An engineering analysis of systems containing time-varying elements
by Philip Alan Buckley

A thesis submitted to the graduate faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Electrical Engineering
Montana State University

Abstract:
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Contents of the thesis are as follows: First, electrical networks containing sinusoidally varying capacitors and excited by sinusoidal sources are investigated. By a unique block diagram representation these networks are visualized as feedback systems and from the diagrams a logical and natural series solution is derived in the complex frequency domain from a basic cause and effect relationship. Individual terms of the series contain the contributions of the various frequencies present in the solution. Second, two sufficient conditions for the absolute and uniform convergence of the series solutions are developed, one valid for the solution after steady-state conditions have been reached, the other valid for the total time solution including transient response. A different series for the steady-state solution is obtained by method of harmonic balance and the convergence criteria for this series is compared with the convergence criteria for the original series. Third, the method of analysis is applied to the general case of equations with time-varying coefficients and the method is extended to include the following cases: (l) nonsinusoidal yet periodic variation of system parameters, (2) more than one time-varying element in the system, (3) systems in which all the elements are time-varying, (4) arbitrary time variations of system parameters, and (5) time-varying systems excited by modulated signals.

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ABSTRACT

In this thesis a general method of analysis is developed for single input, single output, linear, time-varying systems by use of familiar Laplace transform techniques. The method yields the pertinent information necessary for the engineering utilization of any system containing time-dependent parameters.

Contents of the thesis are as follows: First, electrical networks containing sinusoidally varying capacitors and excited by sinusoidal sources are investigated. By a unique block diagram representation these networks are visualized as feedback systems and from the diagrams a logical and natural series solution is derived in the complex frequency domain from a basic cause and effect relationship. Individual terms of the series contain the contributions of the various frequencies present in the solution. Second, two sufficient conditions for the absolute and uniform convergence of the series solutions are developed, one valid for the solution after steady-state conditions have been reached, the other valid for the total time solution including transient response. A different series for the steady-state solution is obtained by method of harmonic balance and the convergence criteria for this series is compared with the convergence criteria for the original series. Third, the method of analysis is applied to the general case of equations with time-varying coefficients and the method is extended to include the following cases: (1) nonsinusoidal yet periodic variation of system parameters, (2) more than one time-varying element in the system, (3) systems in which all the elements are time-varying, (4) arbitrary time variations of system parameters, and (5) time-varying systems excited by modulated signals.

CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1 INTRODUCTION

Most of the techniques that engineers apply to the solution of linear systems with constant parameters fail to be of much value to them in the solution to problems dealing with systems with varying parameters. If the parameters vary in such a way as to be functions of one or more of the system's dependent variables, then the system is immediately nonlinear and the method of solution is sensitive to the type and severity of the nonlinearity. If, on the other hand, the parameters vary in such a way as to be functions of the system's independent variable (most often time), then the system remains linear and is described by a linear differential equation with varying coefficients. In problems of the latter type, the engineer is equipped with several tools which he commonly applies to constant parameter systems which are also valid and powerful tools for the solution of systems with variable parameters. One of the most important of these is the principle of superposition.

It would be ideal if all systems with time-varying parameters could be characterized in complete mathematical detail by the appropriate differential equation and solved to yield all the necessary information. However, apart from simple first order differential equations with varying coefficients, little is known about such solutions. In an effort to obtain some sort of answer, the engineer usually attempts to simplify the system model or to solve the problem by numerical techniques, neither of which may yield the desired qualitative or quantitative result or insight. It would be desirable if there were other techniques available which an engineer could apply to obtain useful information from time-varying parameter systems without having to resort to doubtful approximations or numerical techniques. In the following section, works from the technical literature directed toward this end are discussed.

1.2 LITERATURE REVIEW

Recorded history of solutions to time-varying equations begins with the publication, in 1868, of the solutions to the Mathieu equation (50). In 1877, Hill (37) generalized the Mathieu equation and in 1883 Floquet (28) published a general treatment of linear differential equations with periodic coefficients. Mathieu and Hill equations are special cases in the Floquet theory (52). In 1887, by means of Hill's analysis, Lord Rayleigh studied the Melde experiment, which concerned the maintenance of vibrations in a body by forces of double frequency (47).

Variable parameter system analysis attracted scant attention among engineers and physicists prior to 1912. A possible explanation was the lack of physical application and the analytical difficulties. Whittaker (85) in 1912, and van der Pol (77) in 1928, studied the stability of solutions obtained by Mathieu, Hill, and Floquet. The phenomenon of parametric excitation was investigated in 1931 by Mandelstam and Papalexi (49) and later by Lazarev (44), Migulin (55), Barrow (8,9,10), and Gorelik (34,35).

In 1917 Nichols (60), and in 1921 Carson (18) investigated systems such as the variable capacitor microphone and the induction generator with variable mutual inductance. Carson showed that a system described by a differential equation with varying coefficients could also be written as an integral equation of the Volterra type, possessing an infinite series solution, which unfortunately and unexplainedly diverged in many cases of practical interest. Carson neglected to give any criteria relative to conditions under which his series solution would converge. Later in the same paper Carson suggested yet another series for steady-state solutions but did not develop the technique in detail. Bolle (15), in 1955, furthered this technique, commonly referred to as Harmonic Balance, by using phasor notation to obtain the first steady-state solution usable to an engineer. Later, Desoer (24) in 1959 and Sandberg (66) in 1964 contributed to Bolle's technique.

A parallel path of development has taken place since 1922, due to the increasing interest in frequency modulation. Analysis was done on constant parameter systems with variable frequency excitation and on the similarity between these and time-varying systems. Carson and Fry (19) in 1937, studied the conditions for a quasi-steady-state to apply, and their work was extended by several others including Armstrong (2), Stumpers (75), Weiner (80), and Baghdady (4,5). In 1964, Weiner and Leon (81,82) actually considered the quasi-steady-state plus distortion response of a simple resistor and time-varying capacitor network excited by an FM signal.

From 1950 to 1960, Zadeh (87,88,89,90,91,92,93,94,95,96) published many papers dealing with time-varying and stochastic parameter systems. In particular, he contributed a great deal to the theory of time-varying systems in the specific area of time-varying weighting functions (impulse responses) and their transforms. The papers of Zadeh contribute much to the theory of time-varying systems but the material contained therein is little comfort in that it does not provide a usable engineering analysis unless the system parameters vary slowly with time.

In 1960, Leon (45) reduced a second order differential equation describing a network containing a single sinusoidally varying element to a second order difference equation by transform techniques. In 1962, Schweizer (71,72) using phasor methods, solved essentially the same sort of problem. Confined to the steady-state response, both authors obtained an infinite series solution. The coefficients of the series were continued fractions. Although Leon developed convergence criteria for the continued fractions and estimated the truncation error, no physical significance is attributed to them.

Analytical techniques commonly used in the analysis of nonlinear systems have often also applied to linear time-varying systems with much success (97,98). Perturbation methods, for example, when suitably applied to the problems discussed within this thesis, yield results in the time-domain very similar to the frequency domain results obtained in this thesis.

1.3 PURPOSE OF THE THESIS

The purpose of the thesis is to contribute a sound engineering method of analysis applicable to linear systems containing elements which are functions of the system's independent variable. The types of systems considered are single input, single output, lumped linear systems containing parameters which are arbitrary deterministic functions of time.

A key term in the statement of the purpose of the thesis is the adjective 'engineering'. To be of any engineering value, the method of analysis must provide a solution in such a form that the quantities of engineering value are readily accessible and the numerical results require as little human effort as possible. The method should be mathematically sound and limitations, if any, should be clearly defined and related to the system's physical parameters whenever possible.

CHAPTER 2

A SERIES SOLUTION FOR LINEAR TIME-VARYING SYSTEMS

2.1 INTRODUCTION

Equations characterizing linear time-varying systems are similar to those characterizing linear time-invariant systems with the exception that in the time-varying case, one or more of the coefficients of the equation are functions of time. Figure 2.1 illustrates a time-varying linear system with a single input x(t) and a single output y(t).

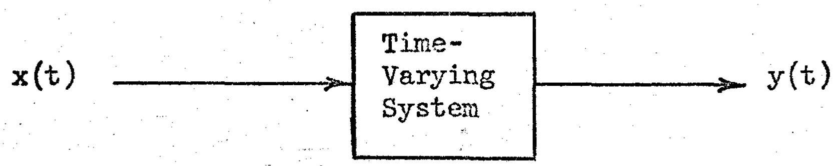

Figure 2.1 Single Input, Single Output Time-Varying System

The linear time-varying system shown in Figure 2.1 is assumed to be characterized by equations of the following form:

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \dots + a_o(t)y(t)$$

$$= b_m(t)x^{(m)}(t) + b_{m-1}(t)x^{(m-1)}(t) + \dots + b_o(t)x(t) \qquad (2.1)$$

Any set of n linearly independent functions $y_1(t)$, $y_2(t)$, ..., $y_n(t)$ satisfying the homogeneous, or unforced, part of equation 2.1, but not necessarily satisfying any particular boundary conditions, is termed a fundamental set of solutions or a set of basis functions. Any solution satisfying the homogeneous or unforced part of equation 2.1 must be a linear combination of $y_1(t)$, $y_2(t)$, ..., $y_n(t)$. Therefore, once any set of basis functions is known, the particular solution satisfying a given set of n boundary conditions can be found by algebraic manipulations of the known basis functions and their derivatives (1).

The classical approach to the solution of time-varying differential equations such as equation 2.1 hinges on the determination of a set of basis functions for the unforced portion of the solution, and the use of the superposition, or convolution, integral to determine the response of the system to particular inputs. The major problem herein is that the basis functions are generally difficult to determine and no general procedure exists for determining a set of basis functions for an arbitrary linear differential equation with time-varying coefficients. If a set of basis functions for a particular linear time-varying differential equation is not known (i.e., the differential equation doesn't belong to a class of equations for which the basis functions are known) or cannot be determined, then the solution to the equation cannot be obtained by the classical method and another technique must be found.

2.2 LAPLACE TRANSFORM METHOD

The use of Laplace transforms for analyzing linear time-invariant systems is attractive because a differential equation may be transformed into an algebraic equation and algebraic equations as a class tend to be much easier to solve than do differential equations. On the other hand, the use of Laplace transforms in this usual way for analyzing linear time-varying systems is not so attractive. The reason lies in the fact that the transformed equation is rarely an algebraic equation as it is when analyzing time-invariant systems. Most often the resulting transformed equation is a differential equation in the complex variable s, or it may be a difference equation, or it may be a combination of both -- a differential-difference equation. In addition, even the resulting transformed equations generally contain varying coefficients and may or may not be easier to solve than the original equation. Fortunately, these difficulties in the use of the Laplace transform for analyzing time-varying systems may be circumvented by the method to be introduced in Section 2.3.

There may be, however, other difficulties associated with the use of Laplace transforms as an equation solving technique whether applied to linear time-varying or time-invariant systems. First of all, the Laplace transform must exist for each term $a_i(t)x^{(i)}(t)$ of the differential equation, which means that the integral

$$\int_0^t a_i(t)x^{(i)}(t)e^{-st}dt \tag{2.2}$$

must converge for some positive, real value of s (46). It is quite possible that equations could be encountered for which the integral 2.2

would diverge for one or more terms of the equation, with the result that the Laplace transform method would fail to yield a solution.

Another difficulty in the use of the Laplace transform method is that after the differential equation is Laplace transformed and the resulting equation solved, the inverse transform of the result must be obtained. This inversion process may, depending upon the complexity of the transformed expression, involve lengthy calculations although the procedure in most cases is straightforward (7,46).

Despite the several difficulties mentioned above, there are still some advantages of the Laplace transform method over and above its sometime ability to yield a simpler-to-solve transformed equation. The first of these is that the transformed variables contain valuable information as to their frequency characteristics that is lost when the differential equations are solved by the classical method of Section 2.1. It is this information about frequency characteristics that is necessary so that the designer may determine such important information as the frequencies present in both the input and output of the system, the bandwidth of the system, the frequency dependent system gain, and the figure of merit Q of the system.

The second additional advantage is that when the inverse of the transformed solution is obtained, the solution is complete. Both the transient solution and the steady-state solution are obtained simultaneously. There is no need, as in the classical method of Section 2.1, to find a set of basis functions to determine the unforced solution and then to evaluate the particular convolution integral to obtain the steady-state solution. The Laplace transform method obtains both parts of the solution simultaneously.

2.3 A SERIES SOLUTION FOR TIME-VARYING NETWORKS

Often, not all the coefficients of a time-varying linear differential equation vary with time. In some cases, only a few or perhaps just a single coefficient actually is time dependent and even then the coefficient may vary about a fixed value; in which case it may be characterized by the sum of a constant part and a time-varying part, such as

$$a(t) = a_o + a_1(t) \tag{2.3}$$

where $a(t)$ is the total coefficient, a_o is the constant part, and $a_1(t)$ is the time-varying part of the coefficient.

As a specific instance of the sort of time-varying differential equation described above, consider the time-varying electrical network shown in Figure 2.2.

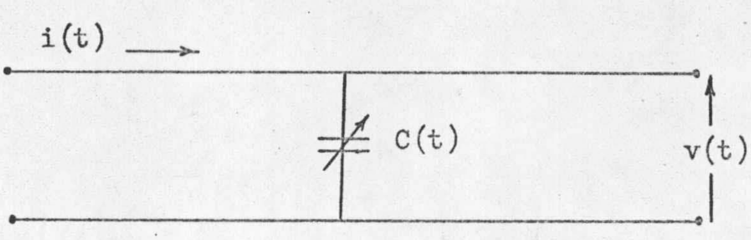

Figure 2.2 Time-Varying Electrical Network

The network contains only a single capacitor varying at a sinusoidal rate of w_c radians per second according to the relation

$$C(t) = C_o + C_1 \cos(w_c t) \tag{2.4}$$

In a physically realizable network the total capacitance would never be negative so the additional requirement that $C_o > C_1$ would guarantee that the equation 2.4 represents a realizable network. Equation 2.4 could also be visualized as describing the time variations of the capacitor network shown in Figure 2.3.

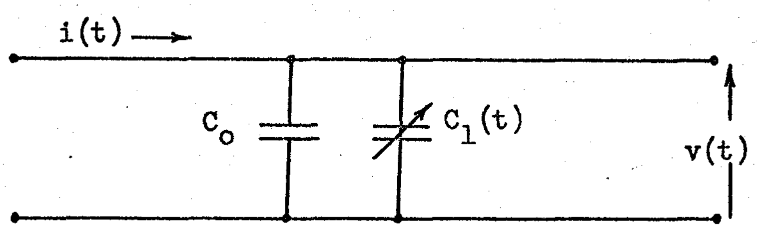

Figure 2.3 Capacitor Network Described by Equation 2.4

The network of Figure 2.3 consists of a fixed capacitor C_o and a time-varying capacitor $C_1(t)$, where $C_1(t)$ is as defined in equation 2.5.

$$C_1(t) = C_1 \cos(w_c t) \tag{2.5}$$

Therefore, it can be seen that the networks of Figures 2.2 and 2.3 are mathematically identical. As shown in both figures, the networks are

are excited by a current $i(t)$, and the problem is to determine the output voltage response $v(t)$. Both networks are described by the time-varying differential equation 2.6.

$$i(t) = \frac{d}{dt}\left[\, C(t)v(t) \,\right] \qquad v(t=0) = 0$$

$$= C_o \frac{dv(t)}{dt} + C_1 \frac{d}{dt}\left[\cos(w_c t) \cdot v(t)\right] \tag{2.6}$$

The first term on the right side of the above equation represents the current flowing through capacitor C_o in Figure 2.3, and the second term represents the current flowing through capacitor $C_1(t)$ and is designated $i_c(t)$. Equation 2.6 may then be rewritten as equation 2.7.

$$i(t) = C_o \frac{dv(t)}{dt} + i_c(t) \tag{2.7}$$

Equation 2.7 may then be Laplace transformed and written as

$$I(s) - I_c(s) = C_o sV(s) \tag{2.8}$$

or

$$V(s) = \frac{1}{sC_o}\left[I(s) - I_c(s)\right] = Z(s)\left[I(s) - I_c(s)\right] \tag{2.9}$$

$Z(s)$, as it appears in equation 2.9, is the impedance of the fixed part of the network of Figure 2.3. In this case, it is the impedance of the capacitor C_o. Furthermore, equation 2.9 suggests the following very interesting interpretation. The voltage response V(s) of a time-varying network to a current I(s) is exactly the same as the voltage response V(s) of the <u>fixed</u> part of the same network to the new forcing current $I_T(s)$ shown in equation 2.10.

$$I_T(s) = I(s) - I_c(s) \qquad (2.10)$$

This fact becomes apparent after examining equation 2.9. Figure 2.4 portrays the meaning of equation 2.9 in block diagram form.

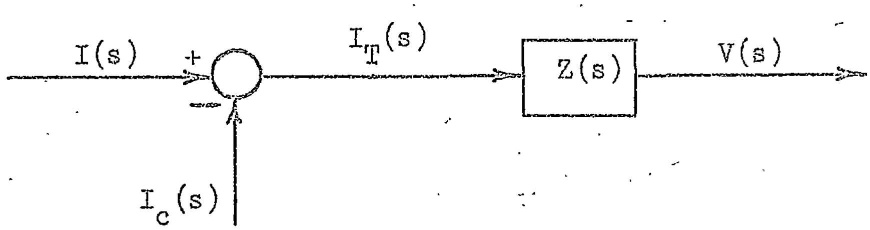

Figure 2.4 Block Diagram Model of Equation 2.9

The current $i_c(t)$ which flows through the time-varying capacitor is a function of the voltage v(t) across the capacitor. This means that $I_c(s)$ must necessarily be some specific function of V(s), and let this functional relationship be represented by the symbol F.

$$I_c(s) = F\left[V(s)\right] \qquad (2.11)$$

The transformed current $I_c(s)$ may be obtained by performing upon $V(s)$ the operation signified by F. The symbol F, then, is not a transfer function in the usual sense of the word; rather, it represents an operation to be performed.

