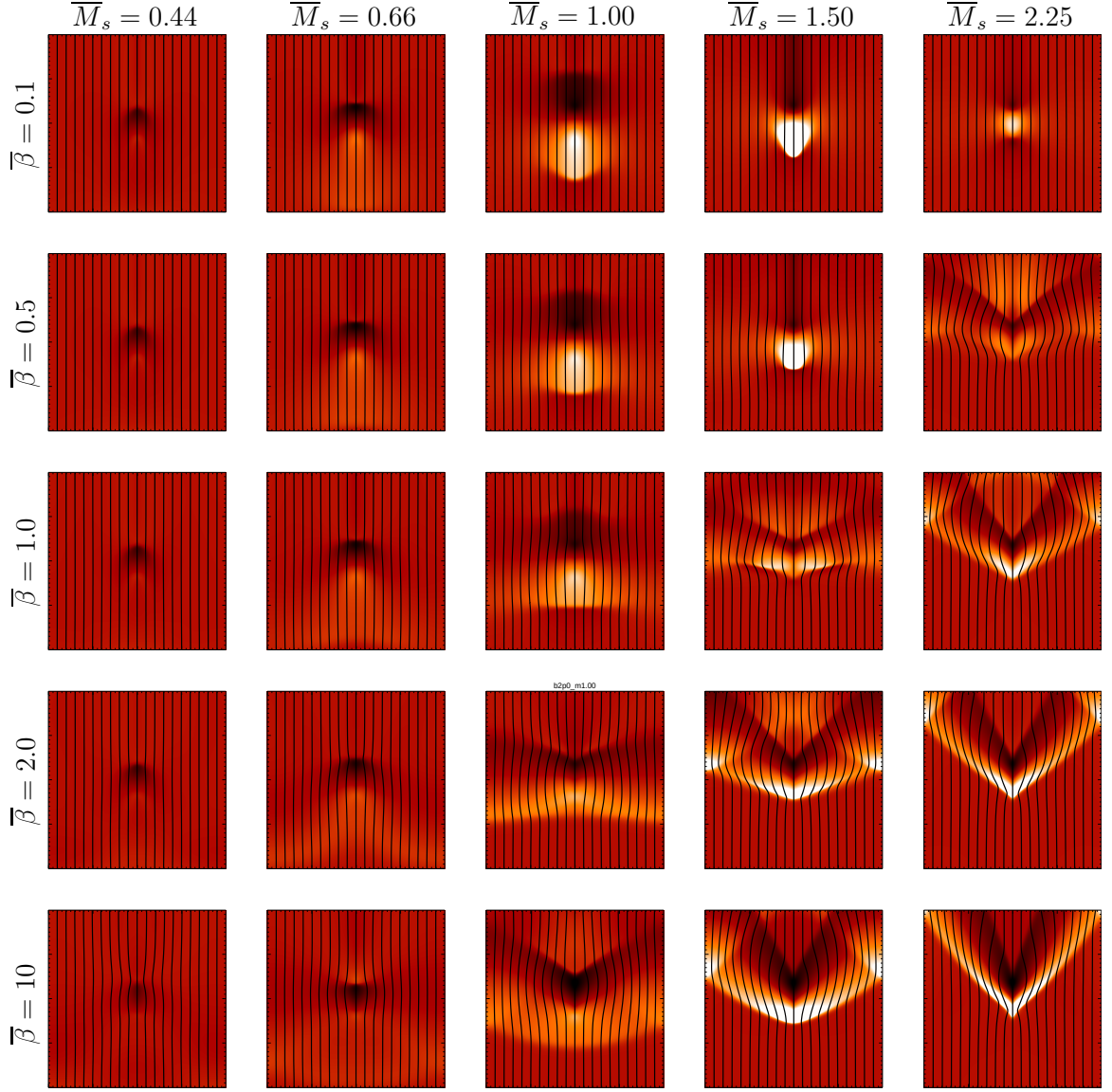


Figure 3.13. A subset of the various simulation runs is shown with density in red-yellow color scale and overlaid field lines. These are depicted in the native simulation coordinates, which allow for the variations of the magnetic field to be more easily distinguished. $\overline{\beta}$ ranges from 0.1 to 10 while $\overline{M}_s$ ranges from 0.44 to 2.25. Each frame is taken at the end of the simulation run at a time of $t = 20 t_s$.

supersonic nozzle flow is in evidence in the top row, which corresponds to $\bar{\beta} = 0.1$. In the second row, where ($\bar{\beta} = 0.5$), the behavior is almost identical until we reach $\bar{M}_s \approx 2.25$, when the system suddenly becomes ballistic. Checking the value of Alfvén Mach number in this case we find that $\bar{M}_A = 1.1$, demonstrating that the transition out of the magnetically confined slow shock regime does, in fact, depend critically on $\bar{M}_A$, and can lead to either supersonic nozzle flow or ballistic fast mode shocks.

Perhaps more surprising than the existence of a low- β , fluid dominated regime, is the resilience of magnetically confined slow shocks to increased values of β . This behavior, which we developed under the assumption of an arbitrarily rigid magnetic field, is clearly exhibited in the $\bar{\beta} = 0.5, \bar{M}_s = 1.5$ case, and remains in evidence for $\bar{\beta} = 1.0, \bar{M}_s = 1.0$ and even up to $\bar{\beta} = 2.0, \bar{M}_s = 0.66$, though careful inspection is required to reveal the behavior for subsonic flows, which are only weakly shocked. We now understand the resilience of this solution to be related to the inaccessibility of the fast magnetosonic mode and inability of slow magnetosonic shocks to form stationary solutions.

The complete findings of our parameter survey are given in Table 3.1, with $\bar{M}_s$ increasing to the right and $\bar{\beta}$ increasing downward, exactly as in Figure 3.13. In cases where the behavior was not clearly indicative of one regime over another we have listed both. Seemingly, the transition between each regime is well described by the values of $\bar{\beta}$, $\bar{M}_s$ and $\bar{M}_A$.

$\bar{\beta}$	$\bar{M}_s$	8/27	4/9	2/3	1	3/2	9/4	27/8
0.0	Regime $\bar{M}_A$	L^- 0.0	S_c, L^- 0.0	S_c 0.0	S_c 0.0	S_c 0.0	L^+ 0.0	L^+ 0.0
0.1	Regime $\bar{M}_A$	L^- 0.1	S_c, L^- 0.1	S_c 0.1	S_c 0.2	S_c 0.3	L^+ 0.5	L^+ 0.8
0.5	Regime $\bar{M}_A$	L^- 0.1	S_c, L^- 0.2	S_c 0.3	S_c 0.5	S_c 0.8	S_b 1.1	S_b 1.7
1.0	Regime $\bar{M}_A$	L^- 0.2	S_c, L^- 0.3	S_c 0.5	S_c, S_b 0.7	S_b 1.1	S_b 1.6	S_b 2.4
2.0	Regime $\bar{M}_A$	L^- 0.3	L^- 0.4	S_c, L^- 0.7	S_b 1.0	S_b 1.5	S_b 2.2	S_b 3.4
10	Regime $\bar{M}_A$	L^- 0.7	L^- 1.0	L^- 1.5	S_b 2.2	S_b 3.4	S_b 5.0	S_b 7.5
∞	Regime $\bar{M}_A$	L^- ∞	L^- ∞	L^- ∞	S_b ∞	S_b ∞	S_b ∞	S_b ∞

Table 3.1. The various behaviors are listed for the parameter space of $\bar{\beta}$ (represented in successive rows) and $\bar{M}_s$ (represented in adjacent columns). For each, the asymptotic Alfvén Mach number, $\bar{M}_A$, is given along with an indication of which behavior was in evidence. If the behavior was not clearly aligned with one particular regime then two were listed. These are indicated in short-hand notation as follows: L^- : sub-sonic nozzle flow; L^+ : supersonic nozzle flow; S_c : magnetically confined slow shocks; S_b : ballistic fast shocks.

In order to better lead the eye, we have constructed a diagram of the parameter space that is relevant to the model. In Figure 3.14 the various behaviors are shown as regions of the logarithmic $\bar{\beta} - \bar{M}_s^2$ energetic phase-space. The Alfvén Mach number is indicated by the three diagonal lines, with $\bar{M}_A \gg 1$ corresponding to the upper right and $\bar{M}_A \ll 1$ corresponding to the lower left. The region to the left of $\log_{10} \bar{\beta} = 0$ and below the $\bar{M}_A = 1$ diagonal is magnetically dominated and leads to the three magnetically aligned flow regimes, with the supersonic nozzle flow corresponding to the only viable supersonic, laminar solution. Ballistic fast-mode shocks form in the

region above $\log_{10} \overline{M}_s^2 = 0$ and to the upper right of $\overline{M}_A = 1$. These become increasingly similar to hydrodynamic shocks in the limit of large- β while the more complex magnetohydrodynamic limit is seen for $\overline{\beta} \leq 1 \leq \overline{M}_A$.

3.5. Discussion

Through the construction of a two-dimensional model with potential-field-aligned conformal coordinates and the variation of plasma parameters and assumed flow conditions – recall that $\overline{\beta}$, $\overline{M}_s$, and $\overline{M}_A$ are the plasma- β , acoustic Mach number and Alfvén Mach number in the asymptotic region, respectively – we have shown that the analytical description of shocks in the vicinity of a retracting flux tube, which was developed under the assumption of zero- β in Scott et al. (2013), can be recovered by numerical simulations for a much broader class of initial conditions; the effect is visible for $\overline{\beta}$ equal to unity or greater, provided that the flow is not significantly supersonic. Furthermore, we have shown that the supersonic, unshocked solution predicted by the nozzle treatment is valid for cases where the flow speed exceeds the maximum value for magnetically confined slow shocks but remains small compared to the Alfvén speed. The behavior in this limit is now understood to be intimately related to the fast magnetosonic Mach number: if the flow speed is less than either the sound speed or the Alfvén speed then the fast magnetosonic wave-mode is inaccessible and the only shocks that can form are slow shocks. These cannot form shock

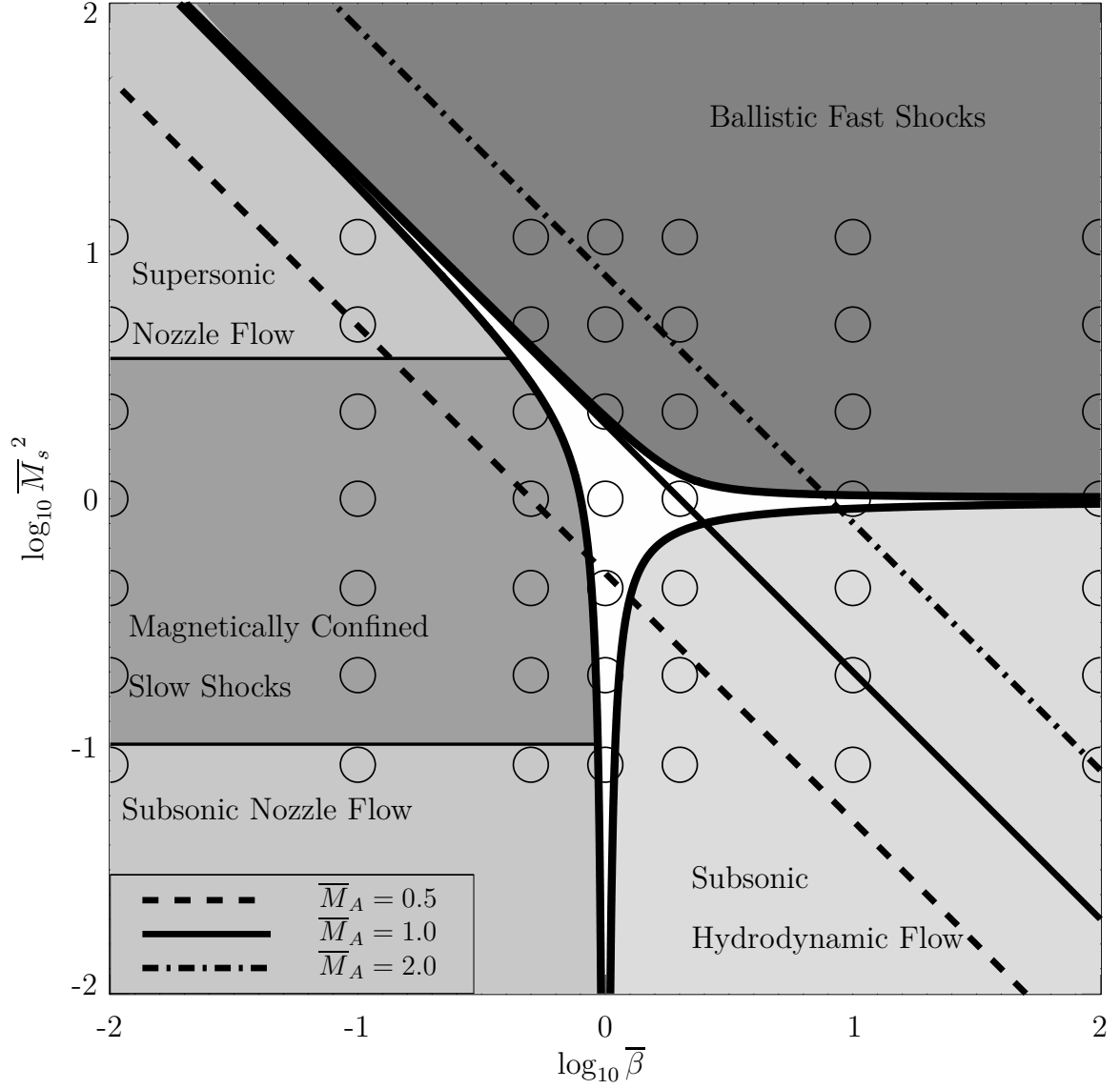


Figure 3.14. The various behavioral regimes are indicated in the $\overline{M}_s^2 - \beta$ parameter space, which is shown in log-scale. The diagonal lines indicate contours of the Alfvén Mach number while the shaded grey areas each indicate a distinct behavioral mode. Individual simulation results are indicated by the circles, which have been used to confirm the behavior in each instance.

cones and can therefore only manifest as non-stationary shock columns, which are, in turn, inaccessible for flow speeds greatly in excess of the sound speed.

We also identified the formation of fast magnetosonic Mach cones in the case of $\overline{M}_s$ and $\overline{M}_A$ greater than unity and developed a description of the asymptotic shock angle, which is qualitatively in agreement with the model. These ballistic shocks, as we call them, were found to exist even in the limit of low- β , which is normally considered magnetically dominated. However, we have observed that while the pressure is small in this limit, the ram pressure is still greater than the magnetic energy density, so the system is actually fluid dominated. Additional work is required to verify the accuracy of this description and we intend to perform additional simulation runs with a larger numerical domain so as to observe the behavior of this solution in the weak shock limit.

These results are broadly applicable to coronal models that depend on the presence of moving elements of magnetic flux, such as have been proposed by Savage et al. (2012) and Longcope et al. (2009), to name a few. The propagation of such an element through a magnetized fluid must impart energy to the fluid and vice versa. The framework that we have developed allows for a quantitative understanding of how the fluid should respond, given certain criteria for the flux tube. And, with a knowledge of the fluid's response, it should be possible to formulate important feedback parameters, such as the drag force that the fluid might exert on the intrusion.

Moreover, these results may help to improve our understanding of the relationship between small-scale, well resolved MHD processes, and bulk emission observations with poorly constrained filling factors; unresolved ballistic elements could create microstructures that alter the local energy budget and a correspondence map between $\overline{\beta}$ and the spatial average of β could be found by integrating over simulation domains such as ours.

Regarding the supra-arcade region, we find that the process of “persistaltic pumping” and the associated formation of collimated shocks and a trailing region of rarefied plasma, as described in Scott et al. (2013), remains a viable mechanism for the formation of SADs, provided that 1) the ambient value of β is not greatly in excess of one, 2) the acoustic Mach number of the retracting flux tube is of order one, and 3) the Alfvén Mach number of the retracting flux tube is less than one. This result provides a mechanism for testing the interpretation of Savage & McKenzie against other models of SADs, such as Guo et al. (2014) and Cécere et al. (2012), as it provides strict criteria for the model mechanism to be applicable; if $\overline{\beta}$ and $\overline{M}_s$ can be measured independently then the presence or absence of collimated slow-mode shocks can be predicted with precision and thereby serve as a proxy for the detection of retracting loops, which may or may not be directly visible. Or, by the same logic, if descending loops are directly detected then the adherence to one of the behavioral modes that

we have described can serve as a constraint on estimates of the ambient plasma parameters; if a shock cone is seen, for example, then it can be assumed that both $\overline{M}_s$ and $\overline{M}_A$ are in excess of one.

Certain limitations to the model exist and should be mentioned. The finite lateral size of the domain creates a rigid wall that, in principle, can deposit energy into the magnetic field and cannot be accounted for. This could be responsible for creating an artificially robust magnetic field and might skew our findings regarding collimated shocks for values of $\overline{\beta}$ near unity. In order to explore this possibility, we will need to repeat the parameter survey on a much larger numerical domain. Additionally, the intrusion itself is and must be rigid in this model since it is an intrinsic aspect of the coordinate system. Some progress can be made in this area by generalizing the model to intrusions of elliptical cross section, and early efforts in that direction have been explored, but in the end only a dynamical model that includes both the retracting flux tube and the surrounding fluid in a single, volume filling domain will be able to resolve this limitation.

In future work we intend to refine this model through the inclusion of adiabatic and viscous heating, thermal conduction, and a more careful treatment of the viscous stress, and the current framework has been developed with these advancements in mind. We also hope to explore scenarios with non-uniform ambient densities and temperatures – most likely through the inclusion of gravitational stratification – as

well as a drag-dependent acceleration term, to mimic the deceleration of the intrusion as a dynamic response to the fluid.

3.6. Acknowledgements

This work would not have been possible without funding from the National Aeronautics and Space Administration under contract NNM07AB07C as well as the Living with a Star grant NNX14AD43G.

4. INFERRING THE MAGNETOHYDRODYNAMIC STRUCTURE OF SOLAR FLARE SUPRA-ARCADE PLASMAS FROM A DATA ASSIMILATED FIELD TRANSPORT MODEL

Contribution of Authors and Co-Authors

Manuscript in Chapter 4

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Co-Author: Dr. David E. McKenzie

Contributions: Helped to conceive of data assimilation technique. Provided feedback of analysis and comments on drafts of the manuscript.

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Contributions: Helped to conceive of data assimilation technique. Provided feedback of analysis and comments on drafts of the manuscript.

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ABSTRACT

Supra-arcade fans are highly dynamic structures that form in the region above post-reconnection flare arcades. In these features the plasma density and temperature evolve on the scale of a few seconds, despite the much slower dynamics of the underlying arcade. Further, the motion of supra-arcade plasma plumes appears to be inconsistent with the low-beta conditions that are often assumed to exist in the solar corona. In order to understand the nature of these highly debated structures it is, therefore, important to investigate the interplay of the magnetic field with the plasma. Here we present a technique for inferring the underlying magnetohydrodynamic processes that might lead to the types of motions seen in supra-arcade structures. Taking as a case study the 2011 October 22 event, we begin with extreme ultraviolet observations and develop a time dependent velocity field that is consistent with both continuity and local correlation tracking. We then assimilate this velocity field into a simplified magnetohydrodynamic simulation, which deals simultaneously with regions of high and low signal to noise ratio, thereby allowing the magnetic field to evolve self-consistently with the fluid. Ultimately, we extract the missing contributions from the momentum equation in order to estimate the relative strength of the various forcing terms. In this way we are able to make estimates of the plasma beta as well as predicting the spectral character and total power of Alfvén waves radiated from the supra-arcade region.

4.1. Introduction

Supra-arcade fan structures have been observed since the beginning of x-ray and EUV imaging of the solar corona. Often referred to as current sheets, these are typically found above the dominant polarity inversion line of solar active regions and are anchored at the apex of an arcade of newly formed flare loops. The fan is, therefore, spatially coincident with the assumed current sheet that defines a rotational discontinuity in the open magnetic field above the arcade. We are careful, however, to distinguish the emissive structure, referred to as a Current Sheet / Thermal Halo

or CSTH by McKenzie (2013), from the actual supra-arcade current sheet, which may be significantly thinner (Seaton & Forbes, 2009) and need not, necessarily, contribute significantly to the supra-arcade emission.

The energy balance in the supra-arcade region has been a topic of interest since the first observations of Supra-Arcade Downflows (SADs) by McKenzie & Hudson (1999). Previously, many authors had suggested that the magnetically dominated forcing typified in active region cores held true for supra-arcade structures as well. However, the dynamics of SADs show evidence of Rayleigh-Taylor type instabilities (Guo et al., 2014), while the oscillations along the edges of supra-arcade density lanes that were considered by Verwichte et al. (2005) may be indicative of Kelvin-Helmholtz wave excitation, both of which suggest a non-trivial contribution from the plasma pressure. Indeed, McKenzie (2013) estimated that the plasma β , which is the ratio of thermal to magnetic energy densities, should be of order unity or greater in the 2011 October 22 event, with significant variation within the structure itself.

It is the aim of this paper to improve our understanding of the dynamics exhibited by supra-arcade fan structures. As a case study, we considered the supra-arcade structure that formed on the NorthWest (NW) limb on 2011 October 22 – the same event considered by Savage et al. (2012), McKenzie (2013), and Hanneman & Reeves (2014). Due to the predominantly north-south alignment of the polarity inversion line and its position on the limb, the line of sight is taken to be normal to the plasma

sheet. This vantage point allows us to estimate the plasma motion and magnetic evolution in the plane of the sky.

The organization of this paper is as follows. In §4.2 we describe our methods for estimating the plasma density and the associated velocity field. §4.3 describes our method for inferring the evolution of the supra-arcade magnetic field. Then, in §4.4 we compare the dynamics of the fluid and the magnetic field in order to estimate the radiated wave spectra, energy balance and, ultimately, plasma pressure. Our results are summarized in §4.5.

