



A microstructure based constitutive theory for granular materials with snow as an example
by Puneet Mahajan

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in
Mechanical Engineering
Montana State University
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Abstract:

Snow is made of ice particles connected together to form chains. Sintering at the points of contact between particles leads to the formation of bonds called necks. The mechanical response of snow is determined by the mechanical behavior of the particles and bonds. Some of the microstructural variables which determine the behavior of snow are particle size, the bond size, the number of bonds per particle and the density of snow. Snow is thermodynamically very active in its natural environment, and depending on meteorological environment the microstructure can vary dramatically both spatially and in time.

In this thesis a constitutive theory is developed to describe the mechanical response of snow in terms of the behavior of ice particles and bonds between them. The principle of virtual work is used to calculate the stresses in the particles and a number of different deformation mechanisms are investigated. Depending on the deformation mechanisms, different equations are used to calculate the strains in particles and bonds. These strains are averaged and scaled to obtain the strains in snow.

Results are presented for medium density snow subject to uniaxial and multiaxial loading. The uniaxial loading results are compared with existing experimental data and show a very good correlation. The application of the theory is shown by solving two problems.

The advantage of the theory lies in its ability to predict multiaxial behavior using parameters obtained from uniaxial tests.

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FOR GRANULAR MATERIALS WITH
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of the requirements for the degree

of

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APPROVAL

of a thesis submitted by

Puneet Mahajan

This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

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ABSTRACT

Snow is made of ice particles connected together to form chains. Sintering at the points of contact between particles leads to the formation of bonds called necks. The mechanical response of snow is determined by the mechanical behavior of the particles and bonds. Some of the microstructural variables which determine the behavior of snow are particle size, the bond size, the number of bonds per particle and the density of snow. Snow is thermodynamically very active in its natural environment, and depending on meteorological environment the microstructure can vary dramatically both spatially and in time.

In this thesis a constitutive theory is developed to describe the mechanical response of snow in terms of the behavior of ice particles and bonds between them. The principle of virtual work is used to calculate the stresses in the particles and a number of different deformation mechanisms are investigated. Depending on the deformation mechanisms, different equations are used to calculate the strains in particles and bonds. These strains are averaged and scaled to obtain the strains in snow.

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The advantage of the theory lies in its ability to predict multiaxial behavior using parameters obtained from uniaxial tests.

CHAPTER 1

INTRODUCTION

Snow

Snow is a granular geologic material with ice as the matrix material. It consists of a system of discrete particles of ice with voids in between them. The particles have varying shapes, orientations and number of bonds connecting each particle to neighboring particles.

The effect of snow on life style and the environment is enormous, and a better understanding of snow is of great use. Snow has recreational uses such as skiing and alpine backpacking. Substantial oil and natural gas reserves are in areas which are covered with snow for at least part of the year. For instance, there are eighteen geologic provinces in the Rocky Mountain region that have at least one significant oil field [Nehring, 1981]. Many of the mining operations are also in these areas. With the increase in activity in these areas has also come an increased exposure to snow avalanches and blowing snow. Buildings and structures in these areas have to be designed to bear impact loads due to avalanches as well as static loads due to snow deposited by winds. It has been determined that deposition of drifting snow on multilevel roofs can cause abnormally high loads on the lower roof [Mackinlay, 1989].

Deposited snow can also form ice dams on roofs. Ice dams are formed by snow melt water which runs under the snow blanket before refreezing on the eaves. These dams can reach two meters in height and can cause hydrostatic pressures which are sufficient to force water through small cracks in the roof. Very deep snow can put heavy loads on lateral walls, due to its creeping and gliding motion. This in turn can totally distort the structure. Snow creep loads are also a major factor in the design of avalanche defense structures. The topic of drifting snow has drawn a lot of interest in recent years, as evidenced by the number of papers on drifting snow presented in the First Conference on Snow Engineering, 1988. A better knowledge of creeping properties of snow is required to make the designs of the buildings in cold regions safer and more economical.

Vehicle travel is adversely affected by snow cover due to wheel track sinkage and slippage. Large volumetric deformations due to compaction by wheels has the effect of reducing the energy available to propel the vehicle forward. Brown [1979b] found that for some tracked vehicles travelling at 20 mph in medium density snow ($300\text{-}400 \text{ kg/m}^3$) the power absorbed by snow cover exceeded 40 percent of the available engine power. Slippage and resulting loss of traction is a much more difficult problem to analyze because both deviatoric and volumetric deformations are involved. This requires an understanding of multiaxial deformation characteristics of snow.

Explosives are often used to break cornices in ski areas and in wind loaded areas above highways. Snow has the ability to absorb a great deal of energy through material compaction, resulting in

tremendous attenuation rates for stress waves, caused by explosives [Brown, 1980b]. It is because of this property that areal explosives are more effective in initiating an avalanche than explosives placed in the snow. The peak pressures of shock waves due to areal detonations are 10-100 times greater than those due to explosives buried in snow [Gubler, 1977]. Areal detonations do not create a crater but rather spread the pressure wave out over the snow surface where the transfer to snow cover is made in a more efficient manner. Stress wave generation and dissipation of energy in snow, by explosives, is also of interest to the army. For instance, inert steel and aluminium projectiles fired at 10^3 m/s into medium density snow are deformed plastically [Swinzow, 1972].

Swinzow [1972] also found that bullets are inherently unstable in snow and tumble, losing much of their kinetic energy in the process. Consequently, these bullets can be stopped by heavy clothing or metal lining installed in a bunker [Johnson, 1977]. Construction of highways, pavements and airstrips to carry heavy wheel loads has also been of interest to the army. Studies have been done on various methods of processing snow so as to increase its density and thereby increase its strength [Abele, Ramseier and Wouri, 1968]. Abele [1965] has looked at the feasibility of subsurface snow transportation in deep snow.

Snow has also been used to augment water supplies in agricultural areas. Water present in clouds which would ordinarily have been lost due to evaporation can be converted to snow using cloud seeding techniques. Orographic clouds may contain supercooled

water droplets that often remain in liquid state at temperatures much colder than freezing due to a scarcity of ice nucleating material [Robitaille, Barlow and Renick, 1986]. In the absence of nucleating material the cloud droplets evaporate in the descending air on the leeward side of the mountain. To prevent this evaporation and initiate the formation of ice particles, silver iodide smoke or dry ice pellets are sprayed in these clouds. The increased snowpack then provides for additional streamflow during spring melt and for increased water supply for municipal and agricultural use.

Snow in its natural state exists at close to its melting point and can readily undergo phenomena such as recrystallization, phase transformation, heat and mass transfer, and sintering which is commonly observed in powder metals at relatively high temperatures. Although a good amount is already known about these phenomena, a still better understanding can be had by studying the same process in snow without limitations imposed by high temperature requirements.

Metamorphism, Sintering and Bond Formation

In its natural environment snow is thermodynamically very active, and undergoes a variety of metamorphic processes including sintering and bonding of grains, temperature gradient metamorphism, melt/freeze processes, and heat and mass transport due to inhomogeneities in physical properties [LaChapelle, 1969]. Heat and mass flux takes place through both the solid and vapor phases. Metamorphic processes are active in snow from the time of

formation as individual ice crystals in the air until it either evaporates back into the atmosphere or melts under warm weather conditions.

While in the atmosphere, ice crystals form by condensation of supercooled water clouds on nuclei such as dust. These ice crystals grow rapidly into often complex and intricate crystalline forms as they fall through the atmosphere, encountering a wide range of temperature and supersaturation. Once formed, these intricate crystals are in an unstable thermodynamic form and tend to metamorphose towards a more stable configuration. Development of an equilibrium form involves the reduction of free area (and therefore excess surface energy), thereby approaching a minimum energy level. This requires the ratio of surface area to grain volume be reduced to minimum, and therefore the grains tend to be large grains with a spherical shape. This is achieved by transfer of material to the region of contact between the particles. Given enough time, snow would consolidate to solid ice to eliminate surface energy.

The controlling mechanism for transfer of material is the difference in vapor pressure at the ice surface due to variations in radius of curvature. Mass migrates from regions of high vapor pressure to low pressure regions through vapor diffusion, i.e., through transfer of water molecules from one part of the crystal and subsequent redeposition as solid on either another part of the snow crystal or on a different crystal. Viscous or plastic flow and surface diffusion also help in material transfer to the contact points [Hobbs, 1974]. Volume diffusion and grain boundary diffusion also contribute

but under some conditions are much less significant than vapor diffusion or sintering.

This process leading to the formation of a rounded equilibrium form is called equitemperature metamorphism, for it proceeds in bodies which are not far from a uniform temperature. The overall strength of snow is also affected during this metamorphic process. The deposition of ice at contact points produces bonds or necks between adjoining grains in a process known as sintering. As the process proceeds, the necks become larger, and the snowpack strengthens.

An externally applied pressure serves to increase the rate of sintering. The mechanism of pressure sintering is analogous to creep kinetics for polycrystalline metals. A number of processes, occurring in either series or parallel, including lattice diffusion, grain boundary diffusion, and dislocation creep have been used to explain the phenomenon. The regime of dominance of each of the above mechanisms depends on the radius of ice particles, density of snow, applied pressure, and temperature. Maeno and Ebinuma [1983] used pressure sintering diagrams to show these various regimes in snow samples with densities varying from 600 kg/m^3 to 900 kg/m^3 . Depending on the radius of the particles, at very low pressure, lattice diffusion or boundary diffusion may be the predominant mechanism. At higher pressures dislocation creep takes over. However, if pressure is very high it is likely to lead to neck fracture.

This normal process of metamorphism is completely altered when large temperature differences exist between adjacent layers.

The steep temperature gradient imposes differences in equilibrium water vapor pressure in the interstitial spaces of adjacent snow layers. Vapor then flows from a high vapor pressure (warmer snow) to a low vapor pressure (colder snow) region. The transfer of vapor is not necessarily accomplished by movement of molecules through the tortuous air passages in snow crystals. Instead a "hand to hand" transfer process is at work. The vapor is sublimated off the top of an ice grain, transported across a pore and deposited on the bottom of a colder grain which is simultaneously having vapor sublimated from its upper surface. As a result, snow has a higher apparent diffusivity than other porous materials which do not have ice as the matrix material, since this hand to hand transfer is not possible. As this process, called temperature gradient metamorphism, proceeds, new crystals begin to take shape and old ones gradually disappear. The new crystals have stepped or ribbed faces and, depending on local conditions, may assume various shapes such as cups, scrolls and columns [LaChapelle, 1969]. This type of metamorphism generally prevails when temperature gradients exceed $10^{\circ}\text{C}/\text{m}$. The strongest, most sustained temperature gradients occur on north facing slopes and in deep shaded gullies as the snow surface is kept cool because of large radiation losses. Temperature gradient layers are more prevalent at higher elevations [Perla and Martinelli, 1976].

Temperature gradients produce very little sintering, and the newly formed crystals are more weakly bonded than those produced by equitemperature metamorphism. The net result is a weakening of snow. If this process is carried to completion, snow develops a very

fragile structure and collapses into cohesionless mass on slight disturbance. This type of snow is called depth hoar and has many undesirable practical effects. It undermines compacted roads or runways, interferes with packing of ski runs, and is a major cause of avalanches.

Approach

In this work we will be concentrating mainly on deformation, due to external loads, of snow formed by equitemperature metamorphism and therefore having well developed bonds. This snow typically has an initial density around 300-350 kg/m³.

Snow is treated as if made of chains of spherical particles joined by areas of much smaller cross section called bonds or necks as shown in Figure 1. Because of their smaller cross section, the necks are subjected to much higher stresses when compared to the ice particle and therefore undergo large deformations. It is the straining of necks which is responsible for deformation of snow, particularly at low stresses.

At higher stresses the necks undergo fracture, and interparticle slip becomes a significant deformation mechanism. It is this relative displacement between the particles which is responsible for strains at large stresses. At these stresses snow behaves somewhat like other granular materials.

Even during deformation the metamorphic processes continue to occur, and there may be some change in number of neighbors of a particle. Also the radii of necks may be changing due to continuing

mass transfer from regions of higher vapor pressure. However, inclusion of these considerations requires much more experimental data than is presently available.

When subjected to compression, ice has its melting point lowered and can undergo pressure melting. This phenomenon is probably occurring in necks during deformation, particularly at elevated temperatures near 0°C . In broken necks it probably facilitates the sliding of particles due to formation of water sublayer. No attempt has been made to include this explicitly in the model.

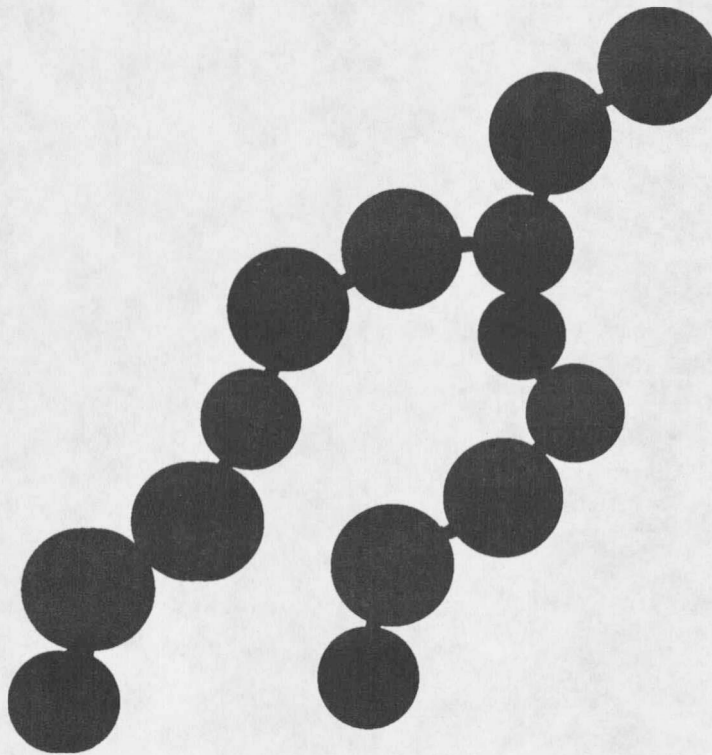


Figure 1. A chain of spherical ice particles connected by necks

Review of Existing Work

The review is divided into three sections. The first section briefly surveys previous investigations into constitutive modelling of snow. Here we concentrate mainly on models based on micromechanical properties. The second section reviews some of the existing literature on the constitutive behavior of ice. The third section is on granular materials and powder metals.

Constitutive Behavior of Snow

Researchers [Yosida, 1955; Bader, 1953, 1962; Abele, 1963] have been describing the behavior of snow for many years, although the modelling methods have undergone a considerable change. Earlier models treated snow as having linear stress strain rate behavior. They made use of linear viscoelastic models to simulate the behavior of snow [Mellor, 1974].

Later models started using nonlinear stress-strain relations but did not attempt a microstructural description for behavior of snow. Brown [1976] used a multiple integral representation to describe the behavior of snow. Brown and Lang [1974] investigated the fracture characteristics of snow, using the principle of continuum thermodynamics. Expressions for the Helmholtz free energy were developed and used to calculate free energy and dissipation during deformation. These variables were then used to characterize a fracture criterion for snow.

Salm [1974] established a constitutive equation for creeping snow in quasi-stationary state by using the principle of least

irreversible force. A dissipation function was developed in a power series of the invariants of the stress tensor. The constitutive equation depended only on this dissipation function. Salm [1967] also investigated the multiaxial behavior, using a constitutive law similar to Hooke's law with strain rates in place of strains. The Lamé's constants were expressed as functions of basic invariants of rate of deformation.

Lately, attempts have been made to describe the response of snow in terms of behavior of ice particles and necks. Significant among these are the works of Feldt and Ballard [1965], Kry [1975], Brown [1979,1980], and Hansen [1985].

Feldt and Ballard [1966] developed a theory of consolidation for laterally confined, age hardened, medium density snow under uniaxial stress (0.0098 to 0.049 MPa). They considered the snow mass to be composed of ice particles joined together by a finite number of ice bonds. The shear component of the axial force (applied to the snow mass) produced a viscous flow in the ice composing the bonds. Pore space decreased due to interparticle glide at the particle boundaries. This interparticle sliding was the major consolidation mechanism in their theory.

Kry [1975] carried out experiments on fine rounded grain snow with a density in the range of 270-340 kg/m³ at stresses low enough so as to avoid any macroscopic rupture during the experiment. He concluded that only a fraction of the grain bonds in a snow sample transmit an applied stress and that the new grain bonds formed during deformation determine the viscoelastic properties. He

ascribed the high strains in snow to bending of chains, which were defined as a series of stress bearing grains. The bending of chains was caused by torques due to eccentricity of the chains. This, however, may not be true because at almost any stress the neighboring chains will prevent bending of each other. Even if bending is taken as the deforming mechanism, the strains in necks required to cause the observed strains in snow will be large and should lead to the breaking of necks.

Brown [1980a] developed a volumetric constitutive law for snow. Pore collapse, intergranular glide and inelastic deformation of intergranular necks were recognized as three major mechanisms responsible for deformation of snow. At low densities the last two mechanisms probably play a predominant role, whereas pore collapse is considered to be important at high densities [Brown, 1979a]. Brown [1980a] calculated compressive stresses and inelastic deformation in the necks and grains in order to calculate the deformation response of the snow to applied stresses. Empirical adjustments were made to account for sliding. The constitutive equation, although it did not take shear effects in the necks into account, approximated the behavior of snow over a wide range of compressive strain rates.

Rice [1971] developed a general internal variable thermodynamic formalism for a class of solids at finite strain exhibiting inelasticity due to specific structural rearrangements on the microscale. Metals deforming plastically due to slip rearrangement of crystallographic planes through dislocation motion

were the primary application of the theory. The rate of change of strain rates with rate of change of internal variables, under restrictions of the Second Law of Thermodynamics were discussed. Rice [1975] discusses the relation between macroscopic deformation and internal structural rearrangements which operate on microscale. Lubliner [1972] used the internal variable theory to establish viscoelastic, viscoplastic, and plastic constitutive laws.

Hansen [1985] has used the theory of Rice [1971] to characterize the multiaxial response of snow to high rate deformation. Some of the internal variables included in his theory were neck radius, density ratio, neck length, number of bonds and intergranular slip distance. Stereological measurements were made to determine the mean value of these state variables. The theory was able to account for many phenomenological properties of snow.

Constitutive Behavior of Ice

Since snow is made of ice particles, the properties of snow depend on the properties of ice. A number of theories exist to describe the mechanical behavior of ice.

Sinha [1979] has proposed a simple thermorheological formulation to describe the uniaxial strain-strain rate-temperature relation for columnar grained ice. This model, consisting of eight parameters, accounts for instantaneous elastic strain, delayed elastic strain and secondary creep. The model takes grain size into consideration and includes delayed elastic strain effects which are proportional to applied stress. The model has been extended to

predict initiation of microcracks and tertiary creep due to crack damage effects.

Spring and Morland [1982, 1983] investigated the viscoelastic fluid and solid relations of the differential and single integral types for nonlinear viscoelastic deformation of ice. Mellor [1982] conjectured that constant stress and constant strain rate responses should be completely dependent, i.e., we should be able to predict constant stress behavior from constant strain rate behavior. While the constant stress and constant strain rate fluid relations of Spring and Morland reflect some dependent and some independent information, the solid relations are fully independent. The differential fluid relation approaches close to the conjecture that constant stress and constant strain rate response should each determine the same constitutive relation. The models treat ice as an isotropic material.

Brown [1987] has developed a theory for sea ice which defines instantaneous strain rates in terms of several microdynamical state variables, which include compressive and tensile mobile dislocation densities and specific microcrack surface area. The time variation of the state variables is described by a set of evolution equations. The deformation mechanisms described in this formulation include elastic strain, dislocation climb, dislocation recoil, and crack damage.

Sunder [1986] has developed an extensive model which uses a nonlinear generalization of a two element Maxwell fluid model. The model describes materially anisotropic behavior and represents a

continuously damaging behavior during the ductile to brittle transition in compression.

Szyszkowski and Glockner [1986, 1987a, 1987b] treated ice as an isotropic, non-linear viscoelastic material. The heredity effects have been included using a Volterra integral. A scalar damage factor has been introduced to describe the deterioration of ice. To simplify the computation, the heredity integral has been approximated by a spring dashpot model, the constants for which have been determined from existing experimental data.

Sjolind [1987] has used thermodynamics and invariant theory to describe multiaxial behavior of snow. The damage caused by microcracks in ice, assumed to be initially orthotropic, is described using independent vector fields of planar microcracks. The model takes into account the anisotropy induced by vector fields of planar microcracks.

Constitutive Behavior of Granular Materials

At high stresses snow exhibits a behavior similar to other granular materials, with sliding of particles relative to one another as the deformation mechanism. The load is supported through the frictional contact between the particles. Normally some rolling accompanies sliding and contributes to the overall mechanical response of snow.

Oda [1974] and Kanatani [1983] have used stochastic theory and variational principles to describe the behavior of soils under multiaxial stresses. Orientation of the grains, their shape, density

ratio, and coordination number are some of the factors affecting the fabric or spatial arrangements of solid particles and associated voids in the granular materials.

Oda [1972] has conducted triaxial compression tests to study the change of orientation of particles, their coordination number and orientation of normals at the contact points between particles. Oda, Konishi and Nemat-Nasser [1980] introduced a density function to characterize three dimensional distribution of contact normals in sand and Oda [1982] extended this to the study of other materials like rocks and gravel. He used a fabric tensor to include position, density, shape and dimensions and orientations of discontinuities such as joint, fault and discrete particle. Oda, Konishi and Nemat-Nasser [1983] concluded that rolling of particles is the major deformation mechanism when interparticle friction is large.

Nemat-Nasser and Mehrabadi [1984] have used a micromechanical model to describe the overall mechanical response of a granular material supporting a load through contact friction. The approach employed is similar to that used to describe the response of polycrystalline metals and composites.

At densities near to close packing the deformation of snow is due to deformation of the ice particles rather than sliding. This situation is similar to that of powder metals. Arzt [1982] studied the densification of spherical powders by cold compaction, hot isostatic pressing and sintering in terms of shrinkage of Voronoi cells associated with initial packing of powder particles. Using simple constitutive laws for low temperature plasticity, diffusion and power

law creep, the effects of increase in coordination number by these mechanisms was assessed. Arzt, Ashby and Easterling [1983] used this model to determine the deformation and densification of snow pack.

CHAPTER 2

KINEMATIC PRINCIPLES

In this chapter, the general principles of continuum mechanics are laid down. These include the kinematics, the definition of strain tensor, principles of balance of mass and momentum, and the principle of virtual work.

Kinematics

In continuum mechanics a body B is thought of as a continuous medium without gaps or voids. It is a set whose elements can be identified as having a one-to-one correspondence with the points of a region B of Euclidean point space. The elements of B are referred to as particles and B is referred to as configuration of B . Kinematics are used to describe the motion of the body without regard to the forces causing this motion.

During motion of a body, the configuration changes with time t . Generally a configuration at $t=t_0$ is chosen as one to refer back to. This configuration is called the "reference configuration". The particles of B are identified by their time independent position relative to the origin O of a chosen coordinate system. Thus,

$$\mathbf{X} = \boldsymbol{\kappa}(\mathbf{X}). \quad (2.1)$$

Here the components of $\mathbf{X}$ are called the referential coordinates X of a representative particle of B and are written as

$$\mathbf{X} = (X_1, X_2, X_3). \quad (2.2)$$

Also the relationship is one-to-one and onto and therefore an inverse exists, so that

$$\mathbf{X} = \kappa^{-1}(\mathbf{x}). \quad (2.3)$$

If, with each value of time t , there exists a unique configuration of body B , the family of configurations is called the motion of B and requires the existence of functions ϕ such that

$$\mathbf{x} = \phi(\mathbf{X}, t)$$

or

$$\mathbf{x} = \phi(\kappa^{-1}(\mathbf{x}), t)$$

or

$$\mathbf{x} = \phi_{\kappa}(\mathbf{X}, t) = \chi(\mathbf{X}, t). \quad (2.4)$$

where $\mathbf{x}$ is the position occupied at time t by the particle X which occupied position $\mathbf{X}$ in the reference configuration. This configuration is called the spatial configuration and components of $\mathbf{x}$ are referred to as spatial coordinates of $\mathbf{X}$.

The Deformation and Velocity Gradients

A deformation is a mapping of a reference configuration into a current deformed configuration. The two point tensor $\mathbf{F}$ is defined such that it operates on an arbitrary infinitesimal vector $d\mathbf{X}$ at $\mathbf{X}$ to associate it with a vector $d\mathbf{x}$ at $\mathbf{x}$ as follows

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}. \quad (2.5)$$

$\mathbf{F}$ is called the deformation gradient tensor. The determinant, $\det \mathbf{F}$, evaluated at $\mathbf{X}$ is assumed to be nonzero for a mapping having continuous derivatives. This is a necessary and sufficient condition for existence of continuously differentiable inverse evaluated at $\mathbf{x}$ in the neighborhood of $\mathbf{x}$. This last condition follows from the Inversion Theorem in Calculus [Fulks, 1978]. The deformation gradient tensor includes information regarding both deformation and rotation. This follows from the polar decomposition theorem [Lancaster and Tismenetsky, 1985] according to which $\mathbf{F}$ can be expressed as

$$\mathbf{F} = \mathbf{R} \mathbf{U} = \mathbf{V} \mathbf{R},$$

where $\mathbf{R}$ is the orthogonal rotation tensor and represents the rigid body rotation. $\mathbf{U}$ and $\mathbf{V}$ are left and right stretch tensors, respectively, and both are symmetric positive definite tensors. Constitutive equations employing the deformation gradient have to be constructed so that they do not predict a stress due to rigid body rotation. This restriction follows from the principle of material objectivity [Billington and Tate, 1981]. The deformation gradient tensor is generally not symmetric and has nine independent components.

When configurations which are continually in a process of change (as in viscous flow) are to be examined, the rates of change become most important. Kinematical variables that describe the instantaneous rate of deformation need to be introduced. The spatial gradient of velocity gives us the tensor $\mathbf{L}$:

$$\mathbf{L} = \text{grad}(\mathbf{v}). \quad (2.6)$$

In Cartesian coordinates this can be written as

$$L_{ij} = \frac{\partial v_i}{\partial x_j} \quad (2.7)$$

L can be written as the sum of a symmetric tensor D and a skew-symmetric tensor W , called the rate-of-deformation and spin tensor, respectively.

$$D = 0.5(L + L^T) \quad (2.8)$$

$$W = 0.5(L - L^T) \quad (2.9)$$

The rate-of-deformation tensor D is independent of rotation. Dependence of a frame indifferent constitutive law on L is expressible in terms of D .

Strain

The strain tensors are defined so that they give the change in squared length of the material line element dX connecting two neighboring particles. The Lagrangian formulation is written as

$$(ds)^2 - (dS)^2 = 2dX \cdot E \cdot dX, \quad (2.10)$$

while the Eulerian formulation is

$$(ds)^2 - (dS)^2 = 2dx \cdot E^* \cdot dx. \quad (2.11)$$

The Lagrangian strain tensor can be expressed in terms of deformation gradient as follows:

$$(ds)^2 = dx \cdot dx = (dX F^T) \cdot (F \cdot dX) = dX \cdot (F^T \cdot F) \cdot dX$$

or

$$(ds)^2 - (dS)^2 = dX \cdot (F^T \cdot F - 1) \cdot dX.$$

Therefore,

$$\begin{aligned} \mathbf{E} &= 0.5(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{1}) \\ &= 0.5(\mathbf{C} - \mathbf{1}), \end{aligned} \quad (2.12)$$

where $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$ is the Green deformation tensor.

Similar equations can be derived for Eulerian strain tensor.

$$\begin{aligned} \mathbf{E}^* &= 0.5(\mathbf{F} \cdot \mathbf{F}^T - \mathbf{1}) \\ \mathbf{E}^* &= 0.5(\mathbf{1} - \mathbf{B}^{-1}), \end{aligned} \quad (2.13)$$

where $\mathbf{B}^{-1}$ is the Cauchy deformation tensor. In index notation, when Cartesian coordinates are used

$$E_{IJ} = 0.5 \left(\frac{\partial x_k}{\partial X_I} \frac{\partial x_k}{\partial X_J} - \delta_{IJ} \right). \quad (2.14)$$

In terms of displacement

$$u_I = x_i - X_I \quad (2.15)$$

$$E_{IJ} = 0.5 \left(\frac{\partial u_I}{\partial X_J} + \frac{\partial u_J}{\partial X_I} + \frac{\partial u_K}{\partial X_I} \frac{\partial u_K}{\partial X_J} \right). \quad (2.16)$$

For infinitesimal strains the product of derivatives can be neglected with respect to the linear terms, resulting with

$$E_{IJ} = 0.5 \left(\frac{\partial u_I}{\partial X_J} + \frac{\partial u_J}{\partial X_I} \right). \quad (2.17)$$

For small strain theory, distinction between Eulerian and Lagrangian strain is generally ignored.

Conservation of Mass

Mass is a measure of the amount of material contained in an arbitrary portion of a body.

Consider an arbitrary configuration B of a body B . Let V be a set of points in B occupied by the particles in arbitrary subset V of B . Denoting the mass associated with V by the nonnegative real number M , the mass $M(V)$ possesses the following properties:

1. $M(V_1 \cup V_2) = M(V_1) + M(V_2)$ for all pairs V_1 and V_2 of B such that $V_1 \cap V_2 = \{\phi\}$
2. $M(V) \rightarrow 0$ as the volume of V tends to zero, i.e., $\|V\| \rightarrow 0$

In continuum physics mass is regarded as continuously distributed in space, and hence the properties 1 and 2 imply the existence of a bounded scalar field ρ , defined on B [Fulks, 1978]. Thus,

$$M(V) = \int_V \rho dV$$

where ρ is the density of the material of which body B is composed.

3. Let $dM(V)/dt$ be the rate of change of mass $M(R)$ of the material R occupying an arbitrary material region R_t at the time t in the current configuration B_t . Balance of mass requires that this rate of change be equal to zero, i.e.,

$$\frac{d}{dt} \int_{R_t} \rho dV = 0. \quad (2.18)$$

On using Reynold's transport theorem we get

$$\int_{R_t} \left(\frac{D}{Dt} \rho + \rho \operatorname{div} \mathbf{v} \right) dV = 0.$$

Since this holds for all arbitrary regions R_t , it follows that

$$\left(\frac{D}{Dt} \rho + \rho \operatorname{div} \mathbf{v} \right) = 0,$$

$$\left(\frac{\partial}{\partial t} \rho + \operatorname{div}(\rho \mathbf{v})\right) = 0. \quad (2.19)$$

Billington and Tate [1981] have provided the following kinematic relationship between the deformed and reference volume:

$$dv = \det \mathbf{F} dV. \quad (2.20)$$

Using this equation the law of mass conservation can be written as

$$\rho_r = \rho (\det \mathbf{F}), \quad (2.21)$$

where ρ_r is the material density in the reference configuration.

Balance of Momentum

There are two principles of balance of momentum

- a) balance of linear momentum,
- b) balance of angular momentum.

Balance of Linear Momentum. This principle states that the rate of change of linear momentum is equal to the applied force, i.e.,

$$\dot{\mathbf{p}} = \frac{d}{dt} \int_{R_t} \rho(\mathbf{V}) dV = \mathbf{F}, \quad (2.22)$$

where $\mathbf{p}$ is the linear momentum of the body and $\mathbf{F}$ is the resultant force on the body.

The external forces acting on the body are of two types:

- a) body forces which act on every material element throughout the body and described by a vector field $\mathbf{f}$, referred to as body force per unit mass.
- b) surface forces which act on the surface area of a volume element and can be described by a vector $\mathbf{t}$, referred to as surface traction.

The total force F is defined as

$$F = \int_{R_t} \rho f \, dv + \int_{\partial R_t} t \, da. \quad (2.23)$$

The vector t can be expressed as

$$t = T^T \cdot n. \quad (2.24)$$

Here T is the stress tensor. The principle of linear momentum in the integral form using equations 2.23 and 2.24 is

$$\frac{d}{dt} \int_{R_t} \rho v \, dv = \int_{R_t} \rho f \, dv + \int_{\partial R_t} t \, da.$$

Using the Reynold's transport theorem and principle of mass conservation, the above equation gives

$$\int_{R_t} \rho \left(\frac{Dv}{Dt} - f \right) dv = \int_{\partial R_t} T^T \cdot n \, da.$$

Using the divergence theorem, the above equation can be rearranged to give

$$\int_{R_t} \left[\rho \left(\frac{Dv}{Dt} - f \right) - \text{div} T \right] dv = 0.$$

This holds for all arbitrary material regions R_t . Therefore, the local form of balance of linear momentum is

$$\rho \frac{Dv}{Dt} = \text{div} T + \rho f. \quad (2.25)$$

Balance of Angular Momentum. The angular momentum h is defined by the relation

$$h = \int_{R_t} (x \times v) \rho \, dv, \quad (2.26)$$

where $\mathbf{x}$ is the position of a representative point of R_t relative to origin o . The principle of angular momentum then states that the rate of change of angular momentum is equal to the applied torque Γ . In the integral form this can be expressed as

$$\frac{d}{dt} \int_{R_t} \rho (\mathbf{x} \times \mathbf{v}) \, dv = \int_{R_t} \rho (\mathbf{x} \times \mathbf{f}) \, dv + \int_{\partial R_t} (\mathbf{x} \times \mathbf{t}) \, da,$$

where the right-hand side is equal to the torque. Substituting equation 2.25 for $\mathbf{t}$ and rearranging after using equation 2.26 and Reynold's transport theorem, we have

$$\int_{R_t} \rho \mathbf{x} \times \left(\frac{D\mathbf{v}}{Dt} - \mathbf{f} \right) \, dv = \int_{\partial R_t} \mathbf{x} \times (\mathbf{T}^T \cdot \mathbf{n}) \, da.$$

Using the vector identity

$$\mathbf{a} \times \mathbf{b} \times \mathbf{c} = (\mathbf{b} \otimes \mathbf{c} - \mathbf{c} \otimes \mathbf{b}) \mathbf{a},$$

we get

$$\begin{aligned} & \int_{R_t} \rho \left[\mathbf{x} \otimes \left(\frac{D\mathbf{v}}{Dt} - \mathbf{f} \right) - \left(\frac{D\mathbf{v}}{Dt} - \mathbf{f} \right) \otimes \mathbf{x} \right] \, dv \\ &= \int_{\partial R_t} \left[\mathbf{x} \otimes (\mathbf{T}^T \cdot \mathbf{n}) - (\mathbf{T}^T \cdot \mathbf{n}) \otimes \mathbf{x} \right] \, da. \end{aligned}$$

Applying the divergence theorem this can be rearranged as

$$\left\{ \mathbf{x} \otimes \left[\rho \left(\frac{D\mathbf{v}}{Dt} - \mathbf{f} \right) - \text{div} \mathbf{T} \right] - \left[\rho \left(\frac{D\mathbf{v}}{Dt} - \mathbf{f} \right) - \text{div} \mathbf{T} \right] \otimes \mathbf{x} + \mathbf{T} - \mathbf{T}^T \right\} \, dv.$$

By way of momentum balance the terms in the square braces sum to zero, and we get

$$\mathbf{T} = \mathbf{T}^T. \quad (2.27)$$

This establishes the symmetry of stress tensor, in the absence of any couple stresses.