With the concept of the operator F understood, a feedback path containing the element marked F may be inserted in the block diagram describing the network.

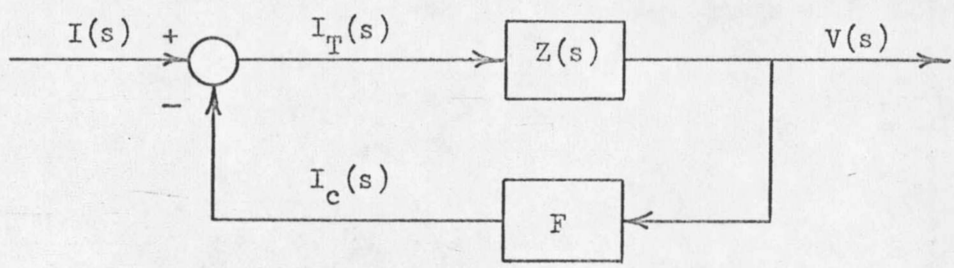

Figure 2.5 Block Diagram Model With Feedback Path

For this specific network, the operation designated by F may be determined by Laplace transforming equation 2.12, which relates $i_c(t)$ to $v(t)$

$$i_c(t) = C_1 \frac{d}{dt}\left[v(t)\cos(w_c t)\right] , \quad v(t=0) = 0$$

$$= \tfrac{1}{2}C_1 \frac{d}{dt}\left[v(t)e^{jw_c t} + v(t)e^{-jw_c t} \right] , \qquad (2.12)$$

to obtain equation 2.13 which relates $I_c(s)$ to $V(s)$.

$$I_c(s) = \tfrac{1}{2}sC_1 \left[V(s+jw_c) + V(s-jw_c) \right]$$

$$= Y(s) \left[V(s+jw_c) + V(s-jw_c) \right] \tag{2.13}$$

Here the operation indicated by F means take the sum of V(s) shifted up in frequency by jw_c and V(s) shifted down in frequency by jw_c and multiply this sum by the admittance function $\tfrac{1}{2}sC_1$.

Suppose, in Figure 2.5 above, a current I(s) is applied to the system, and also suppose that the feedback path is, for the moment, open circuited. At the output will appear a voltage, call it $V_o(s)$.

$$V(s) = V_o(s) = I(s)Z(s) \tag{2.14}$$

If the feedback path is then reconnected, at the output will appear another voltage $V_1(s)$ due to the current $I_{co}(s)$ caused by $V_o(s)$.

$$I_{co}(s) = F\left[V_o(s)\right] \tag{2.15}$$

$$V(s) = V_o(s) + V_1(s) \tag{2.16}$$

$$V_1(s) = -I_{co}(s)Z(s) \tag{2.17}$$

This new voltage $V_1(s)$ will cause yet another current $I_{c1}(s)$ which, in turn, will cause another voltage $V_2(s)$ to appear at the output of the system.

$$I_{c1}(s) = F\left[V_1(s)\right] \tag{2.18}$$

$$V(s) = V_o(s) + V_1(s) + V_2(s) \tag{2.19}$$

$$V_2(s) = -I_{c1}(s)Z(s) \tag{2.20}$$

As this process is repeated again and again, the total transformed output voltage $V(s)$ is arrived at by an infinite number of terms, as indicated in equation 2.21, and illustrated in Figure 2.6.

$$V(s) = V_o(s) + V_1(s) + V_2(s) + \ldots + V_n(s) + \ldots \tag{2.21}$$

The above series expression for $V(s)$ enables one to gain special insight into the solution because each term of the series has the following significance. $V_o(s)$ is the response of the fixed part of the system to an external current stimulus $I(s)$.

$$V_o(s) = I(s)Z(s) \tag{2.22}$$

$V_1(s)$ is the response of the fixed part of the system to a current $I_{co}(s)$.

$$V_1(s) = -I_{co}(s)Z(s) \tag{2.23}$$

$V_2(s)$ is the response of the fixed part of the system to a current $I_{c1}(s)$.

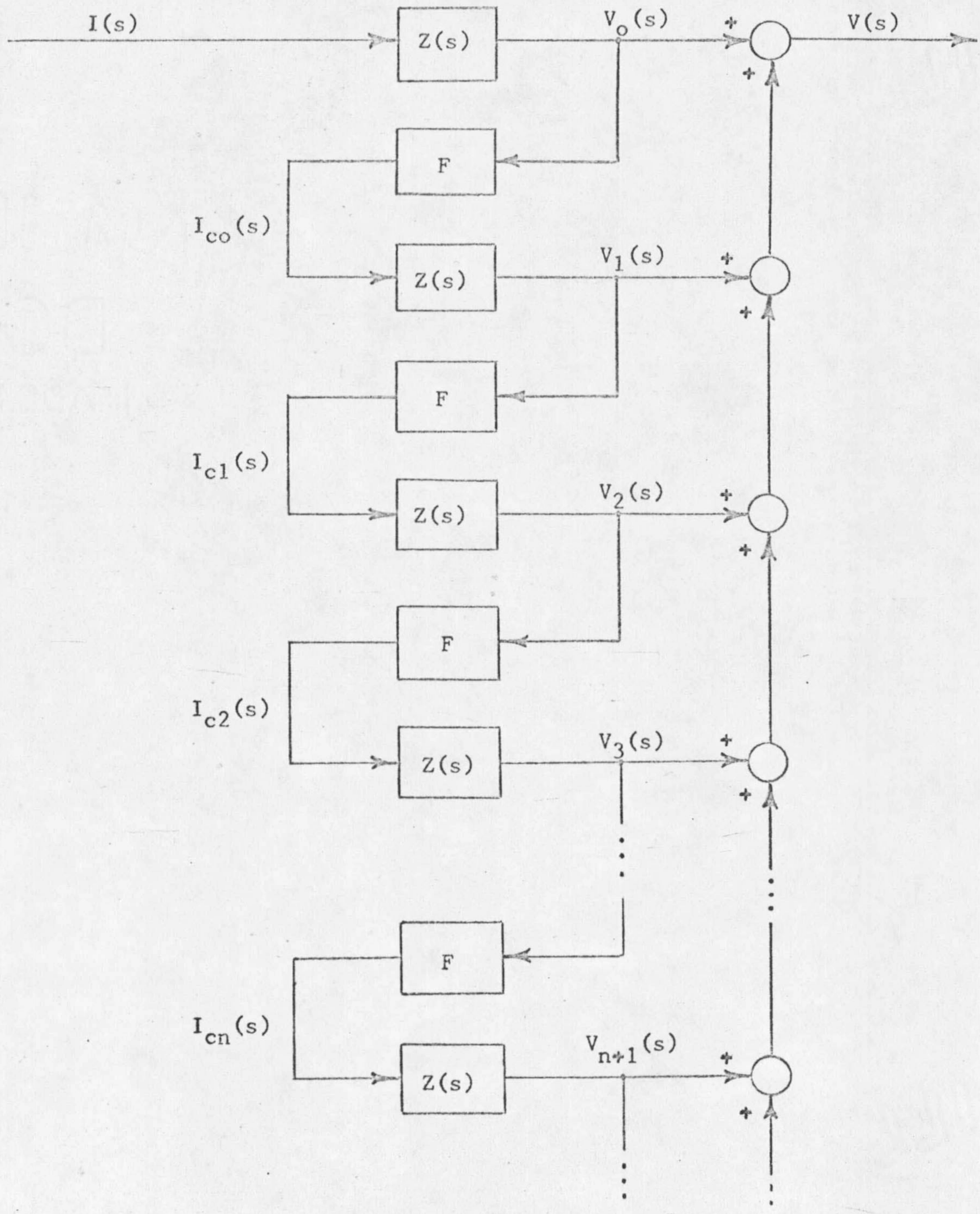

Figure 2.6 An Illustration of the Process By Which the Series Solution is Generated.

$$V_2(s) = -I_{c1}(s)Z(s) \tag{2.24}$$

And, in general, $V_{n+1}(s)$ is the response of the fixed part of the system to a current $I_{cn}(s)$.

$$V_{n+1}(s) = -I_{cn}(s)Z(s) \tag{2.25}$$

Using the procedure described above, it is now possible to obtain a series solution for linear, time-varying, differential equations, provided that each term of the equation is Laplace transformable. In addition, this series solution not only provides the steady-state solution but the total solution, transient included, and each term of the series has the useful physical significance explained above.

2.4 EXAMPLES OF THE APPLICATION OF THE SERIES SOLUTION

For the first of three examples, consider the network in Figure 2.7, introduced in Section 2.3.

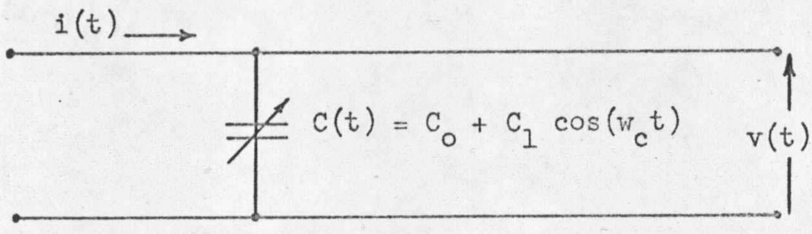

Figure 2.7 A Time-Varying Electrical Network

This network is described by the time-varying differential equation 2.26.

$$i(t) = C_o \frac{dv(t)}{dt} + C_1 \frac{d}{dt}\Big[\cos(w_c t) \cdot v(t)\Big], \qquad v(t=0) = 0$$

$$= C_o \frac{dv(t)}{dt} + \tfrac{1}{2}C_1 \frac{d}{dt}\Big[v(t)e^{jw_c t} + v(t)e^{-jw_c t}\Big] \tag{2.26}$$

Equation 2.27 is obtained by Laplace transforming equation 2.26.

$$I(s) = sC_oV(s) + \tfrac{1}{2}sC_1\Big[V(s+jw_c) + V(s-jw_c)\Big] \tag{2.27}$$

The above equation suggests the use of the following complete system block diagram.

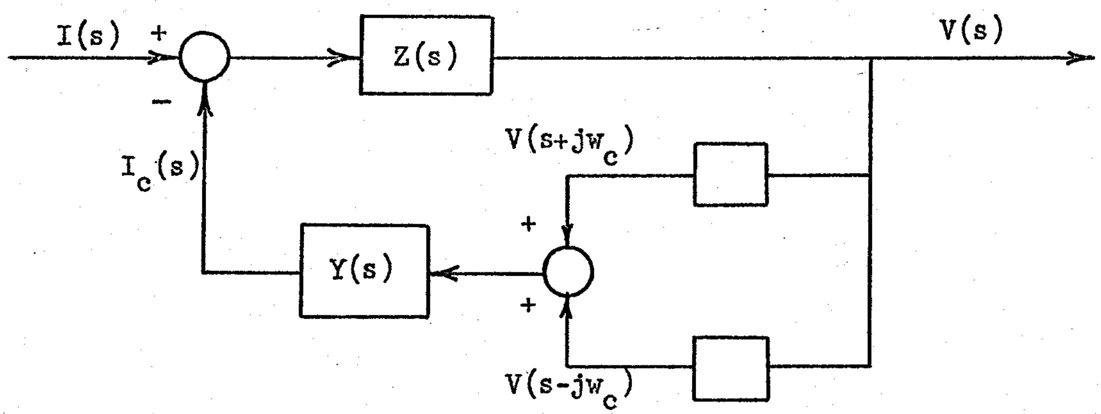

Figure 2.8 Complete Block Diagram of the System Described by Equation 2.27

The impedance $Z(s)$ shown in the forward path of Figure 2.8 is equal to the impedance of the fixed part of the capacitor in Figure 2.7.

$$Z(s) = \frac{1}{sC_o} \tag{2.28}$$

The admittance in the feedback path of Figure 2.8 is defined by equation 2.29.

$$Y(s) = \tfrac{1}{2}sC_1 \tag{2.29}$$

For conciseness in the following equations, a transfer function $G(s)$ is defined by equation 2.30.

$$G(s) = Z(s)Y(s) \tag{2.30}$$

Following the procedure set forth in Section 2.3, the individual terms of the series solution may be determined.

$$V(s) = V_o(s) + V_1(s) + V_2(s) + \ldots + V_n(s) + \ldots \tag{2.31}$$

Suppose, in Figure 2.8 above, a current $I(s)$ is applied to the system, and also suppose that the feedback path is for the moment open circuited. At the output will appear a voltage; call it $V_o(s)$.

$$V_o(s) = Z(s)I(s) \tag{2.32}$$

If the feedback path is then reconnected, at the output will appear another voltage $V_1(s)$ due to the current $I_{co}(s)$ caused by $V_o(s)$.

$$I_{co}(s) = Y(s)\left[\; V_o(s+jw_c) + V_o(s-jw_c) \;\right] \tag{2.33}$$

$$\begin{aligned} V_1(s) &= -Z(s)I_{co}(s) \\ &= -G(s)\left[\; V_o(s+jw_c) + V_o(s-jw_c) \;\right] \\ &= -G(s)\left[\; Z(s+jw_c)I(s+jw_c) + Z(s-jw_c)I(s-jw_c) \;\right] \end{aligned} \tag{2.34}$$

When the voltage $V_1(s)$ appears at the output it will cause yet another current $I_{c1}(s)$ which, in turn, will cause another voltage $V_2(s)$ to appear at the output of the system.

$$I_{c1}(s) = Y(s)\left[\; V_1(s+jw_c) + V_1(s-jw_c) \;\right] \tag{2.35}$$

$$\begin{aligned} V_2(s) &= -Z(s)I_{c1}(s) \\ &= -G(s)\left[\; V_1(s+jw_c) + V_1(s-jw_c) \;\right] \\ &= G(s)\Big[\; G(s+jw_c)\left[\; Z(s+j2w_c)I(s+j2w_c) + Z(s)I(s) \;\right] \\ &\qquad + G(s-jw_c)\left[\; Z(s)I(s) + Z(s-j2w_c)I(s-j2w_c) \;\right] \Big] \end{aligned} \tag{2.36}$$

The above process should be repeated until the desired number of terms of the series solution is obtained. Of course, the more accuracy one desires, the greater is the number of terms of the series that must be calculated.

From the results obtained in this example thus far it is apparent that each successive term of the series introduces two additional frequency components into the output voltage. Table I illustrates that the output voltage V(s) will contain the frequency of the input current I(s) plus an infinite number of the sum and difference frequencies between the frequency of the input current and the frequency of the capacitor variation. Also apparent in this table is the fact that the odd numbered terms of the series contribute only odd sideband frequency components and the even terms contribute only even sideband frequency components.

Up to this point in the example, the solution for the output voltage v(t) has been in terms of a general input current i(t) to a network with unspecified circuit values. In order that a more complete picture of the performance of this time-varying system be obtained, v(t) was calculated for the particular i(t) and the particular set of circuit parameters shown below.

$$i(t) = \cos(w_o t) \text{ amps.}$$

$$w_o = 2\pi \ (10^7) \text{ rad./sec.}$$

$$w_c = 2\pi \ (10^5) \text{ rad./sec.}$$

$$C(t) = C_o + C_1 \cos(w_c t) \text{ farads}$$

$$C_o = 300 \text{ mmf.} \qquad C_1 = 75 \text{ mmf.}$$

TERMS IN SERIES	FREQUENCY COMPONENTS PRESENT
$V_o(s)$	s
$V_1(s)$	$s+jw_c$ $s-jw_c$
$V_2(s)$	$s+j2w_c$ s $s-j2w_c$
$V_3(s)$	$s+j3w_c$ $s+jw_c$ $s-jw_c$ $s-j3w_c$
$V_4(s)$	$s+j4w_c$ $s+j2w_c$ s $s-j2w_c$ $s-j4w_c$
$\vdots$	$\vdots$
$V_n(s)$	$s+jnw_c$ $s+j(n-2)w_c$ ... ... $s+jw_c$ $s-jw_c$ ... ... $s-j(n-2)w_c$ $s-jnw_c$
$V_{n+1}(s)$	$s+j(n+1)w_c$ $s+j(n-1)w_c$ .... $s+j2w_c$ s $s-j2w_c$ ... $s-j(n-1)w_c$ $s-j(n+1)w_c$

Table I Terms of the Series Solution V(s) and Their Respective Frequency Components.

Two computer programs were written for the S.D.S. Sigma 7 computer which calculated the envelope of the response voltage for the input current i(t) and the circuit parameters shown above. The first of the two computer programs calculates the first four terms of the series for V(s) by the method described above and obtains v(t), the inverse transform of V(s) by evaluating the residues of V(s) using the Heaviside expansion method (7). In order to have a means for checking the accuracy of the series solution, the second computer program solved the time-varying differential equation 2.26 numerically for v(t) by the Runge-Kutta method (40). The results of the computer programs for this particular circuit are shown in Figure 2.9.