4.2. Observed Plasma Evolution

The 2011 October 22 event occurred on the northwest limb and was observed by the Atmospheric Imaging Assembly (AIA) onboard the Solar Dynamics Observatory (SDO) (Lemen et al., 2012) at 12 second cadence and 0.6 arcsec (435 Km) per pixel resolution in each of the seven Extreme Ultra Violet (EUV) channels ranging from 94 Å to 335 Å. We retrieved the data from all seven EUV passbands so that we might disambiguate the plasma motion from possible temperature effects. The data were retrieved using the `vso_search` and `vso_get` routines (Hill et al., 2009), which are available through *SolarSoft* (Freeland & Handy, 1998). Each image was cropped using the `read_sdo` routine and then prepped and normalized via `aia_prep`. The

resulting image cube contains a little over three hours of uninterrupted high cadence, multi-spectral data.

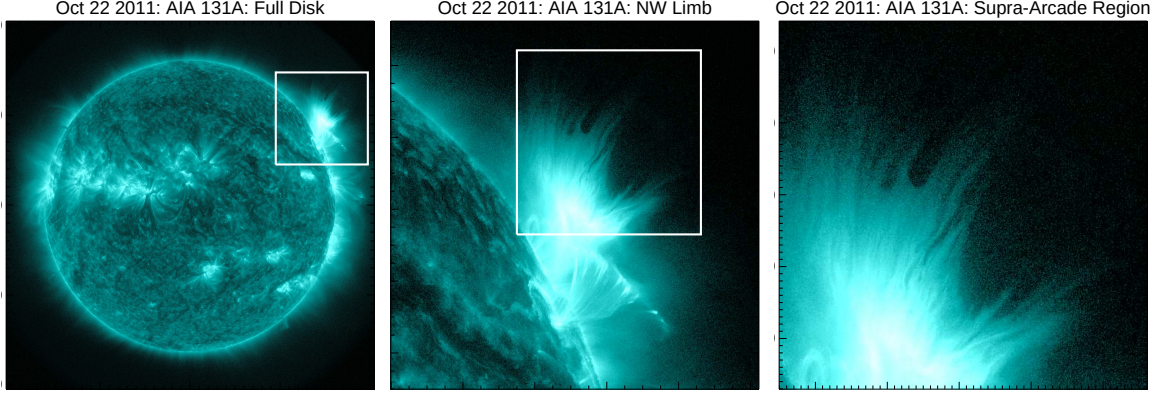


Figure 4.1. A representative frame from the 2011 October 22 event on the Northwest limb. The event is shown with three different fields of view in the AIA 131 Å channel, which is dominated by Fe XXI at approximately $10^7 K$.

For this event, we assumed that the supra-arcade fan, which exhibits strong emission in the 131 Å passband (Figure 4.1), consists of a halo of plasma surrounding a current sheet, which supports a 180 degree rotational discontinuity within an otherwise smooth, predominantly radial magnetic field. The column depth is assumed to be uniform and small compared to the length scale for variation of the magnetic field along the line of sight, and we are therefore able to treat the problem two dimensionally, with the direction along the line of sight understood to be ignorable and the bulk velocity and magnetic fields lying purely in the plane of the sky.

4.2.1. DEM Inversion

We begin our analysis with the Differential Emission Measure (DEM), which we determined using the Fast Iterative Regularized Differential Emission Measure (FIR-DEM) inversion code developed and described by Plowman et al. (2013). Omitting the 304 Å channel (private communication, Joe Plowman, 2014), the inversion was performed with the six remaining channels for a reduced field of view (512 x 512 pix) at full resolution, with a cadence of 12 seconds spanning from 11:35 UT to 15:00 UT.

Inspection of the spatially averaged DEM in the supra-arcade fan confirmed the results of Hanneman & Reeves (2014), in which the entire fan structure exhibits relatively stable emission below $T \sim 10^{6.5}K$, and a highly dynamic contribution from plasma with temperatures of $10^{6.5}K$ and above. We interpret this result to mean that the lower temperature contribution is due to a uniform, low density, relatively cool plasma that extends significantly along the line of sight (i.e. the foreground and background corona), while the high temperature contribution is the result of plasma that is confined to the supra-arcade plasma sheet, with a much higher density and a relatively short column depth.

Anticipating that the dynamics in the 131 Å channel would be associated with the supra-arcade plasma sheet, which we wanted to isolate from the rest of the diffuse corona, we considered only the high temperature contribution to the DEM. In order to isolate this higher density plasma we multiplied the complete DEM, $\epsilon(T)$, by a

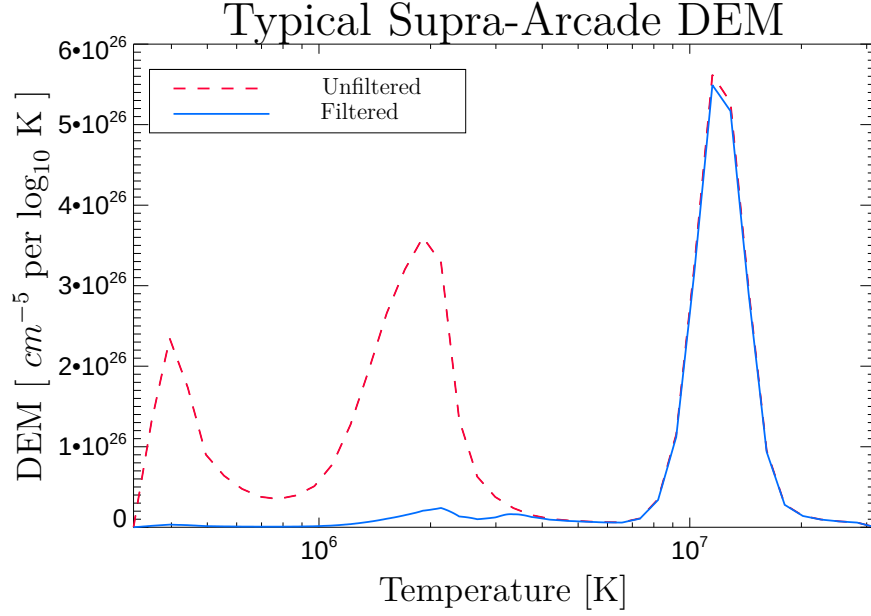


Figure 4.2. A representative example of the spatially averaged, filtered and unfiltered DEMs are depicted in solid blue and dashed red, respectively. This example is taken during the height of supra-arcade activity, at approximately 13:00 UT. Note the suppression of emission measure below $\log_{10} T \approx 6.5$. The filtered DEM retains only the high temperature plasma, which is assumed to be the primary contributor to the supra-arcade fan emission.

high temperature filter,

$$f(T) = \frac{1}{2} + \tan^{-1} 8\pi(\log_{10} T - 6.5), \quad (4.1)$$

which increases smoothly from zero to one over an interval of about 1 MK, centered on $\log_{10} T = 6.5$. We then constructed the total filtered column emission measure, ϵ_f , by integrating the filtered DEM across all temperatures;

$$\epsilon_f = \int dT \epsilon(T) f(T). \quad (4.2)$$

This integrated emission is taken as the total emission measure associated with the supra-arcade region, with minimal contribution from the diffuse corona. An example

of the filtered and unfiltered DEM are depicted in Figure 4.2. Figure 4.3 depicts the total column emission measure with and without the high-temperature filter.

Ultimately we are interested in the plasma density; however, for the time being it suffices to focus on the electron number density, n_e , which is assumed to be related to the column emission measure through

$$\epsilon_f = n_e^2 d. \quad (4.3)$$

Here we have used d to indicate the (unknown) emission column depth of the supra-arcade fan. Without specifying d , we can still track the plasma evolution through the *column depth normalized electron density*,

$$\tilde{n}_e = \sqrt{\epsilon_f} = n_e \sqrt{d}, \quad (4.4)$$

from which the actual plasma density can later be extracted by assigning a value to d and converting from electron number to ion mass.

4.2.2. Continuity

With (a proxy for) the density in hand, we may investigate the plasma velocity, $\mathbf{u}$, through the continuity equation, which can be stated in terms of $\tilde{n}_e$ as

$$\partial_t \tilde{n}_e + \nabla \cdot (\tilde{n}_e \mathbf{u}) = 0. \quad (4.5)$$

Here we have assumed that the column depth, d , is spatially and temporally invariant. The expression above relates the change in density to the divergence of $\tilde{n}_e \mathbf{u}$, which,

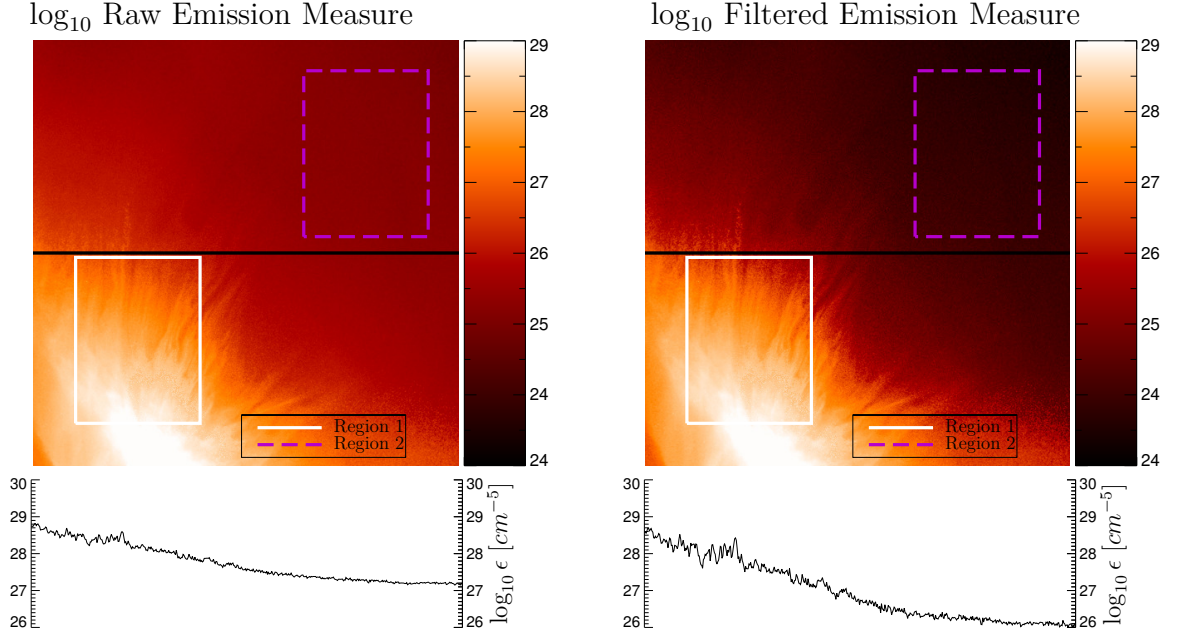


Figure 4.3. The total emission measure (top left panel) and total filtered emission measure (top right panel) exhibit similar features but differ by the diffuse background emission in high altitudes above the limb. The lower two panels show emission measure plotted along the horizontal black cuts above. By removing the ambient contribution, the contrast is enhanced in the supra-arcade region and the resultant density is expected to better match the assumption of a simple slab structure.

like any two dimensional (2D) vector field, can be expressed as the sum of irrotational and solenoidal components. Taking

$$\tilde{n}_e \mathbf{u} = \nabla f + \hat{\mathbf{z}} \times \nabla g, \quad (4.6)$$

the irrotational component must satisfy

$$\nabla^2 f = -\partial_t \tilde{n}_e \quad (4.7)$$

from Equation (4.5). The fluid velocity is then given by

$$\mathbf{u} = \frac{1}{\tilde{n}_e} (\nabla f + \hat{\mathbf{z}} \times \nabla g) \quad (4.8)$$

where ∇g is, as yet, unspecified.

Since $\hat{\mathbf{z}} \times \nabla g$, is unconstrained by continuity, consideration of the electron density is only sufficient to partially specify the velocity; in order to fully determine the velocity we must invoke an additional constraint. In an effort to incorporate all available information about the system, we turn to Fourier Local Correlation Tracking (FLCT), which provides a means for quickly estimating the apparent velocity of intensity features (Fisher & Welsch, 2008). As with most optical flow solutions FLCT assumes an advective evolution of the form

$$\partial_t I + \mathbf{v} \cdot \nabla I = 0, \quad (4.9)$$

and the associated velocity field ($\mathbf{v}$) is found through minimization of the residual in Equation (4.9), where I is the image intensity and is equivalent to ρ in our application. This assumption departs from the behavior of the continuity equation through exclusion of the term $\rho \nabla \cdot \mathbf{u}$, which allows for the local depletion or enhancement of density features with no displacement of the overall structure. As such, the FLCT derived velocity is non-conservative and therefore, while it does excel at characterizing the apparent motion of optical features (i.e. gradients), it is fundamentally inconsistent with the continuity equation.

Despite this limitation, the FLCT derived velocity fields have the benefit of exhibiting much of the behavior that is apparent to the eye. We therefore choose the solenoidal contribution to Equation (4.8) such that the resultant velocity is as much

like the FLCT derived optical flow as possible, while still satisfying the continuity condition, as constrained through Equation (4.7). This is accomplished by developing a functional related to the variance of the density-weighted difference between the two velocities, integrated over the Region Of Interest (ROI), i.e.

$$W(\tilde{n}_e; \mathbf{u}; \mathbf{v}) = \int_{ROI} \tilde{n}_e (\mathbf{u} - \mathbf{v})^2 da, \quad (4.10)$$

where da ($= dx dy$) is the differential area element in the image plane. By rewriting $\mathbf{u}$ in terms of its potentials we find an equivalent form for the aforementioned functional:

$$W(\tilde{n}_e; f, g; \mathbf{v}) = \int_{ROI} \frac{1}{\tilde{n}_e} (\nabla f + \hat{\mathbf{z}} \times \nabla g - \tilde{n}_e \mathbf{v})^2 da. \quad (4.11)$$

Demanding that W be stationary under variations of g , with $\tilde{n}_e$, f and $\mathbf{v}$ held fixed, we can then choose g in order to minimize the value of this functional in each frame, with W guaranteed to be a minimum since no maximum value for the variance can exist.¹ The Euler-Lagrange equation for g is then:

$$\nabla^2 g = \hat{\mathbf{z}} \cdot \frac{1}{\tilde{n}_e} \nabla \tilde{n}_e \times \nabla f + \frac{1}{\tilde{n}_e} \nabla \tilde{n}_e \cdot \nabla g + \hat{\mathbf{z}} \cdot (\tilde{n}_e \nabla \times \mathbf{v}). \quad (4.12)$$

Inspection reveals that the above expression is identical to the requirement that $\nabla \times (\mathbf{u} - \mathbf{v}) = 0$. That is, the velocity field that differs the least from the FLCT derived velocity, and still satisfies continuity, has a curl that is identical to that of the FLCT derived velocity.

¹Any assumed maximum value can be exceeded by globally increasing the value of ∇g .

The right side of Equation (4.7) was estimated by first smoothing the column emission measure in time using a three frame box-car average, which helped to suppress fast moving artifacts from the data, and then subtracting the smoothed value of adjacent frames. We then solved for f in the Fourier domain, where the solution and its Fourier transform, $\hat{f}$, are given by

$$\hat{f}(\mathbf{k}, t) = \frac{-1}{k^2} \int_{\mathbf{x}} e^{i\mathbf{k} \cdot \mathbf{x}'} \partial_t \tilde{n}_e \, d\mathbf{x}' \quad (4.13)$$

and

$$f(\mathbf{x}, t) = \frac{1}{2\pi} \int_{\mathbf{k}} e^{-i\mathbf{k}' \cdot \mathbf{x}} \hat{f} \, d\mathbf{k}', \quad (4.14)$$

with f set to zero along the boundaries and $\hat{f}(\mathbf{k} = 0) \equiv 0$.

The dependence of Equation (4.12) on both $\nabla^2 g$ and ∇g prohibits a simple inversion and so a solution must be found through iterative relaxation. The algorithm that we used in determining the solution is as follows:

$$S_i = \hat{\mathbf{z}} \cdot \frac{1}{\tilde{n}_e} \nabla \tilde{n}_e \times \nabla f + \frac{1}{\tilde{n}_e} \nabla \tilde{n}_e \cdot \nabla g_i + \hat{\mathbf{z}} \cdot (\tilde{n}_e \nabla \times \mathbf{v}) \quad (4.15)$$

is constructed given the current value of $g = g_i$. Then, from S_i we construct g_s such that

$$\nabla^2 g_s = S_i. \quad (4.16)$$

Finally, g_{i+1} is given by

$$g_{i+1} = g_s R + g_i (1 - R), \quad (4.17)$$

with $R \lesssim 0.5$ controlling the rate and stability of convergence. Each solution for g_s is found by solving Equation (4.16) in the Fourier domain, as with solutions for f , and the overall solution for g is said to have converged when the global average of the eccentricity between successive steps falls below a critical value of

$$\left\langle \frac{g_i - g_s}{g_i + g_s} \right\rangle \leq 10^{-4}. \quad (4.18)$$

This scheme appears to converge significantly faster than finite difference relaxation methods, likely owing to the fact that the intermediate solutions, g_s , contain information about every pixel within the numerical domain (a feature of the Fourier transform), whereas for finite difference methods each pixel only receives information about its nearest neighbors. A representative example of the complete velocity inversion is shown in Figure 4.4.

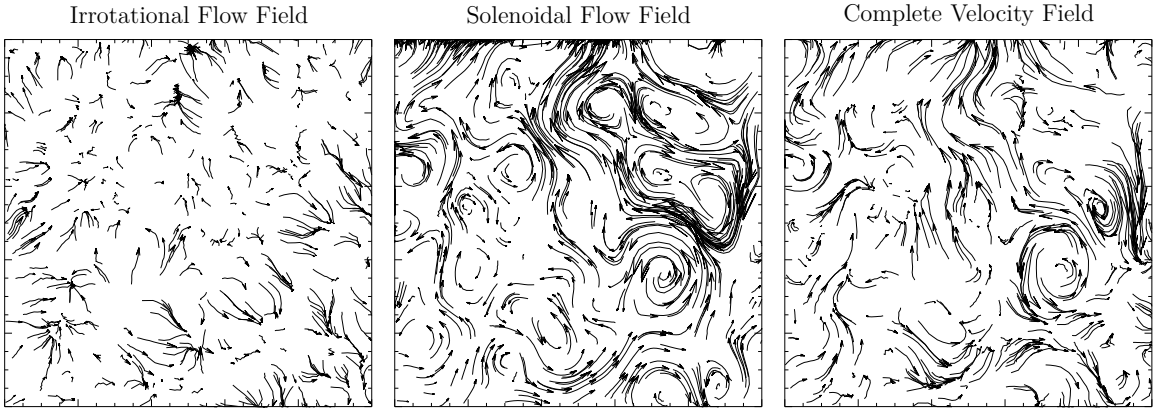


Figure 4.4. The contributions to $\tilde{n}_e \mathbf{u}$ are given by ∇f (left panel) and $\hat{\mathbf{z}} \times \nabla g$ (middle panel). Together they form the continuity consistent velocity field that most closely resembles the FLCT derived optical flow (right panel).

4.3. Data Assimilated Field Transport

We have outlined a method for obtaining the density and flow velocity within that portion of the fan with well-observed emission. The flows there, shown in Figure 4.4, have a qualitative appearance suggestive of non-negligible plasma β . That is to say, the forces generating those flows appear to demand pressure gradients in addition to the Lorentz force. In order to make this claim more rigorous we will use the measured flows to infer the form of the magnetic field and then demonstrate that the resulting Lorentz force is insufficient to achieve the observed forcing of the plasma. We refer to this technique as a Data Assimilated Field Transport (DAFT) model, since it is predicated on the idea of using observational data, which are assimilated into the model, to transport the magnetic field lines.

Outside the region with well-observed emission, which we will call the DAFT region, there is another region, in which the Signal to Noise Ratio (SNR) for EUV emission is relatively low. For reasons that will become apparent, we refer to this as the Alfvén region. While the density inversion described in the previous section works well in the DAFT region and is largely insensitive to the details of the Alfvén region, the density and velocity within the Alfvén region are not well constrained by these methods. The location of the low SNR cutoff is both spatially and temporally dynamic so any rigid boundary that contains all of the interesting dynamics in the DAFT region and none of the Alfvén region would be difficult to treat. Therefore,

rather than attempt to isolate only the DAFT region, we allow both regions to occupy portions of our data cube. As such, any method for inferring their coevolution must somehow account for their interaction.

Simultaneous treatment of both regions begins with the momentum equation and induction equation of ideal MHD:

$$\rho(\partial_t + \mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla P + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B}, \quad (4.19)$$

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}), \quad (4.20)$$

where we have neglected viscosity, resistivity and gravity. Since observations have provided the velocity in the DAFT region, which we refer to as $\mathbf{u}_{\text{obs}}$, we will use these equations to derive the pressure and magnetic field there. The velocity in the Alfvén region, called $\mathbf{u}_A$ and not to be confused with the Alfvén speed, V_a , is not constrained by observations and must therefore be derived, along with the magnetic field, using these equations. The result will be a composite velocity field, $\mathbf{u}$ combining these two regimes, and a magnetic field, $\mathbf{B}$ that spans the entire domain.

4.3.1. Composite MHD

Just as with the velocity treatment in §4.2.2, we assume that the magnetic field is inherently 2-D and can therefore be written in the form of a flux function, φ , such that

$$\mathbf{B} = B_0 \hat{\mathbf{z}} \times \nabla \varphi, \quad (4.21)$$

where $\mathbf{z}$ is taken as a direction of symmetry, out of the plane. The uncurled form of the induction equation is then

$$\partial_t \varphi = -\mathbf{u} \cdot \nabla \varphi. \quad (4.22)$$

Since $\mathbf{u}_{\text{obs}}(\mathbf{x}, t)$ is fully specified in the DAFT region, any initial field configuration is sufficient to completely specify the magnetic field at all future times within that region. We need only choose the initial configuration that best describes the system.