Principle of Virtual Work

The principle of virtual work is obtained using variational principles. It is so named because the work computed is the fictitious work produced by a set of statically admissible forces and stresses which work on a set of infinitesimal kinematically admissible displacements. It is an alternative way of expressing equilibrium equations. Statically admissible stress distributions are ones which satisfy equilibrium on the interior of the body and the boundary conditions wherever boundary tractions are specified. Similarly, by kinematic admissibility of displacements it means that displacements are differentiable functions and satisfy the boundary conditions. Both the stresses and displacements may not be the actual distributions in the deformed body. In Cartesian coordinates if t_i is the external surface traction and f_i the body force per unit volume, both of which remain unchanged during virtual deformation, then the virtual work is

$$\delta W_e = \int_{R_t} \left[T_{ij} \frac{\partial \delta u_i}{\partial x_j} + \left(\frac{\partial T_{ij}}{\partial x_j} + f_i \right) \delta u_i \right] dV. \quad (2.28)$$

We transform the first integral into a volume integral by using the divergence theorem to obtain

$$\delta W_e = \int_{R_t} \left[T_{ij} \frac{\partial \delta u_i}{\partial x_j} + \left(\frac{\partial T_{ij}}{\partial x_j} + f_i \right) \delta u_i \right] dV.$$

By the equilibrium equation (accelerations are assumed negligible) the term in the parentheses vanishes

$$\delta W_e = \int_{R_t} \left[T_{ij} \frac{\partial \delta u_i}{\partial x_j} \right] dV. \quad (2.29)$$

Also

$$\frac{\partial(\delta u_i)}{\partial x_j} = \delta \epsilon_{ij} + \delta \omega_{ij} \quad (2.30)$$

where

$$\delta \epsilon_{ij} = 0.5 \left(\frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i} \right) \quad \text{and} \quad \delta \omega_{ij} = 0.5 \left(\frac{\partial \delta u_i}{\partial x_j} - \frac{\partial \delta u_j}{\partial x_i} \right),$$

are virtual strains and rotations associated with infinitesimal virtual displacement distribution. Using these results in equation 2.29 we get

$$\delta W_e = \int_{R_t} [T_{ij} \delta \epsilon_{ij}] dV \quad (2.31)$$

where the identity

$$T_{ij} \delta \omega_{ij} = 0$$

has been used. The right-hand side of equation 2.31 is called the internal virtual work and equation 2.31 is written as

$$\delta W = \delta W_e + \delta W_i = 0$$

where δW is the total virtual work.

This equation shows that equation 2.31 is a necessary condition for equilibrium. By following the above derivation in reverse, it can be shown that equation 2.31 is also an essential condition for equilibrium. The principle of virtual displacements combines the necessary and sufficiency conditions for equilibrium and states: A system is in equilibrium if and only if total external virtual work

equals total internal work for every kinematically admissible virtual displacement.

In its present form the principle of virtual work is applicable to all systems immaterial of whether energy is conserved or not. If a strain energy function, U , exists so that

$$T_{ij} = \frac{\partial U}{\partial \epsilon_{ij}}$$

then the right-hand side of equation 2.31 can be written as $\delta \int_{R_t} U \, dV$

and the principle of virtual work becomes

$$\delta \int_{R_t} U \, dV = \int_{R_t} f_i \delta u_i \, dV + \int_{\partial R_t} t_i \delta u_i \, ds = 0.$$

If $f_i = -\frac{\partial G}{\partial u_i}$ and $t_i = -\frac{\partial g}{\partial u_i}$, then

$$\delta \int_{R_t} U \, dV + \delta \int_{R_t} G \, dV + \delta \int_{\partial R_t} g \, ds = 0$$

or

$$\delta P = 0 \tag{2.32}$$

where

$$P = \int_{R_t} (U + G) \, dV + \int_{\partial R_t} g \, ds,$$

is called the potential energy of the system. According to equation 2.32, if the displacements satisfy the boundary conditions and the equilibrium equation, then the potential energy acquires a stationary value. It can be further shown that this stationary value is a minima [Fung, 1965].

One of the applications of the principle of virtual work is dummy displacement method. The dummy displacement method is used to determine loads at a given point on a deformable body under stress so that the system is in equilibrium. Many problems in theory of elasticity can be formulated using the minimum potential energy principle [Fung, 1965]. For approximate solutions of problems involving creep of structures, an equation similar to equation 2.31 is used with strain rates replacing strains [Boyle and Spence, 1983].

CHAPTER 3

STRESSES IN NECKS

Snow in its undeformed state is made up of ice particles connected by areas called necks or bonds. Since these necks are regions of much smaller cross section they are subjected to much higher stresses than the ice particles and are therefore regions of large deformation. Whenever neck fracture occurs, sliding between the particles adds to the deformation process. The deformation of snow is therefore an average of deformation of the unbroken necks and of relative sliding between particles with broken necks. To find the deformation of the ice necks or relative sliding between the particles we need to know the stresses which are applied directly to the necked regions.

The phenomenon of sliding between particles is the cause of deformation in many granular materials. Oda [1974] and Kanatani [1983] have used probability principles to determine stresses at contact points of sand grains. The same principles can also be applied to ice particles in snow. The derivation presented below for the stresses at contact points (in the absence of necks) uses the principle of virtual work and is based on the work of Kanatani [1983].

Suppose the material, made up of rigid spheres, is subject to a macroscopically uniform stress. The contact forces vary from particle

to particle. All contact forces are superposed on a hypothetical representative particle whose radius is the average radius of the particle. If the number of particles is sufficiently large, the contact force distribution on the representative particle is approximated by a continuous function of the contact direction determined by $\mathbf{n}$, the outward unit normal vector at the contact point. Let $D(\mathbf{n})d\mathbf{n}$ be the number of contact points in the differential solid angle $d\mathbf{n}$ divided by the number of particles. Then

$$N = \oint D(\mathbf{n})d\mathbf{n}, \quad (3.1)$$

where N is the coordination number (number of contacts per grain).

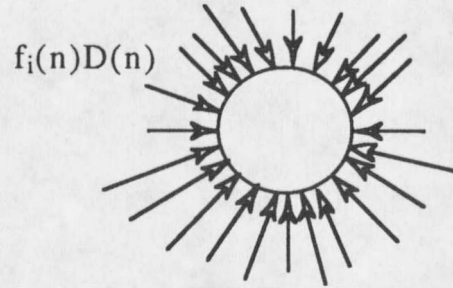


Figure 2. The force distribution on a representative particle.

Let $f_i(\mathbf{n})D(\mathbf{n})d\mathbf{n}$ be the total force acting in the differential solid angle $d\mathbf{n}$ divided by N . Then by equilibrium of force and torque,

$$\oint f_i(\mathbf{n})D(\mathbf{n})d\mathbf{n} = 0 \quad (3.2)$$

$$\oint f_{[i}(\mathbf{n})n_{j]}D(\mathbf{n})d\mathbf{n} = 0 \quad (3.3)$$

where $f_{[i}(\mathbf{n})n_{j]}$ is the skew part of $f_i(\mathbf{n})n_j$.

Consider the uniform linear deformation

$$x_i = A_{ij} X_j, \quad (3.4)$$

which moves the material point X_i to the point x_i during the deformation. In terms of the displacement u_i ,

$$u_i = x_i - X_i \quad (3.5)$$

$$u_i = J_{ij} X_j. \quad (3.6)$$

The distortion tensor (or the displacement gradient) J_{ij} is split into symmetric and skew symmetric parts

$$J_{ij} = e_{ij} + r_{ij} \quad (3.7)$$

$$e_{ij} = J_{(ij)} \text{ and } r_{ij} = J_{[ij]} \quad (3.8)$$

where e_{ij} is the symmetric and r_{ij} the skew symmetric part.

Further, assuming all particles to be rigid, Kanatani imposes a virtual displacement

$$\zeta_i(n) = a J_{ij} n_j, \quad (3.9)$$

which distorts the spherical particles into an ellipsoid. Since the particles are rigid, only rotations and translations are permissible mechanisms of displacement. However, virtual deformations are hypothetical and can be imposed [Kanatani, 1981]. The contact forces are assumed not to change during virtual deformation and the virtual work done by the contact forces on the representative particle is

$$\delta W = \oint f_i \zeta_i Dn = a J_{ij} \oint f_i n_j Dn = a e_{ij} \oint f_i n_j Dn, \quad (3.10)$$

where the last equality follows from the torque balance.

If γ is the solid volume fraction (neglecting the volume of necks)

$$\gamma = \frac{\frac{4}{3} \pi a^3}{\text{Volume of snow}} = \frac{\rho_{\text{snow}}}{\rho_{\text{ice}}},$$

then the virtual work done per unit volume is

$$\delta W = \frac{3\gamma e_{ij}}{4\pi a^2} \oint f_{(i} n_{j)} D dn. \quad (3.11)$$

Equating this to the virtual work per unit volume done on the virtual strain e_{ij} by stress t_{ij}

$$\delta W = t_{ij} e_{ij}, \quad (3.12)$$

where t_{ij} is the stress on snow, we get

$$t_{ij} = \frac{3\gamma}{4\pi a^2} \oint f_{(i} n_{j)} D dn. \quad (3.13)$$

The contact force density $f_i D$ is then expanded into a series of spherical harmonics, and only the first two terms are retained.

$$f_i D = A_i + B_{ij} n_j. \quad (3.14)$$

This is now substituted into the force equilibrium equation 3.2 to obtain

$$\oint (A_i + B_{ij} n_j) dn = 0 \quad (3.15)$$

$$A_i = 0 \quad \text{and} \quad \oint (B_{ij} n_j) dn = 0, \quad (3.16)$$

where the fact $\oint n_j dn = 0$ has been used.

The equilibrium equation 3.3 for the torque gives

$$\oint B_{[ikn_k n_j]} dn = 0, \quad (3.17)$$

which requires B_{ij} to be symmetric, i.e.,

$$B_{[ij]} = 0 \quad (3.18)$$

and equation 3.14 reduces to

$$f_i D = B_{ij} n_j. \quad (3.19)$$

Substituting equation 3.19 into 3.13

$$t_{ij} = \frac{3\gamma}{4\pi a^2} \oint B_{ikn_k n_j} dn$$

$$t_{ij} = \frac{3\gamma}{4\pi a^2} B_{ik} \left(\frac{4}{3} \delta_{jk} \right)$$

$$B_{ij} = \frac{a^2}{\gamma} t_{ij}$$

$$f_i D = \frac{a^2}{\gamma} t_{ij} n_j. \quad (3.20)$$

Here f_i is the force on the contact with normal n_i . In the development presented later this is replaced by $(\sigma_i^n \cdot A_T/N)$ where σ_i^n is the stress vector acting on a grain at the contacts and A_T is the total area of the necks or contacts, N is the coordination number. A_T/N is therefore the average neck cross sectional area.

Also D can be written as

$$D = P \cdot N, \quad (3.21)$$

where P is the probability that two particles will form a contact. For the isotropic case P is independent of α and β . Substituting equation 3.21 into equation 3.1 for this case we get

$$1 = P \int_0^\Pi \int_0^{2\Pi} \sin\beta \, d\alpha \, d\beta,$$

or $P = 1/4\Pi$. Sometimes under action of gravity the particles may be deposited symmetrically about the vertical axis. Then P is a function of β alone and can be written as $P(\beta)$ and this symmetry is known as axial symmetry. When granular materials are sheared under nonequal principal stresses, any initial symmetry soon disappears and rhombic symmetry is induced [Oda, Konishi and Nemat-Nasser, 1980]. For rhombic symmetry

$$P(\alpha, \beta) = P(\alpha, \Pi - \beta) = P(\Pi - \alpha, \beta) = P(\Pi + \alpha, \Pi + \beta).$$

Here we will consider only the isotropic case.

On substituting equation 3.21 into equation 3.20 we get

$$\sigma_i^n \cdot A_T P = \frac{a^2}{\gamma} t_i^n,$$

or in vector notation as

$$\sigma^n A_T P = \frac{a^2}{\gamma} t^n. \quad (3.22)$$

A local coordinate system can be set up at the contact point and the stress vector at the contact point or the ice neck can be resolved into three components along the coordinate axis of this local coordinate system. This coordinate system is illustrated in Figure 3. The distance of any point from the center of sphere is given by

$$\mathbf{x} = r(\cos\alpha\sin\beta\mathbf{e}_1 + \sin\alpha\sin\beta\mathbf{e}_2 + \cos\beta\mathbf{e}_3). \quad (3.23)$$

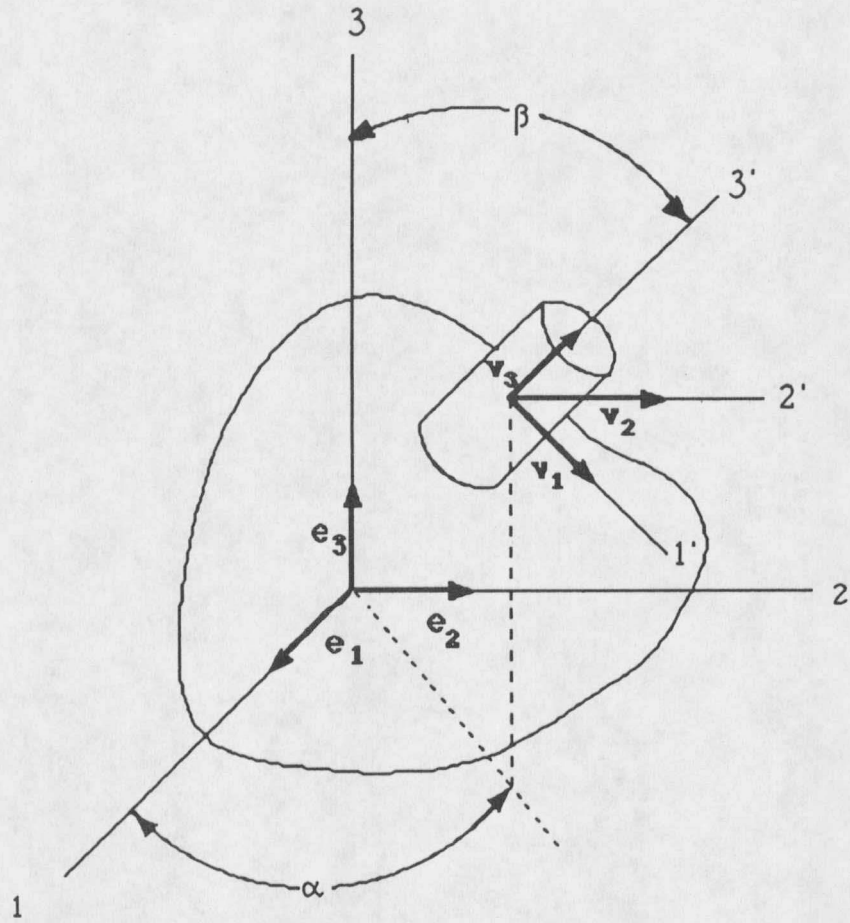


Figure 3. Coordinate system. 1-2-3 global coordinate system centered at the ice particle center. 1'-2'-3' neck coordinate system centered at the center of neck.

The unit vectors in the three directions are obtained by differentiating equation 3.23 with respect to β , α and a and dividing by the magnitude of x . The unit vectors are

$$v_1 = \cos\alpha \cos\beta e_1 + \sin\alpha \sin\beta e_2 - \sin\beta e_3$$

$$v_2 = -\sin\alpha e_1 + \cos\alpha e_2$$

$$v_3 = \cos\alpha \sin\beta e_1 + \sin\alpha \sin\beta e_2 + \cos\beta e_3. \quad (3.24)$$

The vector v_3 is same vector as n in equation 3.22 above.

The components of stress on the face with normal $\mathbf{n}$ are given as

$$\begin{aligned}\sigma_{33} &= \mathbf{v}_3 \cdot \boldsymbol{\sigma}^{\mathbf{n}} \\ \sigma_{13} &= \mathbf{v}_1 \cdot \boldsymbol{\sigma}^{\mathbf{n}} \\ \sigma_{23} &= \mathbf{v}_2 \cdot \boldsymbol{\sigma}^{\mathbf{n}}.\end{aligned}\tag{3.25}$$

Multiplying both sides of equation 3.22 by $\sin\beta d\alpha d\beta$ gives

$$\boldsymbol{\sigma}^{\mathbf{n}} A_T \cdot P \sin\beta d\alpha d\beta = t^n \frac{a^2}{\gamma} \sin\beta d\alpha d\beta.\tag{3.26}$$

To get the components of the stress tensor σ_{ij} , we multiply with appropriate normal, e.g., if we multiply both sides by n_3 , we get σ_{33} .

$$(\boldsymbol{\sigma}^{\mathbf{n}} \cdot \mathbf{v}_3) A_T \cdot P \sin\beta d\alpha d\beta = t^n \frac{a^2}{\gamma} \sin\beta d\alpha d\beta \cdot \mathbf{v}_3\tag{3.27}$$

Integrating both sides between proper limits, we obtain the average value of σ_{33} over the integration range defined by the limits. If averaging is done over the area of contact lying between the solid angle imbedded by angles α to $\alpha+\Delta\alpha$ and β to $\beta+\Delta\beta$, we obtain

$$\begin{aligned}(\sigma_{33})_{av} &= \frac{\int_{\beta}^{\beta+\Delta\beta} \int_{\alpha}^{\alpha+\Delta\alpha} A_T \cdot P \sin\beta d\alpha d\beta}{\int_{\beta}^{\beta+\Delta\beta} \int_{\alpha}^{\alpha+\Delta\alpha} A_T \cdot P \sin\beta d\alpha d\beta} \\ &= \frac{\int_{\beta}^{\beta+\Delta\beta} \int_{\alpha}^{\alpha+\Delta\alpha} t^n \cdot \mathbf{v}_3 \frac{a^2}{\gamma} \sin\beta d\alpha d\beta}{\int_{\beta}^{\beta+\Delta\beta} \int_{\alpha}^{\alpha+\Delta\alpha} A_T \cdot P \sin\beta d\alpha d\beta}.\end{aligned}\tag{3.28}$$

Similar expressions can be found for the other two components σ_{13} and σ_{23} .

In equation 3.28 the factor $a^2/\gamma A_T$ reflects the influence of the radius of the neck, density ratio and area of the ice particle. Rather than treating all of these as separate variables, this equation shows that they can be grouped as one parameter. Snow samples having different densities, grain and neck radii can still end up having the same stress at the contact point, provided $a^2/\gamma A_T$ is the same for the different samples.

In the above derivation the volume of necks has not been taken into account. However, the volume of the necks is negligible compared to the volume of ice particles and, therefore, neglecting the volume of necks is a reasonable approximation.

The particles have been assumed to be spheres of uniform diameter and the contact points between the particles are taken as circles. In an actual snow sample neither of these conditions is likely to be met. Also, at the contact point there can be sharp variations in curvature as the necks merge into ice particles. This can cause stress concentration at the contact, leading to much higher stresses in the necks than calculated using the above relations. To compensate for these factors, the stress in equation 3.28 is multiplied by a constant based on experimental results.

CHAPTER 4

DEFORMATION MECHANISMS AND STRAINS

The strain in snow is due either to the strain in the necks or due to relative sliding of particles with respect to each other once the necks have been broken. Experiments carried out at MSU [Brown, 1988] and by Kry [1975] show that snow does not collapse even when the stresses are high enough to cause neck breaking and subsequent sliding. From this it was concluded that even when sliding of necks is taking place the original chain structure still remains intact. To determine at what stress or at what stage of deformation neck fracture occurs at a significant level, acoustic emission data has been used [Brown, 1988]. Even prior to fracturing of necks, the phenomenon underlying deformation of ice necks may not be the same at high stress as at low stress. This chapter goes into the various deformation mechanisms and determination of corresponding strain rates.

Deformation and Fracturing of Necks

Acoustic emissions have been used in mechanical testing of ice. They are generally associated with the appearance of microcracks in the material and can provide a measure of the extent of

microfracturing which occurs during deformation. In case of snow, ice necks are dimensionally very small (approximately 0.01 to 0.1 mm). Therefore, the propagation of microcracks in necks would require a very short time to cause bond fracture. It is therefore reasonable to assume that acoustic emission count in snow corresponds closely to fracturing of necks. Most of the acoustic emission activity in snow, as in ice, takes place during the initial stages of deformation [St. Lawrence and Cole, 1983].

Experiments [Brown, 1988] were carried out at constant uniaxial compressive stress ranging over 0.004 MPa to 0.024 MPa in steps of 0.004 MPa in the "Cold Regions Lab" at MSU. For stresses less than or equal to 0.004 MPa no acoustic emission activity was observed. At stresses equal to or greater than 0.008 MPa acoustic emission activity was observed and acoustic emission count increased with increasing stress.

Since no acoustic emission occurs at stresses lower than 0.004 MPa, no neck fracturing can be assumed. In this case the deformation of snow can be attributed to deformation of intergranular necks, as has been suggested in Chapter 3. The principal stresses in the necks for these values of stress on the snow can be found to be 0.7 MPa by using equation 3.28.

Most of the constitutive laws for ice are developed using experimental data at high values of stresses. At these stresses microcracking and subsequent deterioration of the ice strength sets in. Dislocation motion and dislocation density changes are often used

to describe the observed strain rates, although microcrack damage should also be included in the formulation.

Most of the past experimental work is on ice with grain size larger than 0.7 mm. On the other hand, ice particles in snow are aggregates of much smaller ice nuclei. It is therefore likely that most constitutive laws for ice are not applicable at low stresses when no microcracking is taking place.

Superplasticity

In ice it has been observed that crack nucleation does not take place until the stress has exceeded a limiting value. Gold [1972] did not observe any cracking in ice subjected to compressive stress of 0.5 MPa. Sinha [1984] cites the experimental results of Burdwick for randomly oriented columnar grained ice with grain size of 0.7 mm subject to initial tensile stress of 0.7 MPa. Specimens were seen to last for four days without fracturing. Burdwick also observed 100% extensions in some of the specimens. Sinha has conjectured the possibility of existence of superplasticity in ice due to favorable conditions such as high temperature, low stress and fine grain size.

In superplasticity the material deforms extensively at elevated temperatures under low stress levels without risk of rupture. Superplasticity is exhibited most commonly in materials with fine equiaxed grain size at temperatures greater than 0.5 times the melting temperature [Langdon, 1982]. In these materials the strain rate and stress are related by

$$\sigma = k\dot{\epsilon}^m \quad (4.1)$$

where m is the strain sensitivity factor and k is a constant. The constant m has a value close to 0.5 (roughly between 0.3 to 0.8). The majority of experimental data suggests that grain boundary sliding is the dominant mechanism in superplastic deformation processes. Experimental observations also show a strong inverse dependence of strain rate on grain size and a small change in grain shape even after the material has undergone large deformations [Alden, 1975]. A number of theories exist to describe the phenomenon of superplasticity [Arieli and Mukherjee, 1982]. According to these theories, grain boundary sliding is accompanied by accommodation at the grain interfaces or within the grains themselves. The small change in grain shape is accounted for by rotation of the grains due to different velocities at the grain interface; as a result the grain which elongated in one direction previously now elongates in the other direction. The existence of rotation requires varying sliding rates at the interfaces, which implies the existence of at least two phases (as in alloys). These theories can explain superplasticity in ice necks only if some impurities or inclusions are present at the grain boundaries. They may not be able to explain the 100% elongation of ice observed by Burdwick [Sinha, 1984]. Superplasticity has also been observed in some pure metals.

As stated above, the conditions required for superplasticity are

- a) temperature of testing greater than 0.5 times the melting temperature,
- b) strain rate sensitivity of approximately 0.5 (if $n=2$),
- c) small equiaxed grains.

Snow does satisfy the first two requirements. Snow generally exists at temperatures much greater than 0.5 times the melting temperature. Also for ice the exponent n varies from 1.5 to 4 as stress goes from a lower to a higher value. Since we are looking at deformation of necks, we need to know if they are made up of equiaxed polycrystalline grains. Necks, we know, are formed due to the sintering of ice grains. This is similar to sintering and bond development in powder metal compacts subjected to pressure and high temperature. The bonding process there, as in snow, involves diffusion of atoms leading to development of grain boundaries. This development of grain boundaries takes place at those sites where there is intimate physical contact between particles [Hirschhorn, 1969]. In metals, as a result, when a compact is sintered there is a transition in structure. The original particle boundaries can no longer be observed, and instead the structure becomes similar to that of metal in a wrought and annealed condition except that it contains pores. It consists of an array of equiaxed grains separated by grain boundaries [Lenel, 1980]. A similar phenomenon in snow can lead to production of equiaxed ice grains.

In powder metals grain recrystallization followed by grain growth takes place. In some cases grain growth can be significant enough so as to make a single crystal 3-4 particles long. The presence of a second phase (material different from original) and grain boundary grooves impede grain growth [Lenel, 1980]. In sintered snow at low densities porosity is high and could possibly impede grain growth. Also sintering could take place without grain growth as

it is taking place at the same temperature at which the original particles were formed (unlike metals where sintering temperature are much higher compared to the normal temperature at which metals exist).

It is generally believed that each snow particle is a single crystal of ice and contact between two particles takes place at a common grain boundary. No investigation has been done to determine if, like metals, a number of grain boundaries are formed at the point of contact. If a single grain boundary exists at the point of contact of particles, grain boundary sliding accompanied by grain boundary diffusion is responsible for superplastic deformation. In case the point of contact has many grain boundaries, i.e., the neck is polycrystalline, in addition to grain boundary sliding and grain boundary diffusion some grain rotation may also take place. Both grain boundary diffusion and grain rotation prevent the formation of voids, which could lead to fracture of the neck at the grain boundaries. Some intragranular dislocation activity may also be taking place [Arieli and Mukherjee, 1982].

From the above discussion it can be concluded that if the stresses in a neck are low enough, then superplastic deformation can take place in an ice neck.

Sinha [1979] has incorporated grain size effect in his uniaxial constitutive law for ice. In this model the recoverable portion of creep strain was assumed to be a delayed elastic effect associated with grain boundary sliding. However, no grain boundary sliding has been associated with steady state creep strains.

If the stress in the neck is high, then superplastic deformation does not take place and ordinary constitutive laws should be applicable for steady state creep. However, for primary creep, grain boundary sliding still could be important because of small grain size as discussed by Sinha [1979]. If strains in the neck reach a critical value, then fracturing of necks takes place. Unlike superplastic deformation, here the grain boundary sliding is not accompanied by any accommodating grain boundary diffusion. As a result, void formation takes place at the grain boundaries [Chen and Machlin, 1956; Gifkins, 1956]. Coalescence of the voids leads to fracturing of the neck. Once this fracturing takes place, interparticle sliding becomes the major deforming mechanism. Neck deformation could still be taking place in the broken necks, but its contribution to the global strains in snow is much smaller than that due to sliding.

Thus, in a snow sample subjected to high stress levels there may be necks which are subjected to a high stress and undergo fracturing and subsequent sliding. On the other hand, there may be suitably oriented necks which are subjected to a low stress and therefore undergo superplastic deformation.

Equations for Ice

To describe the behavior of ice necks under combined compressive and shear stress a multiaxial constitutive law is required. A number of models exist to describe this behavior and a brief review of these was given in Chapter 1. The model used in this paper is based on work of Szyszkowski and Glockner [1986, 1987a,

1987b]. In this model ice is treated as an isotropic, non-linear viscoelastic material. The heredity effects have been included using a Volterra integral.

In general the response of ice is described by a relation of the form

$$e(t) = \int_0^t F[\sigma(\tau), t] d\tau, \quad (4.2)$$

where $e(t)$ and $\sigma(t)$ denote the strain and stress, respectively, while F denotes the viscoelastic properties of the material as a function of stress history, from $\tau=0$ to $\tau=t$. Effects of ageing have been neglected.

Some uniaxial creep test data for ice can be described reasonably well, for primary and secondary stages of creep, by assuming equations of the form

$$e(t) = \frac{\sigma(t)}{E} + \int_0^t F[\sigma(\tau)] \cdot L(t-\tau) d\tau, \quad (4.3)$$

in which the elastic response is characterized with E , the Young's modulus. F defines the nonlinear viscosity and $L(t)$ represents the fading memory function. The function F is chosen as

$$F(\sigma) = \sigma |\sigma|^{n-1}, \quad (4.4)$$

where n is a material constant indicating viscous non-linearity.

The memory function is assumed to be

$$L(t) = \frac{1}{v_1} j(t) + \frac{1}{v_2}, \quad (4.5)$$

in which v_1 and v_2 are material constants and $j(t)$ satisfies the conditions

$$j(0) = 1, j(t) \rightarrow 0 \text{ for } t \rightarrow \infty \text{ and } \frac{dj}{dt} < 0.$$

All the parameters associated with viscosity, namely n , v_1 , v_2 and $j(t)$, are determined from analysis of creep test data. The total strain rate can then be expressed as the sum of three parts:

$$\dot{\epsilon} = \dot{\epsilon}^e + \dot{\epsilon}^c + \dot{\epsilon}^p, \quad (4.6)$$

where the elastic strain rate is

$$\dot{\epsilon}^e = \frac{\dot{\sigma}}{E}, \quad (4.7)$$

and the delayed elastic and plastic creep rates are, respectively

$$\dot{\epsilon}^c = \frac{1}{v_1} \frac{d}{dt} \int_0^t [\sigma(\tau)]^n j(t-\tau) d\tau \quad (4.8)$$

$$\dot{\epsilon}^p = \frac{1}{v_2} [\sigma(t)]^n. \quad (4.9)$$

The above relations can be obtained from a scalar function $P(\sigma)$ in the form

$$\begin{aligned} P(\sigma) = \int_0^\sigma \dot{\epsilon} d\sigma &= \frac{d}{dt} \left[\frac{\sigma^2}{2E} \right] + \frac{1}{v_1} \frac{d}{dt} \int_0^t \frac{[\sigma(t)]^{n+1}}{n+1} j(t-\tau) d\tau \\ &+ \frac{1}{v_2} \frac{[\sigma(t)]^{n+1}}{n+1}, \end{aligned} \quad (4.10)$$

$$\dot{\epsilon} = \frac{dP}{d\sigma}. \quad (4.11)$$

For a multiaxial case, the equation analogous to 4.11 is

$$\dot{\epsilon}_{ij} = \frac{dP}{d\sigma_{ij}}. \quad (4.12)$$

The potential P has been determined so as to satisfy the invariance requirements of the theory of constitutive relations. The elastic response is assumed to be a function of first stress and second stress deviator invariants, while viscous terms are functions of the second invariant of the stress deviator. The effects of the third invariant have been neglected. This assumption is often made in ice mechanics because of lack of sufficient multiaxial experimental test data required to describe the dependence of P on both second and third invariants. Morland and Spring [1981] point out Glen's observation that dependence of P on the third invariant may be necessary. The dependence of the viscous term on only the second stress deviator invariant implies that viscous response is incompressible. The potential P for the multiaxial case has the form

$$P(\sigma_{ij}) = \frac{d}{dt} \left[\frac{\sigma_m^2}{2K} + \frac{S^2}{6G} \right] + \frac{1}{v_1} \frac{d}{dt} \int_0^t F[S(\tau)] j(t-\tau) d\tau \\ + \frac{1}{v_2} F[S(t)], \quad (4.13)$$

where $F(S)$ is an expression for Norton's power creep law

$$F(S) = B \frac{S^{(n+1)}}{(n+1)}. \quad (4.14)$$

In equation 4.13 σ_m is the mean stress, K and G are respectively the bulk modulus and shear modulus. Szyszowski and Glockner [1987a] developed the following form for strain rate

$$\dot{\epsilon}_{ij} = \frac{(1+\nu)}{E} \dot{\sigma}_{ij} - \frac{\nu}{E} \dot{\sigma}_{kk} \delta_{ij} + \frac{1}{\nu_1} \frac{d}{dt} \int_0^t [\tilde{\sigma}_{ij}(\tau)]^n j(t-\tau) d\tau + \frac{\tilde{\sigma}_{ij}^n}{\nu_2} \quad (4.15)$$

where

$$\tilde{\sigma}_{ij} = s_{ij} \left\{ 1.5 \frac{S}{|s_{ij}|} \right\}^{1/n-1} \quad (4.16)$$

$$S^2 = 1.5 s_{ij} s_{ij}. \quad (4.17)$$

The constant B has been absorbed into ν_1 and ν_2 . In equation 4.15 ν is the Poisson's ratio. s_{ij} is the deviatoric stress given by the relation

$$s_{ij} = \sigma_{ij} - \sigma_{kk} \delta_{ij}.$$

The strain rate is decomposed into elastic, recoverable creep and plastic components.

$$\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^e + \dot{\epsilon}_{ij}^c + \dot{\epsilon}_{ij}^p \quad (4.18)$$

The expressions for various components as observed from equation 4.15 are

$$\dot{\epsilon}_{ij}^e = \frac{(1+\nu)}{E} \dot{\sigma}_{ij} - \frac{\nu}{E} \dot{\sigma}_{kk} \delta_{ij} \quad (4.19)$$

$$\dot{\epsilon}_{ij}^c = \frac{1}{\nu_1} \frac{d}{dt} \int_0^t [\tilde{\sigma}_{ij}(\tau)]^n j(t-\tau) d\tau \quad (4.20)$$

$$\dot{\epsilon}_{ij}^p = \frac{\tilde{\sigma}_{ij}^n}{\nu_2}. \quad (4.21)$$

Instead of directly solving the integral in equation 4.20, Szyszkowski and Glockner have approximated the equations 4.20 and 4.21 by a generalized Kelvin body (Figure 4) in series with a nonlinear dashpot.

The equations for the Kelvin body are written as

$$\dot{e}_{ij}^c = \frac{(\tilde{\sigma}_{ij}')^n}{v_1} \quad (4.22)$$

$$(\tilde{\sigma}_{ij}')^n + c\tilde{\sigma}_{ij}'(\tilde{\sigma}_{ij})^{n-1} = E_1 e_{ij}^c \quad (\text{no summation}) \quad (4.23)$$

$$\tilde{\sigma}_{ij}' + \tilde{\sigma}_{ij}'' = \tilde{\sigma}_{ij}, \quad (4.24)$$

where $\tilde{\sigma}_{ij}$ is the effective viscous stress tensor defined above, $\tilde{\sigma}_{ij}'$ is the component of effective stress in the spring and $\tilde{\sigma}_{ij}''$ is the component of effective stress in the dashpot. The second term on the left-hand side in equation 4.23 ensures that at zero stress the spring does not have infinite stiffness. It does not seem to play a major role in numerical calculations.