It should be emphasized at this point that in Figure 2.9, it is the envelope of the response voltage that is shown and not the total response voltage v(t). Notice that the series solution containing just three terms closely approximates the Runge-Kutta solution, and when the series contains four terms there is no visible difference between the series solution and the Runge-Kutta solution.

Figure 2.10 is a normalized illustration of the frequency content of the input current I(s) and the output voltage V(s) after the transients due to the input have subsided. Those frequencies contained within the dashed lines in Figure 2.10(b) are the frequency components present in the first four terms of the series solution shown in Figure 2.9. It is noteworthy that an illustration of the frequency characteristics such as this may only be obtained by the series method and is

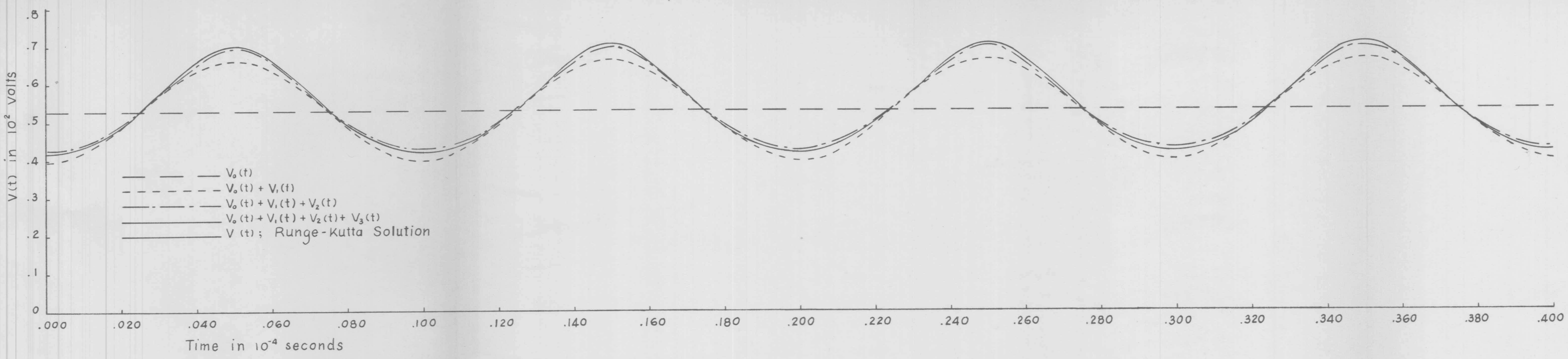

Figure 2.9 Plot of the Envelope of the Response Voltage of a Single Time-Varying Capacitor Excited by an Input Current $i(t) = \cos(\omega_0 t)$

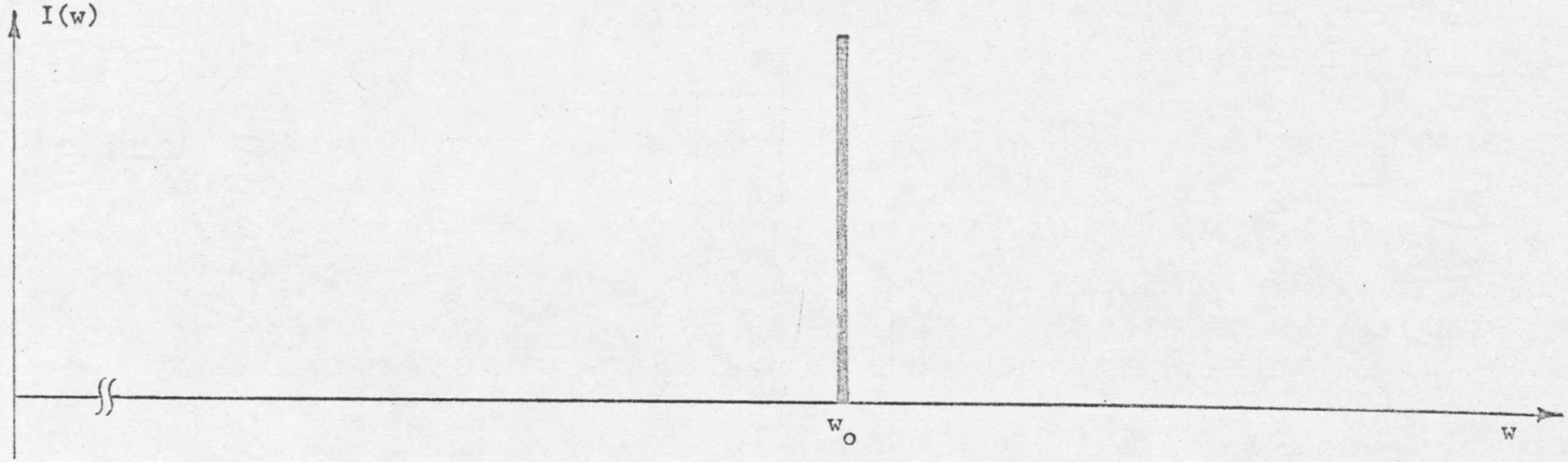

Figure 2.10(a) Normalized Plot of the Frequency Content of i(t).

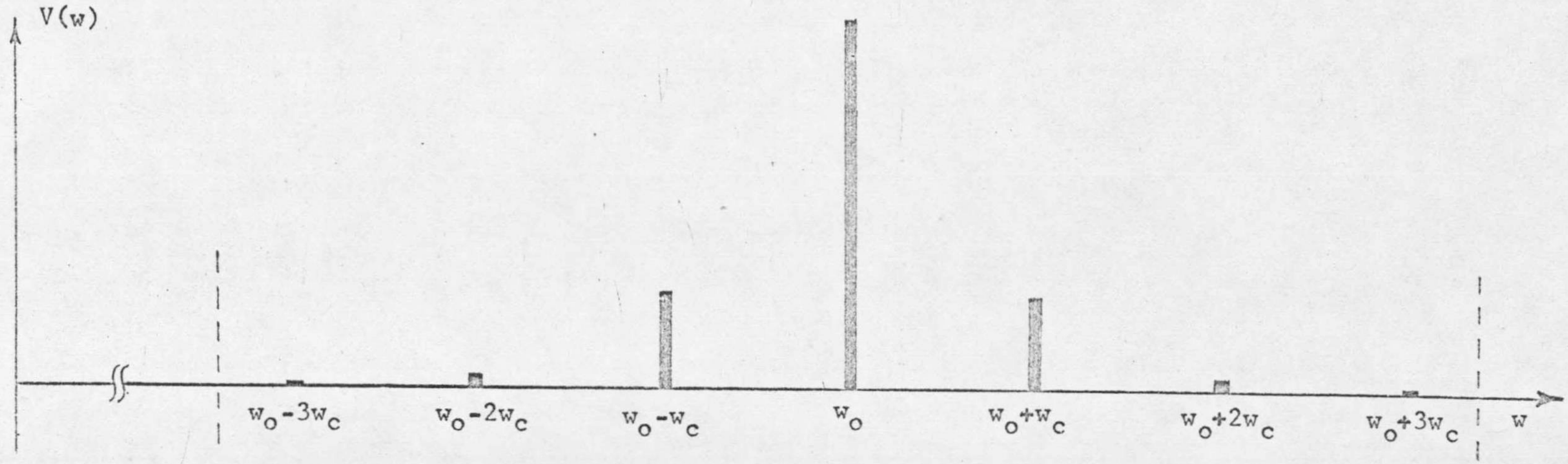

Figure 2.10(b) Normalized Plot of the Frequency Content of v(t).

not at all possible by using the Runge-Kutta method.

As a second example, consider the network shown in Figure 2.11. This network is similar to the one in the first example except that there is now a fixed resistor R in parallel with the time-varying capacitor C(t). The problem remains the same: Determine the voltage response v(t) to a driving current i(t).

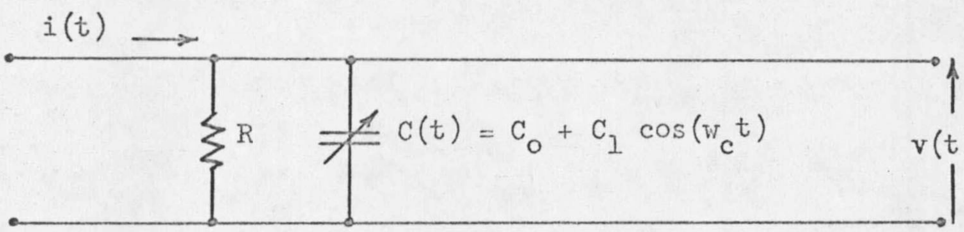

Figure 2.11 Network containing Fixed Resistor and Time-Varying Capacitor

This network is described by the time-varying differential equation 2.36.

$$
\begin{aligned}
i(t) &= \frac{1}{R} v(t) + \frac{d}{dt}\left[C(t)v(t) \right] \qquad (2.36) \\
&= \frac{1}{R} v(t) + C_o \frac{dv(t)}{dt} + C_1 \frac{d}{dt}\left[\cos(w_c t) \cdot v(t) \right] \\
&= \frac{1}{R} v(t) + C_o \frac{dv(t)}{dt} + \tfrac{1}{2} C_1 \frac{d}{dt}\left[v(t)e^{jw_c t} + v(t)e^{-jw_c t} \right]
\end{aligned}
$$

The procedure to be followed to obtain a solution v(t) for the above differential equation is exactly the same procedure explained in

Section 2.3 and used to obtain a solution in the previous example. The first step is to Laplace transform equation 2.36 to obtain equation 2.37.

$$I(s) = \left[\frac{1}{R} + sC_o\right] V(s) + \tfrac{1}{2}sC_1\left[V(s+jw_c) + V(s-jw_c)\right] \qquad (2.37)$$

The above transformed equation suggests the use of the following complete system block diagram.

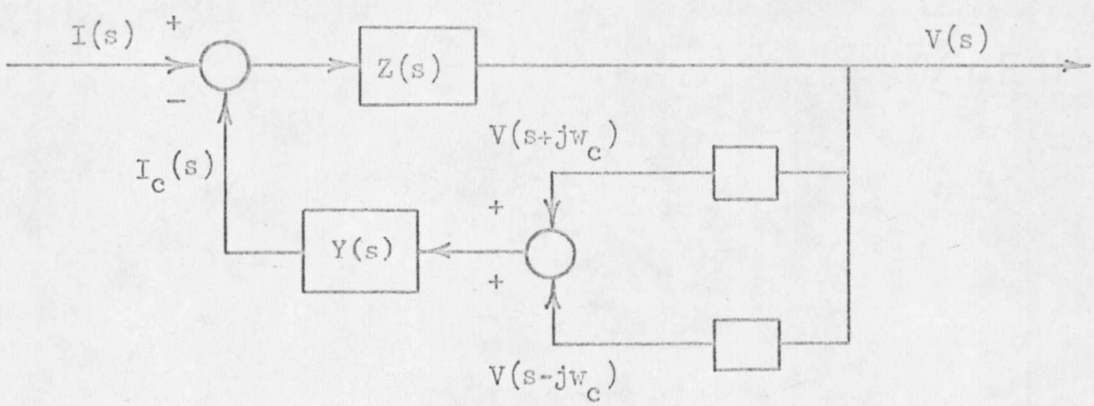

Figure 2.12 Complete Block Diagram of the System Described by Equation 2.37

The block diagram of Figure 2.12 is identical to the block diagram for the single capacitor shown in Figure 2.8 except in the above diagram the impedance of the fixed part of the network Z(s) has changed due to the introduction of the fixed resistance. All the remaining parameters are the same as in Figure 2.8.

$$Z(s) = \frac{1}{C_o} \cdot \frac{1}{s + \frac{1}{RC_o}} \tag{2.38}$$

As before, for conciseness, let G(s) be defined by equation 2.39.

$$G(s) = Y(s)Z(s) \tag{2.39}$$

Following the same procedure as before, it is possible to obtain the individual terms of the series solution.

$$V(s) = V_o(s) + V_1(s) + V_2(s) + \ldots + V_n(s) + \ldots \tag{2.40}$$

Suppose, in Figure 2.12 above, a current I(s) is applied to the system, and also suppose that the feedback path is, for the moment, open circuited. At the output will appear a voltage; call it $V_o(s)$.

$$V_o(s) = Z(s)I(s) \tag{2.41}$$

If the feedback path is then reconnected, at the output will appear another voltage $V_1(s)$ due to the current $I_{co}(s)$ caused by $V_o(s)$.

$$I_{co}(s) = Y(s)\left[V_o(s+jw_c) + V_o(s-jw_c)\right] \tag{2.42}$$

$$\begin{aligned} V_1(s) &= -Z(s)I_{co}(s) \\ &= -G(s)\left[V_o(s+jw_c) + V_o(s-jw_c) \right] \\ &= -G(s)\left[Z(s+jw_c)I(s+jw_c) + Z(s-jw_c)I(s-jw_c)\right] \end{aligned} \tag{2.43}$$

When the voltage $V_1(s)$ appears at the output it will cause yet another current $I_{c1}(s)$ which, in turn, will cause another voltage $V_2(s)$ to appear at the output of the system.

$$I_{c1}(s) = Y(s)\left[V_1(s+jw_c) + V_1(s-jw_c) \right] \tag{2.44}$$

$$\begin{aligned} V_2(s) &= -Z(s)I_{c1}(s) \\ &= -G(s)\left[V_1(s+jw_c) + V_1(s-jw_c) \right] \\ &= G(s)\Big[G(s+jw_c)\left[Z(s+j2w_c)I(s+j2w_c) + Z(s)I(s) \right] \\ &\quad + G(s-jw_c)\left[Z(s)I(s) + Z(s-j2w_c)I(s-j2w_c) \right] \Big] \end{aligned} \tag{2.45}$$

The above process should be repeated until the desired number of terms of the series solution is obtained. As in the previous example, each successive term of the series introduces two additional frequency components in the output voltage and Table I is again appropriate as it illustrates the terms of the series solution and their respective frequency components.

As a numerical example of this sort of time-varying network, v(t) was calculated for the particular i(t) and the particular set of circuit parameters shown below.

$$i(t) = \cos(w_o t) \text{ amps.}$$

$$w_o = 2\pi\ (10^7) \text{ rad./sec.}$$

$$w_c = 2\pi\ (10^5) \text{ rad./sec.}$$

$$C(t) = C_o + C_1 \cos(w_c t) \text{ fds.}$$

$$C_o = 300 \text{ mmf.} \qquad C_1 = 75 \text{ mmf.}$$

$$R = 265 \text{ ohms.}$$

Again, as in the first example, the first four terms of the series for the response voltage V(s) were evaluated by digital computer and this solution was compared to the computer solution of the differential equation 2.36 by the Runge-Kutta method. The plots of the envelope of the response voltage v(t) are shown in Figure 2.13. Notice the series solution containing just three terms closely approximates the Runge-Kutta solution, and when the series contains four terms there is no visible difference between the series solution and the Runge-Kutta solution.

Figure 2.14 is a normalized illustration of the frequency content of the input current I(s) and the output voltage V(s) after the transients due to the input have died away. Those frequencies contained within the dashed lines in Figure 2.14(b) are the frequency components present in the first four terms of the series solution shown in Figure 2.13.

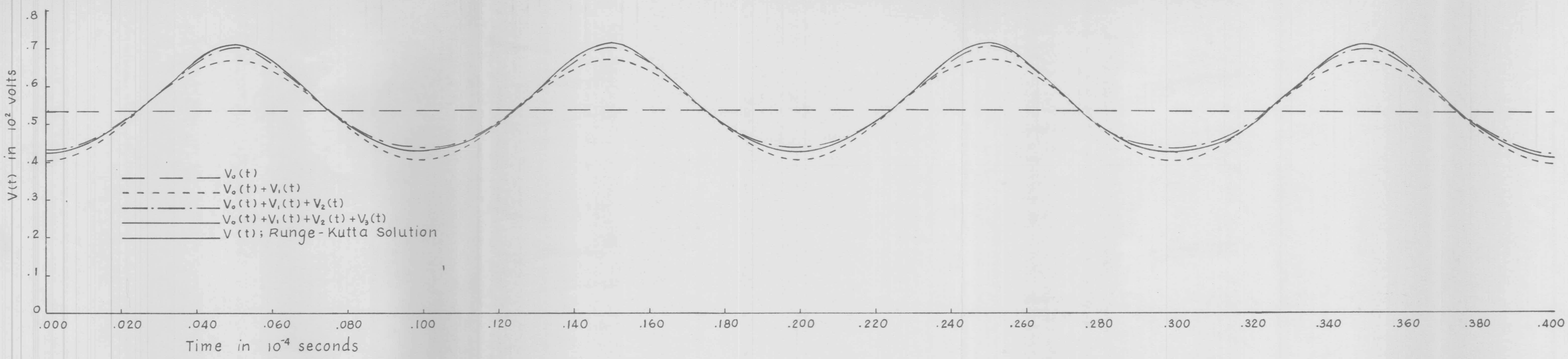

Figure 2.13 Plot of the Envelope of the Response Voltage of a Network Containing a Fixed Resistor and a Time-Varying Capacitor to an Input Current $i(t) = \cos(\omega_0 t)$

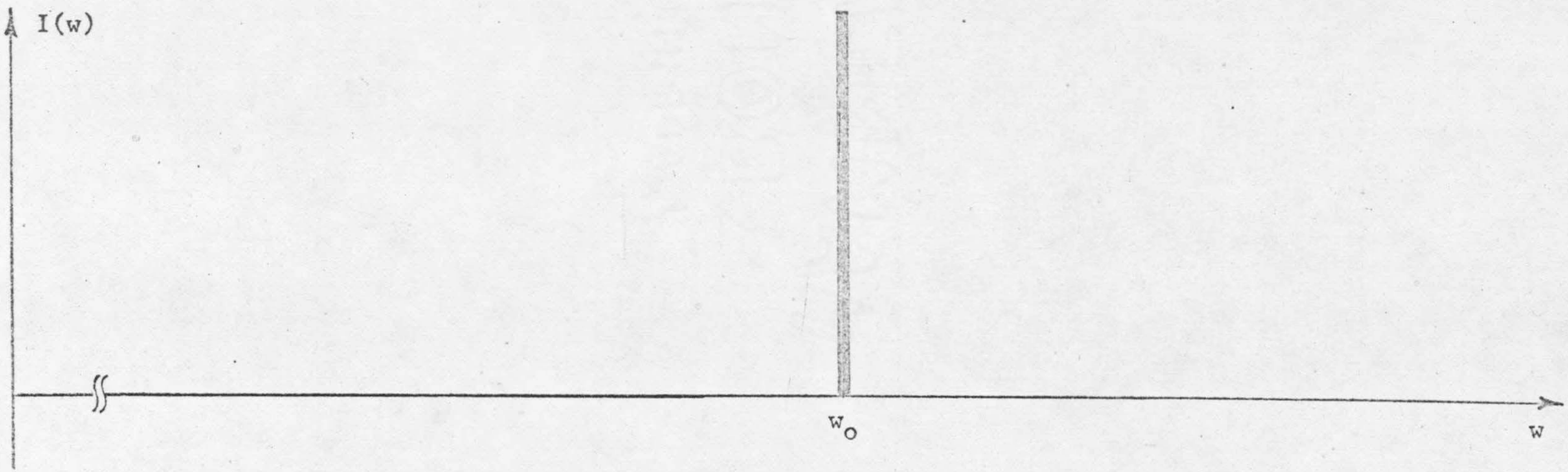

Figure 2.14(a) Normalized Plot of the Frequency Content of i(t) After Transients of v(t) Due to the Input Have Subsided.