The velocity in the Alfvén region will depend on the magnetic field and the interaction with the DAFT region, as well as the choice of boundary conditions at the edges of the domain. In the Alfvén region, we expect the pressure term to contribute little to the forcing of the plasma. Further, since the velocity is not constrained by observation, we cannot determine both the magnetic field and the pressure from equations (4.19) and (4.22). We therefore rewrite the momentum equation for Alfvén region in a form that will be dominated by Alfvénic waves:

$$(\partial_t + \mathbf{u}_A \cdot \nabla) \mathbf{u}_A = \frac{V_a^2}{B^2} (\nabla \times \mathbf{B}) \times \mathbf{B}. \quad (4.23)$$

By omitting the pressure, and replacing the density with $B^2/(4\pi V_a^2)$, the Alfvén speed, V_a , may be specified independently of the background magnetic field strength so that, when linearized and expanded about a smoothly varying equilibrium, as in the WKB limit, the fast magnetosonic wave mode naturally emerges with an isotropic wave speed, while the slow magnetosonic mode is suppressed. Note that in the case of

propagation parallel to $\mathbf{B}_0$, the fast magnetosonic wave is degenerate with the shear Alfvén wave, hence the moniker Alfvénic.

In order to connect the DAFT and Alfvén regimes into a single solution we use a weighted average for the velocity evolution. We define a boolean mask, H , such that $H = 1$ where the SNR $\gg 1$ and $H = 0$ where the SNR ≤ 1 . H is also set to zero for a 10 pixel buffer along the edges of the numerical domain so that the observed velocities do not conflict with the boundary conditions. From inspection, the SNR scales with the emission measure, which falls off with height above the arcade, so the transition from $H = 1$ to $H = 0$ tends to occur on a band that extends from the upper left to lower right of the simulation domain, as seen in Figure 4.5.

A selection function, $\tilde{H}$, is then developed by iteratively smoothing H while holding all of the zero values fixed. $\tilde{H}$ therefore satisfies the same requirements as H , but ramps progressively from 0 to 1 over a spatial scale of ~ 10 pixels. The combined evolution of the velocity field everywhere in the numerical domain is then given by

$$\partial_t \mathbf{u} = \tilde{H} \partial_t \mathbf{u}_{obs} + (1 - \tilde{H}) \partial_t \mathbf{u}_A \quad (4.24)$$

with $\partial_t \mathbf{u}_A$ determined as in Equation (4.23) and $\partial_t \mathbf{u}_{obs}$ given by

$$\partial_t \mathbf{u}_{obs} = \frac{\mathbf{u}_{obs}(t = t_i) - \mathbf{u}_{obs}(t = t_{i-1})}{t_i - t_{i-1}}. \quad (4.25)$$

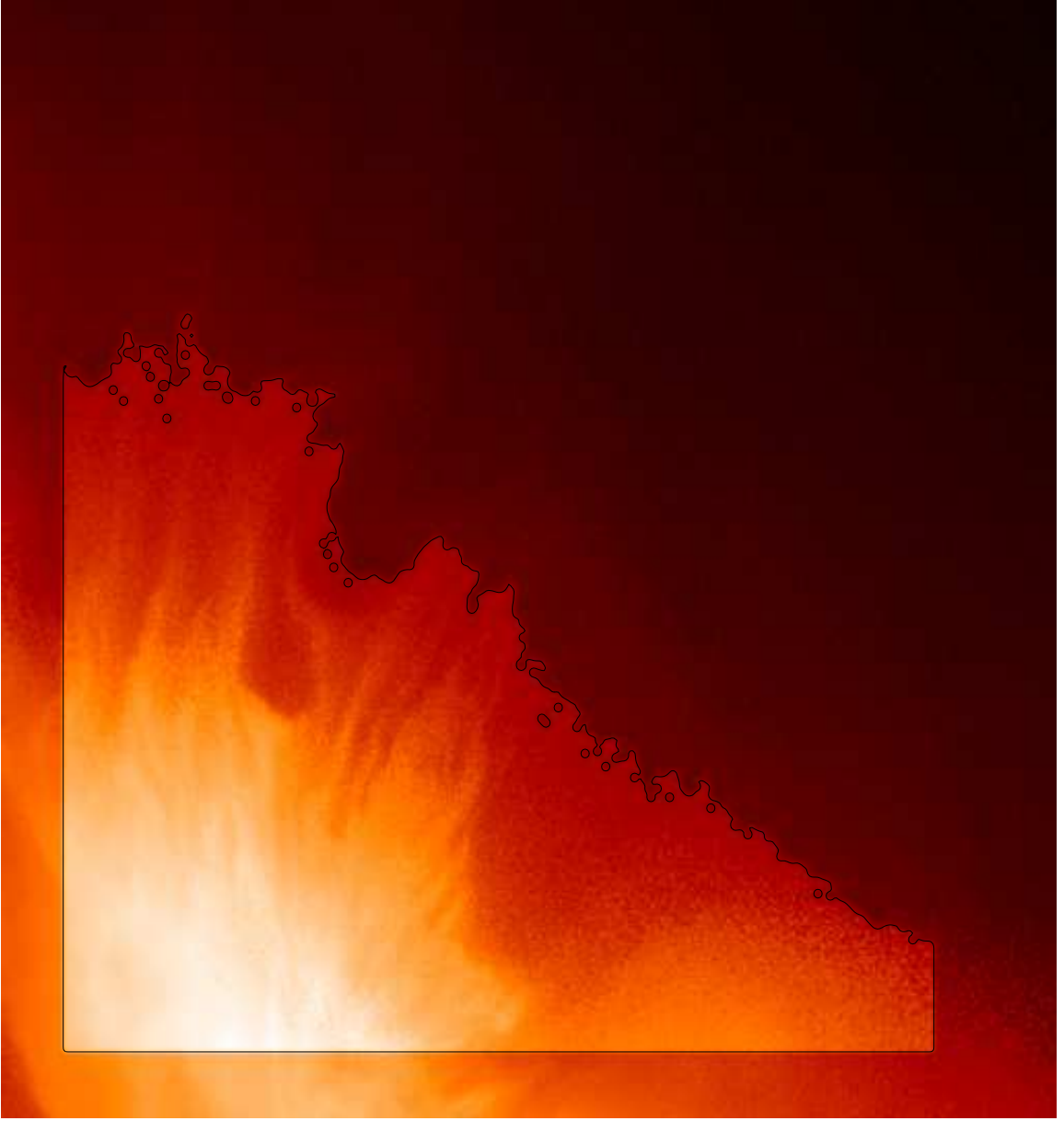


Figure 4.5. In the image above $\tilde{n}_e$ (shown in red-yellow brightness) has been multiplied by $H + 1$ so that the DAFT region, where $\tilde{H} \approx 1$, is visibly enhanced relative to the Alfvén region, where $\tilde{H} \approx 0$. The black contour shows the $\tilde{H} = 0.5$ level, indicating the shape of the transition band, which evolves from frame to frame as the density, and therefore the location of the SNR threshold, evolves.

For the 2011 October 22 event it was observed that the SNR of the filtered column emission measure (ϵ_f) approaches unity as the signal drops below about 15% of the mean, which value was ultimately adopted for the threshold value of H .

This coupling of the DAFT region to the Alfvén region serves two purposes. First, it breaks up the behavior of the boundary conditions into two elements, with the Alfvén region serving as an intermediate layer. Since the physics of the Alfvén region and the details of the coupling are easily characterized this introduces fewer artifacts than might be expected from an abrupt boundary. And since, as we shall see, the behavior in the Alfvén region is dominated by wave modes, its interaction with the boundaries on the exterior of the numerical domain is likewise simpler and better behaved than if the DAFT region were to abut those same boundaries. The second purpose for the coupling is to attempt to model how a highly dynamic supra-arcade structure might stimulate the evolution of the extended corona, which point we will return to in §4.4.

Given an initial configuration for the magnetic field, the aforementioned versions of the induction and momentum equations provide a prescription for advancing the system. It remains only to set the boundary conditions. At the bottom (sunward edge) of the domain we have chosen to anchor the field to the boundary. This is done to respect the assumption that the field lines emanate from a magnetically

dominated region in the lower corona, which is unaffected by the motion of plasma in the supra-arcade region.

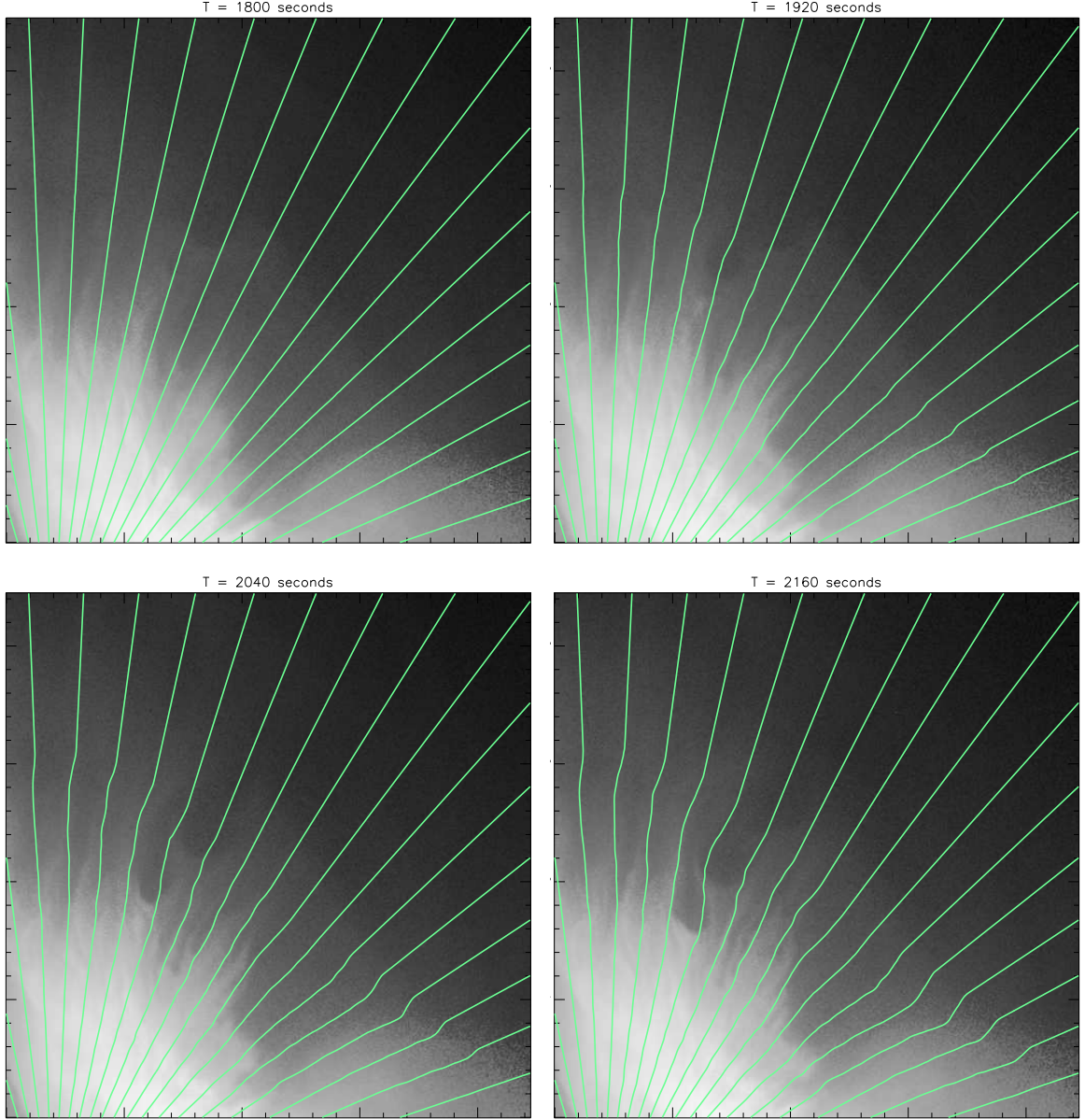


Figure 4.6. The combined solution for the magnetic field and the velocity are depicted in four near-consecutive frames at 120 second cadence. Field lines are shown in green and plasma density is shown in ($\log_{10}$) grey-scale.

For the other three boundaries we choose a non-reflecting boundary condition.² This is accomplished by setting the velocity at the boundary to be consistent with an outward-propagating, transverse, fast magnetosonic wave, as described by the linearized MHD wave equations. The associated wave eigenvector is given by

$$\delta \mathbf{u} = -(\mathbf{B}_0 \cdot \hat{\mathbf{n}}) \frac{V_a}{B_0^2} \delta \mathbf{b}_\perp, \quad (4.26)$$

where $\hat{\mathbf{n}}$ is the normal direction to the boundary and

$$\delta \mathbf{b}_\perp = -\frac{1}{B_0^2} \mathbf{B}_0 \times (\mathbf{B}_0 \times \mathbf{B}) \quad (4.27)$$

is the component of the magnetic field perpendicular to the initial background field, $\mathbf{B}_0$. Imposing this condition on the velocity at the boundary ensures that any outward traveling, transverse waves will escape through the boundaries of the numerical domain, rather than being reflected as would be the case for Neumann, Dirichlet, or even mixed boundary conditions. So long as $|\delta \mathbf{b}_\perp| \ll |\mathbf{B}_0|$ this scheme performs well. It does, however, become unstable as the perturbation grows, owing to the nonlinearity of the system. We therefore use a weighted average based on the relative strength of the perturbation, with an open boundary condition employed in the limit that $|\delta \mathbf{b}_\perp| \sim |\mathbf{B}_0|$.

These boundary conditions complete the prescription for evolving the fluid velocity and magnetic field forward in time from an initial condition. Unfortunately, direct

²Both compressional and transverse fast magnetosonic modes are expected within the Alfvén region, but the compressional mode propagates perpendicular to the background magnetic field and, therefore, parallel to the boundary in most cases.

measurements of coronal magnetic fields are unavailable in the supra-arcade regions, where extrapolations are insufficiently resolved and polarimetry is photon starved. We must therefore infer, from inspection of the emission features, a likely initial field configuration. This is done by observing the alignment of spiky supra-arcade features and assuming that they are aligned with the large-scale orientation of the magnetic field. Then, since nothing depends on the overall magnetic field strength (as yet), the initial field configuration is taken to be a potential field, of arbitrary strength, whose field lines best coincide with the spiky arcade features that will later appear.

The initial condition, as well as several subsequent states of the system can be seen in Figure 4.6. In the Alfvén region, V_a has been set to 30 km s^{-1} in this example, in order to increase the apparent amplitude of transverse oscillations in the Alfvén region, which are difficult to distinguish as V_a is increased. Of particular note are the pockets of low density plasma that descend downward through the left hand portion of the image domain, and the way the field lines move to accommodate the plasma motion. The transverse oscillations that have been observed along the edges of SADs have been previously characterized by Verwichte et al. (2005) and it seems likely that the corresponding motion of the magnetic field is related to this motion through the well known equations of magnetohydrodynamics. Here, the coupling through the induction equation has been made rigorous. But to fully understand the system, we

must also inspect the influence of the Lorentz force as it appears in the momentum equation; and for that we must attempt to infer the strength of the magnetic field.

4.3.2. Magnetic Field Strength

In the preceding sections we developed a solution for the evolving magnetic field without ever specifying the overall field strength. This was a consequence of the fact that the induction equation is independent of the overall strength the magnetic field as well as the imposed definition for the plasma density in the Alfvén region – the density there is proportional to $|\mathbf{B}|^2$ so the momentum equation for that region is independent of $|\mathbf{B}|$. As such, the induction equation and the Alfvén momentum equation both depend only on the structure of the magnetic field. The only dependence on the strength of the magnetic field is in the momentum equation in the DAFT region, which we have avoided solving. However, if we suppose that the momentum equation that governs this region is balanced then we may gain further insight by comparing the plasma acceleration to the Lorentz force and thereby estimating the contribution from the magnetic field.

Returning to Equation (4.19) for the evolution of $\mathbf{u}_{obs}$, the only unknowns are the global scaling for the density (i.e. d), the overall magnetic field strength, and the pressure. As with the continuity equation the density may be written in terms of $\tilde{n}_e$ and d , while the Lorentz force can be expressed as the product of the overall magnetic

field strength and a scale free structure function,

$$\hat{\mathbf{F}}_L = -\nabla\varphi\nabla^2\varphi. \quad (4.28)$$

Here we have taken the spatial and temporal mean of $|\nabla\varphi|$ to be one so that the definition for the magnetic field in Equation (4.21) gives the typical field strength as $\overline{B}^2 = B_0^2$. Since the plasma pressure is unknown, we proceed by first taking the curl of Equation (4.19). The resulting expression depends only on the overall magnetic field strength, the column depth, and terms that are already known to us. That is,

$$\nabla \times (\tilde{n}_e (\partial_t + \nabla \cdot \mathbf{u})\mathbf{u}) = \frac{B_0^2 \sqrt{d}}{m_i} \nabla \times \hat{\mathbf{F}}_L, \quad (4.29)$$

where $m_i (= 1.15 m_p)$ is the typical ion mass, m_p is proton mass and 1.15 is the approximate ratio of protons and neutrons to unbound electrons in the solar corona so that the plasma density is given by

$$\rho = m_i n_e = 1.15 m_p \tilde{n}_e / \sqrt{d}. \quad (4.30)$$

In order to achieve an exact balance in Equation (4.29) we would need to normalize the field strength to the line of sight depth independently for every pixel in every time step. However, since the structure of the magnetic field is already known, only the global field strength may be specified. Thus the line of sight depth would have to be spatially and temporally variable, which is inconsistent with the assumptions made in §4.2.2. Equation (4.29) is therefore taken not as a local prescription but rather as

a mean field normalization. That is, the line of sight depth is related to the *typical* magnetic field strength through

$$B_0^4 = \frac{1}{d} \left\langle \left(\frac{\nabla \times (m_i \sqrt{\epsilon}) D_t \mathbf{u}}{\nabla \times \hat{\mathbf{F}}_L} \right)^2 \right\rangle_{\mathbf{x}, t}, \quad (4.31)$$

where the brackets imply averaging in space and time and we have used the short-hand notation for the advective derivative, $D_t \equiv \partial_t + \mathbf{u} \cdot \nabla$.

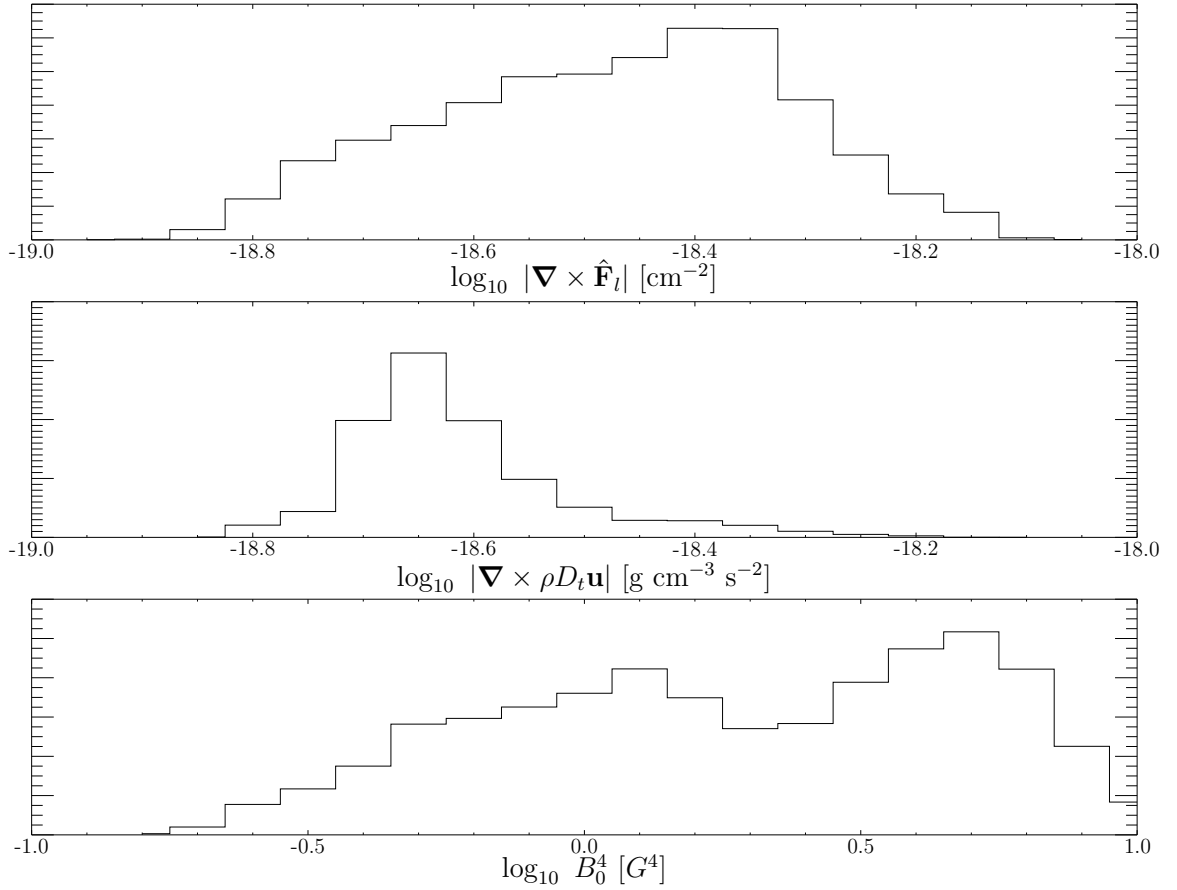


Figure 4.7. The distribution of values for the curl of the scale-free Lorentz force and the inertial response of the plasma are shown in the top and middle panels, respectively. The distribution of B_0^4 , shown in the lower panel, is found from the square of their ratio. A column depth of 10^3 km is assumed in every case.

