A continuum mixture theory with an application to turbulent snow, air flows and sedimentation
by Rand Alan Decker

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Civil Engineering
Montana State University

Abstract:

A theory, based on the Newtonian balance equations is developed for a generalized mixture of n constituents which respond as a whole in a "fluid" manner. These constituents are capable of inter-constituent exchanges of mass, linear momentum and moment of momentum. The physical requirement that the mixture behavior be the sum of its constituent behaviors leads to a set of restriction on each of the constituent balance equations.

This theory is specialized for the case of a snow and air mixture flow. A constitutive assumption is made concerning the transfer of linear momentum between the air and snow phases of the flow. The resulting equations of motion for the snow phase are expanded to include the effects of a turbulent mixture flow. A constitutive assumption is made for the turbulent variables of the snow phase in terms of the intensity of shearing in the airflow. The resulting turbulent momentum balance equation for the snow phase contains a set of terms which could be characterized as apparent or turbulent buoyancies. As a consequence of the constitutive assumption the magnitude of this set of terms is large where the shearing of gradients of the airflow are large.

The system of non-linear partial differential equations resulting from the turbulent equations of motion for the snow phase are approximated by finite difference techniques. Solutions for the snow phase velocity and density fields are investigated for a variety of one and two dimensional airflow regimes.

The snow phase velocity and density field solutions are then compared with the observed phenomena of snow and air mixture flows over flat surfaces and over the crest of a mountain ridge. Lastly the accumulation rates of deposited snow on the immediate lee slope of a mountain ridge are compared with observations.

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Rand Alan Decker

This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citation, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

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This work is dedicated to the "first" professor in the author's life, his father: Dr. R. W. Decker whose belief in the power of the intellect inspired this effort more than any other.

VITA

Rand Alan Decker was born April 18, 1954 in Urbana, Illinois. He is the son of Mrs. and Dr. R. W. Decker. Mr. Decker graduated from Hanover High School, Hanover, New Hampshire in 1972. He received his Bachelor of Science degree in Geological Engineering from the College of Mines at the University of Utah in 1977.

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ABSTRACT

A theory, based on the Newtonian balance equations is developed for a generalized mixture of n constituents which respond as a whole in a "fluid" manner. These constituents are capable of inter-constituent exchanges of mass, linear momentum and moment of momentum. The physical requirement that the mixture behavior be the sum of its constituent behaviors leads to a set of restriction on each of the constituent balance equations.

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CHAPTER 1

INTRODUCTION

The desire to create physical theories which rationally describe the mechanical behavior of the observed environment is a prime motivation in the study of Newtonian or classical mechanics.

The evolution of mechanics has evolved to the point where the systems under theoretical consideration can be represented as mixtures of constituents which are capable of inter-constituent transfers of mass, linear momentum and moment of momentum. The range of mechanical behavior displayed by various types of mixtures is large. One limiting case is that of a mixture or composite of elastic solids. At the other extreme is the case of a mixture of constituents which respond as a whole in a "fluid" like manner, even though some of the constituents may not be fluid elements. There is a continuous variation of mixtures between these limiting cases.

The present state of the art in mixture mechanics is the theoretical investigation of these limiting cases. These investigations involve the development of internally consistent theories which arise from the balance principles and thermodynamic principles of classical Newtonian mechanics. These theories may then be investigated, usually by numerical approximation techniques, for solutions. If solutions do exist then their application or degree of correlation with an observed phenomenon may be affected.

This investigation contains these three elements. A mixture theory is developed for a system of constituents which respond as a whole in a fluid manner. This theory is then applied to the specific phenomenon of turbulent sedimentation of snow in an atmospheric flow. The systems of partial differential equations which arise from the turbulent equations of motion for the snow phase are then solved by finite difference numerical techniques. The resulting solutions for the snow phase density fields and velocity fields for one dimensional and two dimension mixture flows are then compared qualitatively with observations. Lastly, theoretical snow phase accumulation distributions and rates for the two dimensional mixture flow are compared qualitatively with observations of wind-aided snow accumulation on the immediate lee side of a transverse mountain ridge.

Historical Perspectives: Mixture Theories

J. Clerk Maxwell made the first in-depth investigation of the Newtonian mechanics of a system of particles. Maxwell's mixture was a system of variable sized spheres in a frictionless medium. These spheres are allowed to interact with each other via conservative or elastic collisions. Arguing from the necessary probabilistic forms of a functional for this system of discrete particles Maxwell derived the Kinetic Theory of Gases (Maxwell, 1860). This very eloquent theory has since been verified to be an excellent physical analogy of gas behavior.

During the first half of the twentieth century the concepts of the Newtonian mechanics for continuous systems have been highly refined, verified and summarized (Truesdell & Toupin, 1960). Initially mixture theories were developed for elastic composites and consolidating porous solids. These works have been summarized (Boit, 1963). From these foundations the theories of continuous systems of flowing and reacting constituents were argued (Bowen, 1976; Eringen & Ingram, 1965; Green & Naghdi, 1965). The development of internally consistent mixture theories continues presently. A large portion of recent research efforts is aimed at developing constitutive relationships for chemically reacting thermo-mechanical mixtures. These efforts are inhibited by the form and restrictions which a constitutive relationship must take to satisfy the entropy inequality (deGroot & Mazur, 1962; Green & Naghdi, 1971; Prigogine & Defay, 1954).

Investigations into the existence of solutions to the accepted continuum mixture theories for multi-phase flows and comparison of these solutions to observed mixture flow phenomena are all fairly recent (Decker & Brown, 1983, 1985; Drew, 1975; Hill, Bedford & Drumheller, 1980; McTique, 1981, 1983).

Historical Perspective: Wind-Aided Snow and Air Mixture Flows

The investigation of wind-aided snow sedimentation, known variously in the literature as blowing snow or drifting snow was initiated as an element of the original polar regions geophysical studies. In this early literature there is no distinction in the definition between blowing snow and drifting snow, the latter also being a snow and air mixture flow. However, in lay terms drifting snow would imply the accumulation or sedimentation of snow in the lee of structures or land forms. The bulk of this pioneering research was directed at gauging and deriving emperical expressions for snow and air mixture flows over flat terrain (Budd, Dingle & Radok, 1966; Dyunin, 1954, 1954, 1959; Kobayashi, 1972; Mellor, 1965; Radok, 1968, 1977). These early works were often shown to compare favorably with the emperical expressions derived for sand and air mixture flows (Bagnold, 1941).

It was recognized that, if the snow and air mixture flow over flat terrain is to be one dimensional, the mass fraction of the snow phase in transport must be equilibrated with respect to changes in horizontal distance. The horizontal distances required for a snow and air mixture flow to equilibrate have been gauged (Takeuchi, 1980).

The salient material of this early work in one dimension snow and air mixture flows has been summarized (Schmidt, 1982).

The initial research into two dimensional snow and air mixture flows was aimed at determining emperical expressions for the spatial patterns of deposited snow in natural environments and adjacent to structures (Alford, 1980; Tabler, 1975, 1980). Further, rational theories using the trajectories of discrete snow particles have been derived and solutions for the resulting depositional patterns presented (Schmidt & Randloph, 1981).

Two dimensional snow and air mixture flows in the immediate lee of a transverse mountain ridge have been investigated emperically and rationally only recently (Decker & Brown, 1983, 1985; Föhn & Meister, 1983). These investigations have considered the snow phase density and velocity fields in the mixture flow and the spatial patterns of deposited or accumulated snow on the lee slope.

Lastly, it should be noted that successful efforts at scale modeling of snow and air mixture flows have been made for one, two and three dimensional flows (Anno & Konishi, 1981; Anno, 1984, 1984; Iversen, 1980; Wuebben, 1978).

CHAPTER 2

MIXTURE THEORY: THE THEORETICAL DEVELOPMENT

Definitions

Consider a continuous material composed of n constituents. This continuum mixture is in shear (viscous) flow. Due to the possibility of phase changes or inter-constituent reactions the mass density of any single constituent: α may vary with time. However we define the instantaneous mixture density ρ at a point as the sum of the instantaneous constituent densities at that point, where ρ_α is the mass per unit mixture volume for the α constituent.

$$\rho = \sum_{\alpha=1}^{n} \rho_\alpha \tag{1}$$

We define the mixture velocity to be the material time derivative of a "parcel" of mixture as:

$$\frac{d}{dt}\underline{x} = \dot{\underline{x}} \tag{2}$$

The diffusion velocity is defined as the velocity difference between any single constituent α with respect to the mixture velocity.

$$\underline{u}_\alpha = \grave{\underline{x}}_\alpha - \dot{\underline{x}} \tag{3}$$

Where $\grave{\underline{x}}_\alpha$ is the velocity of the α constituent. In this development both the overdot and back-facing overscore represent the material derivative with respect to either the mixture or a specific constituent, respectively.

We impose the condition that the sum of all constituent linear momenta must be equal to the linear momentum of the mixture.

$$\sum_{a=1}^{n} \rho_a \grave{\underline{x}}_a = \rho \dot{\underline{x}} \tag{4}$$

Note that this implies that the sum of all constituent diffusion linear momenta must equal zero.

$$\sum_{a=1}^{n} \rho_a \grave{\underline{x}}_a = \sum_{a=1}^{n} \rho_a (\underline{u}_a + \dot{\underline{x}}) = \rho \dot{\underline{x}} \tag{5}$$

or

$$\sum_{a=1}^{n} \rho_a \underline{u}_a = \rho \dot{\underline{x}} - \sum_{a=1}^{n} \rho_a \dot{\underline{x}} = \underline{0}$$

At this point it is of interest to examine a set of relationships with respect to any differentiable function of the mixture, Γ. Consider the time derivative of Γ with respect to the motion of the mixture.

$$\dot{\Gamma} = \frac{\partial}{\partial t} \Gamma + \dot{\underline{x}} \cdot \underline{\vec{\nabla}} \Gamma \tag{6}$$

Unless otherwise stated the gradient operator is with respect to the spatial coordinate system. Likewise the time derivative of Γ with respect to the motion of the a constituent is:

$${}_a\grave{\Gamma} = \frac{\partial}{\partial t} \Gamma + \grave{\underline{x}}_a \cdot \underline{\vec{\nabla}} \Gamma \tag{7}$$

Note that Equations 6 and 7 imply that

$$\dot{\Gamma} = {}_a\grave{\Gamma} \text{ if and only if } \grave{\underline{x}}_a = \dot{\underline{x}} \tag{8}$$

This is a physically consistent result.

In a similar manner let Γ_a be any differentiable function of the a constituent. Then the substantive derivative of Γ_a with respect to the motion of the mixture is:

$$\dot{\Gamma}_a = \frac{\partial}{\partial t} \Gamma_a + \dot{\underline{x}} \cdot \underline{\vec{\nabla}} \Gamma_a \tag{9}$$

Likewise the substantive derivative of Γ_a with respect to the motion of the a constituent is:

$$ {}_{a}\grave{\Gamma}_{a} = \frac{\partial}{\partial t}\Gamma_{a} + \grave{\underline{x}}_{a} \cdot \vec{\underline{\nabla}}\,\Gamma_{a} \tag{10}$$

Subtracting Equations 6 from 7 results in:

$$ \dot{\Gamma}_{a} - \dot{\Gamma} = (\grave{\underline{x}}_{a} - \dot{\underline{x}}) \cdot \vec{\underline{\nabla}}\,\Gamma = \underline{u}_{a} \cdot \vec{\underline{\nabla}}\,\Gamma \tag{11}$$

Similarly, subtracting Equations 9 from 10 results in:

$$ {}_{a}\grave{\Gamma}_{a} - \dot{\Gamma}_{a} = (\grave{\underline{x}}_{a} - \dot{\underline{x}}) \cdot \vec{\underline{\nabla}}\,\Gamma_{a} = \underline{u}_{a} \cdot \vec{\underline{\nabla}}\,\Gamma_{a} \tag{12}$$

That is, the difference in derivatives for any mixture (or for any a constituent) function with respect to mixture motion or a constituent motion is due solely to the diffusion velocity.

This development will be an exploration of the various constituent and mixture field equations which arise as a consequence of Newtonian balance laws.

Balance of Mass: Continuity

Consider Figure 1. There exists an arbitrary fixed closed region R characterized by surface S totally bounded within the continuous mixture body B. A surface element, dS_R, is characterized by surface normal $\underline{n}$, and there exists at dS_R an instantaneous a constituent velocity $\grave{\underline{x}}_{a}$ and an instantaneous mixture velocity $\dot{\underline{x}}$ with respect to a spatial Cartesian coordinate frame. The velocity difference is the diffusion velocity of the a constituent: $\underline{u}_{a}$.

It is possible to write an expression for the balance of mixture mass ρ within region R

$$ \int_{R} \frac{\partial}{\partial t}\rho \, dV = -\int_{S} \underline{n} \cdot \rho\dot{\underline{x}} \, dS_{R} \tag{13}$$

The leftside term is the time rate of change of mixture mass in region R. The rightside term is the convective flux of mixture mass from region R through the surface of the region S and by convention is negative. By an application of divergence (Gauss') theorem the rightside term may be rewritten as an integral over the volume of region R.

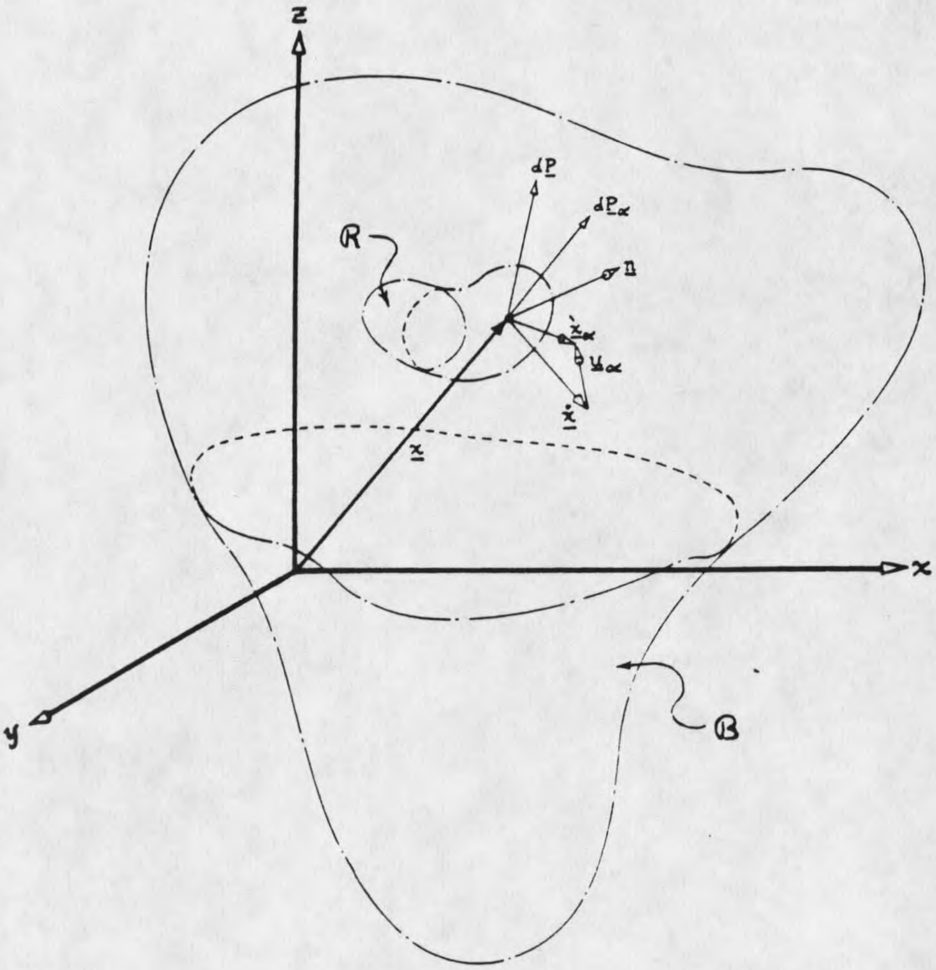

Figure 1. An arbitrary region R totally bounded by the continuous body of mixture B where a surface element dS_R is characterized by normal $\underline{n}$, a constituent velocity $\underline{\dot{x}}_a$, diffusion velocity $\underline{u}_a$ mixture velocity $\underline{\dot{x}}$, a constituent pressure $d\underline{P}_a$ and mixture pressure $d\underline{P}$.

$$\int_R \frac{\partial}{\partial t} \rho \, dV = -\int_R \vec{\underline{\nabla}} \cdot \rho \dot{\underline{x}} \, dV \tag{14}$$

or, rearranging

$$\int_R \left\{ \frac{\partial}{\partial t} \rho + \vec{\underline{\nabla}} \cdot \rho \dot{\underline{x}} \right\} dV = 0 \tag{15}$$

Sense region R is totally arbitrary, as long as it remains bounded in the body of mixture B this requires that for the integral Equation 15 to be equal to zero that the integrand itself must be identically equal to zero.

$$\frac{\partial}{\partial t} \rho + \vec{\underline{\nabla}} \cdot \rho \dot{\underline{x}} = 0 \tag{16}$$

This is the balance of mass or continuity equation for the mixture and is recognizable as being identical to the continuity equation for single phase flow.

Similarly it is possible to write an expression for the balance of a constituent mass ρ_a within region R.

$$\int_R \frac{\partial}{\partial t} \rho_a dV = -\int_S \underline{n} \cdot \rho_a \dot{\underline{x}}_a dS_R + \int_R \hat{c}_a \, dV \tag{17}$$

The additional, second rightside term characterizes the mass supplied to or taken from the a constituent by the n-1 other constituents comprising the mixture. This positive or negative mass supply to the a constituent may be due to phase changes and/or chemical reaction with the n-1 other constituents of the mixture. An application of divergence theorem and rearranging of Equation 17 results in:

$$\int_R \left\{ \frac{\partial}{\partial t} \rho_a + \vec{\underline{\nabla}} \cdot \rho_a \dot{\underline{x}}_a - \hat{c}_a \right\} dV = 0 \tag{18}$$

Again, based on the arbitrary nature of region R it follows that:

$$\frac{\partial}{\partial t} \rho_a + \vec{\underline{\nabla}} \cdot \rho_a \dot{\underline{x}}_a = \hat{c}_a \tag{19}$$

This is the continuity equation for the a constituent of an n constituent mixture.

Clearly Equation 19 represents a set of n equations, each one being the continuity equation for a specific constituent.

If we impose the physical requirement that the mixture behavior be the sum of the constituent behaviors and apply this requirement to the continuity equations then:

$$\frac{\partial}{\partial t}\,\rho + \underline{\vec{\nabla}} \cdot \rho\,\dot{\underline{x}} = \frac{\partial}{\partial t}\sum_{a=1}^{n} \rho_a + \underline{\vec{\nabla}} \cdot \sum_{a=1}^{n} \rho_a \grave{\underline{x}}_a - \sum_{a=1}^{n} \hat{c}_a \tag{20}$$

Recalling Equations 1 and 4, the definitions of mixture mass and mixture linear momentum, respectively, then Equation 20 reduces to:

$$\sum_{a=1}^{n} \hat{c}_a = 0 \tag{21}$$

That is, the sum of the inter-constituent mass supplies must be zero.

It is of interest to examine a relationship between the mass of a specific constituent: a and the motion of the mixture. Recall Equations 9 and 10 and consider the difference:

$$\begin{aligned}\dot{\rho}_a - {}_a\grave{\rho}_a &= \dot{\underline{x}} \cdot \underline{\vec{\nabla}}\,\rho_a - \grave{\underline{x}}_a \cdot \underline{\vec{\nabla}}\,\rho_a \\ &= -\underline{u}_a \cdot \underline{\vec{\nabla}}\,\rho_a\end{aligned} \tag{22}$$

Note that the transient terms have cancelled. Rearranging Equation 19, the continuity equation for the a constituent in light of Equation 10 allows for the substitution of

$${}_a\grave{\rho}_a = -\rho_a \underline{\vec{\nabla}} \cdot \grave{\underline{x}}_a + \hat{c}_a \tag{23}$$

into Equation 22, which after rearranging and the identically zero addition and subtraction of the term: $\rho_a \underline{\vec{\nabla}} \cdot \dot{\underline{x}}$ results in:

$$\begin{aligned}\dot{\rho}_a &= -\rho_a \underline{\vec{\nabla}} \cdot \grave{\underline{x}}_a + \hat{c}_a - \underline{u}_a \cdot \underline{\vec{\nabla}}\rho_a + \rho_a \underline{\vec{\nabla}} \cdot \dot{\underline{x}} - \rho_a \underline{\vec{\nabla}} \cdot \dot{\underline{x}} \\ &= -\rho_a \underline{\vec{\nabla}} \cdot (\grave{\underline{x}}_a - \dot{\underline{x}}) - \underline{u}_a \cdot \underline{\vec{\nabla}}\,\rho_a - \rho_a \underline{\vec{\nabla}} \cdot \dot{\underline{x}} + \hat{c}_a \\ &= -\rho_a \underline{\vec{\nabla}} \cdot \underline{u}_a - \underline{u}_a \cdot \underline{\vec{\nabla}}\,\rho_a - \rho_a \underline{\vec{\nabla}} \cdot \dot{\underline{x}} + \hat{c}_a \\ &= -\underline{\vec{\nabla}} \cdot \rho_a \underline{u}_a - \rho_a \underline{\vec{\nabla}} \cdot \dot{\underline{x}} + \hat{c}_a\end{aligned} \tag{24}$$

or

$$\dot{\rho}_a + \rho_a \underline{\vec{\nabla}} \cdot \underline{\dot{x}} = -\underline{\vec{\nabla}} \cdot \rho_a \underline{u}_a + \hat{c}_a$$

In light of Equation 16, the continuity equation for the mixture we can form the following quotient about the second leftside term of Equation 24

$$\dot{\rho}_a + \frac{\dot{\rho}\, \rho_a \underline{\vec{\nabla}} \cdot \underline{\dot{x}}}{-\rho\, \underline{\vec{\nabla}} \cdot \underline{\dot{x}}} = -\underline{\vec{\nabla}} \cdot \rho_a \underline{u}_a + \hat{c}_a \tag{25}$$

or

$$\dot{\rho}_a - \frac{\dot{\rho}\, \rho_a}{\rho} = -\underline{\vec{\nabla}} \cdot \rho_a \underline{u}_a + \hat{c}_a$$

or

$$\rho\, \dot{\overline{\left(\frac{\rho_a}{\rho}\right)}} = -\underline{\vec{\nabla}} \cdot \rho_a \underline{u}_a + \hat{c}_a$$

where the overbar implies differentiation of both factors in the quotient. Note that the quotient is an expression for the instantaneous dimensionless concentration at a point for the a constituent. Also take note that if there is no diffusion of the a constituent with respect to the mixture ($\underline{u}_a = 0$) and no mass exchange with the a constituent ($\hat{c}_a = 0$) then the instantaneous concentration at a point of the a constituent must be constant. This result is physically consistent and intuitively satisfying.

At this point in the theoretical development it is of interest to generate a generalized identity which is a consequence of the physical requirement that the mixture behavior be sum of the constituent behaviors. Then for any function of the mixture, Γ

$$\rho \Gamma = \sum_{a=1}^{n} \rho_a \Gamma_a \tag{26}$$

or

$$\Gamma = \sum_{a=1}^{n} \frac{\rho_a}{\rho} \Gamma_a$$

where the derivatives with respect to the mixture of both sides of Equation 26:

$$\dot{\Gamma} = \sum_{a=1}^{n} \dot{\overline{\left(\frac{\rho_a}{\rho}\Gamma_a\right)}} = \sum_{a=1}^{n} \left\{ \dot{\overline{\left(\frac{\rho_a}{\rho}\right)}}\Gamma_a + \frac{\rho_a}{\rho}\dot{\Gamma}_a \right\} \tag{27}$$

or

$$\rho\dot{\Gamma} = \sum_{a=1}^{n} \left\{ \rho\,\dot{\overline{\left(\frac{\rho_a}{\rho}\right)}}\Gamma_a + \rho_a\dot{\Gamma}_a \right\}$$

By substitution of Equation 25 and the identically zero addition and subtraction of the term: ${}_a\grave{\Gamma}_a$ into Equation 27 results in:

$$\rho\dot{\Gamma} = \sum_{a=1}^{n} \left\{ (-\underline{\vec{\nabla}} \cdot \rho\underline{u}_a + \hat{c}_a)\Gamma_a + \rho_a(\dot{\Gamma}_a + {}_a\grave{\Gamma}_a - {}_a\grave{\Gamma}_a) \right\} \tag{28}$$

or, in light of Equations 9 and 10 and noting that the transient terms arising from Equations 9 and 10 cancel:

$$\rho\dot{\Gamma} = \sum_{a=1}^{n} \left\{ (-\underline{\vec{\nabla}} \cdot \rho\underline{u}_a + \hat{c}_a)\Gamma_a + \rho_a\,({}_a\grave{\Gamma}_a - \dot{\underline{x}}_a \cdot \underline{\vec{\nabla}}\,\Gamma_a + \dot{\underline{x}} \cdot \underline{\vec{\nabla}}\,\Gamma_a) \right\}$$

$$= \sum_{a=1}^{n} \left\{ -\Gamma_a\underline{\vec{\nabla}} \cdot \rho\underline{u}_a + \Gamma_a\hat{c}_a + \rho_a\,{}_a\grave{\Gamma}_a - \rho_a\underline{u}_a \cdot \underline{\vec{\nabla}}\,\Gamma_a \right\}$$

$$= \sum_{a=1}^{n} \left\{ \rho_a\,{}_a\grave{\Gamma}_a + \Gamma_a\hat{c}_a - \underline{\vec{\nabla}} \cdot \Gamma_a\rho_a\underline{u}_a \right\}.$$

Balance of Linear Momentum

Consider Figure 1. It is possible to write an expression for the balance of mixture linear momentum $\rho\dot{\underline{x}}$ within region R.

$$\int_R \frac{\partial}{\partial t}\rho\,\dot{\underline{x}}\,dV = -\int_{S_R}(\underline{n}\circ\dot{\underline{x}})(\rho\dot{\underline{x}})\,dS + \int_{S_R} d\underline{P} + \int_R \rho\underline{b}\,dV \tag{29}$$

The leftside term is the time rate of change of mixture linear momentum in region R. The rightside terms are, respectively: the convective flux of mixture linear momentum from region R through the surface of the region: S, the addition or subtraction of mixture linear momentum in region R due to mixture surface forces on the surface of the region and the addition or subtraction of mixture linear momentum in region R due to body forces in the region.