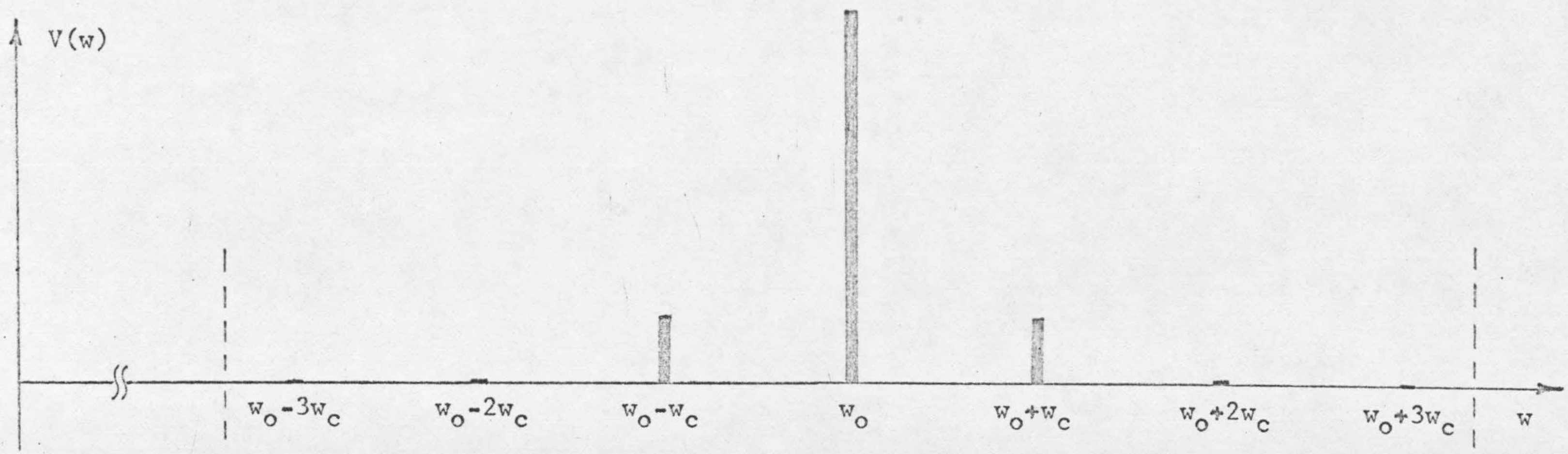

Figure 2.14(b) Normalized Plot of the Frequency Content of v(t) After the Transients Due to the Input Have Subsided.

As the last of the series of three examples, consider the network in Figure 2.15. The network contains fixed resistance R, fixed inductance L, and time-varying capacitance C(t).

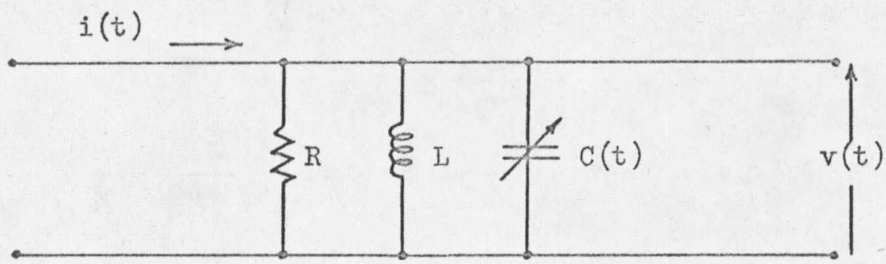

Figure 2.15 Time-Varying Network Containing Fixed Resistance, Fixed Inductance, and Time-Varying Capacitance

As before, the problem is to determine the response voltage v(t) to an input current i(t), and the procedure is the same as was used in both previous examples.

The network in Figure 2.15 is characterized by the time-varying differential equation 2.46.

$$i(t) = \frac{1}{R}\,v(t) + \frac{1}{L}\int_0^t v(t)dt + \frac{d}{dt}\Big[\,C(t)v(t)\Big], \quad v(t=0) = 0$$

$$= \frac{1}{R}\,v(t) + \frac{1}{L}\int_0^t v(t)dt + C_o\,\frac{dv(t)}{dt} + C_1\,\frac{d}{dt}\Big[v(t)\,\cos(w_c t)\Big]$$

$$= \frac{1}{R}\,v(t) + \frac{1}{L}\int_0^t v(t)dt + C_o\,\frac{dv(t)}{dt}$$

$$+ \quad \tfrac{1}{2}C_1\,\frac{d}{dt}\Big[v(t)e^{jw_c t} + v(t)e^{-jw_c t}\Big] \qquad (2.46)$$

Equation 2.47 is the Laplace transformed version of equation 2.46.

$$I(s) = \left[\frac{1}{R} + sC_o + \frac{1}{sL}\right]\cdot V(s) + \tfrac{1}{2}sC_1\Big[\,V(s+jw_c) + V(s-jw_c)\Big] \qquad (2.47)$$

The above transformed equation suggests the use of the following complete system block diagram.

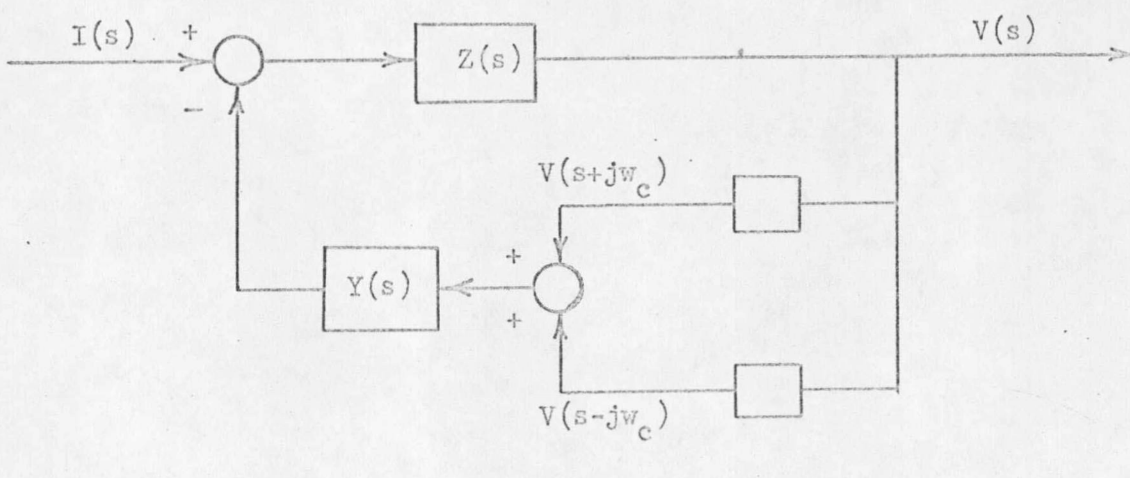

Figure 2.16 Complete Block Diagram of the System Described by Equation 2.47.

The block diagram of Figure 2.16 is identical to both the block diagrams arrived at in the two previous examples except that the impedance of the fixed part of the network Z(s) has changed due to the introduction of the fixed inductance.

$$Z(s) = \frac{1}{C_o} \cdot \frac{s}{s^2 + \frac{1}{RC_o} s + \frac{1}{LC_o}} \tag{2.48}$$

As before, let G(s) be defined by equation 2.49.

$$G(s) = Z(s)Y(s) \tag{2.49}$$

Following the same procedure as before, it is possible to obtain the individual terms of the series solution.

$$V(s) = V_o(s) + V_1(s) + V_2(s) + \ldots + V_n(s) + \ldots \tag{2.50}$$

Suppose, in Figure 2.17 above, a current I(s) is applied to the system, and also suppose that the feedback path is for the moment open circuited. At the output will appear the voltage $V_o(s)$.

$$V_o(s) = Z(s)I(s) \tag{2.51}$$

If the feedback path is then reconnected, at the output will appear another voltage $V_1(s)$ due to the current $I_{co}(s)$ caused by $V_o(s)$.

$$I_{co}(s) = Y(s) \left[V_o(s+jw_c) + V_o(s-jw_c) \right] \tag{2.52}$$

$$V_1(s) = -Z(s)I_{co}(s) \tag{2.53}$$

$$= -G(s)\left[V_o(s+jw_c) + V_o(s-jw_c) \right]$$

$$= -G(s)\left[Z(s+jw_c)I(s+jw_c) + I(s-jw_c)I(s-jw_c) \right]$$

When the voltage $V_1(s)$ appears at the output it will cause yet another current $I_{c1}(s)$ which, in turn, will cause another voltage $V_2(s)$ to appear at the output of the system.

$$I_{c1}(s) = Y(s)\left[V_1(s+jw_c) + V_1(s-jw_c) \right] \tag{2.54}$$

$$V_2(s) = -Z(s)I_{c1}(s) \tag{2.55}$$

$$= G(s)\left[V_1(s+jw_c) + V_1(s-jw_c) \right]$$

$$= G(s)\Big[G(s+jw_c)\left[Z(s+2jw_c)I(s+2jw_c)+ Z(s)I(s)\right]$$

$$+ G(s-jw_c)\left[Z(s-2jw_c)I(s-2jw_c)+ Z(s)I(s)\right]\Big]$$

The above process should be repeated until the desired number of terms of the series solution is obtained. As in the previous two examples, each successive term of the series introduces two additional frequency components in the output voltage and Table I is again appropriate as it illustrates the terms of the series and their respective frequency components.

As a numerical example of the solution of this type of network, the following values were used in the calculations.

$$i(t) = \cos(w_o t) \text{ amps.}$$

$$w_o = 2\pi \, (10^7) \text{ rad./sec.}$$

$$w_c = 2\pi \, (10^5) \text{ rad./sec.}$$

$$C(t) = C_o + C_1 \cos(w_c t) \text{ farads}$$

$$C_o = 300 \text{ mmf.}$$

$$C_1 = 12.37 \text{ mmf.}$$

$$R = 26{,}525 \text{ ohms.}$$

$$L = 0.8443 \text{ mh.}$$

As a result of the particular circuit parameters shown above, the fixed part of the network (the part consisting of R, L, and C_o) is in resonance with the frequency of the input current.

$$w_r = \sqrt{\frac{1}{LC_o} - \frac{1}{(2RC_o)^2}} = 2\pi \, (10^7) \text{ rad./sec.} \qquad (2.56)$$

Because the network capacitance is not constant and does vary with time, the resonant frequency of the network $w_r(t)$ also changes according to equation 2.57.

$$w_r(t) = \sqrt{\frac{1}{LC(t)} - \frac{1}{\left[2RC(t)\right]^2}} \quad \text{rad./sec.} \qquad (2.57)$$

The resonant frequency $w_r(t)$ varies around w_o from a high of 6.41 Mhz. to a low of 6.16 Mhz.

The figure of merit Q for the fixed part of the network is given by equation 2.58.

$$Q = \frac{R}{w_o L} = 500 \qquad (2.58)$$

The value of this figure of merit Q for the entire time-varying network obviously is not constant on account of the changing capacitance but nevertheless varies less than four percent away from the value shown in equation 2.58.

The bandwidth of the total network also is not constant and is given by equation 2.59.

$$B.W. = \frac{w_r(t)}{Q(t)} \approx \frac{w_o}{Q} = 4\pi\ (10^4)\ \text{rad./sec.} \qquad (2.59)$$

The purpose of introducing the quantities of resonant frequency, figure of merit, and bandwidth is not to aid in the solution for V(s) but rather to give some qualitative insight into the operation of this particular network. For example, since a Q of 500 is extremely high, one would expect this network to have an extremely narrow bandwidth.

As a direct consequence of this fact, small variations in the value of the capacitance would tend to have a relatively large effect in detuning the network away from the resonant center frequency w_r, and this would cause large variations in the impedance of the network at the input frequency w_o.

Again, as in the first and second examples, the first four terms of the series for the response voltage V(s) were evaluated by digital computer and this solution was compared to the computer solution of the differential equation 2.46 by the Runge-Kutta method. The plots of the envelope of the response voltage v(t) are shown in Figure 2.17.

It is quite apparent in this example that the series solution did not yield the correct solution as given by the results of the Runge-Kutta method, and in fact, the series diverged rapidly. These results immediately pose several questions that demand to be answered: "What were the differences in the three examples that caused the solutions to the first two to converge and the series solution to the third example to diverge?" Also, "If the reasons can be determined, how can one predict when the series solution will yield useful results and when it will not?" These questions and several more related questions will be examined in Chapter 3.

Figure 2.17 Plot of the Envelope of the Response Voltage of a Network Containing Fixed Resistor, Fixed Inductance, and Time-Varying Capacitor to an Input Current $i(t) = \cos(W_0 t)$

CHAPTER 3

PROBLEMS OF CONVERGENCE OF THE SERIES SOLUTION

3.1 INTRODUCTION

Proceeding through the derivation of the series solution in Chapter 2, one is apt to feel that because of the natural way the series is derived, each successive term being a function of the network and the preceding term, that the series is bound to converge provided the network being analyzed is physically realizable. Such is not the case, however, as was clearly illustrated in the last example of Chapter 2, in which the network was physically realizable and yet the series solution diverged.

It is the purpose of this chapter to resolve the above difficulty and also to develop appropriate convergence criteria for each of the series solutions in the three examples of Chapter 2. Also to be developed by means of the concept of harmonic balance, is a different series for the steady-state solution of time-varying, but not necessarily linear systems.

3.2 ORIGIN OF THE CONVERGENCE PROBLEMS

Consider equation 3.1 which is the equation in the complex frequency domain which described the time-varying systems in all three examples of Chapter 2.

$$V(s) = Z(s)I(s) - G(s)\left[V(s+jw_c) + V(s-jw_c) \right] \qquad (3.1)$$

In the time domain, which is more appropriate than the complex frequency domain for investigating convergence, equation 3.1 is equivalent to integral equation 3.2.

$$v(t) = \int_0^t z(t-x)i(x)dx - 2\int_0^t g(t-x)\cos(w_c x)v(x)dx \qquad (3.2)$$

The solution to equation 3.1 was found to be a series

$$V(s) = \sum_{n=0}^{\infty} V_n(s) , \qquad (3.3)$$

where
$$V_0(s) = I(s)Z(s) , \qquad (3.4)$$

and
$$V_{n+1}(s) = -G(s)\left[V_n(s+jw_c) + V_n(s-jw_c)\right] \quad n = 0,1,2, \ldots \qquad (3.5)$$

One would expect the solution to equation 3.2 to be a series, the time domain equivalent of series 3.3.

$$v(t) = \sum_{n=0}^{\infty} v_n(t) \qquad (3.6)$$

where
$$v_0(t) = \int_0^t z(t-x)i(x)dx , \qquad (3.7)$$

and
$$v_{n+1}(t) = -2\int_0^t g(t-x)\cos(w_c x)v_n(x)dx \quad n = 0,1,2, \ldots \qquad (3.8)$$

To insure that the series for v(t) is indeed a solution to equation 3.2, equations 3.6 and 3.7 should be substituted in equation 3.2.

$$v(t) = v_o(t) - 2\int_o^t g(t-x)\cos(w_c x)\sum_{n=o}^{\infty} v_n(x)dx \tag{3.9}$$

In order to reduce equation 3.9 to equation 3.6 and thus show the validity of the series for $v(t)$ as a solution to equation 3.2, it is necessary that the operations of summation and integration be interchanged in equation 3.9. This interchange is permissible if the series portion of equation 3.9 is uniformly convergent in the time interval of interest, in this case $0 \leq t \leq T$, where T is any positive number.