Typical values for the magnitude of the contributions to Equation (4.29) can be seen in Figure 4.7. In order to estimate the exact value of this normalization we considered averages over only the interior of the DAFT region, where the Alfvén momentum equation makes no contribution. This resulted in an overall magnetic field strength of

$$B_0 = 1.2 \text{ G} \times d_6^{-1/4}, \quad (4.32)$$

where $d = d_6 \times 10^3 \text{ km}$. Comparing this result with the findings of other authors, Savage et al. (2010) estimated the thickness of a similar fan structure in the so called cartwheel flare of 2008 April 9 to be on the order of $5 \times 10^3 \text{ km}$. If this value is used for d , the corresponding magnetic field strength is $\sim 1.2 \text{ G} \sqrt[4]{5} \approx 0.8 \text{ G}$. Conversely, McKenzie (2013) performed a potential field extrapolation for the same 2011 October 22 event that we consider here and found that the typical magnetic field strength in the fan structure was in the range of 4 - 12 G for the potential field. Thus, our measured value, given the assumed column depth, is somewhat lower than previous estimates, but not unreasonably so.

4.4. Results

From the previous section we have already estimated a normalization for the magnetic field strength in this event, which depends weakly on the column depth and is typically of the order a few G for column depths on the order of a few 10^3 km . From

this we can begin to estimate magnetic signatures that events such as this might be expected to produce. We shall focus on three aspects – namely, the temporal power spectrum of radiated Alfvén waves, the Poynting flux at various locations within the simulation domain, and the plasma pressure necessary to account for any forcing not consistent with the magnetic field.

4.4.1. Radiated Alfvén Waves

One advantage of the hybrid momentum equation is that we are able to simultaneously treat the evolution of both the highly dynamic plasma immediately above the arcade and the response of the plasma in the more extended corona. This allows us to address the question of how a magnetically dominated, slowly evolving extended corona might respond to being driven by a higher density region with complex behavior involving the motion of magnetic elements and supra-arcade downflows.

In the model, this transition is handled by the selection function, $\tilde{H}$. In the majority of the domain $\tilde{H}$ has an integer value of either zero or one, but in the *transition layer*, where $0 < \tilde{H} < 1$, a superposition of Equations (4.25) and (4.23) approximates the type of behavior that might be expected in a region where the pressure and density fall off rapidly and a highly dynamic region with non-trivial plasma pressure behaves as a driver for a more rarefied region, which responds by radiating energy away in the form of Alfvén waves.

First let us consider the Temporal Power Spectral Density (TPSD) of both the magnetic and velocity fields, averaged over two regions of interest (Region 1 and Region 2), corresponding to $\tilde{H} = 1$ (well within the DAFT region) and $\tilde{H} = 0$, well within the Alfvén region. The exact locations of Regions 1 and 2 are indicated by the solid white and dashed purple boxes in Figure 4.3. The TPSD for each of the fields in the region of interest is given in terms of the Fourier transform of the windowed field, averaged over the region of interest. That is, if

$$\hat{b}_i(f) = \int e^{2\pi i f t'} \hat{\mathbf{x}}_i \cdot \mathbf{B}(\mathbf{x}, t') w(t') dt' \quad (4.33)$$

is the Fourier transform of the i^{th} component of the magnetic field multiplied by a temporal window function, $w(t)$, then the corresponding TPSD, averaged over some subregion R_j , is given by

$$p_{[b,i,j]}(f) = \left\langle |\hat{b}_i(f)|^2 \right\rangle_{\mathbf{x} \in R_j}. \quad (4.34)$$

For our analysis, we considered only the x-component of the velocity and magnetic fields, as it was observed that the behavior in y was identical. In practice, we applied a Hann window function for $w(t)$, generated via `HANNING.PRO` from the standard IDL distribution.

In exploring the effect of varying the specified Alfvén speed in Equation (4.23), it was observed that a critical frequency arises in the TPSDs in Region 2, above which frequency the TPSDs roll off and contain very little power. This frequency

was found to correspond to the Alfvén crossing time of the transition band – that is $f_{crit} \sim V_a/\lambda_H$, where λ_H is the characteristic width of the $\tilde{H}$ transition – and we have hypothesized that it corresponds to an impedance matching issue of sorts. By setting $V_a = 10^3 \text{ km s}^{-1}$ the critical frequency was pushed past the Nyquist frequency and this artificial roll-off was eliminated from the TPSDs in Region 2, at which point the spectra appeared to become independent of the choice of Alfvén speed. For this reason, all subsequent analysis is related to simulation runs with this choice of Alfvén speed.

Comparing the power spectra of the two fields in Figure 4.8 we find that while the velocity TPSD in Region 1 is largely flat above frequencies of 10^{-3} Hz , the response of the magnetic field in this region shows that $p_{[b,1,x]} \sim f^{-2}$. We attribute this to a fundamental aspect of the induction equation. Consider that, to first order, Equation (4.20) is equivalent to

$$\partial_t \varphi_1 = -\mathbf{u} \cdot \nabla \varphi_0, \quad (4.35)$$

where φ_0 and φ_1 are the equilibrium magnetic flux function and the associated perturbation, respectively. If we take the equilibrium field to be radial, as it was in the previous section, then this expression becomes

$$\partial_t \varphi_1 \propto -u_\theta \frac{1}{r}. \quad (4.36)$$

Then, because differentiation in the time domain is equivalent to multiplication by angular frequency ($\omega = 2\pi f$) in the Fourier domain, we can write the spatial average

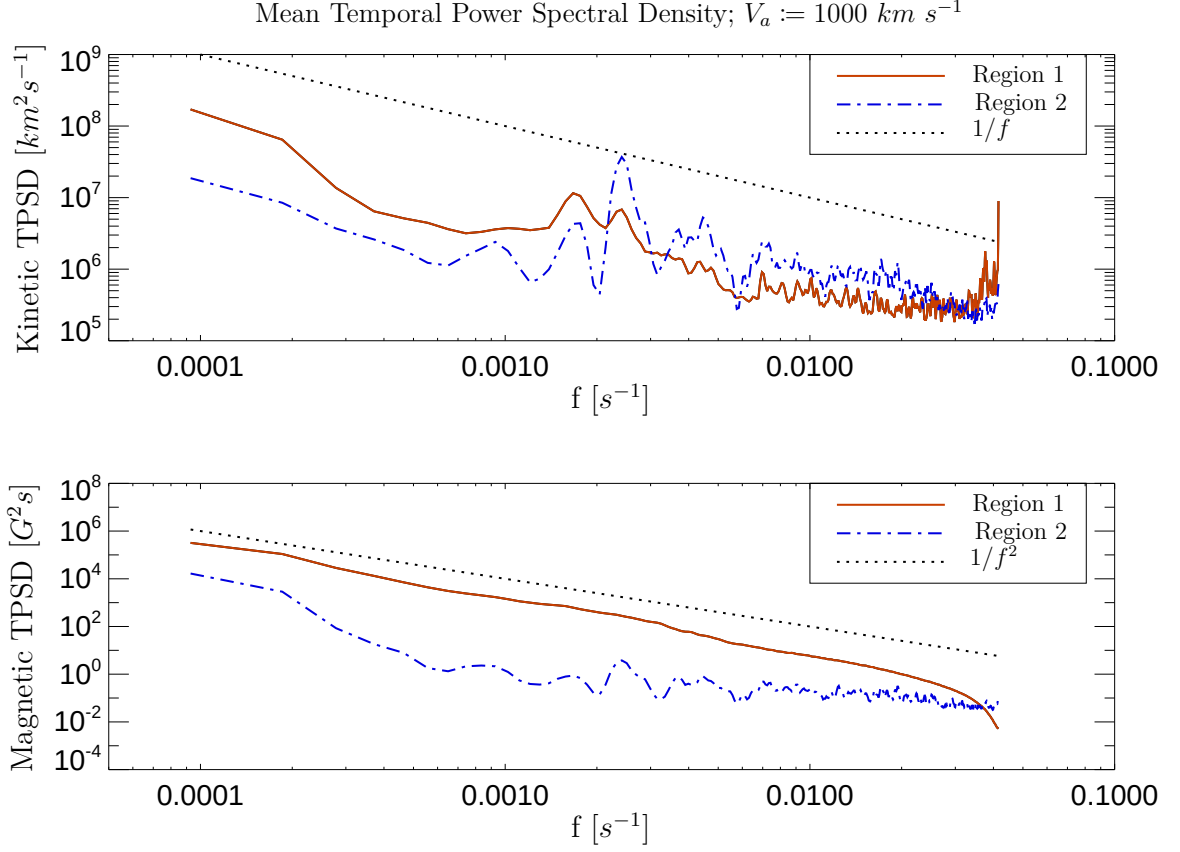


Figure 4.8. The temporal power spectral density of one component of the velocity and the corresponding component of the magnetic field are shown for each of the two regions (1 and 2) as solid-red and dashed-blue curves in the top and bottom panels, respectively. The fine-dashed line indicates a power law of f^{-2} in the case of the magnetic field and f^{-1} in the case of the velocity.

of the Fourier transform of φ_1 as

$$\langle \hat{\varphi}_1^2 \rangle \propto \omega^{-2} \langle \hat{u}_\theta^2 \rangle. \quad (4.37)$$

Apparently, if the spatial average of the Fourier transform of the perpendicular velocity field (u_θ in this case) is vaguely “white”, then the induction equation will integrate it into a “pink” magnetic field spectrum. Of course, in a system with a

momentum equation that continually responds to the changing magnetic field this would likely be suppressed since u_θ would itself be a function of φ_1 . But in Region 1, where the velocity was derived with no dependence on the magnetic field, the noise within the velocity data is decoupled from the momentum equation. The induction equation treats the velocity noise as a source term, which is then integrated, and the result is a magnetic power spectrum that decays as a power law of the frequency.

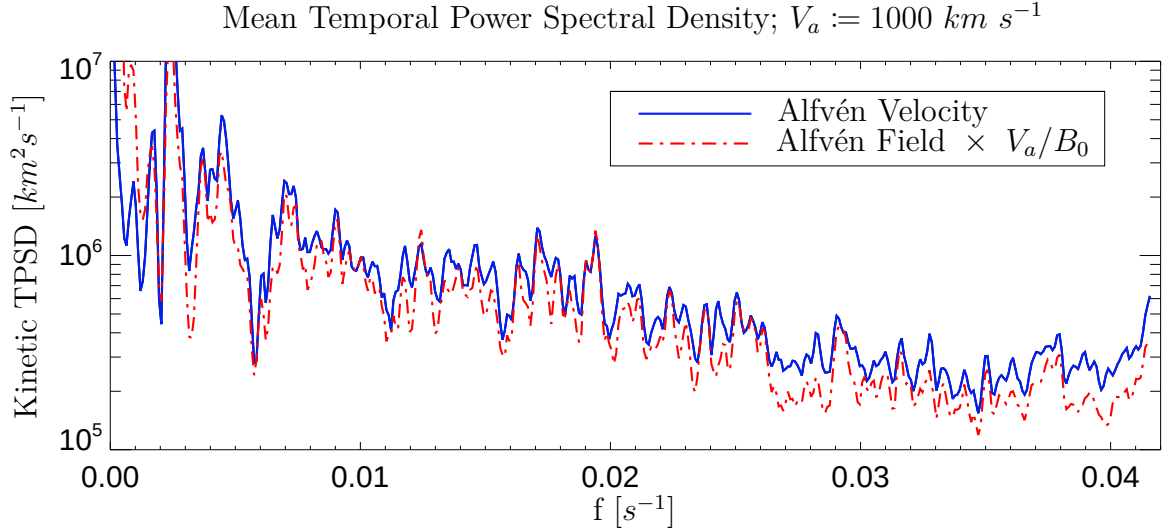


Figure 4.9. The temporal power spectral density of one component of the magnetic field and the corresponding component of velocity are shown for Region 2 in dashed-red and solid-blue, respectively. The magnetic TPSD has been scaled by the ratio of the Alfvén speed and the background magnetic field strength.

To further understand the system, consider the general slope of the power spectra in Region 2. Note the approximate coincidence of many of the peaks in the two spectra in the top panel of Figure 4.8. Also, from the bottom panel of the same figure we can see that the magnetic field in Region 2 inherits most of its spectral profile not

from the magnetic field in Region 1, but rather from the velocity field in Region 1. Apparently, while the magnetic field passes through the transition layer, *it is, ultimately, the velocity field in the DAFT region that dictates the evolution of the Alfvén region.*

Lastly, we consider the relationship between the magnetic field and the velocity in the Alfvén region. Since Equation (4.23) contains no dependence on either ρ or p , only fast magnetosonic waves are expected in the Alfvén region. The fast magnetosonic eigenmodes for propagation parallel and perpendicular to $\mathbf{B}_0$ are given by

$$\delta b_{\perp} = \pm \delta u_{\perp} \frac{B_0}{V_a} \quad \longleftrightarrow \quad (\mathbf{k} \times \mathbf{B}_0 = 0); \quad (4.38)$$

$$\delta b_{\parallel} = \pm \delta u_{\parallel} \frac{B_0}{V_a} \quad \longleftrightarrow \quad (\mathbf{k} \cdot \mathbf{B}_0 = 0), \quad (4.39)$$

respectively, where $\mathbf{k}$ is the wave-vector. Note that, regardless of $\mathbf{k}$, $\delta \mathbf{b} \cdot \mathbf{k} = 0$ while the velocity perturbation is always perpendicular to $\mathbf{B}_0$.

Since the background magnetic field is constant in time, the power spectra for the two fields in this region should be proportional if the response is linear, which is not altogether obvious considering that Equation (4.23) is non-linear in $\mathbf{B}$. Inspection of Figure 4.9 shows that, indeed, with the exception of the extreme low frequency end of the spectrum, the TPSDs of the two fields in the Alfvén region track almost exactly. The fact that the constant of proportionality is (almost exactly) V_a/B_0 is due to the fact that we have sampled, not $b_{\perp}$ and $u_{\perp}$, but rather $\hat{\mathbf{x}} \cdot \mathbf{B}$ and $\hat{\mathbf{x}} \cdot \mathbf{u}$, each of which contain contributions from both modes. The proportionality between these

two components is therefore an average over transverse and compressional modes. But since each of those modes is related through V_a/B_0 , so too is the proportionality between the sampled components of the velocity and the magnetic field. Therefore we cannot claim that one mode contains more energy than the other.

4.4.2. Poynting Flux

Next we considered the Poynting Flux, defined for our purposes according to

$$\mathbf{S} = \frac{1}{4\pi} \mathbf{B} \times (\mathbf{u} \times \mathbf{B}). \quad (4.40)$$

Since the background magnetic field is given as a radial field, with its origin located just below the region of interest (see Figure 4.6), we have chosen to parameterize the “outward” energy flux in terms of the radial component of the Poynting Flux, S_r , defined consistently with the same coordinate system as used for the magnetic field. That is, escaping energy is quantified through

$$S_r = \frac{1}{4\pi r} (x\partial_x\varphi + y\partial_y\varphi)(u_x\partial_x\varphi + u_y\partial_y\varphi). \quad (4.41)$$

This estimation of the radial energy flux is shown in Figure 4.10 for a representative frame, approximately 15 minutes after the system is initiated. Note that within the DAFT region (especially within Region 1) both positive (anti-sunward) and negative (sunward) energy flux are in evidence. However, near and above the transition layer (indicated by the two nested bands of solid black) the radiated energy is heavily dominated by positive flux, owing to the preference of the system to radiate energy

into the lower density, weaker field of Region 2, and ultimately out through the non-reflecting boundary.

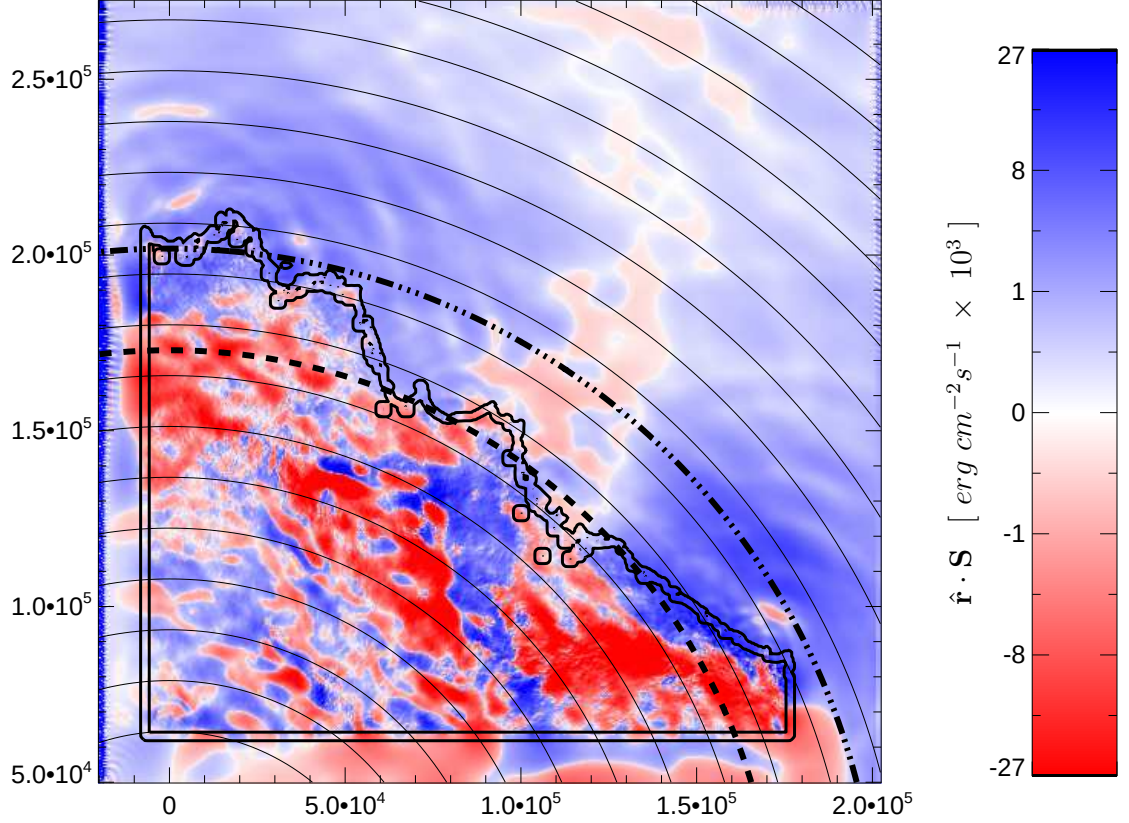


Figure 4.10. The radial Poynting Flux is depicted in the red-blue color table with blue indicating outwardly radiated energy. The concentric black rings indicate successive radii while the two bolt concentric rings indicate the contours of the $r \approx 1.7 \times 10^5$ km and $r \approx 2.0 \times 10^5$ km levels in dashed and dot-dashed respectively. The concentric contours that appear within the two circular dashed bands indicate different levels of $\tilde{H}$, demonstrating the complexity and approximate location of the transition band where $\tilde{H} \approx 0.5$.

To further explore this behavior, we have divided the simulation domain into concentric regions of increasing radius, defined consistently with the initial magnetic

field. These bands are depicted as the regions between concentric arcs in Figure 4.10. We then calculated the mean value for the Poynting Flux in each band, as well as the total flux through each band. These are shown in Figure 4.11. For these estimates, a mean field strength of 1.2 G has been used, equivalent to a column depth of 10^3 km. Since the Poynting flux scales with the square of the magnetic field, these estimates are inversely proportional to the square root of the column depth.

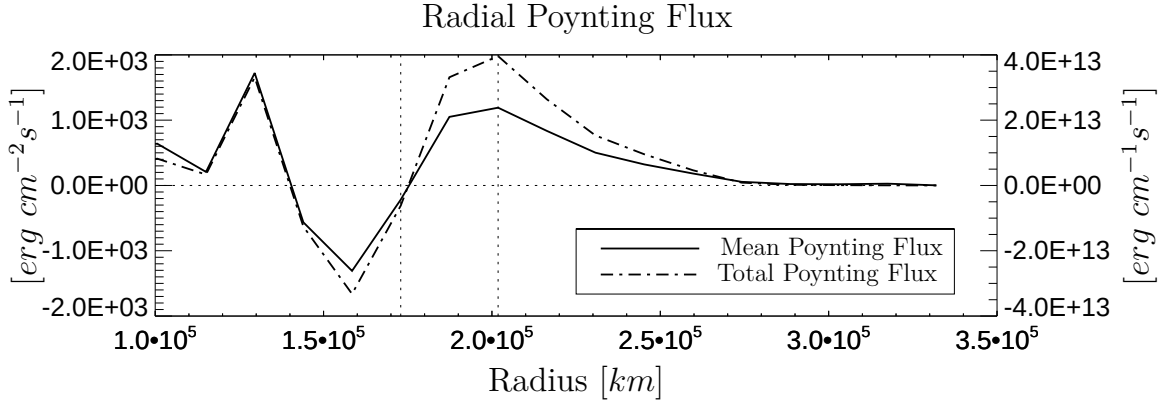


Figure 4.11. The mean and total Poynting flux through bands of successively larger radii are depicted by the solid curve (scale to left) and dashed curve (scale to right) respectively. The vertical dashed lines indicate the location of the $r \approx 1.7 \times 10^5$ km and $r \approx 2.0 \times 10^5$ km bands, which in turn indicate the locations of zero net radiated power and the peak value of the outward Poynting flux, respectively.

From Figure 4.11 we see that, for a column depth of 10^3 km, the typical Poynting Flux is on the order of 10^3 erg cm⁻² s⁻¹. If we instead assume a deeper column depth of 10^4 km, this value decreases by roughly a factor of three. And while there is a strong negative contribution below radii of approximately 1.7×10^5 km, the net flux below 2.0×10^5 km is quite small, and above that level the mean flux in each

concentric band is positive definite, and decays with radius, as would be expected for conservation of transmitted power.

The total flux through each band, multiplying the mean flux by the area of each band and then dividing by its radial width (an approximation of the arc length), demonstrates a similar behavior. And when we compared the details of where the radial Poynting flux becomes positive definite, we found a close correspondence to the $\tilde{H} \approx 0.5$ transition band, indicating that this layer behaves as a source surface for radiated Alfvén energy.