Note that the term characterizing the mixture surface forces can be expressed as a mixture stress or traction vector, $\underline{t}^{(n)}$, and then recall that any mixture stress vector on a surface with surface normal $\underline{n}$ can be expressed as the product of the surface normal $\underline{n}$ and the local mixture stress tensor $\underset{\sim}{T}$.

$$\int_{S_R} d\underline{P} = \int_{S_R} \underline{t}^{(n)}\, ds = \int_{S_R} n \underset{\sim}{T}\, ds \tag{30}$$

Applying divergences theorem to Equation 30 results in:

$$\int_{S_R} n \underset{\sim}{T}\, ds = \int_{R} \vec{\underline{\nabla}} \cdot \underset{\sim}{T}\, dV \tag{31}$$

The term characterizing the convective flux of mixture linear mometum may be rewritten about a dyadic product of the mixture velocities and divergences theorem may then be applied to the resulting expression.

$$\int_{S_R} (n \cdot \dot{\underline{x}})(\rho \dot{\underline{x}})\, ds = \int_{S_R} \underline{n}(\rho \dot{\underline{x}}\dot{\underline{x}})\, ds = \int_{R} \vec{\underline{\nabla}} \cdot (\rho \dot{\underline{x}}\dot{\underline{x}})\, dV \tag{32}$$

Substitution of Equations 31 and 32 into 29 and collecting all terms under a single integral results in:

$$\int_{R} \left\{ \frac{\partial}{\partial t} \rho \dot{\underline{x}} + \vec{\underline{\nabla}} \cdot (\rho \dot{\underline{x}}\dot{\underline{x}}) - \vec{\underline{\nabla}} \cdot \underset{\sim}{T} - \rho \underline{b} \right\} dV = 0 \tag{33}$$

Since region R is totally arbitrary this requires that:

$$\frac{\partial}{\partial t} \rho \dot{\underline{x}} + \vec{\underline{\nabla}} \cdot (\rho \dot{\underline{x}}\dot{\underline{x}}) - \vec{\underline{\nabla}} \cdot \underset{\sim}{T} - \rho \underline{b} = 0 \tag{34}$$

Consider the following reorganization of the first and second terms of Equation 34:

$$\begin{aligned}\frac{\partial}{\partial t}\rho\underline{\dot{x}} + \underline{\vec{\nabla}}\cdot(\rho\underline{\dot{x}}\underline{\dot{x}}) &= \frac{\partial}{\partial t}\rho\underline{\dot{x}} + \underline{\dot{x}}(\underline{\vec{\nabla}}\cdot\rho\underline{\dot{x}}) + \rho\underline{\dot{x}}(\underline{\vec{\nabla}}\cdot\underline{\dot{x}}) \\ &= \overset{\bullet}{\overline{(\rho\underline{\dot{x}})}} + \rho\underline{\dot{x}}(\underline{\vec{\nabla}}\cdot\underline{\dot{x}}) \\ &= \rho\underline{\ddot{x}} + \dot{\rho}\underline{\dot{x}} + \rho\underline{\dot{x}}(\underline{\vec{\nabla}}\cdot\underline{\dot{x}}) \\ &= \rho\underline{\ddot{x}} + \dot{x}(\dot{\rho} + \rho\underline{\vec{\nabla}}\cdot\underline{\dot{x}}) \\ &= \rho\underline{\ddot{x}}\end{aligned} \tag{35}$$

Where the factor in parenthesis is recognizable as Equation 16, the continuity equation for the mixture and is identically zero. Substituting this result into Equation 34 leads to:

$$\rho\underline{\ddot{x}} = \underline{\vec{\nabla}}\cdot\underset{\sim}{T} + \rho\underline{b} \tag{36}$$

This is the momentum balance equation for the mixture and is recognizable as being identical to the momentum balance equation for single phase flow.

Similarly it is possible to write an expression for the balance of a constituent linear momentum $\rho_a\underline{\dot{x}}_a$ within region R.

$$\int_R \frac{\partial}{\partial t}\rho_a\underline{\dot{x}}_a = -\int_{S_R}(\underline{n}\cdot\underline{\dot{x}}_a)(\rho\underline{\dot{x}}_a)\,ds + \int_{S_R} d\underline{P}_a + \int_R\left\{\rho_a\underline{b}_a + \hat{c}_a\underline{\dot{x}}_a + \underline{\hat{p}}_a\right\}dV \tag{37}$$

The additional terms: $\hat{c}_a\underline{\dot{x}}_a$ and $\underline{\hat{p}}_a$ characterize respectively the a constituent linear momentum addition or subtraction in region R due to mass supplied to or from the a constituent by the n-1 other constituents of the mixture and a constituent linear momentum supplied to or from the other constituents by means other than mass exchange, such as particulate collisions or fluid drag.

Note that the term characterizing the a constituent surface forces on the surface of region R can be expressed:

$$\int_{S_R} d\underline{P}_a = \int_{S_R}\underline{t}_a^{(n)}\,ds = \int_{S_R}\underline{n}\,\underset{\sim}{T}_a\,ds \tag{38}$$

where $\underline{t}_\alpha^{(n)}$ and $\underset{\sim}{T}_\alpha$ are the α constituent traction vector and α constituent stress tensor, respectively. Applying divergence theorem to Equation 38 results in:

$$\int_{S_R} \underline{n}\,\underset{\sim}{T}_\alpha\, ds = \int_R \vec{\underline{\nabla}} \cdot \underset{\sim}{T}_\alpha\, dV \tag{39}$$

The term characterizing the convective flux of α constituent linear momentum may be rewritten about a dyadic product of the α constituent velocities and divergence theorem may then be applied to the resulting expression.

$$\int_{S_R} (\underline{n}\cdot\dot{\underline{x}}_\alpha)(\dot{\underline{x}}_\alpha \rho_\alpha)\, ds = \int_{S_R} \underline{n}(\rho_\alpha \dot{\underline{x}}_\alpha \dot{\underline{x}}_\alpha)\, ds = \int_R \vec{\underline{\nabla}} \cdot (\rho_\alpha \dot{\underline{x}}_\alpha \dot{\underline{x}}_\alpha)\, dV \tag{40}$$

Substitution of Equations 39 and 40 into 37 and collecting all terms under a single integral results in:

$$\int_R \left\{ \frac{\partial}{\partial t}\rho_\alpha \dot{\underline{x}}_\alpha + \vec{\underline{\nabla}} \cdot (\rho_\alpha \dot{\underline{x}}_\alpha \dot{\underline{x}}_\alpha) - \vec{\underline{\nabla}} \cdot \underset{\sim}{T}_\alpha - \rho_\alpha \underline{b}_\alpha - \hat{c}_\alpha \dot{\underline{x}}_\alpha - \hat{\underline{p}}_\alpha \right\} dV = 0 \tag{41}$$

Since region R is totally arbitrary this requires that:

$$\frac{\partial}{\partial t}\rho_\alpha \dot{\underline{x}}_\alpha + \vec{\underline{\nabla}} \cdot (\rho_\alpha \dot{\underline{x}}_\alpha \dot{\underline{x}}_\alpha) - \vec{\underline{\nabla}} \cdot \underset{\sim}{T}_\alpha - \rho_\alpha b_\alpha - \hat{c}_\alpha \dot{\underline{x}}_\alpha - \hat{\underline{p}}_\alpha = 0 \tag{42}$$

Consider the following reorganization of the first, second and fifth terms of Equation 42:

$$\begin{aligned}
\frac{\partial}{\partial t}\rho_\alpha \dot{\underline{x}}_\alpha + \vec{\underline{\nabla}} \cdot (\rho_\alpha \dot{\underline{x}}_\alpha \dot{\underline{x}}_\alpha) - \hat{c}_\alpha \dot{\underline{x}}_\alpha &= \frac{\partial}{\partial t}\rho_\alpha \dot{\underline{x}}_\alpha + \dot{\underline{x}}_\alpha(\vec{\underline{\nabla}} \cdot \rho_\alpha \dot{\underline{x}}_\alpha) + \rho_\alpha \dot{\underline{x}}_\alpha(\vec{\underline{\nabla}} \cdot \dot{\underline{x}}_\alpha) - \hat{c}_\alpha \dot{\underline{x}}_\alpha \\
&= \dot{\overline{\rho_\alpha \dot{\underline{x}}_\alpha}} + \rho_\alpha \dot{\underline{x}}_\alpha(\vec{\underline{\nabla}} \cdot \dot{\underline{x}}_\alpha) - \hat{c}_\alpha \dot{\underline{x}}_\alpha \\
&= \rho_\alpha \ddot{\underline{x}}_\alpha + \dot{\rho}_\alpha \dot{\underline{x}}_\alpha + \rho_\alpha \dot{\underline{x}}_\alpha(\vec{\underline{\nabla}} \cdot \dot{\underline{x}}_\alpha) - \hat{c}_\alpha \dot{\underline{x}}_\alpha \\
&= \rho_\alpha \ddot{\underline{x}}_\alpha + \dot{\underline{x}}_\alpha(\dot{\rho}_\alpha + \rho_\alpha \vec{\underline{\nabla}} \cdot \dot{\underline{x}}_\alpha - \hat{c}_\alpha) \\
&= \rho_\alpha \ddot{\underline{x}}_\alpha
\end{aligned} \tag{43}$$

The factor in parenthesis is recognizable as Equation 19, the continuity equation for the α constituent and written in this form is identically zero. Substitution of this result into Equation 42 leads to:

$$\rho_a \underline{\ddot{x}}_a = \vec{\underline{\nabla}} \cdot \underset{\sim}{T}_a + \rho_a \underline{b}_a + \underline{\hat{p}}_a \tag{44}$$

This is the balance of linear momentum equation for the a constituent.

Recall that as early as Equation 19, the continuity equation for the a constituent we have imposed the physical requirement that the mixture behavior must be the linear or unweighted sum of the constituent behaviors. We have applied this requirement to specific behaviors or quantities such as mass, linear momentum and the mass balance equations, the latter which resulted in the physically consistent result that the sum of the constituent mass supplies will be zero.

In light of these successful impositions of the "summed behavior" requirement we logically presume to impose this requirement on the balance of linear momentum.

Consider the mixture external body force term from Equation 36 and the a constituent external body force term from Equation 44 and let:

$$\rho \underline{b} = \sum_{a=1}^{n} \rho_a \underline{b}_a \tag{45}$$

Consider now that if the sum of the balance of linear momentums of the constituents is equal to the balance of linear momentums of the mixture then, via Equations 36 and 44

$$\sum_{a=1}^{n} \vec{\underline{\nabla}} \cdot \underset{\sim}{T}_a + \sum_{a=1}^{n} \rho_a \underline{b}_a + \sum_{a=1}^{n} \underline{\hat{p}}_a - \sum_{a=1}^{n} \rho_a \underline{\ddot{x}}_a = \vec{\underline{\nabla}} \cdot \underset{\sim}{T} + \rho \underline{b} - \rho \underline{\ddot{x}} \tag{46}$$

Via Equation 28, if we let $\Gamma = \underline{\dot{x}}$ and ${}_a\Gamma_a = \underline{\dot{x}}_a$ we obtain:

${}_{\grave{a}}\Gamma_a = \underline{\ddot{x}}_a$ results in:

$$\rho \underline{\ddot{x}} = \sum_{a=1}^{n} \rho_a \underline{\ddot{x}}_a + \sum_{a=1}^{n} \underline{\dot{x}}_a \hat{c}_a - \sum_{a=1}^{n} \vec{\underline{\nabla}} \cdot (\rho_a \underline{\dot{x}}_a \underline{u}_a) \tag{47}$$

Substitution of Equation 47 into 46 in light of Equation 45 results in:

$$\sum_{a=1}^{n} \vec{\underline{\nabla}} \cdot \underset{\sim}{T}_a + \sum_{a=1}^{n} \underline{\hat{p}}_a = \vec{\underline{\nabla}} \cdot \underset{\sim}{T} - \sum_{a=1}^{n} \underline{\dot{x}}_a \hat{c}_a + \sum_{a=1}^{n} \vec{\underline{\nabla}} \cdot (\rho_a \underline{\dot{x}}_a \underline{u}_a) \tag{48}$$

Recalling Equation 3 and substituting into Equation 48 leads to:

$$\sum_{a=1}^{n} \vec{\underline{\nabla}} \cdot \underset{\sim}{T}_a + \sum_{a=1}^{n} \hat{\underline{p}}_a = \vec{\underline{\nabla}} \cdot \underset{\sim}{T} - \dot{\underline{x}} \sum_{a=1}^{n} \hat{c}_a - \sum_{a=1}^{n} \hat{c}_a \underline{u}_a$$

$$+ \dot{\underline{x}} \cdot \vec{\underline{\nabla}} \sum_{a=1}^{n} \rho_a \underline{u}_a + \sum_{a=1}^{n} \vec{\underline{\nabla}} \cdot \rho \underline{u}_a \underline{u}_a \tag{49}$$

Note that by Equations 6 and 21 Equation 49 further reduces to

$$\sum_{a=1}^{n} \vec{\underline{\nabla}} \cdot \underset{\sim}{T}_a + \sum_{a=1}^{n} \hat{\underline{p}}_a = \vec{\underline{\nabla}} \cdot \underset{\sim}{T} + \sum_{a=1}^{n} \vec{\underline{\nabla}} \cdot \rho_a \underline{u}_a \underline{u}_a - \sum_{a=1}^{n} \hat{c}_a \underline{u}_a \tag{50}$$

We may deduce from the order of the various terms that:

$$\sum_{a=1}^{n} (\hat{\underline{p}}_a + \hat{c}_a \underline{u}_a) = 0 \tag{51}$$

If there is no mass exchange between constituents ($\sum_{a=1}^{n} \hat{c}_a = 0$) then we are left with the physically consistent result that the sum of the interconstituent linear momentum supplies must be zero ($\sum_{a=1}^{n} \hat{\underline{p}}_a = 0$). Further, also note that the mixture stress tensor is the sum of the partial stress tensors of the constituents and the sum of the convective flux of constituent linear momenta.

$$\underset{\sim}{T} = \sum_{a=1}^{n} (\underset{\sim}{T}_a - \rho_a \underline{u}_a \underline{u}_a) \tag{52}$$

By definition of the dyadic product the term $\rho_a \underline{u}_a \underline{u}_a$, the sum of terms $\sum_{a=1}^{n} \rho_a \underline{u}_a \underline{u}_a$ must be symmetric.

Balance of Moment of Momentum

Again, consider Figure 1. It is possible to write an expression for the balance of moment of momentum, about the origin of the spatial frame for the a constituent with region R.

$$\int_R \frac{\partial}{\partial t}(\underline{x} \times \rho_a \dot{\underline{x}}_a)\,dV$$

$$= -\int_{S_R} (\underline{x} \times \rho_a \dot{\underline{x}}_a)(\dot{\underline{x}}_a \cdot \underline{n})\,ds + \int_{S_R} \underline{x} \times d\underline{P}_a + \int_R \Big\{ \underline{x} \times (\rho_a \underline{b}_a + \hat{c}_a \dot{\underline{x}}_a + \hat{\underline{p}}_a)$$

$$+ \hat{\underline{m}}_a \Big\}\, dV \tag{53}$$

The additional term of $\hat{\underline{m}}_a$ characterizes the moment supply to and from the a constituent via couple interaction with the n-1 other constituents of the mixture.

Consider the first rightside term of Equation 53 (the flux of a constituent moment of momentum through the surface of R). Application of the divergence theorem can result with:

$$\int_{S_R} (\underline{x} \times \rho_a \dot{\underline{x}}_a)(\dot{\underline{x}}_a \cdot n)\,ds = \int_{S_R} ((\dot{\underline{x}}_a \times \rho_a \dot{\underline{x}}_a)\dot{\underline{x}}_a)\underline{n}\,ds$$

$$= \int_R ((\underline{x} \times \rho_a \dot{\underline{x}}_a)\dot{\underline{x}}_a) \cdot \overleftarrow{\underline{\nabla}}\,dV \tag{54}$$

Consider now the second rightside term of Equation 53 (the moment of momentum from the surface of R due to the surface forces of the a constituent) and note that by Equations 38:

$$\int_{S_R} \underline{x} \times d\underline{P}_a = \int_{S_R} \underline{x} \times \underline{t}_a^{(n)} ds = \int_{S_R} \underline{x} \times \underline{n}\,\underset{\sim}{T}_a ds = \int_{S_R} \underline{x} \times \underset{\sim}{T}_a^T \underline{n}\,ds \tag{55}$$

By application of the identity $\underline{a} \times (\underset{\sim}{Q}\,\underline{b}) = (\underline{a} \times \underset{\sim}{Q})\underline{b}$ for all vectors $\underline{a}$ and second order tensors $\underset{\sim}{Q}$ and the divergence theorem, Equation 55 may be written:

$$\int_{S_R} \underline{x} \times \underset{\sim}{T}_a^T \underline{n}\,ds = \int_{S_R} (\underline{x} \times \underset{\sim}{T}_a^T)\,\underline{n} = \int_R (\underline{x} \times \underset{\sim}{T}_a^T) \cdot \overleftarrow{\underline{\nabla}}\,dV \tag{57}$$

Hence, the balance of moment of momentum for the a constituent can be reorganized under a single volume integral:

$$\int_R \Big\{ \frac{\partial}{\partial t}(\underline{x} \times \rho_a \dot{\underline{x}}_a) + ((\underline{x} \times \rho_a \dot{\underline{x}}_a)\,\dot{\underline{x}}_a) \cdot \overleftarrow{\underline{\nabla}} - (\underline{x} \times \underset{\sim}{T}_a^T) \cdot \overleftarrow{\underline{\nabla}}$$

$$- \underline{x} \times (\rho_a \underline{b}_a + \hat{c}_a \dot{\underline{x}}_a + \hat{\underline{p}}_a) - \hat{\underline{m}}_a \Big\}\,dV = 0 \tag{58}$$

Since region R is totally arbitrary this requires that:

$$\frac{\partial}{\partial t}(\underline{x} \times \rho_a \dot{\underline{x}}_a) + ((\underline{x} \times \rho_a \dot{\underline{x}}_a)\, \dot{\underline{x}}_a) \cdot \overleftarrow{\underline{\nabla}} - (\underline{x} \times \underset{\sim}{T}_a^{\mathrm{T}}) \cdot \overleftarrow{\underline{\nabla}}$$

$$- \ \underline{x} \times (\rho_a \underline{b}_a + \hat{c}_a \dot{\underline{x}}_a + \hat{\underline{p}}_a) - \hat{\underline{m}}_a = 0 \tag{59}$$

Now, by application of the identity

$$(\underline{a} \times \underset{\sim}{Q}^{\mathrm{T}}) \cdot \overleftarrow{\underline{\nabla}} = \underline{a} \times (\underset{\sim}{Q}^{\mathrm{T}} \cdot \overleftarrow{\underline{\nabla}}) + \underline{Q}_{\mathrm{A}} \tag{60}$$

for all vectors $\underline{a}$ and tensors $\underset{\sim}{Q}$. $\underline{Q}_{\mathrm{A}}$ is the axial vector of tensor $\underset{\sim}{Q}$ and has the components:

$$\begin{aligned} Q_{A_1} &= Q_{23} - Q_{32} \\ Q_{A_2} &= Q_{31} - Q_{13} \\ Q_{A_3} &= Q_{12} - Q_{21} \end{aligned} \tag{61}$$

the third term of Equation 59 can be rewritten

$$(\underline{x} \times \underset{\sim}{T}_a^{\mathrm{T}}) \cdot \overleftarrow{\underline{\nabla}} = \underline{x} \times (\underset{\sim}{T}_a^{\mathrm{T}} \cdot \overleftarrow{\underline{\nabla}}) + \underline{T}_{a_{\mathrm{A}}} \tag{62}$$

Also since $\underline{x}$ is a time independent position variable, the first and second terms of Equation 59 can be reorganized such that:

$$\frac{\partial}{\partial t}(\underline{x} \times \rho_a \dot{\underline{x}}_a) + ((\underline{x} \times \rho_a \dot{\underline{x}}_a)\dot{\underline{x}}_a) \cdot \overleftarrow{\underline{\nabla}} = \underline{x} \times \left\{ \frac{\partial}{\partial t} \rho_a \dot{\underline{x}}_a + \rho_a \dot{\underline{x}}_a \cdot \overleftarrow{\underline{\nabla}}\, \dot{\underline{x}}_a + \dot{\underline{x}}_a \cdot \overleftarrow{\underline{\nabla}}\, \rho_a \dot{\underline{x}}_a \right\}$$

$$= \underline{x} \times \left\{ \overline{\rho_a \dot{\underline{x}}_a}^{\,\prime} + \dot{\underline{x}}_a \cdot \overleftarrow{\underline{\nabla}}\, \rho_a \dot{\underline{x}}_a \right\} = \underline{x} \times \left\{ \rho_a \ddot{\underline{x}}_a + \dot{\rho}_a \dot{\underline{x}}_a + \rho_a \dot{\underline{x}}_a \overrightarrow{\underline{\nabla}} \cdot \dot{\underline{x}}_a \right\} \tag{63}$$

Hence in light of Equations 62 and 63, Equation 59, the local form of the balance of moment of momentum equation, can be rewritten:

$$\underline{x} \times \left\{ \rho_a \ddot{x}_a - \underset{\sim}{T}_a^{\mathrm{T}} \cdot \overleftarrow{\underline{\nabla}} - \rho_a \underline{b}_a - \hat{\underline{p}}_a \right.$$

$$\left. + \ (\dot{\rho}_a + \rho_a \overrightarrow{\underline{\nabla}} \cdot \dot{\underline{x}}_a - \hat{c}_a)\, \dot{\underline{x}}_a \right\} - \underline{T}_{a_{\mathrm{A}}} - \hat{\underline{m}}_a = 0 \tag{64}$$

The first four terms of Equation 64 constitute the balance of linear momentum of the a constituent (Equation 44) while the fifth through seventh terms is the expression for continuity of mass for the a constituent (Equation 19). Both sets of terms, written in these forms sum identically to zero leaving the result:

$$\underline{T}_{a_A} + \underline{\hat{m}}_a = 0 \tag{65}$$

or in component form

$$\begin{aligned} T_{a_{23}} - T_{a_{32}} &= \hat{m}_{a_1} \\ T_{a_{31}} - T_{a_{13}} &= \hat{m}_{a_2} \\ T_{a_{12}} - T_{a_{21}} &= \hat{m}_{a_3} \end{aligned}$$

This result implies that any given constituent or partial stress tensor is not symmetric except in the case of no inter-constituent moment of momentum or couple interaction ($\underline{\hat{m}}_a = 0$).

If we accept the axiom of balance of moment of momentum for the mixture to be:

$$\underset{\sim}{T} = \underset{\sim}{T}^{T} \tag{66}$$

then, by Equation 52, the definition of the mixture stress tensor:

$$\sum_{a=1}^{n} (\underset{\sim}{T}_a - \rho_a \underline{u}_a \underline{u}_a) = \sum_{a=1}^{n} (\underset{\sim}{T}_a^{T} - \rho_a \underline{u}_a \underline{u}_a) \tag{67}$$

or

$$\sum_{a=1}^{n} \underset{\sim}{T}_a = \sum_{a=1}^{n} \underset{\sim}{T}_a^{T}$$

In words, the balance of moment of momentum principle shows that the per constituent or partial stress tensors may not be symmetric but that the sum of the partial stress tensors is symmetric.

In summary we can tabulate for the mixture and on a per constituent basis the results of this mixture theory formulation:

Continuity (Balance of Mass)

$$\frac{\partial}{\partial t}\rho + \underline{\vec{\nabla}} \cdot \rho \dot{\underline{x}} = 0 \qquad \text{(mixture)} \qquad (68)$$

$$\frac{\partial}{\partial t}\rho_a + \underline{\vec{\nabla}} \cdot \rho_a \dot{\underline{x}}_a = \hat{c}_a \qquad (a \text{ constituent}) \qquad (69)$$

$$\text{subject to } \sum_{a=1}^{n} \hat{c}_a = 0 \qquad (70)$$

Balance of Linear Momentum

$$\rho \ddot{\underline{x}} = \underline{\vec{\nabla}} \cdot \underset{\sim}{T} + \rho \underline{b} \qquad \text{(mixture)} \qquad (71)$$

$$\rho_a \ddot{\underline{x}}_a = \underline{\vec{\nabla}} \cdot \underset{\sim}{T}_a + \rho_a \underline{b}_a + \hat{\underline{p}}_a \qquad (a \text{ constituent}) \qquad (72)$$

$$\text{subject to } \sum_{a=1}^{n} (\hat{\underline{p}}_a + \hat{c}_a \underline{u}_a) = 0 \qquad (73)$$

and

$$\underset{\sim}{T} = \sum_{a=1}^{n} (\underset{\sim}{T}_a - \rho_a \underline{u}_a \underline{u}_a) \qquad (74)$$

Balance of Moment of Momentum

$$\underset{\sim}{T} = \underset{\sim}{T}^T \qquad \text{(mixture)} \qquad (75)$$

$$\underline{T}_{a_A} + \hat{\underline{m}}_a = 0 \qquad (a \text{ constituent}) \qquad (76)$$

$$\text{subject to: } \sum_{a=1}^{n} \underset{\sim}{T}_a = \sum_{a=1}^{n} \underset{\sim}{T}_a^T \qquad (77)$$

CHAPTER 3

THE THEORY OF TURBULENT, WIND-AIDED SNOW SEDIMENTATION

In the previous chapter a mixture theory is developed, through the momentum balance equations for a mixture of n constituents which respond as a whole in a fluid manner.

It will be of interest to apply this theory to the well documented but mechanically complex phenomena of turbulent wind-aided snow sedimentation. This investigation is made for two purposes, one: to determine whether or not solutions to the theory exist and two: to detemine if these solutions model or approximate in a qualitative sense certain observed aspects of the phenomena. However, in this next chapter and preliminary to any applications it will be necessary to derive the turbulent equations of motion for the snow phase of the mixture flow.

The Equations of Motion for a Mixture of Snow and Air

The pertinent equations of motion for a mixture of snow entrained in an atmospheric flow are summarized below (Decker and Brown, 1983). The subscript s and a are indicative of the snow phase and air phase respectively of the mixture.