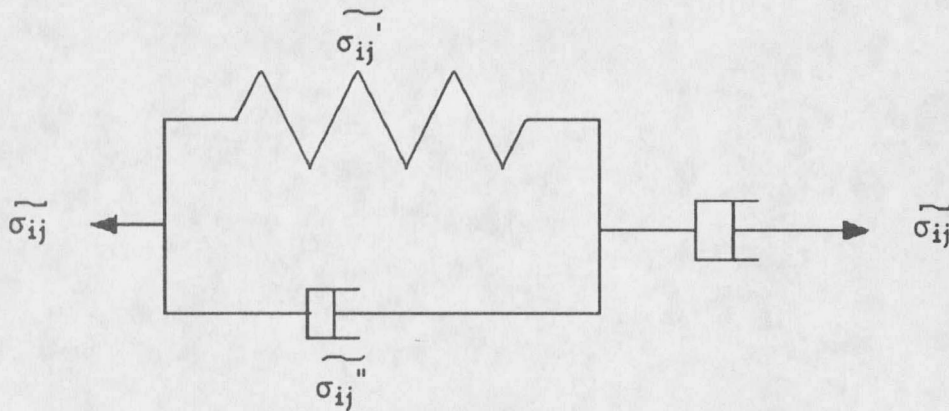


Figure 4. Spring dashpot model for constitutive relation of ice.

To make the model simulate the behavior of ice, the constants c and E_1 must be such that $j_m(t)$ of the model matches $j(t)$. It is found that j_m meets the conditions

$$j_m(0) = j(0) = 1$$

$$j_m(t) \rightarrow 0, j(t) \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (4.25)$$

Two additional restrictions are imposed on the model to determine c and E_1 in terms of material properties. They are

$$\left. \frac{dj}{dt} \right|_{t=0} = \left. \frac{dj_m}{dt} \right|_{t=0} \quad (4.26)$$

$$\int_0^\infty j(t) dt = \int_0^\infty j_m(t) dt. \quad (4.27)$$

Using equations 4.22 to 4.27 Szyszkowski and Glockner have developed the following equations for E_1 and c :

$$c = \frac{\lambda}{1-\lambda}; E_1 = \frac{v_1}{t_1} \frac{c}{n}; \lambda = \frac{v_2 t_1 n}{v_1 t_0}. \quad (4.28)$$

Here

$$t_1 = -1 / \left. \frac{dj}{dt} \right|_{t=0} \quad (4.29)$$

and

$$t_0 = \frac{v_2}{v_1} \int_0^\infty j(\tau) d\tau. \quad (4.30)$$

Equations 4.18 is integrated with respect to time to give the strains. The strains in the ice necks are next transformed to the snow coordinate system by the equation

$$E_{ij} = Q_{ir}^T e_{rs} Q_{sj}. \quad (4.31)$$

Q is an orthogonal transformation matrix carrying the local coordinate system centered at the neck to the global coordinate system centered at ice particle center (Figure 3).

On multiplying both sides of equation 4.31 by $\sin\beta d\alpha d\beta$, we obtain

$$E_{ij}\sin\beta d\alpha d\beta = Q_{ir}^T \epsilon_{rs} Q_{sj} \sin\beta d\alpha d\beta. \quad (4.32)$$

The reason for doing this becomes clear in equation 4.38 below.

Interparticle Sliding

At high stresses microcracks develop in ice to cause tertiary creep and ultimately fracture. Szyszkowski and Glockner [1986, 1987b] have introduced a damage function to account for this. The approach we take is slightly different.

For an ice specimen in tension, failure is often defined as the instant when nucleation of microcracks is initiated. In compression, on the other hand, it is the stress required for the propagation of microcracks. The prediction of first crack occurrence (crack nucleation) under uniaxial compressive loading is based on the hypothesis that the crack nucleates due to lateral tensile strain resulting from the Poisson effect of elasticity and material incompressibility. The first crack is postulated to occur when the lateral tensile strain equals the strain for tensile fracture at the same instantaneous strain rate. If this limiting strain is reached, the material can still continue to sustain a compressive load but loses its ability to take any lateral tensile loads. The failure strain thus depends on the strain rate and a graphical relation between the two is given by Ting and Sunder [1985]. For strain rates between $10^{-3}/s$ to $10^{-6}/s$ the tensile fracture strain varies from 3×10^{-3} to 6×10^{-3} . In a neck there are both shear and compressive stresses. For

determining when failure takes place we calculate the principle tensile strain, and if this exceeds 5×10^{-3} , we assume the neck has lost the ability to carry load (principal tensile stress) and therefore fractures. After the occurrence of fracture, deformations in snow are due to intergranular sliding. In reality, necks in different areas will slide at different angles; however, no experimental data which has looked at neck fracturing exists. It was assumed that fracturing takes place in the direction of maximum shear. This gave a wider variation in the viscous Poissonic effect for snow than experimentally observed. Therefore, empirical relations were found for components of sliding velocity in directions tangential and normal to the point of contact so as to make the axial and lateral strains in snow as determined by the theory match those from experiments. The tangential and normal velocities and displacements are calculated using the equation

$$\dot{\eta} = c_1(\tau) |\tau|^{n-1} (1/\text{disp}^m) \quad (4.33)$$

$$\dot{\xi} = c_1(t_{\text{nor}}) |t_{\text{nor}}|^{n-1} (1/\text{disp}^m) c_2, \quad (4.34)$$

where $\dot{\xi}$ and t_{nor} are respectively the velocity and the component of traction in the direction normal to the grain surface at the neck. $\dot{\eta}$ is the tangential velocity, τ is the component of traction in the shearing direction, and disp is the relative displacement between the grains at a particular time. The expression $1/\text{disp}^m$ appears because as the sliding displacement increases the particles form new contacts with other particles and this impedes further relative displacement. The constants c_1 , c_2 , n , and m were determined from the

experimental data for a single experiment. The constants c_1 and c_2 depend on the thickness and viscosity of the liquid-like sublayer formed between the necks due to pressure melting [Hobbs, 1974]. The theory was then checked by comparing it to experimental data and was found to work very well.

If $\dot{\eta}$ and $\dot{\xi}$ are the components of velocity in the neck coordinate system, then the velocity gradient L in snow associated with these is

$$L_{ij} = \frac{1}{a}(\dot{\eta}s_i n_j + \dot{\xi}n_i n_j), \quad (4.35)$$

where s is the unit vector normal to n , in the sliding direction [Nemat-Nasser and Mehrabadi, 1984], and a is the radius of the ice particle. The symmetric part gives us the rate-of-deformation tensor which for small strain theory is the same as strain rate tensor. The unsymmetrical part gives the spin tensor. The strain rate is

$$\dot{E}_{ij} = \frac{1}{a} \left(\frac{1}{2} \dot{\eta} (s_i n_j + s_j n_i) + \dot{\xi} n_i n_j \right). \quad (4.36)$$

The above equation can be integrated with respect to time to give strains. On multiplying both sides of equation 4.36 (after time integration) with $\sin\beta d\alpha d\beta$ we get

$$E_{ij} \sin\beta d\alpha d\beta = \frac{1}{a} \left[\frac{1}{2} \eta (s_i n_j + s_j n_i) + \xi n_i n_j \right] \sin\beta d\alpha d\beta. \quad (4.37)$$

The strain, $(E_{ij})_{av}$ in snow is the average of strain in all necks.

$$(E_{ij})_{av} \int_0^{2\pi} \int_0^{\pi} \sin\beta d\alpha d\beta = \int_0^{2\pi} \int_0^{\pi} E_{ij} \sin\beta d\alpha d\beta, \quad (4.38)$$

where E_{ij} on the right-hand side is obtained from equation 4.31 or

4.36, depending on whether it is a broken neck or a neck undergoing superplastic deformation.

Numerical Scheme and Results

For calculation purposes equation 4.38 must be integrated numerically. The representative particle is divided into 72 regions so that the angles α and β vary by $\pi/6$. Increasing the number of divisions improves the results, although at the expense of increased computational time. To solve the nonlinear equation 4.23 Brent's method [Press et al., 1988] has been used. Euler's method was used for integrating differential equations 4.19, 4.21 and 4.23. The average stress, for the case of uniaxial loading in z or 3 direction was calculated using equation 3.28. For this case only stresses σ_{23} and σ_{33} are nonzero, and σ_{13} is zero. Simpson's rule was used for double integration. However, when the same load is applied in other directions, i.e., along 1-axis or 2-axis, the same value of the strains is not obtained. For load along the 2-axis or 1-axis, σ_{13} is no longer zero, and although σ_{23} and σ_{33} are orthogonal at every point, their average vector values are not orthogonal. As a result, the strains in the necks, calculated using these average values, are not orthogonal either. This produces errors when strains in snow are calculated using equation 4.38. To overcome this, all stresses and strains were calculated at an average value of angles α and β .

Since the primary creep strain rates became negligible after 10 hours of initial load application, they were not calculated after 10 hours.

A concentration factor of 1.75 has been used to increase the neck stresses over the values calculated using equation 3.28 above. This takes into account the effect of sudden changes in cross section at points of contact between grains.

Very little data for values of coordination number or neck and grain radii are available. For snow of density 270-350 kg/m³ Hansen [1985] and Edens [1989] have cited values of coordination number between 2 to 3. Alley [1986] measured a value of approximately 3 for coordination number of snow of density 300 kg/m³. For tension tests, which were mostly carried out on snow of density 270 kg/m³, we assume the initial coordination number is 2.3. For compression tests the density ranged from 329 to 350 kg/m³ and we used a coordination number of 2.5 to 3.0. The radius of the ice particles and necks was taken from Kry's paper [1975]. In order to obtain realistic values for strains in snow from those of necks, we have to scale the neck strains (to take into account the ice particle which is being treated as undeformed). The scaling factor is approximately $(L/(2a+L))$ where a is the particle radius and L is the length of the neck. No information is available in Kry's paper on length of necks. We take the scaling factor as 5. Hansen [1985] has taken this factor as 10, but the size of the ice particles in that study is much larger than Kry's and 5 appears to be a more suitable approximation.

Szyszkowski and Glockner [1987b] suggest a value 168 and 5040 for v_1 and v_2 , which appear in equation 4.15. At a stress of .004 MPa, these values of constants v_1 and v_2 when used in equation 4.15 give strains in snow much smaller than those from the experimental

data at MSU. To obtain results which match experimental results for very low stresses, v_1 and v_2 have to be lowered by a factor of almost 20. It is assumed at low stresses superplastic deformation of ice is taking place, and therefore the constant v_2 has a lower value than suggested in Szyszkowski and Glockner [1987b]. During primary creep at low stresses and small grain size, grain boundary sliding makes a big contribution to the strains. Unlike superplastic deformation, this is not accompanied by accommodation of grains. This was discussed above in the section Deformation and Fracturing of Bonds. For smaller grain sizes the constant v_1 is likely to have a much lower value than that suggested by Szyszkowski and Glockner [1987b]. The two constants v_1 and v_2 were calculated from experimental data at 0.004 MPa and had values of 10.8 and 235, respectively. The value of v_1 used here is approximately equal to the value used by Sinha [1979] for ice of grain size 0.1 mm. These values were used in theoretical calculations for all other stresses where either superplastic deformation or interparticle glide is occurring. The constant j_∞ was taken to be 800 sec instead of the value of 460 sec used by Szyszkowski and Glockner [1987b]. In cases where the principal tensile stress in the neck exceeds 0.7 MPa, we take v_2 equal to 5040, as no superplastic deformation takes place prior to fracturing at high stresses.

The constant c_1 , in equations 4.33 and 4.34, has a value of 3.906×10^{-3} . The constant c_2 depends on the ratio of shear to compressive stress at the point of contact and is chosen to give the experimentally observed Poissonic effect. If the absolute value of the

ratio of shear to compressive stress is less than 0.5, then c_2 is equal to 0.3. Otherwise it has a value of 0.40. The constant n has a value of 1.8. The constant m varies with displacement of particles. As the strain increases, particles develop more contacts and an increasing resistance to further movement. This seems to account for the strain hardening in snow. The constant m was obtained, using a regression fit, as a function of effective strain in snow. For the uniaxial compressive stress case, the results for various values of loads are shown in Figures 5-10.

The parameters c_1 , c_2 and m have slightly different values for tensile stress states. For tension c_1 is 5.468×10^{-3} and c_2 is approximately 0.1. For tension the Poissonic effect is slightly higher and the constants have to be adjusted to account for this in case of uniaxial tension tests. A possible reason why these constants for tension are different from those in compression may be due to the difference in densities of the snow samples used for the compression and tension tests. For instance, the initial density of snow in tension tests was approximately 270 kg/m^3 , whereas for compression tests the density was close to 330 kg/m^3 . Also, in tension the volume is increasing and the number of contacts itself may be decreasing or remaining the same. This may also explain why snow in tension fails much more quickly than when in compression. With a low initial coordination number and with increasing volume, this coordination number could decrease to the value of two, the minimum required to maintain a chain. Lower initial density also indicates that equitemperature metamorphism has not reached completion, and

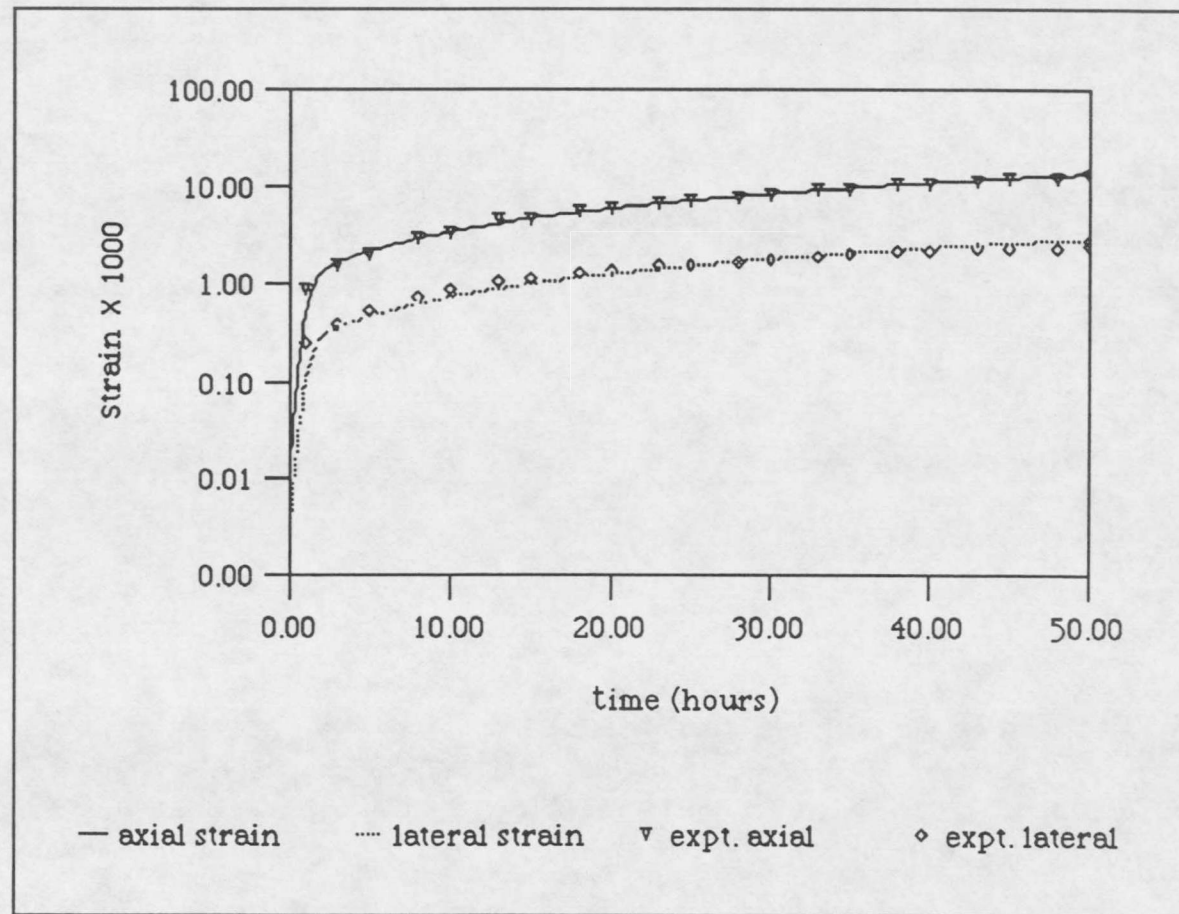


Figure 5. Strain vs time for a uniaxial compressive stress of 0.004 MPa. $a=0.254$ mm, $A_T=0.022$ mm², $\gamma=2.72$, temperature=-10°C.

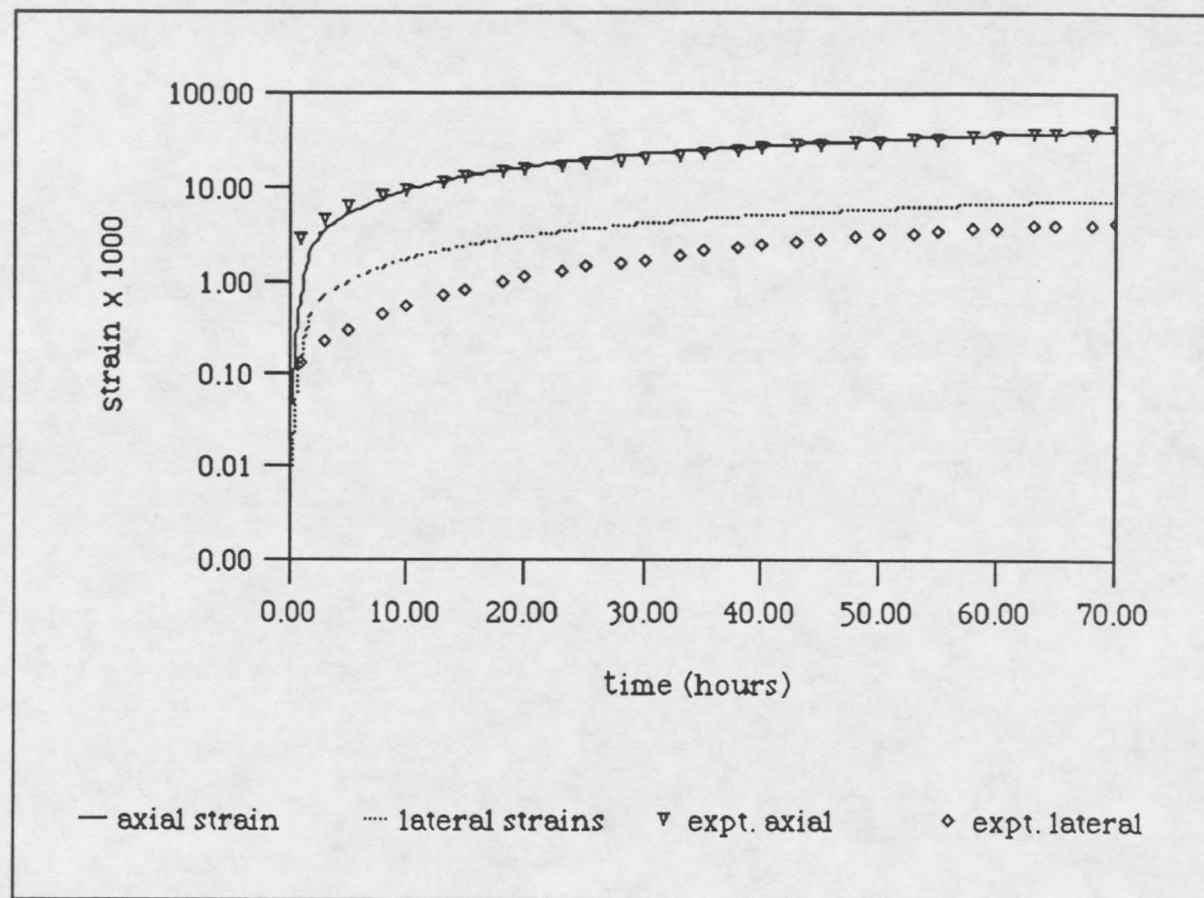


Figure 6. Strain vs time for a uniaxial compressive stress of 0.008 MPa. $a=0.254$ mm, $A_T=0.022$ mm², $\gamma=2.75$, temperature=-10°C .

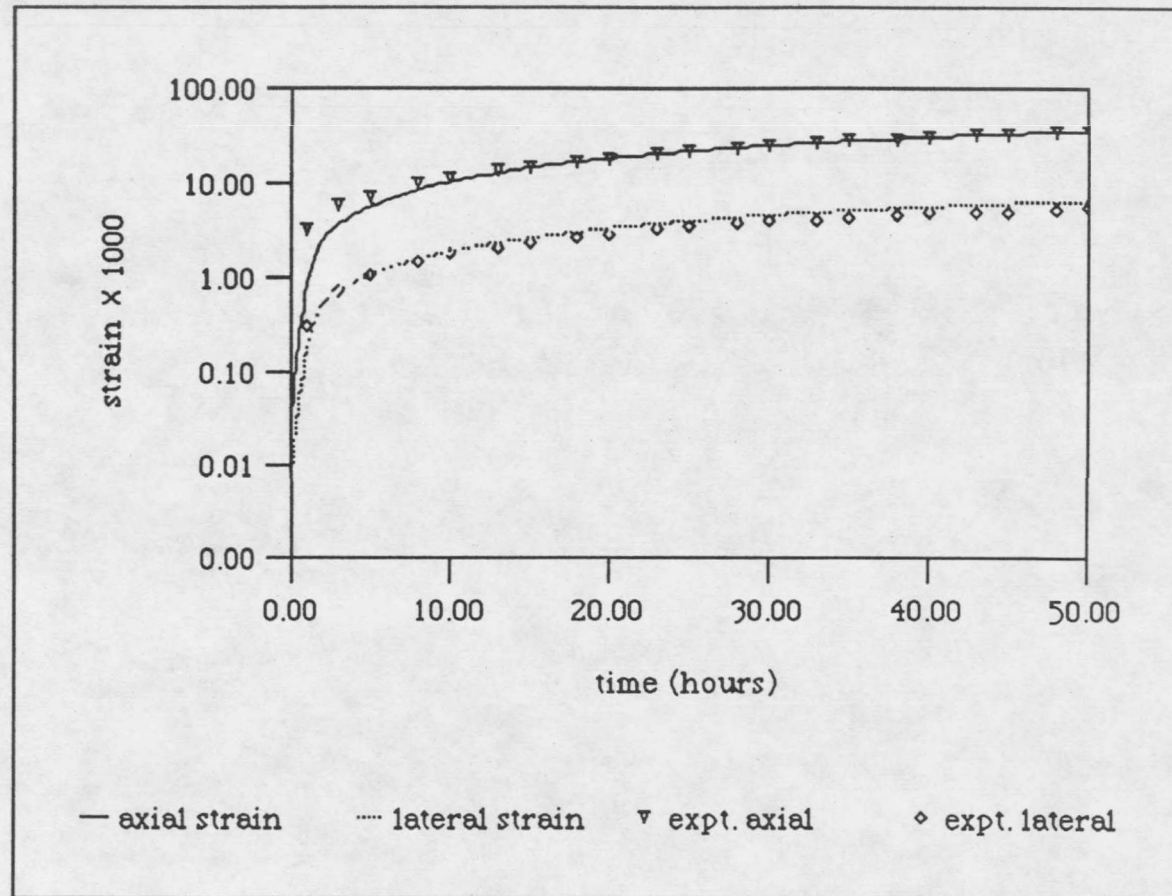


Figure 7. Strain vs time for a uniaxial compressive stress of 0.012 MPa. $a=0.254$ mm, $A_T=0.028$ mm², $\gamma=2.62$, temperature=-10°C.

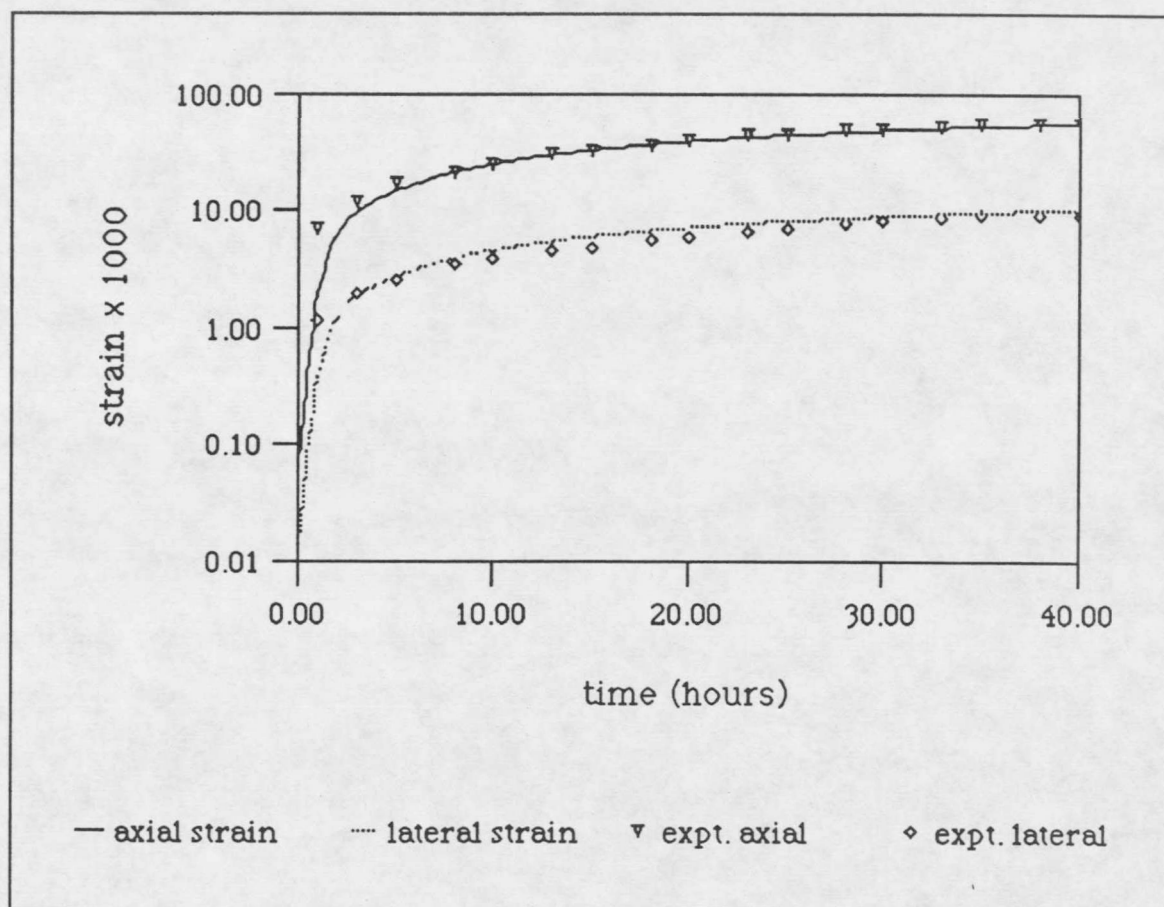


Figure 8. Strain vs time for a uniaxial compressive stress of 0.016 MPa. $a=0.254$ mm, $A_T=0.022$ mm², $\gamma=2.75$, temperature=-10°C.

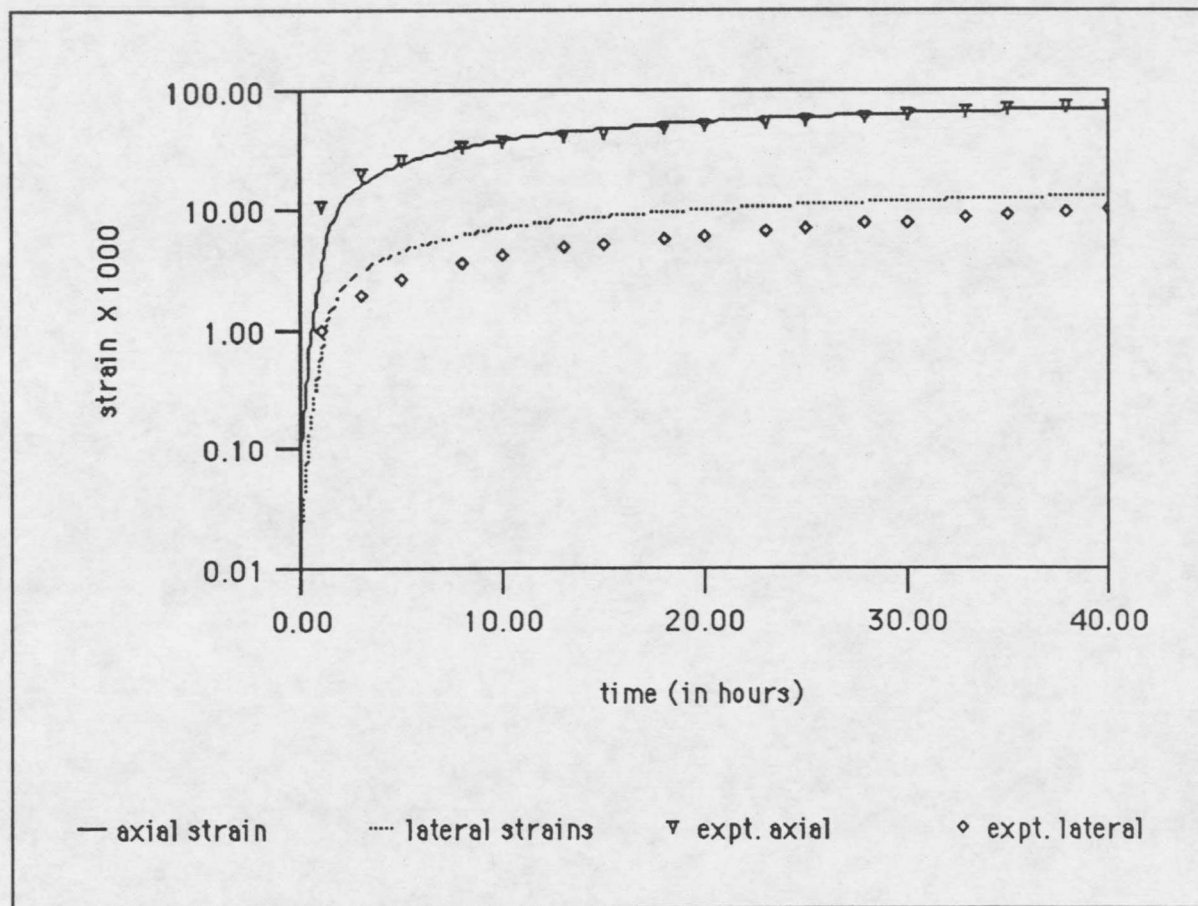


Figure 9. Strain vs time for uniaxial compressive stress of 0.024 MPa. $a=0.254$ mm, $A_T=0.024$ mm², $\gamma=2.67$, temperature=-10°C.

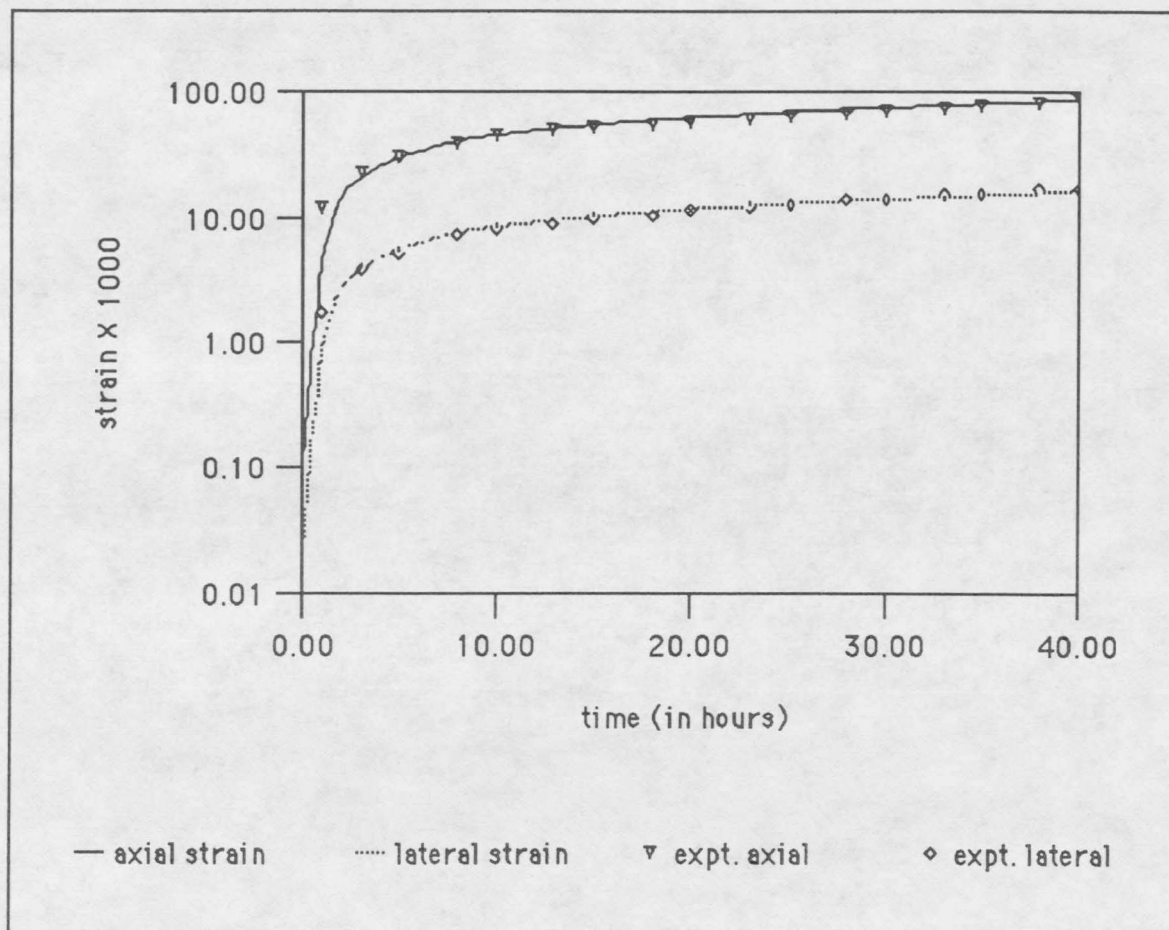


Figure 10. Strain vs time for a uniaxial compressive stress of 0.028 MPa. $a=0.254\text{mm}$, $A_T=0.024\text{ mm}^2$, $\gamma=2.67$, temperature= -10°C .

necks therefore have a smaller radius than for higher initial densities. This conjecture (smaller radius) receives support from the experimental data for two tension tests carried out on snow of density 352 kg/m^3 . These tests were carried out at stresses of 0.008 and 0.016 MPa and no failure was observed even after 70 hours of testing, a phenomenon more commonly observed in compression tests. The results from tension tests for uniaxial stress appear in Figures 11-13.

Another difference between compression and tension is seen when the stresses cause fracturing after some plastic deformation has taken place in the necks. The necks in a snow sample in compression reach the fracture strain at a later stage as compared to the same snow sample in tension. This is because in the former case necks have a lower principal tensile stress and therefore the time required to reach the fracture strain is greater than that for snow sample in tension. This difference in strains becomes very small as the time of deformation becomes large.