Unfortunately, since the arc length of each band decreases above radii of 2.0×10^5 km, this does little to demonstrate whether power is conserved through successive bands.³ This may be due to imperfections in the non-reflecting boundary conditions, which can be seen along the edges of the numerical domain in Figure 4.10. However, if we take the peak value for the total flux and multiply by the estimated column depth of 10^3 km we find that the upper bound for the radiated power from this region is approximately 4×10^{21} erg s⁻¹, much lower than the energy budget of a typical solar active region (Tarr et al., 2013), but perhaps large enough to be detectable in the solar wind. This estimate of the total radiated power scales as $B_0^2 \times d \propto \sqrt{d}$, so if the assumed column depth is increased to 10^4 km the total radiated power is approximately 1.2×10^{22} erg s⁻¹.

³These bands are inscribed within a square domain and typically meet the edge of the domain at an oblique angle.

4.4.3. Pressure and Plasma Beta

Having scaled the magnetic field strength to the column depth we may revisit the momentum equation, the divergence of which is

$$\begin{aligned}\nabla^2 P &= \nabla \cdot \left[\frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} - \rho D_t \mathbf{u} \right] \\ &= B_0^2 \nabla \cdot \hat{\mathbf{F}}_L - m_i \nabla \cdot \sqrt{\frac{\epsilon_f}{d}} D_t \mathbf{u},\end{aligned}\tag{4.42}$$

where the pressure, P , is given in terms of units of the magnetic field strength, i.e. B_0^2 . Since each term on the right hand side of Equation (4.42) is already known, the plasma pressure is well constrained and Equation (4.42) can be solved through multiplication by $\mathbf{k}$ in the Fourier domain using the same techniques as in §4.2, with the pressure gradients set to zero on the boundaries in order to give maximum freedom to the solution in the interior.

Because Equation (4.42) contains no zeroth or first order derivatives of P , the solution is only unique up to an additive harmonic function. That is, any function $h(\mathbf{x})$ satisfying $\nabla^2 h = 0$ can be added to P without affecting Equation (4.42) and we are forced to place an additional constraint on the system in order to find the pressure itself. We therefore choose to consider the smallest possible pressure that is everywhere non-negative within the DAFT region. This is found by first determining the minimum value of P from within Region 1 and then subtracting this from the global value of P in every frame. This pressure, while not a unique solution to Equation (4.42), is the lowest possible thermal energy density that is physically

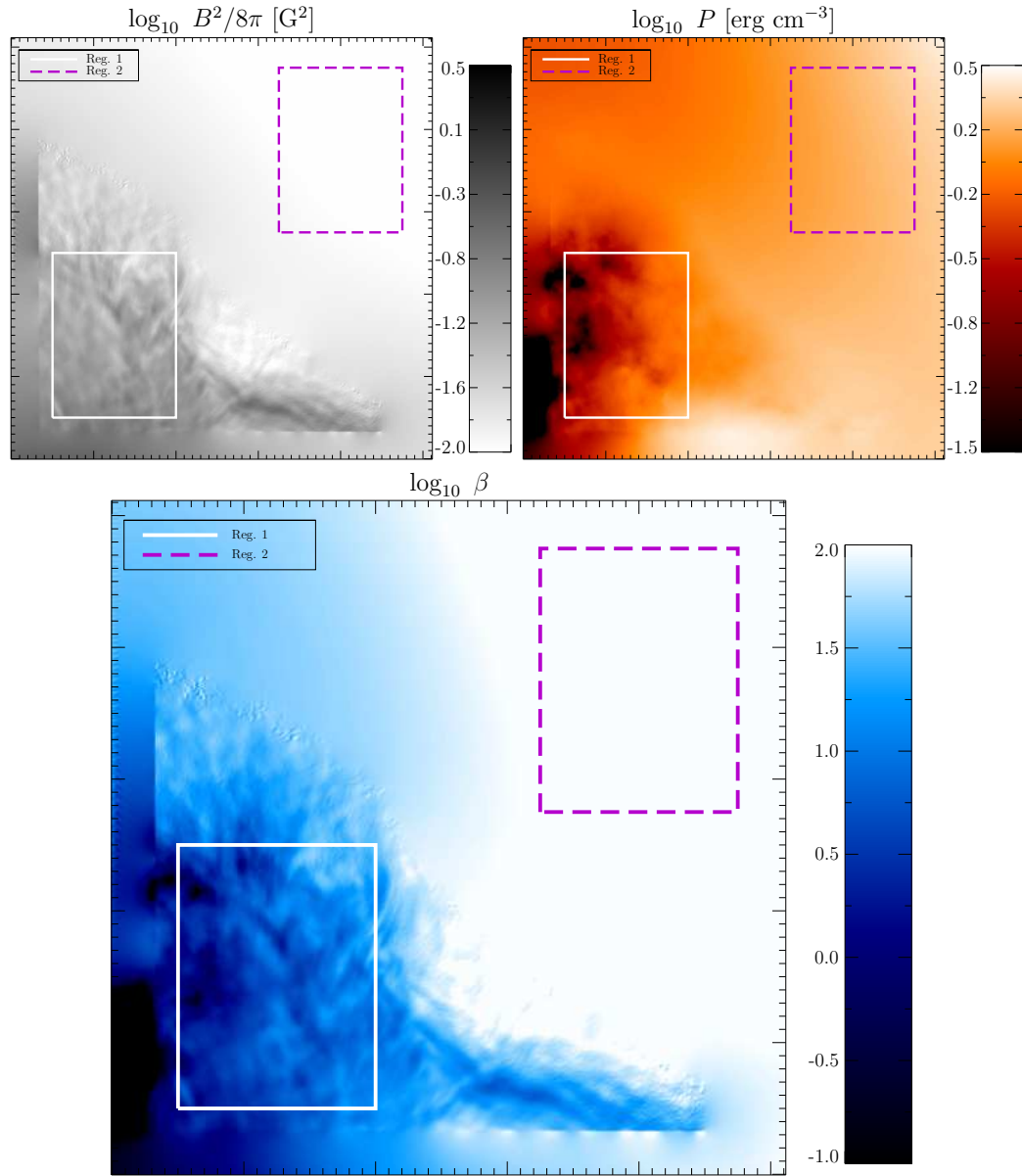


Figure 4.12. The two contributions to the total pressure (i.e. the magnetic energy density and thermal energy density), are shown in the top left and top right panels. The lower panel shows the plasma β . The solid white box indicates a subregion corresponding to Region 1 that is entirely within the DAFT regime while the dashed purple box indicates a subregion of Region 2, which is entirely in the Alfvén regime.

meaningful without exploring the entire parameter space of harmonic functions that could be added to P . Figure 4.12 shows the magnetic and thermal energy densities (P_b and P), assuming a column depth of 10^3km , as well as the plasma β , which is defined in the usual way:

$$\beta = \frac{8\pi P}{B^2}. \quad (4.43)$$

While specifying the lowest value of the pressure within the DAFT region gives physical meaning to the pressure (i.e. the minimum possible thermal energy density), it has the unfortunate side effect of giving surprisingly large gas pressure values, and therefore β values, in the Alfvén region, high above the arcade. This would at first seem to be in conflict with our assumptions about the Alfvénic momentum equation in this region, per Equation (4.23). However, comparing Equations (4.23) and (4.19) we see that the simplified momentum equation of the Alfvén region assumes only that the pressure *gradient* is small. From the top right panel of Figure 4.12 it is clear that the pressure inversion described in this section correctly recognizes that the momentum equation is already balanced without contributions from ∇P , and therefore estimates the pressure in this region to be very smoothly varying, so as to contribute little to Equation (4.42), despite the relatively large pressure in that region.

On the whole, the plasma pressure that results from this calculation is typically less structured than the magnetic energy density – a natural consequence of being

related through the laplacian operator. And it is worth noting that since any change in the assumed column depth will rescale the overall magnetic field strength as $B_0 \sim d^{-1/4}$ the plasma pressure and magnetic energy density both scale as $P \sim B_0^2 \sim d^{-1/2}$. Conversely, the plasma beta, which appears to be similarly structured to the magnetic field, is independent of assumed column depth and is, therefore, the only true *measured* value from this calculation. Histograms of each of these values are shown in Figure 4.13. Note that the peak value of the $\log_{10} \beta$ distribution corresponds to a slightly higher value than would be found through comparison of the peaks of the other two distributions. This is a fundamental aspect of logarithmic distributions; that is, if $n(\beta)$ is a distribution in β , then the corresponding logarithmic distribution is $n(\log_{10} \beta) = \beta n(\beta)$, and the peak of this distribution is necessarily shifted to higher β values.

4.5. Discussion

This work represents an effort to estimate the otherwise poorly constrained plasma parameters in the supra-arcade region. We have used inversion techniques to estimate the plasma velocity, and then ultimately evolve the magnetic field from some initial configuration. We then estimated the missing contributions to the momentum equation in the region where the observed plasma velocity was used. This allowed for normalization of the typical magnetic field strength to the assumed column depth

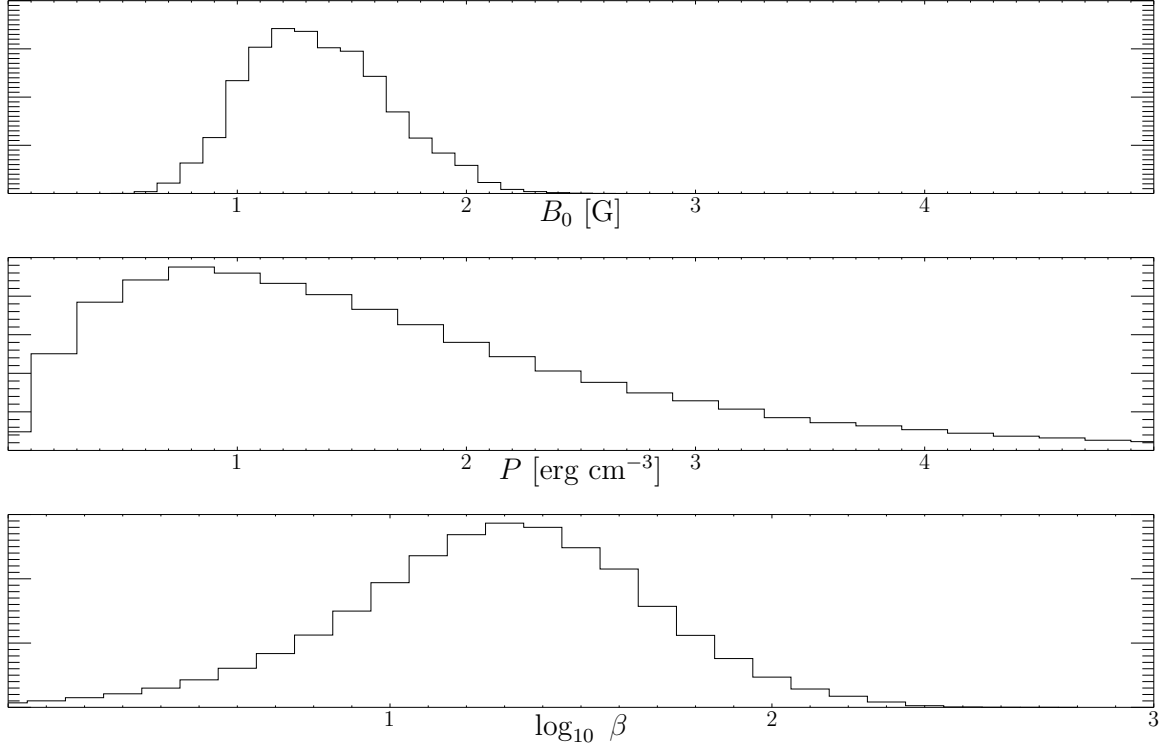


Figure 4.13. Histograms of the magnetic field strength, thermal energy density, and $\log_{10} \beta$ are shown for the interior sub-region, corresponding to Region 1 in Figure (4.12). The $\log_{10} \beta$ distribution is taken from the ratio of values in the upper two distributions.

of the supra-arcade fan. We found that *for a column depth of $\sim 10^3$ km the resulting magnetic field strength is ~ 1.2 G* and that this magnetic field strength scales inversely with the fourth root of the column depth.

We then went on to characterize the temporal power spectral density of both the velocity and the magnetic field. *We found that the wave spectrum in the Alfvén region closely follows that of the velocity source in the DAFT region*, independent of the f^{-1} scaling of the magnetic field in the DAFT region. We attribute this f^{-1} behavior to a decoupling of the induction equation in this region, which does not appear to influence

the response of the Alfvén region. This estimate, while not directly applicable to estimates of solar-wind wave spectra, does provide insight into how the dynamics of low-lying coronal features might generate MHD waves that propagate into the extended corona and ultimately into the solar wind. Since the velocity TPSD in the Alfvén region inherits much of its shape from the velocity TPSD in the DAFT region, and the magnetic PSD in the Alfvén region mimics the corresponding velocity TPSD almost exactly, we suggest that one might be able to infer the spectrum of radiated Alfvén energy directly from observed velocities without making a detailed model of the magnetic field.

Considering the Poynting flux from this region, we find that where the pressure plays a strong role (in the DAFT region) the energy flux shows little evidence of a preferred direction, but as soon as the signals from the assimilated velocity reach the transition layer, there is a strong preference toward outwardly radiated energy. Further we found that *the typical Poynting flux just above the transition layer is of the order* $10^3 \text{ erg cm}^{-2}\text{s}^{-1}$ and that integrating both along the arc length and the assumed column depth gives *an estimate for the power radiating from this region in the range of* $4 \times 10^{21} \text{ erg s}^{-1} - 1.2 \times 10^{22} \text{ erg s}^{-1}$.

Lastly, we attempted to characterize the plasma pressure in the DAFT region. For this we chose to report the smallest possible pressure that is consistent with the plasma evolution in each frame while still corresponding to a positive definite

– and therefore physically meaningful – thermal energy density. In Figure 4.13 the distribution of possible values for the magnetic field strength and plasma pressure are shown, assuming a column depth of $d = 10^3$ km, as well as $\log_{10} \beta$, which is independent of d . The peaks for these distributions are $|\mathbf{B}| \approx 1.2$ G and $P \approx 0.8$ erg cm $^{-3}$, which correspond to a typical value of $\beta \approx 13$. The typical value of $\log_{10} \beta$, which more aptly represents the range of possible behavioral regimes⁴, has a peak at $\log_{10} \beta \approx 1.3$, *corresponding to* $\beta \approx 20$.

This estimate of β would seem to place the supra-arcade region firmly in the fluid-dominated regime; however, there are a few caveats. First, it should be noted that the distribution of values is rather broad, so that within the region of interest there are pockets in which β could be significantly higher or lower. Moreover, the value that we have estimated is predicated on the assumption that the smallest possible value of p is zero and every other value is therefore larger than this value. Since the pressure is relatively complex and depends strongly on estimates of the magnetic field strength, which in turn depend strongly on the observed density and corresponding velocity inversion, it is reasonable to assume that the lowest value could be artificially low, which would bias every other value to be artificially high.

This could be further exacerbated by our inability to include additional forcing terms, such as microscopic turbulent processes or out-of-plane magnetic influences.

⁴The behavioral regime for $0 < \beta < 1$ is as rich as for $1 < \beta < \infty$. This is best captured by $\log_{10} \beta$, which maps the two regimes onto domains of equal size.

For example, in the Savage & McKenzie interpretation of SADs, the retracting flux tubes penetrate the supra-arcade fan perpendicular to the surface normal. If such a structure indeed exists in this region, it is independent of the magnetic field that we have modeled, and any energy that said structure might deposit would necessarily appear as a pressure contribution, thereby biasing our result to artificially large pressure estimates and, correspondingly, artificially large β estimates. We therefore emphasize that these values should be considered not as a strict measurement, but rather as a likely upper bound.

Additionally, we suggest that while β may not be as high as we have estimated here (of order 10 - 20), our model has demonstrated that the supra-arcade region is certainly not consistent with a low- β environment; the transverse motions and swirling evolution of the magnetic field simply cannot be accounted for without the inclusion of some form of plasma pressure. We find it likely, therefore, that the supra-arcade environment corresponds to a plasma- β of order unity. This value has been employed in a number of numerical simulations of late (Cécere et al., 2012; Guo et al., 2014; Cécere et al., 2015). Our own more recent work (Scott et al., in preparation: 2016) extends the efforts of Scott et al. (2013) showing that these results are also valid for $\beta \lesssim 1$. In the coming years it will be important, therefore, to test each of these models against different β regimes; only in this way will we ever be able to

truly pinpoint the energy distribution of the supra-arcade region and, simultaneously, explain the fundamental processes behind the evolution of supra-arcade features.

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5. REFLECTIONS

The plasma in the supra-arcade region above a coronal arcade is subject to a number of influencing factors, but the two main drivers seem to be the plasma pressure and the Lorentz force, and the relative strength of these two factors is a topic of much debate. On the one hand, the detailed dynamics of the supra-arcade region seem to suggest significant contributions from both of these influences. And yet, a detailed investigation of the plasma β , which describes the strength of the magnetic field relative to the thermal energy density, has not been previously offered; McKenzie (2013) did estimate β from a Potential Field Source Surface extrapolation, but such a study cannot resolve the dynamics within a supra-arcade structure, and therefore has limited applicability.

While others have characterized certain observational aspects of the supra-arcade region (Verwichte et al., 2005; Savage & McKenzie, 2011; Hanneman & Reeves, 2014) and attempted to explain or even reproduce certain dynamical aspects of supra-arcade plasmas (Guo et al., 2014; Cassak et al., 2013; Cécere et al., 2012; Savage et al., 2012), no previous study has looked carefully at how these models are affected by changes to the preexisting conditions within the supra-arcade, or whether those conditions can be scrutinized in light of the results of their models. The content presented in the foregoing chapters represents an effort to alleviate these omissions and can practically be viewed as two distinct and separate projects, each aimed at

improving our understanding of the plasma properties and influencing aspects of the supra-arcade region.

The first of these studies begins in Chapter 2 and considers the effects of a moving magnetic element on the surrounding plasma and thereby makes rigorous the interpretation of Supra-Arcade Downflows (SADs) as given by Savage et al. (2012), as well as providing contextual criteria for the formation of SADs, given the assumptions in the model. This is accomplished by demonstrating how a moving magnetic element can lead to the formation of non-stationary, magnetically aligned shocks, which, in turn, lead to the development of a rarefied plasma column. This process, which we dubbed “peristaltic pumping” provides the first connection between magnetohydrodynamics (MHD) and the interpretation of SADs as the wakes formed by retracting flare loops.

The investigation that is described in Chapter 3 continues in this effort by providing numerical validation of the analytical model in Chapter 2 as well as extending the scope of that model to a broader regime of MHD. And while verification of the previous analytical result is important, the unanticipated formation of fast magnetosonic shock cones in the low- β regime is itself a discovery of great significance. Through consideration of these low- β “ballistic” shocks, we came to realize that, despite the significance that is often ascribed to β , it is ultimately the acoustic and Alfvénic Mach numbers that are of fundamental importance to the formation of magnetosonic

shocks; the value of β can be recovered from the ratio of these values but is, ultimately, less important.

The study that is described in Chapter 4 is, by contrast, somewhat at odds with the former. Models that attempt to describe supra-arcade structures must typically assume an initial configuration, so without prior knowledge of the ambient conditions in the supra-arcade, any attempt to invoke such a model is necessarily begging the question that it attempts to answer. In this work, we therefore attempted to approach the problem from a different direction. By assuming only the broad structure of the supra-arcade system, with the various forcing terms left to be inferred from within the model, we set out to see if a self consistent description could be found.

To this end, we began with only the multi-spectral EUV observations. Using these data we were able to map the differential emission measure and associated plasma density in the supra-arcade, which immediately revealed the nature of the apparent motion; the evolving structures that we see in the short-wave EUV passbands are indicative of the plasma density evolution and are largely independent of temperature. From these density structures we inferred the plasma velocity and, subsequently, the forward evolution of the magnetic field from an assumed initial configuration. From these we were then able to estimate the strength of the magnetic field and the structure of the plasma pressure and, therefrom, the plasma β in the supra-arcade.

The resulting values for the density, velocity, pressure, and magnetic field, are, admittedly, flawed; the slab geometry that we assumed and the lack of a unique solution in several of our inversion techniques prohibit even a broad estimate of the uncertainty in this model. And yet, the value of the exercise as a pedagogical tool cannot be overstated since the task of incorporating observational data into a numerical model, a task that is inherently ill-posed, is critical to the reliability of numerical models. And while the estimates that came out of this model, specifically β and the spectral content of the velocity and magnetic fields, cannot be interpreted as measurements, per se, the study demonstrates the complexity of the supra-arcade region fairly unambiguously, as well as the potential power of data-assimilated models.

On the whole these projects, together, constitute significant steps toward improving our understanding of the supra-arcade region and, thus, the corona as a whole; supra-arcade structures provide a unique opportunity to test MHD models against a laboratory that is (largely) free of line-of-sight observational ambiguities, which typically plague coronal observations. Yet there are lingering questions and the need for continued investigation is clear.

Might it be possible to expand the data-assimilated model to identify energy sources that are not accounted for within the model? In Chapter 4 we sought to extract the missing terms in the momentum equation and constrain the plasma pressure. But what if some of the forcing terms are separate from the fluid? Suppose

that there were retracting, post-reconnection flux tubes creating the rarified features that we identify as SADs. By our own model these would not be part of the two-dimensional system that is the supra-arcade. Might it be possible to identify these features as anomalous sites of energy deposition?

One of the main results of the DAFT model was the extraction of an estimate of β in a region where the magnetic field cannot be directly measured. Are there other analysis techniques that could be applied to these structures that would reveal similar or contradictory estimates? One possibility lies in the transverse oscillations along the edges of low emission columns. The supra-arcade region is often observed to oscillate coherently but we also see local oscillations that seem to propagate along the edges of SADs. Perhaps these are the result of a Kelvin-Helmholtz instability. If the model that we presented in Chapter 2 is an accurate description of the conditions in the supra-arcade then there should be velocity shear and density stratification perpendicular to the magnetic field. An analysis of the Kelvin-Helmholtz instability in this regime, which will likely depend on β , seems like a natural direction in which to take this research.