Continuity

$$\frac{\partial}{\partial t} \rho_s + \vec{\underline{\nabla}} \cdot \rho_s \underline{\dot{x}}_s = \hat{c}_s \qquad (78)$$

$$\frac{\partial}{\partial t} \rho_s + \vec{\underline{\nabla}} \cdot \rho_s \underline{\dot{x}}_a = \hat{c}_a \qquad (79)$$

and

Balances of Linear Momentum

$$\rho_s \ddot{\underline{x}}_s = \vec{\underline{\nabla}} \cdot \underset{\sim}{T}_s + \rho_s \underline{b}_s + \hat{\underline{p}}_s \tag{80}$$

$$\rho_a \ddot{\underline{x}}_a = \vec{\underline{\nabla}} \cdot \underset{\sim}{T}_a + \rho_a \underline{b}_a + \hat{\underline{p}}_a \tag{81}$$

Note that $\dot{\underline{x}}_s = \underline{u}_s$ and $\dot{\underline{x}}_a = \underline{u}_a$. These are the velocity fields for the snow phase and air phase respectively and should not be confused with diffusion velocities as previously defined.

Consider that if the intrinsic time scale of the snow sedimentation process ($>$ 1.0 hr) is large compared to the inertial time scale ($\sim$ 1.0 sec) of the process, then the transient effects may be neglected and the process could be considered steady. This assumption is consistent with other attempts to apply a mixture theory to sedimentation processes (Drew, 1975; McTique, 1981, 1983). Further, consider that if approximately 40% of the total entrained snow in transport over a flat surface sublimates into the air phase over a distance of 3 km (Schmidt, 1982), then for the transport distances of this investigation (10 m), it is reasonable to neglect the effects of inter-phase mass exchange. Further, it is assumed that the only body force acting on either the snow or air phase is the gravitational potential. However, it should be noted that the formation of snow cornices does occur in blowing snow environments on the immediate lee of mountain slopes and there is good evidence to support the hypothesis that snow cornices form as a consequence of very large electrostatic potentials (Latham & Montagne, 1970). Lastly, it is assumed that if the mass fraction of snow in transport remains small ($<$ 10%) compared to the total mass of the mixture. Therefore the components of the partial stress tensor of the snow phase are negligible relative to those of the air phase. This assumption is consistent with those postulated for dispersed multi-phase flows (McTique, 1983) and is analogous to the statement that as long as the mass fraction of snow in transport remains small the streamlines of the airflow will remain essentially unchanged from those of single phase airflow, i.e., the snow phase does not have a significant effect upon the air phase flow.

In light of these assumptions the equations of motions for the snow phase when the motion of the air phase is known are:

$$\underline{\vec{\nabla}} \cdot \rho_s \underline{u}_s = 0 \qquad \text{Continuity} \tag{82}$$

and

$$\rho_s \underline{u}_s \cdot \underline{\vec{\nabla}}\, \underline{u}_s = \rho_s \underline{g} + \hat{\underline{p}}_s \qquad \text{Balance of Linear Momentum} \tag{83}$$

Momentum Supply Between the Air and Snow Phases

At this point it is necessary to make a constitutive assumption for the momentum supply or transfer terms: $\hat{\underline{p}}_a$ and $\hat{\underline{p}}_s$ between the air phase and snow phase of the mixture flow. Let:

$$\hat{\underline{p}}_a = -\hat{\underline{p}}_s = \rho_s D\,(\underline{u}_s - \underline{u}_a) \tag{84}$$

That is, the momentum supply between the air and snow phases is equal and opposite in sign and is dependent on the velocity difference between the phases and a drag coefficient: D with dimensions 1/time. This satisfies the restriction of Equation 51 on the balance of linear momentum when mass supply: $\hat{c}_s = \hat{c}_a = 0$. Further, momentum supply between the phases is dependent on the local mass density of the snow phase, which is physically consistent inasmuch as mass density of the snow phase approaches zero so does the momentum supply between the phases.

Equation 84 characterizes the transfer of momentum between the snow and air phase of the mixture flow. It is effectively a description of the drag between the air and snow phases. It would be possible to describe this term as the difference of some polynomial value of these velocities. The difference of the squares of the snow phase and air phase velocities would be a logical increase in the degree of theoretical complexity. This would lead to an increase in the degree of non-linearity of the resulting systems of partial differential equations to be solved for the snow phase velocity field. Since this system of partial differential equations will be solved by numerical approximation techniques the instability of the subsequent algebraic equation system may also increase. Also, a velocity squared drag or momentum transfer term may not be an objective or frame indifferent constitutive assumption. Lastly, it will be shown that the theory in fact oversuspends the snow phase within the mixture flow. Increasing the magnitude of the momentum transfer between the phases via a velocity squared drag term would increase the oversuspension of the snow phase in the mixture flow.

For any constitutive assumption the magnitude and direction of the term must remain objective or retains its values regardless of the frame of reference, i.e., a valid constitutive assumption must be indifferent to any time dependent orthogonal change of reference frame. Therefore now consider that for any time dependent orthogonal transformation of coordinate frame $\underset{\sim}{Q}$ where $\dot{\underline{x}}_s$ and $\dot{\underline{x}}_a$ are the snow and air velocities with respect to the original coordinate frame and $\dot{\underline{x}}_s{}^*$ and $\dot{\underline{x}}_a{}^*$ are these velocities with respect to the new coordinate frame such that:

$$\underline{x}_s{}^* = \underset{\sim}{Q}\,\underline{x}_s + \underline{c} \tag{85}$$

and

$$\underline{x}_a{}^* = \underset{\sim}{Q}\,\underline{x}_a + \underline{c} \tag{86}$$

Where $\underline{c}$ is a time dependent translation of the coordinate frame.

Then:

$$\underline{u}_s{}^* = \dot{\underline{x}}_s{}^* = \overset{\cdot}{\overline{(\underset{\sim}{Q}\,\underline{x}_s + \underline{c})}} = \underset{\sim}{Q}\,\dot{\underline{x}}_s + \dot{\underset{\sim}{Q}}\,\underline{x}_s + \dot{\underline{c}} \tag{87}$$

and

$$\underline{u}_a{}^* = \dot{\underline{x}}_a{}^* = \overset{\cdot}{\overline{(\underset{\sim}{Q}\,\underline{x}_a + \underline{c})}} = \underset{\sim}{Q}\,\dot{\underline{x}}_a + \dot{\underset{\sim}{Q}}\,\underline{x}_a + \dot{\underline{c}} \tag{88}$$

Therefore

$$\underline{u}_s^* - \underline{u}_a^* = \underset{\sim}{Q}\,\dot{\underline{x}}_s + \dot{\underset{\sim}{Q}}\,\underline{x}_s + \dot{\underline{c}} - \underset{\sim}{Q}\,\dot{\underline{x}}_a - \dot{\underset{\sim}{Q}}\,\underline{x}_a - \dot{\underline{c}}$$

$$= \underset{\sim}{Q}\,(\dot{\underline{x}}_s - \dot{\underline{x}}_a) + \dot{\underset{\sim}{Q}}\,(\underline{x}_s - \underline{x}_a) \qquad (89)$$

Since $\underline{x}_s = \underline{x}_a$ (i.e., the position vectors to the point where the momentum transfer is effected are the same) then

$$\underline{u}^*_s - \underline{u}^*_a = \underset{\sim}{Q}\,(\underline{u}_s - \underline{u}_a) \qquad (90)$$

Hence, the constitutive assumptions on the supply or transfer of momentum between the air and snow phases is objective or frame indifferent (Malvern, 1969).

In summary, the equations of motion for the snow phase of the mixture flow are:

$$\underline{\vec{\nabla}} \cdot \rho_s \underline{u}_s = 0 \qquad \text{Continuity} \qquad (91)$$

and

$$\rho_s \underline{u}_s \cdot \underline{\vec{\nabla}}\, \underline{u}_s = \rho_s \underline{g} - \rho_s D(\underline{u}_s - \underline{u}_a) \qquad \text{Balance of Linear Momentum} \qquad (92)$$

The Turbulent Equations of Motion for the Snow Phase of the Mixture Flow

Consider the dimensionless Reynolds's number, classically a ratio or measure of the inertial forces to the viscous forces of a flow. For large Reynold's numbers, when inertia is the predominant force of a flow these flows are observed to be turbulent. For a variety of engineering applications a large amount of research has been done on determining the "critical" Reynold's number for which a laminar flow may become unstable and go through transition to turbulent. At Reynold's number much greater than the critical Reynold's number the flow will be fully turbulent.

By definition the Reynold's number is:

$$R = \frac{V_{ref}\, L_{ref}}{\nu} \qquad (93)$$

where V_{ref} is a reference or characteristic velocity, L_{ref} is a characteristic length and ν is the kinematic viscosity of the fluid. Obviously the magnitude of the Reynold's number for any given flow is controlled in part by the researcher's perception of what constitutes a "characteristic" set of velocities and lengths.

In the specific case of atmospheric flows there is a large volume of research investigating the domains of these characteristic dimensions. However, irregardless of the V_{ref} and L_{ref} chosen for an atmospheric flow $\nu_{air} \simeq 1.3 \times 10^{-5}$ m²/s. Consequently, over the resulting range of Reynold's numbers, the magnitudes of these Reynold's numbers are quite large and these flows are considered fully turbulent (Britter, Hunt, & Richards, 1981; Bradley, 1980; Jackson & Hunt, 1975; Plate, 1971).

The equations of motion of the snow phase of the mixture flow can be decomposed and expanded to include the effect of a turbulent mixture flow.

Adopting the standard Reynold's description (Hinze, 1975) of a turbulent variable for the snow phase results in:

$$\underline{u}_s = \overline{\underline{u}}_s + \underline{u}'_s \tag{94}$$

$$\underline{u}_a = \overline{\underline{u}}_a + \underline{u}'_a \tag{95}$$

$$\rho_s = \overline{\rho}_s + \rho'_s \tag{96}$$

That is, the instantaneous value of any turbulent variable is the sum of its mean or time averaged value and its turbulent fluctuations, denoted by the overbar and overscore, respectively. Any product of mean and fluctuating variables may be subject to additional time averaging and must conform to the following conditions, where for any time dependent variables g and f:

$$\overline{\overline{f}} = \overline{f} \tag{97}$$

$$\overline{f} \pm \overline{g} = \overline{f \pm g} \tag{98}$$

$$\overline{\overline{f}g} = \overline{f}\,\overline{g} \tag{99}$$

$$\overline{\underline{\vec{\nabla}} f} = \underline{\vec{\nabla}}\,\overline{f} \tag{100}$$

$$\overline{f'} = 0 \tag{101}$$

$$\overline{f'g'} \neq \overline{f'}\,\overline{g'} = 0 \tag{102}$$

$$\overline{(f')^{2n-1}} = 0 \qquad n = 1, 2, 3, \ldots \tag{103}$$

$$\overline{(f')^{2n}} \neq 0 \qquad n = 1, 2, 3, \ldots \tag{104}$$

Substituting Equations 94 and 96 into the continuity equation (Equation 91) for the snow phase of the mixture flow results in:

$$\underline{\vec{\nabla}} \cdot (\overline{\rho}_s + \rho'_s)(\underline{\overline{u}}_s + \underline{u}'_s) = 0 \tag{105}$$

or

$$\underline{\vec{\nabla}} \cdot \left\{ \overline{\rho}_s \underline{\overline{u}}_s + \overline{\rho}_s \underline{u}'_s + \rho'_s \underline{\overline{u}}_s + \rho'_s \underline{u}'_s \right\} = 0$$

When a subsequent time average of Equation 105 is taken the terms $\overline{\overline{\rho}_s \underline{u}'_s}$ and $\overline{\rho'_s \underline{\overline{u}}_s}$, by Equation 101 are identically zero, resulting in:

$$\underline{\vec{\nabla}} \circ \left\{ \overline{\rho}_s \underline{\overline{u}}_s + \overline{\rho'_s \underline{u}'_s} \right\} = 0 \tag{106}$$

Now consider that if the turbulent fluctuations of a variable are joint-normally distributed then the conservation of mass at a point will be independent of the turbulence and the turbulent continuity equations will be:

$$\underline{\vec{\nabla}} \cdot \rho_s \underline{u}_s = \underline{\vec{\nabla}} \cdot \overline{\rho}_s \underline{\overline{u}}_s = 0 \tag{107}$$

This requires that:

$$\underline{\vec{\nabla}} \cdot \overline{\rho' \underline{u}'} = 0 \tag{108}$$

In the absence of any information about the nature of a turbulent flow the assumption that the turbulence is joint-normally distributed is a simplication but is also the only logical description available (Oral communication, J. T. Oden). The resultant turbulent continuity equation (Equation 107) is identical with that derived by Drew (1975) and McTique (1981) for the particulate phase of a multiphase turbulent flow.

Substitution of Equations 94 through 96 into the linear momentum balance equation (Equation 92) for the snow phase of the mixture flow results in:

$$\left\{(\bar{\rho}_s + \rho'_s)(\underline{\bar{u}}_s + \underline{u}'_s)\right\} \cdot \underline{\vec{\nabla}}\,(\underline{\bar{u}}_s + \underline{u}'_s)$$

$$= (\bar{\rho}_s + \rho'_s)\underline{g} - (\bar{\rho}_s + \rho'_s)\,D\,(\underline{\bar{u}}_s + \underline{u}'_s - \underline{\bar{u}}_a - \underline{u}'_a) \qquad (109)$$

or

$$(\bar{\rho}_s\underline{\bar{u}}_s + \bar{\rho}_s\underline{u}'_s + \rho'_s\underline{\bar{u}}_s + \rho'_s\underline{u}'_s) \circ (\underline{\vec{\nabla}}\,\underline{\bar{u}}_s + \underline{\vec{\nabla}}\,\underline{u}'_s)$$

$$= \bar{\rho}_s\underline{g} + \rho'_s\underline{g} - \bar{\rho}_s D(\underline{\bar{u}}_s - \underline{\bar{u}}_a) - \bar{\rho}_s D(\underline{u}'_s - \underline{u}'_a)$$

$$- \rho'_s D(\underline{\bar{u}}_s - \underline{\bar{u}}_a) - \rho'_s D(\underline{u}'_s - \underline{u}'_a)$$

When the left side is expanded and a subsequent time average of Equation 109 is taken, by Equations 101 and 103 the terms $\overline{\bar{\rho}_s\underline{\bar{u}}_s \cdot \underline{\vec{\nabla}}\,\underline{u}'_s}$, $\overline{\bar{\rho}_s\underline{u}'_s \cdot \underline{\vec{\nabla}}\,\underline{\bar{u}}_s}$, $\overline{\rho'_s\underline{\bar{u}}_s \cdot \underline{\vec{\nabla}}\,\underline{\bar{u}}_s}$, $\overline{\rho'_s\underline{u}'_s \cdot \underline{\vec{\nabla}}\,\underline{u}'_s}$, $\overline{\rho'_s\underline{g}}$, $\overline{\bar{\rho}_s D(\underline{u}'_s - \underline{u}'_a)}$ and $\overline{\rho'_s D(\underline{\bar{u}}_s - \underline{\bar{u}}_a)}$ are identically zero resulting in:

$$\bar{\rho}_s\underline{\bar{u}}_s \cdot \underline{\vec{\nabla}}\,\underline{\bar{u}}_s = \bar{\rho}_s\underline{g} - \rho_s D(\underline{\bar{u}}_s - \underline{\bar{u}}_a) - \underline{F}_T \qquad (110)$$

where

$$\underline{F}_T = \overline{\bar{\rho}_s\underline{u}'_s \cdot \underline{\vec{\nabla}}\,\underline{u}'_s} + \overline{\rho'_s\underline{\bar{u}}_s \cdot \underline{\vec{\nabla}}\,\underline{u}'_s} + \overline{\rho'_s\underline{u}'_s} \cdot \underline{\vec{\nabla}}\,\underline{\bar{u}}_s + \overline{\rho'_s D(\underline{u}'_s - \underline{u}'_a)}$$

Note that if $\rho'_s \ll \bar{\rho}_s$, $\underline{u}'_s \ll \underline{\bar{u}}_s$, $\underline{u}'_a \ll \underline{\bar{u}}_a$ and $\underline{u}'_s \simeq \underline{u}'_a$ then the fourth term of $\underline{F}_T$ may be assumed to be negligibly small relative to the other terms.

The Constitutive Assumption for Turbulent Fluctuating Variables in Terms of Mean Flow Variables

In order to render the system determinate, a constitutive assumption relating the turbulent fluctuation of a given variable in terms of a mean flow variable must be made. If the bulk of the mixture flow inertia and all the shear of the mixture flow are intrinsic to the air phase then it would be consistent to relate the turbulent fluctuations of the snow phase variables to a mean flow variable of the air. Consider then the following constitutive assumption for ρ'_s and $\underline{u}'_s$.

$$\rho'_s = \frac{\sqrt{2}\,\bar{\rho}_s\,\underline{\epsilon}}{3} \cdot \underline{1}\sqrt{II_{\underset{\sim}{\bar{\Delta}}_a}} \tag{111}$$

$$\underline{u}'_s = \frac{\underline{\gamma}}{\sqrt{2}} \sqrt{II_{\underset{\sim}{\bar{\Delta}}_a}} \tag{112}$$

where:

$$\underset{\sim}{\bar{\Delta}}_a = \begin{bmatrix} 0 & \dfrac{\partial \bar{u}_a}{\partial y} & \dfrac{\partial \bar{u}_a}{\partial z} \\[2ex] \dfrac{\partial \bar{v}_a}{\partial x} & 0 & \dfrac{\partial \bar{v}_a}{\partial z} \\[2ex] \dfrac{\partial \bar{\omega}_a}{\partial x} & \dfrac{\partial \bar{\omega}_a}{\partial y} & 0 \end{bmatrix} \tag{113}$$

$$II_{\underset{\sim}{\bar{\Delta}}_a} = \frac{1}{2}\,\underset{\sim}{\bar{\Delta}}_{a_{ij}}\,\underset{\sim}{\bar{\Delta}}_{a_{ij}}\,,\ i \neq j \tag{114}$$

$II_{\underset{\sim}{\bar{\Delta}}_a}$ is the second scalar invariant of the deviatoric mean airflow gradient tensor. $\bar{u}_a$, $\bar{v}_a$ and $\bar{\omega}_a$ are the components of the mean airflow velocities. $\underline{\epsilon}$ is a vector valued function with the dimensions of time and $\underline{\gamma}$ is a vector valued function with the dimensions of length.

Note that the constitutive assumption for the turbulent fluctuation is, by definition of a scalar valued variable objective (Malvern, 1969). The form given by Equation 112 must be proven frame indifferent. For any orthogonal transformation of coordinate frame

$\underset{\sim}{Q}$ that $\underline{\gamma}^* = \underset{\sim}{Q}\,\underline{\gamma}$ where the starred quantity is the value of that variable in the transformed coordinate frame.

$$\underline{u}'_s{}^* = \frac{\underline{\gamma}^*}{\sqrt{2}} \sqrt{II_{\underset{\sim}{\bar{\Lambda}}_a}}$$

$$= \underset{\sim}{Q} \frac{\underline{\gamma}}{\sqrt{2}} \sqrt{II_{\underset{\sim}{\bar{\Lambda}}_a}} \qquad (115)$$

Note that $\underline{u}'_s{}^*$ transforms like a frame indifferent vector value function and hence the constitutive assumption on $\underline{u}'_s$ is objective.

For the special case of one-dimensional mixture flow $(\frac{\partial}{\partial x} = \frac{\partial}{\partial z} = \bar{v}_a = \bar{\omega}_a = 0)$ the constitutive assumptions for ρ'_s and u'_s reduce to

$$\rho'_s = \frac{\rho_s(\epsilon_x + \epsilon_y + \epsilon_z)}{3} \sqrt{(\frac{\partial \bar{u}_a}{\partial y})^2} = \frac{\rho_s(\epsilon_x + \epsilon_y + \epsilon_z)}{3} \left| (\frac{\partial \bar{u}_a}{\partial y}) \right| \qquad (116)$$

$$u'_s = \frac{\gamma_x}{2} \sqrt{(\frac{\partial \bar{u}_a}{\partial y})^2} = \frac{\gamma_x}{2} \left| (\frac{\partial \bar{u}_a}{\partial y}) \right| \qquad (117)$$

These results are analogous to the one-dimensional phenomenologically derived Prandtl mixing length theory for turbulent single phase flow (Hinze, 1975; Schlicting, 1979).

By substitution of the constitutive assumptions for ρ'_s and $\underline{u}'_s$ into the terms of F_T the turbulent balance of linear momentum equation for the snow phase of the mixture flow results

$$\bar{\rho}_s \underline{\bar{u}}_s \cdot \underline{\vec{\nabla}}\, \underline{\bar{u}}_s = \rho_s \underline{g} - \rho_s D(\underline{\bar{u}}_s - \underline{\bar{u}}_a) - F_T$$

where

$$F_T = \bar{\rho}_s \Big\{ \frac{\underline{\gamma}}{\sqrt{2}} \sqrt{II_{\underset{\sim}{\bar{\Lambda}}_a}} \cdot \underline{\vec{\nabla}}\, \frac{\underline{\gamma}}{\sqrt{2}} \sqrt{II_{\underset{\sim}{\bar{\Lambda}}_a}} + (\frac{\sqrt{2}\underline{\epsilon}}{3} \cdot \underline{1} \sqrt{II_{\underset{\sim}{\bar{\Lambda}}_a}}) \underline{\bar{u}}_s \cdot \underline{\vec{\nabla}}\, \frac{\underline{\gamma}}{\sqrt{2}} \sqrt{II_{\underset{\sim}{\bar{\Lambda}}_a}}$$

$$+ ((\frac{\sqrt{2}\underline{\epsilon}}{3} \cdot \underline{1} \sqrt{II_{\underset{\sim}{\bar{\Lambda}}_a}}) \frac{\underline{\gamma}}{\sqrt{2}} \sqrt{II_{\underset{\sim}{\bar{\Lambda}}_a}}) \cdot \underline{\vec{\nabla}}\, \underline{\bar{u}}_s \Big\} \qquad (117)$$

Equation 117, the turbulent balance of linear momentum equation along with the turbulent continuity equation below:

$$\vec{\underline{\nabla}} \cdot \bar{\rho}_s \underline{\bar{u}}_s = 0 \tag{118}$$

form the turbulent equations of motion for the snow phase of the mixture flow.

For the case of a two dimensional mixture flow $(\frac{\partial}{\partial z} = \omega_a = \omega_s = 0)$ the constitutive assumptions on ρ'_s and $\underline{u}'_s$ reduce to

$$\rho'_s = \frac{\sqrt{2}\ \bar{\rho}_s \underline{\epsilon}}{3} \cdot \underline{1} \sqrt{\left\{ \left(\frac{\partial \bar{u}_a}{\partial y}\right)^2 + \left(\frac{\partial \bar{v}_a}{\partial x}\right)^2 \right\}} \tag{119}$$

and

$$\underline{u}'_s = \frac{\gamma}{2} \sqrt{\left\{ \left(\frac{\partial \bar{u}_a}{\partial y}\right)^2 + \left(\frac{\partial \bar{v}_a}{\partial x}\right)^2 \right\}} \tag{120}$$

Consider the following upper bound approximation to the factor:

$$\sqrt{\left\{ \left(\frac{\partial \bar{u}_a}{\partial y}\right)^2 + \left(\frac{\partial \bar{v}_a}{\partial x}\right)^2 \right\}} = \left(\left|\frac{\partial \bar{u}_a}{\partial y}\right| + \left|\frac{\partial \bar{v}_a}{\partial x}\right| \right) \tag{121}$$

The error inherent in this approximation can be analyzed by considering the binomial expansion:

$$\begin{aligned}\sqrt{\left\{ \left(\frac{\partial \bar{u}_a}{\partial y}\right)^2 + \left(\frac{\partial v_a}{\partial x}\right)^2 \right\}} &= \left(\frac{\partial \bar{u}_a}{\partial y} + \frac{\partial \bar{v}_a}{\partial x}\right) \sqrt{\left\{ 1 - \frac{2\left(\frac{\partial \bar{u}_a}{\partial y}\right)\left(\frac{\partial \bar{v}_a}{\partial x}\right)}{\left(\frac{\partial \bar{u}_a}{\partial y} + \frac{\partial \bar{v}_a}{\partial x}\right)^2} \right\}} \\ &= \left(\frac{\partial \bar{u}_a}{\partial y} + \frac{\partial \bar{v}_a}{\partial x}\right) \left\{ 1 - \frac{\left(\frac{\partial \bar{u}_a}{\partial y}\right)\left(\frac{\partial \bar{v}_a}{\partial x}\right)}{\left(\frac{\partial \bar{u}_a}{\partial y} + \frac{\partial \bar{v}_a}{\partial x}\right)^2} + \frac{\left(\frac{\partial \bar{u}_a}{\partial y}\right)^2 \left(\frac{\partial \bar{v}_a}{\partial x}\right)^2}{2(2!)\left(\frac{\partial \bar{u}_a}{\partial y} + \frac{\partial \bar{v}_a}{\partial x}\right)^4} + \ldots \right\}\end{aligned} \tag{122}$$

Inasmuch as all terms of the expansion are positive after the first sum in the expansion truncation at the second sum will result in a measure of the maximum error. Note that the error in this approximation is large when gradients of the airflow are small. However when gradients of the airflow are small the terms of $\underline{F}_T$ are also small and the error to ρ'_s and

$\underline{u}'_s$ will not be of consequence. Further, when gradients of the airflow are large the terms of $\underline{F}_T$ will be large but the error to ρ'_s and $\underline{u}'_s$ will be minimized.

By evoking the concept of a joint-normally distributed turbulence it is possible to write $\underline{\gamma} = \gamma \, \underline{\underline{1}}$ and $\underline{\epsilon} = \epsilon \, \underline{1}$.

Lastly, the turbulent balance of linear momentum equation (Equation 118) can have ρ_s divided out on a termwise basis. This results in the non-conservative form of the turbulent balance of linear momentum equation, which is uncoupled from the turbulent continuity equation.

