

Judging the relative qualities and merits of Galerkins approximate solutions to a dyamically loaded beam system
by Thomas Michael Hanson

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE in Aerospace and Mechanical Engineering
Montana State University
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Abstract:
Galerkin's method is applied to a beam structure that is forced to nonlinear behavior by a dynamic load. Nonlinearities in the system include a nonlinear stress-strain relation and consideration of geometry changes due to large deflections.

Several trial deflection shapes are assumed as approximate solutions of the problem and these trial shapes are combined in various manners in an effort to produce a better quality solution. All resulting solutions are studied in the light of three criteria that are postulated in an attempt to define the relative merits and qualities of approximate solutions.

It is concluded that although the criteria are good guidelines to finding reasonable solutions, they are not strict in defining a good quality solution.

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JUDGING THE RELATIVE QUALITIES AND MERITS OF GALERKIN'S APPROXIMATE SOLUTIONS TO A DYNAMICALLY LOADED BEAM SYSTEM

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A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree

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MASTER OF SCIENCE

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Bozeman, Montana

December, 1970

ACKNOWLEDGMENT

The author is indebted to the United States Army Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland, for providing the financial aid necessary for the completion of this work.

The continuing and valuable assistance of Dr. D. O. Blackketter is greatly appreciated.

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ABSTRACT

Galerkin's method is applied to a beam structure that is forced to nonlinear behavior by a dynamic load. Nonlinearities in the system include a nonlinear stress-strain relation and consideration of geometry changes due to large deflections.

Several trial deflection shapes are assumed as approximate solutions of the problem and these trial shapes are combined in various manners in an effort to produce a better quality solution. All resulting solutions are studied in the light of three criteria that are postulated in an attempt to define the relative merits and qualities of approximate solutions.

It is concluded that although the criteria are good guidelines to finding reasonable solutions, they are not strict in defining a good quality solution.

CHAPTER I: INTRODUCTION

Earlier works concerned with problems of nonlinear deflections have usually only considered single component systems such as beams, plates, or shells. It is of interest to apply techniques used in solving these single component problems to problems involving a system of such components.

Solution techniques very often used on nonlinear problems are finite difference methods [7,8][1] and Galerkin's method.[1,2,4,5] A popular method for both linear and nonlinear structures is the finite element method.[10,17] Comparisons of the finite difference and Galerkin methods have been made [3] but criteria for judging the relative qualities of solutions as given by Galerkin's method have only been hypothesized. Convergence proofs for certain linear problems are sometimes stated, [9] however, little is known about the convergence for nonlinear problems. If one is confronted with two or more approximate solutions to a problem, it is of particular interest to be able to discuss the relative merits and qualities of each solution.

Previous investigations of nonlinear, dynamically loaded systems have included a study of a blast loaded cantilever beam [1] and of a blast loaded plate.[2] Large deflections of thin elastic beams have been studied [3] and Galerkin methods have been applied

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[1] Numbers in brackets refer to literature consulted.

to nonlinear problems of finite elastodynamics.[11] Considerable work has been done utilizing rigid-plastic theory which is valid only for moderately large deflections.[4,5,6]

This paper studies a simple structural system of beams forced to large displacements by a blast load. Galerkin's modified method is used to reduce the nonlinear partial differential equations of motion to ordinary differential equations which can be solved by numerical methods. Nonlinearities in the system arise from geometrical changes of large deflections and consideration of a nonlinear stress-strain relation. The effects of various one-term approximate solutions are studied and the deflection shapes are combined in an attempt to improve the resulting answer. Methods of combining the assumed deflection shapes are investigated and compared. All results are studied in an effort to define the relative qualities of the various trial solutions.

The purpose of this paper will be to test the hypothesis that the quality of an assumed solution to a nonlinear problem solved by Galerkin's technique can be judged in part by a combination of three criteria. These criteria are:

Criterion 1: The quality of an assumed solution can be measured by the magnitude of the absolute area under the equation residual curve. This is a measure of the "closeness" of the equation residual to zero as implied by Finlayson and Scriven in their dis-

cussion of weighted residual methods.[9]

Criterion 2: The larger the resulting deflection of the system, the better the assumed solution. This is suggested by Anderson [11] in that an assumed solution to a linear system, other than the exact solution will result in an increased system stiffness and therefore smaller deflections of the system. This is stating that forcing the deflection of a system in a shape other than the true deflection shape will require more energy than forcing the deflection in the exact shape.

Criterion 3: The more uniform the distribution of the equation residual about zero, the better the assumed solution. Koszuta[1] states that a good assumed solution will exhibit a more uniform distribution over a greater portion of the solution interval.

Trial solutions to the beam system are studied and discussed in considerstion of the above criteria.