Even leaving these tangential topics aside, there is much more that can be done to continue the work that we began in Chapter 3. To start, the size of our simulation domain should be expanded in order to fully capture the various behavioral modes without wondering whether the boundary conditions might be steering the behavior

in one direction or another. In the case of the ballistic fast shocks, in particular, the transverse extent of the numerical domain was clearly insufficient as the shock can be clearly seen to reflect from the boundary before the shock amplitude has fallen into the weak-shock limit. Moreover, while the current implementation relies on reflection across the center of the domain, the model could be expanded to allow for asymmetries and extended to higher Reynolds numbers in order to accommodate an analysis of Kármán vortex sheets and validation of the Kelvin Helmholtz instability in the context of SADs.

Even beyond improving the accuracy of our findings to date, there is a wealth of knowledge that can be had by imbedding this model into more complex regimes. The inclusion of a stratified atmosphere, realistic treatments of viscous effects and thermal conduction, incorporated acceleration to track the kinematics of the flux tube – these are just a few examples of the ways that this model could be expanded. But, ultimately the models presented here all suffer from the same limitation; they are constrained to a lower dimensional space. Eventually these findings will have to be considered in light of three dimensional models, which will, in turn, be interpreted by way of analogy to these simpler, two dimensional structures.

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APPENDIX A

DERIVATIONS

A.1. The Governing Equations of Magnetohydrodynamics

In order to study the dynamics of highly ionized fluids we require a model that accounts for the combined evolution of the fluid with the electric and magnetic fields – a hybrid theory that is commonly referred to as Magnetohydrodynamics or MHD. In this formulation the plasma is treated as an ideal gas with the usual approximations made for developing a bulk fluid velocity and temperature from the moments of the particle velocity distribution. As with any fluid theory, there are extensions to non-Maxwellian scenarios and departures from Local Thermodynamic Equilibrium (LTE vs. non-LTE), but we will not go into these. The combined fluid / field system can be described by the following macroscopic state variables:

$$\mathbf{u}_i : \quad \text{bulk ion velocity;} \quad (\text{A.1})$$

$$\mathbf{u}_e : \quad \text{bulk electron velocity;} \quad (\text{A.2})$$

$$n_i : \quad \text{ion number density;} \quad (\text{A.3})$$

$$n_e : \quad \text{electron number density;} \quad (\text{A.4})$$

$$T_i : \quad \text{ion temperature;} \quad (\text{A.5})$$

$$T_e : \quad \text{electron temperature;} \quad (\text{A.6})$$

$$\mathbf{B} : \quad \text{magnetic Field;} \quad (\text{A.7})$$

$$\mathbf{E} : \quad \text{electric Field;} \quad (\text{A.8})$$

$$\mathbf{j}_e : \quad \text{electric current density;} \quad (\text{A.9})$$

$$\sigma_e : \quad \text{electric charge density.} \quad (\text{A.10})$$

From these quantities we may begin to reduce the complexity of the system. In the solar corona we typically consider that the plasma is fully ionized and that the dominant species is hydrogen, with every free electron associated with an ionization state of hydrogen. Thus, the ion number density is taken to be equal to the electron number density and the mass density can be approximated as

$$\rho = n_e m_i, \quad (\text{A.11})$$

where m_i is the typical ion mass (per free electron). This could just as well be given in terms of the ion number density, but quantum resonance spectroscopy typically deals with electron energy level transitions, so the electron number density is more closely related to observations. The bulk velocity is taken to be equivalent to the ion velocity whereas the electric current is related to the difference in the two velocities;

$$\mathbf{u} = \frac{\mathbf{u}_i n_i m_i + \mathbf{u}_e n_e m_e}{n_i m_i + n_e m_e} \approx \mathbf{u}_i, \quad (\text{A.12})$$

$$\mathbf{j}_e = q_e (n_e \mathbf{u}_e - n_i \mathbf{u}_i), \quad (\text{A.13})$$

with m_e and m_i indicating the electron and ion masses and q_e representing the electron charge.

In terms of these variables the governing equation for the fluid velocity is

$$\rho \partial_t \mathbf{u} = -\rho (\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla p + \nabla \cdot \underline{\underline{\sigma}} + \mathbf{f}, \quad (\text{A.14})$$

where $\underline{\sigma}$ is the viscous stress and γ is the adiabatic index. The volumetric force density, $\mathbf{f}$ can incorporate any external force, including gravitational effects, but the critical term that distinguishes MHD from hydrodynamics is the Lorentz force,

$$\mathbf{f}_L = \frac{1}{c} \mathbf{j}_e \times \mathbf{B}. \quad (\text{A.15})$$

Here the contribution from the electric field has been neglected owing to the fact that the plasma is assumed to be a very good conductor and therefore to remain charge-neutral, i.e. $\sigma_e = q_e(n_e - n_i) = 0$. The current density is related to the magnetic field through Ampère's law, which can be simplified by neglecting the contribution from the displacement current;

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j}_e + \frac{1}{c} \partial_t \mathbf{E} \approx \frac{4\pi}{c} \mathbf{j}_e. \quad (\text{A.16})$$

This approximation is valid provided that the characteristic time scale for the system is longer than $\tau_E = uL/c^2$, where L and u are the typical speed and length and c is the speed of light.¹ In this case, the Lorentz force is totally independent of the electric field so that

$$\mathbf{f}_L = \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B}. \quad (\text{A.17})$$

With the inclusion of the Lorentz force (and later with a modification to the viscous stress) the dependence of the fluid on the macroscopic electric and magnetic

¹This claim can be verified by reconstructing the electric field once the complete system of MHD equations has been formulated.

fields is complete. However, we must still address the evolution of the fields. We begin with Faraday's law,

$$\partial_t \mathbf{B} = -c \nabla \times \mathbf{E}, \quad (\text{A.18})$$

and Ohm's law,

$$\mathbf{j}_e = \sigma \mathbf{E}', \quad (\text{A.19})$$

where σ is the electrical conductivity, not to be confused with the volumetric charge density, σ_e , and $\mathbf{E}'$ is the electric field in the center of mass frame, i.e. the frame that is co-moving with the fluid. The electric field in the lab-frame is related through

$$\mathbf{E}' = \mathbf{E} + \frac{1}{c} \mathbf{u} \times \mathbf{B}, \quad (\text{A.20})$$

which can be found through the general form of the Lorentz transformation in the limit that $u/c \ll 1$. Combining these expressions we can eliminate the electric field entirely so that

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \eta \nabla \times (\nabla \times \mathbf{B}), \quad (\text{A.21})$$

where we have again invoked the argument of a small displacement current to relate the current density to the magnetic field and we have defined the magnetic diffusivity, given by

$$\eta = \frac{c^2}{4\pi\sigma}. \quad (\text{A.22})$$

In principle these replacements close the system. The continuity and energy equation for the fluid are unchanged from hydrodynamics. The fluid is coupled to the

magnetic field through the momentum equation and the effects of the electric field have been accounted for by proxy. The magnetic field, in turn, evolves under the influence of the fluid and the current density and electric field may be recovered at any time from a combination of Equations (A.19) and (A.20). Of course, variations of these expressions can be found and it is often useful to write them in conservative form or to track certain quantities explicitly. However, for our purposes the entire system of governing equations is as follows:

$$\partial_t \rho = -\nabla \cdot \rho \mathbf{u}, \quad (\text{A.23})$$

$$\partial_t p = -\nabla \cdot p \mathbf{u} + (\gamma - 1)p \nabla \cdot \mathbf{u} + (\gamma - 1)\dot{Q}, \quad (\text{A.24})$$

$$\rho \partial_t \mathbf{u} = -\rho(\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla p + \nabla \cdot \underline{\underline{\sigma}} + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} + \mathbf{f}_{ext}, \quad (\text{A.25})$$

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{b}) - \eta \nabla \times (\nabla \times \mathbf{B}), \quad (\text{A.26})$$

where $\dot{Q}$ represents the volumetric heating due to viscosity and thermal conduction and $\mathbf{f}_{ext}$ is left as a general place holder for external forces, such as gravitational effects or an accelerated reference frame. Note that both the viscous stress and viscous heating will ultimately depend on the rate of strain, which will in turn depend on the anisotropy that is introduced by the magnetic field. Thus, while the response of the fluid to the magnetic field is only made explicit through $\mathbf{f}_L$, both the viscosity and volumetric heating terms will have to be treated carefully in the context of MHD.

A.2. Conformal Representation of Magnetohydrodynamics

In order to avoid distracting the reader we have foregone careful derivation of the fluid equations as shown in §3.2. We provide those derivations here for completeness. As a reminder, f and g are coordinates analogous to x and y , with their gradients serving the role of basis vectors. We will assume and take considerable advantage of the fact that

$$\nabla^2 f = \nabla^2 g = \nabla f \cdot \nabla g = 0 \quad (\text{A.27})$$

and further employ our definition of the metric for this coordinate system:

$$m = |\nabla f| = |\nabla g|. \quad (\text{A.28})$$

The cartesian limit can be recovered at any time by setting $m \rightarrow 1$, but leaving it general affords us an understanding of all possible coordinate choices provided that the basis vectors are orthogonal and that their magnitudes are equal.

Ignoring gravitational effects and external forces the four vector equations of ideal MHD are:

$$\partial_t \rho = -\nabla \cdot \rho \mathbf{u}, \quad (\text{A.29})$$

$$\partial_t p = -(\nabla \cdot p \mathbf{u} + (\gamma - 1)p \nabla \cdot \mathbf{u}) + (\gamma - 1)(\dot{Q}_T + \dot{Q}_\sigma), \quad (\text{A.30})$$

$$\partial_t \mathbf{u} = -(\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{1}{\rho} \left(\frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} + \nabla \cdot \underline{\underline{g}} - \nabla p \right), \quad (\text{A.31})$$

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}), \quad (\text{A.32})$$

where ρ , p , $\mathbf{u}$, and T are the density, pressure, velocity and temperature of the fluid, γ is the adiabatic index, $\mathbf{B}$ is the magnetic field. $\dot{Q}_T$ and $\dot{Q}_\sigma$ are the heating due to thermal conduction and viscosity, respectively and $\underline{\sigma}$ is the viscous stress.

We shall treat each of the above equations by projecting their vector components onto the conformal basis vectors and then reducing each expression until only scalar entries remain. The resulting expressions are as follows

A.2.1. Continuity

As the continuity equation is already a scalar we need only expand it in terms of f and g .

$$\begin{aligned}
\partial_t \rho &= \nabla \cdot \rho \mathbf{u} \\
&= \nabla \cdot \rho u_f \nabla f + \nabla \cdot \rho u_g \nabla g \\
&= \nabla f \cdot \rho u_f + \nabla g \cdot \nabla \rho u_g + \rho u_f \nabla^2 f + \rho u_g \nabla^2 g \\
&= (\nabla f \cdot \nabla f) \partial_f \rho u_f + (\nabla g \cdot \nabla g) \partial_g \rho u_g \\
&= m^2 (\partial_f \rho u_f + \partial_g \rho u_g).
\end{aligned} \tag{A.33}$$

A.2.2. Induction

Beginning with the traditional form of the induction equation we can substitute our definition for the magnetic field in terms of the flux function, which leads to

$$\nabla \times (\mathbf{u} \times (\hat{\mathbf{z}} \times \nabla \varphi)) = \partial_t \hat{\mathbf{z}} \times \nabla \varphi = -\nabla \times \hat{\mathbf{z}} \partial_t \varphi. \tag{A.34}$$

Uncurling this expression and selecting the $\hat{\mathbf{z}}$ component we are then left with

$$\begin{aligned}
\partial_t \varphi &= -\hat{\mathbf{z}} \cdot [\mathbf{u} \times (\hat{\mathbf{z}} \times \nabla \varphi)] \\
&= -\mathbf{u} \cdot [\hat{\mathbf{z}} \times (\nabla \varphi \times \hat{\mathbf{z}})] \\
&= -\mathbf{u} \cdot [(\hat{\mathbf{z}} \cdot \hat{\mathbf{z}}) \nabla \varphi - (\hat{\mathbf{z}} \cdot \nabla \varphi) \hat{\mathbf{z}}] \\
&= -\mathbf{u} \cdot \nabla \varphi.
\end{aligned} \tag{A.35}$$

From there the component form is found analogously to the continuity equation as shown in §A.2.1:

$$\begin{aligned}
\partial_t \varphi &= -(u_g \nabla g + u_f \nabla f) \cdot (\nabla g \partial_g + \nabla f \partial_f) \varphi \\
&= -(\nabla g \cdot \nabla g + \nabla f \cdot \nabla f)(u_g \partial_g + u_f \partial_f) \varphi \\
&= -m^2(u_g \partial_g + u_f \partial_f) \varphi.
\end{aligned} \tag{A.36}$$

A.2.3. Energy

For the full form of the energy equation we will lump the viscous heating and thermal conduction into a general volumetric heating term,

$$\dot{Q} = \dot{Q}_T + \dot{Q}_\sigma. \tag{A.37}$$

The time rate of change of the pressure, less these volumetric heating terms, is

$$\begin{aligned}
\partial_t p - (\gamma - 1) \dot{Q} &= -(\nabla \cdot p \mathbf{u} + (\gamma - 1) p \nabla \cdot \mathbf{u}) \\
&= -\nabla \cdot (p u_f \nabla f + p u_g \nabla g) - (\gamma - 1) p \nabla \cdot (u_f \nabla f + u_g \nabla g)
\end{aligned}$$

$$\begin{aligned}
&= -\nabla f \cdot \nabla p u_f - \nabla g \cdot \nabla p u_g - (\gamma - 1)p(\nabla f \cdot \nabla u_f + \nabla g \cdot \nabla u_g) \\
&= -m^2 (\partial_f p u_f + \partial_g p u_g - (\gamma - 1)p(\partial_f u_f + \partial_g u_g)). \tag{A.38}
\end{aligned}$$

Then, with $\dot{Q}_T$ and $\dot{Q}_\sigma$ as given in §A.3 and §A.4, the prescription for updating the pressure is

$$\partial_t p = -m^2 ((\partial_f p u_f + \partial_g p u_g) + (\gamma - 1)p(\partial_f u_f + \partial_g u_g)) + (\gamma - 1)\dot{Q}. \tag{A.39}$$

A.2.4. Momentum

The momentum equation is significantly more involved than the other three and writing it out in component form requires consideration of each term independently. Restating and explicitly naming the first, second and third terms on the Left-Hand Side (LHS) and Right-Hand Side (RHS), respectively, the full momentum equation is

$$\overbrace{\rho \partial_t \mathbf{u}}^{\text{LHS:1}} + \overbrace{\rho(\mathbf{u} \cdot \nabla) \mathbf{u}}^{\text{LHS:2}} = \overbrace{\frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}}^{\text{RHS:1}} - \overbrace{\nabla p}^{\text{RHS:2}} + \overbrace{\nabla \cdot \underline{\underline{\sigma}}}_{\text{RHS:3}}. \tag{A.40}$$

For now we will forego the details of $\nabla \cdot \underline{\underline{\sigma}}$, which are treated separately in §A.4.

To get scalar equations for the remaining terms we'll form their inner products with $(1/\rho)\nabla g$ and $(1/\rho)\nabla f$, starting with the g component of the first term of the LHS.

We shall omit treatment of the f components which can be found by permutation of f with g .

LHS:1 - Explicit Rate of Change – The g component of the explicit time dependence of $\mathbf{u}$ is given by

$$\begin{aligned}
 \frac{1}{\rho} \nabla g \cdot \rho \partial_t \mathbf{u} &= \nabla g \cdot \partial_t (u_g \nabla g + u_f \nabla f) \\
 &= \nabla g \cdot \nabla g \partial_t u_g \\
 &= m^2 \partial_t u_g.
 \end{aligned} \tag{A.41}$$

LHS:2 - Advective Derivative – The g component of the advective portion of the derivative is

$$\begin{aligned}
 \frac{1}{\rho} \nabla g \cdot \rho (\mathbf{u} \cdot \nabla) \mathbf{u} &= \nabla g \cdot [(\mathbf{u} \cdot \nabla) \mathbf{u}] \\
 &= \nabla g \cdot \left[\frac{1}{2} \nabla (\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u}) \right] \\
 &= \nabla g \cdot \frac{1}{2} (\nabla g \partial_g [\mathbf{u} \cdot \mathbf{u}] + \nabla f \partial_f [\mathbf{u} \cdot \mathbf{u}]) - \nabla g \cdot [\mathbf{u} \times (\nabla \times \mathbf{u})] \\
 &= \frac{1}{2} m^2 \partial_g (\mathbf{u} \cdot \mathbf{u}) - (\nabla \times \mathbf{u}) \cdot (\nabla g \times \mathbf{u})
 \end{aligned} \tag{A.42}$$

We'll treat the second term saperately.

$$\begin{aligned}
 \nabla \times \mathbf{u} &= \nabla \times (\nabla g u_g + \nabla f u_f) \\
 &= (\nabla \times \nabla g) u_g + (\nabla \times \nabla f) u_f + \nabla g \times \nabla u_g + \nabla f \times \nabla u_f \\
 &= \nabla g \times \nabla u_g + \nabla f \times \nabla u_f \\
 &= \nabla g \times (\nabla g \partial_g u_g + \nabla f \partial_f u_g) + \nabla f \times (\nabla g \partial_g u_f + \nabla f \partial_f u_f) \\
 &= (\nabla f \times \nabla g) (\partial_g u_f - \partial_f u_g) \\
 &= \hat{\mathbf{z}} m^2 (\partial_g u_f - \partial_f u_g)
 \end{aligned} \tag{A.43}$$

and

$$\begin{aligned}
\nabla g \times \mathbf{u} &= \nabla g \times (u_f \nabla f + u_g \nabla g) \\
&= \nabla g \times u_f \nabla f \\
&= u_f \nabla g \times \nabla f \\
&= \hat{\mathbf{z}} u_f m^2
\end{aligned} \tag{A.44}$$

So, picking up from where we left off and substituting $m^2(u_f^2 + u_g^2)$ for $\mathbf{u} \cdot \mathbf{u}$ we have

$$\begin{aligned}
\frac{1}{\rho} \nabla g \cdot \rho(\mathbf{u} \cdot \nabla) \mathbf{u} &= \frac{1}{2} m^2 \partial_g (m^2(u_g^2 + u_f^2)) - m^4 u_f (\partial_g u_f - \partial_f u_g) \\
&= m^4 (u_g \partial_g u_g + u_f \partial_f u_g) + m^3 (u_g^2 + u_f^2) \partial_g m.
\end{aligned} \tag{A.45}$$

RHS:1 - Lorentz Force – The g component of the Lorentz force is fairly straight

forward with the substitution $\hat{\mathbf{z}} \times \nabla \varphi = -\nabla \times \hat{\mathbf{z}} \varphi$.

$$\begin{aligned}
\frac{1}{\rho} \nabla g \cdot ((\nabla \times \mathbf{B}) \times \mathbf{B}) &= \frac{1}{\rho} \nabla g \cdot ((\nabla \times (-\nabla \times \hat{\mathbf{z}} \varphi)) \times (\hat{\mathbf{z}} \times \nabla \varphi)) \\
&= \frac{1}{\rho} \nabla g \cdot (-(\nabla(\nabla \cdot \hat{\mathbf{z}} \varphi) - \nabla^2 \hat{\mathbf{z}} \varphi) \times (\hat{\mathbf{z}} \times \nabla \varphi)) \\
&= \frac{1}{\rho} \nabla g \cdot ((\nabla^2 \hat{\mathbf{z}} \varphi) \times (\hat{\mathbf{z}} \times \nabla \varphi)) \\
&= \frac{1}{\rho} \nabla g \cdot ((\hat{\mathbf{z}} \nabla^2 \varphi \cdot \nabla \varphi) \hat{\mathbf{z}} - (\hat{\mathbf{z}} \nabla^2 \varphi \cdot \hat{\mathbf{z}}) \nabla \varphi) \\
&= \frac{1}{\rho} \nabla g \cdot (-\nabla^2 \varphi \nabla \varphi) \\
&= -\frac{1}{\rho} (\nabla g \cdot \nabla \varphi) \nabla \cdot (\nabla g \partial_g + \nabla f \partial_f) \varphi
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{\rho}(\nabla g \cdot \nabla g \partial_g \varphi)(\nabla g \cdot \nabla g \partial_g^2 + \nabla f \cdot \nabla f \partial_f^2)\varphi \\
&= -\frac{1}{\rho}m^4(\partial_g \varphi)(\partial_g^2 + \partial_f^2)\varphi.
\end{aligned} \tag{A.46}$$

RHS:2 - Pressure – The g component of the pressure term is trivial.

$$\begin{aligned}
\frac{1}{\rho}\nabla g \cdot \nabla p &= \frac{1}{\rho}\nabla g \cdot \nabla g \partial_g p \\
&= \frac{1}{\rho}m^2 \partial_g p.
\end{aligned} \tag{A.47}$$

The complete expressions are then given by

$$\begin{aligned}
\partial_t u_f &= -m^2(u_f \partial_f + u_g \partial_g)u_f - m(u_f^2 + u_g^2)\partial_f m - \frac{1}{\rho}\partial_f p \\
&\quad + \frac{1}{\rho}m^2(\partial_f \varphi)(\partial_f^2 + \partial_g^2)\varphi + \frac{1}{m^2\rho}\nabla f \cdot (\nabla \cdot \underline{a})
\end{aligned} \tag{A.48}$$

and

$$\begin{aligned}
\partial_t u_g &= -m^2(u_f \partial_f + u_g \partial_g)u_g - m(u_f^2 + u_g^2)\partial_g m - \frac{1}{\rho}\partial_g p \\
&\quad + \frac{1}{\rho}m^2(\partial_g \varphi)(\partial_f^2 + \partial_g^2)\varphi + \frac{1}{m^2\rho}\nabla g \cdot (\nabla \cdot \underline{a})
\end{aligned} \tag{A.49}$$

with $\nabla g \cdot (\nabla \cdot \underline{a})$ and $\nabla f \cdot (\nabla \cdot \underline{a})$ given below.