In light of the above for a two dimensional mixture flow, the components of the non-conservative turbulent momentum balance equation are

$$u_s \frac{\partial u_s}{\partial x} + v_s \frac{\partial u_s}{\partial y} = g_x - D(u_s - u_a) - F_{T_x} \qquad \text{x-comp}$$

$$u_s \frac{\partial v_s}{\partial x} + v_s \frac{\partial v_s}{\partial y} = g_y - D(v_s - v_a) - F_{T_y} \qquad \text{y-comp} \tag{123}$$

where

$$F_{T_x} = \epsilon\gamma C \frac{\partial u_s}{\partial x} + \epsilon\gamma C \frac{\partial u_s}{\partial y} + \epsilon\gamma A u_s + \epsilon\gamma B v_s + \frac{\gamma^2}{4} (A+B)$$

$$F_{T_y} = \epsilon\gamma C \frac{\partial v_s}{\partial x} + \epsilon\gamma C \frac{\partial v_s}{\partial y} + \epsilon\gamma A u_s + \epsilon\gamma B v_s + \frac{\gamma^2}{4} (A+B) \tag{124}$$

and

$$A = \left|\frac{\partial v_a}{\partial x}\right| \left(\left|\frac{\partial^2 v_a}{\partial x^2}\right| + \left|\frac{\partial^2 u_a}{\partial x \partial y}\right|\right) + \left|\frac{\partial u_a}{\partial y}\right| \left(\left|\frac{\partial^2 v_a}{\partial x^2}\right| + \left|\frac{\partial^2 u_a}{\partial x \partial y}\right|\right)$$

$$B = \left|\frac{\partial v_a}{\partial x}\right| \left(\left|\frac{\partial^2 v_a}{\partial x \partial y}\right| + \left|\frac{\partial^2 u_a}{\partial y^2}\right|\right) + \left|\frac{\partial u_a}{\partial y}\right| \left(\left|\frac{\partial^2 v_a}{\partial x \partial y}\right| + \left|\frac{\partial^2 u_a}{\partial y^2}\right|\right) \qquad (125)$$

$$C = \left|\frac{\partial v_a}{\partial x}\right| \left(\left|\frac{\partial v_a}{\partial x}\right| + \left|\frac{\partial u_a}{\partial y}\right|\right) + \left|\frac{\partial u_a}{\partial y}\right| \left(\left|\frac{\partial v_a}{\partial x}\right| + \left|\frac{\partial u_a}{\partial x}\right|\right)$$

u_s, v_s, ω_s, u_a, v_a, ω_a are the components, in rectangular cartesian coordinates of the snow and air phase velocity vectors: $\underline{u}_s$ and $\underline{u}_a$. The overscore has been dropped, since now all variables are in terms of the mean values.

These component equations can be reorganized into the following system of non-linear partial differential equations:

$$\begin{bmatrix} (u_s + \epsilon\gamma C) & 0 \\ 0 & (u_s + \epsilon\gamma C) \end{bmatrix} \frac{\partial}{\partial x} \begin{Bmatrix} u_s \\ v_s \end{Bmatrix} + \begin{bmatrix} (v_s + \epsilon\gamma C) & 0 \\ 0 & (v_s + \epsilon\gamma C) \end{bmatrix} \frac{\partial}{\partial y} \begin{Bmatrix} u_s \\ v_s \end{Bmatrix} +$$

$$\begin{bmatrix} (D + \epsilon\gamma A) & B \\ A & (D + \epsilon\gamma B) \end{bmatrix} \begin{Bmatrix} u_s \\ v_s \end{Bmatrix} = \begin{Bmatrix} g_x + Du_a - \frac{\gamma^2}{4} (A+B) \\ g_y + Dv_a - \frac{\gamma^2}{4} (A+B) \end{Bmatrix} \qquad (126)$$

The two dimensional turbulent continuity equation is summarized below:

$$\frac{\partial}{\partial x} \rho_s u_s + \frac{\partial}{\partial y} \rho_s v_s = 0 \qquad (127)$$

For the case of one dimensional mixture flow ($\frac{\partial}{\partial x} = \frac{\partial}{\partial z} = \omega_a = v_a = \omega_s = 0$) the component form of the momentum balance equation for the snow phase reduces to the following system of non-linear partial differential equations:

$$(v_s + \epsilon\gamma c) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial}{\partial y} \begin{Bmatrix} u_s \\ v_s \end{Bmatrix} + \begin{bmatrix} D & \epsilon\gamma B \\ 0 & (D + \epsilon\gamma B) \end{bmatrix} \begin{Bmatrix} u_s \\ v_s \end{Bmatrix} = \begin{Bmatrix} g_x + Du_a - \frac{\gamma^2}{4} B \\ g_y - \frac{\gamma^2}{4} B \end{Bmatrix} \tag{128}$$

The turbulent continuity equation for the snow phase reduces to:

$$\frac{\partial}{\partial y} \rho_s v_s = 0 \tag{129}$$

Lastly, by examining the nonconservative form of the turbulent balance of linear momentum equation for the case of still air ($\omega_a = v_a = u_a = 0$) mixture motion the coefficient: D of the momentum supply or transfer term may be analyzed. Consider:

$$v_s \frac{\partial v_s}{\partial y} + Dv_s = g_y \tag{130}$$

or

$$\frac{\partial v_s}{\partial y} = \frac{g_y}{v_s} - D \tag{131}$$

If g_y is the gravitational acceleration (-9.81 m/sec^2) and the snow phase mixture motion has a domain of *constant* still air fall velocities of -1.0 to -0.5 m/sec then D will have a range from 9.81 to 19.6 1/sec. For all following calculations D is set at 13.0 1/sec. This corresponds to a still air fall velocity of -0.75 m/sec.

CHAPTER 4

SOLUTIONS OF THE SNOW PHASE EQUATIONS OF MOTION FOR A ONE DIMENSIONAL AIRFLOW

In the previous chapter the turbulent equations of motion for the snow phase of an atmospheric mixture flow of snow and air were derived.

It is of interest to solve these equations of motion for the snow phase velocity and density profiles as function of height above the solid surface for a one dimensional atmospheric boundary layer flow. These solutions can be compared with observations concerning the nature of the snow phase velocity and density profiles for snow sedimentation flows over flat surfaces (Budd, Dingle & Radok, 1965; Mellor, 1965; Kobayashi, 1972; Schmidt, 1977; Takeuchi, 1980).

Consider a one dimensional turbulent atmospheric boundary layer airflow of the form (Plate, 1971):

$$\underline{u}_a = u_a = \frac{u^*}{k} \ln \frac{y}{y_o} \tag{132}$$

with the following gradients

$$\frac{\partial u_a}{\partial y} = \frac{u^*}{ky}$$

$$\frac{\partial^2 u_a}{\partial y^2} = -\frac{u^*}{ky^2} \tag{133}$$

where u^*, k and y_o are the friction velocity, Von Karman's constant and the surface roughness height respectively.

For this specific case of a one dimensional airflow the system of partial differential equations (Equation 128) for the snow phase may be solved by finite difference techniques for the component snow phase velocities as functions of height above the solid boundary.

Consider the forward difference approximation to $\frac{\partial}{\partial y}$ (Ames, 1977; Smith, 1978) applied to the snow phase momentum balance partial differential equation system. j is the cell location description on a vertical column solution domain of m cells and G is the cell height. The snow phase subscript, s, has been dropped.

$$(v_j^{n-1}+\epsilon\gamma c_j)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}(u_{j+1}^n-u_j^n)/G\\ (v_{j+1}^n-v_j^n)/G\end{Bmatrix}+\begin{bmatrix}D & \epsilon\gamma B_j\\ 0 & (D+\epsilon\gamma B_j)\end{bmatrix}\begin{Bmatrix}u_j^n\\ v_j^n\end{Bmatrix}$$

$$=\begin{Bmatrix}Du_{a_j}-\frac{\gamma^2}{4}B_j\\ g_y-\frac{\gamma^2}{4}B_j\end{Bmatrix}\qquad\begin{matrix}n=1,2,3\ldots\\ j=1,2,3,\ldots,m\end{matrix}\tag{134}$$

The non-linear term is treated quasi-linearly by retaining its value from the n−1 iteration during the nth solution iteration procedure. Further we have defined the y coordinate axis to be parallel to the gravitational potential. The above discrete system of linear algebraic equations can be reorganized:

$$\frac{v_j^{n-1}+\epsilon\gamma C_j}{G}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}u_{j+1}^n\\ v_{j+1}^n\end{Bmatrix}+\begin{bmatrix}D-(v_j^{n-1}+\epsilon\gamma C_j) & \epsilon\gamma B_j\\ 0 & (D+\epsilon\gamma B_j)-\dfrac{(v_j^{n-1}+\epsilon\gamma C_j)}{G}\end{bmatrix}\begin{Bmatrix}u_j^n\\ v_j^n\end{Bmatrix}$$

$$=\begin{Bmatrix}Du_{a_j}-\frac{\gamma^2}{4}B_j\\ g_y-\frac{\gamma^2}{4}B_j\end{Bmatrix}\qquad\begin{matrix}n=1,2,3\ldots\\ j=1,2,3,\ldots,m\end{matrix}\tag{135}$$

By using the one dimensional airflow velocity description of Equations 132 and 133, defining $g_y = -9.81$ m/sec^2, D = 13.0 1/sec, setting boundary condition values for u_{m+1} and N_{m+1} and defining ϵ and γ, primarily to maintain computational stability renders Equa-134 a determinant system of 2 m algebraic equations in 2 m unknown snow phase velocity components.

ONEDEE is a Fortran code (see Appendix for listing) designed to solve Equation 134 by Gauss-Seidel iteration (Ames, 1977; Smith, 1978).

The ONEDEE solution for the snow phase is depicted in Figure 2. Note that the snow phase contains sufficient inertia to have a positive horizontal impact velocity at the snow surface, where the air velocity goes to zero. This is consistent with the mechanisms of saltation flow or impact induced restitution of the snow phase from the solid surface back into the mixture flow (Kobayashi, 1972).

In a naturally occurring snow-air mixture flow the bulk of the snow phase in transport over a flat surface is a product of saltation flow. However, this theory requires that some component of the snow in transport be a product of the apparent or turbulent buoyancy of the snow phase in regions of large airflow gradients. In other words, even when there is no snow phase restitution from the solid surface there should still be a discernibly, variable snow phase density profile vs. height. Snow density profiles for snow-air mixture flows adjacent open water (i.e., a non-restituting lower boundary) have been measured (Takeuchi, 1980). There is a non-uniform snow phase density profile where the difference in density maximum and minimum for these profiles are approximately one order of magnitude less than that for saltation flows (Oral communication, M. Takeuchi).

The turbulent continuity equation for the snow phase for a one-dimensional airflow (Equation 129) has the solution:

$$\rho_s v_s = \text{constant} \tag{136}$$

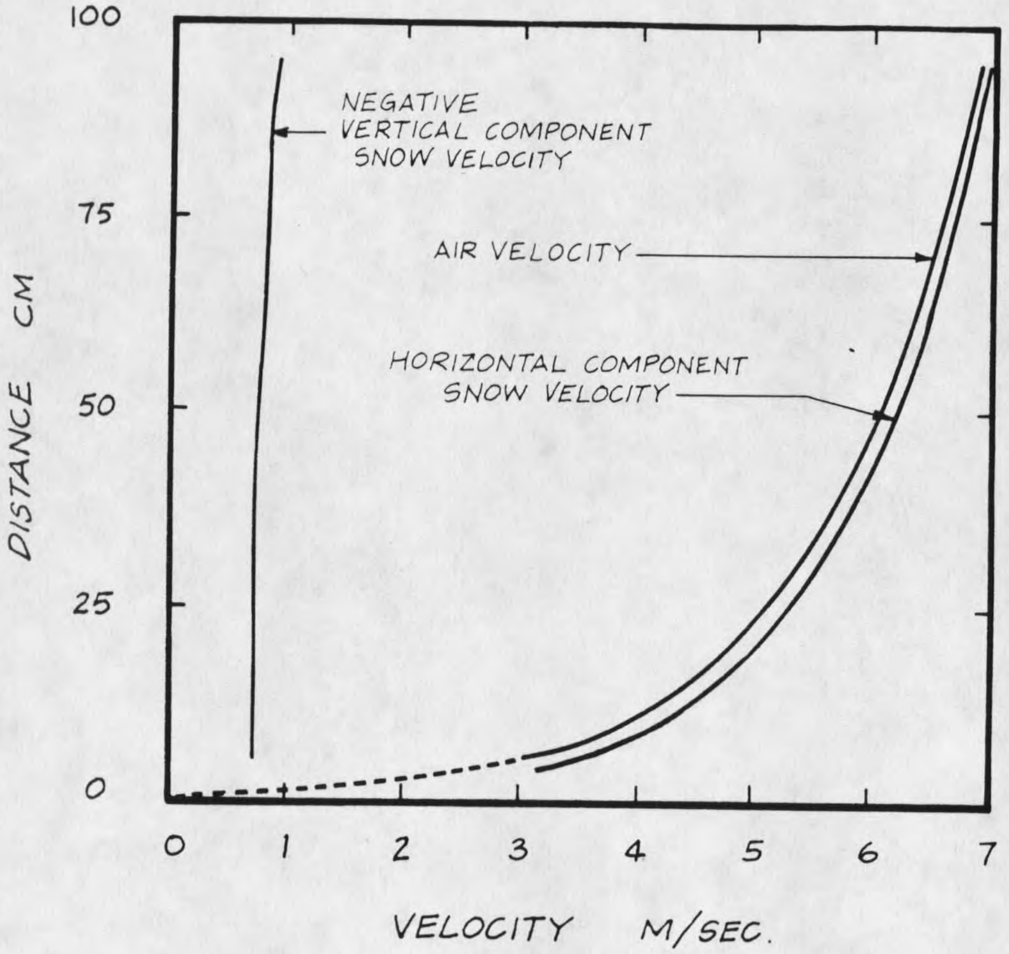

Figure 2. Component snow phase and airflow velocities vs. height above the surface u_a (10 m) = 10 m/sec, u_s (1 m) = u_a (1 m) = 7.0 m/sec, v_s (1 m) = -1.0 m/sec, γ = 0.6 m, ϵ = 0.6 sec, u^* = 0.33 m/sec, k = 0.25, y_o = 0.005 m.

If we define the snow phase mass density at the top of the boundary layer flow to be 1.0×10^{-4} kg/m^3 then the snow phase density profile corresponding to the snow phase vertical or fall velocity components of Figure 2 can be calculated. These data are presented in Figure 3. It is satisfying to note that a theoretical snow phase density profile does exist and has a range of 1.5×10^{-5} kg/m^3.

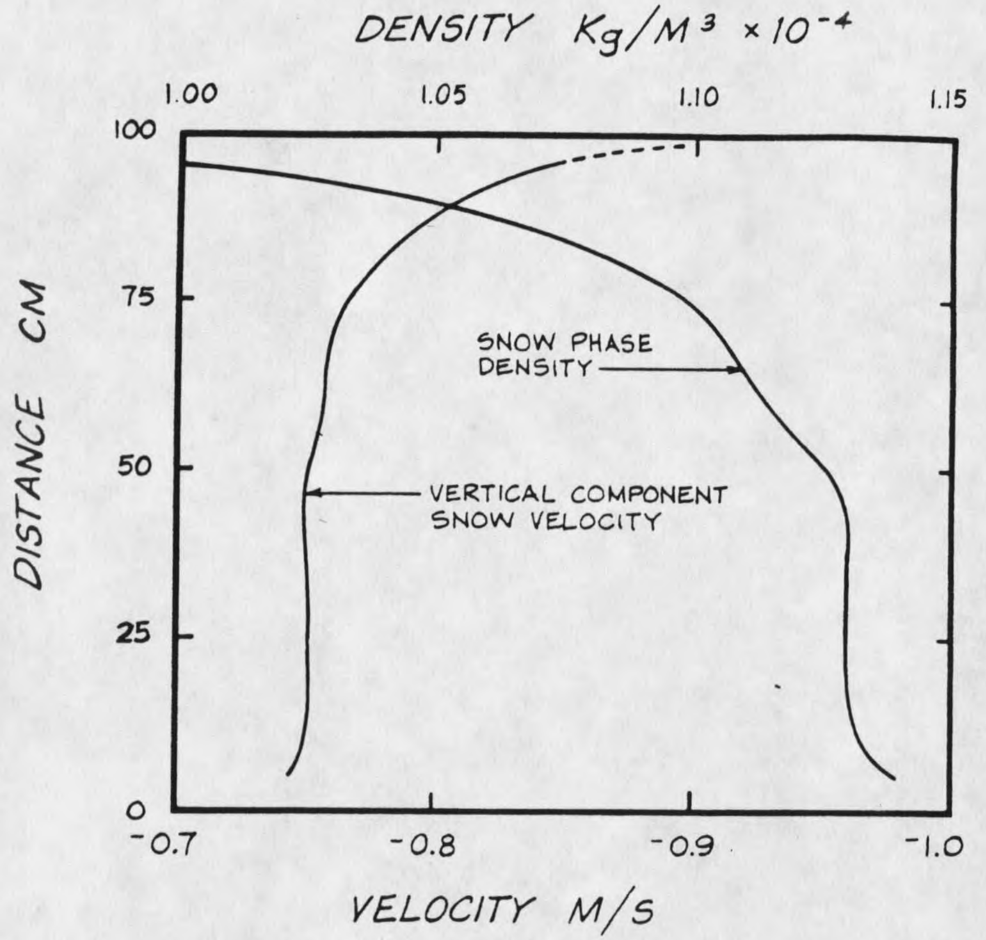

Figure 3. Snow phase fall velocity and corresponding density profile for the one dimensional airflow of Figure 2, ρ_s (1 m) = 1.0×10^{-4} kg/m^3.

CHAPTER 5

SOLUTIONS OF THE SNOW PHASE EQUATIONS OF MOTION FOR TWO DIMENSIONAL AIRFLOW ON THE IMMEDIATE LEESLOPE OF A MOUNTAIN RIDGE

In the previous chapter the turbulent equations of motion for the snow phase of a snow and air mixture flow were derived. If the airflow for a specific two-dimensional geometry can be determined, then the system of partial differential equations (Equation 126) for the momentum balance equations for the snow phase can be solved for the velocity field. Once the snow phase velocity field is determined, these results can then be used to solve for the snow phase mass density field using the continuity equation (Equation 127) for the snow phase. The results can then be qualitatively compared to observations. Lastly, using the snow phase velocity and density fields adjacent a solid surface the distribution and rate of wind-aided snow accumulation can be found and qualitatively compared with observation (Decker & Brown, 1985).

As a phenomenon the wind-aided deposition of snow on the immediate leeslope of a mountain ridge has been studied by a number of researchers for two purposes. First, wind-aided snow deposition on mountain slopes is the single most important factor in the production of direct action or new snow avalanching (Mellor, 1968; Perla & Martinelli, 1976). Second, inasmuch as snow is a significant mechanism of hydrologic storage the areal extent of snow distribution in the alpine has been studied, including the area of above average seasonal snow accumulations on leeslopes due to wind-aided snow sedimentation (Montagne et al., 1968; Alford, 1980).

The ability of snow in mixture flow to attenuate light and decrease visibility has been postulated and confirmed to be a linear function of local mass density of the snow phase in the flow (Budd, Dingle & Radok, 1965; Mellor, 1965; Schmidt, 1977). On the immediate leeslope of a mountain ridge during periods of mixture flow a characteristic plume develops. Figure 4 is a photograph of a set of typical leeslope snow plumes taken on the crest of the Bridger Mountains, Montana. The mixture flow is left to right and closely normal to the axis of the ridge crest. It should be noted that these plumes also exist during snow-air mixture flow periods when there is additions to the snow phase due to atmospheric precipitation, but in general these plumes are visibly obscured. However, the existence of snow phase mass flux profiles at ridge crests during storm periods has been verified and these data have been used for boundary conditions in the solution of the systems of partial differential equations derived from turbulent equations of motion of the snow phase of the mixture flow (Decker & Brown, 1985; Föhn, 1980).

Additionally, wind-aided snow accumulation depth profiles down the immediate leeslope of mountain ridges have been taken (Decker, 1982; Föhn & Meister, 1983). These profiles often show an accumulation maximum at or near ridge crests which then decreases over the next 10 meters to the average snow accumulation depth for that storm. In addition, these profiles sometimes display an accumulation minimum, less than the average accumulation depth at or near 10 meters down the leeslope. It is these asymmetric "wedges" of wind-aided snow accumulation which may release from the leeslope as a wind-slab snow avalanche. Figure 5 is a photograph of an artificially (explosives) released wind-slab avalanche at Bridger Bowl Ski Area, Bridger Mountains, Montana as the slab or wedge of wind-aid snow accumulation first begins to accelerate down the leeslope but before it has mechanically broken up.

Figure 4. Snow plumes on the Bridger mountain ridge crest, Montana, photo credit: S. Challender.

Figure 5. An avalanching slab of wind-aided snow accumulation on the immediate lee slope of the Bridger mountain ridge, Montana, photo credit: D. Richmond.

The Discretized Leeslope Environment

Figure 6 characterizes the discretized environment on the immediate leeslope of an idealized two-dimensional section of a mountain ridge. It is over this domain that solutions for the systems of partial differential equations derived from the turbulent equations of motion for the snow phase of the mixture flow will be generated. Further, the snow accumulation distribution and accumulation rates on the leeslope surface will be investigated.

The 90° angle in the ridge crown is an artifice of the geometry of solution domains required for airflow modelling; recalling that knowing an airflow regime within this domain is a prerequisite to any solution of the system of partial differential equations derived from the turbulent equations of motion for the snow phase of the mixture flow.

Airflow Models

Four different models for the airflow regimes through this two-dimensional domain are investigated. Two computational solutions to the Navier-Stokes equations for single phase flow, an empirical solution for a half-jet with viscous flow and empirically derived data from a 16 mm movie of smoke laden airflow filmed on the Bridger mountain ridge, Montana. The smoke test movie and hence all subsequent airflow regimes are calculated for a free stream windspeeds of 10 m/sec.

Figure 7 characterizes the airflow velocity field within the discretized lesslope domain as computed from the Fortran code: SOLA (Hirt, Nichols & Romero, 1975). SOLA solves the incompressible transient Navier-Stokes equations in two dimensions for a user specified fluid viscosity; in this specific case, that of air at 0°C. SOLA has been successfully used to model airflow on the immediate leeslope of mountain ridges where the airflow regime has been modified by the presences of a "jet-roof" structure (Dawson & Lang, 1979).

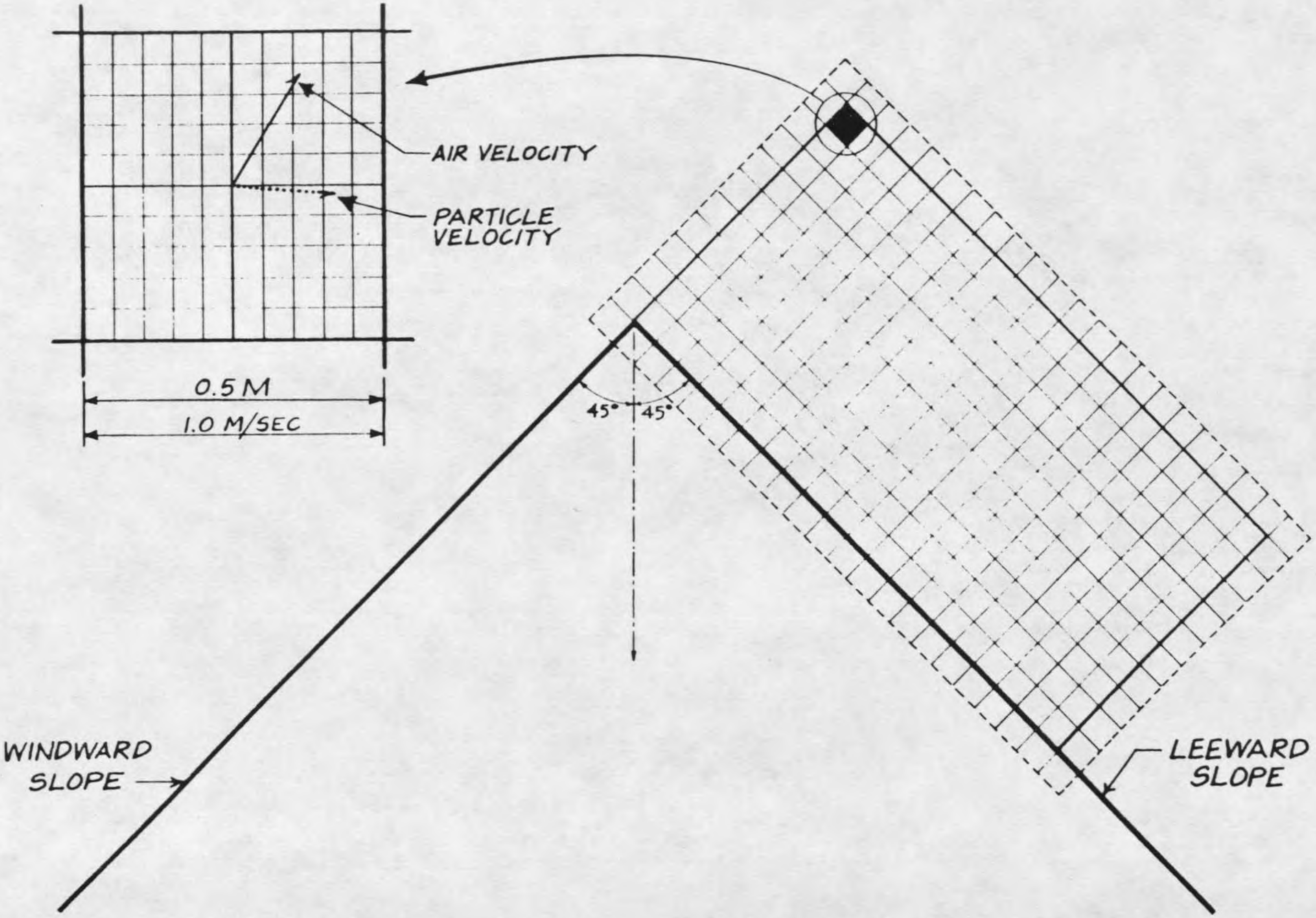

Figure 6. The discretized solution domain on the lee slope of a two dimensional model ridge.