Assume for the moment that the interchange in summation and integration is justified. Performing the interchange yields equation 3.10.

$$\begin{aligned} v(t) &= v_o(t) - \sum_{n=o}^{\infty} 2\int_o^t g(t-x)\cos(w_c x)v_n(x)dx \\ &= v_o(t) - 2\int_o^t g(t-x)\cos(w_c x)v_o(x)dx \\ &\quad + 2\int_o^t g(t-x)\cos(w_c x)v_1(x)dx \\ &\quad - \dots + (-1)^n\, 2\int_o^t g(t-x)\cos(w_c x)v_{n-1}(x)dx + \dots \end{aligned} \tag{3.10}$$

Comparing the terms of series 3.10 to the terms of series 3.6, the resulting conclusion is that the series for $v(t)$ is truly a solution to equation 3.2 on $0 \leq t \leq T$, where T is any positive number.

The problem yet remains to prove the uniform convergence of series 3.6 for $0 \leq t \leq T$ in order to justify the interchange of the operations of summation and integration in equation 3.9.

3.3 DEVELOPMENT OF THE CONVERGENCE CRITERIA

The method to be employed to show the uniform convergence of series 3.6 is the Weierstrass M-test for infinite series (86). Consider a series of positive constants

$$M = \sum_{n=0}^{\infty} M_n \, , \tag{3.11}$$

such that for all t in the interval $0 \leq t \leq T$, where T is some positive number:

$$M_n \geq \left| v_n(t) \right| \, , \quad n = 0,1,2, \ldots \, . \tag{3.12}$$

If the series for M is term by term greater than the series for $v(t)$, and if the series for M can be shown to converge, then the series for $v(t)$ must also converge uniformly and absolutely. The problem now is to define a suitable series for M and to show that it converges.

Define the terms of the series for M as follows:

$$M_o = \sup_t \left[\left| v_o(t) \right| \right] = \sup_t \left[\left| \int_o^t z(t-x)i(x)dx \right| \right] \tag{3.13}$$

$$\begin{aligned} M_n &= \sup_t \left[\left| v_n(t) \right| \right] \\ &= \sup_t \left[\left| 2\int_o^t g(t-x)\cos(w_c x)v_{n-1}(x)dx \right| \right] \qquad n = 1,2, \ldots \end{aligned} \tag{3.14}$$

D'Alembert's Ratio Test may now be applied to the series for M to investigate its convergence (86).

For sufficiently large n,

$$\frac{M_{n+1}}{M_n} = \frac{\sup_t \left[\left| 2\int_0^t g(t-x)\cos(w_c x) v_n(x)dx \right| \right]}{M_n}$$

$$= \frac{\sup_t \left[\left| 2\int_0^t g(x)\cos\left[w_c(t-x)\right] v_n(t-x)dx \right| \right]}{M_n}$$

$$\leq \frac{\sup_t \left[2\int_0^t \left| g(x)\right| \cdot \left|\cos\left[w_c(t-x)\right]\right| \cdot \left| v_n(t-x)\right| dx \right]}{M_n}$$

$$\leq \frac{\sup_t \left[2\int_0^t \left| g(x)\right| \cdot M_n \, dx \right]}{M_n}$$

$$= \sup_t \left[2\int_0^t \left| g(x)\right| dx \right] < 1 \qquad (3.15)$$

Inequality 3.15 is a sufficient condition for the convergence of series 3.11 and also a sufficient condition for the uniform and absolute convergence of series 3.6 over the interval $0 \leq t \leq T$. Although a sufficient condition, it is not a necessary one. There may exist series solutions that violate the condition of inequality 3.15 and yet are convergent series solutions.

Now that the sufficient condition for the uniform convergence of the series solutions has been developed, it is appropriate that it be applied to the three examples of Chapter 2 in order to gain some insight as to the physical significance of the convergence criterion just derived.

The first example was a single capacitor network excited by $i(t) = \cos(w_o t)$ and the capacitor varied as $C(t) = C_o + C_1 \cos(w_c t)$.

$$G(s) = Y(s)Z(s) = \frac{sC_1}{2} \cdot \frac{1}{sC_o} = \frac{C_1}{2C_o} \tag{3.16}$$

$$g(t) = \frac{C_1}{2C_o} \delta(t) \tag{3.17}$$

where $\delta(t)$ is the Dirac delta function (61). The criteria for uniform convergence of the series for $v(t)$ is very simple:

$$\frac{C_1}{2C_o} \sup_t \left[\int_o^t 2\delta(x)dx \right] = \frac{C_1}{C_o} < 1 \tag{3.18}$$

In this example, when C_1 is less than C_o, the series 3.6 will converge uniformly, as was seen to be true in the example.

The second example was a resistor-capacitor network with the same input and same capacitor variation as the first example.

$$G(s) = Y(s)Z(s) = \frac{sC_1}{2} \cdot \frac{1}{C_o} \cdot \frac{1}{s+k} = \frac{C_1}{2C_o} \cdot \frac{s}{s+k} \tag{3.19}$$

$$g(t) = \frac{C_1}{2C_o}\left[\delta(t) - ke^{-kt} \right] \tag{3.20}$$

where
$$k = \frac{1}{RC_o} . \tag{3.21}$$

The criteria for uniform convergence of the series for $v(t)$ becomes:

$$\frac{C_1}{2C_o} \sup_t \left[2\int_0^t \left| \delta(x) - ke^{-kx} \right| dx \right]$$

$$\leq \frac{C_1}{2C_o} \sup_t \left[2\int_0^t \left[\delta(x) + ke^{-kx} \right] dx \right]$$

$$= \frac{C_1}{C_o} \sup_t \left[1 - e^{-kx} + 1 \right]$$

$$= \frac{2C_1}{C_o} < 1 \tag{3.22}$$

In this example, when C_1 is less than $\frac{1}{2}C_o$, the series for $v(t)$ is guaranteed to converge uniformly.

The third example was a resistor-inductor-capacitor network with the same input and the same capacitor variation as in the two preceding examples.

$$G(s) = Y(s)Z(s) = \frac{sC_1}{2} \cdot \frac{1}{C_o} \cdot \frac{s}{s^2 + 2ks + w_o^2}$$

$$= \frac{C_1}{2C_o} \cdot \frac{s^2}{s^2 + 2ks + w_o^2} \tag{3.23}$$

$$g(t) = \frac{C_1}{2C_o}\left[\ \delta(t) - Ae^{-kt}\sin(w_r t + B)\ \right] \tag{3.24}$$

$$k = \frac{1}{2RC_o} \tag{3.25}$$

$$w_r = \sqrt{w_o^2 - k^2} \tag{3.26}$$

$$A = \sqrt{4k^2 + \left[\frac{w_o^2 - 2k^2}{w_r}\right]^2} \tag{3.27}$$

$$B = \text{Arc tan}\left[\frac{2kw_r}{w_o^2 - 2k^2}\right] \tag{3.28}$$

The criteria for uniform convergence of the series for $v(t)$ becomes:

$$\frac{C_1}{2C_o} \sup_t \left[2\int_o^t \left| \delta(x) - Ae^{-kx} \sin(w_r x + B) \right| dx \right]$$
$$\leq \frac{C_1}{2C_o} \sup_t \left[2\int_o^t \left[\delta(x) + Ae^{-kx} \left| \sin(w_r x + B) \right| \right] dx \right] \tag{3.29}$$

The right hand side of inequality 3.29 may be further simplified to yield inequality 3.30.

$$\frac{C_1}{2C_o} \sup_t \left[2\int_o^t \left[\delta(x) + Ae^{-kx} \right] dx \right]$$
$$\leq \frac{C_1}{C_o} \sup_t \left[1 - \frac{A}{k}\left[e^{-kt} - 1 \right] \right]$$
$$= \frac{C_1}{C_o} \left[1 + \frac{A}{k} \right] < 1 \tag{3.30}$$

In terms of the values of the network parameters in the third example of Chapter 2, the series for $v(t)$ is guaranteed to be uniformly convergent on $0 \leq t \leq T$, where T is some positive number, if C_1 is less than $.001C_o$. Observing the values assigned to both C_1 and C_o in this example, ($C_1 = 12.4$ mmf. and $C_o = 300$ mmf.), it is obvious that they fall far short of satisfying the convergence criteria and consequently it is not surprising that the series solution diverged.

It would be reassuring to consider an example similar to the third example of Chapter 2, but with the value of C_1 chosen such that the convergence criterion is satisfied. However, when the value of C_1 is taken as .30 mmf., a value which just satisfies the convergence criteria, the total variation of the capacitance with time is so small as to be considered practically constant. There is little consolation in the fact that the series solution does actually converge because the capacitance varies so little that the waveform for v(t) appears to be the response of a time-invariant network. As a result, one is apt to conclude that the series solution for this sort of time-varying network only converges when the time-varying parameters change so little that the network, for most practical purposes, can be considered time-invariant.

Fortunately, such is not the case. It must be remembered that the previously developed convergence criteria are sufficient conditions only. It is entirely possible to have a value of C_1 which violates the convergence criteria and yet discover that the corresponding series solution converges. As an example, if the value of C_1 is chosen as 7.5 mmf., a value which is 25.0 times larger than specified by the convergence criteria, and the other parameters are the same as the third example in Chapter 2, the series solution for this network converges. The plot of the envelope of the response voltage v(t) is shown in Figure 3.1. It is apparent in this figure that the series solution is converging toward the correct solution as obtained by the Runge-Kutta method, but the convergence is not nearly so rapid as in the first and

Figure 3.1 Plot of the Envelope of the Response Voltage of the Fixed Resistor and Inductor, and Time-Varying Capacitor Network of Example 3, Chapter 2, with C_1 = 7.5 mmf, and All Other Parameters Unchanged.

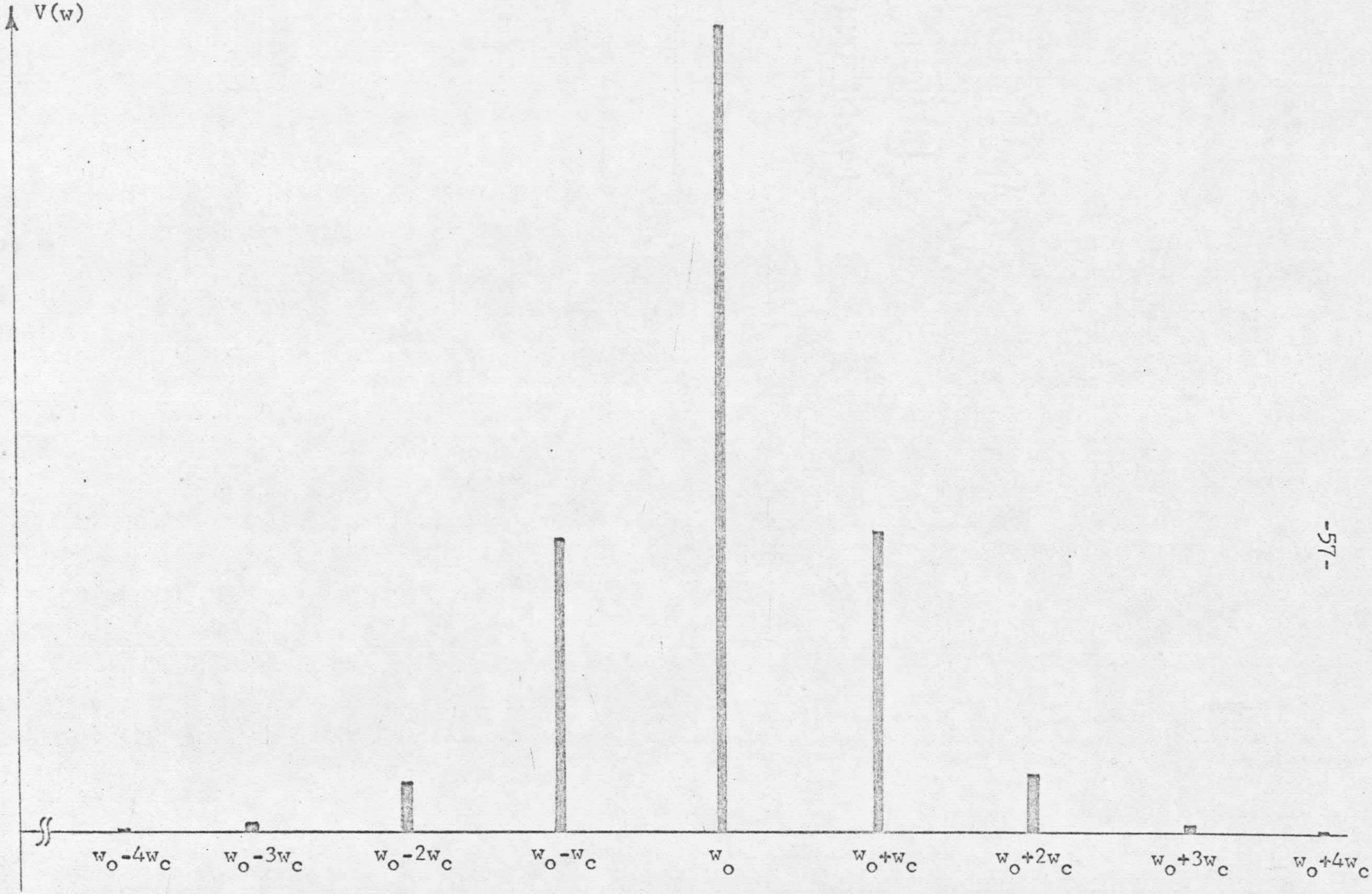

Figure 3.2 Normalized Plot of the Frequency Content of v(t) After Transients Due to the Input Have Subsised.

second examples of Chapter 2. In this case, it would be necessary to consider more terms in the series solution to obtain the accuracy that was obtained previously.

Figure 3.2 is a normalized illustration of the frequency content of the output voltage v(t) after the transients due to the input have subsided. Those frequencies contained within the figure are the frequency components present in the first four terms of the series solution shown in Figure 3.1.

3.4 CONVERGENCE CRITERIA FOR STEADY-STATE SERIES SOLUTION

In the previous section, sufficient conditions for the convergence of the series solution to equation 3.1 were developed. These conditions apply to the total solution, both transient and steady-state solution, for t in the interval $0 \leq t \leq T$, where T is any positive number. Although transient response is certainly important in many applications, there are solutions in which either the steady-state response is all that is desired or the transient response decays so quickly that its effect may be neglected. In such a situation the method of solution is unchanged; it remains the same as explained in Chapter 2. There may be, however, some slackening in the convergence criteria for steady-state response when compared to the criteria for steady-state plus transient solution developed in Section 3.3.

The series solution for the total response is given by equation 3.31.

$$v(t) = \sum_{n=o}^{\infty} v_n(t) \tag{3.31}$$

Each $v_n(t)$ in the above series may be considered as being composed of a transient response $v_n(t)_{tr}$ which eventually becomes insignificant as time becomes large, and a steady-state response $v_n(t)_{ss}$ which is the periodic waveform obtained after sufficient time has elapsed to allow the transient response to become insignificant.

$$v(t) = \sum_{n=o}^{\infty} v_n(t) = \sum_{n=o}^{\infty} \left[v_n(t)_{tr} + v_n(t)_{ss} \right] \tag{3.32}$$

Since series 3.31 has been shown to be uniformly convergent on $0 \leq t \leq T$, where T is any positive number, it would be possible to write

$$v(t) = \sum_{n=o}^{\infty} v_n(t) = \sum_{n=o}^{\infty} v_n(t)_{tr} + \sum_{n=o}^{\infty} v_n(t)_{ss} \tag{3.33}$$

if it can be shown that either of the two series on the right hand side of equation 3.33 is uniformly convergent on $0 \leq t \leq T$.

Before investigating the convergence of the series describing the steady-state solution,

$$v(t)_{ss} = \sum_{n=o}^{\infty} v_n(t)_{ss} \tag{3.34}$$

it is necessary to define the individual terms of the series. Figure 3.3 illustrates the process by which the individual terms are obtained. The system is excited by a steady-state exponential source and the remaining quantities in the figure have been defined in Chapter 2.