CHAPTER II: SYSTEM MODEL FORMULATION

2.1 The System

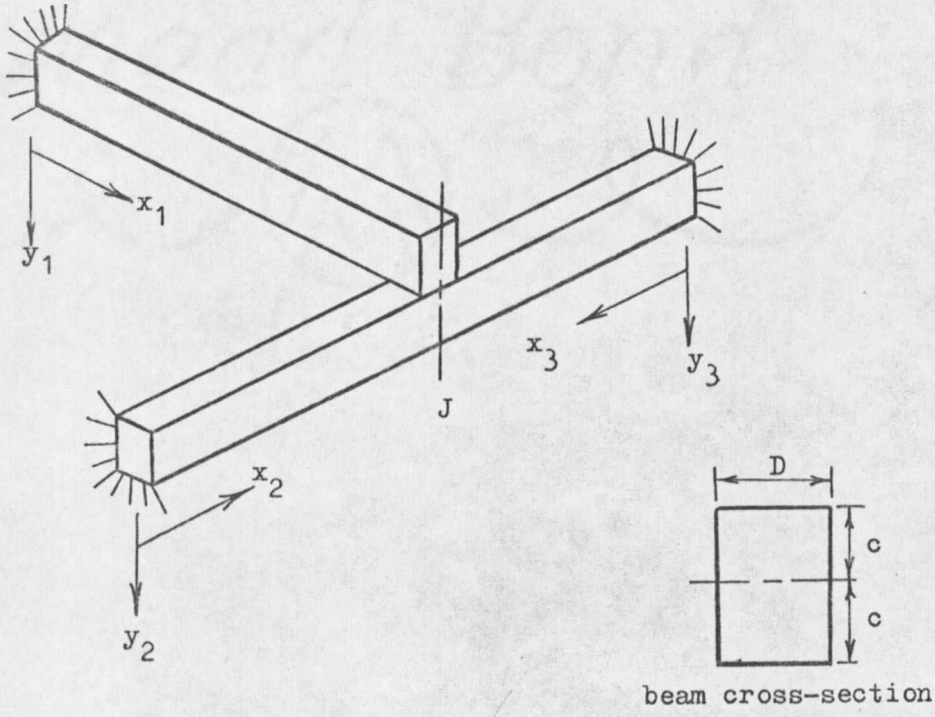


Figure 1. The Beam System

An example problem that will test the postulates given in Chapter I and that will fulfill the requirement that Galerkin's method be applicable to a beam system is shown in Figure 1. It consists of a fixed-simply supported beam resting on the midspan of a fixed-fixed beam. It will be convenient to consider the fixed-fixed beam as two seperate beams joined at J so that the coordinate directions, x_1, y_1, x_2, y_2, and x_3, y_3, each describe a different beam. Beams 1,

2 and 3 all have the same length, L, and the same cross-section. Respective material properties are identical for all the beams and are assumed homogeneous throughout each beam.

2.2 The Nonlinear Beam Equation

Consider a general beam element with loading w(t) as shown in Figure 2.

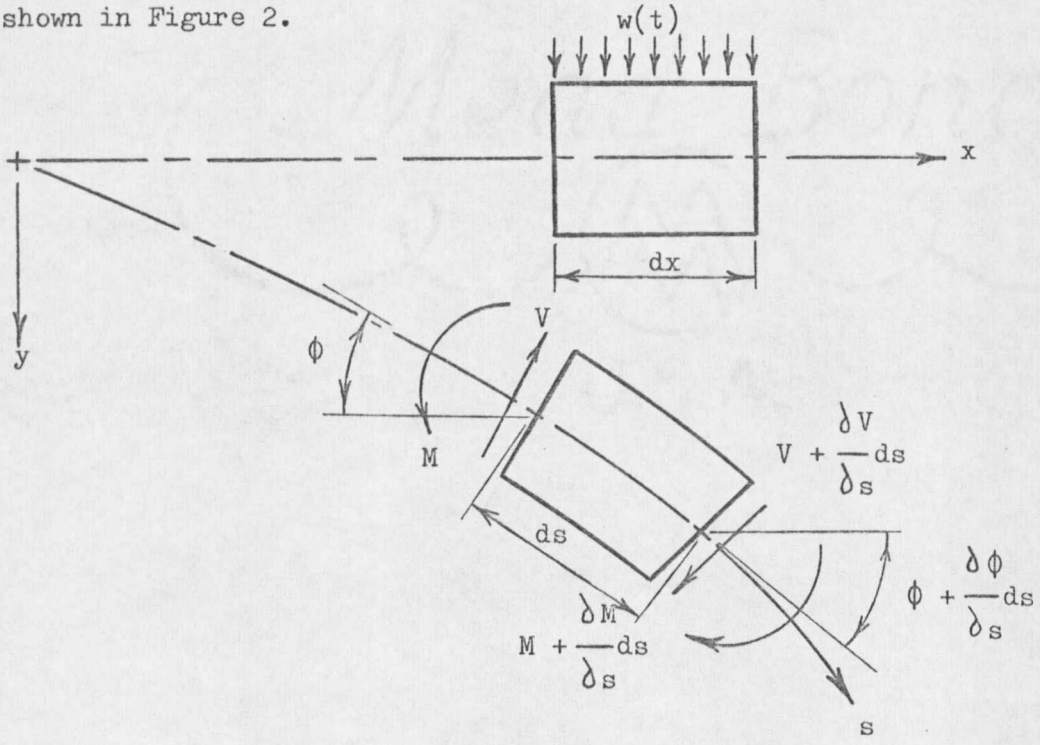


Figure 2. A General Beam Element

If it is assumed that the beam element is not in a gravitational field and if rotational inertia is neglected, then the equation of motion of the element can be written as: (See Appendix A)

$$w(t)dx - \left\{ \frac{\partial^2 M}{\partial x^2} \cos^2\phi - \frac{\partial M \partial \phi}{\partial x \partial x} \sin 2\phi \right\} dx - \gamma \frac{\partial^2 y}{\partial t^2} dx = 0 \qquad (1)$$

Equation (1) is the governing equation in terms of the element's displacement, $y(x,t)$, the internal moment, M, the rotation, ϕ, and the forcing function, $w(t)$. γ is the weight per unit length of the material.

The stress-strain relation of an actual beam material can be approximated, by inspection, with the function:

$$\sigma = \frac{1}{b} \tan^{-1}(a\epsilon) \qquad (2)$$

where ϵ is the strain and σ is the stress. The constants a and b are chosen such that the stress-strain curve approximates that of a true material.

If the neutral plane of the beam remains a neutral plane and if plane sections remain plane, then the strain in a fiber a distance z from the neutral plane can be written in terms of the change in rotation with respect to x, $\frac{\partial \phi}{\partial x}$. The internal moment is expressed in terms of the stress on a cross-section and equation (1) can be written in the following form: (Refer to Appendix A)

$$w(t) - F[y(x,t)] - \gamma \ddot{y} = 0 \qquad (3)$$

where $F[y(x,t)]$ is a nonlinear differential operator on $y(x,t)$ and

involves $y(x,t)$ and it's first four spatial derivatives.