A.3. More on Thermal Conduction

Assuming that the plasma behaves as an ideal gas, the temperature is related to the gas pressure through the ideal gas law,

$$p = \frac{k_B}{m_i} \rho T, \tag{A.50}$$

where p is the pressure, T is the temperature, ρ is the density, k_B is Boltzman's constant, and m_i is the typical ion mass. The volumetric heating due to thermal conduction is given by

$$\dot{Q}_T = \nabla \cdot (\underline{k} \cdot \nabla T), \quad (\text{A.51})$$

where $\underline{k}$ is the thermal conductivity tensor. In the case if a highly ionized, hot plasma

$$\underline{k} = k_{\parallel} \hat{\mathbf{b}} \hat{\mathbf{b}}, \quad (\text{A.52})$$

with the field-aligned conductivity given by $k_{\parallel} = k_0 T^{5/2}$ (Fitzpatrick, 2015).

As a simplification, consider the case where the thermal conduction is isotropic.

The above expression then reduces to

$$\begin{aligned} \dot{Q}_T^{(\text{isotropic})} &= \nabla \cdot (\underline{k}^{(\text{isotropic})} \cdot \nabla T) \\ &\approx k \nabla^2 T \\ &\approx km^2 (\partial_f^2 + \partial_g^2) T. \end{aligned} \quad (\text{A.53})$$

Even in a fluid dominated regime (high- β) the anisotropy of the magnetic field precludes this isotropic treatment for high temperature plasma. However, it can be a useful computational tool under certain circumstances.

The more correct treatment is given by

$$\begin{aligned} \nabla \cdot (\underline{k} \cdot \nabla T) &= \nabla \cdot (k_{\parallel} \hat{\mathbf{b}} \hat{\mathbf{b}} \cdot \nabla T) \\ &= \nabla \cdot (k_{\parallel} \hat{\mathbf{b}} (\hat{\mathbf{b}} \cdot \nabla) T). \end{aligned} \quad (\text{A.54})$$

Here, again, we may simplify matters to inform intuition or in the case of a slowly varying magnetic field. Taking $\hat{\mathbf{b}}$ to be specially uniform we have

$$\begin{aligned}\dot{Q}_T^{(\text{WKB})} &\approx k_{||}(\hat{\mathbf{b}} \cdot \nabla) \hat{\mathbf{b}} \cdot \nabla T \\ &\approx k_{||} \partial_{||}^2 T\end{aligned}\tag{A.55}$$

where, $\partial_{||} \equiv \hat{\mathbf{b}} \cdot \nabla$ is the directional derivative along the magnetic field. Again, this limit is of some utility but, since the magnetic field is generally varying, the full treatment is more complex.

Recalling our definition of the magnetic flux function we have

$$\begin{aligned}\dot{Q}_T &= k_{||} \nabla \cdot (\hat{\mathbf{b}} \hat{\mathbf{b}} \cdot \nabla T) \\ &= k_{||} \nabla \cdot \left(\frac{\hat{\mathbf{z}} \times \nabla \varphi}{|\nabla \varphi|^2} ((\hat{\mathbf{z}} \times \nabla \varphi) \cdot \nabla) T \right) \\ &= k_{||} \nabla \cdot \left(\frac{\hat{\mathbf{z}} \times \nabla \varphi}{B^2} \hat{\mathbf{z}} \cdot (\nabla \varphi \times (\nabla f \partial_f + \nabla g \partial_g) T) \right) \\ &= k_{||} \nabla \cdot \left(\frac{\hat{\mathbf{z}} \times \nabla \varphi}{B^2} \hat{\mathbf{z}} \cdot ((\nabla f \partial_f + \nabla g \partial_g) \varphi \times (\nabla f \partial_f + \nabla g \partial_g) T) \right) \\ &= k_{||} \nabla \cdot \left(\frac{\hat{\mathbf{z}} \times \nabla \varphi}{B^2} \hat{\mathbf{z}} \cdot (\nabla f \times \nabla g \partial_f \varphi \partial_g + \nabla g \times \nabla f \partial_g \varphi \partial_f) T \right) \\ &= k_{||} \nabla \cdot \frac{\hat{\mathbf{z}} \times \nabla \varphi}{B^2} m^2 (\partial_f \varphi \partial_g - \partial_g \varphi \partial_f) T \\ &= k_{||} (\hat{\mathbf{z}} \times \nabla \varphi) \cdot \nabla \frac{m^2}{B^2} (\partial_f \varphi \partial_g - \partial_g \varphi \partial_f) T \\ &= k_{||} m^2 (\partial_f \varphi \partial_g - \partial_g \varphi \partial_f) \frac{m^2}{B^2} (\partial_f \varphi \partial_g - \partial_g \varphi \partial_f) T\end{aligned}\tag{A.56}$$

where we have used the fact that $\nabla \cdot \mathbf{B} = \nabla \cdot (\hat{\mathbf{z}} \times \nabla \varphi) = 0$ and written the magnitude of the magnetic field in short hand as $B^2 = |\mathbf{B}|^2 = |\nabla \varphi|^2 = m^2 ((\partial_f \varphi)^2 + (\partial_g \varphi)^2)$.

Note that for a uniform magnetic field we can freely set $\partial_g \varphi = 0$ and $m^2(\partial_f \varphi)^2 = B^2$, where g is taken as the coordinate along the magnetic field. In this case, $\partial_g^2 T \equiv \partial_{||}^2 T$ and the above expression reduces to identically to that of Equation (A.55). With this prescription for the thermal contribution to the volumetric heating, Equation (A.39) is nearly complete. Only $\dot{Q}_\sigma$ remains to be considered and for that we must take a closer look at the viscosity.

A.4. More on Bulk Viscosity

The viscous stress tensor is given by the contraction of the fourth rank viscosity tensor, $\mu^{(4)}$, with the second rank rate of strain tensor, $\underline{\underline{W}}$. The dominant contribution to the viscosity tensor is

$$\mu^{(4)} \approx \mu_{||} \left(\frac{3}{2} \hat{\mathbf{b}} \hat{\mathbf{b}} - \frac{1}{2} \mathbb{I} \right) \hat{\mathbf{b}} \hat{\mathbf{b}}, \quad (\text{A.57})$$

with $\mu_{||} = \mu_0 T^{5/2}$ (Fitzpatrick, 2015), while the rate of strain tensor is generally written

$$\underline{\underline{W}} = \nabla \mathbf{u} + (\nabla \mathbf{u})^T - \frac{2}{3} (\nabla \cdot \mathbf{u}) \mathbb{I}. \quad (\text{A.58})$$

Written out, the viscous stress is then

$$\underline{\underline{\sigma}} \approx \mu_{||}^{(4)} : \underline{\underline{W}} = \mu_{||} \left(\frac{3}{2} \hat{\mathbf{b}} \hat{\mathbf{b}} - \frac{1}{2} \mathbb{I} \right) \hat{\mathbf{b}} \hat{\mathbf{b}} : \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T - \frac{2}{3} (\nabla \cdot \mathbf{u}) \mathbb{I} \right). \quad (\text{A.59})$$

From this expression, it is convenient to define the $\hat{\mathbf{b}}\hat{\mathbf{b}}$ component of the rate of strain tensor as

$$W_{bb} = \hat{\mathbf{b}}\hat{\mathbf{b}} : \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T - \frac{2}{3}(\nabla \cdot \mathbf{u})\mathbb{I} \right). \quad (\text{A.60})$$

This term will appear in our treatment of both the viscous heating and the bulk viscosity so we'll take a moment now to work it out.

A.4.1. Projected Rate of Strain

We begin by simplifying the definition of W_{bb} with the fact that $\hat{\mathbf{b}}\hat{\mathbf{b}} : (\nabla \mathbf{u})^T = \hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla \mathbf{u}$, which follows from the symmetry of $\hat{\mathbf{b}}\hat{\mathbf{b}}$ without any explicit knowledge of $\nabla \mathbf{u}$. We can then write

$$\begin{aligned} W_{bb} &= \hat{\mathbf{b}}\hat{\mathbf{b}} : \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T - \frac{2}{3}(\nabla \cdot \mathbf{u})\mathbb{I} \right) \\ &= \hat{\mathbf{b}} \cdot \left(2\hat{\mathbf{b}} \cdot (\nabla \mathbf{u}) - \frac{2}{3}(\nabla \cdot \mathbf{u})\hat{\mathbf{b}} \right) \\ &= \underbrace{2\hat{\mathbf{b}} \cdot (\hat{\mathbf{b}} \cdot (\nabla \mathbf{u}))}_{W_{bb}^{(1)}} - \underbrace{\frac{2}{3}(\nabla \cdot \mathbf{u})}_{W_{bb}^{(2)}}. \end{aligned} \quad (\text{A.61})$$

Working in pieces, with our previous definition for $\mathbf{B}$ and the coordinates f and g , the first term is

$$\begin{aligned} W_{bb}^{(1)} &= 2\hat{\mathbf{b}} \cdot (\hat{\mathbf{b}} \cdot \nabla \mathbf{u}) \\ &= \frac{2\hat{\mathbf{b}}}{B} \cdot \{ (\nabla g \partial_f \varphi - \nabla f \partial_g \varphi) \cdot (\nabla g \partial_g + \nabla f \partial_f) (\nabla g u_g + \nabla f u_f) \} \\ &= \frac{2(\nabla f \partial_g \varphi - \nabla g \partial_f \varphi)}{B^2} \cdot \{ (m^2 \partial_g \varphi \partial_f - m^2 \partial_f \varphi \partial_g) (\nabla g u_g + \nabla f u_f) \} \end{aligned}$$

$$\begin{aligned}
&= \frac{2m^2}{B^2} \begin{pmatrix} +\partial_g \varphi \nabla f \cdot \partial_g \varphi \partial_f (\nabla g u_g + \nabla f u_f) \\ -\partial_g \varphi \nabla f \cdot \partial_f \varphi \partial_g (\nabla g u_g + \nabla f u_f) \\ -\partial_f \varphi \nabla g \cdot \partial_g \varphi \partial_f (\nabla g u_g + \nabla f u_f) \\ +\partial_f \varphi \nabla g \cdot \partial_f \varphi \partial_g (\nabla g u_g + \nabla f u_f) \end{pmatrix} \\
&= \frac{2m^2}{B^2} \begin{pmatrix} +(\partial_g \varphi)^2 \nabla f \cdot (\nabla g \partial_f u_g + u_g \partial_f \nabla g + \nabla f \partial_f u_f + u_f \partial_f \nabla f) \\ -\partial_g \varphi \partial_f \varphi \nabla f \cdot (\nabla g \partial_g u_g + u_g \partial_g \nabla g + \nabla f \partial_g u_f + u_f \partial_g \nabla f) \\ -\partial_f \varphi \partial_g \varphi \nabla g \cdot (\nabla g \partial_f u_g + u_g \partial_f \nabla g + \nabla f \partial_f u_f + u_f \partial_f \nabla f) \\ +(\partial_f \varphi)^2 \nabla g \cdot (\nabla f \partial_g u_f + u_f \partial_g \nabla f + \nabla g \partial_g u_g + u_g \partial_g \nabla g) \end{pmatrix} \\
&= \frac{2m^2}{B^2} \begin{pmatrix} +(\partial_g \varphi)^2 \left(u_g \nabla f \cdot \partial_f \nabla g + m^2 \partial_f u_f + \frac{1}{2} u_f \partial_f m^2 \right) \\ -\partial_g \varphi \partial_f \varphi \left(u_g \nabla f \cdot \partial_g \nabla g + m^2 \partial_g u_f + \frac{1}{2} u_f \partial_g m^2 \right) \\ -\partial_g \varphi \partial_f \varphi \left(u_f \nabla g \cdot \partial_f \nabla f + m^2 \partial_f u_g + \frac{1}{2} u_g \partial_f m^2 \right) \\ +(\partial_f \varphi)^2 \left(u_f \nabla g \cdot \partial_g \nabla f + m^2 \partial_g u_g + \frac{1}{2} u_g \partial_g m^2 \right) \end{pmatrix} \\
&= \frac{2m^2}{B^2} \begin{pmatrix} +(\partial_g \varphi)^2 (m \partial_f b u_f + u_g \nabla \cdot \nabla g - u_g \nabla g \cdot \partial_g \nabla g) \\ -\partial_g \varphi \partial_f \varphi (m \partial_g b u_f + u_g \partial_g \nabla f \cdot \nabla g - u_g \nabla g \cdot \partial_g \nabla f) \\ -\partial_g \varphi \partial_f \varphi (m \partial_f b u_g + u_f \partial_f \nabla g \cdot \nabla f - u_f \nabla f \cdot \partial_f \nabla g) \\ +(\partial_f \varphi)^2 (m \partial_g b u_g + u_f \nabla \cdot \nabla f - u_f \nabla f \cdot \partial_f \nabla f) \end{pmatrix} \\
&= \frac{2m^2}{B^2} \begin{pmatrix} +(\partial_g \varphi)^2 \left(m \partial_f m u_f + u_g \nabla^2 g - \frac{1}{2} u_g \partial_g m^2 \right) \\ -\partial_g \varphi \partial_f \varphi (m \partial_g m u_f - u_g \nabla \cdot \nabla f + u_g \nabla f \cdot \partial_f \nabla f) \\ -\partial_g \varphi \partial_f \varphi (m \partial_f m u_g - u_f \nabla \cdot \nabla g + u_f \nabla g \cdot \partial_g \nabla g) \\ +(\partial_f \varphi)^2 \left(m \partial_g m u_g + u_f \nabla^2 f - \frac{1}{2} u_f \partial_f m^2 \right) \end{pmatrix} \\
&= \frac{2m^2}{B^2} \begin{pmatrix} +(\partial_g \varphi)^2 (m \partial_f m u_f - m u_g \partial_g m) \\ -\partial_g \varphi \partial_f \varphi (m \partial_g m u_f + m u_g \partial_f m) \\ -\partial_g \varphi \partial_f \varphi (m \partial_f m u_g + m u_f \partial_g m) \\ +(\partial_f \varphi)^2 (m \partial_g m u_g - m u_f \partial_f m) \end{pmatrix}. \tag{A.62}
\end{aligned}$$

The second part of W_{bb} is much simpler:

$$\frac{2}{3} \nabla \cdot \mathbf{u} = \frac{2}{3} m^2 (\partial_g u_g + \partial_f u_f). \tag{A.63}$$

So, combining these and simplifying a little bit, the $\hat{\mathbf{b}}\hat{\mathbf{b}}$ component of the rate of strain is

$$W_{bb} = \frac{2m^2}{|\nabla \varphi|^2} \begin{pmatrix} +(\partial_g \varphi)^2 (m \partial_f m u_f - m u_g \partial_g m) \\ +(\partial_f \varphi)^2 (m \partial_g m u_g - m u_f \partial_f m) \\ -\partial_g \varphi \partial_f \varphi (\partial_f m^2 u_g + \partial_g m^2 u_f) \end{pmatrix} - \frac{2}{3} m^2 (\partial_g u_g + \partial_f u_f). \tag{A.64}$$

This expression will come up again so we'll refer back to it.

A.4.2. Viscous Heating

The viscous contribution to the volumetric heating is

$$\dot{Q}_\sigma = \underline{\underline{\sigma}} : \nabla \mathbf{u}, \quad (\text{A.65})$$

where $\nabla \mathbf{u}$ is the gradient of the fluid velocity with matrix elements $\partial_i u_j$. Making use of our definition for the $\hat{\mathbf{b}}\hat{\mathbf{b}}$ component of $\underline{\underline{W}}$ as W_{bb} from Equation A.64 we may simplify $\dot{Q}_\sigma$ considerably:

$$\begin{aligned} \dot{Q}_\sigma &= \underline{\underline{\sigma}} : \nabla \mathbf{u} \\ &= \mu_{||} \left(\frac{3}{2} \hat{\mathbf{b}}\hat{\mathbf{b}} - \frac{1}{2} \mathbb{I} \right) W_{bb} : \nabla \mathbf{u} \\ &= \frac{\mu_{||}}{2} W_{bb} (3\hat{\mathbf{b}}\hat{\mathbf{b}} - \mathbb{I}) : \nabla \mathbf{u} \\ &= \frac{\mu_{||}}{2} W_{bb} (3\hat{\mathbf{b}} \cdot ((\hat{\mathbf{b}} \cdot \nabla) \mathbf{u}) - \nabla \cdot \mathbf{u}) \\ &= \frac{\mu_{||}}{2} W_{bb} (3(\hat{\mathbf{b}} \cdot \nabla)(\hat{\mathbf{b}} \cdot \mathbf{u}) - 3\mathbf{u} \cdot ((\hat{\mathbf{b}} \cdot \nabla) \hat{\mathbf{b}}) - \nabla \cdot \mathbf{u}). \end{aligned} \quad (\text{A.66})$$

These can be worked out individually. The last is

$$\nabla \cdot \mathbf{u} = m^2 (\partial_f u_f + \partial_g u_g) \quad (\text{A.67})$$

and the first is

$$\begin{aligned} (\hat{\mathbf{b}} \cdot \nabla)(\hat{\mathbf{b}} \cdot \mathbf{u}) &= \frac{1}{B} ((\hat{\mathbf{z}} \times \nabla \varphi) \cdot \nabla) \left(\frac{1}{B} (\hat{\mathbf{z}} \times \nabla \varphi) \cdot \mathbf{u} \right) \\ &= \frac{1}{B} m^2 (\partial_f \varphi \partial_g - \partial_g \varphi \partial_f) \frac{1}{B} \hat{\mathbf{z}} \cdot (\nabla \varphi \times \mathbf{u}) \\ &= \frac{m}{B} (\partial_f \varphi \partial_g - \partial_g \varphi \partial_f) \frac{m}{B} (u_g \partial_f \varphi - u_f \partial_g \varphi). \end{aligned} \quad (\text{A.68})$$

The middle term in $\dot{Q}_\sigma$ is somewhat more involved.

$$\begin{aligned}
\mathbf{u} \cdot ((\hat{\mathbf{b}} \cdot \nabla) \hat{\mathbf{b}}) &= \mathbf{u} \cdot \left(\frac{1}{2} \nabla (\hat{\mathbf{b}} \cdot \hat{\mathbf{b}}) - \hat{\mathbf{b}} \times (\nabla \times \hat{\mathbf{b}}) \right) \\
&= (\nabla \times \hat{\mathbf{b}}) \cdot (\hat{\mathbf{b}} \times \mathbf{u}) \\
&= \hat{\mathbf{b}} \cdot (\nabla \times (\hat{\mathbf{b}} \times \mathbf{u})) + \nabla \cdot (\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \times \mathbf{u})) \\
&= \left(\begin{aligned} &\hat{\mathbf{b}} \cdot (\nabla \times (\frac{\dot{z}}{B} (u_f \nabla f \cdot \nabla \varphi + u_g \nabla g \cdot \nabla \varphi))) \\ &+ \nabla \cdot (\hat{\mathbf{b}} \times (\frac{\dot{z}}{B} (u_f \nabla f \cdot \nabla \varphi + u_g \nabla g \cdot \nabla \varphi))) \end{aligned} \right) \\
&= \left(\begin{aligned} &\hat{\mathbf{b}} \cdot (\nabla \times (\frac{m^2 \dot{z}}{B} (u_f \partial_f \varphi + u_g \partial_g \varphi))) \\ &+ \nabla \cdot (\hat{\mathbf{b}} \times (\frac{m^2 \dot{z}}{B} (u_f \partial_f \varphi + u_g \partial_g \varphi))) \end{aligned} \right) \\
&= \left(\begin{aligned} &-\frac{\nabla \varphi}{B} \cdot (\nabla \frac{m^2}{B} (u_f \partial_f \varphi + u_g \partial_g \varphi)) \\ &+ \nabla \cdot (\frac{\nabla \varphi}{B} \frac{m^2}{B} (u_f \partial_f \varphi + u_g \partial_g \varphi)) \end{aligned} \right) \\
&= \left(\left(\nabla \cdot \frac{\nabla \varphi}{B} \right) \frac{m^2}{B} (u_f \partial_f \varphi + u_g \partial_g \varphi) \right) \\
&= \left(\frac{\nabla^2 \varphi}{B} - \nabla \varphi \cdot \frac{\nabla B}{B^2} \right) \left(\frac{m^2}{B} (u_f \partial_f \varphi + u_g \partial_g \varphi) \right) \\
&= \frac{m^4 (u_f \partial_f + u_g \partial_g) \varphi}{B^2} \left((\partial_f^2 + \partial_g^2) \varphi - \frac{(\partial_f \varphi \partial_f + \partial_g \varphi \partial_g) B}{B} \right), \quad (\text{A.69})
\end{aligned}$$

where we have used the fact that $\nabla(\hat{\mathbf{b}} \cdot \hat{\mathbf{b}}) = \nabla(1) = 0$.

Combining these terms, the full form of the viscous heating is

$$\dot{Q}_\sigma = \frac{3}{2} \mu_{||} W_{bb} \left(\begin{aligned} &\frac{m}{B} (\partial_f \varphi \partial_g - \partial_g \varphi \partial_f) \frac{m}{B} (u_g \partial_f \varphi - u_f \partial_g \varphi) - \frac{1}{3} m^2 (\partial_f u_f + \partial_g u_g) \\ &- \frac{m^4 (u_f \partial_f + u_g \partial_g) \varphi}{B^2} \left((\partial_f^2 + \partial_g^2) \varphi - \frac{(\partial_f \varphi \partial_f + \partial_g \varphi \partial_g) B}{B} \right) \end{aligned} \right). \quad (\text{A.70})$$

This term, with $\dot{Q}_T$ from the previous section completes the volumetric heating, $\dot{Q}$ and thus closes the momentum equation in §A.2.3.

A.4.3. Divergence of the Viscous Stress

For the sake of simplicity and owing to the need for unphysically large viscosity values in order to maintain the stability in numerical simulations, it is often useful to consider an isotropic viscous stress, namely,

$$\begin{aligned}\nabla \cdot \underline{\underline{\sigma}}^{(\text{isotropic})} &= \nabla \cdot \mu \nabla \mathbf{u} \\ &\approx \mu \nabla \cdot \nabla \mathbf{u}.\end{aligned}\tag{A.71}$$

Then, projecting against our coordinate basis we have

$$\frac{1}{\rho} \nabla g \cdot (\mu \nabla^2 \mathbf{u}) \approx \frac{1}{\rho} m^3 (\partial_f^2 + \partial_g^2) m u_g,\tag{A.72}$$

with the f component found similarly or through simple permutation of f with g .