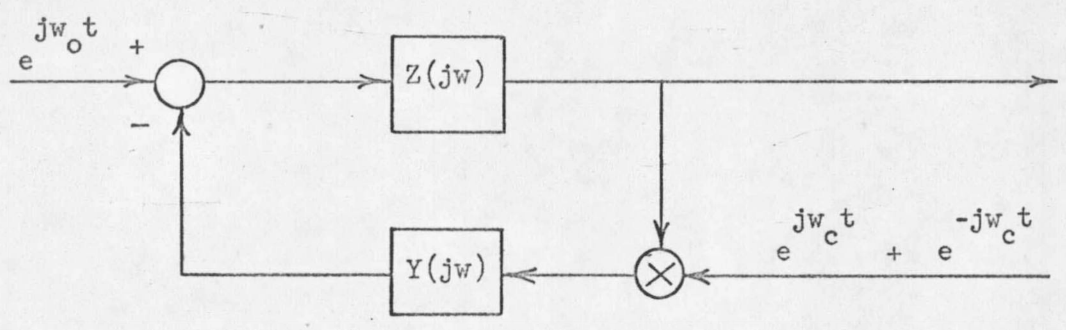

Figure 3.3 Block Diagram Model Illustrating the Process by Which the Steady-State Series is Obtained

The terms of series 3.34 are defined by the recursion relationships:

$$v_o(t)_{ss} = Z(jw_o)e^{jw_o t} \tag{3.35}$$

$$v_{n+1}(t)_{ss} = -G(jw)v_n(t)_{ss}\left[e^{jw_c t} + e^{-jw_c t}\right] \tag{3.36}$$

$$n = 0,1,2,\ \ldots$$

Again use the Weierstrass M-test and define positive constants G_m and V_m as follows:

$$G_m = \sup_{w,k}\left[\,\left|G\left[j(w+kw_c)\right]\right|\,\right] \qquad \begin{array}{c}k \text{ is an integer}\\ -\infty < k < \infty\end{array} \tag{3.37}$$

$$V_m = \sup_{t,w}\left[\,\left|Z(jw)e^{jwt}\right|\,\right] \tag{3.38}$$

Let P be a series of positive constants,

$$P = \sum_{n=0}^{\infty} P_n \tag{3.39}$$

such that

$$P_o = V_m \geq \left|v_o(t)_{ss}\right| \tag{3.40}$$

and

$$P_{n+1} = G_m P_n\left|e^{jw_c t} + e^{-jw_c t}\right| \geq \left|v_{n+1}(t)_{ss}\right| \tag{3.41}$$

$$n = 0,1,2,\ \ldots$$

Now that a series has been defined which is term by term greater than or equal $v(t)_{ss}$, it remains to show that the series P does indeed converge. The convergence may be shown by D'Alembert's Ratio Test.

$$\frac{P_{n+1}}{P_n} = \frac{G_m P_n \left| e^{jw_c t} + e^{-jw_c t} \right|}{P_n}$$

$$\leq 2G_m < 1 \quad \text{for convergence of P.} \qquad (3.42)$$

If G_m as defined by equation 3.37 is less than one-half, then the series for P converges, and consequently the series for $v(t)_{ss}$ converges uniformly on $0 \leq t \leq T$, where T is any positive number.

In order to investigate the physical significance of the above convergence criteria for the steady-state response, let us examine its application to the three examples introduced in Chapter 2 whose convergence criteria for the total response were developed in Section 3.3.

In the first example, G_m is as defined by equation 3.40.

$$G_m = \sup_{w,k} \left[\frac{C_1}{2C_o} \right] = \frac{C_1}{2C_o} \qquad (3.43)$$

For the convergence of the steady-state response in this first example, it is required by inequality 3.42 that the value of the capacitance C_1 be less than the value of C_o. Referring back to equation 3.18, in Section 3.3, it can be seen that the criterion for convergence of the steady-state response is the same as the criterion for the total response of the network. The result is not surprising. Since the network contains but a single varying capacitance, there can be no transient response, only a steady-state response.

The second example concerned a resistor-capacitor network.

$$G_m = \sup_{w,k} \left[\left| G\left[j(w+kw_c) \right] \right| \right] \qquad \begin{matrix} k \text{ is an integer} \\ -\infty < k < \infty \end{matrix}$$

$$= \sup_{w,k} \left[\frac{C_1}{2C_o} \left| \frac{j(w+kw_c)}{j(w+kw_c) + \frac{1}{RC_o}} \right| \right] = \frac{C_1}{2C_o} \tag{3.44}$$

For convergence of the steady-state response in this example, it is required by inequality 3.42 that the value of the capacitance C_1 again be less than the value of C_o. This convergence criterion for the steady-state response is more lenient, by a factor of two, than the requirement on the value of C_1 for the convergence of the total response.

The third and last example dealt with a very high Q resistor-inductor-capacitor network.

$$G_m = \sup_{w,k} \left[\left| G\left[j(w+kw_c) \right] \right| \right] \qquad \begin{matrix} k \text{ is an integer} \\ -\infty < k < \infty \end{matrix}$$

$$= \sup_{w,k} \left[\frac{C_1}{2C_o} \left| \frac{\left[j(w+kw_c) \right]^2}{\left[j(w+kw_c) \right]^2 + \frac{1}{RC_o}\left[j(w+kw_c) \right] + \frac{1}{LC_o}} \right| \right]$$

$$= 250 \frac{C_1}{C_o} \tag{3.45}$$

For convergence of the steady-state response in this third example, it is required by inequality 3.42 that the value of the capacitance C_1 be less than 1/500 the value of C_o. This convergence criterion for the steady-state response represents a slackening in the requirements on C_1 by 50.0 percent of the requirement on C_1 for the convergence of the total response.

In this section it has been shown that if only the steady-state response of a time-varying network is desired, the method of solution is the same as developed in Chapter 2, but at the same time, the requirements on the parameters of the network to guarantee the uniform convergence of the series solution have been more lenient than the requirements for the convergence of the series for the total solution. It must be remembered, however, that all the convergence criteria developed for steady-state solutions and for total solutions thus far, are only sufficient conditions for the uniform convergence of the series solution. Violation of the convergence criteria in no way guarantees divergence of the series solution.

3.5 ANOTHER SERIES FOR STEADY-STATE SOLUTIONS

In the previous section, convergence criteria were developed for steady-state series solutions in which the series were derived by the method of Chapter 2. The series describing steady-state response was a special case of the series which described the total response but in which t was allowed to become sufficiently large so that all the transients became insignificant. In this section, a different series for the steady-state solution is to be developed by a technique different from the one described in Chapter 2. The method of deriving the new steady-state series solution is called Harmonic Balance (15).

Consider equation 3.46, which is the equation in the complex frequency domain which has described all of the time-varying systems considered thus far.

$$V(s) = I(s)Z(s) - I_c(s)Z(s)$$

$$= I(s)Z(s) - G(s)\left[V(s+jw_c) + V(s-jw_c) \right] \qquad (3.46)$$

Equation 3.46 also describes the familiar block diagram shown in Figure 3.4.

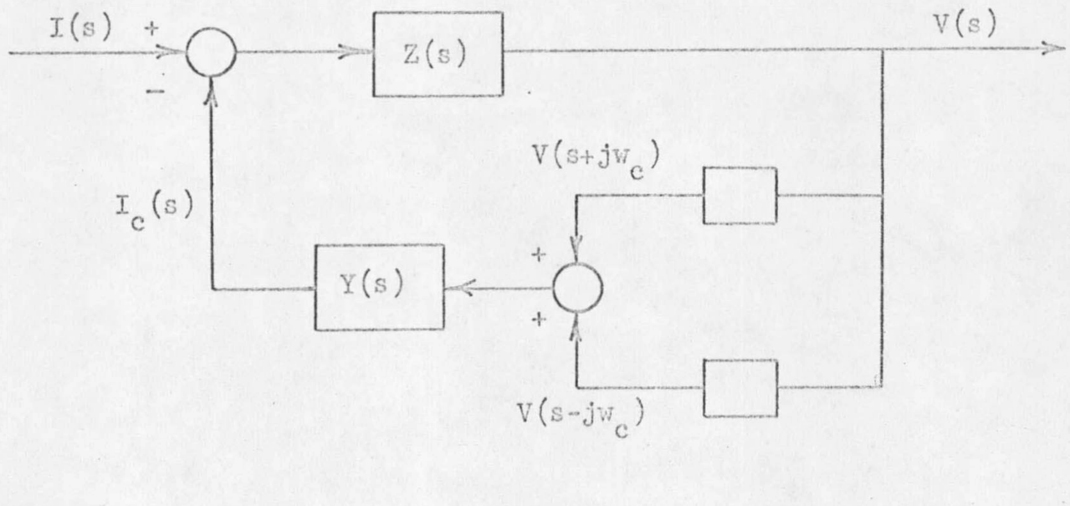

Figure 3.4 Block Diagram Model of Equation 3.46

It has been shown in Chapter 2 that the output voltage V(s) will contain the frequency of the input, w_o, together with frequencies of the sum and difference between w_o and the capacitor variation frequency w_c, such as shown below.

$$
\begin{array}{cc}
 & w_o \\
w_o - w_c & w_o + w_c \\
w_o - 2w_c & w_o + 2w_c \\
w_o - 3w_c & w_o + 3w_c \\
\vdots & \vdots
\end{array}
$$

Since the output voltage will contain an infinite number of frequencies as shown above, consider a series description of V(w) as shown in equation 3.47.

$$V(w) = \sum_{n=-\infty}^{\infty} V_n(w+nw_c) \qquad (3.47)$$

Transformed into the time domain, equation 3.47 is equivalent to equation 3.48,

$$v(t) = \sum_{n=-\infty}^{\infty} A_n e^{j(w+nw_c)t}, \tag{3.48}$$

where the coefficients A_n are, in general, complex constants which must be determined.

Substituting series 3.47 into equation 3.46 results in equation 3.49.

$$\sum_{n=-\infty}^{\infty} V_n(w+nw_c) = I(w)Z(w) - G(w)\left[\sum_{n=-\infty}^{\infty} V_n\left[w+(n+1)w_c\right]\right.$$

$$\left. + \sum_{n=-\infty}^{\infty} V_n\left[w+(n-1)w_c\right]\right] \tag{3.49}$$

The following two transform function pairs aid in the simplification of the above equation.

$$\sum_{n=-\infty}^{\infty} V_n\left[w+(n+1)w_c\right] \longleftrightarrow \sum_{n=-\infty}^{\infty} A_n e^{j\left[w+(n+1)w_c\right]t}$$

$$\longleftrightarrow \sum_{n=-\infty}^{\infty} A_{n-1} e^{j(w+nw_c)t} \tag{3.50}$$

$$\sum_{n=-\infty}^{\infty} V_n \left[w+(n-1)w_c \right] \longleftrightarrow \sum_{n=-\infty}^{\infty} A_n e^{j\left[w+(n-1)w_c \right] t}$$

$$\longleftrightarrow \sum_{n=-\infty}^{\infty} A_{n+1} e^{j(w+nw_c)t} \tag{3.51}$$

If it is assumed again, as it was assumed before, that the input is a sinusoid of frequency w_o,

$$i(t) = e^{jw_o t}, \tag{3.52}$$

then equation 3.49 may be transformed into the time domain to yield

$$\sum_{n=-\infty}^{\infty} A_n e^{j(w_o + nw_c)t} = Z(w_o)\, e^{jw_o t}$$

$$- \sum_{n=-\infty}^{\infty} G(w_o + nw_c) A_{n-1} e^{j(w_o + nw_c)t}$$

$$- \sum_{n=-\infty}^{\infty} G(w_o + nw_c) A_{n+1} e^{j(w_o + nw_c)t} \tag{3.53}$$

Equation 3.53 presupposes that series 3.48 is a uniformly convergent series, a topic which will be discussed later in this section. In order to force series 3.48 to indeed be the steady-state solution, equate the coefficients on both sides of equation 3.53 which correspond to like frequencies.

$$A_n = -G(w_o + nw_c)\left[A_{n-1} + A_{n+1}\right] \qquad \text{for all integers } n \neq 0 \quad -\infty < n < \infty \tag{3.54-a}$$

$$A_o = Z(w_o) - G(w_o)\left[A_{-1} + A_1\right] \qquad n = 0 \tag{3.54-b}$$

The following definitions simplify the notation in subsequent equations.

$$Z(n) = Z(w_o + nw_c) \tag{3.55}$$

$$Y(n) = Y(w_o + nw_c) \tag{3.56}$$

$$G(n) = Y(n)Z(n) \tag{3.57}$$

Utilizing the above notational changes, equations 3.54 may be rewritten as follows:

$$A_o = Z(0) - G(0)\left[A_{-1} + A_1\right] \tag{3.58-a}$$

$$A_1 = -G(1)\left[A_o + A_2\right] \qquad A_{-1} = -G(-1)\left[A_o + A_{-2}\right] \tag{3.58-b}$$

$$A_2 = -G(2)\left[A_1 + A_3\right] \qquad A_{-2} = -G(-2)\left[A_{-1} + A_{-3}\right] \tag{3.58-c}$$

$$A_3 = -G(3)\left[A_2 + A_4\right] \qquad A_{-3} = -G(-3)\left[A_{-2} + A_{-4}\right] \tag{3.58-d}$$

$$\vdots \qquad\qquad \vdots$$

$$A_n = -G(n)\left[A_{n-1} + A_{n+1}\right] \tag{3.58-e}$$

$$\vdots$$

The above equations represent a double infinity of algebraic equations in the unknown complex quantity A_n, where n represents the integers, $-\infty < n < \infty$. It is impractical to even begin to solve these equations for the A_n unless the recursion may be terminated at some sufficiently large value of n, call it N. This is in effect saying that the infinite series solution, equation 3.48, may be well approximated by a finite series of terms.

$$v(t) = \sum_{n=-\infty}^{\infty} A_n e^{j(w_o + nw_c)t} \approx \sum_{n=-N}^{N} A_n e^{j(w_o + nw_c)t} \qquad (3.59)$$

Since, in all real physical systems, the amplitude of the response decreases as the frequency becomes large, the above approximation may be well justified for some value N, chosen sufficiently large so as to yield the desired accuracy. Also, if the infinite number of terms in equation 3.59 is truly a good approximation of series 3.48, then the problems associated with uniform convergence of the series portion of equation 3.53 are avoided since a finite sum of bounded terms must also be bounded and therefore convergent.

The problem now has been reduced to finding the values of the A_n where $-N \leqq n \leqq N$ while assuming that all the remaining A_n, where $|n| \geqq N$, are equal to zero.

$$A_N = -G(N)\left[A_{N-1} + \overset{0}{\cancel{A_{N+1}}}\right] \qquad (3.60\text{-a})$$

$$A_{-N} = -G(-N)\left[\overset{0}{\cancel{A_{-N-1}}} + A_{-N+1}\right] \qquad (3.60\text{-b})$$

Equations 3.58 with $\left[-(N-1) \leq n \leq N-1\right]$ together with equations 3.60 represent 2N+1 algebraic equations in 2N+1 complex unknowns A_n. Therefore, the A_n may be determined from this set of equations.

As an illustration of the procedure followed in solving for the values of the A_n, consider the case in which a value for N of 4 is justified. There are then nine algebraic equations and nine unknown A_n. Begin by using equation 3.60-a.

$$A_4 = -G(4)A_3 \qquad (3.61)$$

Equation 3.58 yields:

$$A_3 = -G(3)\left[A_2 + A_4\right] \qquad (3.62)$$

Now substitute equation 3.61 into equation 3.62 and solve for A_3 in terms of A_2.

$$A_3 = -G(3)\left[A_2 - G(4)A_3\right]$$

$$= \frac{-G(3)}{1 - G(3)G(4)} A_2 \qquad (3.63)$$

Now substitute equation 3.63 into equation 3.58 and solve for A_2 in terms of A_1.

$$A_2 = -G(2)\left[A_1 + A_3\right]$$

$$= -G(2)\left[A_1 - \frac{G(3)A_2}{1 - G(3)G(4)}\right]$$

$$= \frac{-G(2)}{1 - \dfrac{G(2)G(3)}{1 - G(3)G(4)}} \cdot A_1 \tag{3.64}$$

Now substitute equation 3.64 into equation 3.58 and solve for A_1 in terms of A_o.

$$A_1 = -G(1)\left[A_o + A_2\right]$$

$$= -G(1)\left[A_o - \frac{G(2)}{1 - \dfrac{G(2)G(3)}{1 - G(3)G(4)}} \cdot A_1\right]$$

$$= \frac{-G(1)}{1 - \dfrac{G(1)G(2)}{1 - \dfrac{G(2)G(3)}{1 - G(3)G(4)}}} \cdot A_o \tag{3.65}$$

Because of the symmetry in equation 3.58 for the positive and negative values of n, A_{-1} is given by equation 3.66.

$$A_{-1} = \frac{-G(-1)}{1 - \dfrac{G(-1)G(-2)}{1 - \dfrac{G(-2)G(-3)}{1 - G(-2)G(-4)}}} \cdot A_o \tag{3.66}$$

By substituting equations 3.65 and 3.66 into equation 3.58-a, the value of A_o may be solved for in closed form.

$$A_o = Z(0) - G(0)\left[A_{-1} + A_1 \right]$$

$$= Z(0) + \cfrac{G(0)G(1)\,A_o}{1 - \cfrac{G(1)G(2)}{1 - \cfrac{G(2)G(3)}{1 - G(3)G(4)}}} + \cfrac{G(0)G(-1)\,A_o}{1 - \cfrac{G(-1)G(-2)}{1 - \cfrac{G(-2)G(-3)}{1 - G(-3)G(-4)}}}$$

$$= \cfrac{Z(0)}{1 - \cfrac{G(0)G(1)}{1 - \cfrac{G(1)G(2)}{1 - \cfrac{G(2)G(3)}{1 - G(3)G(4)}}} - \cfrac{G(0)G(-1)}{1 - \cfrac{G(-1)G(-2)}{1 - \cfrac{G(-2)G(-3)}{1 - G(-3)G(-4)}}}} \tag{3.67}$$

Once the value of A_o has been found by the above procedure, the remaining values of the A_n may be found by using the value of A_o in equation 3.67 to solve for A_1 and A_{-1} in equations 3.65 and 3.66, respectively, and then using these values of A_1 and A_{-1} to solve for A_2 and A_{-2} by equations 3.64. Continuing this procedure it is possible to find all the remaining A_n using the previously calculated A_{n-1}.