The results of the calculations appearing in Figures 5-13 match the results from experiments once the material parameters have been properly chosen. For a grain radius of 0.508 mm, neck radius of 0.053 mm, coordination number equal to 2.5 and density ratio equal to 2.75 the Young's modulus is 270.63 MPa. For the same density ratio Mellor [1974] gives Young's modulus of snow obtained from dynamical testing between 100 to 225 MPa. This is about 30 times as large as values obtained from static testing. No information about the average size of grains or necks is provided in Mellor's paper. Large

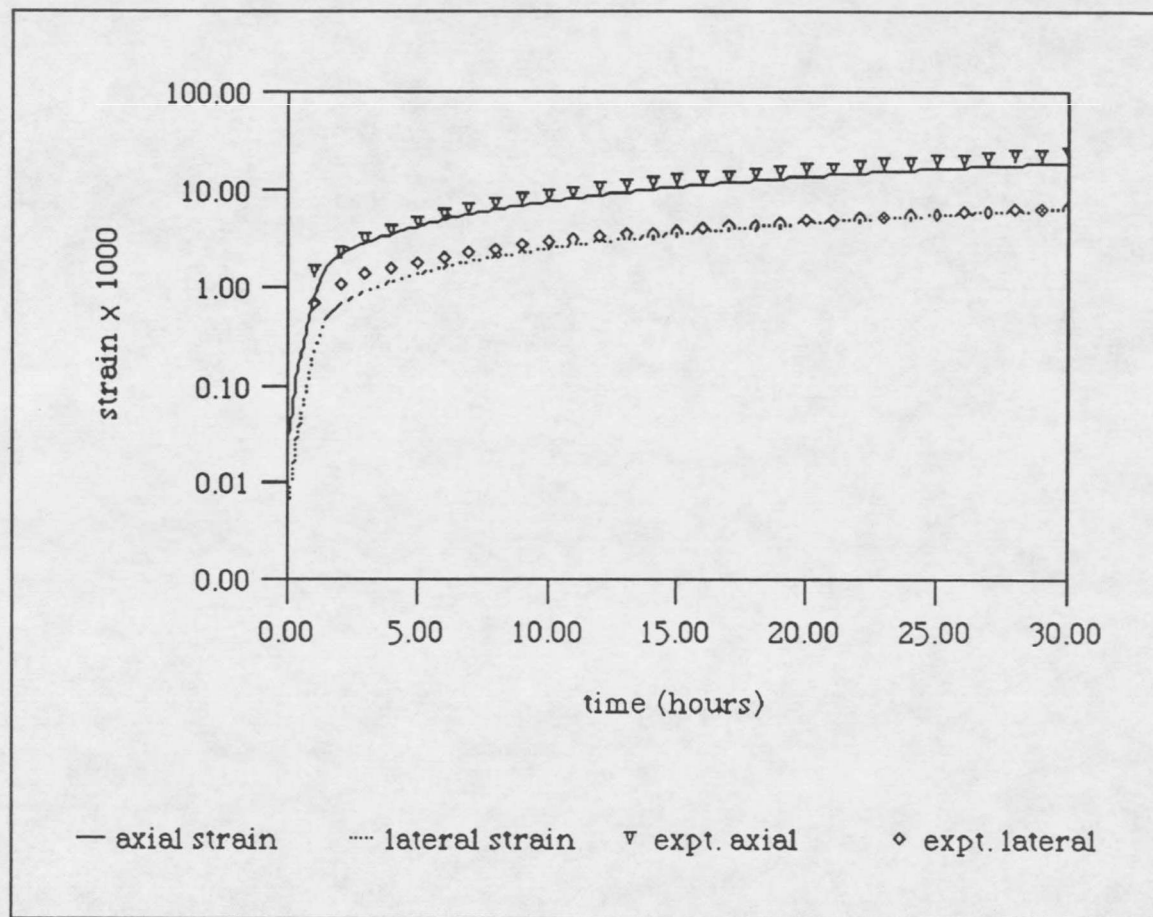


Figure 11. Strain vs time for a uniaxial tensile stress of 0.004 MPa. $a=0.225$ mm, $A_T=0.020$ mm², test temperature=-10°C, $\gamma=2.67$.

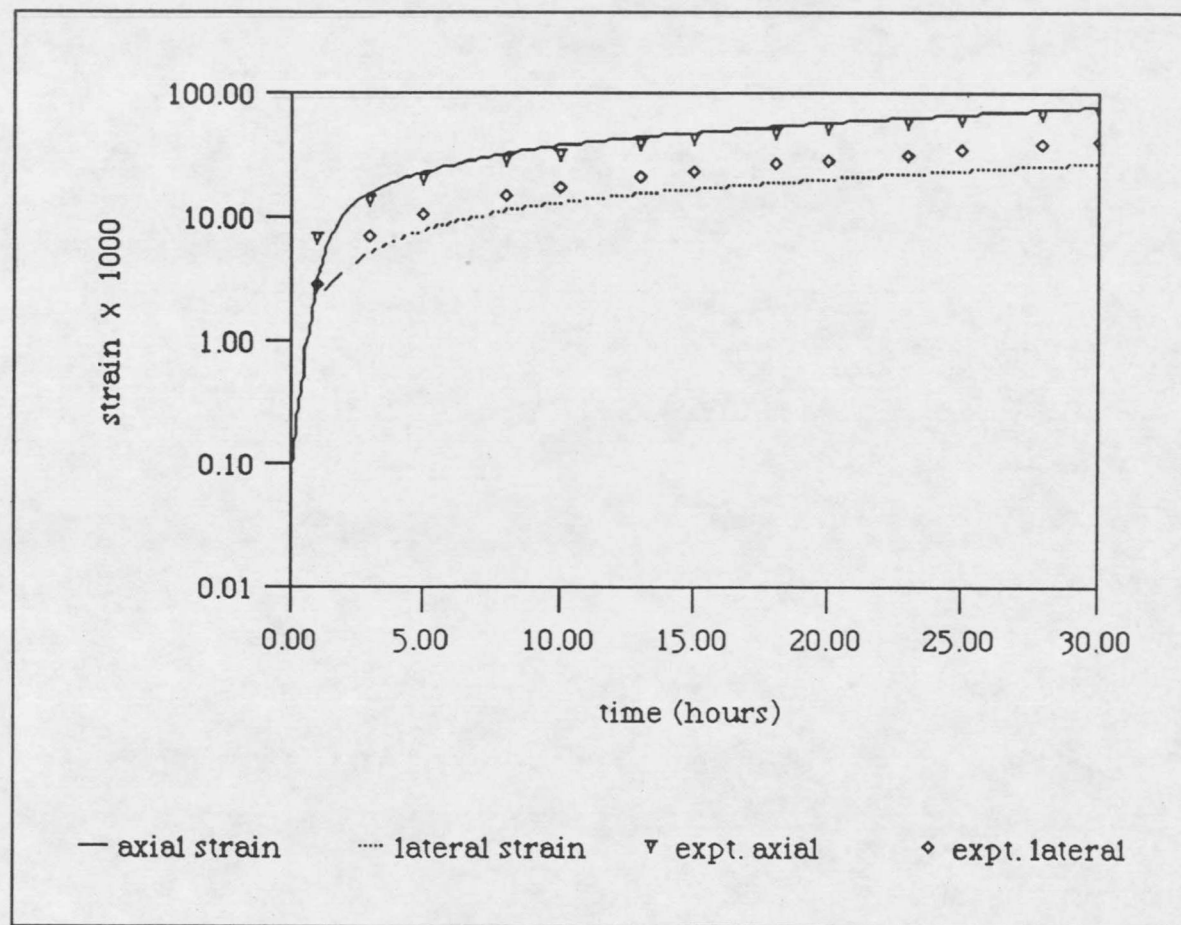


Figure 12. Strain vs time for a uniaxial tensile stress of 0.012 MPa. $a=0.225$ mm, $A_T=0.020$ mm², $\gamma=2.67$, temperature=-10°C.

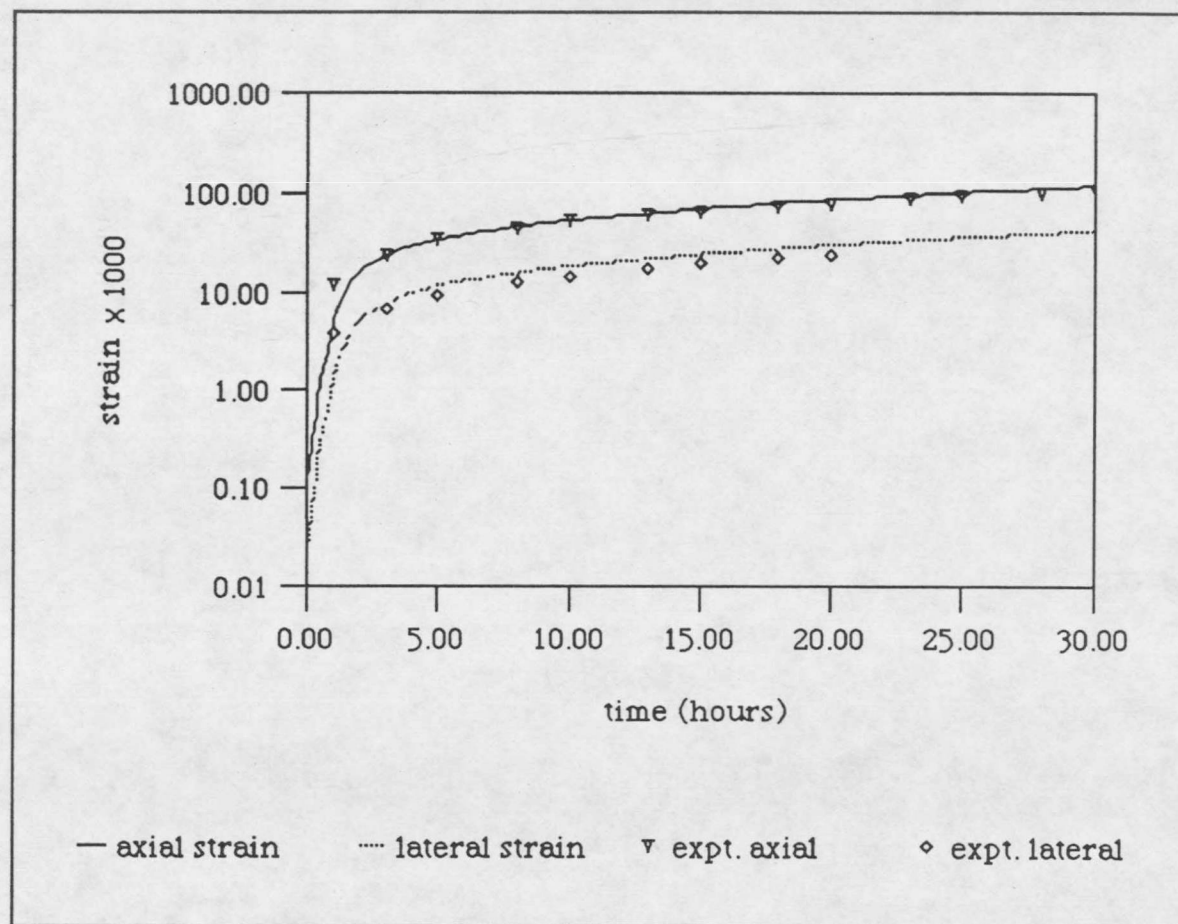


Figure 13. Strain vs time for a uniaxial tensile stress of 0.016 MPa. $a=0.225$ mm, $A_T=0.020$ mm², $\gamma=2.67$, temperature=-10°C.

necks, smaller grain radius or a higher coordination number all serve to make snow stiffer. This may account for the wide range observed in Mellor's work. The dependence of Young's modulus on neck size, grain size, density ratio and coordination number is partially determined by the factor $\frac{a^2}{\gamma} A_T$ in equation 3.28 for stress in necks.

The value of Young's modulus chosen for ice was about 5400 MPa from Szyszkowski and Glockner [1987b]. This seems to be on the low side. However, during formation of necks the contact between particles may be not be perfect and some air may get entrapped in the necks. This entrapped air causes the value of Young's Modulus to be reduced [Rice, 1977], and therefore the value of 5400 MPa seems reasonable.

The ratio of lateral strains to axial strains, in most of the experimental data, varies from 0.15 to 0.2 for uniaxial compression tests and from 0.2 to 0.3 for uniaxial tension tests. The theoretical results give the value of ratio of lateral strain to axial strain as approximately 0.18 for compression and 0.23 for tension. An interesting feature of lateral to axial strain ratio is that it does not change much even when the underlying deformation phenomenon is entirely different. At a compressive stress of 0.004 MPa when superplastic deformation of necks is the sole deforming mechanism we have almost the same lateral to axial strain ratio as for a snow sample subjected to 0.028 MPa with interparticle sliding as the only deformation mechanism. This is because at low stresses the lateral deformation of the necks is into the pore space and does not

contribute to the snow strains. For sliding particles there are no such lateral strains, and therefore the value of lateral to axial strain in this case is also almost the same. A better understanding about this ratio could be obtained if some information as to the plane of fracturing of ice under multiaxial loading were available.

The major advantage of this formulation is its ability to calculate strains under multiaxial loading without requiring any additional constants other than the ones used for uniaxial tests. This reduces the need to perform extensive experiments normally required to find additional constants arising in multiaxial constitutive laws. For multiaxial loading the constants used were those used in uniaxial compression tests. The results from these are plotted in Figures 14-18. In Figure 14 the results from hydrostatic state of stress are shown. A 5.65% change in volume occurs over thirty hours. This is in accordance with the compressible nature of snow. At present no experimental data exists for such tests. Figure 15 shows results from a test in which the stresses are very low and strains are due to superplastic deformation of necks rather than sliding of broken necks.

At very low stresses (0.004 MPa) the strain rates vary almost linearly with stress. Even at higher stresses (0.008-0.028 MPa) the nonlinearity is not large as long as the coordination number, density, grain and neck size remain nearly constant. In Figure 18, where $t_{11} = -0.008$ MPa and $t_{12} = t_{21} = 0.008$ MPa, the strains e_{22} and e_{33} are no longer equal. Also, when shear stresses t_{13} and t_{23} are applied along with normal stress t_{33} , shear strain e_{12} is observed although no shear

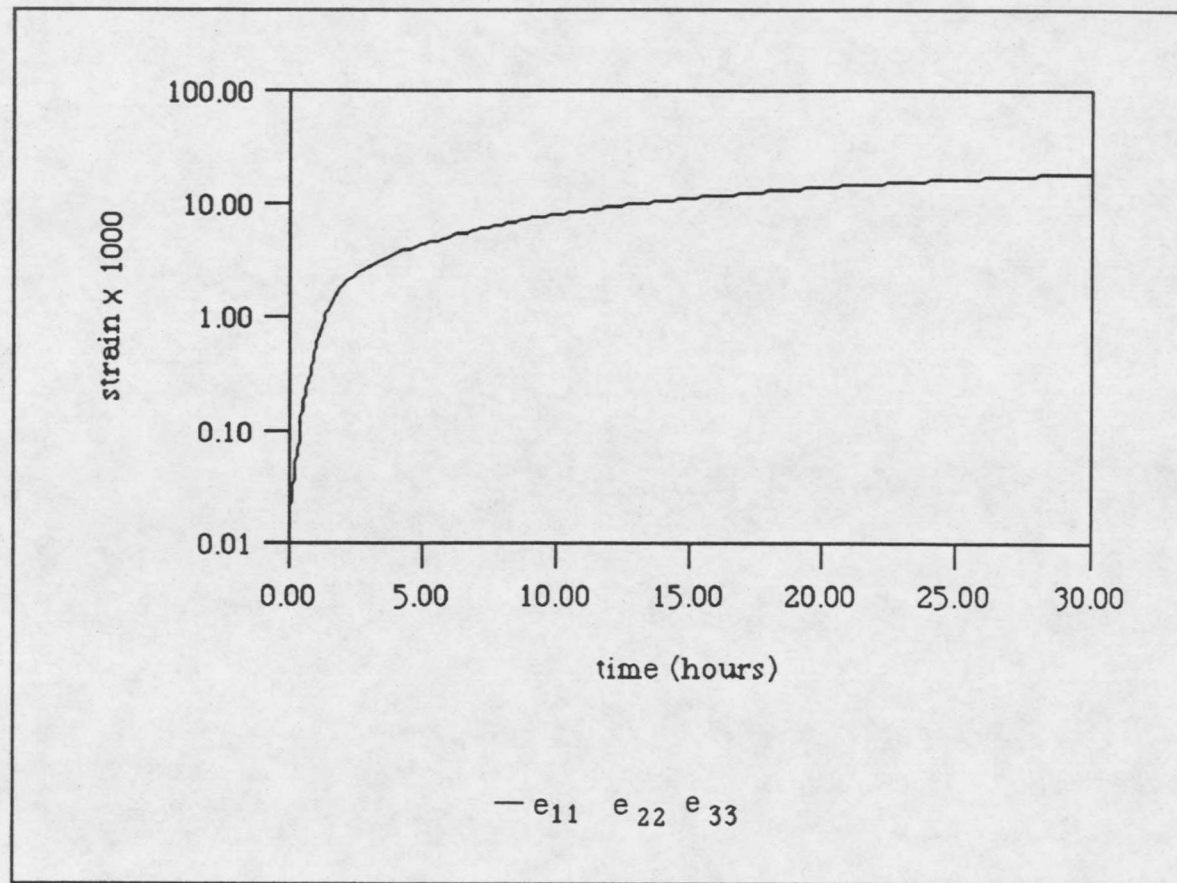


Figure 14. Strain vs time for a hydrostatic state of stress. $t_{11} = t_{22} = t_{33} = -0.008$ MPa. $a=0.254$ mm, $A_T=0.022$ mm², $\gamma=2.72$, temperature= -10°C .

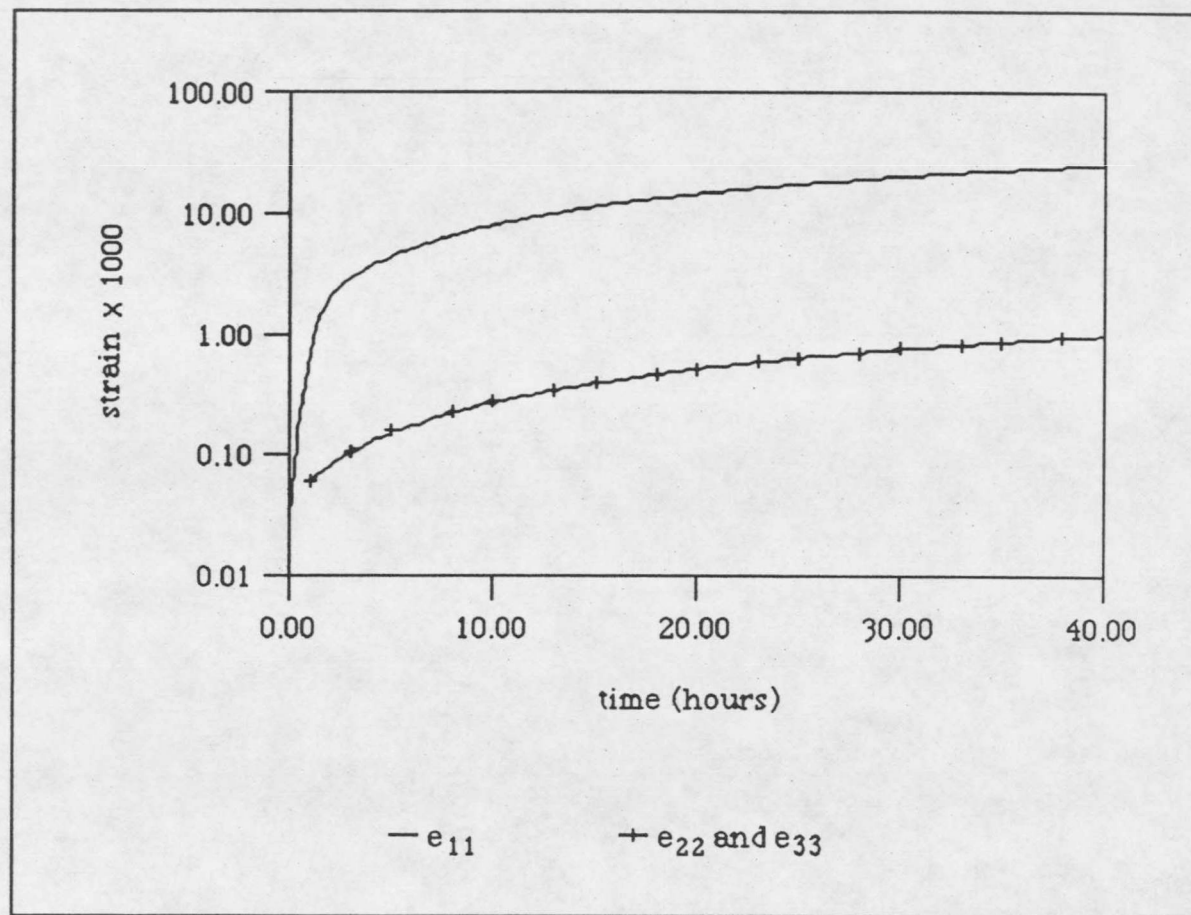


Figure 15. Strain vs time for $t_{11} = -0.008$ MPa, $t_{22} = t_{33} = -0.002$ MPa. $a=0.254$ mm, $A_T=0.022$ mm², $\gamma=2.72$, temperature=-10°C.

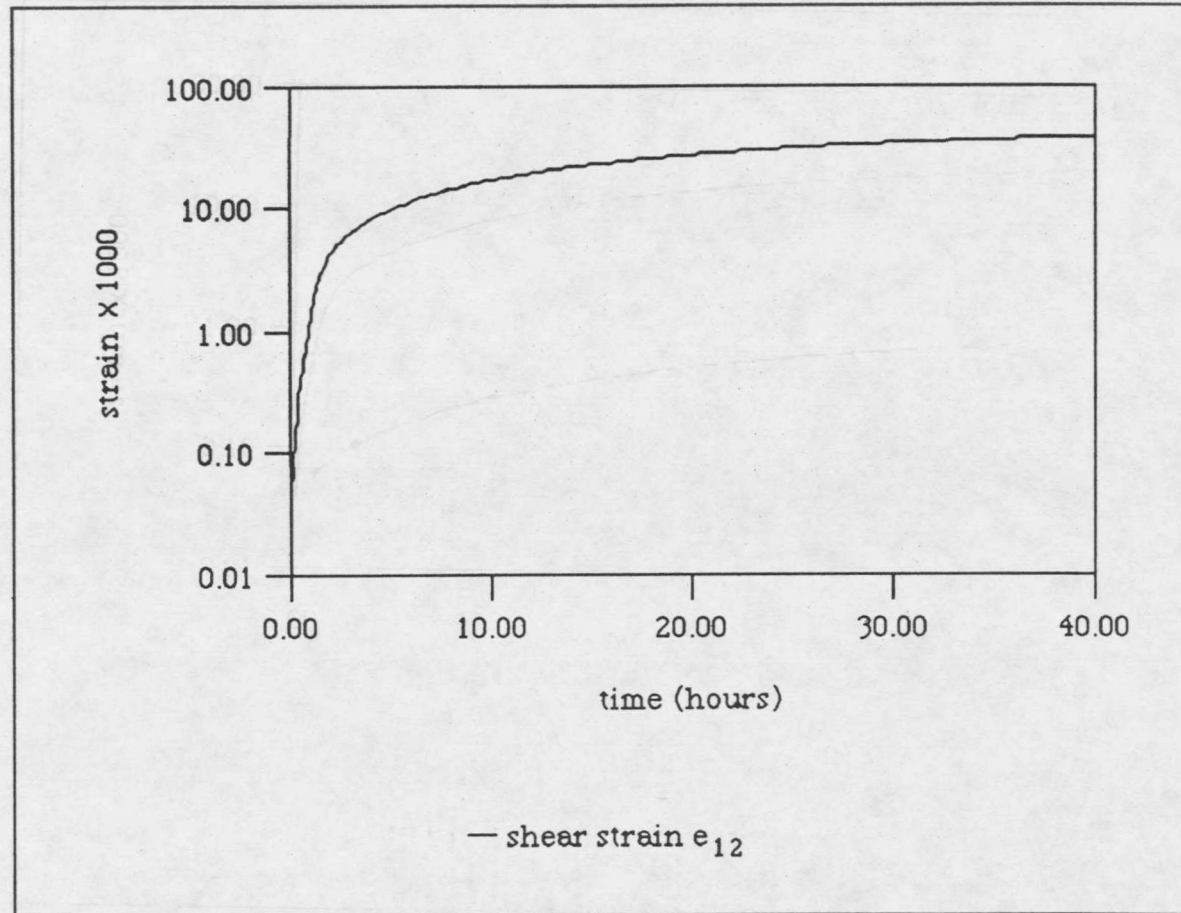


Figure 16. Strain vs time for shear stress $t_{12} = t_{21} = 0.0098$ MPa. $a=0.254$ mm, $A_T=0.022$ mm², $\gamma=2.72$, temperature= -10°C .

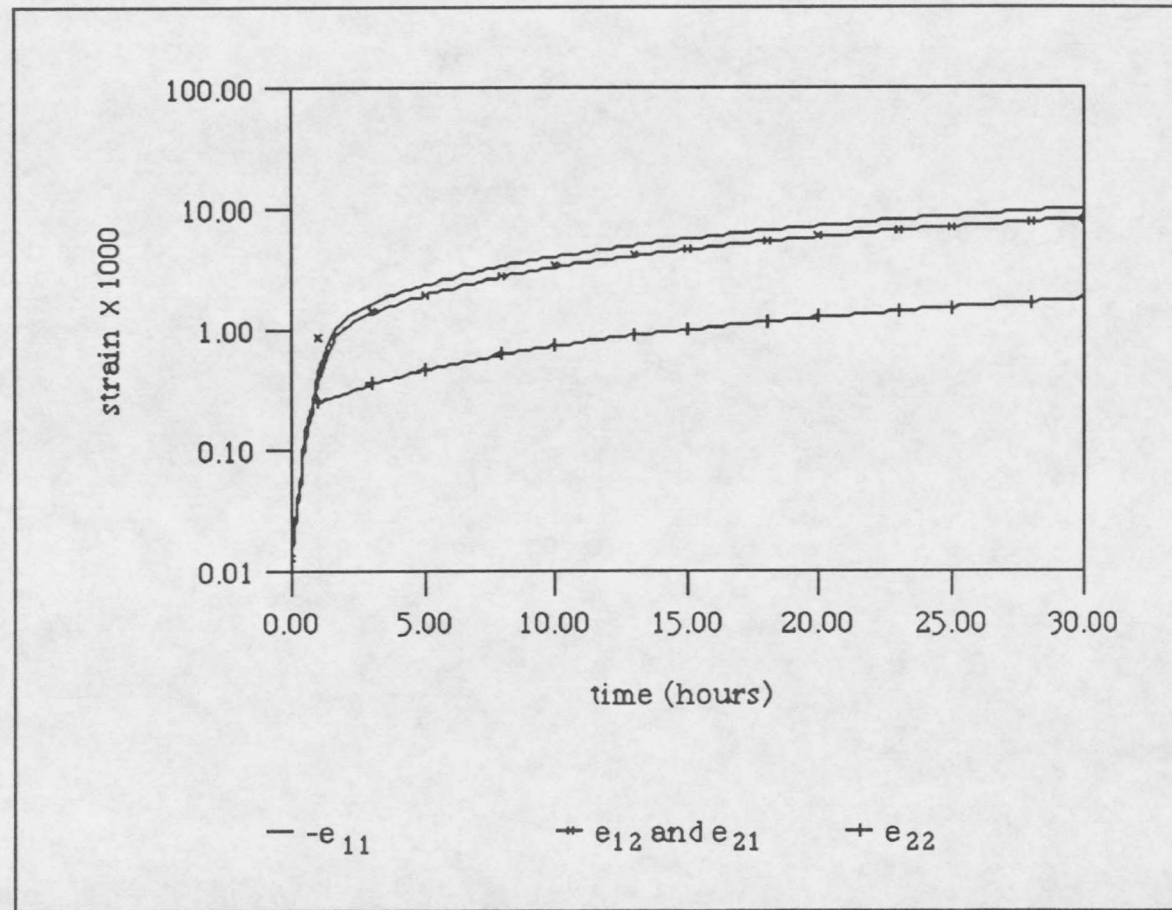


Figure 17. Strain vs time for $t_{11} = -0.004$ MPa, $t_{12} = t_{21} = 0.0028$ MPa. $a=0.254$ mm, $A_T=0.022$ mm², $\gamma=2.72$, temperature= -10°C .

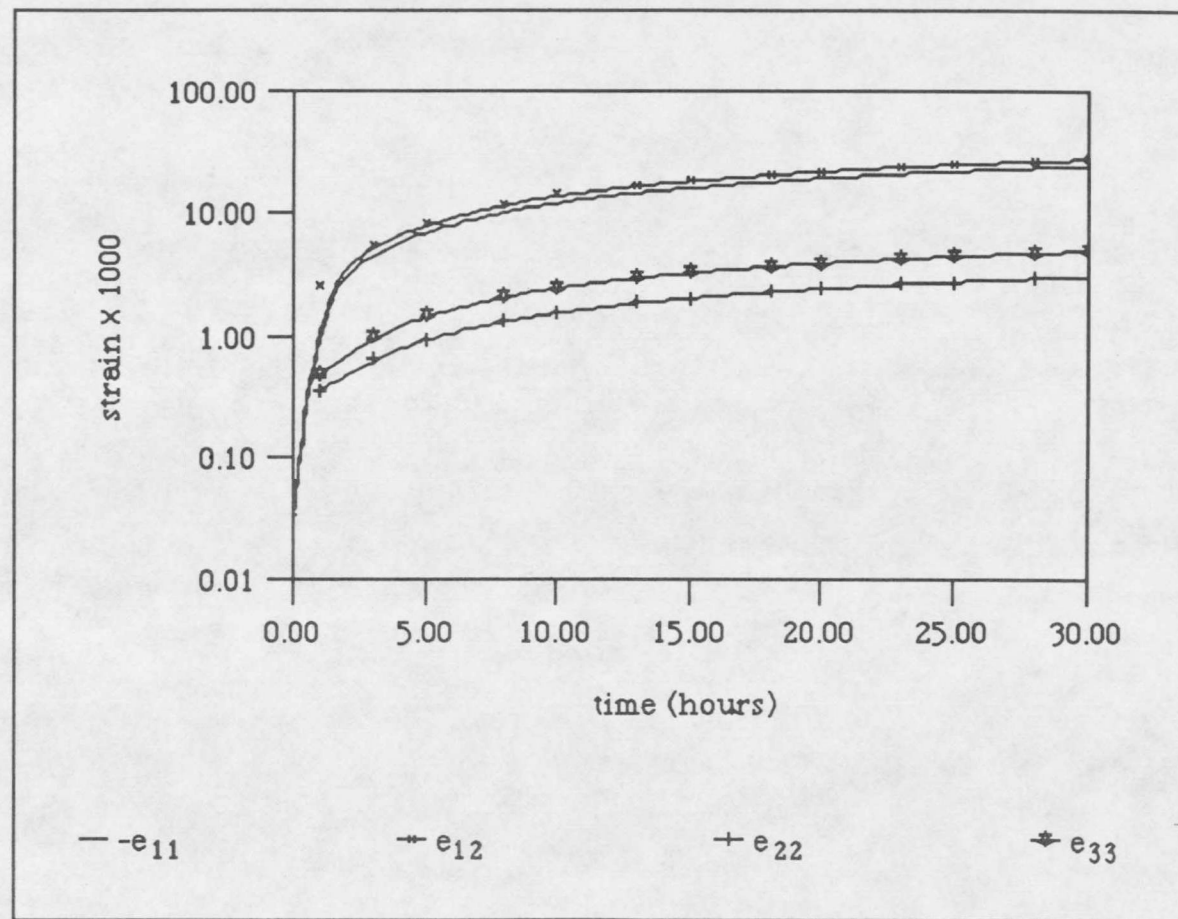


Figure 18. Strain vs time for $t_{11} = -0.008$ MPa, $t_{12} = t_{21} = 0.008$ MPa. e_{33} differs significantly from e_{22} . $a=0.254$ mm, $A_T=0.022$ mm², $\gamma=2.72$, temperature=-10°C.

stress was applied in this direction. The value of this shear strain is one order of magnitude smaller than the strains in the directions in which stress was applied. In multiaxial tests when both shear stress and normal compressive stress are present the relation between deformation rate and stress is of the form:

$$\dot{E}_{ij} = C_0 + C_1 t_{ij} + C_2 t_{ik} t_{kj}$$

Since the dependence on the third term on right-hand side is rather small, as is apparent from relatively small value of e_{12} in the test mentioned above, the relation between deformation rate and stress can be approximately written as

$$\dot{E}_{ij} = C_0 + C_1 t_{ij}$$

In the above equation C_0 is a function of first and second principal invariant of stress tensor and second invariant of strain tensor. C_1 is a function of second principal invariant of stress and second invariant of strain tensor. For both C_0 and C_1 the dependence on second invariant of strain tensor can be represented quite accurately by a second order polynomial. However, no simple dependence could be found with respect to the other two variables.

CHAPTER 5

APPLICATIONS

In this chapter we discuss two applications of the constitutive law. The first application is bending of a cantilever beam loaded by a uniformly loaded pressure. Simple beam bending theory, with plane sections remaining plane, has been used. The second problem deals with the settlement of a foundation into a snow cover. This problem is approximated by having part of the upper surface loaded by a uniformly distributed load. Plane strain has been assumed for this problem, and therefore a three dimensional state of stress exists. The method followed in solving the two problems is a commonly used technique for solving problems involving creep of structures [Boyle and Spence, 1983]; Penny and Marriott, 1971].

The method for solving any creep problem is closely linked to the elastic solution procedure. This relation between creep and elastic solutions derives from the fact that, after initial loading, creep strains arising during the passage of time require that stresses change by elastic straining to accommodate these changes [Boyle and Spence, 1983].

We begin by formulating the general boundary value problem. Consider a solid body occupying a volume V bounded by a surface S . The basic field equations are

Equilibrium:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0. \quad (5.1)$$

Strain displacement:

$$e_{ij} = 0.5 \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \quad (5.2)$$

It is assumed that over part of the surface S tractions are prescribed, while over the remainder of S displacements are imposed. The boundary conditions are then

$$\begin{aligned} u_i &= \bar{u}_i & \text{on } S_u \\ \sigma_{ij} n_j &= \bar{p}_i & \text{on } S_p. \end{aligned} \quad (5.3)$$

The field equations and boundary conditions are completed by the constitutive relations. It is assumed that the strain can be decomposed into elastic and creep parts

$$e_{ij} = e_{ij}^e + e_{ij}^c. \quad (5.4)$$

The elastic strain is related to the stress through the Hooke's law

$$e_{ij}^e = C_{ijkl} \sigma_{kl}. \quad (5.5)$$

Here C_{ijkl} is the elastic modulus.

Substituting the above relation into equation 5.5, we get

$$\sigma_{ij}^e = M_{ijkl}(e_{kl} - e_{kl}^c), \quad (5.6)$$

where M_{ijkl} is the compliance tensor.

In a structure experiencing creep according to a time hardening law there is a stage of stress redistribution, resulting from interaction between elastic and creep strains, before steady state

stresses are attained [Penny and Marriot, 1971]. What happens is that as time progresses the stress distribution, which is purely elastic initially, changes until it reaches a steady state distribution. When stresses become steady (constant for constant load), the strain and displacement rates are also steady [Boyle and Spence, 1983].