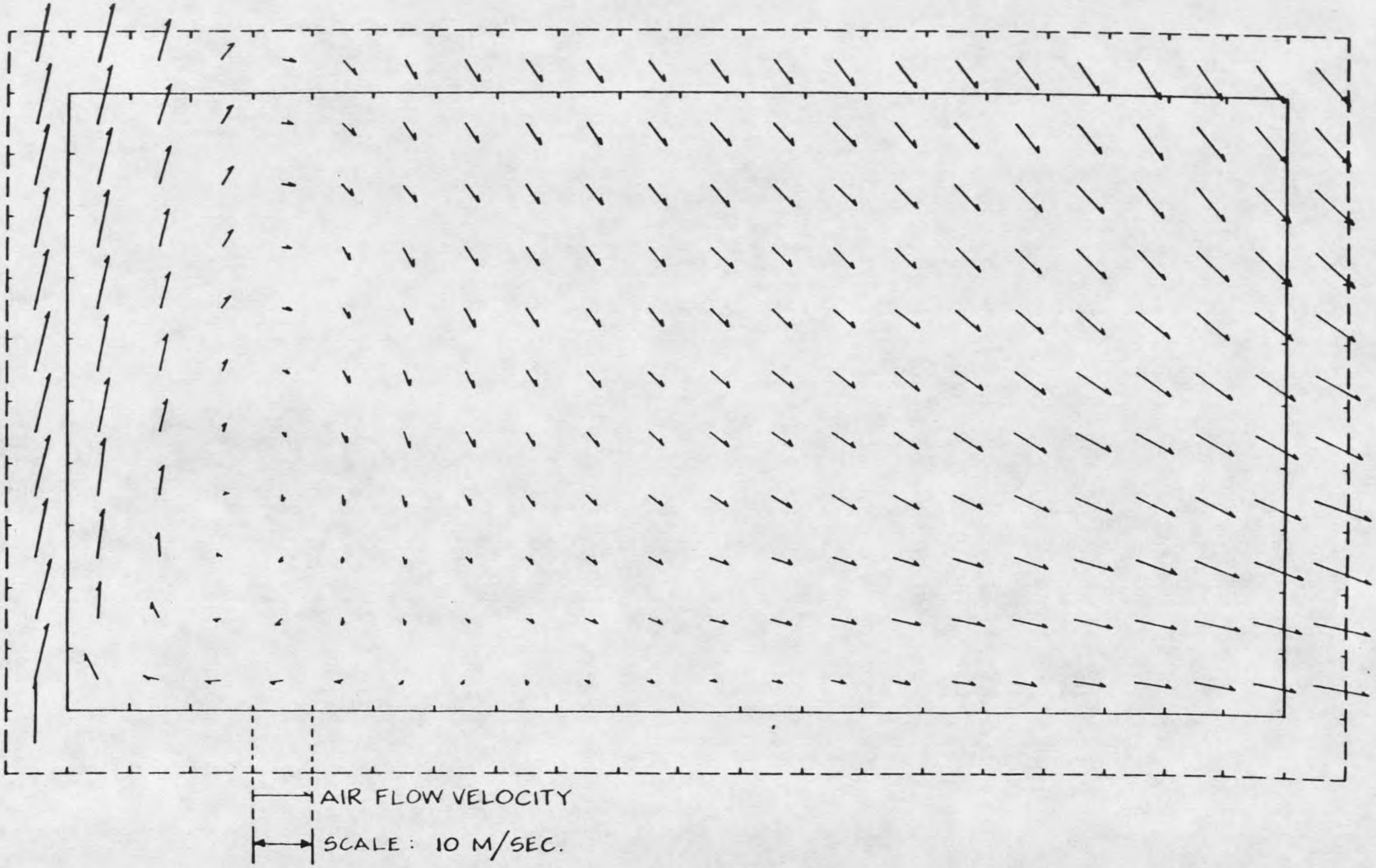

Figure 7. Airflow velocity field within the discretized solution domain as computed by SOLA.

Figure 8 characterizes the airflow velocity field within the discretized leeslope domain as computed from the Fortran code: SMAC (Amsden & Harlow, 1970). SMAC solves the two-dimensional incompressible transient Navier-Stokes equations. A biviscous modification was used to model snow avalanche debris motion (Dent & Lang, 1980; Dent, 1982).

The airflow velocity fields calculated by SOLA and SMAC are for a backward (lee) facing 90° step with gravity at 45° to the step. SOLA was calculated for an unconfined upper boundary and a no-slip lower boundary. The airflow velocities depicted along the left boundaries of Figures 7 and 8 as calculated by SOLA and SMAC, respectively are internal and intrinsic to the specific solutions and are not fixed boundary conditions. It is disconcerting to note that two codes which are designed to solve the same equation system produce such a disparity of results.

Figures 9 and 10 are, respectively, a still photo of the airflow velocity field and the flow field as derived empirically from a 16 mm movie of a smoke laden airflow taken on Bridger mountain ridge, Montana.

Figure 11 contains the airflow regimes as calculated from an empirically derived expression for two-dimensional viscous half-jet flow (Yuu, Yasukouchi, Hirosawa & Jotaki, 1978). JET is a Fortran code (see Appendix for listing) which calculates the values of the component velocities of the airflow within the discretized leeslope solution domain.

Snow Phase Velocity Field

For these airflow regimes and this specific two dimensional solution domain geometry, the system of partial differential equations (Equation 126) for the turbulent momentum balance for the snow phase may be solved by finite difference techniques.

Consider the asymmetric difference approximation (forward in the leeslope normal or y coordinate direction and backward in the leeslope parallel or x coordinate direction) to $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial x}$ (Ames, 1977; Smith, 1978) applied to the snow phase momentum balance

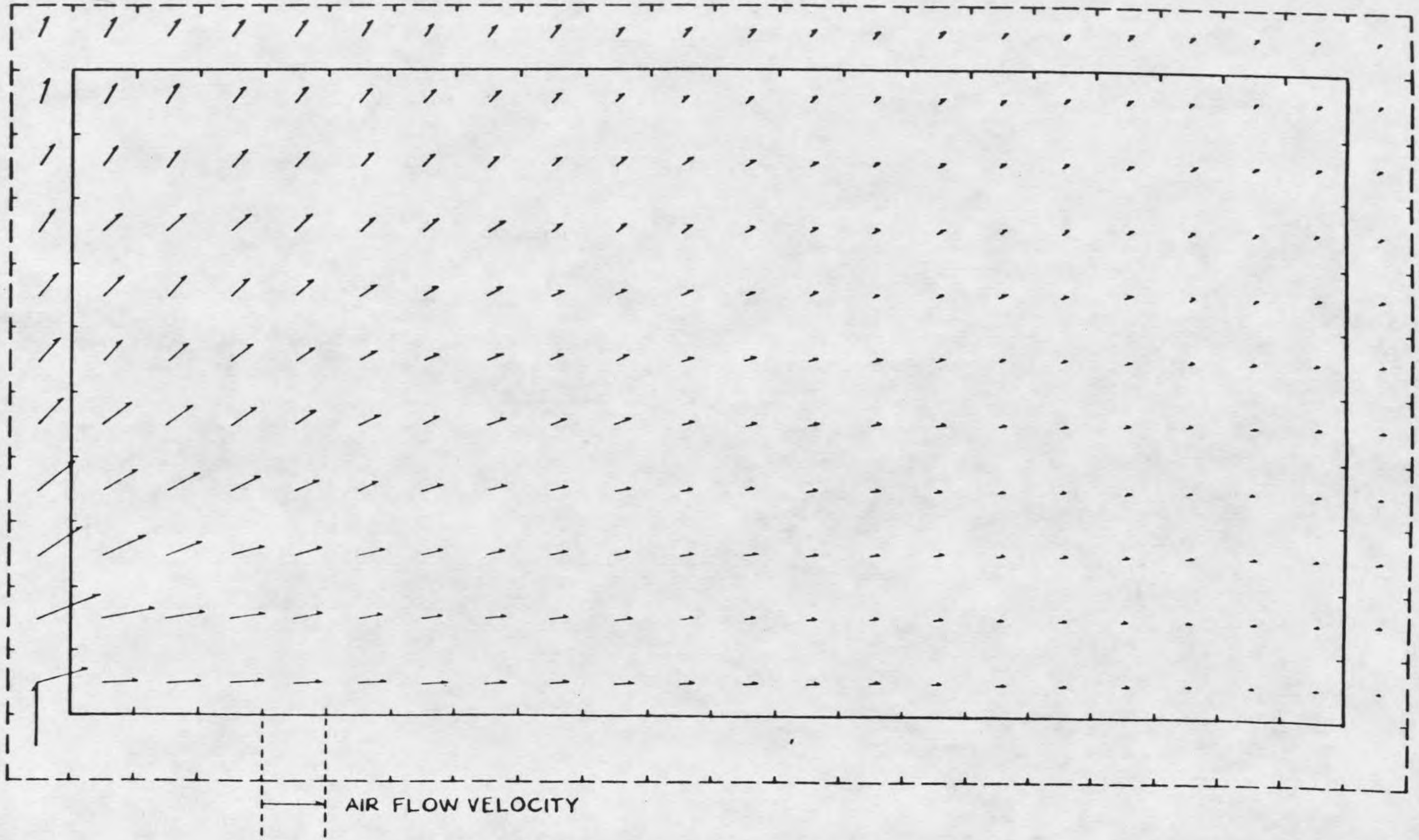

Figure 8. Airflow velocity field with the discretized solution domain as computed by SMAC.

Figure 9. Smoke laden airflow over the Bridger mountain ridge, Montana.

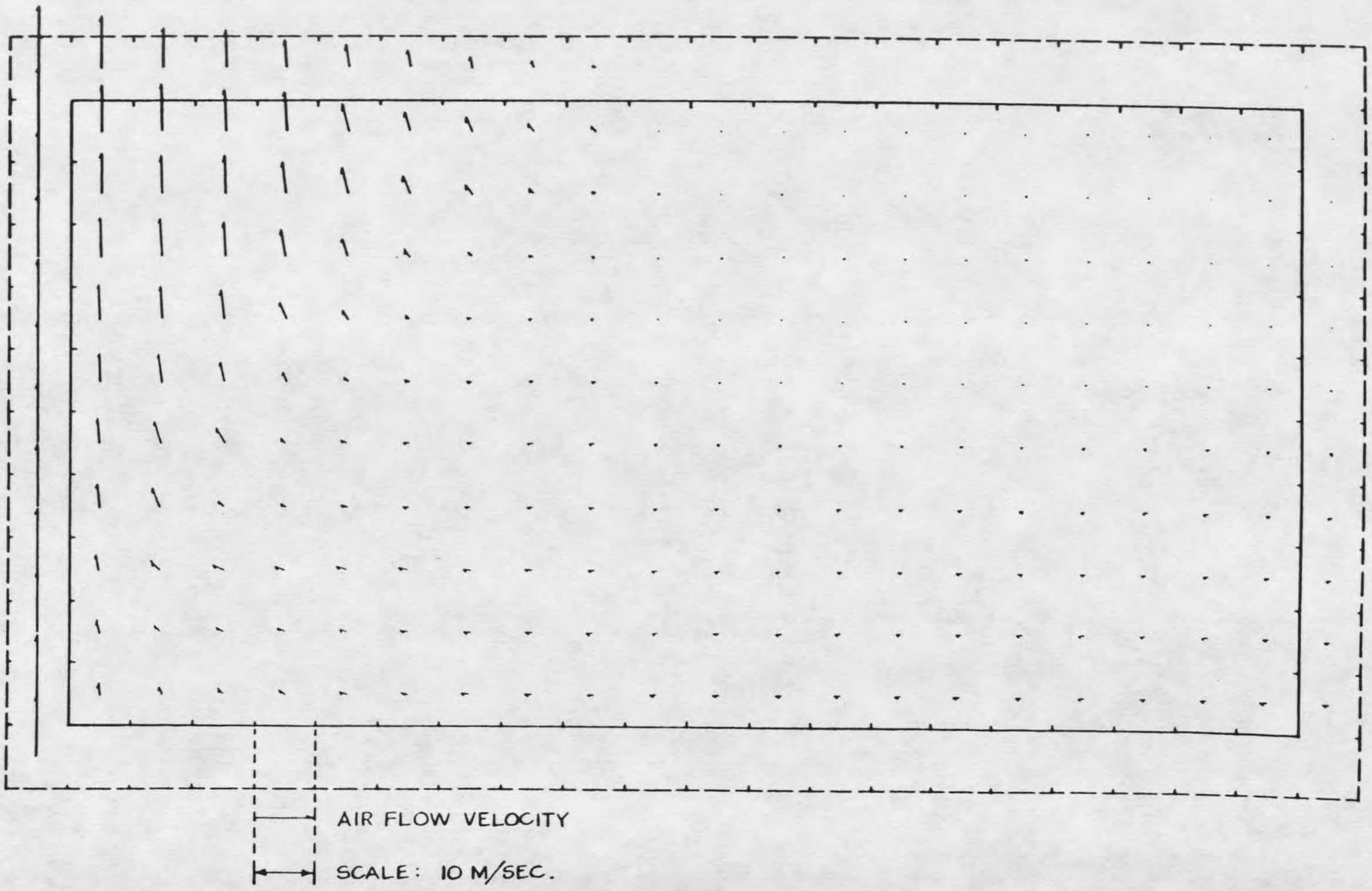

Figure 10. Airflow velocity field within the discretized solution domain as determined from motion pictures of a smoke laden airflow.

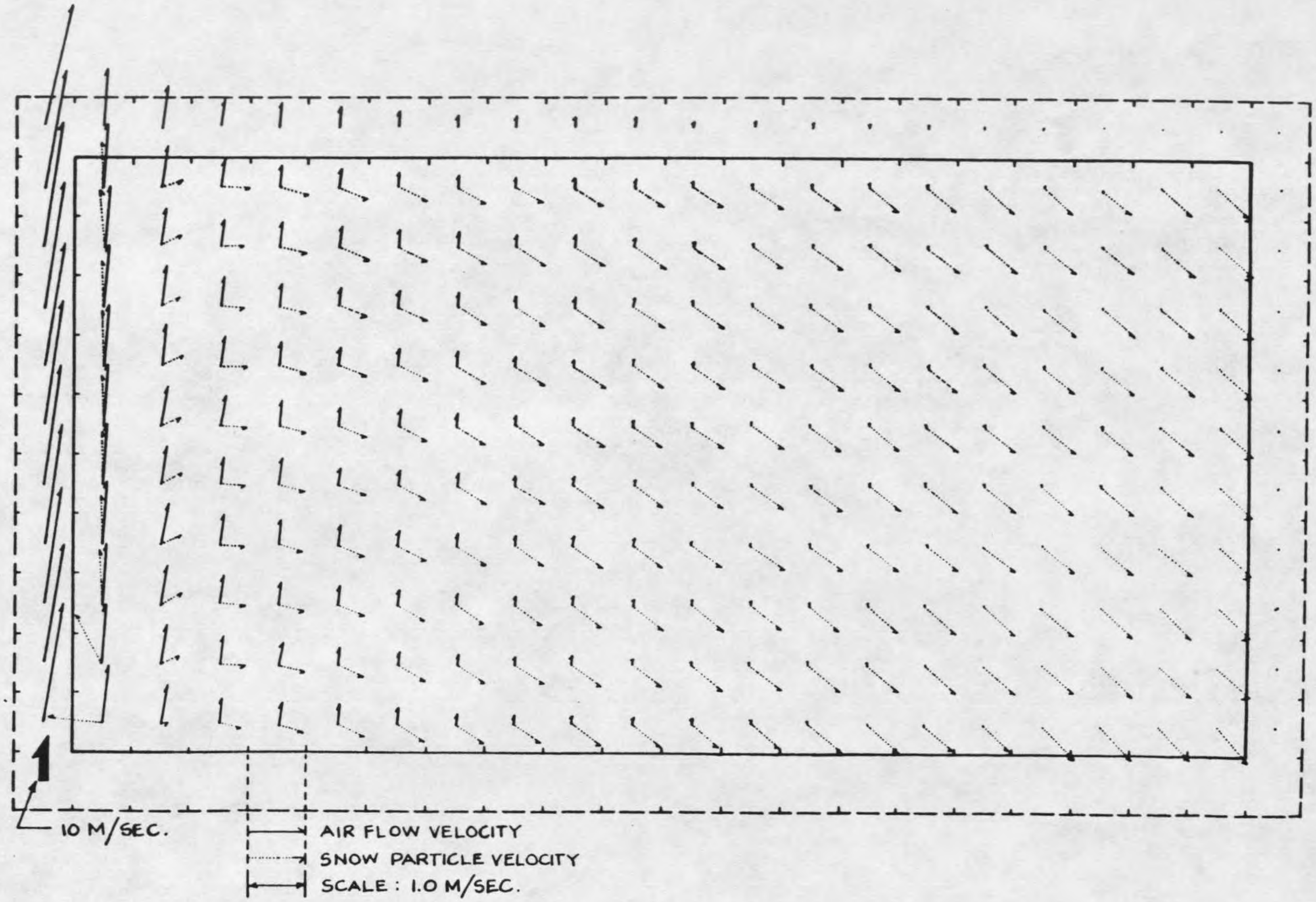

Figure 11. Airflow velocity field within the discretized solution domain as computed by JET and the corresponding snow phase velocity field, $\gamma = 1.4$ m, $\epsilon = 1.4$ secs.

partial differential equation system. i and j are respectively the x coordinate and y coordinate cell location. G and H are respectively the x coordinate and y coordinate cell dimensions. The snow phase subscript s label has been dropped for the sake of brevity.

$$
\begin{aligned}
&(u^{n-1}_{i,j}+\epsilon\gamma C_{i,j})\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix} u^{n}_{i,j}-u^{n}_{i-1,j}/G \\ v^{n}_{i,j}-v^{n}_{i-1,j}/G \end{Bmatrix}\\
&+(v^{n-1}_{i,j}+\epsilon\gamma C_{i,j})\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix} u^{n}_{i,j+1}-u^{n}_{i,j}/H \\ v^{n}_{i,j+1}-v_{i,j}/H \end{Bmatrix}\\
&+\begin{bmatrix}(D+\epsilon\gamma A_{i,j}) & \epsilon\gamma B_{i,j}\\ \epsilon\gamma A_{i,j} & (D+\epsilon\gamma B_{i,j})\end{bmatrix}\begin{Bmatrix} u^{n}_{i,j} \\ v^{n}_{i,j} \end{Bmatrix}=\begin{Bmatrix} g_x+Du_{a_{i,j}}-\dfrac{\gamma^2}{4}(A_{i,j}+B_{i,j}) \\ g_y+Dv_{a_{i,j}}-\dfrac{\gamma^2}{4}(A_{i,j}+B_{i,j}) \end{Bmatrix}
\end{aligned}
\tag{137}
$$

$$
\begin{aligned}
n &= 1,2,3\ldots\\
i &= 1,2,3,\ldots,k\\
j &= 1,2,3,\ldots,L
\end{aligned}
$$

The non-linear terms are treated quasi-linearly by retaining their values from the n–1 iteration during the nth solution iterate. The above discrete system of algebraic equations can be reorganized:

$$
\begin{aligned}
&\frac{v^{n-1}_{i,j}+\epsilon\gamma C_{i,j}}{H}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix} u^{n}_{i,j+1} \\ v^{n}_{i,j+1} \end{Bmatrix}\\
&+\begin{bmatrix}(D+\epsilon\gamma A_{i,j})-\dfrac{(v^{n-1}_{i,j}+\epsilon\gamma C_{i,j})}{H}+\dfrac{(u^{n-1}_{i,j}+\epsilon\gamma C_{i,j})}{G} & \epsilon\gamma B_{i,j}\\ \epsilon\gamma A_{i,j} & (D+\epsilon\gamma B_{i,j})-\dfrac{(v^{n-1}_{i,j}+\epsilon\gamma C_{i,j})}{H}+\dfrac{(u^{n-1}_{i,j}+\epsilon\gamma C_{i,j})}{G}\end{bmatrix}\begin{Bmatrix} u^{n}_{i,j} \\ v^{n}_{i,j} \end{Bmatrix}
\end{aligned}
$$

$$- \frac{u_{i,j}^{n-1} + \epsilon\gamma C_{i,j}}{G} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} u_{i-1,j}^{n} \\ v_{i-1,j}^{n} \end{Bmatrix} = \begin{Bmatrix} g_x + D u_{a_{i,j}} - \frac{\gamma^2}{4} (A_{i,j} + B_{i,j}) \\ g_y + D v_{a_{i,j}} - \frac{\gamma^2}{4} (A_{i,j} + B_{i,j}) \end{Bmatrix} \quad (138)$$

$n = 1, 2, 3 \ldots$

$i = 1, 2, 3, \ldots, k$

$j = 1, 2, 3, \ldots, L$

By using one of the four airflow velocity regimes (SLA, SMC, MVE or JET), defining the components of the gravitational accleration vector, D = 13.0 1/sec, setting the component velocity boundary conditions on the snow phase velocity field on the left and top boundaries of the solution domain equal to the airflow velocities and defining ϵ and γ, primarily to maintain computational stability renders Equation 138 a determinant system of 2 kL algebraic equations in 2 kL unknown snow phase velocity field components.

DURBY is a Fortran code (see Appendix for listing) designed to solve Equation 138 by Gauss-Seidel iteration (Ames, 1977; Smith, 1978).

Consider the DURBY solution for the snow phase velocity field, along with the corresponding JET airflow field depicted on Figure 11. For the boundary conditions (leftside and top of the solution domain) the snow phase velocity components are set equal to the corresponding discretized airflow velocity components. It is of interest to note that only the airflow regime models: SMC and JET as inputs to DURBY produced computationally stabile solutions for the snow phase velocity field. Considering Figures 7, 8, 10 and 11 note that both the SLA and MVE airflow regimes have centers of vorticity within the solution domain where as both the SMC and JET airflow regimes could be characterized as divergent or diffuser type flows. This leads to the postulate that airflow models with vorticity produce a singular coefficient matrix for the system of algebraic equations by violating the requirement that the coefficient matrix be irreducible (Ames, 1977). Furthermore, using SMC or JET airflow regimes as input and choices of ϵ and γ less than approximately 0.1 m or sec or greater than approximately 10 m or sec produced non-convergent DURBY solutions for the snow phase velocity field which are respectively either harmonic between zero and positive infinity or monotonically increasing to positive infinity.

The Snow Phase Density Field

Consider the following expansion of the two dimensional turbulent continuity equation (Equation 139) for the snow phase of the mixture flow:

$$\rho_s \frac{\partial u_s}{\partial x} + u_s \frac{\partial \rho_s}{\partial x} + \rho_s \frac{\partial v_s}{\partial y} + v_s \frac{\partial \rho_s}{\partial y} = 0 \tag{139}$$

Note that this form of the turbulent continuity equation can be reorganized, with the snow phase sub s label has been dropped:

$$\rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} = 0 \tag{140}$$

Consider now the following asymmetric difference approximation to $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial x}$ (Ames, 1977; Smith, 1978) applied to the above snow phase turbulent continuity equation.

$$\rho_{i,j} \left\{ \frac{(u_{i,j} - u_{i-1,j})}{G} + \frac{(v_{i,j+1} - v_{i,j})}{H} \right\} + u_{i,j} \frac{(\rho_{i,j} - \rho_{i-1,j})}{G} + v_{i,j} \frac{(\rho_{i,j+1} - \rho_{i,j})}{H} = 0$$

$i = 1, 2, 3, \ldots, k$ (141)

$j = 1, 2, 3, \ldots, L$

This system of algebraic equations can be reorganized:

$$\frac{v_{i,j}}{H} \rho_{i,j+1} + \left\{ \frac{(u_{i,j} - u_{i-1,j})}{G} + \frac{(v_{i,j+1} - v_{i,j})}{H} + \frac{u_{i,j}}{G} - \frac{v_{i,j}}{H} \right\} \rho_{i,j} - \frac{u_{i,j}}{G} \rho_{i-1,j} = 0$$

$i = 1, 2, 3, \ldots, k$ (142)

$j = 1, 2, 3, \ldots, L$

By using the discretized values of the snow phase velocity components from a DURBY solution and setting the boundary conditions on the snow phase mass density along the left and top boundaries of the solution domain renders Equation 142 a determinant system of kL algebraic equations in kL unknown snow phase densities.

FLAX is a Fortran code (see Appendix for listing) designed to solve Equation 142 by Gauss-Seidel iteration (Ames, 1977; Smith, 1978).

Consider the FLAX solution for the snow phase density field depicted on Figure 12. This snow phase density field corresponds to the snow phase velocity field of Figure 11. For the boundary conditions the snow phase density was allowed to decrease logrithmically from 1.65×10^{-3} kg/m^3 to 6.5×10^{-4} kg/m^3 along the left boundary and held constant along the top boundary at 6.5×10^{-4} kg/m^3. In the absence of wind-aided snow accumulation (i.e., a "still air" snow fall) a snow phase density of 6.5×10^{-4} kg/m^3 would accumulate at a rate of 1.0 cm/hr at a depositional density of 175 kg/m^3.

Note that the snow phase density field of Figure 12 corresponds well quantitatively to the distribution of snow in the photographs of ridge top snow plumes depicted in Figure 4.

It is of interest to note that even though both SMC and JET airflow models would produce computationally stabile DURBY snow phase velocity field solutions only those DURBY solutions corresponding to JET airflow models would produce computationally stabile (all positive densities) FLAX snow phase density field solutions.