2.3 The Boundary Conditions

Since the beam system in question is described by three fourth order, partial differential equations of the form of equation (3), it will be necessary to define twelve boundary conditions. Of these twelve conditions, six will be given by the "forced boundary conditions" and six more will arise from the "natural boundary conditions". These twelve boundary conditions can be written as:

$$\text{Displacement at the fixed ends:} \begin{cases} y_1(0,t) = 0 \\ y_2(0,t) = 0 \\ y_3(0,t) = 0 \end{cases}$$

$$\text{Slope at the fixed ends:} \begin{cases} \dfrac{\delta y_1(0,t)}{\delta x_1} = 0 \\ \dfrac{\delta y_2(0,t)}{\delta x_2} = 0 \\ \dfrac{\delta y_3(0,t)}{\delta x_3} = 0 \end{cases} \qquad (4)$$

$$\text{Displacements at J:} \begin{cases} y_1(L,t) = y_2(L,t) \\ y_2(L,t) = y_3(L,t) \end{cases}$$

$$\text{Slope at J:} \quad \frac{\delta y_2(L,t)}{\delta x_2} + \frac{\delta y_3(L,t)}{\delta x_3} = 0$$

Moments at J:

$$\begin{cases} M[y_1(L,t)] = 0 \\ M[y_2(L,t)] = M[y_3(L,t)] \end{cases}$$

Shears at J:

$$V[y_1(L,t)] + V[y_2(L,t)] + V[y_3(L,t)] = 0$$

$M[y_i(L,t)]$ and $V[y_i(L,t)]$ are nonlinear differential operators on the $y_i(x,t)$ and represent the nonlinear expressions of moment and shear respectively. They are discussed in detail in Appendix B.

2.4 Definition of the Forcing Function

The forcing function is taken as an approximation to an actual bomb blast curve [1,10] and is shown in Figure 4.

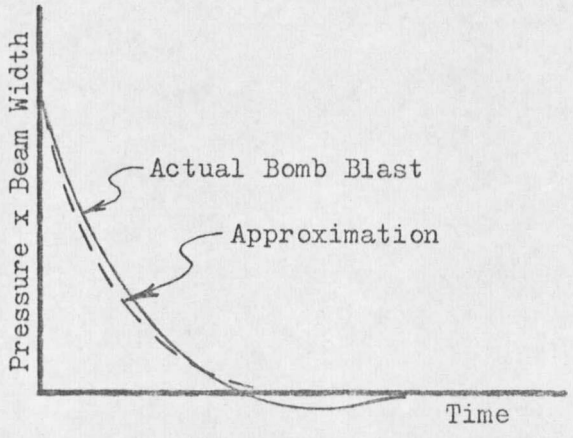


Figure 3. The Forcing Function

The approximation to the load on the beam is given by:

$$w(t) = \alpha e^{-\beta t} \tag{5}$$

where α was chosen to be of sufficient magnitude to insure deflection of the system such that a portion of the material was strained into the nonlinear range of the stress-strain relation and that geometric relations became nonlinear. The time decay constant, β, was chosen on the basis of experimentation [10] and previous usage by Koszuta.[1] The blast source is assumed far from the beam system such that the pressure distribution is uniform over the system.

CHAPTER III: THE SOLUTION METHOD

3.1 Modified Galerkin's Method

The Galerkin method is a form of the general weighted residual method as described by Finlayson and Scriven.[9] These methods can be used to rapidly yield approximate solutions to difficult linear and nonlinear problems.

In particular, if one is confronted with a system whose governing equation is of the form:

$$L[y(x,t)] + \frac{\delta^2 y(x,t)}{\delta t^2} = 0 \qquad (6)$$

where $L[y(x,t)]$ is a general differential operator involving spatial derivatives of $y(x,t)$. And if the solution domain is described by the boundary conditions:

$$B[y(v,t)] = 0 \qquad v = \text{boundary values of } x \qquad (7)$$

where $B[y(v,t)]$ is a general differential operator involving spatial derivatives of $y(x,t)$. The procedure is to assume a solution of the form:

$$y^*(x,t) = q_1(t)\theta_1(x) + q_2(t)\theta_2(x) + \ldots q_n(t)\theta_n(x) \qquad (8)$$

where the $\theta_i(x)$'s are prescribed functions which satisfy all the "forced boundary conditions". The modified Galerkin method does not require the $\theta_i(x)$'s to necessarily satisfy the "natural boundary conditions". Substituting the assumed solution, equation (8), into the

governing equation (6) yields:

$$L[y^*(x,t)] + \frac{\partial^2 y^*}{\partial t^2} = R(x,t)$$

where $R(x,t)$ is the residual resulting from the fact that the approximate solution was not the exact solution.

Substituting the assumed solution into the equations of the boundary conditions, equation (7), gives:

$$B[y^*(v,t)] = E(v,t)$$

where $E(v,t)$ is the boundary residual resulting from the unsatisfied boundary conditions.

Galerkin's modified method will require the sum of the equation residual and the boundary residual to be orthogonal to the assumed functions, $\theta_i(x)$, over the solution domain. This condition can be written as:

$$\int_{\text{domain}} R(x,t)\theta_i(x)dx + E(v,t)\theta_i(v) = 0 \quad i=1,n \tag{9}$$

The result of equation (9) is n simultaneous ordinary differential equations in the time mode. They involve $q_i(t)$ and the second derivative of $q_i(t)$ with respect to time. Equation (9) states that the weighted averages of the errors in the solution are zero over the solution domain. If one had chosen the exact solution as

the assumed solution to the problem, then the equation residual, $R(x,t)$, and the boundary residual, $E(v,t)$, would both be identically zero over the entire range of solution.

3.2 Galerkin's Method Applied to the Beam System

As stated before, the beam system under study can be thought of as a three beam system as shown in Figure 1. It is a set of three homogeneous beams coupled by a common joint at J and described by the twelve boundary conditions of equations (4).