In the case of snow at low stresses, the coordination number does not change significantly with time, and no strain hardening takes place. Therefore, a steady state should eventually be reached. At high stresses, strain hardening results due to increased concentration of neighboring particles which increase resistance to further motion of the particle. Under strain hardening conditions a steady state of stress may not be reached unless the constitutive law is of a very special type. This is shown by the following relation for uniaxial case

$$\frac{de}{dt} = \frac{\partial e}{\partial \sigma} \frac{d\sigma}{dt} + \frac{\partial e}{\partial t}$$

Now if stresses are steady, the first term on the right-hand side is zero. The second term is independent of the conditions anywhere else in the component. If the first term is zero, then the second term will not satisfy compatibility unless strains are restricted in some way. Penny and Marriot [1971] have shown this restriction to be of the form

$$e = f_1(\sigma)f_2(t).$$

Since snow at high stresses does not follow this type of law, a steady state is not likely to be reached.

In both problems it has been assumed that all the necks fracture after a certain time. This assumption is used because there are areas where ice necks, at a particular solid angle, have a principal stress of, for example, 0.7 MPa at one grid point, and at the next grid point the principal neck stress, for the same solid angle, is 0.705 MPa. In the theory we have assumed that for stresses less than or equal to 0.7 MPa superplastic deformation takes place, whereas for stresses greater than this value deformation is due to sliding of particles. Therefore, at the first grid point the neck is undergoing a superplastic deformation whereas at the second grid point there is relative sliding between the particles. This results in discontinuous strain rates (because the strain rates for superplastic deformation and sliding are different) and also leads to a situation where regions of low stress have higher strain rates as compared to regions with higher stress (superplastic strain rate at 0.7 MPa is higher than sliding rate at the same stress). The solution becomes unstable rather quickly if this assumption is not made. For the beam it was assumed to require five hours for all necks to fracture. By this time all the necks in the high stress region would have fractured anyway. This constitutes about 25% of the total grain necks. Since these fractured necks will determine the time snow structure takes to collapse, the error introduced by assuming the fracturing of other necks should not be significant. For the foundation problem approximately two hours were required for most of the ice necks in the region right beneath the footing to fracture.

Cantilever Beam

We consider a cantilever beam fixed at one end and subject at its top edge to a uniformly distributed load of the same magnitude as the weight of the beam. Let p be the uniformly distributed load per unit length. Let x denote the distance along the beam with the fixed end at $x = 0$ and free end at $x = l$. The problem is analyzed using simple beam bending theory: let M denote the bending moment such that $M > 0$ for a beam concave downward, and let κ and w be respectively the curvature and the deflection, positive upwards.

The equilibrium equation is

$$M = \int_A \sigma z dA = \int_{-d_2}^{d_1} \sigma b z dz = 0.5 p (l-x)^2. \quad (5.7)$$

Here z is the vertical distance from the centerline varying from $-d_2$ to d_1 . The height of the beam is $d_2 + d_1$. The width of the beam is b .

The strain displacement equation is

$$\kappa = - \frac{d^2 w}{dx^2}. \quad (5.8)$$

with the boundary conditions at the clamped end $x=0$ as

$$w = 0, \quad \frac{dw}{dx} = 0. \quad (5.9)$$

Based on the assumption that originally plane sections remain plane after bending, the strain e is given by

$$e = \kappa z. \quad (5.10)$$

Also

$$e = e^e + e^c = \frac{\sigma}{E} + e^c, \\ \sigma = E (e - e^c) = E (\kappa z - e^c). \quad (5.11)$$

Substituting equation 5.11 into the equilibrium equation 5.7 gives

$$\int_{-d_2}^{d_1} E (\kappa z - e^c) b z dz = 0.5 p (1-x)^2 \quad (5.12)$$

or

$$\kappa = \frac{1}{EI} \int_{-d_2}^{d_1} e^c b z dz + \frac{0.5}{EI} p (1-x)^2. \quad (5.13)$$

At time $t=0$ the first term on the left-hand side of the last equation is zero. Equation 5.13 is solved to obtain the curvature at time $t=0$, and the result is then substituted into equation 5.11 to determine the elastic stress solution. This is used to calculate the creep strains, which are substituted in equation 5.13 to obtain the curvature at the end of time step. The above procedure is now repeated for all successive times t greater than zero.

For calculation purposes a rectangular beam 0.25 m high, 0.25 m wide and 1.5 m long was taken. The maximum elastic stress was about 0.1 Mpa, and it was assumed that after five hours all necks break. By this time strains in some necks are high enough to cause fracturing as explained above. The stress distribution, in the form of contour plots, by 30 hours is shown in Figure 19.

To solve for the tip deflection, equation 5.8 subject to boundary condition 5.9 was integrated using forward Euler method. Figure 20 shows the tip deflection as a function of time. The tip deflection increases linearly with time, and by the end of 30 hours the tip deflection is about 5 cm. Figure 21 shows the variation of the maximum stress in the beam with time. The maximum stress first

decreases with time, reaches a minimum and, once strain hardening starts taking place, begins to increase. The rate of increase is much slower than the initial rate of decrease of maximum stress.

Boyle and Spence [1983] have solved a similar problem but have not presented any results. They have, however, given results for displacements for a beam subject to pure bending moment. The displacements vary linearly with time and the behavior, at least qualitatively, is similar to the one we have shown here for snow. They have used Norton's power law to relate stress and strain rate and have not taken primary creep into consideration.

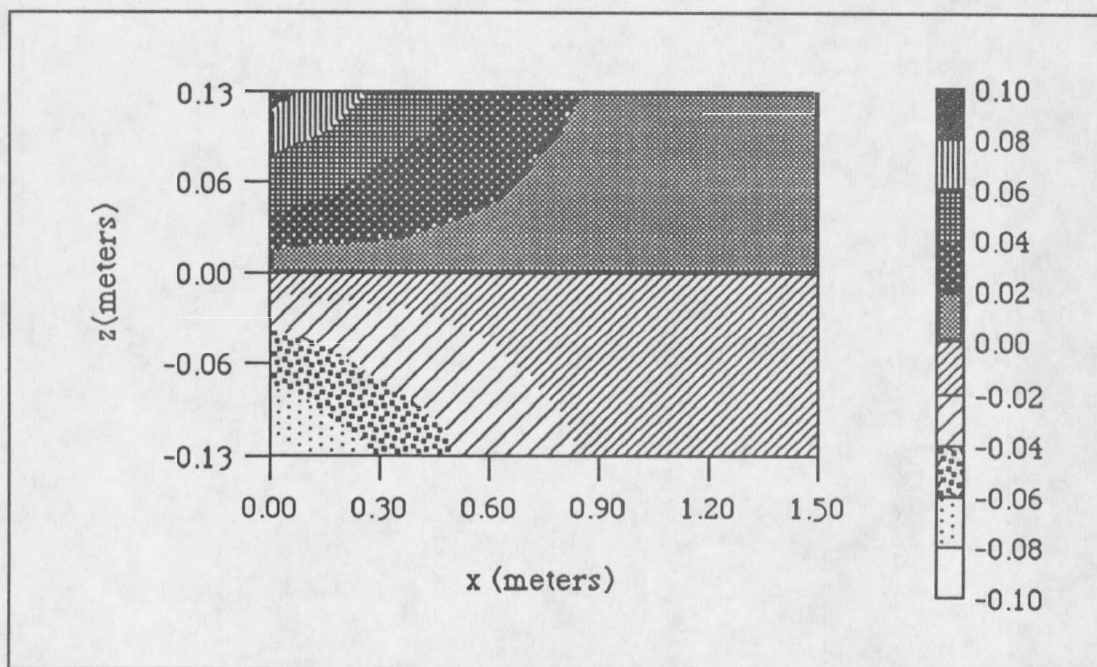


Figure 19. Stress contours for cantilever beam. The stresses are in MPa.

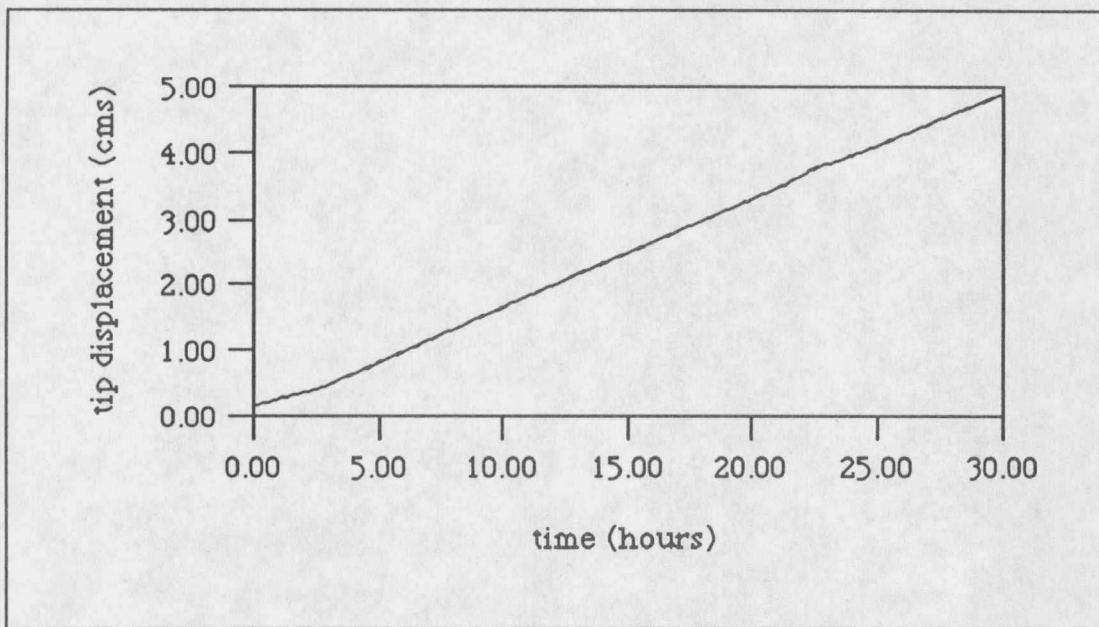


Figure 20. Variation of tip displacement with time for cantilever beam.

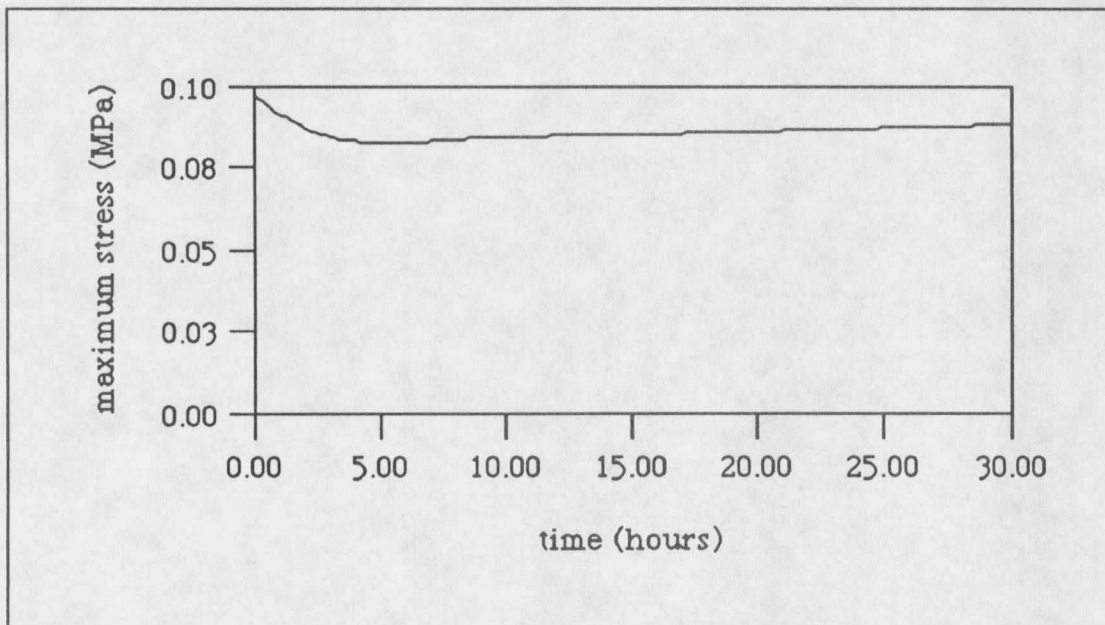


Figure 21. Variation of maximum stress with time for cantilever beam.

Settlement of Foundation

The second problem solved is that of settlement of a foundation into a snowcover. The settlement of foundations can rarely be accurately predicted because of effect of temperature on mechanical properties. Nevertheless, a theoretical analysis permits one to obtain an idea of the magnitude and distribution of stress and displacement of the snowcover supporting the foundation.

It has been experimentally observed that for symmetrical loading, the vertical and horizontal extent of the deformation zone (pressure bulb) in a foundation extends to about two times the width of the footing [Reed, 1966]. For simplicity we consider a rectangular area underneath the footing as shown in Figure 22. The dimensions of this rectangle are larger than the observed size of the pressure bulb.

The stress strain relation for isotropic elastic material is

$$\sigma_{ij} = \lambda e_{kk}^c + 2\mu e_{ij}^c,$$

where λ and μ are Lamé's constants. Using equation 5.6 we can write this last relation as

$$\sigma_{ij} = \lambda(e_{kk} - e_{kk}^c)\delta_{ij} + 2\mu(e_{ij} - e_{ij}^c)$$

$$\sigma_{ij} = \lambda\left(\frac{\partial u_k}{\partial x_k} - e_{kk}^c\right)\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - 2e_{ij}^c\right). \quad (5.14)$$

Substituting this into the equilibrium equation we get

$$\mu(\nabla^2 u_j) + (\lambda + \mu)\nabla_j \nabla_i u_i + \rho b_j = \lambda \frac{\partial e_{kk}^c}{\partial x_j} + 2\mu \frac{\partial e_{ij}^c}{\partial x_i}. \quad (5.15)$$

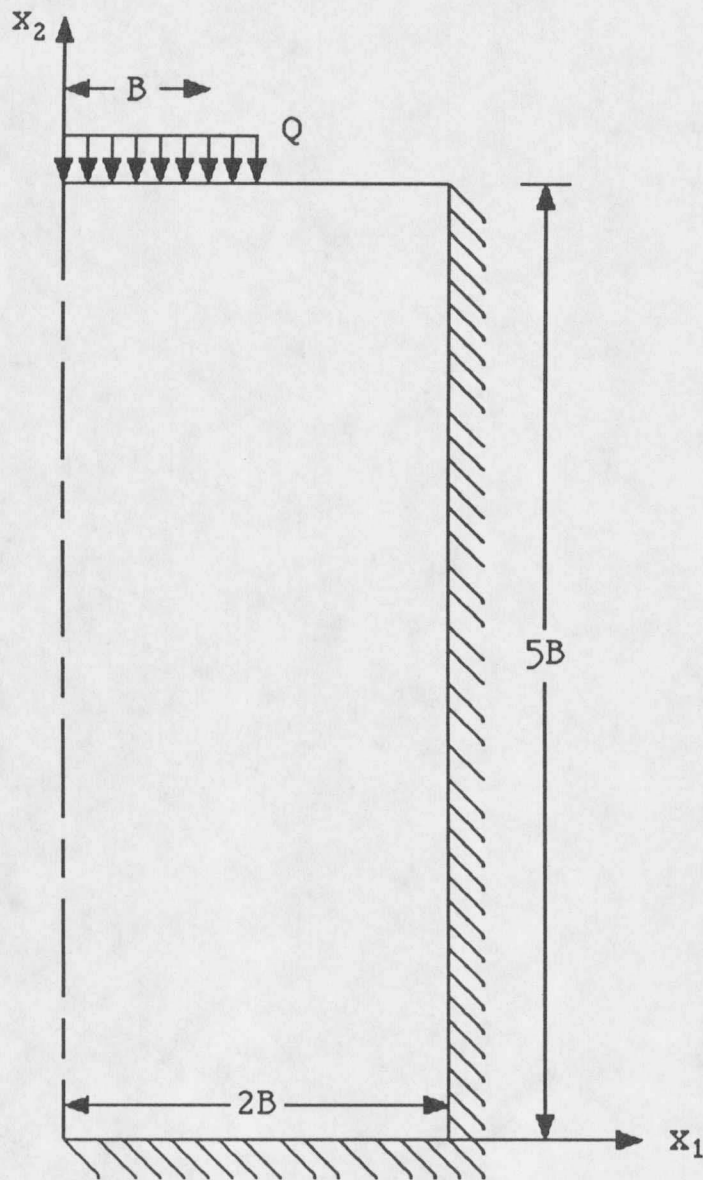


Figure 22. Foundation with a stress of Q MPa. $B=.45\text{m}$. Symmetry with respect to x_1 allows only half of the foundation and snow cover region to be analyzed.

For the case of plane strain, the reduced Navier's equations are

$$\begin{aligned}
 & G[V^2 u_1 + \frac{1}{1-2\nu} \frac{\partial}{\partial x_1} (\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2})] + \rho b_1 \\
 &= \frac{E\nu}{(1+\nu)(1-2\nu)} (\frac{\partial e_{11}^c}{\partial x_1} + \frac{\partial e_{22}^c}{\partial x_1} + \frac{\partial e_{33}^c}{\partial x_1}) + 2G(\frac{\partial e_{11}^c}{\partial x_1} + \frac{\partial e_{21}^c}{\partial x_2}) \quad (5.16)
 \end{aligned}$$

$$\begin{aligned}
 & G[V^2 u_2 + \frac{1}{1-2\nu} \frac{\partial}{\partial x_2} (\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2})] + \rho b_2 \\
 &= \frac{E\nu}{(1+\nu)(1-2\nu)} (\frac{\partial e_{11}^c}{\partial x_2} + \frac{\partial e_{22}^c}{\partial x_2} + \frac{\partial e_{33}^c}{\partial x_2}) + 2G(\frac{\partial e_{22}^c}{\partial x_2} + \frac{\partial e_{12}^c}{\partial x_1}). \quad (5.17)
 \end{aligned}$$

If the right-hand side was zero, then we get Navier's equations of elasticity.

$$G[V^2 u_1 + \frac{1}{1-2\nu} \frac{\partial}{\partial x_1} (\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2})] + \rho b_1 = 0$$

$$G[V^2 u_2 + \frac{1}{1-2\nu} \frac{\partial}{\partial x_2} (\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2})] + \rho b_2 = 0$$

The boundary conditions are:

Top face:

$$\begin{array}{lll}
 x_2 = 2.25 \text{ m} & Q = Q_0 & 0 \leq x_1 \leq 0.45 \text{ m} \\
 & Q = 0 & x_1 > 0.45 \text{ m} \\
 & t_{xy} = 0 & 0 \leq x_1 \leq 0.9 \text{ m}.
 \end{array}$$

Using the above relations in the constitutive relation (equation 5.14) we have

$$Q = (\lambda + 2\mu) \left(\frac{\partial u_2}{\partial x_2} - e_{22}^c \right) + \lambda \left(\frac{\partial u_1}{\partial x_1} - e_{11}^c \right) - \lambda e_{33}^c \quad (5.18)$$

$$\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) = 2e_{12}^c \quad (5.19)$$

Axis of symmetry:

$$\frac{\partial u_2}{\partial x_1} = 0, u_1 = 0 \quad 0 \leq x_1 \leq 0.9 \text{ m}, x_2 = 0. \quad (5.20)$$

Bottom and right vertical side:

$$u_1 = 0, u_2 = 0 \quad 0 \leq x_1 \leq 0.9 \text{ m}, x_2 = 0 \\ \text{and } x_1 = 0.9 \text{ m}, 0 \leq x_2 \leq 2.25 \text{ m}. \quad (5.21)$$

If we substitute the boundary conditions for the top face into equations 5.16 and 5.17, we obtain

$$G\left[\left(1 + \frac{1}{1-2\nu}\right) \frac{\partial^2 u_1}{\partial x_1^2} + \left(1 - \frac{1}{1-2\nu}\right) \frac{\partial^2 u_2}{\partial x_2^2}\right] \\ = \frac{Ev}{(1+\nu)(1-2\nu)} \left(\frac{\partial e_{11}^c}{\partial x_1} + \frac{\partial e_{22}^c}{\partial x_1} + \frac{\partial e_{33}^c}{\partial x_1}\right) + 2G\left(\frac{\partial e_{21}^c}{\partial x_1} + \frac{\partial e_{11}^c}{\partial x_2}\right) - \frac{2}{1-2\nu} \frac{\partial e_{12}^c}{\partial x_2} \quad (5.22)$$

$$G\left[\left(1 + \frac{1}{1-2\nu}\right) \frac{\partial^2 u_2}{\partial x_2^2} + \left(1 - \frac{1}{1-2\nu}\right) \frac{\partial^2 u_1}{\partial x_1^2}\right] + \rho b_2 \\ = \frac{Ev}{(1+\nu)(1-2\nu)} \left(\frac{\partial e_{11}^c}{\partial x_2} + \frac{\partial e_{22}^c}{\partial x_2} + \frac{\partial e_{33}^c}{\partial x_2}\right) + 2G\left(\frac{\partial e_{22}^c}{\partial x_2} + \frac{\partial e_{12}^c}{\partial x_1}\right) - \frac{2}{1-2\nu} \frac{\partial e_{21}^c}{\partial x_1} \quad (5.23)$$

The symmetry boundary condition, on substitution into the equation 5.16, acquires the form

$$G\left[\left(1 + \frac{1}{1-2\nu}\right) \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_1^2}\right] + \rho b_2$$

$$= \frac{Ev}{(1+\nu)(1-2\nu)} \left(\frac{\partial e_{11}^c}{\partial x_2} + \frac{\partial e_{22}^c}{\partial x_2} + \frac{\partial e_{33}^c}{\partial x_2} \right) + 2G \left(\frac{\partial e_{22}^c}{\partial x_2} + \frac{\partial e_{12}^c}{\partial x_1} \right). \quad (5.24)$$

The above differential equations with boundary conditions are solved using the finite difference method. Central finite differences have been used throughout. A grid with mesh refinement close to the top face has been used.

Although not obvious, these equations are time dependent. The time dependence appears because the creep strains in the above equations change with time. The first step in solving these equations involves finding the elastic solution at time $t=0$. For this the creep terms on the right-hand side are set to zero. The stresses from this elastic solution are then used to calculate the creep strain rates using equations developed in Chapter 4. Once the creep strains have been calculated from these creep rate equations, they are substituted in the above differential equations which are solved for displacements at end of time step. The new stresses at the end of time step are next calculated. Due to the rather complex nature of the creep equations, it is not possible to use implicit methods to integrate the creep strain rate equations and explicit methods have to be used. This, unfortunately, poses a big limitation on the size of time step.

Using Kry's data [Kry, 1975] the stresses in some of the ice necks are about 50 times the stress applied to the snow. Further, the stresses are raised to an exponent of 1.6 or 1.8 to calculate the strains. Therefore, even a small change in the stress applied to the snow can cause large changes in strains and stresses in necks and therefore in snow. For the numerical scheme to be stable, extremely

short time steps have to be used. To overcome this problem the grain data reported by Hansen [1985] was used. For a compressive stress of 8×10^{-3} MPa with a time step of 120 seconds, a steady state of stress is not reached even after 300 iterations, although the rate of change of stress does exhibit a decreasing trend. For instance, the rate of change of largest stress decreases from 1.5×10^{-4} MPa/hr at the beginning to 3×10^{-5} MPa/hr by 300th iteration. Since this rate of change is small, it was decided to assume a steady state of stress after 300 time steps.

For Q equal to 0.1 MPa there are areas under the loading where stresses are high enough to cause the neck fracturing over a very small period of time. On the other hand, there are areas away from loading at which the stresses are low enough that, according to the theory developed, superplastic deformation should be taking place in necks. As has been explained earlier, this leads to a situation where regions of low stress have higher strain rates as compared to regions with higher stresses. Also, this leads to jumps in strain rates. It was therefore assumed that necks remain intact for first fifty iterations, deforming according to the creep law for high stresses. After fifty iterations all necks were assumed to have broken, and relative sliding of particles was taken as the sole deforming mechanism for the whole problem.

At this load it was found that with a time step of 120 seconds the stresses continue to decrease, as was expected, for 150 iterations. However, after that the high stresses change negligibly, but the low stresses, instead of decreasing, begin increasing and the problem

becomes unstable, as indicated by large changes in lateral stresses and a decrease in vertical displacement instead of an increase. It is possible that by use of a still smaller time step this problem can be remedied, but this would also result in increased computational time. This instability seems to be associated with the strain hardening in the constitutive equation becoming significant. Most of the multiaxial creep problems are solved using implicit methods and even then a very short time step has to be used. So the instability with the use of the explicit method for strain hardening problem is not surprising. It was decided to take the stresses at the end of 150 iterations as the stresses to be used for all future iterations. By imposing this restriction we are violating the equilibrium equation. However, the change in maximum stresses by the 150th iteration are rather small and this assumption may not be unreasonable.

For low compressive stress of 8×10^{-3} MPa the normalized stress ($\text{stress}/(8 \times 10^{-3})$) contours are shown in Figure 23 and the displacement contours at the end of sixty hours are shown in Figure 24. A steady state of stress was assumed to have been reached by seven hours. Figures 25 and 26 show normalized stress distribution along a horizontal section. From these and Figure 27 it becomes apparent the stresses are much higher closer to the axis of symmetry and fall to a very low value as one moves away from this axis. Also, the stresses beneath the loaded area at 0.45 m from the bottom are about 65% of the stresses at 0.45 m from top. Figures 27 and 28 illustrate the stress distribution along a vertical section. For a section at the axis of symmetry the stresses first decrease with height and

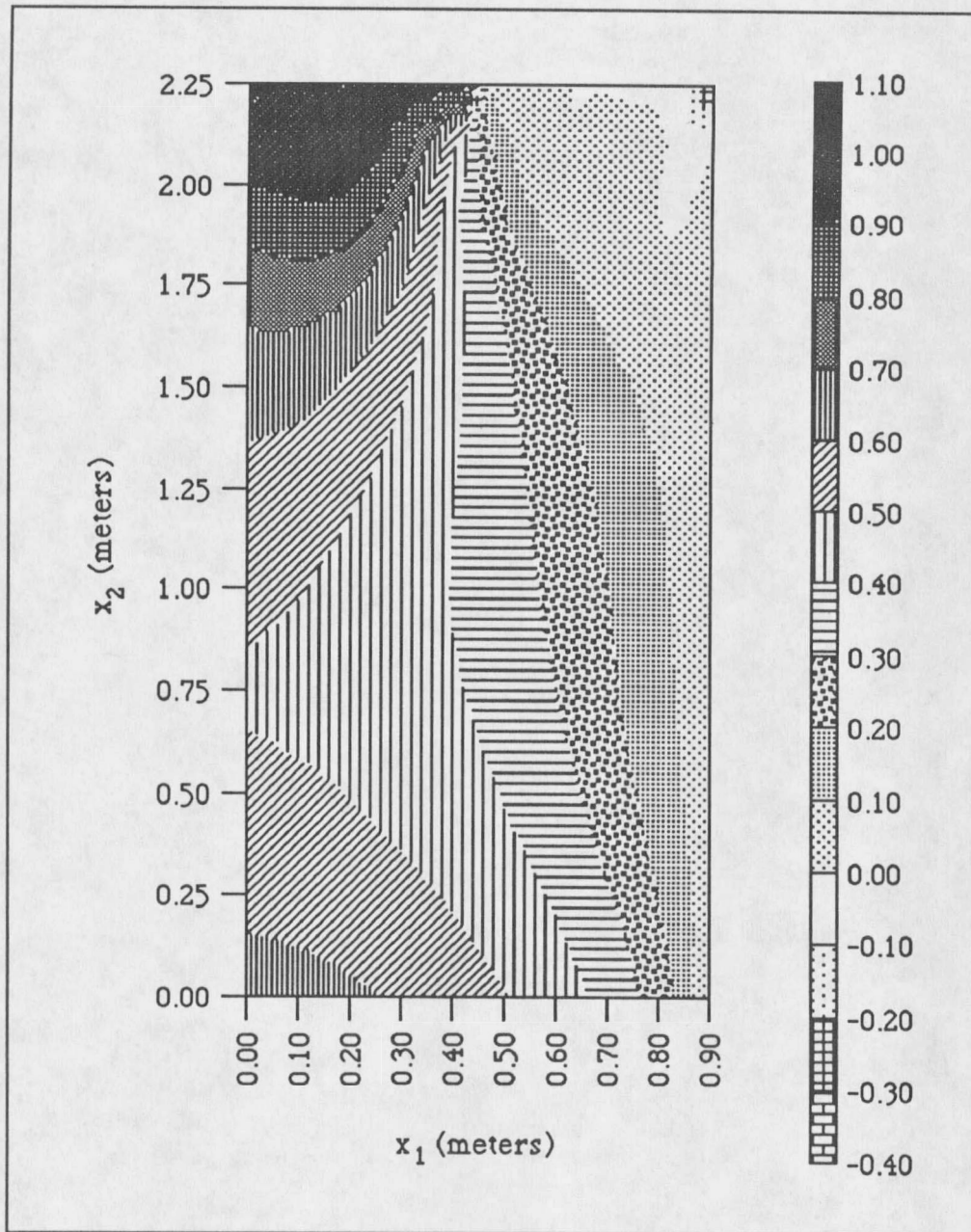


Figure 23. Normalized stress contours by seven hours for $Q=0.008\text{MPa}$. The stresses were assumed to have reached a steady value at this time.

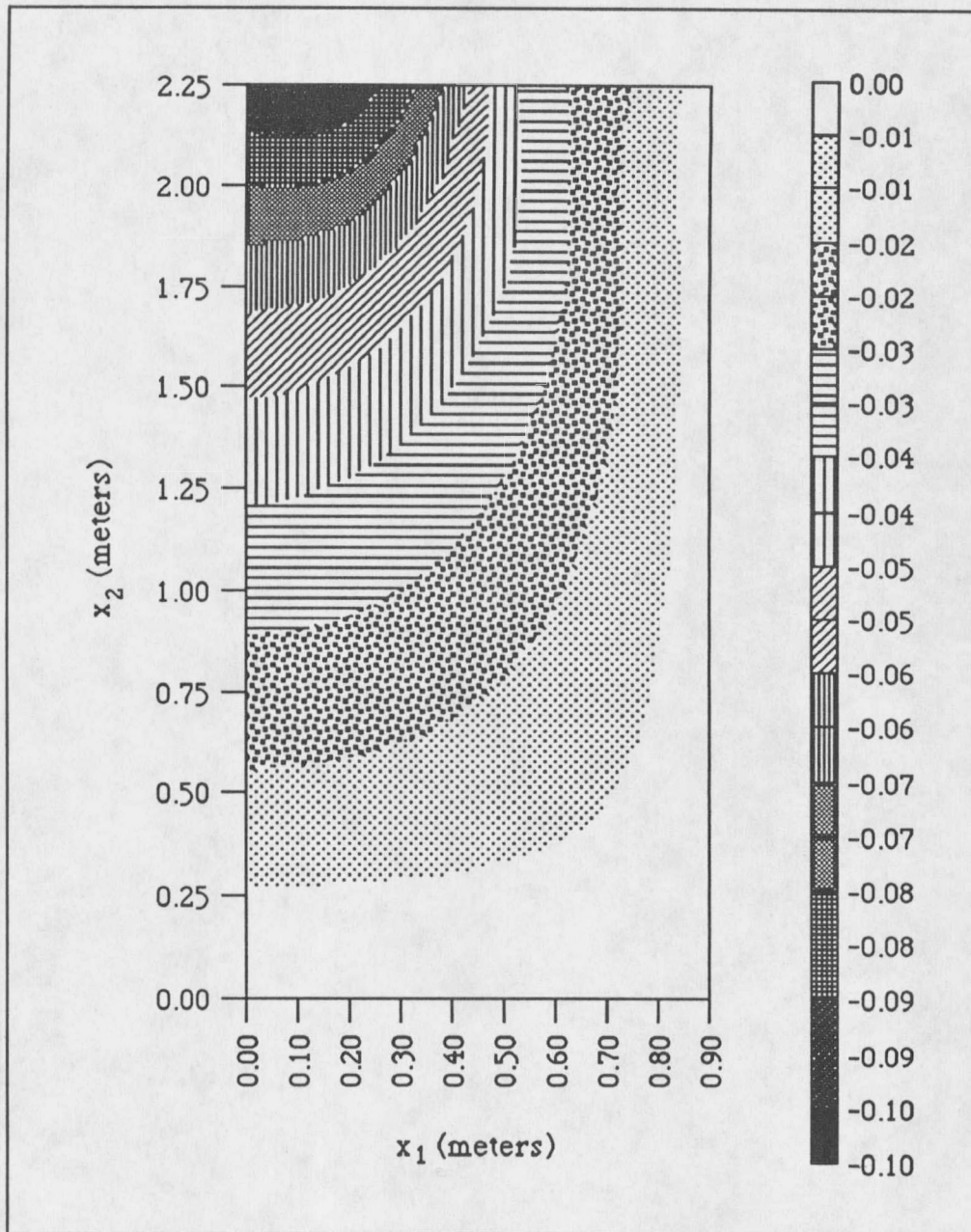


Figure 24. Displacement contours by sixty hours. $Q=0.008$ MPa.

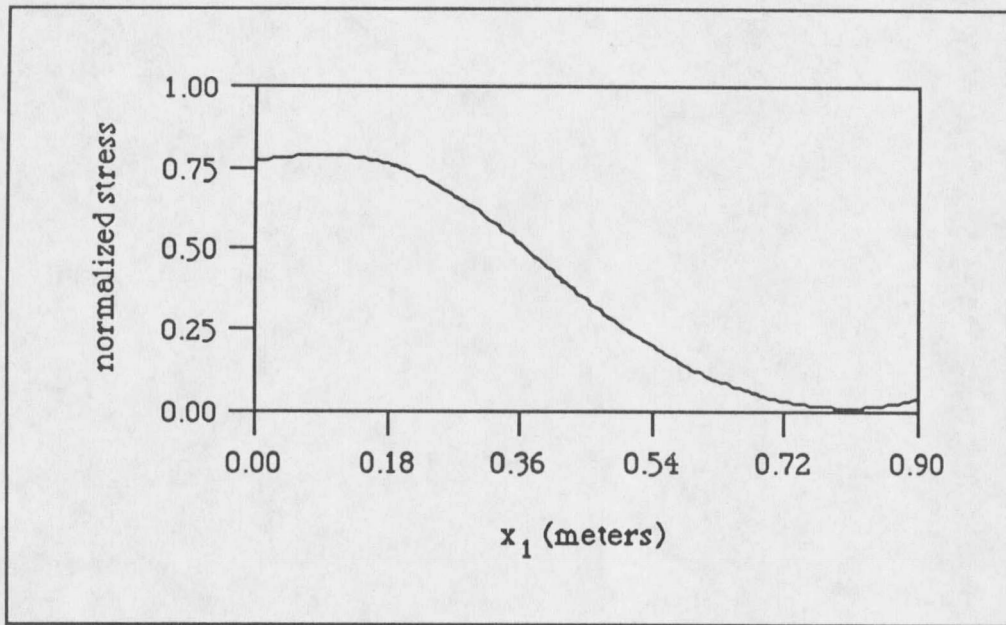


Figure 25. Normalized stress distribution at a horizontal section 0.45 m from the top. $Q=0.008$ MPa.