Wind-Aided Snow Accumulation on the Leeslope

At any point along the leeslope surface, the rate of wind-aided snow accumulation must be less than or, in the absence of surface erosion, equal to the slope normal flux of snow to the leeslope surface due to the snow-air mixture flow.

$$v_S \rho_S = -\ \text{(snow accumulation rate)(accumulated snow density)} \qquad (143)$$

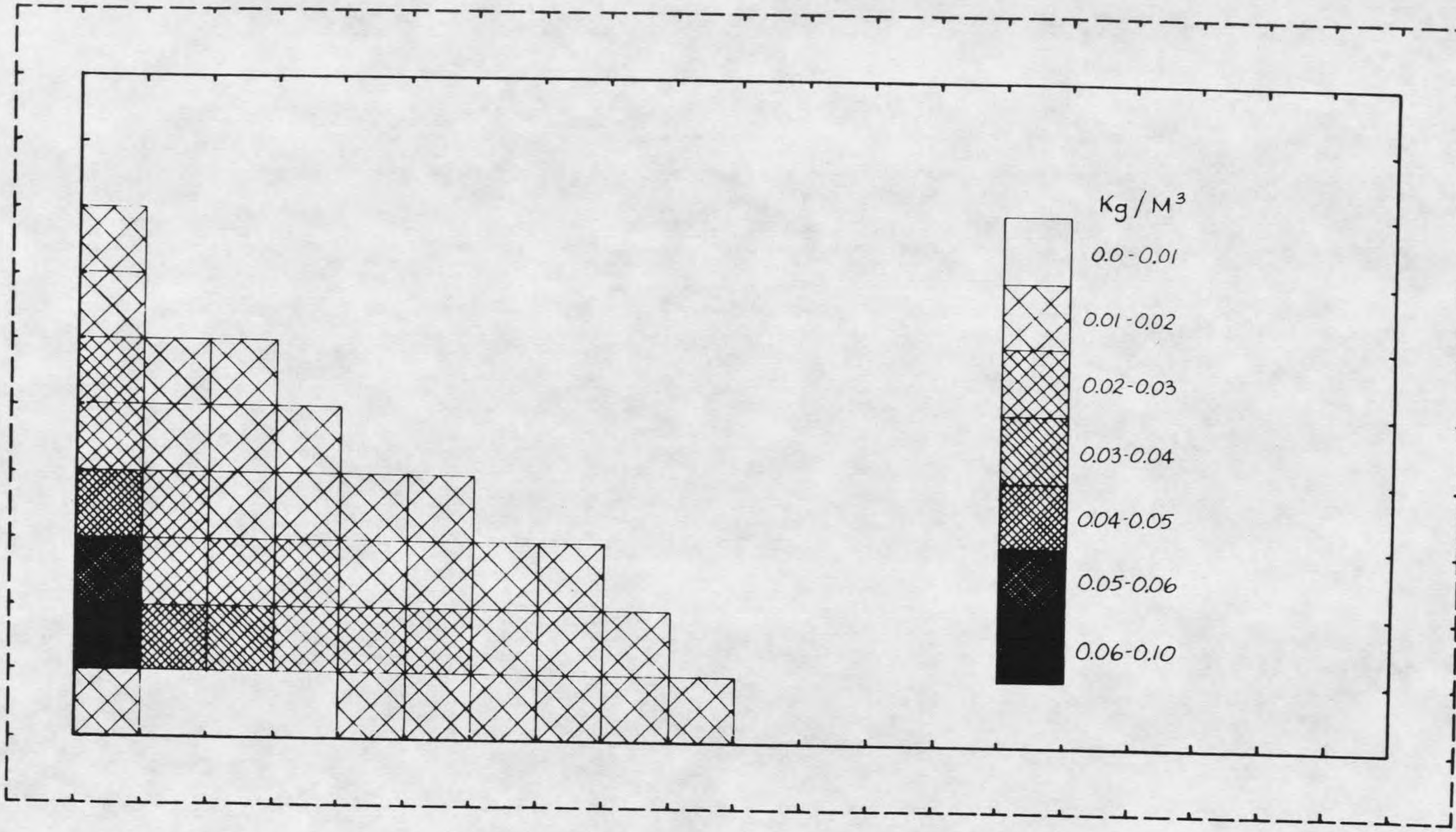

Figure 12. Snow phase density field corresponding to the airflow and snow phase velocity field of Figure 11.

If we assume a constant accumulated or deposited snow density of 175 kg/m^3 and v_s and ρ_s values at the leeslope surface from DURBY and FLAX solutions, respectively, then the wind-aided snow accumulation rates as functions of position along (down) the leeslope may be calculated.

SLAB is a Fortran code (see Appendix for listing) designed to calculate the wind-aid snow accumulation rates vs. distance down the leeslope.

Figure 13 depicts the wind-aided snow accumulation rates vs. distance down the leeslope for a DURBY snow phase velocity field solution corresponding to a JET airflow model with freestream velocity of 10 m/sec and three FLAX snow phase density field solutions for three separated snow phase density boundary conditions. Note that these theoretical wind-aided snow accumulation rates would produce asymmetric "wedges" of deposited snow with an accumulation maximum at ∿ 4 m down the leeslope from the ridge-crest. Further, by 10 m down the leeslope the excess or wind-aided accumulation has decreased to that value which would be expected outside the wind-aided accumulation zone.

Figure 14 depicts the SLAB wind-aided snow accumulation rates vs. distance down the leeslope for a DURBY snow phase velocity field solution corresponding to a JET airflow model of freestream velocity of 5 m/sec and three FLAX snow phase density field solutions. It is disconcerting to note that the theory, at this freestream airflow velocity has over-suspended the snow phase to the point that the maximum wind-aided accumulation rates do not exceed the still air accumulation rates intrinsic to the FLAX snow phase density field boundary conditions. However, the loci of the theoretical wind-aided snow accumulation maximums have shifted closer to the ridgecrest as compared with those of Figure 13, where the freestream airflow velocity is 10 m/sec. Further, these accumulation rates produce a wind-aided accumulation minimum at ∿ 7 m down the leeslope.

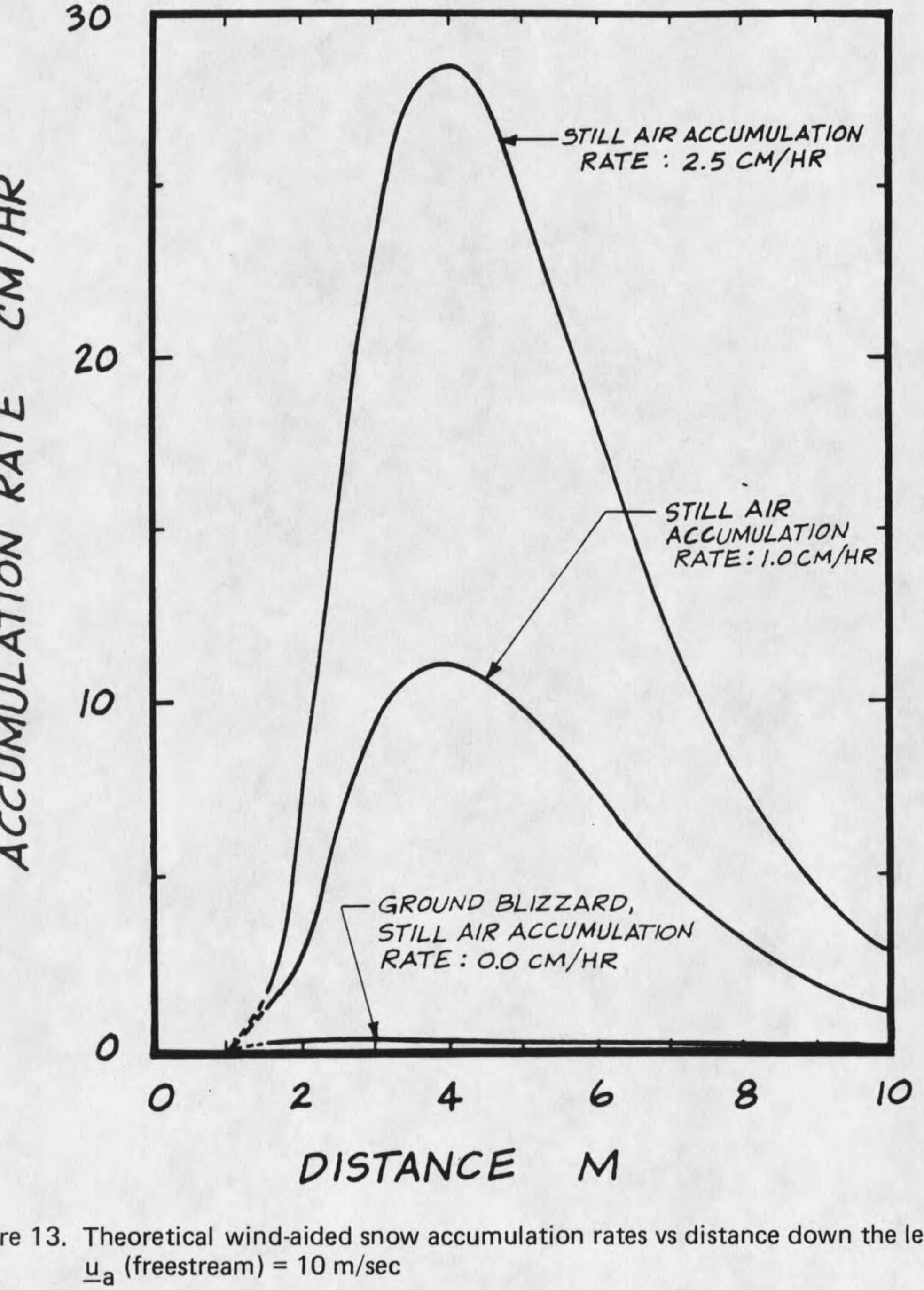

Figure 13. Theoretical wind-aided snow accumulation rates vs distance down the lee slope, $\underline{u}_a$ (freestream) = 10 m/sec

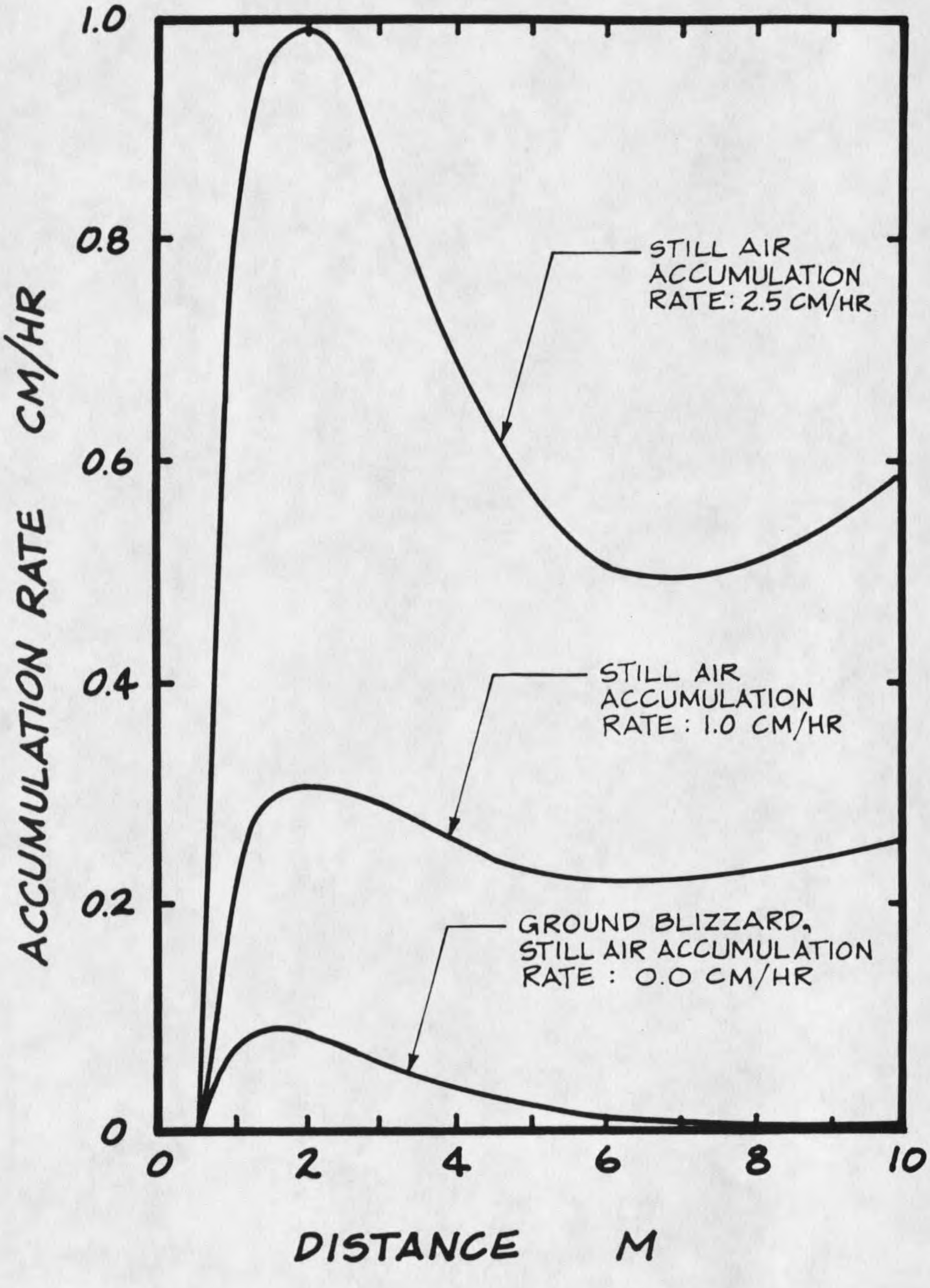

Figure 14. Theoretical wind-aided snow accumulation rates vs distance down the lee slope, $\underline{u}_a$ (freestream) = 5 m/sec

Figure 15 depicts the new snow accumulation for a model storm of 4 hour duration using the theoretical wind-aided snow accumulation rates of Figure 12. Superimposed on Figure 15 is a set of new snow accumulation profiles from the Bridger ridge, Montana.

Despite the fact that the theory oversuspends the amount of snow in the snow-air mixture flow, it is gratifying to note that the theory correctly predicts the location of the wind-aided snow accumulation maximum and models, in a qualitative fashion the apparent effect that the wind-aided snow accumulation maximum shifts towards the ridgecrest as the freestream airflow velocity decreases.

Further, the theory does reproduce the secondary effect of a wind-aided snow accumulation minimum at or near 10 m down the leeslope.

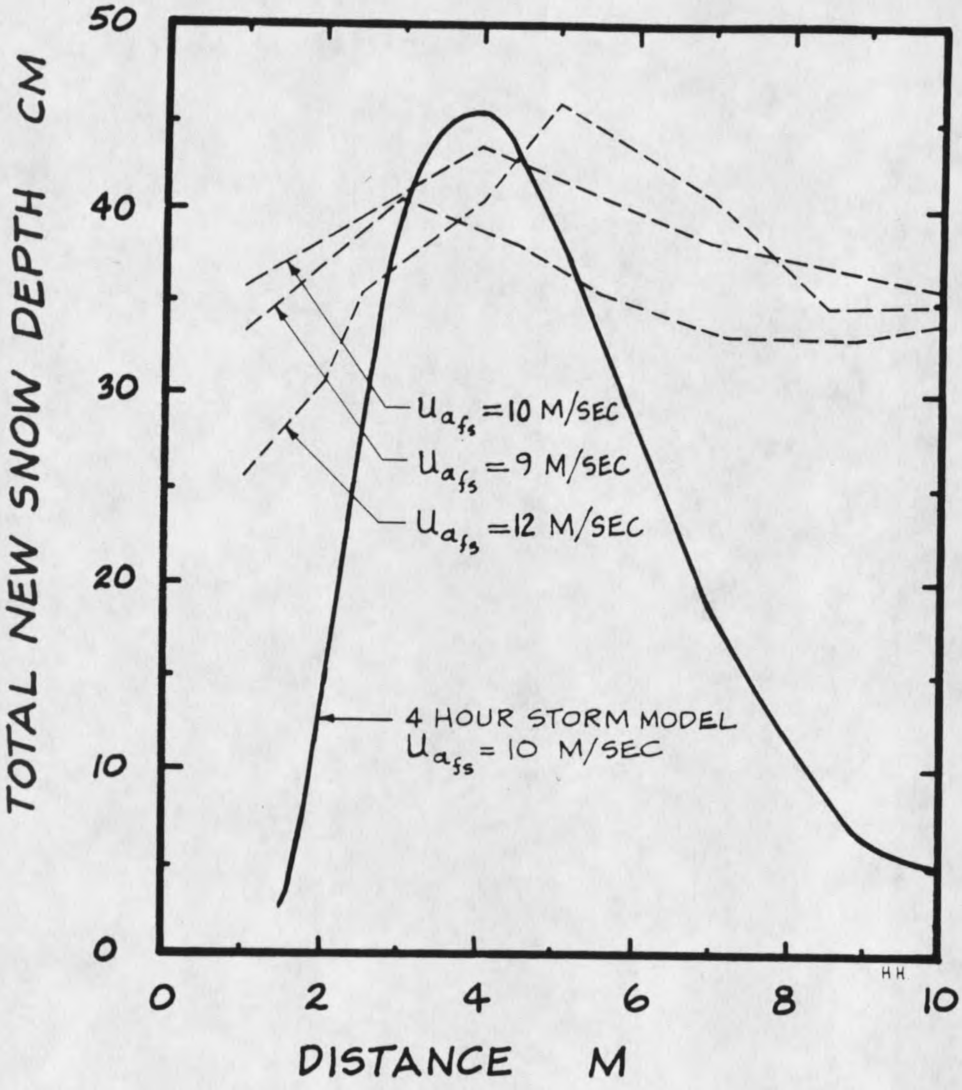

Figure 15. Theoretical and measured wind-aided snow accumulation vs distance down the lee slope for a model storm of 4 hour duration, nonwind-aided snow accumulation rate of 1 cm/hr.

CHAPTER 6

SUMMARY AND CONCLUSIONS

In Chapter 2 a mixture theory was developed, through the momentum balance equations for a mixture of n constituent which respond as a whole in a "fluid" like manner. The constituents of this mixture are capable of inter-constituent exchanges of mass, linear momentum and moment of momentum. Imposed on this theory was the physical requirement that on a specific and balance equation basis the mixture behavior must be equal to the sum of the constituent behaviors. This requirement resulted in a set of restrictions on the per constituent balance equations. These balance equations and restrictions were summarized in Equations 68 through 77.

In the preface of the Mathematical Principles of Natural Philosophy, the source of the balance equations used in this and virtually all classical mechanics theories Sir Isaac Newton states: "The error is not in the art, but in the artificer. For if one could work with perfect accuracy, he would be a perfect mechanic" (Newton, 1687). In other words the unbiased theory of the balance equations is wholly objective but by our very nature we impose subjective elements into these theories. The "sum behavior" requirement of Chapter 2 is an example of this subjectification of a mechanical theory which lead to some very satisfying intuitive and physical results.

In Chapter 3 the equations of motion for the snow phase of a snow and air mixture flow were derived (Equations 82 and 83). The balance of linear momentum equation contained a constitutive assumption for the momentum supply or transfer between the air and snow phases of the mixture.

Under the rationale that single phase atmospheric flows are considered fully turbulent the equations of motion for the snow phase were expanded to include the effect of turbulence. This expansion produced a set of terms which could be characterized as apparent or turbulent buoyancies. Due to our choice of constitutive assumptions for the turbulent variables in terms of mean flow variables these apparent or turbulent buoyancies were large where the intensity of the shear flow of the air phase was large. The turbulent expansion and subsequent constitutive assumptions resulted in the turbulent equations of motion for the snow phase of the mixture flow, summarized in Equation 117.

The continued subjectification of this mechanical theory by turbulent expansion is certainly less than perfect. Consider that when discussing the justification for treating only the intensity of a turbulent stress without regard to its algebraic sign Dr. Herman Schlichting states: "The qualifying word 'mostly' in the above context expresses the fact that the appearance of particles for which u' has the opposite sign to the above is not completely excluded but is nevertheless much less frequent" (Schlichting, 1979). Furthermore when introducing the mathematically necessary concept of an isotropic turbulence, that is a turbulence which is objective with respect to changes of coordinate frame Dr. J. O. Hinze states: "It is a hypothetical type of turbulence, because no actual turbulent flow shows true isotropy" (Hinze, 1975). Obviously the need to include the effects of turbulence in any flow theory are gained at the expense of mechanical objectivity.

In Chapter 4 solutions for the snow phase velocity field and density field are determined by numerical techniques for a one dimensional airflow regime. These results compare well qualitatively with observed inertial effects and density profiles for snow and air mixture flows adjacent a flat surface.

In Chapter 5 solutions for the snow phase velocity field and density field are determined by numerical techniques for a two dimensional airflow regime in the immediate lee of a transverse ridge. These results compare well qualitatively with observed optical effects of snow plumes in natural snow and air mixture flow environments. Further, the theoretical distribution and accumulation rates of wind-aided snow distribution on the immediate leeslope are investigated. The theory correctly reproduces to wind-aided snow accumulation rates and geometries with the exception that the theory in general oversuspends the snow phase in the airflow. However, recall that in Equation 121 we effectively maximized the magnitude of the apparent or turbulent buoyancy terms. Hence, that the theory tends to oversuspend the snow phase should not be unexpected.

Lastly, in the formulation of the constitutive assumptions for the turbulent fluctuations of the snow phase velocity and density there arose two physical constants γ and ϵ whose dimensions were length and time, respectively. There is a very strong intuitive desire to relate these to a turbulent mixing length and mixing time analogous to Prantl's mixing length for turbulent velocity fluctuations for single phase flows (Schlichting, 1979). γ and ϵ were used primarily to maintain computational stability of the system of algebraic equations resulting from the numerical solution techniques. However it is gratifying to note that if γ and ϵ are to be conceptualized as turbulent mixing parameters that their magnitudes (0.01 m or sec to 10 m or sec) are of reasonable physical scale.

In conclusion: it is this author's opinion that the formulation and exploration of mixture theories purely for the purpose of testing for the existence of and degree of correlation between theoretical solutions and a given reality was fully warranted and successful. It is now left to the more clever and needy to formulate the arena of their future civilized and useful application.

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APPENDIX

Fortran Code Listings: ONEDEE, DURBY, FLAX, JET and SLAB

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                U   U  E      M   M  R R    B   B  00  0  00  0    1
                U   U  E      M   M  R  R   B   B  0   0  0   0    1
                UUUUU  EEEEE  M   M  R   R  BBBB    000    000    111

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          000000    NN      NN  EEEEEEEEEE  DDDDDDDD    EEEEEEEEEE  EEEEEEEEEE
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        00      00  NN      NN  EE          DD      DD  EE          EE
        00      00  NNNN    NN  EE          DD      DD  EE          EE
        00      00  NNNN    NN  EE          DD      DD  EE          EE
        00      00  NN  NN  NN  EEEEEEEE    DD      DD  EEEEEEEE    EEEEEEEE
        00      00  NN  NN  NN  EEEEEEEE    DD      DD  EEEEEEEE    EEEEEEEE
        00      00  NN    NNNN  EE          DD      DD  EE          EE
        00      00  NN    NNNN  EE          DD      DD  EE          EE
        00      00  NN      NN  EE          DD      DD  EE          EE
        00      00  NN      NN  EE          DD      DD  EE          EE
          000000    NN      NN  EEEEEEEEEE  DDDDDDDD    EEEEEEEEEE  EEEEEEEEEE
          000000    NN      NN  EEEEEEEEEE  DDDDDDDD    EEEEEEEEEE  EEEEEEEEEE