If one assumes a deflection shape, $\Theta(x_i)$, for each of the beams in Figure 1, then a one-term solution can be written:

$$\begin{aligned} &\text{a)} \quad y_1^*(x_1,t) = q(t)\Theta(x_1) \\ &\text{b)} \quad y_2^*(x_2,t) = q(t)\Theta(x_2) \\ &\text{c)} \quad y_3^*(x_3,t) = q(t)\Theta(x_3) \end{aligned} \tag{10}$$

where $\Theta(x_1)$, $\Theta(x_2)$, and $\Theta(x_3)$ are the deflection shapes for beams 1, 2, and 3 respectively and must satisfy at least the "forced boundary conditions". The function $q(t)$ is unique and describes the displacements of the beams in the time mode.

If the trial solution is substituted into the equation of motion, equation (3), the resulting residual condition consists of errors over each beam:

$$
\begin{aligned}
&\text{a)}\quad R_1(x,t) = w(t) - F[y_1^*] - \gamma \ddot{y}_1^* \\
&\text{b)}\quad R_2(x,t) = w(t) - F[y_2^*] - \gamma \ddot{y}_2^* \\
&\text{c)}\quad R_3(x,t) = w(t) - F[y_3^*] - \gamma \ddot{y}_3^*
\end{aligned}
\tag{11}
$$

If the deflection shapes do not satisfy the "natural boundary conditions" at the system joint, then errors in the moment and shear at J must be accounted for.

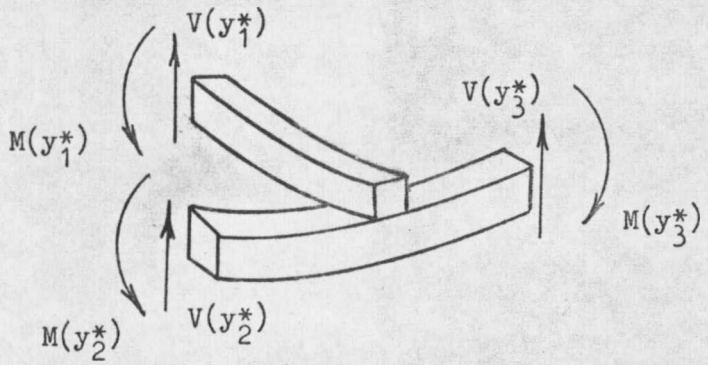


Figure 4. The Beam End Element

With the aid of Figure 3 and retaining the same sign convention as used in the equation of motion, the shear and moment errors can be written as:

$$
\begin{aligned}
&\text{a)}\quad E_m^1 = -M[y_1^*(L,t)] \\
&\text{b)}\quad E_m^{23} = M[y_2^*(L,t)] - M[y_3^*(L,t)]
\end{aligned}
\tag{12}
$$

+ c) $E_v = V[y_1^*(L,t)] + V[y_2^*(L,t)] + V[y_3^*(L,t)]$

Because of symmetry in beams 2 and 3, the same deflection shape will be assumed for these two beams and hence equation (12b) is identically zero.

Substituting the residual conditions, equations (11), and the remaining non-zero boundary condition, equation (12c), into equation (9) will yield an ordinary differential equation in the parameter q(t). This orthogonality condition may be expressed as:

$$\int_{\text{system}} [R_1\theta_1 + R_2\theta_2 + R_3\theta_3]dx + E_v\theta_{x=L} = 0 \qquad (13)$$

And the resulting equation in q(t) is:

$$\bar{A}e^{-\beta t} - G(q,\theta_1,\theta_2,\theta_3) - \bar{M}\ddot{q} = 0 \qquad (14)$$

where:

$$\bar{A} = \int_0^L \alpha(\theta_1 + \theta_2 + \theta_3)dx$$

$$G(q,\theta_1,\theta_2,\theta_3) = \int_0^L (F[q,\theta_1]\theta_1 + F[q,\theta_2]\theta_2 + F[q,\theta_3]\theta_3)dx - E_v\theta_{x=L}$$

$$\bar{M} = \int_0^L \gamma(\theta_1^2 + \theta_2^2 + \theta_3^2)dx$$

and $\theta_1 = \theta(x_1)$; $\theta_2 = \theta(x_2)$; $\theta_3 = \theta(x_3)$

Equation (14) can be written in the general form of a differential equation:

$$\dot{q}(t) = H[t,q] \tag{15}$$

where $H[t,q]$ is a nonlinear operator on $q(t)$. Equation (15) is solved with the initial conditions:

Displacements: $y_i^*(x_i,0) = q(0)\theta(x_i) = 0$

Velocities: $\dot{y}_i^*(x_i,0) = \dot{q}(0)\theta(x_i) = 0$

Accelerations: $\ddot{y}_i^*(x_i,0) = \ddot{q}(0)\theta(x_i) = 0$

Equation (15) can be reduced to two first order differential equations and then be solved by one of a number of numerical methods. In particular, Runge-Kutta methods [16,17] are valuable to propagate the first three values of $q(t)$. With the first four values of $q(t)$ known (including initial conditions), Milne's method [16,17] with corrector equations can be used to carry the solution to the first maximum value of $q(t)$. Since hysteresis losses are not accounted for in the equation of motion, the solution is not valid beyond a point where the strain in any part of the system begins to relax. This point is maximum $q(t)$.

The switch is made from the Runge-Kutta method to Milne's method in the interest of increased computer efficiency.

Note that if the assumed deflection shapes, $\theta(x_i)$, are normalized to 1 at the system joint, J, then $q(t)$ will represent the displacement of the system at J. For the cases studied, the maximum displacement always occurs at J.

With the maximum value of q(t) known, the residual equations (11 a, b, c), can be plotted as a function of the x_i at maximum q(t) and studied to determine the quality of the assumed solution.