From the expressions which define the respective A_n, such as equation 3.67, it is quite easy to see what the result would be of increasing the value of N. It would simply extend the continued fraction expansions which define the A_n, and it is not necessary to resolve equations 3.58 and 3.60 again for the A_n once the recursion relationships have been established as they were in equations 3.60 through 3.67.

$$A_n = \cfrac{-G(n)\,A_{n-1}}{1 - \cfrac{G(n)G(n+1)}{1 - \cfrac{G(n+1)G(n+2)}{\ddots \atop 1 - G(N-1)G(N)}}} \qquad n > 0 \tag{3.68}$$

$$A_n = \cfrac{-G(-n)\,A_{n+1}}{1 - \cfrac{G(-n)G(-n-1)}{1 - \cfrac{G(-n-1)G(-n-2)}{1 - \cfrac{G(-n-2)G(-n-3)}{\ddots \atop 1 - G(-N+1)G(-N)}}}} \qquad n < 0 \tag{3.69}$$

Equations 3.68 and 3.69 together with equation 3.58 define all the A_n for any finite value of N.

The next point of interest is what would happen if the above finite continued fractions were allowed to become infinite continued fractions. In this case the problem of the uniform convergence of the continued fraction coefficients and of the series solutions as a whole must be reconsidered because the previous assumptions concerning the convergence of a finite number of terms does not hold if the number of terms is allowed to become infinite.

First, to begin the investigation of the infinite continued fractions, make the following definitions

$$H(n) = \frac{1}{G(n)} \tag{3.70}$$

$$K(0) = \frac{1}{Z(0)} \tag{3.71}$$

Equation 3.67 may now be rewritten incorporating the above definitions.

$$A_o = \cfrac{K(0)}{H(0) - \cfrac{1}{H(1) - \cfrac{1}{H(2) - \cfrac{1}{\ddots}}} - \cfrac{1}{H(-1) - \cfrac{1}{H(-2) - \cfrac{1}{\ddots}}}} \tag{3.72}$$

In order to eliminate some of the unwieldiness of the equations containing continued fractions, consider the following commonly used abbreviated notation (45):

$$\left[b_o, \frac{a_n}{b_n}\right]_1^{\infty} = b_o + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{\ddots}}}} \tag{3.73}$$

$$\left[\frac{a_n}{b_n}\right]_1^{\infty} = \left[0, \frac{a_n}{b_n}\right]_1^{\infty} \tag{3.74}$$

Using the above notational changes, equation 3.72 may now be rewritten as equation 3.75.

$$A_o = \frac{K(0)}{H(0) + \left[\frac{-1}{H(n)}\right]_1^{\infty} + \left[\frac{-1}{H(-n)}\right]_1^{\infty}} \tag{3.75}$$

In order that equation 3.75 be meaningful, it is necessary that each of the continued fractions therein must converge to a finite limit. In showing the convergence of continued fractions, the following definition and theorem are of considerable worth (57).

Definition: Continued fractions of the form of equation 3.73 which satisfy the following conditions:

(1) $a_n \neq 0 \quad n = 1, 2, \ldots$

(2) $\lim_{n \to \infty} a_n = a$

(3) $\lim_{n \to \infty} b_n = b$

are called periodic in the limit.

The following theorem applies to continued fractions which are periodic in the limit:

Theorem: The condition

$$\lim_{n \to \infty} \left[\sup_n \left| C_n \right| \right] \leq g \qquad (3.76)$$

is sufficient for the continued fraction

$$E = \left[b_o, \frac{C_n s}{1} \right]_1^\infty \qquad (3.77)$$

to be convergent in the circle $|s| < 1/4g$ (excepting possibly at certain poles) to an analytic function.

Two corollaries follow immediately from the above theorems.

Corollary 1: The continued fraction

$$F = \left[b_o, \frac{1}{b_n}\right]_1^{\infty} \tag{3.78}$$

converges if

$$|b_n| > 2 \quad \text{for } n = 1, 2, \ldots .$$

Corollary 2: The continued fraction

$$J = \left[b_o, \frac{a_n}{1}\right]_1^{\infty} \tag{3.79}$$

converges if

$$0 < |a_n| < 1/4 \quad \text{for } n = 1, 2, \ldots .$$

The above theorem and corollaries are presented here without proof. For the interested reader, the proof for the above theorem and corollaries together with a complete discussion of continued fractions may be found in the references (45,57,79).

Continued fractions of the sort shown in equation 3.72 are periodic in the limit for all three of the $G(w)$ considered previously in the examples of Chapter 2. This fact justifies the application of Corollary 1 to the determination of the convergence requirements for the continued fraction coefficients.

It is required by Corollary 1 that the following inequalities must be satisfied for convergence of the continued fractions in equations 3.67, 3.68, and 3.69.

$$|H(n)| < 2 \qquad (3.80\text{-a})$$
$$|H(-n)| < 2 \qquad n = 1,2, \ldots \qquad (3.80\text{-b})$$

$$|G(n)| = \left|\frac{1}{H(n)}\right| < \tfrac{1}{2} \qquad (3.81\text{-a})$$
$$|G(-n)| = \left|\frac{1}{H(-n)}\right| < \tfrac{1}{2} \qquad n = 1,2, \ldots \qquad (3.81\text{-b})$$

Notice that inequalities 3.81 impose essentially the same restriction upon $G(w)$ as did inequality 3.42 which gave the requirements that $G(w)$ must satisfy in order to guarantee the uniform convergence of the series for the steady-state response developed by the method of Chapter 2. Inequalities 3.81 give the restrictions upon $G(w)$ for the convergence of the continued fraction coefficients of the series developed by the method of Harmonic Balance.

In addition, if the series itself were to be extended to an infinite number of terms, it would be necessary also to establish criteria for the convergence of the infinite series. However, on account of the reasons stated earlier for justifying a series containing a finite number of terms, it is unnecessary to extend the series to an infinite number of terms. Therefore further investigation into the convergence of the infinite series would be without benefit for practical applications.

3.6 CONCLUSION

In Chapter 3, the following items have been accomplished:

1. Reasons for the divergence of some series solutions in Chapter 2, and the convergence of others were explained to be due to the previously incorrect assumption of the uniform convergence of the series solution.

2. Requirements on the value of capacitance C_1 were established to guarantee the uniform convergence of the series solutions for the three examples of Chapter 2 and for a general transfer function $G(s)$.

3. A series solution for steady-state response was derived from the series developed in Chapter 2, and convergence criteria were developed for this series.

4. A different series for steady-state response was derived by the method of Harmonic Balance. The series contained continued fraction coefficients and requirements for the convergence of these continued fractions were established.

Due to the results of this chapter, it is now possible to use with confidence the method of series solution for analyzing time-varying networks. The method has been developed in Chapter 2 and requirements for uniform convergence of series solutions for the three examples of Chapter 2 have been defined in Chapter 3. To develop meaningful convergence criteria for series solutions for general time-varying systems and to relate these criteria to the magnitude of variation of an arbitrary system parameter is a pointless task. Convergence criteria, in any particular case, may be developed by forming an integral equation which corresponds to equation 3.2 for the previous examples. Then following the same sequence of operations, suitable requirements on the system parameters may be found. If only steady-state response is desired, two series solutions are now available, both possessing similar requirements for convergence. What remains to be accomplished is the extension of this method of series solution to: (1) systems containing more than one time-varying element, and (2) systems containing elements which vary in a nonsinusoidal manner.

CHAPTER 4

GENERALIZATION OF THE SERIES METHOD OF SOLUTION

4.1 INTRODUCTION

In the preceding chapters attention was directed almost exclusively to time-varying networks whose inputs and parameter variations were both sinusoids. Networks such as these do represent some of the most interesting and practical problems facing designers today, but there are yet other systems with varying parameters that are of great interest and whose solutions yield to slight variations of the methods developed in the previous chapters. A few of the examples considered in this chapter are of the following types.

(1) Systems in which the parameter variations are periodic but not necessarily sinusoidal.

(2) Systems in which the parameter variations are well defined but are not periodic.

(3) Systems in which more than one parameter is a function of time.

(4) Systems in which every parameter of the system is a function of time.

(5) Systems which are excited by modulated signals.

All of the above types are particular cases of the general single input, single output time-varying system shown in Figure 4.1.

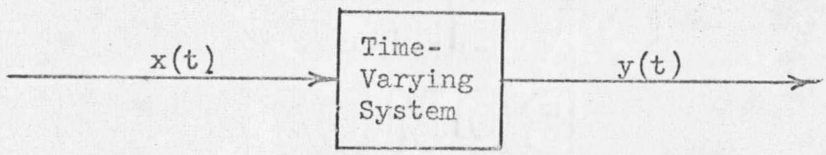

Figure 4.1 Model of General Time-Varying System

Systems such as shown in the model of Figure 4.1 are described by time-varying differential equations such as equation 4.1.

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \ldots + a_o(t)y(t) = b_m(t)x^{(m)}(t) + b_{m-1}(t)x^{(m-1)}(t) + \ldots + b_o(t)x(t) \tag{4.1}$$

Special cases of the above differential equation will be examined in the following sections.

4.2 SYSTEMS WITH A PERIODICALLY VARYING PARAMETER

Systems which contain periodically time-varying parameters are perhaps the most encountered of all time-varying systems. In fact, the three examples of Chapter 2 all had a sinusoidally varying parameter, which is a special case of periodically varying parameters.

Consider the following differential equation which describes a system containing a single periodically varying coefficient $a_o(t)$. It is not necessary that the coefficient be $a_o(t)$. It could as well have been any of the a_i.

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \dots + a_1 y^{(1)}(t) + a_o(t) y(t)$$

$$= b_m x^{(m)}(t) + b_{m-1} x^{(m-1)}(t) + \dots + b_o x(t) \qquad (4.2)$$

The above differential equation refers to a time-varying system such as represented by the model in Figure 4.1, in which the excitation is $x(t)$, and $y(t)$ is the response. The equation may be interpreted in a slightly different light as follows. If the time-varying term $a_o(t)y(t)$ on the left hand side of the equation is transferred to the right hand side of the equation, then $y(t)$ may be visualized as the response of a time-invariant system to a modified excitation composed of $x(t)$ together with some specific function of $a_o(t)y(t)$. For convenience of notation, let this function of $a_o(t)y(t)$ be designated as shown in equation 4.3.

$$x_1(t) = \Psi\left[a_o(t)y(t)\right] \qquad (4.3)$$

Figure 4.2 illustrates graphically the meaning of the above statements.

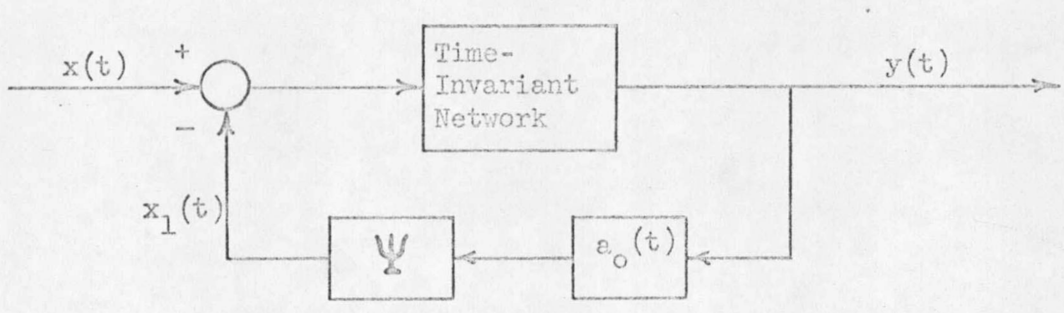

Figure 4.2 A Block Diagram Interpretation of Equation 4.2

In terms of the fixed parameters in equation 4.2, the transfer function G(s) of the time-invariant system of Figure 4.2 would be as shown in equation 4.4, where s is the complex frequency variable.

$$G(s) = \frac{\sum_{k=o}^{m} b_k s^k}{\sum_{k=1}^{n} a_k s^k}\ , \quad m < n \tag{4.4}$$

After being transformed into the complex frequency domain, equation 4.3 may also be rewritten in terms of the fixed parameters of equation 4.2.

$$X_1(s) = \frac{1}{\sum_{k=1}^{n} a_k s^k} \mathcal{L}\left[a_o(t)y(t)\right]$$

$$= H(s)\,\mathcal{L}\left[a_o(t)y(t)\right] \tag{4.5}$$

$\mathcal{L}$ denotes the Laplace transform operator.

$a_o(t)$ is a periodically varying parameter which perhaps might not even be specified analytically. The parameter might only be graphically portrayed with the consequent result that the Laplace transform might be very difficult to find, and if found, it must then be convolved with Y(s) in order to find $X_1(s)$ (7). All of this represents a very cumbersome procedure which might be greatly simplified if the periodic parameter were expanded as a Fourier series (30,61).

$$a_o(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{jnw_a t} \tag{4.6}$$

w_a is the frequency of parameter variation, T is the period, and the α_n are the complex Fourier coefficients.

$$\alpha_n = \frac{1}{T}\int_o^T a_o(t)e^{-jnw_a t}\, dt \tag{4.7}$$

The complex Fourier series representation for $a_o(t)$ in equation 4.6 is in a very convenient form for obtaining the Laplace transform of the product $a_o(t)y(t)$ to solve for $X_1(s)$ from equation 4.5 (7;46).

$$\mathcal{L}\left[a_o(t)y(t)\right] = \mathcal{L}\left[\sum_{n=-\infty}^{\infty} \alpha_n e^{-jnw_a t}\, y(t)\right]$$

$$= \sum_{n=-\infty}^{\infty} \alpha_n Y(s+jnw_a) \tag{4.8}$$

$$= \alpha_o Y(s) + \sum_{n=1}^{\infty}\left[\alpha_n Y(s+jnw_s) + \alpha_{-n} Y(s-jnw_a)\right]$$

Utilizing the results obtained above, the equation in the complex frequency domain which describes the system shown in Figure 4.2 may be expressed as equation 4.9.

$$Y(s) = G(s)X(s) - W(s)\left[\sum_{n=-\infty}^{\infty} \alpha_n Y(s+jnw_a)\right]. \qquad (4.9)$$

$$W(s) = G(s)H(s) \qquad (4.10)$$

Figure 4.3 illustrates the block diagram model which is described by equation 4.9.

It would be clearly impossible to actually include an infinite number of terms in the feedback path of Figure 4.3. The Fourier series must be truncated after a sufficient number of terms to make a detailed analysis possible. This termination of the Fourier series can be justified by noting that both transfer functions G(s) and H(s), will greatly attenuate high frequency terms and thus reduce to insignificance terms containing frequency components above a certain frequency. The particular frequency, above which higher frequency terms may be neglected, would have to be determined in a particular problem by considering the frequency characteristics of G(s) and H(s) and the contribution of the individual terms of the particular Fourier series involved.

After an appropriate truncation of the Fourier series has been performed it is necessary to solve equation 4.11 for Y(s).

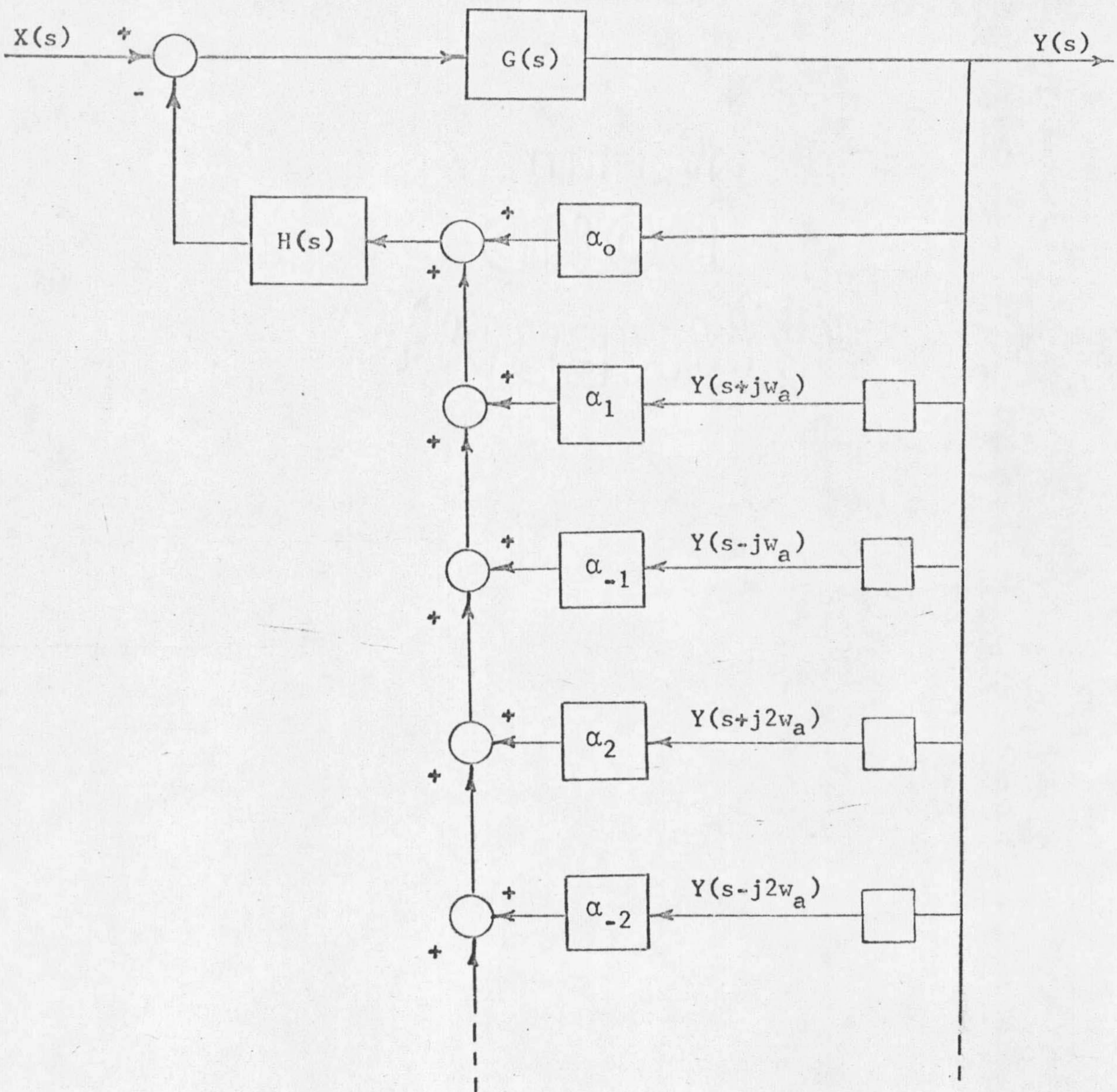

Figure 4.3 Block Diagram Model of Equation 4.9

$$Y(s) = G(s)X(s) - H(s)\left[\sum_{n=-N}^{N} \alpha_n Y(s+jnw_a)\right] \tag{4.11}$$