                                  LL            IIIIII      SSSSSSSS
                                  LL            IIIIII      SSSSSSSS
                                  LL              II      SS
                                  LL              II      SS
                                  LL              II      SS
                                  LL              II      SS
                                  LL              II        SSSSSS
                                  LL              II        SSSSSS
                                  LL              II              SS
                                  LL              II              SS
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                         ....     LL              II              SS
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Figure 16. ONEDEE Fortran Code.

```
                                                               22-Nov-1985 17:2
                                                               25-Aug-1984 21:4
C****       ONEDEE is a Fortran code for evaluating the Momen-
C**** tum Balance, in one space variable for a dispersed,
C**** non-suspenced, non-reacting particulate in a turbulent
C**** one-dimensional fluid flow. Specifically an atmospheric
C**** mixture of snow and air near the ground surface.
C****       ONEDEE solves by Finite Difference techniques.
C****
C****    ****     ****     ****     ****     ****     ****     ****
C****
C**** LOGIC ASSIGNMENTS
C****       INPUT:
C****             1-D AIR VELOCITY PROFILE PARAMETERS AND
C****                NUMERICAL FACTORS: FOR001
C****       OUTPUT:
C****             AIR VELOCITY PROFILE, COMP SNOW VELOCITIES: FOR006
C****
C****    ****     ****     ****     ****     ****     ****     ****
C****
C**** INTIALIZATION, PARAMETERIZATION & DIMENSIONING
C****
      INTEGER CYCLE,CYCMX
      REAL*8 ARXVEL,UC,WC,BC,CC,FC
      REAL*8 SNM1XV,SNM1YV,SNXVEL,SNYVEL
      REAL*8 A,B,UKM1,UK,SUM1,SUM2,TOL,CONX,CONY
      PARAMETER (L=10)
      DIMENSION ARXVEL(L),UC(L),WC(L),BC(L),CC(L),FC(L)
      DIMENSION SNM1XV(L),SNM1YV(L),SNXVEL(L),SNYVEL(L)
      DIMENSION CONX(L),CONY(L)
      DIMENSION A(2*L,2*L),B(2*L),UKM1(2*L),UK(2*L),TOL(2*L)
C****
C****    ****     ****     ****     ****     ****     ****     ****
C****
C**** I/O FORMATS & DATA
C****
      READ(01,12)USTR,VNKC,RUFFY
      READ(01,12)EPSLN,GAMMA,DRGCF
      READ(01,12)BODYX,BODYY,G
      READ(01,18)CYCMX,ITERMX,TOLMX,CNVCRT
      READ(01,20)SNTBCX,SNTBCY
   12 FORMAT(3X,F7.3,3X,F7.3,3X,F7.3)
   18 FORMAT(3X,I3,3X,I3,3X,F7.3,3X,F7.3)
   20 FORMAT(3X,F7.3,3X,F7.3)
   65 FORMAT (A25,1I3)
   70 FORMAT (A50)
   75 FORMAT (6F7.3,1I3)
   80 FORMAT (5F7.3)
   85 FORMAT (12X,1I3,3X,1E12.5,3X,1E12.5,3X,1E12.5)
C****
C****    ****     ****     ****     ****     ****     ****     ****
C****
C**** CALCULATING ARXVEL(L),UC(L),WC(L)
C****
```

```
                                                              22-Nov-1985 17:2
                                                              25-Aug-1984 21:4
C****       ONEDEE is a Fortran code for evaluating the Momen-
C**** tum Balance, in one space variable for a dispersed,
C**** non-suspenced, non-reacting particulate in a turbulent
C**** one-dimensional fluid flow. Specifically an atmospheric
C**** mixture of snow and air near the ground surface.
C****       ONEDEE solves by Finite Difference techniques.
C****
C****    ****      ****      ****      ****      ****      ****      ****
C****
C**** LOGIC ASSIGNMENTS
C****     INPUT:
C****          1-D AIR VELOCITY PROFILE PARAMETERS AND
C****              NUMERICAL FACTORS: FOR001
C****     OUTPUT:
C****          AIR VELOCITY PROFILE, COMP SNOW VELOCITIES: FOR006
C****
C****    ****      ****      ****      ****      ****      ****      ****
C****
C**** INTIALIZATION, PARAMETERIZATION & DIMENSIONING
C****
      INTEGER CYCLE,CYCMX
      REAL*8 ARXVEL,UC,WC,BC,CC,FC
      REAL*8 SNM1XV,SNM1YV,SNXVEL,SNYVEL
      REAL*8 A,B,UKM1,UK,SUM1,SUM2,TOL,CONX,CONY
      PARAMETER (L=10)
      DIMENSION ARXVEL(L),UC(L),WC(L),BC(L),CC(L),FC(L)
      DIMENSION SNM1XV(L),SNM1YV(L),SNXVEL(L),SNYVEL(L)
      DIMENSION CONX(L),CONY(L)
      DIMENSION A(2*L,2*L),B(2*L),UKM1(2*L),UK(2*L),TOL(2*L)
C****
C****    ****      ****      ****      ****      ****      ****      ****
C****
C**** I/O FORMATS & DATA
C****
      READ(01,12)USTR,VNKC,RUFFY
      READ(01,12)EPSLN,GAMMA,DRGCF
      READ(01,12)BODYX,BODYY,G
      READ(01,18)CYCMX,ITERMX,TOLMX,CNVCRT
      READ(01,20)SNTBCX,SNTBCY
   12 FORMAT(3X,F7.3,3X,F7.3,3X,F7.3)
   18 FORMAT(3X,I3,3X,I3,3X,F7.3,3X,F7.3)
   20 FORMAT(3X,F7.3,3X,F7.3)
   65 FORMAT (A25,1I3)
   70 FORMAT (A50)
   75 FORMAT (6F7.3,1I3)
   80 FORMAT (5F7.3)
   85 FORMAT (12X,1I3,3X,1E12.5,3X,1E12.5,3X,1E12.5)
C****
C****    ****      ****      ****      ****      ****      ****      ****
C****
C**** CALCULATING ARXVEL(L),UC(L),WC(L)
C****
```

```
      DO 1000 IL=1,L
      ARXVEL(IL)=(USTR/VNKC)*ALOG(((IL*G)-(G/2))/RUFFY)
      UC(IL)=(USTR/VNKC)*(1/(IL*G))
      WC(IL)=-(USTR*VNKC)*(1/(IL*IL*G*G))
 1000 CONTINUE
      DO 1100 IL=1,L
      BC(IL)=ABS(EPSLN*GAMMA*UC(IL)*WC(IL))
      CC(IL)=ABS(EPSLN*GAMMA*UC(IL)*UC(IL))
      FC(IL)=ABS((GAMMA*GAMMA/2)*UC(IL)*WC(IL))
 1100 CONTINUE
C****
C****    ****    ****    ****    ****    ****    ****    ****
C****
C**** ESTABLISHING AN INTIAL VALUE FOR SNM1(X,Y)V(L)
C****
      DO 1115 IL=1,L
      SNM1XV(IL)=0.0
      SNM1YV(IL)=0.0
 1115 CONTINUE
C****
C****    ****    ****    ****    ****    ****    ****    ****
C****
C**** ZEROING A(2L,2L),B(2L) & SN(X,Y)VEL(L)
C**** AND UPDATING THE CYCLE COUNTER
C****
 4000 CYCLE=CYCLE+1
      DO 1110 IL=1,L
      SNXVEL(IL)=0.0
      SNYVEL(IL)=0.0
 1110 CONTINUE
      DO 1120 IL2=1,2*L
      DO 1120 IL1=1,2*L
      A(IL1,IL2)=0.0
 1120 CONTINUE
      DO 1140 IL3=1,2*L
      B(IL3)=0.0
 1140 CONTINUE
C****
C****    ****    ****    ****    ****    ****    ****    ****
C****
C**** MAIN DIAGONAL: A(2L,2L)
C****
      DO 1200 IL=1,L
      IME=2*IL
      IMO=IME-1
      A(IMO,IMO)=DRGCF-((SNM1YV(IL)+CC(IL))/G)
      A(IME,IME)=DRGCF+BC(IL)-((SNM1YV(IL)+CC(IL))/G)
 1200 CONTINUE
C****
C**** FIRST (UPPER) CODIAGONAL: A(2L,2L)
C****
      DO 1300 IL=1,L
```

```
           IUDE=2*IL
           IUDO=IUDE-1
           A(IUDO,IUDE)=BC(IL)
      1300 CONTINUE
C****
C****      EXTREAME (UPPER) CODIGONALS: A(2L,2L)
C****
           DO 1400 IL=2,L
           IEDLE=2*IL
           IEDLO=IEDLE-1
           IEDSE=IEDLE-2
           IEDSO=IEDLE-3
           A(IEDSO,IEDLO)=(SNM1YV(IL-1)+CC(IL-1))/(G)
           A(IEDSE,IEDLE)=(SNM1YV(IL-1)+CC(IL-1))/(G)
      1400 CONTINUE
C****
C****      COEFF. VECTOR: B(2L)
C****
           DO 1500 IL=1,L
           IBE=2*IL
           IBO=IBE-1
           B(IBO)=BODYX+(DRGCF*ARXVEL(IL))-FC(IL)
           B(IBE)=BODYY-FC(IL)
           IF(IL .LT. L)GO TO 5010
           B(IBO)=B(IBO)-SNTBCX*((SNM1YV(IL)+CC(IL))/(G))
           B(IBE)=B(IBE)-SNTBCY*((SNM1YV(IL)+CC(IL))/(G))
      5010 CONTINUE
      1500 CONTINUE
C****
C****         ****      ****      ****      ****      ****      ****      ****
C****
C****      SOLVING [A](UK)=(B), FOR ONE CYCLE
C****      USING A GUASS-SEIDEL ITERATIVE METHOD
C****
           DO 1600 IUKM1=1,2*L
           UKM1(IUKM1)=0.0
      1600 CONTINUE
           ITER=0
      7010 ITER=ITER+1
           DO 1700 IGS=1,2*L
           SUM1=0.0
           IF(IGS .EQ. 1)GO TO 7020
           IGSM1=IGS-1
           DO 1800 JGS1=1,IGSM1
           SUM1=SUM1-(A(IGS,JGS1)*UK(JGS1))
      1800 CONTINUE
      7020 SUM2=0.0
           IF(IGS .EQ. 2*L)GO TO 7030
           IGSP1=IGS+1
           DO 1900 JGS2=IGSP1,2*L
           SUM2=SUM2-(A(IGS,JGS2)*UKM1(JGS2))
      1900 CONTINUE
```

```
 7030 UK(IGS)=(SUM1+SUM2+B(IGS))/A(IGS,IGS)
 1700 CONTINUE
      DO 2000 IGS=1,2*L
      TOL(IGS)=ABS((UK(IGS)-UKM1(IGS))/UK(IGS))
      IF(TOL(IGS) .GT. TOLMX)GO TO 7040
 2000 CONTINUE
      GO TO 7050
 7040 IF(ITER .GT. ITERMX)GO TO 9000
      DO 2100 IGS=1,2*L
      UKM1(IGS)=UK(IGS)
 2100 CONTINUE
      GO TO 7010
C****
C****    ****    ****    ****    ****    ****    ****    ****
C****
C**** WRITING UK TO SN(X,Y)VEL
C****
 7050 DO 2200 ILC1=1,L
      ILC1E=2*ILC1
      ILC1O=ILC1E-1
      SNXVEL(ILC1)=UK(ILC1O)
      SNYVEL(ILC1)=UK(ILC1E)
 2200 CONTINUE
C****
C****    ****    ****    ****    ****    ****    ****    ****
C****
C**** CHECKING THE CONVERGENCE OF SN(X,Y)VEL
C**** AGAINST SNM1(X,Y)V
C****
      DO 2300 IL=1,L
      CONX(IL)=ABS((SNXVEL(IL)-SNM1XV(IL))/SNXVEL(IL))
      CONY(IL)=ABS((SNYVEL(IL)-SNM1YV(IL))/SNYVEL(IL))
      IF(CONX(IL) .GT. CNVCRT)GO TO 8000
      IF(CONY(IL) .GT. CNVCRT)GO TO 8000
 2300 CONTINUE
      GO TO 9010
C****
C****    ****    ****    ****    ****    ****    ****    ****
C****
C**** UPDATING SNM1(X,Y)V AND CYCLING AGAIN
C****
 8000 CONTINUE
      IF(CYCLE .EQ. CYCMX)GO TO 9020
      DO 2400 IL=1,L
      SNM1XV(IL)=SNXVEL(IL)
      SNM1YV(IL)=SNYVEL(IL)
 2400 CONTINUE
      GO TO 4000
C****
C****    ****    ****    ****    ****    ****    ****    ****
C****
C**** WRITE STATEMENTS FOR CONVERGENT & NON-CONVERGENT SOLUTIONS
```

```
      C****
       9000 WRITE(05,70) 'GUASS-SEIDEL RTN DID NOT CONVERGE'
       9010 WRITE(05,70) 'USTR,VNKC,RUFFY,BODYX,BODYY,G,CYCLE'
            WRITE(05,75) USTR,VNKC,RUFFY,BODYX,BODYY,G,CYCLE
            WRITE(05,70) 'EPSLN,GAMMA,DRGCF,SNTBCX,SNTBCY'
            WRITE(06,80) EPSLN,GAMMA,DRGCF,SNTBCX,SNTBCY
            WRITE(05,70) 'L,SNXVEL,SNYVEL'
            DO 2500 IL=1,L
            WRITE(06,85) IL,ARXVEL(IL),SNXVEL(IL),SNYVEL(IL)
       2500 CONTINUE
            GO TO 9900
       9020 WRITE(06,65) 'DID NOT CONVERGE CYCLE=',CYCLE
       9900 STOP
            END
```

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AAAAAAAAAA    0000000000000000000000000000000000000000000000000000000000000    AAAAA
AAAAAAAAAA    0    Montana State University - SSS - Vax/Vms ver. 4.1    0    AAAAA
AAAAAAAAAA    0000000000000000000000000000000000000000000000000000000000000    AAAAA

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        U   U  EEEE   M   M  RRRR   BBBB   0 0 0  0 0 0    1
        U   U  E      M   M  R R    B   B  00  0  00  0    1
        U   U  E      M   M  R  R   B   B  0   0  0   0    1
        UUUUU  EEEEE  M   M  R   R  BBBB    000    000    111

      DDDDDDDD    UU      UU  RRRRRRRR    BBBBBBBB    YY      YY
      DDDDDDDD    UU      UU  RRRRRRRR    BBBBBBBB    YY      YY
      DD      DD  UU      UU  RR      RR  BB      BB  YY      YY
      DD      DD  UU      UU  RR      RR  BB      BB  YY      YY
      DD      DD  UU      UU  RR      RR  BB      BB    YY  YY
      DD      DD  UU      UU  RR      RR  BB      BB    YY  YY
      DD      DD  UU      UU  RRRRRRRR    BBBBBBBB        YY
      DD      DD  UU      UU  RRRRRRRR    BBBBBBBB        YY
      DD      DD  UU      UU  RR  RR      BB      BB      YY
      DD      DD  UU      UU  RR  RR      BB      BB      YY
      DD      DD  UU      UU  RR    RR    BB      BB      YY
      DD      DD  UU      UU  RR    RR    BB      BB      YY
      DDDDDDDD    UUUUUUUUUU  RR      RR  BBBBBBBB        YY
      DDDDDDDD    UUUUUUUUUU  RR      RR  BBBBBBBB        YY

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                      LL          IIIIII    SSSSSSSS
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                      LL            II    SS
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                      LL            II    SS
                      LL            II      SSSSSS
                      LL            II      SSSSSS
                      LL            II            SS
                      LL            II            SS
                      LL            II            SS
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              ::::    LLLLLLLLLL  IIIIII  SSSSSSSS

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17:53 from queue LP1.

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Figure 17. DURBY Fortran Code.

```
C****      DURBY is a Fortran code for evaluating the Momen-
C**** tum Balance, in two space variables for a dispersed,
C**** non-suspenced, non-reacting particulate in a turbulent
C**** fluid flow. Specifically, an atmospheric mixture of
C**** snow and air near the ground surface.
C****      DURBY attempts to solve, by Finite Difference
C**** techniques for the particle velocity field given a
C**** specified fluid flow regime.
C****
C****    ****     ****     ****     ****     ****     ****     ****
C****
C**** VARIABLE DEFINITIONS
C****
C**** ALL VARIABLES STARTING WITH "I" ARE COUNTING VARIABLES
C**** PARAMETERIZED INTEGER VARIABLES
C****     K; 10:100, NUMBER OF X-COMP DISCRETIZED CELLS
C****     L; 10:100, NUMBER OF Y-COMP DISCRETIZED CELLS
C****     N; 2*K*L, DIMENSION OF THE LINEAR ALGEBRAIC SYSTEM
C**** INTEGER VARIABLES
C****     CYCMX; 1:99, MAXIMUM NUMBER OF COMPUTATIONAL CYCLES ALLOWED
C****     ITERMX; 1:99, MAXIMUM NUMBER OF ITERATIVE CYCLES ALLOWED
C**** REAL  VARIABLES:
C****     G; DISCRETIZED X DIMENSION
C****     H; DISCRETIZED Y DIMENSION
C****     EPSLN; "MIXING"-TIME COEFF ASSOCIATED WITH DENSITY
C****     GAMMA; "MIXING"-LENGTH COEFF ASSOCIATED WITH VELOCITY
C****     DRGCF; DRAG COEFF ASSOCIATED WITH VELOCITY DIFFERENCES
C****     CNVCRT; CONVERGENCE (ERROR) CRITEREA
C****     TOLMX; ITERATION CONVERGENCE CRITERIA
C****     BODY(X,Y); BODY FORCES IN COMPONENTS
C****     SNLBCX(L); LEFT (INFLOW) PARTICLE X-COMP VELOCITY B.C.
C****     SNLBCY(L); LEFT (INFLOW) PARTICLE X-COMP VELOCITY B.C.
C****     SNTBCX(K); TOP PARTICLE X-COMP B.C. VELOCITY
C****     SNTBCY(K); TOP PARTICLE Y-COMP B.C. VELOCITY
C****     ARXVEL(K+2,L+2); FLUID X-COMP VELOCITY
C****     ARYVEL(K+2,L+2); FLUID Y-COMP VELOCITY
C**** REAL "CALCULATED" VARIABLES:
C****     UC(K,L),VC(K,L),WC(K,L),XC(K,L),YC(K,L),ZC(K,L);
C****          DISCRETIZED 1ST & 2ND ORDER GRADIENTS OF
C****          FLUID FLOW
C****     AC(K,L),BC(K,L),CC(K,L),FC(K,L); PRODUCTS AND SUMS
C****          OF UC,VC,WC,XC,YC,ZC
C****     SNXVEL(K,L); PARTICLE X-COMP VELOCITY
C****     SNYVEL(K,L); PARTICLE Y-COMP VELOCITY
C****     SNM1XV(K,L); PARTICLE X-COMP VELOCITY AT THE PREVIOUS
C****          CYCLE
C****     SNM1YV(K,L); PARTICLE Y-COMP VELOCITY AT THE PREVIOUS
C****          CYCLE
C****     A(N,N); COEFF MATRIX OF THE SYSTEM OF ALGEBRAC EQU
C****     B(N); CCEFF VECTOR OF THE SYSTEM OF ALGEBRAC EQU
C****     UK(N) & UKM1(N); INTERMEDIATE SOLUTION VECTORS
C****     TOL(N); GUASS-SIEDEL ITERATION TOLERENCE
```

```
C****     CONX(K,L); CONVERGENCE ERROR X-COMP VELOCITIES
C****     CONY(K,L); CONVERGENCE ERROR Y-COMP VELOCITIES
C****
C****     ****     ****     ****     ****     ****     ****     ****
C****
C**** INTIALIZATION, PARAMETERIZATION & DIMENSIONING
C****
      INTEGER CYCLE,CYCMX
      REAL*8 SNLBCX,SNLBCY,SNTBCX,SNTBCY
      REAL*8 SNXVEL,SNYVEL,SNM1XV,SNM1YV
      REAL*8 ARXVEL,ARYVEL
      REAL*8 UC,VC,WC,XC,YC,ZC
      REAL*8 AC,BC,CC,FC
      REAL*8 CONX,CONY
      REAL*8 A,UK,UKM1,B,TOL,SUM1,SUM2
      PARAMETER (K=20,L=10,N=400)
      DIMENSION SNLBCX(L),SNLBCY(L),SNTBCX(K),SNTBCY(K)
      DIMENSION SNXVEL(K,L),SNYVEL(K,L),SNM1XV(K,L),SNM1YV(K,L)
      DIMENSION ARXVEL(K+2,L+2),ARYVEL(K+2,L+2)
      DIMENSION UC(K,L),VC(K,L),WC(K,L),XC(K,L),YC(K,L),ZC(K,L)
      DIMENSION AC(K,L),BC(K,L),CC(K,L),FC(K,L)
      DIMENSION CONX(K,L),CONY(K,L)
      DIMENSION A(N,N),UK(N),UKM1(N),B(N),TOL(N)
      CYCMX=30
      ITERMX=30
      CNVCRT=0.05
      TOLMX=0.05
C****
C****     ****     ****     ****     ****     ****     ****     ****
C****
C**** INPUT/OUTPUT FORMATS
C****
   10 FORMAT (5F7.3)
   15 FORMAT (2F7.3)
   20 FORMAT (30X,1E12.5,3X,1E12.5)
   60 FORMAT (A30)
   65 FORMAT (A25,1I3)
   70 FORMAT (A50)
   75 FORMAT (2I3,7F7.3,1I3)
   80 FORMAT (A30)
   85 FORMAT (21X,2I3,3X,1E12.5,3X,1E12.5)
C****
C****     ****     ****     ****     ****     ****     ****     ****
C****
C**** READ &/OR WRITE STATEMENTS
C****
      READ(01,10)G,H,EPSLN,GAMMA,DRGCF
      READ(01,15)BODYX,BODYY
      DO 100 IL=1,L
      READ(02,20)SNLBCX(IL),SNLBCY(IL)
  100 CONTINUE
      DO 200 IK=1,K
```

```
      READ(02,20)SNTBCX(IK),SNTBCY(IK)
  200 CONTINUE
      DO 300 IL=1,L+2
      DO 300 IK=1,K+2
      READ(03,20)ARXVEL(IK,IL),ARYVEL(IK,IL)
  300 CONTINUE
C****
C****     ****     ****     ****     ****     ****     ****     ****
C****
C**** CALCULATING THE NON-CYCLING COMPUTATIONAL VARIABLES
C****
      DO 400 IL=1,L
      DO 400 IK=1,K
      UC(IK,IL)=ABS((ARXVEL(IK+1,IL+2)-ARXVEL(IK+1,IL))/(2*H))
      VC(IK,IL)=ABS((ARYVEL(IK+2,IL+1)-ARYVEL(IK,IL+1))/(2*G))
      WC(IK,IL)=ABS((ARXVEL(IK+1,IL+2)-(2*ARXVEL(IK+1,IL+1))+
     1ARXVEL(IK+1,IL))/(H*H))
      XC(IK,IL)=ABS((ARYVEL(IK+2,IL+1)-(2*ARYVEL(IK+1,IL+1))+
     1ARYVEL(IK,IL+1))/(G*G))
      YC(IK,IL)=ABS((ARXVEL(IK+2,IL+2)-ARXVEL(IK+2,IL)-
     1ARXVEL(IK,IL+2)+ARXVEL(IK,IL))/(4*G*H))
      ZC(IK,IL)=ABS((ARYVEL(IK+2,IL+2)-ARYVEL(IK+2,IL)-
     1ARYVEL(IK,IL+2)+ARYVEL(IK,IL))/(4*G*H))
  400 CONTINUE
      DO 500 IL=1,L
      DO 500 IK=1,K
      AC(IK,IL)=(EPSLN*GAMMA)*(VC(IK,IL)*(XC(IK,IL)+YC(IK,IL))+
     1(UC(IK,IL)*(XC(IK,IL)+YC(IK,IL))))
      BC(IK,IL)=(EPSLN*GAMMA)*(VC(IK,IL)*(ZC(IK,IL)+WC(IK,IL))+
     1(UC(IK,IL)*(ZC(IK,IL)+WC(IK,IL))))
      CC(IK,IL)=(EPSLN*GAMMA)*(VC(IK,IL)*(VC(IK,IL)+UC(IK,IL))+
     1(UC(IK,IL)*(VC(IK,IL)+UC(IK,IL))))
  500 CONTINUE
      DO 600 IL=1,L
      DO 600 IK=1,K
      FC(IK,IL)=(GAMMA/(EPSLN*2))*(AC(IK,IL)+BC(IK,IL))
  600 CONTINUE
C****
C****     ****     ****     ****     ****     ****     ****     ****
C****
C**** ESTABLISHING INTIAL VALUES FOR THE CYCLING COMPUTA-
C**** TIONAL VARIABLES
C****
      CYCLE=0
      DO 700 IL=1,L
      DO 700 IK=1,K
      SNM1XV(IK,IL)=0.0
      SNM1YV(IK,IL)=0.0
  700 CONTINUE
C****
C****     ****     ****     ****     ****     ****     ****     ****
C****
```

```
C**** UP-DATING THE COUNTER; CYCLE AND ZEROING-OUT THE
C**** CYCLING COMPUTATIONAL ARRAYS
C****
 5000 CYCLE=CYCLE+1
      DO 800 IL=1,L
      DO 800 IK=1,K
      SNXVEL(IK,IL)=0
      SNYVEL(IK,IL)=0
  800 CONTINUE
      DO 900 IN1=1,N
      DO 900 IN2=1,N
      A(IN2,IN1)=0
  900 CONTINUE
      DO 1000 IN1=1,N
      B(IN1)=0
 1000 CONTINUE
C****
C****    ****     ****     ****     ****     ****     ****     ****
C****
C**** FILL-OUT (INTIALLY) OR UPDATE THE CYCLING COMPUTATIONAL
C**** ARRAYS
C****
C**** MAIN DIAGONAL OF A(N,N)
C****
      DO 1100 IL=1,L
      DO 1100 IK=1,K
      IMDE=(IL-1)*2*K+2*IK
      IMDO=IMDE-1
      A(IMDO,IMDO)=DRGCF+AC(IK,IL)-((CC(IK,IL)+SNM1YV(IK,IL))/H)
     1+((CC(IK,IL)+SNM1XV(IK,IL))/G)
      A(IMDE,IMDE)=DRGCF+BC(IK,IL)-((CC(IK,IL)+SNM1YV(IK,IL))/H)
     1+((CC(IK,IL)+SNM1XV(IK,IL))/G)
 1100 CONTINUE
C****
C**** FIRST UPPER & LOWER CODIAGONALS OF A(N,N)
C****
      DO 1200 IL=1,L
      DO 1200 IK=1,K
      IFCE=(IL-1)*2*K+2*IK
      IFCO=IFCE-1
      A(IFCO,IFCE)=BC(IK,IL)
      A(IFCE,IFCO)=AC(IK,IL)
 1200 CONTINUE
C****
C**** SECOND (LOWER) CODIGONAL OF A(N,N)
C****
      DO 1300 IL=1,L
      DO 1300 IK=2,K
      ISCLE=(IL-1)*2*K+2*IK
      ISCLO=ISCLE-1
      ISCSE=ISCLE-2
      ISCSO=ISCLE-3
```

```
      A(ISCLO,ISCSO)=-((CC(IK,IL)+SNM1XV(IK,IL))/G)
      A(ISCLE,ISCSE)=-((CC(IK,IL)+SNM1XV(IK,IL))/G)
 1300 CONTINUE
C****
C**** EXTREAME (UPPER) CODIAGONAL OF A(N,N)
C****
      DO 1400 IL=2,L
      DO 1400 IK=1,K
      IECLE=(IL-1)*2*K+2*IK
      IECLO=IECLE-1
      IECSE=(IL-2)*2*K+2*IK
      IECSO=IECSE-1
      A(IECSE,IECLE)=((CC(IK,IL-1)+SNM1YV(IK,IL-1))/H)
      A(IECSO,IECLO)=((CC(IK,IL-1)+SNM1YV(IK,IL-1))/H)
 1400 CONTINUE
C****
C**** B(N)
C****
      DO 1500 IL=1,L
      DO 1500 IK=1,K
      ICFVBE=(IL-1)*2*K+2*IK
      ICFVBO=ICFVBE-1
      B(ICFVBO)=(DRGCF*ARXVEL(IK+1,IL+1))-FC(IK,IL)+BODYX
      B(ICFVBE)=(DRGCF*ARYVEL(IK+1,IL+1))-FC(IK,IL)+BODYY
      IF (ICFVBO .NE. ((IL-1)*2*K)+1) GO TO 5030
      B(ICFVBO)=B(ICFVBO)+((CC(IK,IL)+SNM1XV(IK,IL))/(G))*
     1SNLBCX(IL)
 5030 IF (ICFVBE .NE. ((IL-1)*2*K)+2) GO TO 5040
      B(ICFVBE)=B(ICFVBE)+((CC(IK,IL)+SNM1XV(IK,IL))/(G))*
     1SNLBCY(IL)
 5040 IF (ICFVBO .LT. ((L-1)*2*K)+1) GO TO 5050
      B(ICFVBO)=B(ICFVBO)-((CC(IK,IL)+SNM1YV(IK,IL))/(H))*
     1SNTBCX(IK)
 5050 IF (ICFVBE .LT. ((L-1)*2*K)+2) GO TO 1500
      B(ICFVBE)=B(ICFVBE)-((CC(IK,IL)+SNM1YV(IK,IL))/(H))*
     1SNTBCY(IK)
 1500 CONTINUE
C****
C****     ****     ****     ****     ****     ****     ****     ****
C****
C**** ITERATIVELY (GUASS-SIEDEL) SOLVING FOR THE PARTICLE
C**** VELOCITY VECTOR
C****
      DO 3000 IUKM1=1,N
      UKM1(IUKM1)=0.0
 3000 CONTINUE
      ITER=0
 7010 ITER=ITER+1
      DO 3100 IGS=1,N
      SUM1=0.0
      IF (IGS .EQ. 1) GO TO 7020
      IGSM1=IGS-1
```