3.3 Computational Procedure

In the interest of applying the criteria as given in Chapter I for determining the quality of a trial solution, five deflection shapes were chosen as logical trial solutions to the beam system. Each of these were tried as a solution to equation (15), then various combinations were tried and the corresponding residuals were studied.

Each of the solutions consists of a deflection shape for beam 1, and a shape for beams 2 and 3. Because of symmetry in beams 2 and 3, these last two deflection shapes are the same and hence reference will usually be made only to beam 2, realizing that beam 3 is identical.

Two of the trial deflection shapes were chosen as the resulting shapes of a similar linear system under static load configurations. Another shape was chosen as the first eigenfunction of a linear system[1] and the remaining shapes were chosen as arb-

[1] The vibrations solution to the system is a difficult problem in itself. The solution presented here is taken from the yet unwritten thesis by Daniel F. Prill of the Aerospace and Mechanical Engineering Department, Montana State University.

itrary, but logical configurations of the deflected system.

The deflection shapes are:

1. Deflection shape resulting from a uniform static load:

$$\theta_1 = (9x_1^4 - 28x_1^3L + 30x_1^2L^2)/11L^4$$

$$\theta_2 = (9x_2^4 - 40x_2^3L + 42x_2^2L^2)/11L^4$$

2. Deflection shape of the first eigenfunction:

$$\theta_1 = A_1\sin zx_1 + B_1\cos zx_1 + C_1\sinh zx_1 + D_1\cosh zx_1 + E_1$$

$$\theta_2 = A_2\sin zx_2 + B_2\cos zx_2 + C_2\sinh zx_2 + D_2\cosh zx_2 + E_2$$

3. Deflection shape due to a static point load applied at J:

$$\theta_1 = (x_1^3 - 3Lx_1^2)/-2L^3$$

$$\theta_2 = (2x_2^3 - 3Lx_2^2)-L^3$$

4. A cosine function:

$$\theta_1 = 1 - \cos\frac{\pi x_1}{2L}$$

$$\theta_2 = (\cos\frac{\pi x_2}{2L} - 1)/-2$$

5. A sine-cosine function:

$$\theta_1 = 1 - \frac{\pi x_1}{2L} + \frac{\pi^2 x_1^2}{2L^2} + \sin\frac{\pi x_1}{2L} - \cos\frac{\pi x_1}{2L}$$

$$\theta_2 = (\cos\frac{\pi x_2}{2L} - 1)/-2$$

Each of these trial solutions satisfies the "forced

boundary conditions" of the system but does not necessarily satisfy the "natural boundary conditions". Recall that the boundary shear is accounted for in equation (13).

The procedure will be to try each of the sets of deflection shapes in equations (10 a, b, c) and equation (15), solving for the resulting function q(t).

CHAPTER IV: RESULTS AND CONCLUSIONS

4.1 The Residuals

The resulting residual curves for each trial solution given in section 3.3 have been found at the time of the maximum deflection and are shown in Table 1. Listed adjacent to each residual curve are parameters which will be helpful in discussing the criteria hypothesized in Chapter I. These parameters, the maximum value of the residual, the maximum deflection of the system, the absolute area under the residual curve, and the average deflection of the system, have been nondimensionalized as follows:

$$R^{+}_{max} = \frac{R_{max}}{w(0)}$$

$$AR^{+} = \frac{AR}{w(0)L}$$

$$y^{+}(x,t) = \frac{y(x,t)}{L}$$

$$y^{+}_{avg}(x,t) = \frac{\int_{system} y(x,t)\, dx}{\int_{system} dx}$$

where R_{max} is the maximum residual, AR is the area under the residual curve and a + denotes a nondimensional quantity.

With Criterion 1 in mind, and consulting Table 1, it can be seen that trial solution 3, the static point load, best fulfills Criterion 1 with an absolute area of AR^{+}=.157. Solution 5, a

sine-cosine function, on the basis of absolute area, seems to be the poorest quality solution. ($AR^+ = .332$) Note that solutions 1 and 2, the uniform static load ($AR^+ = .255$) and the first eigenfunction ($AR^+ = .252$), have nearly the same absolute areas. Inspection of these two deflection shapes indicates they are nearly identical and therefore, similar results are not surprising.

One method of applying Criterion 2 is to assume that the maximum resulting deflection will indicate a solution closer to the exact solution. Investigation of Table 1 shows that trial solution 1 best fits this definition, ($y^+_{max} = .246$), while solution 2, ($y^+_{max} = .242$), is also apparently a good solution. Trial solution 4, a sine-cosine function, is apparently the poorest with a maximum deflection $y^+_{max} = .209$.

A parameter also having merit in judging Criterion 2 might be one considering the average displacements of the system. comparing this parameter will indicate the relative size of the average displacements of one trial solution over another. These average displacements are listed in Table 1 and it can be seen that little change in fulfilling Criterion 2, either by maximum deflections or by average deflections is indicated.

A parameter describing Criterion 3 is hard to define. Visual inspection and comparison of the residual shapes is presently

Absolute Area	Total Absolute Area AR^+	Maximum Deflection y^+_{max}	Average Deflection y^+_{avg}
.166	.255	.246	.121
.165	.252	.242	.119
.140	.157	.221	.099
.172	.194	.209	.092
.178	.332	.221	.107