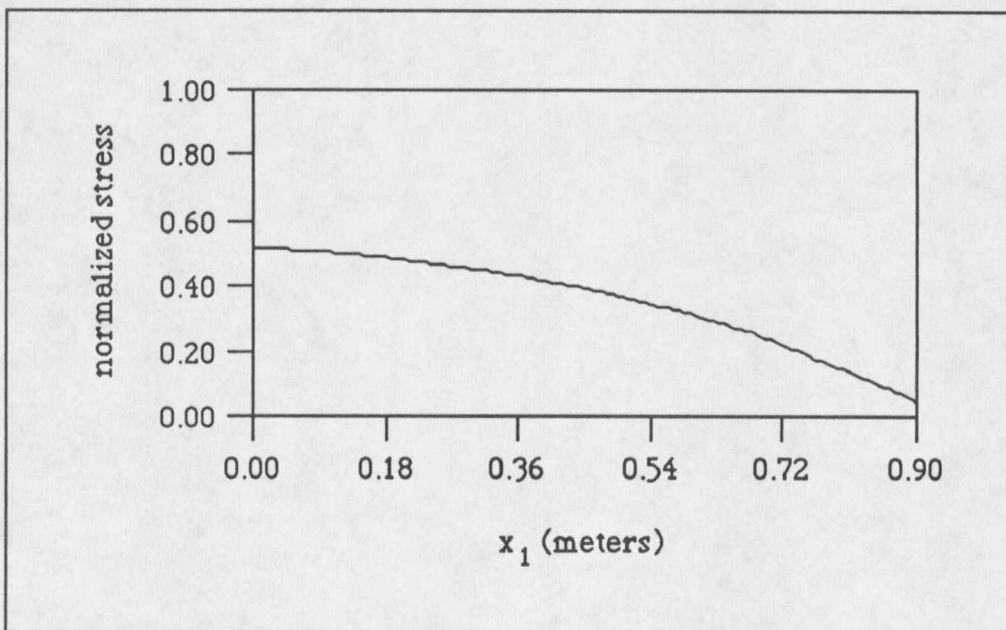


Figure 26. Normalized stress distribution at a horizontal section 0.45 m from the bottom. $Q=0.008$ MPa

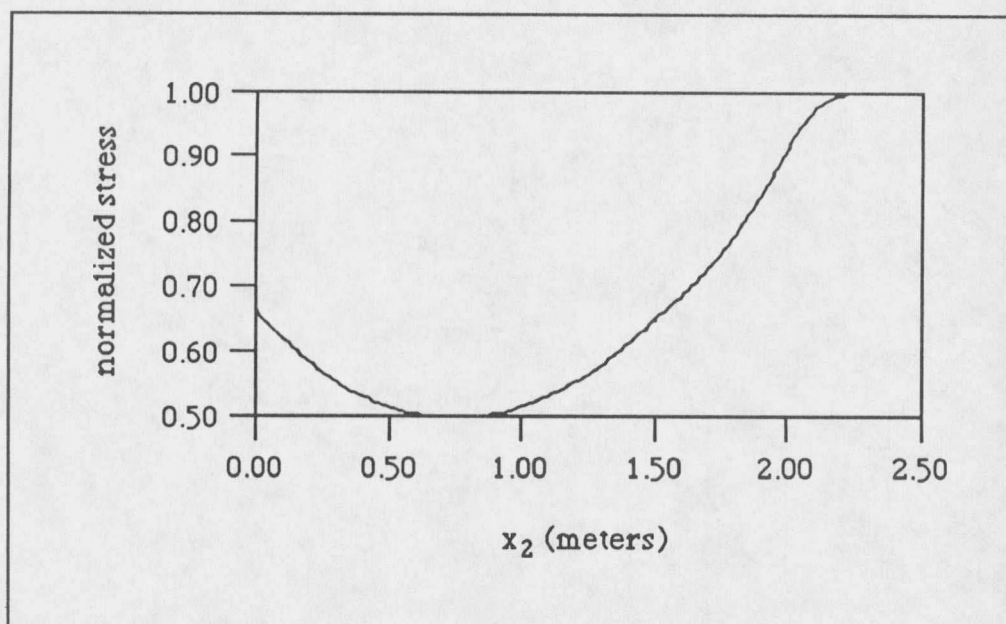


Figure 27. Normalized stress distribution at a vertical section along the axis of symmetry. $Q=0.008$ MPa.

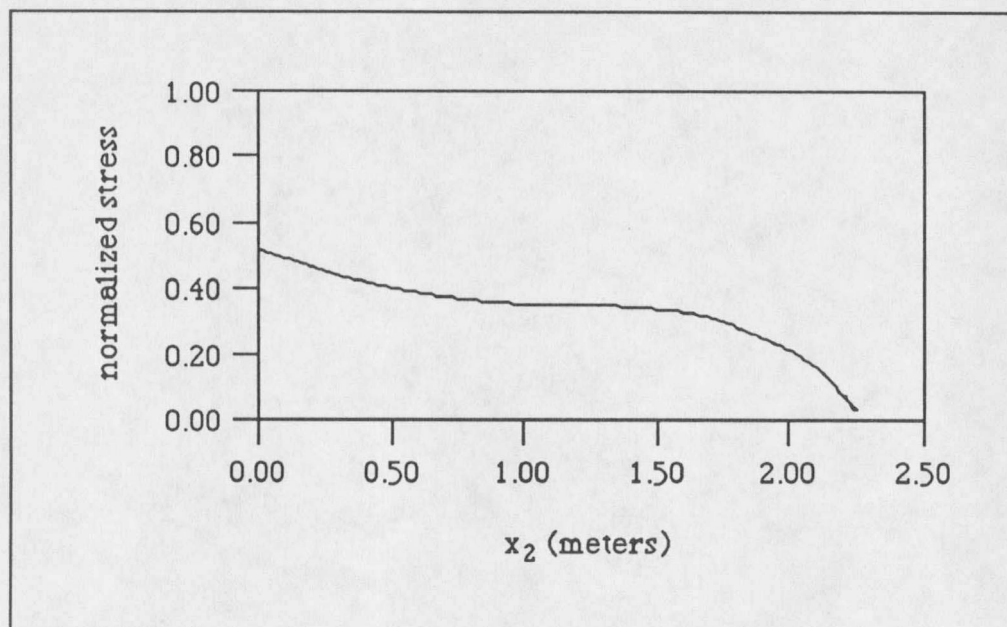


Figure 28. Normalized stress distribution at a vertical section 0.5 m from the axis of symmetry. $Q=0.008$ MPa.

then rise. The stresses are high at the bottom because the body force effect is almost half the value of the applied stress effect. Had the body forces not been taken into account, the stress at the bottom would have been much lower than stress at the top. Figure 28 is plotted for a section 0.5 m from the axis of symmetry. Q is equal to zero at this point. Here the stress at bottom is only about 25% lower than that in Figure 27, probably because the stresses here again are largely due to weight of snow. From Figure 24 it is seen that vertical displacements are highest right under the applied load. At the end of 60 hours the maximum vertical displacement is 0.1 cm. The maximum horizontal displacement is about 0.25 times this value.

For Q equal to 0.1 MPa the body forces are insignificant compared to the external load. The normalized stress contours are shown in Figure 29. The stresses in the section right under the footing show a big drop within the top 0.45 m, as observed in Figure 34. The stresses at 1.6 m are 20% of the value at the top. This is consistent with experimental data, according to which the pressure bulb extends to about two times the width of the footing. Body forces being insignificant compared to load, the stresses at the bottom are less than 10% of the values at the top. This is in sharp contrast to values at the bottom for Q equal to 8×10^{-3} MPa, which were almost 75% of the values at the top. Figures 30 and 31 show normalized stress distribution along two horizontal sections 0.45 m from top and bottom, respectively. Figure 34 shows the stress distribution at a section 0.45 m from axis of symmetry. This shows a fairly constant value for stresses from top to bottom. The displacement contours in

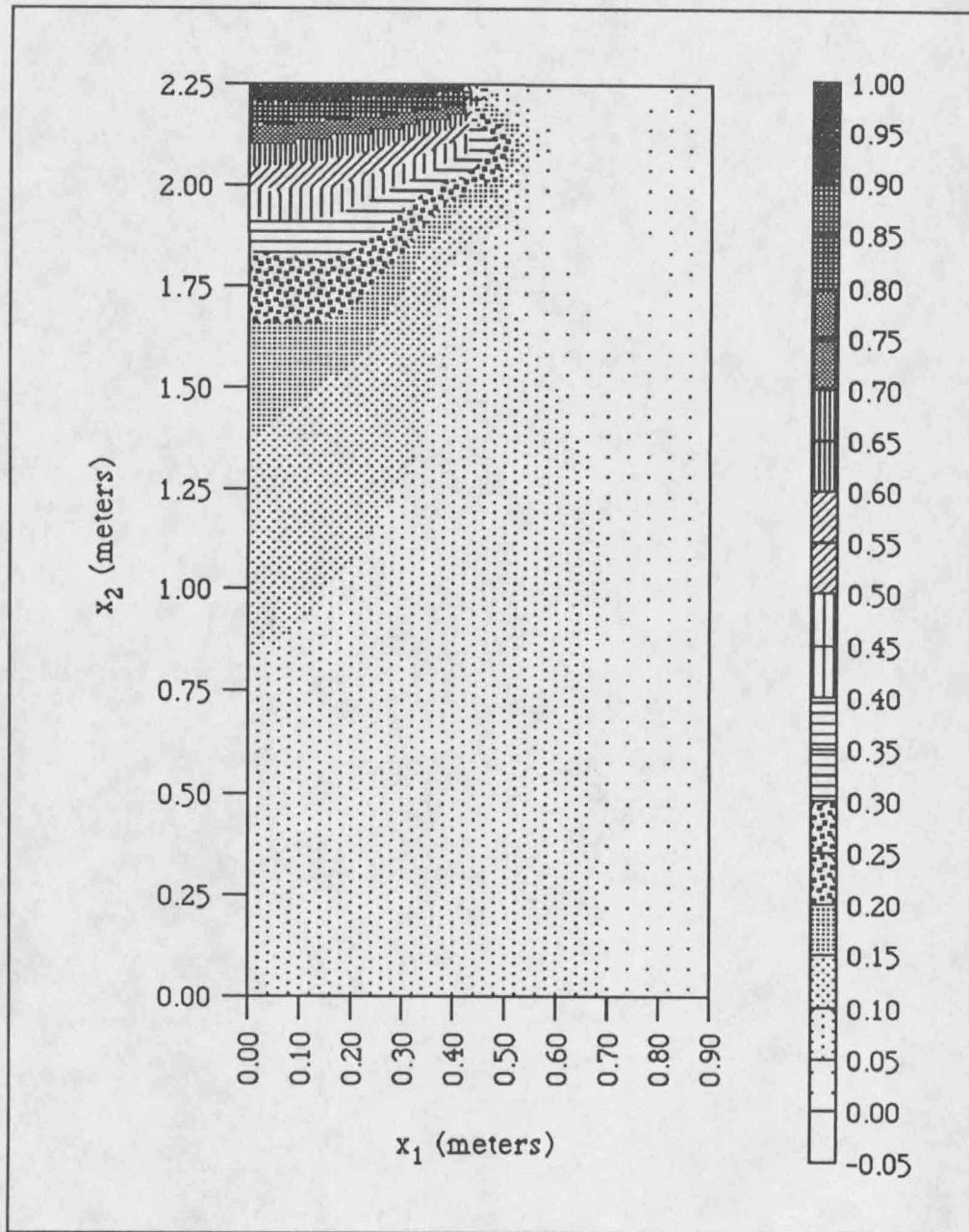


Figure 29. Normalized stress contours by five hours for $Q=0.1$ MPa. The stresses were assumed to have reached a steady value at this time.

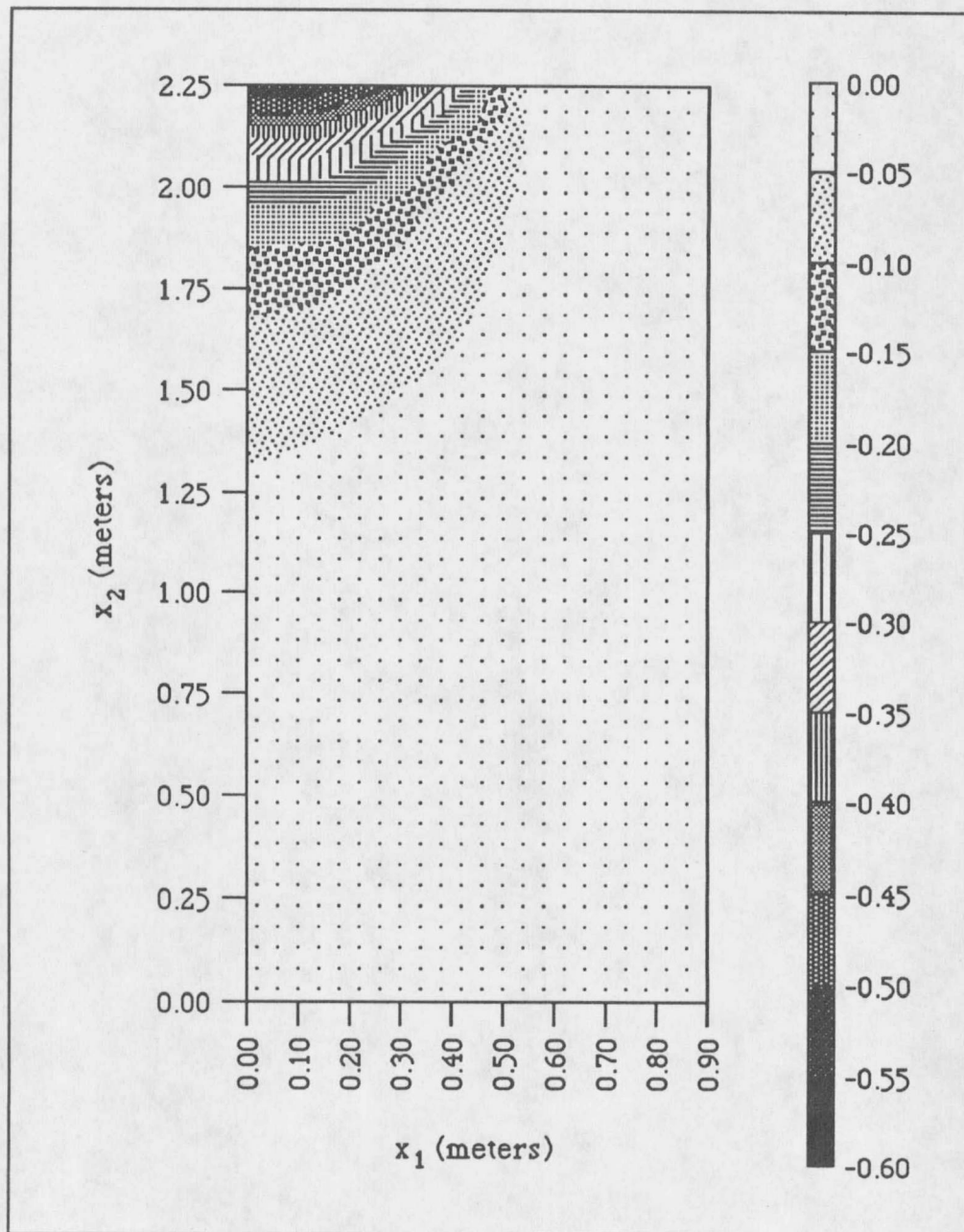


Figure 30. Displacement contours by thirty hours. $Q=0.1$ MPa.

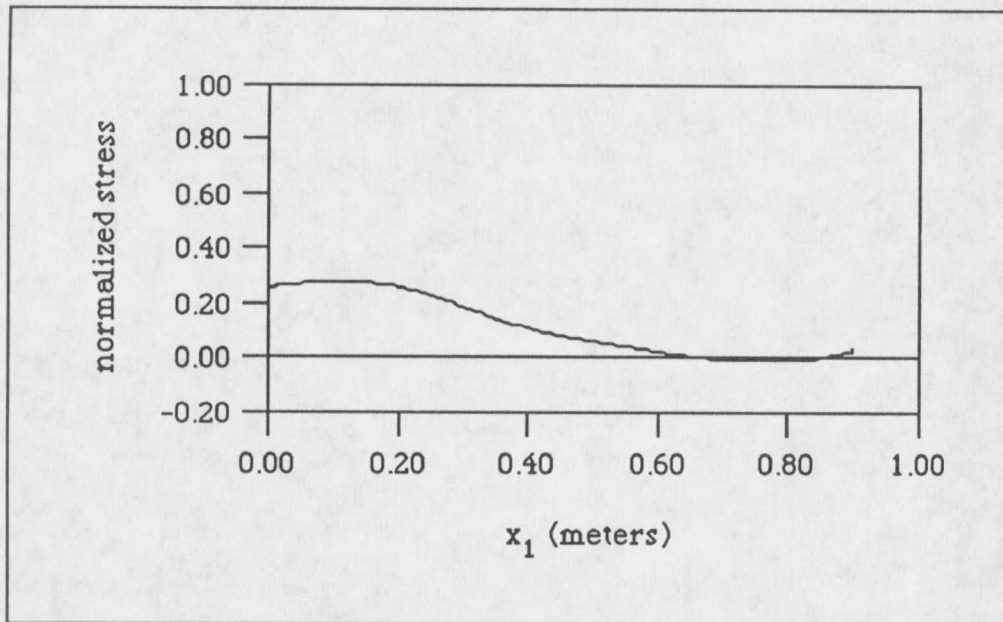


Figure 31. Normalized stress distribution at a horizontal section 0.45 m from top. $Q=0.1$ MPa.

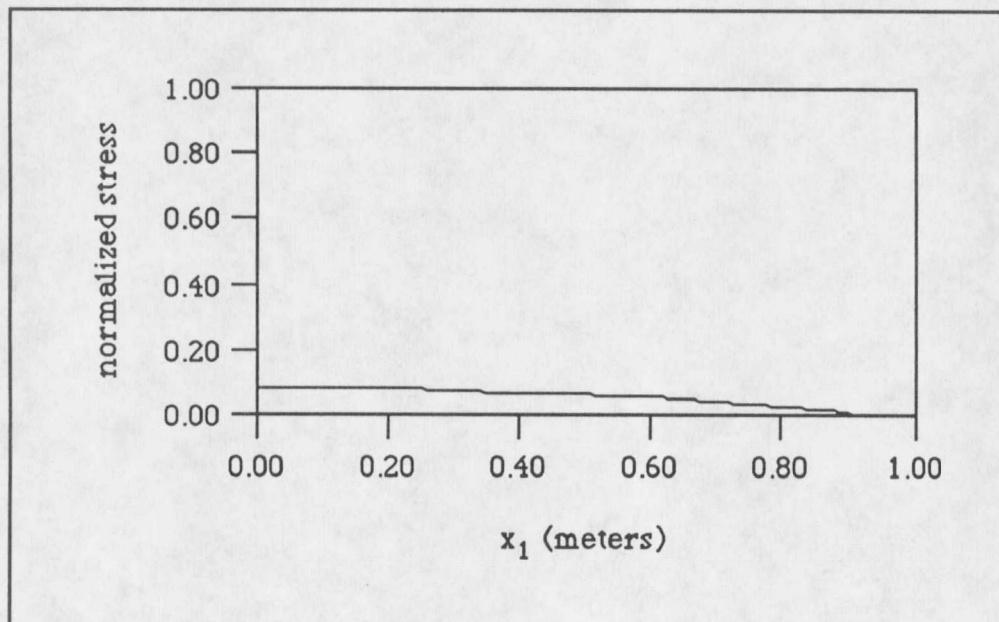


Figure 32. Normalized stress distribution at a horizontal section 0.45 m from the bottom. $Q=0.1$ MPa.

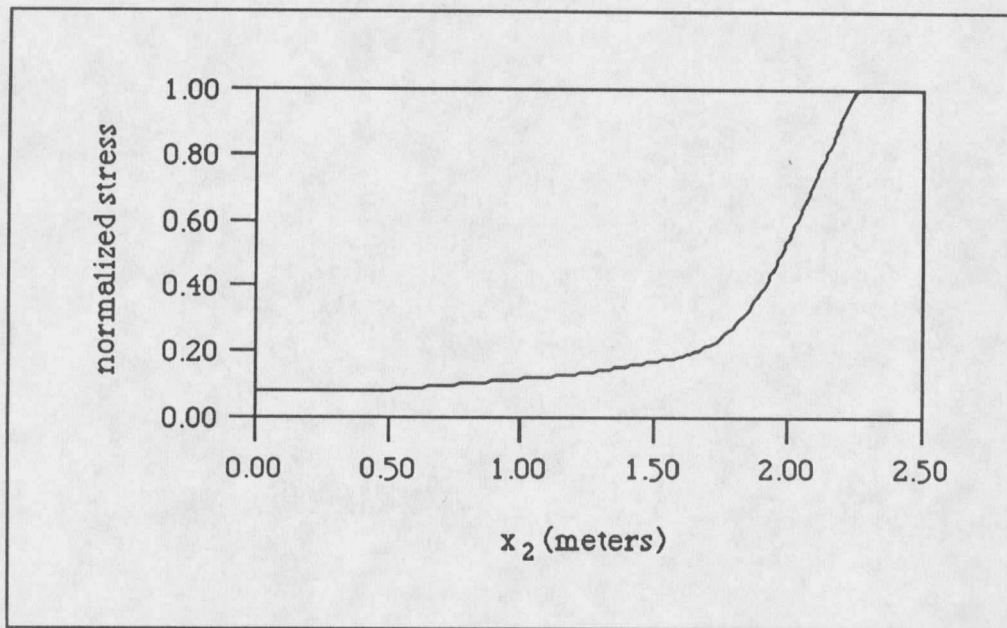


Figure 33. Normalized stress distribution at a vertical section along the axis of symmetry. $Q=0.1$ MPa.

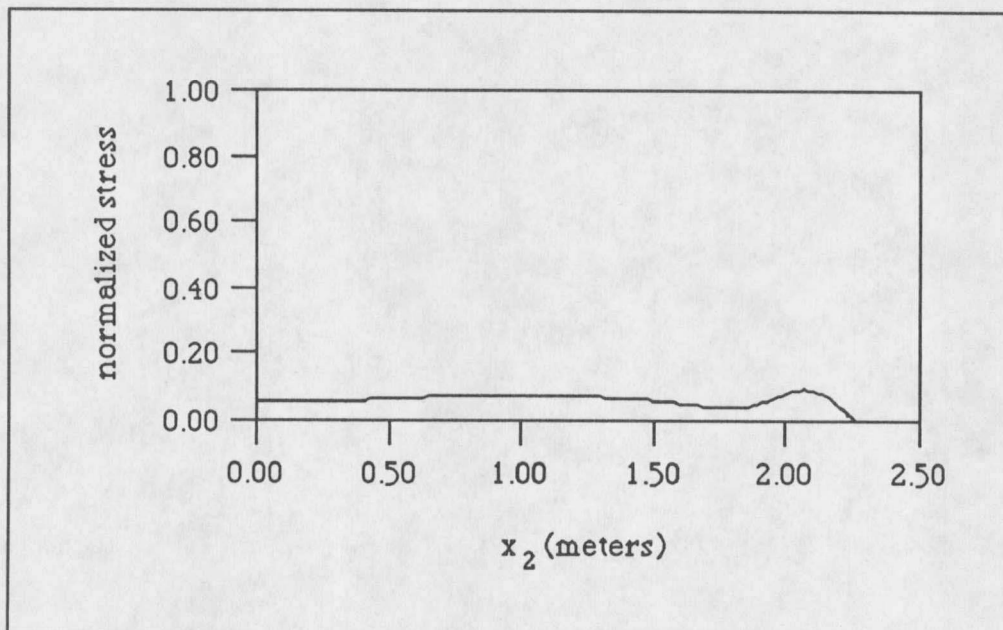


Figure 34. Normalized stress distribution at a vertical section 0.5 m from axis of symmetry. $Q=0.1$ MPa.

Figure 30 show that major displacements are largely restricted to about the top 0.9 m of the foundation. The gradients in vertical displacement in Figure 31 for 8×10^{-3} MPa was more gradual. The maximum displacement at the end of 30 hours was 0.57 cms.

Dandekar [1982] has solved a similar problem using a linear stress-strain rate relation. He takes only steady state creep into account and uses his theory to predict long time behavior of snow. The normalized stress distribution contours for a compressive load of 0.1 MPa shown here are similar to Dandekar's steady state stress contours.

CHAPTER 6

DISCUSSION

A theory has been proposed to explain the multiaxial deformation of snow. The theory is limited to small strains and much more experimental work needs to be done before it can be extended to large strains.

Summary

Snow is made up of ice particles connected by necks. The principle of virtual work was used to calculate the stresses in the ice necks. At low stresses superplastic deformation of these necks was recognized as the mechanism responsible for strains in snow. An existing multiaxial constitutive law for ice was used to model the superplastic deformation of the necks. However, the values of various parameters in the existing model had been predicted on the basis of experiments on polycrystalline ice undergoing dislocation creep. The parameters were adjusted by comparing the results from the theory with the experimental data for a single experiment. The theory was checked against other experimental data and was shown to work well. The theory is able to predict correctly the Young's modulus, strains and ratio of lateral to axial strains in snow. It was

assumed that lateral strains in ice necks do not contribute to the strains in snow.

At high stresses necks break and there is relative sliding between the particles. The rate of sliding and angle of sliding depend on the stresses at the contact where the particle is located. The angle of sliding should be determined by the plane of fracture of ice necks. The constitutive equations of ice are not advanced enough to give this information, and therefore an empirical relation relating the sliding rate to the stresses at the contact points was assumed. The constants were adjusted to make the results from the theory match the experimental data for a single experiment and then checked against other experimental data.

The constants were found to be different for compression and tension. The difference is more prominent for cases where the sliding of grains is the predominant mechanism.

The theory was then applied to two practical problems. Some numerical difficulties were observed while trying to solve problems in which strain hardening became significant. For these applications constants for tension were taken to be the same as those for compression.

Suggestions for Future Work

In this theory, it has been assumed that stresses in necks, coordination number and density ratio do not change. As necks deform or particles undergo sliding there is likely to be some change in the coordination number and stress. In soils and sand the

coordination number is often related to density by an empirical relation. In these materials deformation is time independent, and therefore time is not a variable. The relation between coordination number, density and time for snow needs to be developed to get a better understanding of the deformation mechanism.

The orientation of particles may change with stress state and time. If that is the case, the assumption of isotropic distribution of snow particles cannot be made and a different probability distribution function would have to be chosen. This is likely to lead to the noncoaxiality of stress and strain rates in plane shear, a phenomenon commonly observed in granular material [Mandl and Luque, 1970]. In granular materials the principal stress is seen to rotate as the material becomes increasingly oriented in the direction of shear [Oda and Konishi, 1974].

Some improvement in the theory will also take place as a better comprehension of mechanical behavior of ice evolves. Research in superplasticity should help in modelling the behavior of ice at low stresses more accurately. Extensive multiaxial experimental testing of ice needs to be done to obtain a better understanding of the dependence of the potential function, used to derive the strains, on the third invariant of stress. Progress in understanding of fracture in ice will help in a better determination of angle of sliding of broken necks. In this work we assumed that if strains exceed a particular value, fracturing of necks takes place. If, however, the Griffith criterion [Mellor, 1983] for fracturing is used, the fracturing of necks may not take place as long as stresses in

necks are below a certain critical value, immaterial of what the neck strains are. Since neither superplasticity nor interparticle sliding can then be used to explain the strain rates in snow, it is not clear as to what phenomenon high strain rates can be ascribed. Also, very little experimental data exists on sliding of ice on ice. If these phenomena can be comprehended better, the empirical relations used for sliding can be replaced by more sound mathematical relations. It should then be possible to explain better the experimentally observed variation in ratio of lateral to axial strains.

Grain boundary sliding seems to play a major role in deformation of snow at low stresses. To understand its role in deformation of necks, it needs to be determined whether the necks are polycrystalline or not. It may then be possible to explain the values of some of the coefficients used.

The major drawback of the theory in its present form is the excessive computation time and the instability of solutions once strain hardening comes into play. Therefore, the present theory should be used to derive a simpler constitutive law by expressing strain rates as a function of stress, invariants of stress, invariants of strain and $a^2/\gamma A_t$. This can be done by choosing a work potential which is a function of the above mentioned invariants and following a procedure similar to the one suggested by Lubliner [1972]. The components of the strain tensor correspond to the internal variables in his theory. However, finding the correct form for this potential function is not simple, as this involves a nonlinear multivariable regression analysis. It becomes even more complicated if anisotropy

due to the orientation of particles is introduced into the formulation. An additional vector or tensor function would have to be introduced as one of the independent variables in the description of strain [Boehler, 1987]. The data required for the regression analysis can be obtained from the present theory. If a simpler relation is achieved, then it may be possible to reduce the computation time. It should also be possible to use the implicit method for integrating the differential equation for strain rate. This may alleviate the stability problems associated with integration.

Conclusion

The major advantage of the theory is its ability to predict multiaxial behavior with experimental data from a few uniaxial tests. The principal phenomenon underlying uniaxial and multiaxial deformation are the same. It was found that, depending on stress level, various deformation mechanisms may be responsible for the behavior of snow. Once a better understanding of some of the phenomena is gained, modifications in the theory should lead to an even better prediction of snow behavior while eliminating some of the empirical relations used.

NOTATION AND NOMENCLATURE

Notation

In general, except where explicitly noted in the text, bold upper case symbols indicate second order tensors, bold lower case symbols indicate vectors and scalars are represented by standard print symbols.

Nomenclature

a	radius of particle
A_T	total area of contact
B	inverse of Cauchy deformation tensor
c₁	constant in equation for interparticle sliding
c₂	constant in equation for interparticle sliding
C	Green deformation
D	Probability density function
D	rate-of-deformation
E	Young's modulus
e	components of strain
e^c	delayed elastic strain
e^v	viscous strain
E	Lagrangian strain
E*	Eulerian strain
f	body force
F	function representing viscoelastic properties
F	deformation gradient
G	shear modulus
h	angular momentum

j	viscous compliance
J	displacement gradient
L	velocity gradient
M	mass
n	normal in particle coordinates
N	coordination number
p	load on cantilever beam
p	linear momentum
P	probability of contact
Q	load on the foundation
R	rotation
r	components of rotation
s	deviatoric stress
t	time
t	traction vector
T	stress tensor
tau	shear stress on broken necks
tnor	normal stress on broken necks
u ₁	horizontal displacement of foundation
u ₂	vertical displacement of foundation
U	left stretch
v	velocity
V	volume
V	right stretch
w	cantilever tip deflection
W _e	external work

W_i	internal work
x	position vector in spatial coordinates
X	position vector in referential coordinates
γ	solid volume fraction
ζ	displacement of particle
η	tangential component of sliding
κ	curvature
ξ	normal component of sliding
ρ	density
σ	traction vector at neck
σ	component of stress tensor at neck
$\tilde{\sigma}$	component of effective stress
$\tilde{\sigma}'$	component of effective stress, in the spring
$\tilde{\sigma}''$	component of effective stress in the dashpot
ν	Poisson ratio
$\mathbf{n}$	normal at the neck
ν_1	constant for delayed elasticity in ice
ν_2	constant for viscous creep in ice

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APPENDIX

Figure 35. Computer program for the settlement of foundation.

```

C FOUNDATION PROBLEM
C GRID REFINING CLOSE TO THE TOP FACE
C MATRIX INVERSION FOR APPLICATION OF CONSTITUTIVE LAW
C VARIABLES
C   STIN STRESS ON SNOW
C   T-STRESS IN THE ICE GRAIN
C   TINC INCREMENT OF STRESS IN THE ICE GRAIN
C   EPS-STRAIN IN SNOW USED FOR CALC DEFORMATION GRAD
C   DEFG-DEFORMATION GRAD USED FOR FINDING THE DENSITY OF SNOW
C   KONTER INDEX USED TO DIFFERENTIATE BETWEEN NECKS AND SLIDING
C   GRAINS
C   KKOUNT NO. OF TIME STEPS
C   KK1=KKOUNT
C   STARTX,STARTY STARTING ANGLES FOR AN OCTANT
C   CCRE CREEP STRAIN
C   PCRE CREEP STRAIN AT PREVIOUS TIME LEVEL
C   PESIGM STRESS AT PREVIOUS TIME STEP. FOR CAL ELASTIC
C   STRAINS
C   EPSAV INCREMENTAL STRAIN IN LOCAL COORD.
C   PLS PLASTIC STRAIN
C   PPLS PLASTIC STRAINS AT PREVIOUS TIME STEP
C   KONT
C   DELEPS
C   SIGP,SIGPP USED IN CREEP CAL
C   ESIGM STREE ON NECKS
C   SIGO DEVIATORIC STRESS IN NECKS
C   CRESIM SAME AS ESIGM BUT IS 1-D
C   RESIGM INCREMENT IN NECK STRESS

```

```

COMMON/BLOCK1/T(9),STIN(9),RO,TINC(9),KONTER,KK1
COMMON/BLOCK3/RLEN,DIA,RNEC,TOTL,DR,CN,CONC,RN,RJ,RNU1
COMMON/BLOCK7/DCP(3,3,3,3,8,9),DENO(8,9)
COMMON/BLOC11/EPS(3,3),DISS(8,9,3,400)
COMMON/CREEP2/CCRE(8,9,3,400),RSIGP(8,9,3,400),PCRE(8,9,3)
COMMON/DISP2/PLS(8,9,3,400),PPLS(8,9,3),KONT(8,9,400)
COMMON/DISP/PESIGM(8,9,3,400),EPSAV(8,9,4)
COMMON/DISP4/TOTSTR(8,9,4,400),IFRACT(8,9,400)
COMMON/DISP10/LOAD,EFFST(400),CNOD(400),EFFRAT(400)
DIMENSION C(640,640)
DIMENSION Y(640,640),INDX(640)
DIMENSION RHS(640),DISP(640)
DIMENSION EPSX(320),EPSY(320),EPSXY(320)
DIMENSION U(320),V(320),DECR(3,3)
DIMENSION SIGX(320),SIGY(320),SIGZ(320),SIGXY(320)
DIMENSION CREX(320),CREY(320),CRXY(320),CREZ(320)
DIMENSION PEPSX(320),PEPSY(320),PEPXY(320)
DIMENSION DSIGX(320),DSIGY(320),DSIGXY(320),DSIGZ(320)
DIMENSION DUMEPS(3,3,2,400)