While this formulation can be useful, the correct treatment is found from the divergence of Equation (A.59). Specifically,

$$\begin{aligned}\nabla \cdot \underline{\underline{\sigma}} &= \nabla \cdot (\mu_4 : \underline{\underline{W}}) \\ &\approx \mu_{||} \nabla \cdot \left(\left(\frac{3}{2} \hat{\mathbf{b}} \hat{\mathbf{b}} - \frac{1}{2} \mathbb{I} \right) W_{bb} \right).\end{aligned}\tag{A.73}$$

We can begin to make progress by projecting the divergence of $\underline{\underline{\sigma}}$ onto one of our coordinate basis vectors and then employing the product rule so that

$$\nabla g \cdot (\nabla \cdot \underline{\underline{\sigma}}) = \overbrace{\nabla \cdot (\nabla g \cdot \underline{\underline{\sigma}})}^{\text{TS:1}} - \overbrace{\underline{\underline{\sigma}} : \nabla \nabla g}^{\text{TS:2}},\tag{A.74}$$

where we have identified two important terms, TS:1 and TS:2, one of which relates to the behavior of the fluid, and the other to the curvature of the coordinate system.

Working these out separately, the first term is

$$\begin{aligned}
\text{TS : 1} &= \nabla \cdot \left(\nabla g \cdot \underline{\underline{g}} \right) \\
&= \nabla \cdot \left(\nabla g \cdot \mu^{(0)} \left(\frac{3}{2} \hat{\mathbf{b}} \hat{\mathbf{b}} - \frac{1}{2} \mathbb{I} \right) W_{bb} \right) \\
&= \mu^{(0)} \nabla \cdot \left(\nabla g \cdot \left(\frac{3}{2} \frac{1}{|\nabla \varphi|} (\nabla g \partial_f \varphi - \nabla f \partial_g \varphi) \hat{\mathbf{b}} - \frac{1}{2} \mathbb{I} \right) W_{bb} \right) \\
&= \mu^{(0)} \nabla \cdot \left(\left(\frac{3}{2} \frac{m^2 \partial_f \varphi}{|\nabla \varphi|^2} (\nabla g \partial_f \varphi - \nabla f \partial_g \varphi) - \frac{1}{2} \nabla g \right) W_{bb} \right) \\
&= \mu^{(0)} \nabla \cdot \left(\left(\frac{3}{2} \frac{m^2 (\partial_f \varphi)^2}{|\nabla \varphi|^2} - \frac{1}{2} \right) W_{bb} \nabla g - \frac{3}{2} \frac{m^2 \partial_f \varphi \partial_g \varphi}{|\nabla \varphi|^2} W_{bb} \nabla f \right) \\
&= \mu^{(0)} \left(\nabla g \cdot \nabla \left(\left(\frac{3}{2} \frac{m^2 (\partial_f \varphi)^2}{|\nabla \varphi|^2} - \frac{1}{2} \right) W_{bb} \right) - \nabla f \cdot \nabla \left(\frac{3}{2} \frac{m^2 \partial_f \varphi \partial_g \varphi}{|\nabla \varphi|^2} W_{bb} \right) \right) \\
&= \mu^{(0)} \left(m^2 \partial_g \left(\left(\frac{3}{2} \frac{m^2 (\partial_f \varphi)^2}{|\nabla \varphi|^2} - \frac{1}{2} \right) W_{bb} \right) - m^2 \partial_f \left(\frac{3}{2} \frac{m^2 \partial_f \varphi \partial_g \varphi}{|\nabla \varphi|^2} W_{bb} \right) \right) \quad (\text{A.75})
\end{aligned}$$

and the second term is

$$\begin{aligned}
\text{TS : 2} &= \underline{\underline{g}} : \nabla \nabla g \\
&= \mu^{(0)} \left(\frac{3}{2} \hat{\mathbf{b}} \hat{\mathbf{b}} - \frac{1}{2} \mathbb{I} \right) W_{bb} : \nabla \nabla g \\
&= \mu^{(0)} W_{bb} \left(\frac{3}{2} \hat{\mathbf{b}} \hat{\mathbf{b}} - \frac{1}{2} \mathbb{I} \right) : \nabla \nabla g \\
&= \mu^{(0)} W_{bb} \left(\frac{3}{2} \hat{\mathbf{b}} \hat{\mathbf{b}} : \nabla \nabla g - \frac{1}{2} \mathbb{I} : \nabla \nabla g \right) \\
&= \mu^{(0)} W_{bb} \left(\frac{3}{2} \hat{\mathbf{b}} \frac{\nabla g \partial_f \varphi - \nabla f \partial_g \varphi}{|\nabla \varphi|} : (\nabla f \partial_f + \nabla g \partial_g) \nabla g - \frac{1}{2} \nabla^2 g \right) \\
&= \frac{3}{2} \mu^{(0)} W_{bb} \frac{m^2}{|\nabla \varphi|^2} (\nabla g \partial_f \varphi - \nabla f \partial_g \varphi) \cdot (\partial_f \varphi \partial_g \nabla g - \partial_g \varphi \partial_f \nabla g). \quad (\text{A.76})
\end{aligned}$$

The final contraction in the previous expression must be worked out carefully.

$$(\nabla g \partial_f \varphi - \nabla f \partial_g \varphi) \cdot (\partial_f \varphi \partial_g \nabla g - \partial_g \varphi \partial_f \nabla g)$$

$$\begin{aligned}
&= \begin{pmatrix} (\partial_g \varphi)^2 \nabla f \cdot \partial_f \nabla g + (\partial_f \varphi)^2 \nabla g \cdot \partial_g \nabla g \\ -\partial_g \varphi \partial_f \varphi (\nabla f \cdot \partial_g \nabla g + \nabla g \cdot \partial_f \nabla g) \end{pmatrix} \\
&= \begin{pmatrix} (\partial_g \varphi)^2 (\nabla^2 g - \nabla g \cdot \partial_g \nabla g) + (\partial_f \varphi)^2 \nabla g \cdot \partial_g \nabla g \\ -\partial_g \varphi \partial_f \varphi (\nabla f \cdot \partial_g \nabla g + \nabla g \cdot \partial_f \nabla g) \end{pmatrix} \\
&= \frac{1}{2} \left((\partial_f \varphi)^2 - (\partial_g \varphi)^2 \right) \partial_g m^2 - \partial_g \varphi \partial_f \varphi (-\nabla g \cdot \partial_g \nabla f + \frac{1}{2} \partial_f m^2) \\
&= \frac{1}{2} \left((\partial_f \varphi)^2 - (\partial_g \varphi)^2 \right) \partial_g m^2 - \partial_g \varphi \partial_f \varphi \partial_f m^2. \tag{A.77}
\end{aligned}$$

So, ultimately,

$$\underline{g} : \nabla \nabla g = \frac{3}{4} \mu^{(0)} \frac{m^2 W_{bb}}{|\nabla \varphi|^2} \left(((\partial_f \varphi)^2 - (\partial_g \varphi)^2) \partial_g m^2 - 2 \partial_g \varphi \partial_f \varphi \partial_f m^2 \right) \tag{A.78}$$

and, therefore,

$$\nabla g \cdot (\nabla \cdot \underline{g}) = \mu^{(0)} m^2 \begin{pmatrix} \partial_g \left(\left(\frac{3}{2} \frac{m^2 (\partial_f \varphi)^2}{|\nabla \varphi|^2} - \frac{1}{2} \right) W_{bb} \right) - \partial_f \frac{3}{2} \frac{m^2 \partial_f \varphi \partial_g \varphi}{|\nabla \varphi|^2} W_{bb} \\ -\frac{3}{4} \frac{W_{bb}}{|\nabla \varphi|^2} \left(((\partial_f \varphi)^2 - (\partial_g \varphi)^2) \partial_g m^2 - 2 \partial_g \varphi \partial_f \varphi \partial_f m^2 \right) \end{pmatrix}, \tag{A.79}$$

with W_{bb} defined as in Equation (A.64). In numerical applications it is important to recognize that W_{bb} contains only first order derivatives and so, if evaluated and then utilized in the above expression, will lead to checkerboard-type instabilities. Thus, the above expression must be expanded explicitly to accomodate higher order finite differencing schemes in order to maintain numerical stability.

Uniform Magnetic Field – As a brief aside let's consider the case of a uniform magnetic field. Without loss of generality we may take $\hat{\mathbf{b}}^{(\text{uniform})} = \hat{\mathbf{y}}$ with $\hat{\mathbf{x}}$ taken to be perpendicular to the field. We'll begin by working out the rate of strain projected

onto the magnetic field, W_{yy} :

$$\begin{aligned}
W_{yy} &= 2\hat{\mathbf{y}} \cdot (\hat{\mathbf{y}} \cdot \nabla \mathbf{u}) - \frac{2}{3} \nabla \cdot \mathbf{u} \\
&= 2\hat{\mathbf{y}} \cdot (\hat{\mathbf{y}} \cdot (\hat{\mathbf{y}} \partial_y + \hat{\mathbf{x}} \partial_x)(\hat{\mathbf{y}} u_b + \hat{\mathbf{x}} u_p)) - \frac{2}{3} \partial_x u_x + \partial_y u_y \\
&= 2\hat{\mathbf{y}} \cdot (\hat{\mathbf{y}} \cdot (\hat{\mathbf{y}} \hat{\mathbf{y}} \partial_y u_b + \hat{\mathbf{x}} \hat{\mathbf{x}} \partial_x u_p + \hat{\mathbf{y}} \hat{\mathbf{x}} \partial_y u_p + \hat{\mathbf{x}} \hat{\mathbf{y}} \partial_x u_b)) - \frac{2}{3} \partial_x u_x + \partial_y u_y \\
&= 2\hat{\mathbf{y}} \cdot (\hat{\mathbf{y}} \partial_y u_b + \hat{\mathbf{x}} \partial_y u_p) - \frac{2}{3} \partial_x u_x + \partial_y u_y \\
&= 2\partial_y u_y - \frac{2}{3} (\partial_x u_x + \partial_y u_y) \\
&= \frac{4}{3} \partial_y u_y - \frac{2}{3} \partial_x u_x.
\end{aligned} \tag{A.80}$$

The $\hat{\mathbf{y}}$ component of $\nabla \cdot \underline{\underline{\sigma}}$ is then

$$\hat{\mathbf{y}} \cdot (\nabla \cdot \underline{\underline{\sigma}}) = \nabla \cdot (\hat{\mathbf{y}} \cdot \underline{\underline{\sigma}}) - \underline{\underline{\sigma}} : \nabla \hat{\mathbf{y}} \tag{A.81}$$

$$= \mu_{||} \nabla \cdot \left(\hat{\mathbf{y}} \cdot \left(\frac{3}{2} \hat{\mathbf{y}} \hat{\mathbf{y}} - \frac{1}{2} \mathbb{I} \right) W_{yy} \right) \tag{A.82}$$

$$= \mu_{||} \nabla \cdot (\hat{\mathbf{y}} W_{yy}) \tag{A.83}$$

$$= \mu_{||} \partial_y \left(\frac{4}{3} \partial_y u_y - \frac{2}{3} \partial_x u_x \right) \tag{A.84}$$

$$= \mu_{||} \left(\frac{4}{3} \partial_y^2 u_y - \frac{2}{3} \partial_y \partial_x u_x \right) \tag{A.85}$$

and the $\hat{\mathbf{x}}$ component is similarly found to be

$$\hat{\mathbf{x}} \cdot (\nabla \cdot \underline{\underline{\sigma}}) = \mu_{||} \nabla \cdot \left(\hat{\mathbf{x}} \cdot \left(\frac{3}{2} \hat{\mathbf{y}} \hat{\mathbf{y}} - \frac{1}{2} \mathbb{I} \right) W_{yy} \right) \tag{A.86}$$

$$= \mu_{||} \nabla \cdot \left(-\frac{1}{2} \hat{\mathbf{p}} W_{yy} \right) \tag{A.87}$$

$$= \mu_{||} \frac{1}{2} \partial_x \left(\frac{2}{3} \partial_x u_x - \frac{4}{3} \partial_y u_y \right) \tag{A.88}$$

$$= \mu_{||} \left(\frac{1}{3} \partial_x^2 u_x - \frac{2}{3} \partial_y \partial_x u_y \right). \tag{A.89}$$

Interestingly, the effect of viscosity along the direction of the magnetic field has a contribution from the velocity perpendicular to the field. And the viscosity perpendicular to the field, which many assume to vanish, is, in fact, nearly identical to the field-aligned viscosity, differing only by the leading coefficient to the diagonal term, which is smaller by a factor of 4.

A.5. One Dimensional Departures from Bernoulli's Equation

In Chapter 2 the formation of shocks and rarefaction waves was used to stitch together De Laval's solution for smooth, slowly varying one dimensional flow. Here we will develop the governing equations for these features by relaxing first the inviscid assumption, which allows for violation of Bernoulli's equation in the case of steep gradients, and second by relaxing the time-independent assumption, which allows for self-similarity in the reduced variable, $\phi = l/tC_s$.

A.5.1. Zero-Beta Acoustic Shocks

The formation of steep gradients, which was implied by the infinite acceleration along ill-behaved solutions in Chapter 2, immediately suggests that we consider situations where a very large pressure gradient leads to the formation of shocks which provide a means to connect unrelated inviscid flows. Since we derived the De Laval nozzle equation under the assumption that the viscosity was everywhere small, we

must return to that derivation and consider how it may be modified to account for viscous forces.

Recall that the equation of interest is

$$\rho \mathbf{u} \cdot [(\mathbf{u} \cdot \nabla) \mathbf{u}] + \mathbf{u} \cdot \nabla p = \mathbf{u} \cdot \left[\nabla \cdot \underline{\underline{g}} \right], \quad (\text{A.90})$$

which is valid provided that the density and momentum density are non-varying in time. If these parameters vary rapidly over a short length scale, then variations on the global scale become irrelevant and it is in the rest frame of the shock that the above equation must hold. The following analysis therefor applies in a frame where the shock is at rest.

In arriving at De Laval's solution we divided through by ρ which allowed the left side to be antidifferentiated. However, if we leave the viscous term as it is we will find that it becomes a perfect differential which can then be evaluated far from the viscous region and will therefor contribute nothing. Treating $\mathbf{u}$ as $u\hat{\mathbf{u}}$ we can use the product rule and we find that

$$\begin{aligned} \mathbf{u} \cdot \left[\nabla \cdot \underline{\underline{g}} \right] &= \rho u \hat{\mathbf{u}} [(\mathbf{u} \cdot \nabla) \mathbf{u}] + \mathbf{u} \cdot \nabla p \\ &= \hat{\mathbf{u}} \cdot [(\mathbf{u} \cdot \nabla) \rho u \mathbf{u}] + (\mathbf{u} \cdot \nabla) p \\ &= (\mathbf{u} \cdot \nabla) \rho u^2 - (\mathbf{u} \cdot \nabla) p - \rho u \mathbf{u} [(\mathbf{u} \cdot \nabla) \hat{\mathbf{u}}], \end{aligned} \quad (\text{A.91})$$

where the final term on the right hand side is associated with the curvature of the fluid trajectory.

In general this term does not vanish as it accounts for the force exerted on the fluid by the walls of the duct through which it flows. In order for this term to vanish we must choose an integration volume which is much shorter than the length scale for variations in the flux tube cross section. That is, the shock width (which scales with the kinematic velocity divided by the shock speed) must be small compared to the magnetic field scale size (which scales with r , the distance from the origin). The shock speed through the fluid is typically of order the sound speed so the relevant constraint is

$$Re = \frac{RC_s}{\mu} \gg 1. \quad (\text{A.92})$$

That is, for high fluid Reynolds numbers, the contribution from the curvature will vanish.

In this case, we may neglect the final term and, again recalling that $\hat{u} \cdot \nabla \rightarrow \partial_l$, we find that

$$\partial_l[\rho u^2 + p] = \hat{u} \cdot \left[\nabla \cdot \underline{\underline{\sigma}} \right]. \quad (\text{A.93})$$

Here we note that the primary contribution to the viscous stress comes from $\mu \partial_l u$ so that the right hand side becomes $\partial_l \mu \partial_l u$ which may be integrated exactly to give,

$$\left| [\rho u^2 + p] \right| = |[\mu \partial_l u]|, \quad (\text{A.94})$$

where the double bracket indicates the difference in the enclosed quantity evaluated at the limits of integration. Provided that the viscous stress goes to zero outside of

the shocked interior region the right hand side is again zero and we are left with,

$$\rho u^2 + p = \kappa_p, \quad (\text{A.95})$$

which is a constant. Here we must stress that this is only valid for short length scales where α may be taken to be constant. Replacing p with $C_s^2 \rho$ and dividing through by $C_s \rho u$, which is constant for uniform α we find that

$$M'_1 + \frac{1}{M'_1} = M'_2 + \frac{1}{M'_2}, \quad (\text{A.96})$$

implying that

$$M'_2 = \left\{ M'_1, \frac{1}{M'_1} \right\}, \quad (\text{A.97})$$

where $M'_{\{1,2\}}$ is the mach number of the inflowing and outflowing plasma in the rest frame of the shock. The first solution is consistent with Bernoulli's equation for constant α . The second solution provides a means for constructing a discontinuous flow which conserves mass, energy and momentum in the rest frame of the discontinuity, which is neither a source nor a sink.

A.5.2. Rarefaction Waves

In a rarefaction wave, because the inflowing fluid travels slower than the outflowing fluid, the interior structure necessarily grows in time so that we can no longer expect the interior to conserve mass, energy, or momentum. Instead, we seek a time dependent solution to the fluid equations of the previous section with the caveat that

it should have a single stationary point where the density and velocity are constant in time so that this point can be used as a boundary to match the inflow conditions. We shall perform our analysis in the rest frame of this stationary point, which, as we shall see, is the location of the leading edge of the rarefaction wave. As in the case of Bernoulli's equation we shall assume that the viscosity is negligible so that the momentum equation involves no second order derivatives of $\mathbf{u}$. The rarefaction wave will begin as a discontinuity but will quickly grow and so it is assumed that for all time, after the initial configuration, the following equations are satisfied:

$$\partial_t \rho + \nabla \rho \mathbf{u} = 0; \quad (\text{A.98})$$

$$\rho \partial_t \mathbf{u} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} + C_s^2 \nabla \rho = 0, \quad (\text{A.99})$$

where we have again used the isothermal condition to replace p in terms of ρ .

Dividing everywhere by ρ and antidifferentiating we may express these 1-D fluid equations as

$$0 = \partial_t K + \partial_\chi M_{rw} + M_{rw} \partial_\chi K \quad (\text{A.100})$$

and

$$0 = \partial_t M_{rw} + M_{rw} \partial_\chi M_{rw} + \partial_\chi K, \quad (\text{A.101})$$

where $M_{rw} = u_{rw}/C_s$ is the Mach number of the fluid in the rest frame of the stationary point, $K = \ln(\rho)$, and $\chi = l/C_s$ is the distance from the stationary point, normalized to the acoustic speed.

Since every term involves exactly one derivative we see that this system invites a self-similar solution and will reduce to an ODE with one variable of the form $\phi = \chi/t + \chi_0 \ln(\chi) + t_0 \ln(t)$. Since we desire a solution that is well defined everywhere in χ , we immediately set χ_0 to zero. Further, we must set t_0 to zero so that there will be at least one point where M and K are time independent. Thus, the characteristic variable of this system is

$$\phi = \chi/t. \quad (\text{A.102})$$

Eq. (A.102) can be used to transform Eqs. (A.100) and (A.101) into a pair of coupled ordinary differential equations in ϕ ;

$$-\frac{\phi}{t} \partial_\phi K + \frac{1}{t} \partial_\phi M_{rw} + M_{rw} \frac{1}{t} \partial_\phi K = 0, \quad (\text{A.103})$$

$$-\frac{\phi}{t} \partial_\phi M_{rw} + M_{rw} \frac{1}{t} \partial_\phi M_{rw} + \frac{1}{t} \partial_\phi K = 0. \quad (\text{A.104})$$

Dividing through by t and substituting $\partial_\phi K$ from the second equation, we find that

$$(M_{rw} - \phi)(\phi \partial_\phi M_{rw} - M_{rw} \partial_\phi M_{rw}) + \partial_\phi M_{rw} = 0 \quad (\text{A.105})$$

and thus

$$M_{rw} = \phi \pm 1, \quad (\text{A.106})$$

where the two roots allow for a choice of coordinate direction; and we will choose the upper with no loss of generality.

Recalling our definition of ϕ , we can then write the Mach number and density of the interior solution as

$$M_{rw}(\chi, t) = \frac{\chi}{t} + 1, \quad (\text{A.107})$$

$$\rho(\chi, t) = \rho_0 e^{-\chi/t} \quad (\text{A.108})$$

for $\chi \geq 0$, with both the density and velocity being constant at $\chi = 0$. For $\chi < 0$ the velocity and density are constant and equal to the “upstream” value. Since the velocity of the stationary fluid is necessarily zero in its own reference frame, the Mach number at the leading edge of the rarefaction wave must also be zero in this frame if it is to be continuous. This then dictates that the location of the leading edge must propagate into the stationary fluid at the sound speed so that in the rest frame of the stationary fluid the Mach number and density of the interior solution are given by

$$M(z, t) = \frac{|z - z_e|}{C_s t}, \quad (\text{A.109})$$

$$\rho(z, t) = \rho_0 e^{-\frac{|z - z_e|}{C_s t}}, \quad (\text{A.110})$$

with z_e , the location of the leading edge, given by $z_e = z_0 + C_s t$, where z_0 is its initial value. Note that the behavior in the interior of the rarefaction wave is not affected by the downstream boundary condition. Rather, it is the location of the downstream boundary condition that is dictated by the fluid velocity or density at that boundary.