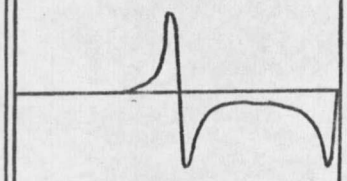
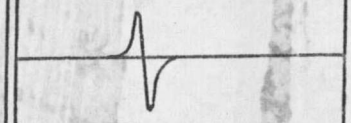
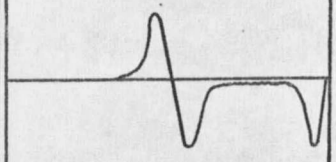
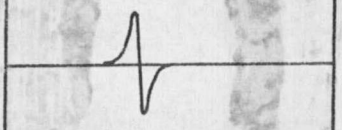
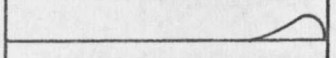
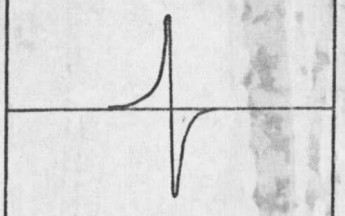
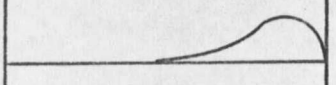
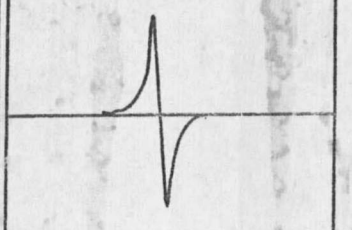
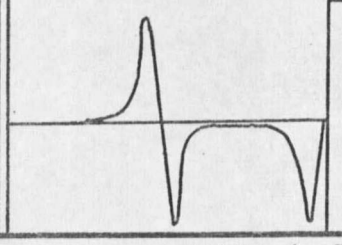
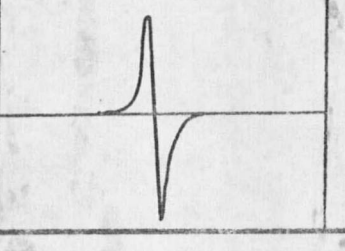
Trial No.	BEAM 1			BEAMS 2 & 3	
	Residual Shape	Maximum Value	Absolute Area	Residual Shape	Maximum Value
1		.366	.088		.234
2		.356	.087		.232
3		.114	.017		1.76
4		.172	.021.		2.
5		1.02	.155		

Table 1. Residuals and Parameters

the best method of applying this criterion. A consideration of the maximum value of the residual curve will be of some aid in comparing the residual shapes. The residual curves for beams 2 and 3 for all the trial solutions seem to be about evenly distributed about zero. Trial solutions 3 and 4 would be ruled the poorest utilizing Criterion 3 and solutions 1, 2 and 5 all have essentially the same residual distribution.

4.2 Residual Averaging

Koszuta [1] has suggested finding two trial solutions which yield residual curves that are similar in shape such that the trial solutions might be added or subtracted in such a manner as to reduce the equation residual. This technique was successfully used by Koszuta to find solutions to a blast loaded cantilever beam. For example, if $S_1(x)$ represents a deflection shape for any one of the beams of the system, and $S_2(x)$ represents another shape, then a new deflection shape, $S(x)$, can be defined by:

$$S(x) = S_1(x)X_1 + S_2(x)X_2$$

where the new assumed solution is:

$$y^*(x,t) = q(t)S(x)$$

and X_1 and X_2 are parameters such that:

$$X_1 + X_2 = 1 \tag{17}$$

In order to uniquely define X_1 and X_2, another equation

is written that will correlate $S_1(x)$ and $S_2(x)$.

One obvious method of correlation is to set the absolute areas of the equation residuals of $S_1(x)$ and $S_2(x)$ to zero. For instance:

$$ARS_1X_1 + ARS_2X_2 = 0 \tag{18}$$

where ARS_1 and ARS_2 are the absolute areas of the equation residuals of $S_1(x)$ and $S_2(x)$ respectively.

Yet another possibility is to set the sum of the maximum value of each residual to zero:

$$RS_1X_1 + RS_2X_2 = 0 \tag{19}$$

where RS_1 and RS_2 are the maximum values of the residuals of $S_1(x)$ and $S_2(x)$ respectively.

Equations (17) and (18) or (17) and (19) now represent two equations and two unknowns from which the parameters X_1 and X_2 can be found. Correlation equations (18) and (19) were used in an attempt to find a method of summing any two trial solutions into one assumed shape which would hopefully yield a higher quality solution.

These techniques were applied to various combinations of the five original deflection shapes and the results are tabulated in Table 2.

Applying Criterion 1 to this data indicates that a combination of trial solutions 3 and 4 related by the maximum residuals

Absolute Area	Total Absolute Area AR^+	Maximum Deflection y^+_{max}	Average Deflection y^+_{avg}	Correlation Method
.166	.246	.212	.132	Absolute Area
.170	.259	.251	.138	Maximum Residual
.354	.392	.181	.097	Maximum Residual
.238	.266	.271	.128	Absolute Area
.073	.081	.245	.112	Maximum Residual

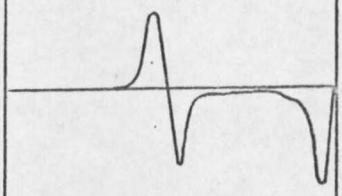
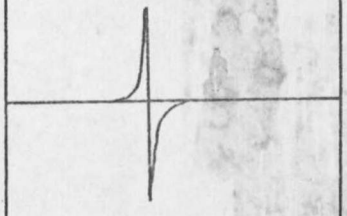
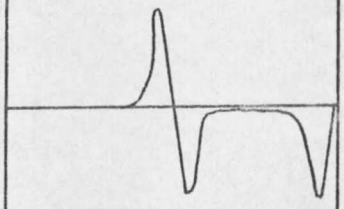
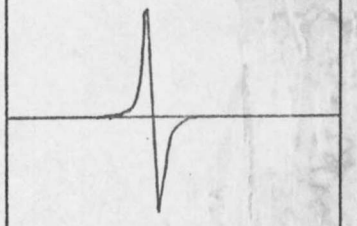
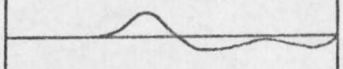
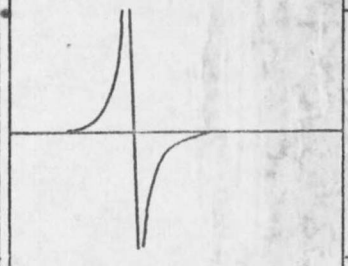
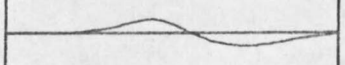
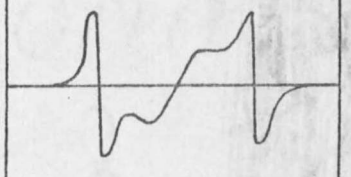
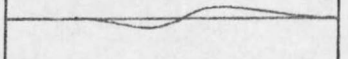
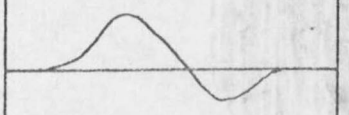
	BEAM 1			BEAMS 2 & 3	
	Residual Shape	Maximum Value	Absolute Area	Residual Shape	Maximu Value
1 & 2		.320	.080		2.46
1 & 2		.390	.089		.280
1 & 5		.102	.038		9.50
3 & 4		.083	.028		1.0
3 & 4		.019	.008		.4