```

```
DIMENSION DECX(320),DECY(320),DECXY(320)
DIMENSION PRINC(400)
```

```
C STR1.DAT STORES STRESS, DIS.DAT DISPLACEMENTS, CRE.DAT STRAINS
C MATIN.DAT STORES INVERSE OF THE MATRIX
  OPEN (3,FILE='STR1.DAT',STATUS='NEW')
  OPEN(5,FILE='DIS.DAT',STATUS='NEW')
  OPEN (6,FILE='CRE.DAT',STATUS='NEW')
  OPEN(4,FILE='MATIN.DAT',STATUS='OLD',FORM='UNFORMATTED')
```

```
C READ GRAIN DATA
C AREA OF GRAIN=2.095
C TOTAL AREA OF NECKS=.339 SQ MM
C RLEN LENGTH OF NECKS
C DIA DIAMETER OF GRAIN
C RNEC RADIUS OF NEC
C CN COORDINATION NUMBER
C CONC CONCENTRATION FACTOR
C RN INDEX FOR STRESS STRAIN RELATION
C RJ AND RNU1 CONSTANTS FOR REVERSIBLE CREEP
  PI=3.14159
  OPEN(11,FILE='GRAIN1.DAT',STATUS='OLD')
  READ(11,*)RLEN,DIA,RNEC,TOTL,DR,CN,CONC,RN
  READ(11,*)RJ,RNU1
  CLOSE(11)
  AREA=CN*PI*RNEC**2
  DUD=(DIA/2)**2/AREA
  DUD1=DUD*DR*CONC
  KKOUNT=1
```

```
  SNOYOU=269.
  RNU=.2
  KKOUNT=1
  BULK=SNOYOU/(3.0*(1-2*RNU))
  RMU=SNOYOU/(2*(1+RNU))
  RLAM=(3*BULK-2.*RMU)/3.
  GRAV=-9.81
  RHO=917.0/DR*1E-6
C INITIALIZE EPS AND DECR
  DO 5 I=1,3
  DO 5 J=1,3
  EPS(I,J)=0.0
  DECR(I,J)=0.
5  CONTINUE
```

```
  OPEN(12,FILE='STRIN1.DAT',STATUS='OLD')
  DO 10 I=1,9
    READ(12,*)STIN(I)
    T(I)=STIN(I)
10  CONTINUE
```

```

DO 20 I=1,3
  WRITE(5,*)STIN(I),STIN(I+1),STIN(I+2)
  WRITE(3,*)STIN(I),STIN(I+1),STIN(I+2)
20 CONTINUE
  WRITE(5,*)RLEN,DIA,RNEC,TOTL,DR,CN,CONC,RN
  WRITE(3,*)RLEN,DIA,RNEC,TOTL,DR,CN,CONC,RN
C NJ TOTAL NUMBER OF NODES
C N NUMBER OF NODES IN Y DIRECTION
  NJ=320
  N=16
  DELTX=.15
  DELTY=.06
  RLOAD=T(1)
  WRITE(5,*)RLOAD,'DX=',DELTX,'DY=',DELTY,NJ
  WRITE(3,*)RLOAD,'DX=',DELTX,'DY=',DELTY,NJ
  CON1=RLAM+2*RMU
  DO 10 J = 1,NJ
    IF(J.GT.256)THEN
      DELTX=.00125
    ELSE
      DELTX=.15
    END IF

    A=2*(1-RNU)/(1-2*RNU)*4*DELTY**2
    B=1/(1-2*RNU)*DELTX*DELTY
    D=4.*DELTX**2
    C0=2*(A+D)
    D1=4.*DELTY**2
    A1=(1.+1./(1.-2.*RNU))*4*DELTX**2
    B1=1./(1.-2.*RNU)*DELTX*DELTY
    C1=2.*(A1+D1)

  C TOP CENTRE
    IF(J.EQ.NJ-N+1)THEN
      A5=(1.+1./(1.-2.*RNU))*DELTY**2
      C2=DELTX/DELTY*RLAM/(RLAM+2*RMU)
      B5=DELTX**2
      K=2*J-1
      C(K,2*J-2*N-1)=2*A5
      C(K,2*J-2*N)=0.
      C(K,2*J-1)=-2*(A5+B5)
      C(K,2*J)=0
      C(K,2*J+1)=2*B5
      C(K,2*J+2)=-C2*A5*2

      K=2*J
      B6=4.*DELTX**2*(1.-RNU)/(1.-2.*RNU)
      A6=2.*DELTY**2
      C(K,2*J)=1

    ELSE IF(J.GE.NJ-N+1.AND.J.LT.NJ)THEN

```

CTOPFACE

```

A2=(1.+1./(1.-2.*RNU))*DELTY**2
B2=(1.-1./(1.-2.*RNU))*DELTX**2
C2=DELTX/DELTY*RLAM/(RLAM+2*RMU)
A3=(1.-1./(1.-2.*RNU))*DELTY**2
B3=(1.+1./(1.-2.*RNU))*DELTX**2

K=2*J-1
C(K,2*J-3)=B2
C(K,2*J-2)=C2*A2
C(K,2*J-1)=-2*(A2+B2)
C(K,2*J)=0
C(K,2*J+1)=B2
C(K,2*J+2)=-C2*A2
C(K,2*J-2*N-1)=2*A2
C(K,2*J-2*N)=0
K=2*J
C(K,2*J-3)=A3*DELTX/DELTY
C(K,2*J-2)=B3
C(K,2*J-1)=0
C(K,2*J)=-2*(A3+B3)
C(K,2*J+1)=-A3*DELTX/DELTY
C(K,2*J+2)=B3
C(K,2*J-2*N)=2*A3

```

C AXIS OF SYMM

```

ELSE IF(J.GT.N.AND.MOD(J,N).EQ.1)THEN
ALP=.00125/.15

```

```

K=2*J-1
A4=(1.+1./(1.-2.*RNU))*DELTY**2
IF(J.EQ.241)THEN
  C(K,2*J+2*N-1)=A4*2/(ALP)/(1+ALP)
  C(K,2*J-2*N-1)=A4*2/(1+ALP)
  C(K,K)=-2*(DELTX**2+A4/(1+ALP)*(1/ALP+1.))
  C(K,2*J+1)=2*DELTX**2
ELSE
  C(K,2*J-2*N-1)=A4
  C(K,2*J-1)=-2*(A4+DELTX**2)
  C(K,2*J+1)=(DELTX**2)*2
  C(K,2*J+2)=0
  C(K,2*J+2*N-1)=(1.+1./(1.-2.*RNU))*DELTY**2
END IF
K=2*J
C(K,2*J)=1.

```

CFIXED FACE

```

ELSE IF(J.GT.N.AND.MOD(J,N).EQ.0)THEN
  A4=(1.+1./(1.-2.*RNU))*DELTY**2
  B2=(1.-1./(1.-2.*RNU))*DELTX**2
  C2=DELTX/DELTY*RLAM/(RLAM+2*RMU)

```

```

K=2*J-1
C(2*J-1,2*J-1)=1.
C(2*J,2*J)=1

```

C INNER GRID POINTS

```

ELSE IF(J.GT.(N+1).AND.J.LE.NJ-N+1)THEN
  IF (J.GT.241.AND.J.LT.256)THEN

```

```

    ALP=.00125/.15
    K=2*J-1
    C(K,2*J-3)=D
    C(K,2*J+1)=D
    C(K,K)=-2*((A)/(ALP+1)*(1+1/ALP)+D)
    C(K,2*J-2*N-1)=2/(1+ALP)*A
    C(K,2*J+2*N-1)=(2/(ALP+1)/ALP)*A
    C(K,2*J+2*N+2)=2/ALP/(1+ALP)*B
    C(K,2*J+2*N-2)=-2/ALP/(1+ALP)*B
    C(K,2*J+2)=2*B/(1+ALP)*(ALP-1/ALP)
    C(K,2*J-2)=-2*B/(1+ALP)*(ALP-1/ALP)
    C(K,2*J-2*N+2)=-2*ALP/(1+ALP)*B
    C(K,2*J-2*N-2)=2*ALP/(1+ALP)*B
    K=2*J
    C(K,2*J)=-2*(A1+D1*(1+1/ALP)/(1+ALP))
    C(K,2*J+2*N)=2/(1+ALP)/ALP*D1
    C(K,2*J-2*N)=2/(ALP+1)*D1
    C(K,2*J+2)=A1
    C(K,2*J-2)=A1
    C(K,2*J+2*N+1)=2/ALP/(1+ALP)*B
    C(K,2*J+2*N-3)=-2/ALP/(1+ALP)*B
    C(K,2*J+1)=2*B/(1+ALP)*(ALP-1/ALP)
    C(K,2*J-3)=-2*B/(1+ALP)*(ALP-1/ALP)
    C(K,2*J-2*N+1)=-2*ALP/(1+ALP)*B
    C(K,2*J-2*N-3)=2*ALP/(1+ALP)*B

```

```

ELSE

```

```

    K=2*J-1
    C(K,2*J-2*N-1)=A
    C(K,2*J-3)=D
    C(K,2*J-1)=-C0
    C(K,2*J+2*N-1)=D
    C(K,2*J+1)=A
    C(K,2*J+2*N+2)=B
    C(K,2*J-2*N-2)=B
    C(K,2*J-2*N+2)=-B
    C(K,2*J+2*N-2)=-B
    K=2*J
    C(K,2*J-2*N)=D1
    C(K,2*J-2)=A1
    C(K,2*J)=-C1
    C(K,2*J+2)=A1
    C(K,2*J+2*N)=D1
    C(K,2*J-2*N-3)=B1
    C(K,2*J-2*N+1)=-B1

```

```

      C(K,2*J+2*N-3)=-B1
      C(K,2*J+2*N+1)=B1
    END IF

    ELSE IF(J.GE.1)THEN
      C(2*J-1,2*J-1)=1
      C(2*J,2*J)=1
    END IF
10    CONTINUE

C INVERT THE MATRIX USING LU DECOMPOSITION
C LUDCMP - LU DECOMPOSITION
C LUBKSB - BACK SUBSTITUTION
    NN=2*NJ
    NP=640
    DO 12 I=1,NN
    DO 11 J=1,NN
      Y(I,J)=0.
11    CONTINUE
      Y(I,I)=1.
12    CONTINUE
    CALL LUDCMP(C,NN,NP,INDX,D)
    DO 13 J=1,NN
      CALL LUBKSB(C,NN,NP,INDX,Y(1,J))
13    CONTINUE
    WRITE(4)Y
    CLOSE(4)

    DO 1000 KOUNT=2,900

C RIGHT SIDE MATRIX
    DO 40 J=1,NJ
      IF(J.LE.256)THEN
        DELTX=.15
      ELSE
        DELTX=.00125
      END IF

C TOP CENTRE
    IF (J.EQ.NJ-N+1)THEN
      A5=(1.+1./(1.-2.*RNU))*DELTY**2
      K=2*J-1
      RH1=-(-3*CREX(J)+4*CREX(J-N)-CREX(J-2*N))/(2*DELTXX)
      RH1=RH1*CON1
      RH2=-(-3*CREY(J)+4*CREY(J-N)-CREY(J-2*N))/(2*DELTXX)
      RH4=-(-3*CREZ(J)+4*CREZ(J-N)-CREZ(J-2*N))/(2*DELTXX)
      RH2=(RH2+RH4)*RLAM
      RH3=(-3*CRXY(J)+4*CRXY(J+1)-CRXY(J+2))/(DELTY)*RMU
      RH=(RH1+RH2+RH3)*DELTXX**2*DELTY**2/RMU
      RLOAD1=RLOAD+CON1*CREX(J)+RLAM*CREY(J)
      RHS(2*J-1)=-A5*2.*DELTXX/(RLAM+2.*RMU)*(RLOAD1)+RH
    
```

RHS(2*J)=0

C TOPFACE

```

ELSE IF(J.GT.NJ-N+1.AND.J.LT.NJ)THEN
  CON2=(-RNU)/(1-2*RNU)*RMU*2
  A2=(1.+1./(1.-2.*RNU))*DELTY**2
  A3=(1.-1./(1.-2.*RNU))*DELTY**2
  K=2*J-1
  RH1=(-(3*CREX(J)+4*CREX(J-N)-CREX(J-2*N))/(2*DELTX)
  RH1=RH1*(RLAM+2*RMU)
  RH2=(-(3*CREY(J)+4*CREY(J-N)-CREY(J-2*N))/(2*DELTX)
  RH4=(-(3*CREZ(J)+4*CREZ(J-N)-CREZ(J-2*N))/(2*DELTX)
  RH2=(RH2+RH4)*RLAM
  RH3=(CRXY(J+1)-CRXY(J-1))/(DELTY)*CON2
  RH=(RH1+RH2+RH3)*DELTX**2*DELTY**2/RMU
  IF(J.LT.NJ-N+1+8)THEN
    RLOAD1=RLOAD
  ELSE
    RLOAD1=0.
  END IF
  RLOAD1=(RLAM+2*RMU)*CREX(J)+RLAM*CREY(J)+RLOAD1
  RHS(K)=-A2*2.*DELTX/(RLAM+2*RMU)*RLOAD1+RH
  K=2*J
  IF(J.EQ.NJ)THEN
    RH1=(3*CREX(J)-4*CREX(J-1)+CREX(J-2))/(2*DELTY)
    RH4=(3*CREZ(J)-4*CREZ(J-1)+CREZ(J-2))/(2*DELTY)
    RH1=(RH1+RH4)*RLAM
    RH2=(3*CREY(J)-4*CREY(J-1)+CREY(J-2))/(2*DELTY)
    RH2=CON1*RH2
  ELSE
    RH4=RLAM*(CREZ(J+1)-CREZ(J-1))/(2*DELTY)
    RH1=RLAM*(CREX(J+1)-CREX(J-1))/(2*DELTY)
    RH1=RH1+RH4
    RH2=CON1*(CREY(J+1)-CREY(J-1))/(2*DELTY)
  END IF
  RH3=(-(3*CRXY(J)+4*CRXY(J-N)-CRXY(J-2*N))/(DELTX)
  RH3=RH3*CON2
  RH=(RH1+RH2+RH3)*DELTX**2*DELTY**2/RMU
  RHS(K)=RH+4*A3*DELTX*CRXY(J)

```

C AXIS OF SYMMETRY

```

ELSE IF(J.GE.N.AND.MOD(J,N).EQ.1)THEN
  K=2*J-1
  CON1=(RLAM+2*RMU)
  IF(J.EQ.241)THEN
    RH1=(CREX(J+N)-CREX(J))/ALP+ALP*(CREX(J)-CREX(J-N))
    RH1=RH1/DELTX/(1+ALP)*CON1
    RH2=(CREY(J+N)-CREY(J))/ALP+ALP*(CREY(J)-CREY(J-N))
    RH4=(CREZ(J+N)-CREZ(J))/ALP+ALP*(CREZ(J)-CREZ(J-N))
    RH2=(RH2+RH4)/DELTX/(1+ALP)*RLAM
  ELSE

```

```

      RH1=CON1*(CREX(J+N)-CREX(J-N))/(2*DELTX)
      RH4=CREZ(J+N)-CREZ(J-N)
      RH2=RLAM*(RH4+(CREY(J+N)-CREY(J-N)))/(2*DELTX)
    END IF
    RH3=RMU/DELTY*(-3*CRXY(J)+4*CRXY(J+1)-CRXY(J+2))
    RH=((RH1+RH2+RH3)*DELTX**2*DELTY**2)/RMU
    RHS(K)=-RHO*GRAV*DELTX**2*DELTY**2/RMU+RH
    RHS(2*J)=0

```

C FIXED FACE

```

    ELSE IF(J.GE.N.AND.MOD(J,N).EQ.0)THEN
      CON1=RLAM+2*RMU
      K=2*J-1
      RHS(2*J-1)=0
      RHS(2*J)=0.

```

C INNER GRID POINTS

```

    ELSE IF(J.GT.(N+1).AND.J.LE.NJ-N+1)THEN
      CON1=RLAM+2*RMU
      K=2*J-1
      IF(J.GT.241.AND.J.LT.256)THEN
        RH1=(CREX(J+N)-CREX(J))/ALP+ALP*(CREX(J)-CREX(J-N))
        RH1=RH1/DELTX/(1+ALP)*CON1
        RH2=(CREY(J+N)-CREY(J))/ALP+ALP*(CREY(J)-CREY(J-N))
        RH4=(CREZ(J+N)-CREZ(J))/ALP+ALP*(CREZ(J)-CREZ(J-N))
        RH2=(RH2+RH4)/DELTX/(1+ALP)*RLAM
      ELSE
        RH1=CON1*(CREX(J+N)-CREX(J-N))/(2*DELTX)
        RH4=CREZ(J+N)-CREZ(J-N)
        RH2=RLAM*(RH4+CREY(J+N)-CREY(J-N))/(2*DELTX)
      END IF
      RH3=RMU*(CRXY(J+1)-CRXY(J-1))/(DELTY)
      RH=(RH1+RH2+RH3)*4*DELTX**2*DELTY**2/RMU
      RHS(K)=-RHO*GRAV*4*DELTX**2*DELTY**2/RMU+RH

      K=2*J
      RH1=CON1*(CREY(J+1)-CREY(J-1))/(2*DELTY)
      RH2=RLAM*(CREX(J+1)-CREX(J-1))/(DELTY*2)
      RH2=RLAM*(CREZ(J+1)-CREZ(J-1))/(DELTY*2)+RH2
      IF(J.GT.241.AND.J.LT.256)THEN
        RH3=(CRXY(J+N)-CRXY(J))/ALP+ALP*(CRXY(J)-CRXY(J-N))
        RH3=RH3/DELTX/(1+ALP)*(2*RMU)
      ELSE
        RH3=RMU/DELTX*(CRXY(J+N)-CRXY(J-N))
      END IF
      RHS(K)=(RH1+RH2+RH3)*4*DELTX**2*DELTY**2/RMU

```

CBOTTOM

```

    ELSE IF(J.GT.1)THEN
      RHS(2*J-1)=0.
      RHS(2*J)=0

```

40 END IF
 CONTINUE

 IM=1
 CALL MATMLT(Y,RHS,DISP,NP,NP,IM)

C WRITE DISPLACEMENT AND STRAINS TO OUTPUT FILES

 DX=0
 K=1
 DO 50 I=1,2*NJ,2
 U(K)=DISP(I)
 V(K)=DISP(I+1)
 IF(MOD(KKOUNT,100).EQ.0)THEN
 IF(K.EQ.1)THEN
 WRITE(5,*)
 WRITE(5,*)KKOUNT
 IF(MOD(KKOUNT,300).EQ.0)THEN
 WRITE(6,*)
 WRITE(6,*)KKOUNT
 END IF
 END IF
 WRITE(5,*)U(K),V(K),K
 WRITE(6,*)K,EPSX(K),EPSY(K),EPSXY(K)
 END IF
 K=K+1
 CONTINUE

C STRESS CALCULATIONS

 N=16
 DO 60 J=1,NJ
 IF(J.LE.256)THEN
 DELTX=.15
 ELSE
 DELTX=.00125
 END IF

C TOPFACE

 IF(J.GT.NJ-N+1.AND.J.LT.NJ)THEN
 EPSY(J)=(V(J+1)-V(J-1))/(2*DELT Y)
 DELUY=(U(J+1)-U(J-1))/(2*DELT Y)
 EPSX(J)=(3*U(J)-4*U(J-N)+U(J-2*N))/(2*DELT X)
 DELVX=(3*V(J)-4*V(J-N)+V(J-2*N))/(2*DELT X)
 EPSXY(J)=0

C TOPFACE CENTER

 ELSE IF(J.EQ.NJ-N+1)THEN
 EPSX(J)=(3*U(J)-4*U(J-N)+U(J-2*N))/(2*DELT X)
 EPSY(J)=V(J+1)/DELT Y
 DELVX=(3*V(J)-4*V(J-N)+V(J-2*N))/(2*DELT X)
 DELUY=0
 EPSXY(J)=.5*(DELVX+DELUY)
 EPSXY(J)=0

C AXIS OF SYMM

```

ELSE IF(J.GT.N.AND.MOD(J,N).EQ.1)THEN
  IF(J.EQ.241)THEN
    EPSX(J)=(U(J+N)-U(J))/ALP+ALP*(U(J)-U(J-N))
    EPSX(J)=EPSX(J)/(DELTX*(1+ALP))
    DELVX=(V(J+N)-V(J))/ALP+ALP*(V(J)-V(J-N))
    DELVX=DELVX/(DELTX*(1+ALP))
  ELSE
    EPSX(J)=(U(J+N)-U(J-N))/(DELTX*2)
    DELVX=(V(J+N)-V(J-N))/(DELTX*2)
  END IF
  DELUY=0
  EPSY(J)=V(J+1)/DELT
  EPSXY(J)=.5*(DELVX+DELUY)

```

C FIXED FACE

```

ELSE IF(J.GE.N.AND.MOD(J,N).EQ.0)THEN
  EPSX(J)=0
  EPSY(J)=(3*V(J)-4*V(J-1)+V(J-2))/(2*DELT)
  DELVX=0.
  DELUY=(3*U(J)-4*U(J-1)+U(J-2))/(2*DELT)
  EPSXY(J)=.5*(DELVX+DELUY)

```

C INNER GRID POINTS

```

ELSE IF(J.GT.N+1.AND.J.LE.NJ-N+1)THEN
  EPSY(J)=(V(J+1)-V(J-1))/(DELT*2)
  DELUY=(U(J+1)-U(J-1))/(DELT*2)
  IF(J.GT.241.AND.J.LT.256)THEN
    EPSX(J)=(U(J+N)-U(J))/ALP+ALP*(U(J)-U(J-N))
    EPSX(J)=EPSX(J)/(DELTX*(1+ALP))
    DELVX=(V(J+N)-V(J))/ALP+ALP*(V(J)-V(J-N))
    DELVX=DELVX/(DELTX*(1+ALP))
  ELSE
    EPSX(J)=(U(J+N)-U(J-N))/(DELTX*2)
    DELVX=(V(J+N)-V(J-N))/(DELTX*2)
  END IF
  EPSXY(J)=.5*(DELUY+DELVX)

```

C BOTTOMFACE

```

ELSE IF(J.GT.1.AND.J.LT.N)THEN
  EPSX(J)=-(3*U(J)-4*U(J+N)+U(J+2*N))/(2*DELTX)
  EPSY(J)=(V(J+1)-V(J-1))/(DELT*2)
  DELUY=(U(J+1)-U(J-1))/(DELT*2)
  DELVX=-(3*V(J)-4*V(J+N)+V(J+2*N))/(2*DELTX)
  EPSXY(J)=.5*(DELVX+DELUY)

```

C BOTTOM LEFT POINT

```

ELSE IF (J.EQ.1)THEN
  EPSX(J)=-(3*U(J)-4*U(J+N)+U(J+2*N))/(2*DELTX)
  EPSY(J)=0.
  DELVX=-(3*V(J)-4*V(J+N)+V(J+2*N))/(2*DELTX)

```

```

      DELUY=0
      EPSXY(J)=.5*(DELUY+DELVX)
      EPSXY(J)=0
    END IF

```

```

      EP0=ABS(RLOAD)/SNOYOU
      DEPX=EPSX(J)-PEPSX(J)-DECX(J)
      DEPY=EPSY(J)-PEPSY(J)-DECY(J)
      DEPMY=EPSXY(J)-PEPMY(J)-DECMY(J)
      DELTA=RNU/(1-2*RNU)*(DEPX+DEPY)
      DSIGX(J)=SNOYOU/(1+RNU)*(DEPX+DELTA)
      DSIGY(J)=SNOYOU/(1+RNU)*(DEPY+DELTA)
      DSIGXY(J)=SNOYOU/(1+RNU)*DEPMY
      DSIGZ(J)=RLAM*(DEPX+DEPY)

```

```

60    CONTINUE

```

```

C FIND THE MAX DSIGX(J)

```

```

      DSMAX=ABS(DSIGX(1))
      DO 70 J=2,NJ-N
        IF (ABS(DSIGX(J)).GT.DSMAX)THEN
          DSMAX=ABS(DSIGX(J))
        END IF

```

```

70    CONTINUE

```

```

      IF (DSMAX.LE.(.0002*ABS(RLOAD)).OR.KKOUNT.GE.150)THEN
        DO 80 J=1,NJ
          DSIGX(J)=0
          DSIGY(J)=0
          DSIGXY(J)=0
          DSIGZ(J)=0

```

```

80    CONTINUE

```

```

      END IF
      DO 90 J=1,NJ
        SIGX(J)=SIGX(J)+DSIGX(J)
        SIGY(J)=SIGY(J)+DSIGY(J)
        SIGXY(J)=SIGXY(J)+DSIGXY(J)
        SIGZ(J)=SIGZ(J)+DSIGZ(J)
        PEPSX(J)=EPSX(J)
        PEPSY(J)=EPSY(J)
        PEPXY(J)=EPSXY(J)
      IF(J.GE.NJ-N+1.AND.J.LT.NJ)THEN
        IF(J.GE.NJ-N+1.AND.J.LT.NJ-N+9)THEN
          RLOAD1=RLOAD
          SIGX(J)=RLOAD1
        ELSE
          SIGX(J)=0.
        END IF
      END IF

```

```

      END IF
      IF((MOD(KKOUNT,50).EQ.0).AND.KKOUNT.LE.150)THEN
        IF(J.EQ.1)THEN
          WRITE(3,*)
          WRITE(3,*)KKOUNT

```

```

      END IF
      WRITE(3,*)J,SIGX(J),SIGY(J),SIGXY(J)
    END IF

90    CONTINUE
81    CONTINUE
      N=16
      DO 100 J=1,NJ
        T(1)=SIGX(J)*DUD1
        T(2)=SIGXY(J)*DUD1
        T(3)=0
        T(4)=SIGXY(J)*DUD1
        T(5)=SIGY(J)*DUD1
        T(6)=0
        T(7)=0
        T(8)=0
        T(9)=SIGZ(J)*DUD1
        DO 110 I=1,9
          IF(ABS(T(I)/DUD1).LE..01*ABS(RLOAD))THEN
            T(I)=0
          END IF
110    CONTINUE
        NODE=J
        CALL TRY(KKOUNT,NODE,DUMEPS,DECR,PRINC)
        CREX(J)=EPS(1,1)
        CREY(J)=EPS(2,2)
        CRXY(J)=EPS(1,2)
        CREZ(J)=EPS(3,3)
        DECX(J)=DECR(1,1)
        DECY(J)=DECR(2,2)
        DECXY(J)=DECR(1,2)
100    CONTINUE
1000  CONTINUE
      END

```

C MATRIX OPERATIONS

```

C    MATRIX MULTIPLICATION
C    T(M,P)=A(M,N)*U(N,P)

```

```

SUBROUTINE MATMLT(A,U,T,M,N,NP)
DIMENSION A(M,N),U(N,NP),T(M,NP)
INTEGER P

```

```

      P=NP
      DO 1 I=1,M
        DO 1 J=1,P
          T(I,J)=0.
1      DO 2 I=1,M
        DO 2 J=1,P
          DO 2 K=1,N

```

```

      T(I,J)=A(I,K)*U(K,J)+T(I,J)
2    CONTINUE
      RETURN
      END

```

```

C-----
C-----
C TRANSPOSE OF A MATRIX

```

```

      SUBROUTINE TRNSPZ(A,V,M,N)

      DIMENSION V(N,M),A(M,N)
      DO 14 I=1,M
      DO 14 J=1,N
14    V(J,I)=A(I,J)
      RETURN
      END

```

```

C-----
C-----
      SUBROUTINE TRY(KKOUNT,NODE,DUMEPS,DECR,PRINC)

```

C SMALL STRAINS. VOL CHANGE=E11+E22+E33
 C EACH GRAIN IS DIVIDED INTO EIGHT OCTANTS AND EACH OCTANT IS DIVIDED
 C INTO 9 REGIONS CALLED QUADS. STRESSES ARE CALCULATED IN EACH OF THE
 C QUADS. THESE ARE USED TO CALCULATE STRAINS IN EACH QUAD. THE STRAINS
 C ARE AVERAGED TO GIVE C STRAINS IN SNOW.
 C STATEMENTS BEGINNING WITH CX ARE USED IF AVERAGING IS DONE BY
 C EQUATIONS GIVEN IN CHAPTER 3 RATHER THAN AT AVERAGE ANGLE OF EACH
 C QUAD.

```

      INTEGER I,QUAD,ROCT,J,K
      COMMON/BLOCK1/T(9),STIN(9),RO,TINC(9),KONTER,KK1
      COMMON/BLOCK2/SUM1(3,3),SUM2(3,3)
      COMMON/BLOCK3/RLEN,DIA,RNEC,TOTL,DR,CN,CONC,RN,RJ,RNU1
      COMMON/BLOCK6/STARTX,STARTY
      COMMON/BLOCK7/DCP(3,3,3,3,8,9),DENO(8,9)
      COMMON/BLOCK9/IN1,JN1,KN1
      COMMON/BLOC11/ EPS(3,3),DISS(8,9,3,400)
      COMMON/CREEP2/CCRE(8,9,3,400),RSIGP(8,9,3,400),PCRE(8,9,3)
      COMMON/DISP/PESIGM(8,9,3,400),EPSAV(8,9,4)
      COMMON/DISP2/PLS(8,9,3,400),PPLS(8,9,3),KONT(8,9,400)
      COMMON/DISP10/RLOAD,EFFST(400),CNOD(400),EFFRAT(400)
      COMMON/DISP3/DELEPS(3,3)
      COMMON/DISP4/TOTSTR(8,9,4,400),IFRACT(8,9,400)
      COMMON/CREEP3/SIGP(9),SIGPP(9)
      COMMON/SLID3/U1,V1,W1
      COMMON/DINT1/Y1A,Y1B
      DIMENSION ESIGM(8,9,3),SIGO(9)
      DIMENSION DIS(3),RESIGM(9),CRESIM(9)
      DIMENSION CREP(3)

```

```

DIMENSION DUMEPS(3,3,2,400),DECR(3,3)
DIMENSION PRINC(400)
DIMENSION TOT(3,3),D(3),V(3,3),DIR(3,3,8,9)
DIMENSION ZETA(8,9,3),THETA(8,9),KCHECK(8,9)

```

C CONSTANTS (FOR ICE)

```

PI=3.14159
RNU=.33
E=5.4E3
RLAMDA=RNU*E/((1+RNU)*(1-2*RNU))
RMU=E/(2*(1+RNU))

```

C INITIALIZE SUM(I,J)=0 BEFORE STARING OF EACH LOOP

```

DO 20 I=1,3
DO 20 J=1,3
SUM1(I,J)=0.
SUM2(I,J)=0.
20 CONTINUE

```

C THE GRAIN IS DIVIDED INTO 72 PARTS. EACH OCTANT HAS BEEN DIVIDED INTO 9
C QUADS.THE QUADS ARE NOT OF EQUAL AREA EG.THE FIRST QUAD IS
C MUCH SMALLER THEN THE NINTH.THE FIRST QUAD GOES FROM BETA = 0 TO PI/6
C AND ALPHA= 0 TO PI/6 THE NINTH QUAD GOES FROM BETA = PI/3 TO PI/2 AND
C ALPHA =PI/3 TO C PI/2.

```

CROCT=NO.OF OCTANT
CQUAD=NO. OF QUAD
C X1A =ALPHA BEGINNING
C X1B =ALPHA ENDING
C Y1A =BETA BEGINNING
C Y1B =BETA ENDING

```

```

K=1
Y1A=0.1E-12
Y1B=PI/6
X1A=0.1E-12
STARTX=X1A
STARTY=Y1A

```

```

DO 40 ROCT=1,8
DO 50 QUAD=1,9
X1B=X1A+PI/6

```

C KKOUNT=1 IS USED TO CALCULATE YOUNG'S MOD FOR SNOW FOR THE
C PARTICULAR COORDINATION NO. IT IS THEN USED IN RINV.FOR
C INITIALIZE INITIAL STRAINS TO BE ZERO
C PESIGM
C PCRE REVERSIBLE STRAN AT THE BEGINNING OF INTEGRATION TIME
C PPLS PLASTIC STRAIN " " " " " "
C CCRE " " " " END. " " " "

```

C PLS PLASTIC STRAIN " " " " " "
  IF (KKOUNT.EQ.2.OR.KKOUNT.EQ.1) THEN

```

```

    DO 55 I=1,3
      PESIGM(ROCT,QUAD,I,NODE)=0.
      CCRE(ROCT,QUAD,I,NODE)=0.
      PCRE(ROCT,QUAD,I)=0.
      PLS(ROCT,QUAD,I,NODE)=0.
      PPLS(ROCT,QUAD,I)=0.
      TOTSTR(ROCT,QUAD,I,NODE)=0.

```

```

55      CONTINUE
      END IF
      K1=KKOUNT

```

```

CX      IF (KKOUNT.EQ.1) THEN
CX      CALCULATE THE DOUBLE INTEGRAL OF PRODUCT OF DIR COSINES
CX      DO 101 IN=1,3
CX      DO 101 JN=1,3
CX      DO 101 KN=1,3
CX      IN1=IN
CX      JN1=JN
CX      KN1=KN
CX      CALL DOUINT(X1A,X1B,Y1A,Y1B,DC,RD1,RD2,RD3)
CX      ERRABS=0.0
CX      ERRREL=.01
CX      IRULE=3
CX      CALL TWODQ(F1,X1A,X1B,G1,H1,ERRABS,ERRREL,IRULE,RD1,ERREST)
CX      ERRABS=0.0
CX      ERRREL=.01
CX      IRULE=3
CX      CALL TWODQ(F2,X1A,X1B,G1,H1,ERRABS,ERRREL,IRULE,RD2,ERREST)
CX      ERRABS=0.0
CX      ERRREL=.01
CX      IRULE=3
CX      CALL TWODQ(F3,X1A,X1B,G1,H1,ERRABS,ERRREL,IRULE,RD3,ERREST)
CX      DCP(IN,KN,JN,1,ROCT,QUAD)=RD1
CX      DCP(IN,KN,JN,2,ROCT,QUAD)=RD2
CX      DCP(IN,KN,JN,3,ROCT,QUAD)=RD3
101    CONTINUE
CX      CALL DOUINT(X1A,X1B,Y1A,Y1B,DENOM,R4,R5,R6)
CX      R4=-1/(4*PI)*(X1B-X1A)*(COS(Y1B)-COS(Y1A))
CX      DENO(ROCT,QUAD)=R4
CX      END IF

```

```

      KONTER=1

```

```

C SUBROUTINE AVSTR CALCULATES THE AVERAGE OF STRESS OVER A QUAD
C RESIGM STORES THESE AVERAGE STRESSES

```

```
CALL AVSTR(X1A,X1B,Y1A,Y1B,RESIGM,ROCT,QUAD)
```

```
C SINCE THE NECK HAS ONLY THREE STRESSES STORE THESE AS ESIGM
```

```
ESIGM(ROCT,QUAD,1)=RESIGM(9)
ESIGM(ROCT,QUAD,2)=RESIGM(6)
ESIGM(ROCT,QUAD,3)=RESIGM(3)
```

```
C CALCULATE THE PRINCIPAL STRESS. IF PRINCIPAL TENSILE STRESS,
C IN A NECK, EXCEEDS
```

```
C .7 MPA AND PRINCIPAL STRAIN EXCEEDS .005 THEN THE NECK FRACTURES.
```

```
TAU=ESIGM(ROCT,QUAD,3)**2+ESIGM(ROCT,QUAD,2)**2
TAU=SQRT(4.*TAU+(ESIGM(ROCT,QUAD,1))**2)
PRINC(NODE)=(ABS(ESIGM(ROCT,QUAD,1))+TAU)/2
```

```
C CRESIM - COMPONENTS OF STRESS TENSOR
```

```
CRESIM(1)=0.
CRESIM(2)=0.
CRESIM(3)=ESIGM(ROCT,QUAD,3)
CRESIM(4)=0.
CRESIM(5)=0.
CRESIM(6)=ESIGM(ROCT,QUAD,2)
CRESIM(7)=CRESIM(3)
CRESIM(8)=CRESIM(6)
CRESIM(9)=ESIGM(ROCT,QUAD,1)
```

```
C DEVIAT CALCULATES DEVIATORIC STRESSES AS DEFINED IN GLOCKNER'S PAPER
```

```
CALL DEVIAT(CRESIM,SIGO,RN)
```

```
C CREEP1 CALCULATES STRAIN INCREMENT DUE TO PRIMARY/REVERSIBLE
C CREEP
```

```
C IFRACT=0 MEANS THE NECK HAS NOT FRACTURED
```

```
C IF NECK IS NOT FRACTURED THEN CALCULATE PLASTIC STRAINS
```

```
C FOR APPLICATIONS WE USE A SLIGHTLY DIFFERENT VERSION.
```

```
C IF(KKOUNT.GT.150)THEN
```

```
IF(IFRACT(ROCT,QUAD,NODE).NE.1)THEN
```

```
DO 3001 IM=1,3
```

```
PCRE(ROCT,QUAD,IM)=0
```

```
CCRE(ROCT,QUAD,IM,NODE)=0
```

```
3001 CONTINUE
```

```
GO TO 5000
```

```
C ELSE IF(KKOUNT.GE.5)THEN
```

```
C
```

```
C
```

```
C
```

```
DO 3002 IM=1,3
```

```
PCRE(ROCT,QUAD,IM)=0
```

```
CCRE(ROCT,QUAD,IM,NODE)=0
```

```
C3002 CONTINUE
```

```
ELSE
```

```

      CALL CREEP1(ROCT,QUAD,K1,SIGO,CRESIM,RN,RJ,RNU1,NODE)
    END IF

```

C PLAS CALCULATES THE PLASTIC STRAIN INCREMENT

```

      CALL PLAS(CRESIM,SIGO,ROCT,QUAD,K1,CRESIM,RN,PRINC,NODE)

```

C DISPL CALCULATES ELASTIC STRAIN INCREMENT AND ADDS
CTHESE TO PLASTIC AND PRIMARY

```

      CALL DISPL(ESIGM,ROCT,QUAD,SRHO,K1,NODE,PRINC)

```

C NTERM TELLS THE SUBROUTINE STOT (USED LATER) WHETHER STRAINS ARE
DUE TO SLIDING OR C DEFORMATION OF NECKS.