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```
      DO 3200 JGS1=1,IGSM1
      SUM1=SUM1-(A(IGS,JGS1)*UK(JGS1))
 3200 CONTINUE
      SUM2=0.0
      IF (IGS .EQ. N) GO TO 7030
 7020 IGSP1=IGS+1
      DO 3300 JGS2=IGSP1,N
      SUM2=SUM2-(A(IGS,JGS2)*UKM1(JGS2))
 3300 CONTINUE
 7030 UK(IGS)=(SUM1+SUM2+B(IGS))/A(IGS,IGS)
 3100 CONTINUE
      DO 3400 IGS=1,N
      TOL(IGS)=ABS((UK(IGS)-UKM1(IGS))/UK(IGS))
      IF (TOL(IGS) .GT. TOLMX) GO TO 7040
 3400 CONTINUE
      GO TO 7050
 7040 IF (ITER .EQ. ITERMX) GO TO 8000
      DO 3500 IGS=1,N
      UKM1(IGS)=UK(IGS)
 3500 CONTINUE
      GO TO 7010
C****
C****     ****     ****     ****     ****     ****     ****     ****
C****
C**** WRITING UK VECTOR TO SN(X,Y)VEL(K,L)
C****
 7050 DO 1600 IL=1,L
      DO 1600 IK=1,K
      IUKE=(IL-1)*2*K+2*IK
      IUKO=IUKE-1
      SNXVEL(IK,IL)=UK(IUKO)
      SNYVEL(IK,IL)=UK(IUKE)
 1600 CONTINUE
C****
C****     ****     ****     ****     ****     ****     ****     ****
C****
C**** CHECKING THE CONVERGENCE OF SN(X,Y)VEL(K,L) AGAINST
C**** ITS VALUE AT THE PREVIOUS CYCLE: SNM1(X,Y)V
C****
      DO 1700 IL=1,L
      DO 1700 IK=1,K
      CONX(IK,IL)=ABS((SNXVEL(IK,IL)-SNM1XV(IK,IL))/
     1SNXVEL(IK,IL))
      CONY(IK,IL)=ABS((SNYVEL(IK,IL)-SNM1YV(IK,IL))/
     1SNYVEL(IK,IL))
      IF (CONX(IK,IL) .GT. CNVCRT) GO TO 5060
      IF (CONY(IK,IL) .GT. CNVCRT) GO TO 5060
 1700 CONTINUE
      GO TO 5070
C****
C****     ****     ****     ****     ****     ****     ****     ****
C****
```

```
      C**** UPDATING THE CYCLING ARRAY; SNM1(X,Y,)V(K,L)
      C**** TO THE VALUE; SN(X,Y)VEL(K,L)
      C****
       5050 CONTINUE
            IF (CYCLE .EQ. CYCMX) GO TO 5080
            DO 1800 IL=1,L
            DO 1800 IK=1,K
            SNM1XV(IK,IL)=SNXVEL(IK,IL)
            SNM1YV(IK,IL)=SNYVEL(IK,IL)
       1800 CONTINUE
      C****
      C****    ****     ****     ****     ****     ****     ****     ****
      C****
      C**** RETURN FOR AN ADDITIONAL COMPUTATIONAL CYCLE
      C****
            GO TO 5000
      C****
      C****    ****     ****     ****     ****     ****     ****     ****
      C****
      C**** WRITE STATEMENTS FOR CONVERGENT & NON-CONVERGENT SOLUTIONS
      C****
       8000 WRITE(06,70) 'GUASS-SIEDEL RTN DID NOT CONV.'
            GO TO 9000
       5070 WRITE(06,60) 'THIS IS A CONVERGENT SOLUTION'
            GO TO 5090
       5080 WRITE(06,65) 'DID NOT CONVERGE CYCMX=',CYCMX
       5090 WRITE(06,70) 'K,L,G,H,EPSLN,GAMMA,DRGCF,BODYX,BODYY,CYCLE'
            WRITE(06,75)K,L,G,H,EPSLN,GAMMA,DRGCF,BODYX,BODYY,CYCLE
            WRITE(06,80) 'K,L,SNXVEL,SNYVEL'
            DO 2000 IL=1,L
            DO 2000 IK=1,K
            WRITE(06,85)IK,IL,SNXVEL(IK,IL),SNYVEL(IK,IL)
       2000 CONTINUE
       9000 STOP
            END
```

```
BBBBBBBBBB    000000000000000000000000000000000000000000000000000000000
BBBBBBBBBB    0   Montana State University - SSS - Vax/Vms ver. 4.1   0    BBBBB
BBBBBBBBBB    000000000000000000000000000000000000000000000000000000000    BBBBB
                                                                           BBBBB

              U   U  EEEEE  M   M  RRRR   BBBB    000     000     1
              U   U  E      MM MM  R   R  B   B  0   0   0   0   11
              U   U  E      M M M  R   R  B   B  0  00   0  00    1
              U   U  EEEE   M   M  RRRR   BBBB   0 0 0   0 0 0    1
              U   U  E      M   M  R R    B   B  00  0   00  0    1
              U   U  E      M   M  R  R   B   B  0   0   0   0    1
              UUUUU  EEEEE  M   M  R   R  BBBB    000     000    111

                 FFFFFFFFFF  LL            AAAAAA    XX          XX
                 FFFFFFFFFF  LL            AAAAAA    XX          XX
                 FF          LL          AA      AA  XX          XX
                 FF          LL          AA      AA  XX          XX
                 FF          LL          AA      AA    XX      XX
                 FF          LL          AA      AA    XX      XX
                 FFFFFFFF    LL          AA      AA      XX  XX
                 FFFFFFFF    LL          AA      AA      XX  XX
                 FF          LL          AAAAAAAAAA    XX      XX
                 FF          LL          AAAAAAAAAA    XX      XX
                 FF          LL          AA      AA  XX          XX
                 FF          LL          AA      AA  XX          XX
                 FF          LLLLLLLLLL  AA      AA  XX          XX
                 FF          LLLLLLLLLL  AA      AA  XX          XX

                             LL          IIIIII    SSSSSSSS
                             LL          IIIIII    SSSSSSSS
                             LL            II    SS
                             LL            II    SS
                             LL            II    SS
                             LL            II    SS
                             LL            II      SSSSSS
                             LL            II      SSSSSS
                             LL            II            SS
                             LL            II            SS
                             LL            II            SS
                     ....    LL            II            SS
                     ....    LLLLLLLLLL  IIIIII    SSSSSSSS
                     ....    LLLLLLLLLL  IIIIII    SSSSSSSS

                          ;;;;        11      000000
                          ;;;;        11      000000
                          ;;;;      1111    00      00
                          ;;;;      1111    00      00
                                      11    00    0000
                                      11    00    0000
                          ;;;;        11    00  00  00
                          ;;;;        11    00  00  00
                          ;;;;        11    0000    00
                          ;;;;        11    0000    00
                          ;;;;        11    00      00
                            ;;        11    00      00
                          ;;      111111      000000
                          ;;      111111      000000

File _DUA1:[UEMRB001]FLAX.LIS;10 (8609,3,0), last revised on 25-NOV-1985 15:32,
26 block sequential file owned by UIC [UEMRB001].  The records are variable leng
with implied (CR) carriage control.  The longest record is 132 bytes.

Job DURBY (2030) queued to SYS$PRINT on 25-NOV-1985 15:39 by user UEMRB001, UIC
[UEMRB001], under account 1 at priority 4, started on printer LP1 on 25-NOV-1985
17:54 from queue LP1.

BBBBBBBBBB    000000000000000000000000000000000000000000000000000000000    BBBBB
BBBBBBBBBB      Digital Equipment Corporation - VAX/VMS Version V4.1  0    BBBBB
BBBBBBBBBB    000000000000000000000000000000000000000000000000000000000    BBBBB
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Figure 18. FLAX Fortran Code.

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                                                      22-Nov-1985 17:2
                                                      25-Aug-1984 21:1
C****       FLAX is a Fortran code for evaluating the Continuity
C**** equation, in two space variables for the resultant
C**** snow velocity field from DURBY.
C****       FLAX solves by Finite Difference technique.
C****
C****   ****     ****     ****     ****     ****     ****     ****
C****
C**** VARIABLE DEFINITIONS
C****
C**** USN(K,L),VSN(K,L),USNC(K,L),VSNC(K,L);
C****     COEFF. CONTAINING SNOW VELOCITIES AND GRADIENTS
C****     OF SNOW VELOCITIES.
C**** RO(K,L); DRIFT SNOW DENSITY
C**** ROLBC(L),RCTBC(K); DRIFT SNOW DENSITY B.C.
C**** URO(M),UROM1(M),SUM1,SUM2,TOL; GUASS-SEIDEL
C**** INTERMEDIATE VALUES DURING ITERATION
C****
C****   ****     ****     ****     ****     ****     ****     ****
C****
C**** LOGIC ASSIGNMENTS
C****
C****     INPUT:
C****           NUMERICAL FACTORS: FOR001
C****           DRIFT SNOW DENSITY B.C.: FOR002
C****           SNOW VELOCITY FIELD FROM DURBY: FOR003
C****           SNOW VELOCITY B.C.: FOR004
C****     OUTPUT:
C****           DRIFT SNOW DENSITY FIELD: FOR006
C****
C****   ****     ****     ****     ****     ****     ****     ****
C****
C**** INTIALIZATION,PARAMETERIZATION & DIMENSIONING
C****
      REAL*8 SNXVEL,SNYVEL,SNLBCX,SNTBCY
      REAL*8 USNC,VSNC,USN,VSN,A,B
      REAL*8 RO,ROLBC,ROTBC,URO,UROM1,SUM1,SUM2,TOL
      PARAMETER (K=20,L=10,M=200)
      DIMENSION SNXVEL(K,L),SNYVEL(K,L),SNLBCX(L),SNTBCY(K)
      DIMENSION USNC(K,L),VSNC(K,L),USN(K,L),VSN(K,L)
      DIMENSION A(M,M),B(M)
      DIMENSION RO(K,L),ROLBC(L),ROTBC(K),URO(M),UROM1(M),TOL(M)
      ITERMX=30
      TOLMX=0.05
C****
C****   ****     ****     ****     ****     ****     ****     ****
C****
C**** INPUT/OUTPUT FORMATS
C****
   15 FORMAT(2F7.3)
   20 FORMAT(30X,1E12.5,3X,1E12.5)
   25 FORMAT(30X,1E12.5)
   35 FORMAT(45X,1E12.5)
```

```
      60 FORMAT(A30)
      70 FORMAT(A50)
      80 FORMAT(A10)
      90 FORMAT(21X,2I3,3X,1E12.5)
C****
C****    ****     ****     ****     ****     ****     ****     ****
C****
C**** READ &/OR WRITE STATEMENTS
C****
         READ(01,15)G,H
         DO 001 IL=1,L
         READ(02,25)ROLBC(IL)
     001 CONTINUE
         DO 100 IK=1,K
         READ(02,25)ROTBC(IK)
     100 CONTINUE
         DO 200 IL=1,L
         DO 200 IK=1,K
         READ(03,20)SNXVEL(IK,IL),SNYVEL(IK,IL)
     200 CONTINUE
         DO 300 IL=1,L
         READ(04,25)SNLBCX(IL)
     300 CONTINUE
         DO 400 IK=1,K
         READ(04,35)SNTBCY(IK)
     400 CONTINUE
C****
C****    ****     ****     ****     ****     ****     ****     ****
C****
C**** CALCULATING THE COMPUTATIONAL VARIABLES
C****
         DO 500 IL=1,L
         DO 500 IK=1,K
         USN(IK,IL)=SNXVEL(IK,IL)/G
         VSN(IK,IL)=SNYVEL(IK,IL)/H
     500 CONTINUE
         DO 600 IL=1,L
         DO 600 IK=1,K
         IF (IK .EQ. 1) GO TO 5000
         USNC(IK,IL)=(SNXVEL(IK,IL)-SNXVEL(IK-1,IL))/G
         GO TO 5010
    5000 USNC(IK,IL)=(SNXVEL(IK,IL)-SNLBCX(IL))/G
    5010 IF (IL .EQ. L) GO TO 5030
         VSNC(IK,IL)=(SNYVEL(IK,IL+1)-SNYVEL(IK,IL))/H
         GO TO 600
    5030 VSNC(IK,IL)=(SNTBCY(IK)-SNYVEL(IK,IL))/H
     600 CONTINUE
C****
C****    ****     ****     ****     ****     ****     ****     ****
C****
C**** ZEROING-OUT THE COMPUTATIONAL ARRAYS
C****
```

```
      DO 700 IA1=1,M
      DO 700 IA2=1,M
      A(IA1,IA2)=0.0
  700 CONTINUE
      DO 800 IB1=1,M
      B(IB1)=0.0
  800 CONTINUE
C****
C****     ****     ****     ****     ****     ****     ****     ****
C****
C**** FILLING-OUT THE COMPUTATIONAL ARRAYS
C****
C**** MAIN DIAGONAL OF A(M,M)
C****
      DO 900 IL=1,L
      DO 900 IK=1,K
      IMD=(IL-1)*K+IK
      A(IMD,IMD)=USNC(IK,IL)+VSNC(IK,IL)+USN(IK,IL)-VSN(IK,IL)
  900 CONTINUE
C****
C**** LOWER CODIAGONAL OF A(M,M)
C****
      DO 1000 IL=1,L
      DO 1000 IK=2,K
      ILC=(IL-1)*K+IK
      ILCM1=ILC-1
      A(ILC,ILCM1)=-USN(IK,IL)
 1000 CONTINUE
C****
C**** UPPER CODIAGONAL OF A(M,M)
C****
      DO 1100 IL=2,L
      DO 1100 IK=1,K
      IUC=(IL-1)*K+IK
      IUCMK=(IL-2)*K+IK
      A(IUCMK,IUC)=VSN(IK,IL-1)
 1100 CONTINUE
C****
C**** B(M)
C****
      DO 1200 IL=1,L
      DO 1200 IK=1,K
      IB=(IL-1)*K+IK
      IF (IB .NE. ((IL-1)*K+1)) GO TO 5040
      B(IB)=USN(IK,IL)*ROLBC(IL)
 5040 IF (IB .LT. ((L-1)*K+1)) GO TO 1200
      B(IB)=B(IB)-VSN(IK,IL)*ROTBC(IK)
 1200 CONTINUE
C****
C****     ****     ****     ****     ****     ****     ****     ****
C****
C**** ITERATIVELY (GAUSS-SEIDEL) SOLVING FOR THE PARTICLE
```

```
C**** DENSITY DISTRIBUTION
C****
      DO 3000 IUROM1=1,M
      UROM1(IUROM1)=0.0
 3000 CONTINUE
      ITER=0
 7010 ITER=ITER+1
      DO 3100 IGS=1,M
      SUM1=0.0
      IF (IGS .EQ. 1) GO TO 7020
      IGSM1=IGS-1
      DO 3200 JGS1=1,IGSM1
      SUM1=SUM1-(A(IGS,JGS1)*URO(JGS1))
 3200 CONTINUE
      SUM2=0.0
      IF (IGS .EQ. M) GO TO 7030
 7020 IGSP1=IGS+1
C  DO 3300 JGS2=IGSP1,M
      DO 3300 JGS2=IGSP1,M
      SUM2=SUM2-(A(IGS,JGS2)*UROM1(JGS2))
 3300 CONTINUE
 7030 URO(IGS)=(SUM1+SUM2+B(IGS))/A(IGS,IGS)
 3100 CONTINUE
      DO 3400 IGS=1,M
      TOL(IGS)=ABS((URO(IGS)-UROM1(IGS))/URO(IGS))
      IF (TOL(IGS) .GT. TOLMX) GO TO 7040
 3400 CONTINUE
      GO TO 7050
 7040 IF (ITER .EQ. ITERMX) GO TO 8000
      DO 3500 IGS=1,M
      UROM1(IGS)=URO(IGS)
 3500 CONTINUE
      GO TO 7010
C****
C****     ****     ****     ****     ****     ****     ****     ****
C****
C**** WRITING URO VECTOR TO RO(K,L)
C****
 7050 DO 1300 IL=1,L
      DO 1300 IK=1,K
      IURO=(IL-1)*K+IK
      RO(IK,IL)=URO(IURO)
 1300 CONTINUE
      GO TO 5050
C****
C****     ****     ****     ****     ****     ****     ****     ****
C****
C**** WRITE STATEMENTS FOR CONVERGENT & NON-CONVERGENT SOLUTIONS
C****
 8000 WRITE(06,70) 'GAUSS-SEIDEL RTN DID NOT CONV.'
      GO TO 9000
 5050 WRITE(06,60) 'THIS IS A CONVERGENT SOLUTION'
```

```
      WRITE(06,80) 'K,L,RO'
      DO 1400 IL=1,L
      DO 1400 IK=1,K
      WRITE(06,90)IK,IL,RO(IK,IL)
 1400 CONTINUE
 9000 STOP
      END
```

```
CCCCCCCCCC    0000000000000000000000000000000000000000000000000000000000    CCCCC
CCCCCCCCCC    0   Montana State University - SSS - Vax/Vms ver. 4.1   0    CCCCC
CCCCCCCCCC    0000000000000000000000000000000000000000000000000000000000    CCCCC

               U   U  EEEEE  M   M  RRRR   BBBB     000    000     1
               U   U  E      MM MM  R   R  B   B   0   0  0   0   11
               U   U  E      M M M  R   R  B   B   0  00  0  00    1
               U   U  EEEE   M   M  RRRR   BBBB    0 0 0  0 0 0    1
               U   U  E      M   M  R R    B   B   00  0  00  0    1
               U   U  E      M   M  R  R   B   B   0   0  0   0    1
               UUUUU  EEEEE  M   M  R   R  BBBB     000    000    111

                                 JJ  EEEEEEEEEE  TTTTTTTTTT
                                 JJ  EEEEEEEEEE  TTTTTTTTTT
                                 JJ  EE              TT
                                 JJ  EE              TT
                                 JJ  EE              TT
                                 JJ  EE              TT
                                 JJ  EEEEEEEE        TT
                                 JJ  EEEEEEEE        TT
                         JJ      JJ  EE              TT
                         JJ      JJ  EE              TT
                         JJ      JJ  EE              TT
                         JJ      JJ  EE              TT
                           JJJJJJ    EEEEEEEEEE      TT
                           JJJJJJ    EEEEEEEEEE      TT

                               LL             IIIIII     SSSSSSSS
                               LL             IIIIII     SSSSSSSS
                               LL               II     SS
                               LL               II     SS
                               LL               II     SS
                               LL               II     SS
                               LL               II       SSSSSS
                               LL               II       SSSSSS
                               LL               II             SS
                               LL               II             SS
                        ....   LL               II             SS
                        ....   LL               II             SS
                        ....   LLLLLLLLLL     IIIIII   SSSSSSSS
                        ....   LLLLLLLLLL     IIIIII   SSSSSSSS

                              ;;;;       11        000000
                              ;;;;       11        000000
                              ;;;;     1111      00      00
                              ;;;;     1111      00      00
                                         11      00    0000
                                         11      00    0000
                              ;;;;       11      00  00  00
                              ;;;;       11      00  00  00
                              ;;;;       11      0000    00
                              ;;;;       11      0000    00
                                ;;       11      00      00
                                ;;       11      00      00
                              ;;       111111      000000
                              ;;       111111      000000

File _DUA1:[UEMRB001]JET.LIS;10 (3597,22,0), last revised on 25-NOV-1985 15:33,
9 block sequential file owned by UIC [UEMRB001].  The records are variable lengt
with implied (CR) carriage control.  The longest record is 132 bytes.

Job DURBY (2030) queued to SYS$PRINT on 25-NOV-1985 15:39 by user UEMRB001, UIC
[UEMRB001], under account 1 at priority 4, started on printer LP1 on 25-NOV-1985
17:55 from queue LP1.

CCCCCCCCCC    0000000000000000000000000000000000000000000000000000000000    CCCCC
CCCCCCCCCC       Digital Equipment Corporation - VAX/VMS Version V4.1  0    CCCCC
CCCCCCCCCC    0000000000000000000000000000000000000000000000000000000000    CCCCC
```

Figure 19. JET Fortran Code.

11-Jul-1984 12:3
11-Jul-1984 12:3

```
C****       JET fills a DURBY solution domain with compo-
C**** nent airflow velocities based on an emprical half-
C**** jet model.
      DIMENSION ARXVEL(22,12),ARYVEL(22,12)
      DIMENSION XTERM1(22,12),XTERM2(22,12),XTERM3(22,12)
      DIMENSION YTERM1(22,12),YTERM2(22,12)
   05 FORMAT(3X,F7.3,3X,F7.3)
   10 FORMAT(3X,F7.3,3X,F7.3,3X,E12.5,3X,E12.5)
   85 FORMAT(21X,2I3,3X,E12.5,3X,E12.5)
      READ(01,05)UJET,SCALE
      FAX1=0.0049/SCALE
      FAX2=0.000000123/(SCALE**3)
      ARXVEL(1,1)=0.0
      ARYVEL(1,1)=UJET
      DO 100 IK=2,22
      IL=1
      ARXVEL(IK,IL)=0.0
      ARYVEL(IK,IL)=0.0
  100 CONTINUE
      DO 200 IL=2,12
      DO 200 IK=1,22
      XTERM1(IK,IL)=(FAX1*3.4*SCALE*IL*0.5)/(IK*0.5)
      XTERM2(IK,IL)=FAX2*(-472)*(((SCALE*IL*0.5)/(IK*0.5))**3)
      XTERM3(IK,IL)=195*(((SCALE*IL*0.5)/(IK*0.5))**4)
      YTERM1(IK,IL)=-630*(((SCALE*IK*0.5)/(IL*0.5))**2)
      YTERM2(IK,IL)=2313*(((SCALE*IK*0.5)/(IL*0.5))**3)
  200 CONTINUE
      DO 300 IL=2,12
      DO 300 IK=1,22
      ARXVEL(IK,IL)=-(UJET/(IL*0.5))*(XTERM1(IK,IL)+XTERM2(IK,IL)+
     1XTERM3(IK,IL))
      ARYVEL(IK,IL)=((UJET*SCALE)/(IK*0.5))*(6.8+YTERM1(IK,IL)+
     1YTERM2(IK,IL))
  300 CONTINUE
      WRITE(06,10)UJET,SCALE,FAX1,FAX2
      DO 400 IL=1,12
      DO 400 IK=1,22
      WRITE(06,85)IK,IL,ARXVEL(IK,IL),ARYVEL(IK,IL)
  400 CONTINUE
      STOP
      END
```

```
EEEEEEEEEE    000000000000000000000000000000000000000000000000000000    EEEEE
EEEEEEEEEE    0    Montana State University - SSS - Vax/Vms ver. 4.1   0    EEEEE
EEEEEEEEEE    000000000000000000000000000000000000000000000000000000    EEEEE

          U   U  EEEEE  M   M  RRRR   BBBB    000    000     1
          U   U  E      MM MM  R   R  B   B  0   0  0   0   11
          U   U  E      M M M  R   R  B   B  0  00  0  00    1
          U   U  EEEE   M   M  RRRR   BBBB   0 0 0  0 0 0    1
          U   U  E      M   M  R R    B   B  00  0  00  0    1
          U   U  E      M   M  R  R   B   B  0   0  0   0    1
          UUUUU  EEEEE  M   M  R   R  BBBB    000    000    111

                  SSSSSSSS  LL            AAAAAA    BBBBBBBB
                  SSSSSSSS  LL            AAAAAA    BBBBBBBB
                SS          LL          AA      AA  BB      BB
                SS          LL          AA      AA  BB      BB
                SS          LL          AA      AA  BB      BB
                SS          LL          AA      AA  BB      BB
                  SSSSSS    LL          AA      AA  BBBBBBBB
                  SSSSSS    LL          AA      AA  BBBBBBBB
                        SS  LL          AAAAAAAAAA  BB      BB
                        SS  LL          AAAAAAAAAA  BB      BB
                        SS  LL          AA      AA  BB      BB
                        SS  LL          AA      AA  BB      BB
                SSSSSSSS    LLLLLLLLLL  AA      AA  BBBBBBBB
                SSSSSSSS    LLLLLLLLLL  AA      AA  BBBBBBBB

                            LL            IIIIII      SSSSSSSS
                            LL            IIIIII      SSSSSSSS
                            LL              II      SS
                            LL              II      SS
                            LL              II      SS
                            LL              II      SS
                            LL              II        SSSSSS
                            LL              II        SSSSSS
                            LL              II              SS
                            LL              II              SS
                    ....    LL              II              SS
                    ....    LL              II              SS
                    ....    LLLLLLLLLL    IIIIII    SSSSSSSS
                    ....    LLLLLLLLLL    IIIIII    SSSSSSSS

                              ;;;;            11      000000
                              ;;;;            11      000000
                              ;;;;          1111    00      00
                                            1111    00      00
                                              11    00    0000
                              ;;;;            11    00    0000
                              ;;;;            11    00  00  00
                              ;;;;            11    00  00  00
                              ;;;;            11    0000    00
                                ;;            11    0000    00
                                ;;            11    00      00
                              ;;              11    00      00
                              ;;          111111      000000
                                          111111      000000


File _DUA1:[UEMRB001]SLAB.LIS;10 (5601,6,0), last revised on 25-NOV-1985 15:31,
7 block sequential file owned by UIC [UEMRB001].  The records are variable lengt
with implied (CR) carriage control.  The longest record is 132 bytes.

Job DURBY (2030) queued to SYS$PRINT on 25-NOV-1985 15:39 by user UEMRB001, UIC
[UEMRB001], under account 1 at priority 4, started on printer LP1 on 25-NOV-1985
17:56 from queue LP1.

EEEEEEEEEE    000000000000000000000000000000000000000000000000000000    EEEEE
EEEEEEEEEE      Digital Equipment Corporation - VAX/VMS Version V4.1   0    EEEEE
EEEEEEEEEE    000000000000000000000000000000000000000000000000000000    EEEEE
```

Figure 20. SLAB Fortran Code.

```
C****       SLAB solves for the particle accumulation rates
C**** [cm/hr] at the solid surface, using the particle fall
C**** velocity from DURBY and the suspended particle
C**** density from FLAX. An accumulation density must
C**** be assumed. In SLAB this accumulation density is
C**** a spatial constant.
      REAL*8 SNYVEL,RO,ACMRT
      PARAMETER (K=20)
      DIMENSION SNYVEL(K),RO(K),ACMRT(K)
      ACMRO=175
C****
C****
   15 FORMAT(45X,1E12.5)
   20 FORMAT(30X,1E12.5)
   90 FORMAT(21X,2I3,3X,1E12.5)
C****
C****
      DO 300 IK=1,K
      READ(01,15)SNYVEL(IK)
  300 CONTINUE
      DO 400 IK=1,K
      READ(02,20)RO(IK)
  400 CONTINUE
C****
C****
      DO 500 IK=1,K
      ACMRT(IK)=(360000*SNYVEL(IK)*RO(IK))/ACMRO
  500 CONTINUE
C****
C****
      DO 600 IK=1,K
      IL=1
      WRITE(06,90)IK,IL,ACMRT(IK)
  600 CONTINUE
      STOP
      END
```