Table 2. Trial Combinations

yields a good solution. ($AR^+ = .081$) The size of the residual has been considerably reduced over that of trial solutions 3 or 4 alone. The combination of trial solutions 1 and 5 has produced the largest absolute area of the residual curves ($AR^+ = .392$) and so, by Criterion 1, is judged the poorest solution. The other combinations listed in Table 2 show no significant change in the absolute area of the residual curve when compared to their counterparts in Table 1.

On the basis of maximum deflections as prescribed by Criterion 2, the combination of trial solutions 3 and 4, correlated by absolute area, appears to be the best quality solution with $y^+_{max} = .271$. On the other hand, if Criterion 2 is applied by the average deflections of the system, it can be seen that a combination of solutions 1 and 2, correlated by the maximum value of the residual, is the best quality solution. ($y^+_{max} = .138$) Other apparently good solutions are given by a combination of trials 1 and 2, correlated by absolute area ($y^+_{max} = .132$), and a combination of trials 3 and 4, correlated by absolute area. ($y^+_{max} = .128$)

Visual inspection of the residual curves in Table 2 will enable the application of Criterion 3. A good residual distribution is apparent in the combination of the trial solutions 3 and 4 correlated by absolute areas. This combination also gave the largest displacements of all the solutions tried.

From the foregoing discussion it is clear that no one factor of the three postulates described in Chapter I can be confirmed as the one positive criterion. Disagreement of all three of the criteria are apparent in Table 3. The trial solution with the smallest absolute area does not have the largest maximum deflection or the largest average deflection. Likewise, the trial solution with the largest maximum displacement does not have a small absolute area, neither does it have the largest average displacement. It does have, however, a well distributed residual and was previously judged a good quality solution on this basis.

It is the conclusion of this study that good approximate solutions to nonlinear problems can be found by Galerkin's technique. Assumed solutions that were tried in this study and that were intuitively logical deflection shapes all gave comparable results. It is postulated that reasonable trial solutions are stable and will give reasonable answers. Solution comparison by the criteria presented here is a valuable guide in the evaluation of assumed solutions to problems solved by Galerkin's methods. It was noted by the author that if intuitively poor trial solutions, or combinations of the solutions, were assumed for the system, the results would be obviously inferior. Invariably the deflections would be very small or negative and the residuals would be exceedingly large. For trial

TRIAL SOLUTION	CORRELATION METHOD	MAXIMUM DEFLECTION	AVERAGE DEFLECTION	TOTAL ABSOLUTE AREA
3 & 4	Abs. Area	.271	.128	.266
1 & 2	Max. Resid.	.251	.138	.259
1	-	.246	.121	.255
3 & 4	Max. Resid.	.245	.112	.081
2	-	.242	.119	.252
3	-	.221	.099	.157
5	-	.221	.107	.332
1 & 2	Abs. Area	.212	.132	.246
4	-	.209	.092	.194
1 & 5	Max. Resid.	.181	.097	.245

Table 3. Parameter Comparison

solutions giving reasonable answers, further judgement of the quality of the solution by the three criteria originally postulated is apparently unpredictable.

4.3 Further Studies

In the interest of evaluation of the deflection magnitude of a blast loaded beam system, it would be helpful to obtain some experimental data to a typical problem. Then, Galerkin's technique as presented here, could be applied to that problem and the results compared. A technique such as high speed photography should be used in the experimentation so that deflection shapes as well as maximum deflection amplitudes could be obtained.

Finlayson and Scriven, in their paper on weighted residuals, [9] state, "...The convergence of successive approximations gives a clue, but not necessarily a definitive one to the reasonableness of the approximations." With this in mind, an assumed solution of more than one term should be applied to the problem presented here. Then, by successively applying more and more trial deflection shapes, the results of a multi-term solution could be studied and the worth of such a statement as the above could be investigated. Perhaps a fourth and more significant criteria could then be added to those postulated in Chapter I.

APPENDIX

APPENDIX A

The Nonlinear Beam Equation

Refering to Figure 2 of the text, the sum of the forces in the vertical direction can be set equal to the mass times the acceleration:

$$+\uparrow \quad w(t)dx - V\cos\phi + (V + \frac{\partial V}{\partial s}ds)\cos(\phi + \frac{\partial \phi}{\partial s}ds) = \rho dx \frac{\partial^2 y}{\partial t^2} \qquad (1)$$

Now by neglecting rotational inertia, the moments can be summed about the right face:

$$-M + (M + \frac{\partial M}{\partial s}ds) + Vds - w(t)dx\frac{dx}{2} = 0 \qquad (2)$$

Neglecting second order terms and realizing that:

$$ds = \frac{dx}{\cos\phi}$$

equation (2) becomes:

$$V = -\frac{\partial M}{\partial x}\cos\phi$$

and the first spatial derivative of the shear can be written as:

$$\frac{\partial V}{\partial x} = -\frac{\partial^2 M}{\partial x^2}\cos\phi + \frac{\partial \phi}{\partial x}\frac{\partial M}{\partial x}\sin\phi$$