C NTERM=1 DEFORMATION

C NTERM=2 SLIDING

```

      NTERM=1

```

```

      ELSE

```

C THE NECKS FRACTURE ONLY AFTER UNDERGOING SOME PRIMARY
C CREEP.THEREFORE FRACTURING TAKES PLACE AFTER KCOUNT IS GREATER
C THAN 2. EPSAV STORES THE TOTAL STRAIN INCREMENTS DUE TO DEFORMATION
C OF NECKS BEFORE FRACTURING.

C IN CASE FRACTURING HAS TAKEN THEN THESE STRAIN INCREMENTS(DUE
C TO DEFORMATION) ARE SET TO ZERO AND WE CALCULATE DIPLACEMENTS DUE TO
C.SLIDING

```

5000      EPSAV(ROCT,QUAD,1)=0.
          EPSAV(ROCT,QUAD,2)=0.
          EPSAV(ROCT,QUAD,3)=0.
          EPSAV(ROCT,QUAD,4)=0.

```

C SHEAR CALCULATES THE DISPLACEMENTS OF THE FRACTURED NECKS.

```

      CALL SHEAR(ROCT,QUAD,ZETA,ESIGM,THETA,KCOUNT,DISS,NODE)

```

```

      NTERM=2

```

```

    END IF

```

C STOT CONVERTS THE STRAIN INCREMENTS FROM LOCAL COORDINATE

C SYSTEM OF THE NECK TO GLOBAL COORDINATE SYSTEM OF SNOW.

C IT ALSO CALCULTES THE AVERAGE STRAIN INCREMENT IN SNOW

```

      CALL STOT(X1A,X1B,Y1A,Y1B,ROCT,QUAD,NTERM,ESIGM,ZETA)

```

```

361    CONTINUE

```

```

    END IF

```

C INCREMENT QUADRANT BY 1 AND CHANGE ALPHA,BETA ACCORDINGLY.

```

      X1A=X1B

```

```

      IF (QUAD.EQ.3)THEN

```

```

        X1A=STARTX

```

```

      Y1A=PI/6+STARTY
      Y1B=PI/6+Y1A
    ELSE IF (QUAD.EQ.6) THEN
      X1A=STARTX
      Y1A=STARTY+PI/3
      Y1B=PI/6+Y1A+.1E-10
    END IF
50    CONTINUE
    C BEFORE WE GO TO FIFTH OCTANT SOME CHANGES IN VALUE OF STARTY HAVE
    C TO BE MADE

```

```

      IF(ROCT.NE.4) THEN
        X1A=STARTX+PI/2
        STARTX=X1A
        IF(ROCT.LT.4) THEN
          Y1A=0.1E-12
          Y1B=PI/6
        ELSE
          Y1A=PI/2
          Y1B=PI/2+PI/6
        END IF
      ELSE
        X1A=.1E-12
        STARTX=.1E-12
        Y1A=PI/2
        Y1B=PI/2+PI/6
        STARTY=Y1A
      END IF
40    CONTINUE

```

```

    C DUMEPS(I,J,1) AVERAGE STRAIN IN SNOW AT BEGINNING OF TIME STEP
    C DUMEPS(I,J,2) " " " " " END " " "
    C DIVISION BY 5 IS DUE TO THE SCALING FACTOR
    C DECR(I,J) IS THE INCREMENT IN SNOW STRAIN

```

```

      DO 365 I=1,3
      DO 365 J=1,3
        DSUM=SUM1(I,J)/(8.)+SUM2(I,J)/(5*1.8)
        DUMEPS(I,J,2,NODE)=DSUM+DUMEPS(I,J,1,NODE)
365    CONTINUE
      DO 375 I=1,3
      DO 375 J=1,3
        EPS(I,J)=DUMEPS(I,J,2,NODE)
        DECR(I,J)=SUM1(I,J)/(8.)+SUM2(I,J)/(1.8*5.)
        DUMEPS(I,J,1,NODE)=DUMEPS(I,J,2,NODE)
375    CONTINUE

```

```

    C CALCULATE EFFECTIVE STRAIN
    DUDE=0.
    DUD=0.
    DO 377 I=1,3

```

```

DO 377 J=1,3
  DUDE=DECR(I,J)*DECR(I,J)+DUDE
  DUD=(DUMEPS(I,J,2,NODE)*DUMEPS(I,J,2,NODE))+DUD
377 CONTINUE
  EFFST(NODE)=SQRT(2./3*(DUD))
  EFFRAT(NODE)=SQRT(DUDE)

END

```

```

C-----
C-----
C STOT CONVERTS STRAIN INCREMENTS IN ICE COORDINATE SYSTEM
C TO STRAIN INCREMENTS IN SNOW COORDINATES, ALSO AVERAGE STRAIN
C INCREMENTS
C ARE CALCULATED.
C

```

```

SUBROUTINE STOT(XA,XB,YA,YB,ROCT,QUAD,,NTERM,ESIGM,ZETA)
  DIMENSION ZETA(8,9,3),Q(3,3)
  COMMON/STOBL/E(3,3)
  COMMON/DISP/PESIGM(8,9,3,400),EPSAV(8,9,4)
  COMMON/SLID5/U2,V2,W2
  COMMON/DISP10/RLOAD,EFFST(400),CNOD(400)
  COMMON/BLOCK2/SUM1(3,3),SUM2(3,3)
  COMMON/BLOCK7/DCP(3,3,3,3,8,9),DENOM(8,9)
  DIMENSION EPS(3,3),DUMEPS(3,3,2,400)
  DIMENSION S(3),RN(3),ESIGM(8,9,3)
  INTEGER ROCT,QUAD

```

```

  PI=3.14159
  TAU1=ESIGM(ROCT,QUAD,3)
  TAU2=ESIGM(ROCT,QUAD,2)
  RESLT=SQRT(TAU1**2+TAU2**2)
  C EPSAV IS THE TOTAL STRAIN INCREMENT CALCULATED IN SUBROUTINE
  C DISLD, FOR NECKS WHICH HAVN'T FRACTURED,

```

```

  IF(NTERM.EQ.1)THEN
    E(1,3)=EPSAV(ROCT,QUAD,3)
    E(2,3)=EPSAV(ROCT,QUAD,2)
    E(3,3)=EPSAV(ROCT,QUAD,1)
    E(1,1)=EPSAV(ROCT,QUAD,4)
    E(2,2)=E(1,1)

```

```

  C IF EFFST EXCEEDS 1.5E-2 THEN STRAIN RATES CALCULATED USING THE
  C EQUATIONS ARE REDUCED BY 10% BECAUSE OF SLIGHTLY INCREASED
  C NUMBER OF CONTACTS.

```

```

  IF(EFFST(NOD).GT.1.5E-2)THEN
    E(3,3)=E(3,3)*.9
    E(1,3)=E(1,3)*.9
    E(2,3)=E(2,3)*.9
  END IF
  E(3,1)=E(1,3)

```

E(3,2)=E(2,3)

C E(1,1) AND E(2,2) ARE EXTENSIONS OF ICE NECK INTO THE PORE SPACE AND
C THEREFORE DO NOT CONTRIBUTE TO STRAINS IN SNOW.FOR SMALL STRAIN C
THEORY

E(1,1)=0

E(2,2)=0

E(1,2)=0.

E(2,1)=0.

C DOUINT IS A DOUBLE INTEGRATION SUBROUTINE WHICH USES SIMPSON'S RULE.
C EPS(I) ARE THE NINE COMPONENTS OF STRAIN IN SNOW

C (IE WE ARE CONVERTING STRAINS FROM NECK COORDINATES TO SNOW
COORDINATES)

C XA,XB ARE THE BEGINNING AND END VALUES OF ALPHA BETWEEN WHICH

C THIS CONVERSION IS BEING DONE.SIMILARLY YA AND YB ARE LIMITS ON BETA

C CONVERSION IN COORDINATE SYSTEM AND AVERAGING ARE TAKING PLACE
TOGETHER.

C DO 10 I=1,9

C EPS(I)=0.0

C10 CONTINUE

C THE DOUBLE INTEGRALS ARE FUNCTIONS OF ANGLES AND NEED TO BE
CALCULATED

C ONLY ONCE.THEY HAVE ALREADY BEEN EVALUATED ABOVE

A=(XA+XB)/2

B=(YA+YB)/2

CALL DRCOS(A,B,Q)

DO 25 I=1,3

DO 25 J=1,3

EPSUM=0.

DO 30 K=1,3

DO 30 L=1,3

DUMM=-(COS(YB)-COS(YA))*(XB-XA)/(4*PI)

EPSUM=EPSUM+Q(I,K)*E(K,L)*Q(J,L)*DUMM

30 CONTINUE

25 EPS(I,J)=EPSUM

OX CALL DOUINT(XA,XB,YA,YB,DC1,EPS(1),EPS(2),EPS(3))

OX CALL DOUINT(XA,XB,YA,YB,DC2,EPS(4),EPS(5),EPS(6))

OX CALL DOUINT(XA,XB,YA,YB,DC3,EPS(7),EPS(8),EPS(9))

N=0

DO 21 I=1,3

DO 21 J=1,3

N=N+1

SUM1(I,J)=SUM1(I,J)+EPS(I,J)

21 CONTINUE

ELSE

C NTERM=2. IN CASE OF SLIDING NECKS WE GO TO THIS PORTION.

U2=ZETA(ROCT,QUAD,1)*5.

```

V2=ZETA(ROCT,QUAD,2)*5.
W2=(ZETA(ROCT,QUAD,3)*5.)
SHEAD=SQRT(U2**2+V2**2)

```

C VECROT1, VECROT2, VECROT3 ARE FUNCTIONS IN DOUINT WHICH
C CALCULATE STRAIN INCREMENTS FROM SLIDING DISPLACEMENT CINCRMENT.

```

CX  CALL DOUINT(XA,XB,YA,YB,VECROT1,EPS(1),EPS(2),EPS(3))
CX  CALL DOUINT(XA,XB,YA,YB,VECROT2,EPS(4),EPS(5),EPS(6))
CX  CALL DOUINT(XA,XB,YA,YB,VECROT3,EPS(7),EPS(8),EPS(9))

```

```

A=(XA+XB)/2
B=(YA+YB)/2
CALL DRCOS(A,B,Q)
IF(RESLT.EQ.0)THEN
  S(1)=0
  S(2)=0
  S(3)=0
ELSE
  S(1)=(TAU1*Q(1,1)+TAU2*Q(1,2))/RESLT
  S(2)=(TAU1*Q(2,1)+TAU2*Q(2,2))/RESLT
  S(3)=(TAU1*Q(3,1)+TAU2*Q(3,2))/RESLT

```

```

END IF

```

```

RN(1)=Q(1,3)
RN(2)=Q(2,3)
RN(3)=Q(3,3)

```

```

L=3

```

```

DO 55 I=1,3

```

```

DO 55 J=1,3

```

```

  EPSUM=0.

```

```

  DUMM=-((COS(YB)-COS(YA)))*(XB-XA)/(4*PI)

```

```

  EPSUM=(.5*(S(I)*RN(J)+S(J)*RN(I))*SHEAD)

```

```

  EPSUM=(EPSUM+(RN(I)*RN(J))*W2)*DUMM

```

```

  EPS(I,J)=EPSUM

```

```

55  CONTINUE

```

```

1000 N=0

```

```

  DO 20 I=1,3

```

```

  DO 20 J=1,3

```

```

    N=N+1

```

```

    SUM2(I,J)=SUM2(I,J)+EPS(I,J)

```

```

20  CONTINUE

```

```

  END IF

```

```

  RETURN

```

```

  END

```

```

C-----

```

```

C-----

```

C CALCLATE DIRECTION COSINES

C THIS ACTUALLY CORRESPONDS TO TRANSPOSE OF Q

```

SUBROUTINE DRCOS(A,B,Q)

```

```

DIMENSION Q(3,3),ST(3,3)

```

```

Q(1,1)=COS(A)*COS(B)
Q(1,2)=-SIN(A)
Q(1,3)=COS(A)*SIN(B)
Q(2,1)=SIN(A)*COS(B)
Q(2,2)=COS(A)
Q(2,3)=SIN(A)*SIN(B)
Q(3,1)=-SIN(B)
Q(3,2)=0.0
Q(3,3)=COS(B)

```

```

RETURN
END

```

C-----

C-----

C CALCULATES AVERAGE SHEAR AND NORMAL STRESSES
C THE PROGRAM CALCULATES THE STRESSES AT AVERAGE ANGLES.
C THE STATEMENTS WITH CX CALCULATE AVERAGE STRESSES USING
C EQUATIONS DERIVED IN THIRD CHAPTER OF THESIS.

```

SUBROUTINE AVSTR(XA,XB,YA,YB,RESIGM,ROCT,QUAD)

```

```

COMMON/BLOCK1/T(9),STIN(9),RO ,TINC(9),KONTER,KK1
COMMON/BLOCK7/DCP(3,3,3,3,8,9),DENO(8,9)
DIMENSION RN(3),STRVEC(3)
DIMENSION RESIGM(9),ST(3,3)

```

```

CX DIMENSION DENO(8,9)
CX EXTERNAL RNOR,DENOM

```

```

INTEGER I,ROCT,QUAD

```

```

PI=3.14159

```

```

C SNOR NORMAL STRESS
C SHEAR1 AND SHEAR2 ARE SHEAR STRESSES
SNOR=0.0
SHEAR1=0.
SHEAR2=0.0

```

C-----

C DOUINT IS A DOUBLE INTEGRATION SUBROUTINE.

```

CX CALL DOUINT (XA,XB,YA,YB,RNOR,SNOR,SHEAR1,SHEAR2)
CX DENO(ROCT,QUAD)=0.0
CX SUM=0.0
CX CALL DOUINT (XA,XB,YA,YB,DENOM,DE0,SUM,SUM)
CX DENO(ROCT,QUAD)=DE0
CX SHEAR1=SHEAR1/DENO(ROCT,QUAD)
CX SHEAR2=SHEAR2/DENO(ROCT,QUAD)
CX SNOR=SNOR/DENO(ROCT,QUAD)

```

```

CX PRODUCTS OF DIRECTION COSINES HAVE BEEN CALCULATED IN THE CX
MAIN PROG

```

CX MULT BY 4PI IS BECAUSE DCP INCLUDES 4*PI WHICH IS NOT THERE FOR CX
STRESSES

C -----

A=(XA+XB)/2
B=(YA+YB)/2
RNO=0.
N=1
DO 5 J=1,3
DO 5 K=1,3
ST(J,K)=T(N)
N=N+1

5 CONTINUE

CX DO 10 I=1,3
CX DO 10 J=1,3
CX RNO=DCP(I,3,J,3,ROCT,QUAD)*ST(I,J)+RNO
CX10 CONTINUE
CX SNOR=RNO/DENO(ROCT,QUAD)
CX SNOR=SNOR*4*PI

CX SHEARA=DCP(1,3,1,1,ROCT,QUAD)*ST(1,1)
CX SHEARA=SHEARA+DCP(2,3,1,1,ROCT,QUAD)*ST(1,2)
CX SHEARA=SHEARA+DCP(3,3,1,1,ROCT,QUAD)*ST(1,3)

CX SHEARB=DCP(1,3,2,1,ROCT,QUAD)*ST(2,1)
CX SHEARB=SHEARB+DCP(2,3,2,1,ROCT,QUAD)*ST(2,2)
CX SHEARB=SHEARB+DCP(3,3,2,1,ROCT,QUAD)*ST(2,3)

CX SHEARC=DCP(1,3,3,1,ROCT,QUAD)*ST(3,1)
CX SHEARC=SHEARC+DCP(2,3,3,1,ROCT,QUAD)*ST(3,2)
CX SHEARC=SHEARC+DCP(3,3,3,1,ROCT,QUAD)*ST(3,3)

CX SHEAR1=(SHEARA+SHEARB+SHEARC)/DENO(ROCT,QUAD)
CX SHEAR1=SHEAR1*4*PI
CX SHEAR2=DCP(1,2,1,3,ROCT,QUAD)*ST(1,1)
CX SHEAR2=SHEAR2+DCP(1,2,2,3,ROCT,QUAD)*ST(1,2)
CX SHEAR2=SHEAR2+DCP(1,2,3,3,ROCT,QUAD)*ST(1,3)
CX SHEAR2=SHEAR2+DCP(2,2,1,3,ROCT,QUAD)*ST(2,1)
CX SHEAR2=SHEAR2+DCP(2,2,2,3,ROCT,QUAD)*ST(2,2)
CX SHEAR2=SHEAR2+DCP(2,2,3,3,ROCT,QUAD)*ST(2,3)
CX SHEAR2=SHEAR2*4*PI/DENO(ROCT,QUAD)

RN(1)=SIN(B)*COS(A)
RN(2)=SIN(B)*SIN(A)
RN(3)=COS(B)

C CALCULATE THE STRESS VECTOR
DO 10 I=1,3

```

10  STRVEC(I)=0.0
    DO 15 I=1,3
    DO 15 K=1,3
        STRVEC(I)=STRVEC(I)+RN(K)*ST(I,K)
15  CONTINUE

C SUM IS THE NORMAL STRESS BEFORE INTEGRATION
    SUM=0.0
    DO 30 I=1,3
    DO 30 J=1,3
        SUM=RN(I)*RN(J)*(ST(I,J))+SUM
30  CONTINUE
    SNOR=SUM*4*PI

    SHEARA=(STRVEC(1))*(COS(B)*COS(A))
    SHEARB=(STRVEC(2))*(SIN(A)*COS(B))
    SHEARC=(STRVEC(3))*(-SIN(B))

    SHEAR1=(SHEARA+SHEARB+SHEARC)*4*PI
    SHEAR2=(STRVEC(1))*(-SIN(A))
    SHEAR2=(STRVEC(2))*COS(A)+SHEAR2
    SHEAR2=SHEAR2*4*PI

    DO 40 I=1,9
        RESIGM(I)=0.0
40  CONTINUE

    RESIGM(9)=SNOR
    RESIGM(6)=SHEAR2
    RESIGM(8)=SHEAR2
    RESIGM(3)=SHEAR1
    RESIGM(7)=SHEAR1

    RETURN
    END

```

C-----

C-----

C CAL DEVIATORIC STRESS AND S(AS DEFINED IN GLOCKNER'S PAPER)

SUBROUTINE DEVIAT(ESIGM,SIGO,m)

DIMENSION ESIGM(9),DEV(9),SIGO(9)

C S MEAN IS THE HYDROSTATIC STRESS

SUM=(ESIGM(1)+ESIGM(5)+ESIGM(9))

SMEAN=1/3.*SUM

C CALCULATE THE DEVIATORIC STRESS

DO 10 I=1,9

```

      IF (I.EQ.1.OR.I.EQ.5.OR.I.EQ.9) THEN
        DEV(I)=ESIGM(I)-SMEAN
      ELSE
        DEV(I)=ESIGM(I)
      END IF
10    CONTINUE

```

C CALCULATION OF S

```

      DUMMY=0.0
      DO 20 I=1,9
        DUMMY=3/2*DEV(I)*DEV(I)+DUMMY
20    CONTINUE
      S=SQRT(DUMMY)

```

C SIGO IS WHAT GLOCHNER DESCRIBES AS VISCOUS STRESS OR THE STRESS ON
C HIS KELVIN MODEL

```

      DO 30 I=1,9
        IF (ABS(DEV(I)).LE.1E-6) THEN
          SIGO(I)=0.0
        ELSE
          SIGO(I)=(1.5*(ABS(S/DEV(I))))**(RN-1))
        END IF
        SIGO(I)=SIGO(I)**(1/RN)*DEV(I)
30    CONTINUE

      RETURN
      END

```

C-----
C-----

C CALCULATE PLASTIC STRAINS . NO DAMAGE IS TAKEN

SUBROUTINE PLAS(RESIGM,SIGO,ROCT,QUAD,KKONT,CRESIM,RN,PRINC,NOD)

```

      COMMON/DISP2/PLS(8,9,3,400),PPLS(8,9,3),KONT(8,9,400)
      COMMON/DISP10/RLOAD,EFFST(400)
      INTEGER I,J,KOUNT,KKOUNT,ROCT,QUAD
      DIMENSION RESIGM(9),SIGO(9),PLSTIC(9),CRESIM(9),PRINC(400)

```

C CONSTANTS

```

      A=1.0
      AETA=15.0
      POW1=1/(AETA+1)
      POW=- (RN)/(AETA+1)
      EP0=.007
      RK=1.805E-7
      IF(ABS(PRINC(NOD)).GT..7) THEN
        RNU2=168.*3600.*28

```

```

ELSE
C SUPERPLASTICITY
  RNU2=168*3600.*1.4
END IF

```

```

GAM=RN
TP=ABS(RLOAD)**RN
IF(KKONT.LE.5)THEN
  DELTIM=30.
ELSE
  DELTIM=120.
END IF

```

C PLS(ROCT,QUAD,I) ARE THE PLASTIC STRAINS FROM PREVIOUS TIME STEP
 C PLASTIC STRAIN IS EXPRESSED AS 9*1 MATRIX RATHER THAN 3*3
 C PLASTIC STRAINS ARE INCOMPRESSIBLE.PLSTIC(1) IS THEREFORE 50% OF
 C PLSTIC(9)

```

  PLSTIC(1)=PLS(ROCT,QUAD,1,NOD)*(-.5)
  PLSTIC(2)=0.
  PLSTIC(3)=PLS(ROCT,QUAD,3,NOD)
  PLSTIC(4)=0.
  PLSTIC(5)=PLSTIC(1)
  PLSTIC(6)=PLS(ROCT,QUAD,2,NOD)
  PLSTIC(7)=PLSTIC(3)
  PLSTIC(8)=PLSTIC(6)
  PLSTIC(9)=PLS(ROCT,QUAD,1,NOD)

```

```

C-----
DO 10 I=1,9
  IF(IEQ.3.OR.IEQ.6.OR.IEQ.9)THEN
    PLAS1=(ABS(SIGO(I)))*RN
    IF(SIGO(I).LT.0)THEN
      PLAS1=-PLAS1
    END IF
    PLAS3=PLAS1/RNU2
  
```

C EFP=0. IN MY VERSION OF PROG.GLOCNER CALLS IT EFFECTIVE PLASTIC
 STRAIN AND TAKES ACCOUNT OF STRAIN HARDENING IN ICE.

```

  IF(EFP.GT.EP0)THEN
    ALPHA=1.
  ELSE
    ALPHA=0.0
  END IF
  PLAS4=(1+ALPHA*(EFP/EP0-1))
  PLSTIC(I)=PLAS3*PLAS4*DELTIM
END IF

```

```

10  CONTINUE
C PPLS STORES PLASTIC STRAINS AT THE BEGINNING OF TIME STEP

```

```

  PPLS(ROCT,QUAD,1)=PLS(ROCT,QUAD,1,NOD)
  PPLS(ROCT,QUAD,2)=PLS(ROCT,QUAD,2,NOD)

```

PPLS(ROCT,QUAD,3)=PLS(ROCT,QUAD,3,NOD)

C PLS STORES PLASTIC STRAINS AT THE END OF TIME STEP

PLS(ROCT,QUAD,1,NOD)=PLSTIC(9)+PPLS(ROCT,QUAD,1)

PLS(ROCT,QUAD,2,NOD)=PLSTIC(6)+PPLS(ROCT,QUAD,2)

PLS(ROCT,QUAD,3,NOD)=PLSTIC(3)+PPLS(ROCT,QUAD,3)

RETURN

END

C-----

C-----

C SUBROUTINE DISPL CALCULATES ELASTIC STRAIN INCREMENTS. ADDS UP
C ELASTIC, REVERSIBLE AND PLASTIC CREEP INCREMENTS TO GIVE EPSAV WHICH
C IS USED IN TRY1.FOR AND STOT. CALCULATES TOTAL STRAIN AT THE END OF
C TIME STEP AND CHECKS FOR FRACTURING OF NECKS(IF PRINCIPAL TENSILE
C STRAIN EXCEEDS .5%)

SUBROUTINE DISPL(ESIGM,ROCT,QUAD,SRHO,K1,NODE,PRINC)

COMMON/BLOCK1/T(9),STIN(9),RO,TINC(9),KONTER

COMMON/BLOC11/EPS(3,3)

COMMON/CREEP2/CCRE(8,9,3,400),RSIGP(8,9,3,400),PCRE(8,9,3)

COMMON/DISP/PESIGM(8,9,3,400),EPSAV(8,9,4)

COMMON/DISP2/PLS(8,9,3,400),PPLS(8,9,3),KONT(8,9,400)

COMMON/DISP3/DELEPS(3,3)

COMMON/DISP10/RLOAD,EFFST(400)

COMMON/DISP4/TOTSTR(8,9,4,400),IFRACT(8,9,400)

DIMENSION STT(3,3),PRCRE(3,3),CRE(3,3),ESIGM(8,9,3),DEPS(3,3)

DIMENSION INDX(3),DPLAS(3,3)

DIMENSION TOT(3,3),V(3,3),D(3),ELAST(3,3),PRINC(400)

INTEGER ROCT,QUAD

C-----

C CONSTANTS FOR ICE

N=3

NP=3

RNU=.33

E=5.4E3

E1=E/(1+RNU)

RATIO=RNU/(1-2*RNU)

C TP=ABS(RLOAD)**RN

DELTIM=30.

C-----

C MANY ARRAYS ARE STORED AS 3-D ARRAYS TO KEEP TRACK OF THE OCTANTS
C AND QUADS THEY BELONG TO..THEY ARE FIRST CONVERTED TO 2-D ARRAYS FOR
C EASE OF CALCULATION.

C PRCRE REVERSIBLE CREEP AT BEGINNING OF TIME STEP

C CRE " " " END " " "
 C THE ABOVE WERE CALCULATED IN SUBROUTINE CREEP1
 C FACTOR,OF -.5 ACCOUNTS FOR INCOMPRESSIBILITY

```

PRCRE(1,1)=-.5*PCRE(ROCT,QUAD,1)
PRCRE(1,2)=0.
PRCRE(1,3)=PCRE(ROCT,QUAD,3)
PRCRE(2,1)=0
PRCRE(2,2)=PRCRE(1,1)
PRCRE(2,3)=PCRE(ROCT,QUAD,2)
PRCRE(3,1)=PRCRE(1,3)
PRCRE(3,2)=PRCRE(2,3)
PRCRE(3,3)=PCRE(ROCT,QUAD,1)
CRE(1,1)=-.5*CCRE(ROCT,QUAD,1,NODE)
CRE(1,2)=0.
CRE(1,3)=CCRE(ROCT,QUAD,3,NODE)
CRE(2,1)=0
CRE(2,2)=CRE(1,1)
CRE(2,2)=-.5*CCRE(ROCT,QUAD,1,NODE)
CRE(2,3)=CCRE(ROCT,QUAD,2,NODE)
CRE(3,1)=CRE(1,3)
CRE(3,2)=CRE(2,3)
CRE(3,3)=CCRE(ROCT,QUAD,1,NODE)

```

C STT IS REQUIRED FOR FIRST TIME STEP IN CASE STRESS IS CONSTANT

```

STT(1,1)=0.0
STT(1,2)=0.
STT(1,3)=ESIGM(ROCT,QUAD,3)-PESIGM(ROCT,QUAD,3,NODE)
STT(2,1)=0
STT(2,2)=0.0
STT(2,3)=ESIGM(ROCT,QUAD,2)-PESIGM(ROCT,QUAD,2,NODE)
STT(3,1)=STT(1,3)
STT(3,2)=STT(2,3)
STT(3,3)=ESIGM(ROCT,QUAD,1)-PESIGM(ROCT,QUAD,1,NODE)

```

C DPLAS PLASTIC STRAIN INCREMENT CALCULATED IN SUBROUTINE PLAST

```

DPLAS(1,1)=-(.5)*(PLS(ROCT,QUAD,1,NODE)-PPLS(ROCT,QUAD,1))
DPLAS(1,2)=0.
DPLAS(1,3)=PLS(ROCT,QUAD,3,NODE)-PPLS(ROCT,QUAD,3)
DPLAS(2,1)=0.
DPLAS(2,2)=DPLAS(1,1)
DPLAS(2,3)=PLS(ROCT,QUAD,2,NODE)-PPLS(ROCT,QUAD,2)
DPLAS(3,1)=DPLAS(1,3)
DPLAS(3,2)=DPLAS(2,3)
DPLAS(3,3)=PLS(ROCT,QUAD,1,NODE)-PPLS(ROCT,QUAD,1)

```

C VOLUMETRIC STRAIN

ELAVOL=(STT(1,1)+STT(2,2)+STT(3,3))*RNU/E

C ADDING UP VARIOUS STRAINS .BEFORE NECK FRACTURE STAGE.

C FOR THE FIRST STEP CAL ONLY ELASTIC STRAINS

DO 10 I=1,3

DO 10 J=1,3

IF (I.EQ.J)THEN

DELTA=1

ELSE

DELTA=0.

END IF

C ELAST IS ZERO FOR KOUNT.GT.1 FOR CONSTANT STRESS

ELAST(I,J)=((1+RNU)/E*STT(I,J)-DELTA*ELAVOL.)

IF (K1.GE.2)THEN

ELAST(I,J)=0

END IF

C SUM UP THE STRAIN INCREMENTS

C DELEPS GIVES THE TOTAL STRAIN INCREMENT

IF(K1.EQ.1)THEN

DELEPS(I,J)=ELAST(I,J)

ELSE IF(K1.LE.3000)THEN

DELEPS(I,J)=(CRE(I,J)-PRCRE(I,J))

DELEPS(I,J)=DELEPS(I,J)+DPLAS(I,J)

END IF

10 CONTINUE

DO 20 I=1,3

20 PESIGM(ROCT,QUAD,I,NODE)=ESIGM(ROCT,QUAD,I)

C CONVERT THE STRAINS TO 3-D ARRAY TO KEEP TRACK OF OCTANT AND QUADS TO
C WHICH THEY BELONG. DELEPS(1,2)=0.0, DELEPS(1,1)=DELEPS(2,2)

EPSAV(ROCT,QUAD,1)=DELEPS(3,3)

EPSAV(ROCT,QUAD,2)=DELEPS(2,3)

EPSAV(ROCT,QUAD,3)=DELEPS(1,3)

EPSAV(ROCT,QUAD,4)=DELEPS(1,1)

C TOTSTR IS THE TOTAL STRAIN IN THE NECK.

C TOTAL STRAIN NEEDS TO BE CALCULATED ONLY IF SUPERPLASTIC
C DEFORMATION IS NOT TAKING PLACE.

IF(IFRACT(ROCT,QUAD,NODE).NE.1)THEN

DUMM1=EPSAV(ROCT,QUAD,1)+ELAST(3,3)

DUMM2=EPSAV(ROCT,QUAD,3)+ELAST(1,3)

DUMM3=EPSAV(ROCT,QUAD,2)+ELAST(2,3)

DUMM4=EPSAV(ROCT,QUAD,4)+ELAST(1,1)

TOTSTR(ROCT,QUAD,1,NODE)=TOTSTR(ROCT,QUAD,1,NODE)+DUMM1

TOTSTR(ROCT,QUAD,3,NODE)=TOTSTR(ROCT,QUAD,3,NODE)+DUMM2

TOTSTR(ROCT,QUAD,2,NODE)=TOTSTR(ROCT,QUAD,2,NODE)+DUMM3

TOTSTR(ROCT,QUAD,4,NODE)=TOTSTR(ROCT,QUAD,4,NODE)+DUMM4

IF (PRINC(NODE).GT..725)THEN

TOT(1,1)=(TOTSTR(ROCT,QUAD,4,NODE))

TOT(1,2)=0.

TOT(1,3)=(TOTSTR(ROCT,QUAD,3,NODE))

```

TOT(2,1)=0.
TOT(2,2)=TOT(1,1)
TOT(2,3)=(TOTSTR(ROCT,QUAD,2,NODE))
TOT(3,1)=TOT(1,3)
TOT(3,2)=TOT(2,3)
TOT(3,3)=(TOTSTR(ROCT,QUAD,1,NODE))

```

```

N=3

```

```

C JACOBI CALCULATES THE EIGEN VALUES AND EIGEN VECTORS.

```

```

CALL JACOBI(TOT,N,N,D,V,NROT)

```

```

C FIND LARGEST EIGEN VALUE

```

```

RMAX=(D(1))
IF((D(2)).GT.RMAX)THEN
  RMAX=(D(2))
END IF
IF((D(3)).GT.RMAX)THEN
  RMAX=(D(3))
END IF
END IF

```

```

C IF MAXIMUM PRINCIPAL TENSILE STRAIN IS GREATER THAN .005 THEN THE
C NECK FRACTURES. IFRACT=1 MEANS NECK HAS FRACTURED.

```

```

IF (RMAX.GT..005)THEN
  IFRACT(ROCT,QUAD,NODE)=1
END IF

```

```

RETURN
END

```

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