Substituting these equations into equation 1 and neglecting second order terms yields:

$$\gamma\, dx \frac{\partial^2 y}{\partial t^2} = w(t)dx + \frac{\partial M}{\partial x}\cos^2\phi$$

$$+ \left[\frac{\partial M}{\partial x}\cos^2\phi - \frac{\partial^2 M}{\partial x^2}\cos^2\phi\, dx + 2\frac{\partial\phi}{\partial x}\frac{\partial M}{\partial x}\sin\phi\cos\phi\, dx \right]$$

By canceling terms and making the proper trigonometric substitution, this equation is written as:

$$w(t) - \frac{\partial^2 M}{\partial x^2}\cos^2\phi - \frac{\partial M}{\partial x}\frac{\partial\phi}{\partial x}\sin 2\phi - \gamma\frac{\partial^2 y}{\partial t^2} = 0 \qquad (3)$$

Equation (3) corresponds to equation (1) in the text and is the equation of motion in terms of the displacement, $y(x,t)$, the internal moment, M, and the rotation, ϕ. It is now necessary to express the moment in terms of the displacement and rotation. Finally, the rotation can be expressed in terms of the displacement.

Assuming that the neutral plane of the beam element remains a neutral plane and that plane sections remain plane, the strain in a fiber a distance z from the neutral plane can be written:

$$\epsilon = z\frac{\partial\phi}{\partial x}$$

If the stress-strain curve is described by:

$$\sigma = \frac{1}{b}\tan^{-1}(a\epsilon)$$

as given in the text, then the internal moment on a cross-section is given by:

$$M = D\int_{-c}^{c} z\,dz$$

where D is the beam width and 2c is the beam height. Performing the integration and taking successive derivatives yields:

$$M = \frac{D}{b}\left[\, c^2\tan^{-1}(a\phi' c) + (\tan^{-1}(a\phi' c) - a\phi' c)/(a\phi'^2)\right] \tag{4}$$

$$\frac{\partial M}{\partial x} = \frac{2D\phi''}{a^2 b}\left[\frac{a\phi' c - \tan^{-1}(a\phi' c)}{\phi'^3}\right] \tag{5}$$

$$\frac{\partial^2 M}{\partial x^2} = \frac{2\,D}{a^2 b}\left[\phi''\frac{3(1 + a^2c^2\phi'^2)\tan^{-1}(a\phi' c) - 3a\phi' c - 2(a\phi' c)^3}{\phi'^4[1 + (a\phi' c)^2]} + \frac{\phi'''\, a\phi' c - \tan^{-1}(a\phi' c)}{\phi'^3}\right] \tag{6}$$

Equations (4), (5), and (6) can be placed in equation (3) to yield the equation of motion in terms of the displacement and the rotation. The rotation and it's first three derivatives can be expressed as:

$$\phi = \tan^{-1}\frac{\partial y}{\partial x}$$

$$\phi' = \frac{y''}{(1 + y'^2)}$$

$$\phi'' = \frac{y'''(1 + y'^2)^2 - 2y'y''^2}{(1 + y'^2)^2}$$

$$\phi''' = \frac{y''''}{(1 + y'^2)} - \frac{(6y'y''y''' + 2y''^3)}{(1 + y'^2)^2} + \frac{8y'^2y''^3}{(1 + y'^2)^3}$$

Now all terms in equation (3) can be expressed entirely in terms of the displacement, $y(x,t)$. It can be readily seen that equation (3) contains $y(x,t)$ and various nonlinear combinations of its first four derivatives. Equation (3) may be expressed as:

$$w(t) - F[y(x,t)] - \gamma\ddot{y} = 0 \qquad (7)$$

where $F[y(x,t)]$ is a nonlinear differential operator on $y(x,t)$. Equation (7) corresponds to equation (3) of the text.

APPENDIX B

Nonlinear Expression of Natural Boundary Conditions

Refering to Appendix A, an equation for the moment was given in equation (4):

$$M = -\frac{D}{b}\left[c^2 \tan^{-1}(a\phi' c) + \frac{\tan^{-1}(a\phi' c) - a\phi' c}{a\phi'^2}\right] \qquad (1)$$

The expression for the shear was derived in Appendix A as:

$$V = -\frac{\delta M}{\delta x}\cos\phi \qquad (2)$$

Recall that the $\frac{\delta M}{\delta x}$ was also given in Appendix A, and the shear can now be written:

$$V = -\frac{2D\phi''}{a^2 b}\left[\frac{a\phi' c - \tan^{-1}(a\phi' c)}{\phi'^3}\right]\cos\phi \qquad (3)$$

It is apparently possible for the expressions for the moment, equation (1), and for the shear, equation (3), to be undefined for cases of ϕ' going to zero.

Recall from Appendix A:

$$\phi' = \frac{y''}{(1 + y'^2)}$$

Clearly ϕ' can be zero if $y''(x,t)$ is zero.

Applying a zero value of ϕ' to the moment, equation (1),

obviously yields an indeterminent form:

$$M = \frac{D}{b}\left[\, 0 + \frac{0-0}{0}\right]$$

However, making successive applications of L'Hospital's rule to the interior part of equation (1) will show that the actual value of the nonlinear moment, when $y''(x,t)$ is zero, is also zero.

If a zero value of ϕ' is applied to the shear, equation (3), another indeterminent form is apparent:

$$V = -\frac{2\phi''D}{a^2 b}\left[\frac{0-0}{0}\right]\cos\phi$$

Again making successive applications of L'Hospital's rule will yield a usable result. The nonlinear shear is just a constant when $y''(x,t)$ goes to zero. The constant is:

$$\lim_{\phi' \to 0} V = -\frac{2}{3}\frac{Da\phi''c^3}{b}$$

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