A solution for Y(s) in the above equation may be found by the successive substitution technique described in Chapter 2 to solve equation 2.27. The solution will again be found to be a series,

$$Y(s) = \sum_{n=o}^{\infty} Y_n(s) \, , \tag{4.12}$$

where

$$Y_o(s) = G(s)X(s) \, , \tag{4.13}$$

and

$$Y_{n+1}(s) = -W(s)\left[\sum_{k=-N}^{N} \alpha_k Y_n(s+jkw_a)\right] \quad k = 1,2, \ldots. \tag{4.14}$$

To investigate the convergence of the series solution of equation 4.12, it is desirable to transform equation 4.9 back into the time domain.

$$y(t) = \int_o^t g(t-z)x(z)dz - \int_o^t w(t-z)a_o(z)y(z)dz \tag{4.15}$$

Equation 4.15 is an integral equation of the Volterra type (1,29,48,76). The time domain equivalent of series 4.12 is a solution to the above integral equation if series 4.16 can be shown to be uniformly convergent in the interval $0 \leq t \leq T$, where T is any positive number.

$$y(t) = \sum_{n=o}^{\infty} y_n(t) \tag{4.16}$$

The method used to show uniform convergence of the above series is identical to the method described in Section 3.3 of Chapter 3. The results of the method indicate that if inequality 4.17 is satisfied, then series 4.16 converges and is a solution to integral equation 4.14.

$$\sup_t \left[\int_o^t \left| w(t-z)a_o(z) \right| dz \right] < 1 \tag{4.17}$$

As an application of the analysis of a time-varying system with a periodically time-varying parameter, consider the practical example shown below.

<u>Example 4.2.1</u>

A thermoelectric tank cooler is used in an industrial process to supply cold water. A sketch of the cooler is shown in Figure 4.4. The tank is insulated from its surroundings and heat is removed by the thermoelectric element at the constant rate Q.

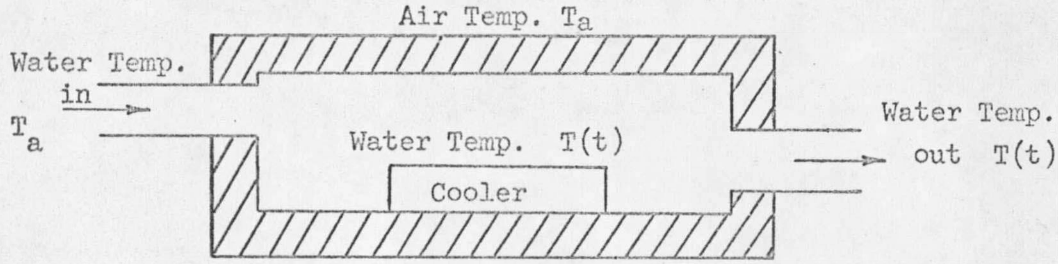

Figure 4.4 Thermoelectric Water Cooler

The demand for cold water is not constant, but is supplied in fixed quantities at regular intervals so that the flow of cold water from the tank occurs as shown in Figure 4.5.

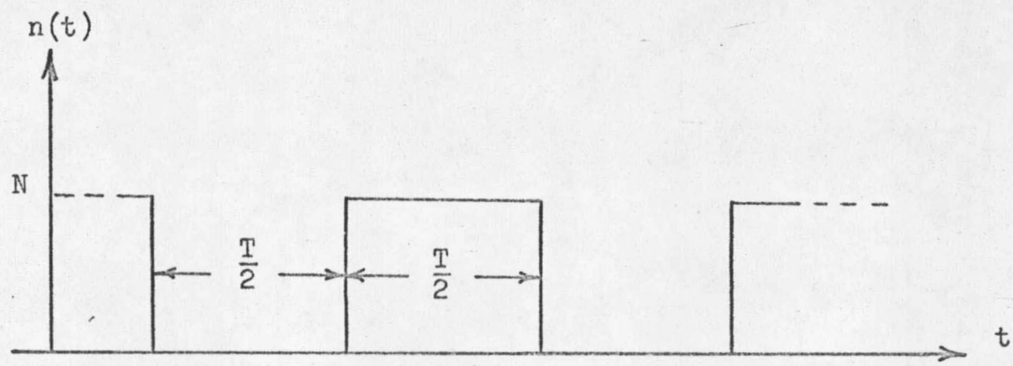

Figure 4.5 Schedule of Flow of Cold Water from the Tank

The simplifying assumptions are as follows:

(1) There is no heat storage in the insulation. This is valid since the specific heat of the insulation is small and water temperature variation is small.

(2) All the water in the tank is at a uniform temperature. This requires perfect mixing of the water.

The equation governing the process described above is given by equation 4.18.

$$Q = C \frac{d}{dt} T(t) + \frac{T_a - T(t)}{R} + n(t)S\left[T_a - T(t)\right] \quad (4.18)$$

Definitions of the system parameters and variation are as follows:

Q = constant heat flow rate into cooling element

$\theta(t)$ = difference between ambient air temperature and the temperature of the water in the tank $= T_a - T(t)$

T_a = temperature of the air surrounding the tank and of the water entering the tank

C = heat capacity of the water in the tank

R = thermal resistance of the insulation

$n(t)$ = water flow from tank at a frequency of w_f

S = specific heat of water

Making the substitution, $\Theta(t) = T_a - T(t)$, and transposing the time-varying term to the left hand side of equation 4.18, results in equation 4.19.

$$Q - n(t)S\Theta(t) = \frac{1}{R}\Theta(t) + C\frac{d\Theta(t)}{dt} \qquad (4.19)$$

The Fourier series representation of n(t) may be found by application of equations and 4.6 and 4.7 to the waveform shown in Figure 4.5.

$$n(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{jkw_f t} \qquad (4.20)$$

$$\alpha_k = \frac{1}{T}\int_0^T n(t)e^{-jkw_f t}\, dt = N\left[\frac{1 - e^{-jk\pi}}{jk2\pi}\right] \qquad (4.21)$$

Using the Fourier series for n(t), equation 4.19 may be transformed into the Laplace domain resulting in equation 4.22.

$$\frac{Q}{s} - S\sum_{k=-\infty}^{\infty} \alpha_k \Theta(s+jkw_f) = \left[\frac{1}{R} + sC\right]\Theta(s) \qquad (4.22)$$

Equation 4.22 may be visualized as describing the block diagram system in Figure 4.6. The truncation of the Fourier series representing n(t) may be performed and justified because of the low-pass frequency charac-

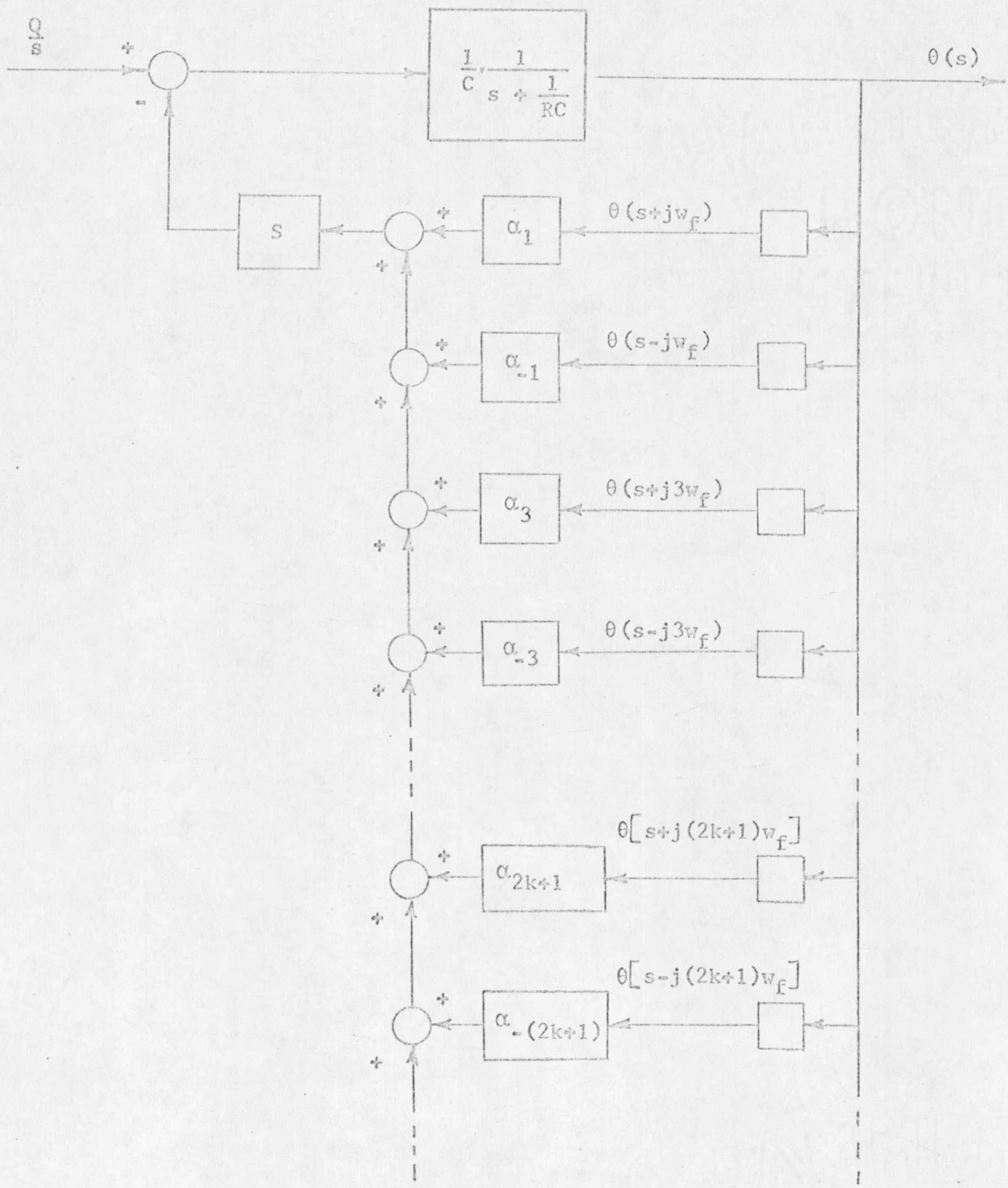

Figure 4.6 Block Diagram Of System Described By Equation 4.21.

teristics of the transfer function in the forward path of Figure 4.6. The point at which the series is to be truncated is a function of the frequency characteristic of this transfer function, the speed with which the Fourier series for $n(t)$ converges, and the desired accuracy of the solution. The solution for $\Theta(s)$ in equation 4.22 is an infinite series,

$$\Theta(s) = \sum_{m=0}^{\infty} \Theta_m(s) , \tag{4.23}$$

where

$$\Theta_o(s) = \frac{Q}{C} \frac{1}{s\left[s + \frac{1}{RC}\right]} \tag{4.24}$$

and

$$\Theta_{m+1}(s) = -\frac{S}{C} \cdot \frac{1}{s + \frac{1}{RC}} \sum_{k=-K}^{K} \alpha_k \Theta_m(s+jkw_f) \quad m = 0,1,2, \ldots \tag{4.25}$$

The series solution 4.23 will converge uniformly in $0 \leq t \leq T$, where T is any positive number, if the following inequality is satisfied. This relationship was derived directly from inequality 4.17.

$$\frac{NS}{C}\sup_{t}\left[\int_{o}^{t}\left| e^{-\frac{z}{RC}} \right| dz \right]$$

$$= \frac{NS}{C}\sup_{t}\left[RC\left[1 - e^{-\frac{t}{RC}} \right] \right]$$

$$= NSR < 1 \qquad (4.26)$$

4.3 SYSTEMS WITH NONPERIODIC PARAMETERS

Often, time-varying systems contain parameters which have time variations which do not repeat themselves on a regular basis. This is especially true in cases in which the process is a "one-shot" affair such as the launching of a satellite to be placed in earth orbit. It is the purpose of this section to investigate time-varying systems of this sort and to attempt to apply to them a variation of the analysis technique developed in Chapter 2.

Consider a special case of the general time-varying differential equation in which one coefficient, for example $a_o(t)$, is a nonperiodic function of time, and all the other coefficients are constants.

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \ldots + a_1 y^{(1)}(t) + a_o(t) y(t)$$

$$= b_m x^{(m)}(t) + b_{m-1} x^{(m-1)}(t) + \ldots + b_o x(t) \qquad (4.27)$$

Following the same procedure as in the preceding section, the time-varying term $a_o(t)y(t)$ is transposed to the opposite side of the equation so that the equation may be visualized as describing the response $y(t)$ of a time-invariant system excited by an input composed of $x(t)$ together with some specific function $a_o(t)y(t)$.

$$x_1(t) \;=\; \Psi\left[a_o(t)y(t)\right] \qquad (4.28)$$

The meaning of the above statement is illustrated by the block diagram model in Figure 4.7.

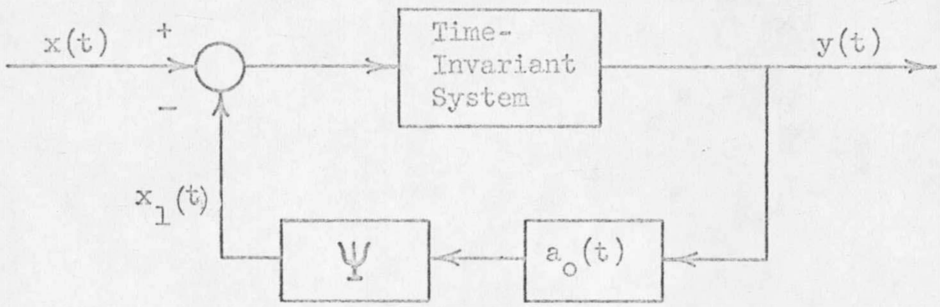

Figure 4.7 A Block Diagram Interpretation of Equation 4.27

In terms of the fixed coefficients in equation 4.27, the transfer function $G(s)$ of the time-invariant system of Figure 4.7, would be as shown in equation 4.29, where s is the complex frequency variable.

$$G(s) = \frac{\sum_{k=o}^{m} b_k s^k}{\sum_{k=1}^{n} a_k s^k} \qquad m < n \tag{4.29}$$

After being transformed into the complex frequency domain, equation 4.28 may also be rewritten in terms of the fixed coefficients of equation 4.27. $\mathcal{L}$ denotes the Laplace transform operator.

$$X_1(s) = \frac{\mathcal{L}\left[a_o(t)y(t)\right]}{\sum_{k=1}^{n} a_k s^k}$$

$$= H(s)\,\mathcal{L}\left[a_o(t)y(t)\right] \tag{4.30}$$

The problem now is to find the Laplace transform of the product $a_o(t)y(t)$. Nonperiodic, time-varying, analytic functions for $a_o(t)$ certainly exist such that the Laplace transform, $\mathcal{L}\left[a_o(t)y(t)\right]$, is easily found. In such a case the procedure to be followed is exactly as explained in Chapter 2.

Example 4.3.1

Consider an example in which $a_o(t) = a_o e^{-rt}$, where r is a real constant.

$$X_1(s) = \frac{\mathcal{L}\left[a_o e^{-rt}\, y(t)\right]}{\sum_{k=1}^{n} a_k s^k}$$

$$= \frac{a_o Y(s+r)}{\sum_{k=1}^{n} a_k s^k} \tag{4.31}$$

The Laplace transform of $a_o(t)y(t)$ in this case was found by elementary Laplace transform theory, and by applying the procedure of Chapter 2, it is soon found that the solution for $Y(s)$ is a series,

$$Y(s) = \sum_{n=o}^{\infty} Y_n(s) , \tag{4.32}$$

where $$Y_o(s) = G(s)X(s) , \tag{4.33}$$

and $$Y_{n+1}(s) = - a_o G(s)H(s)Y_n(s+r) \qquad n = 0,1,2, \ldots \tag{4.34}$$

The solution to the above example was obtained in a straightforward manner because the coefficient $a_o(t)$ was described by a well defined function, which, when multiplied by $y(t)$, led directly to an easily obtained Laplace transform. This case is, of course, an exceptional one. Only rarely could one expect the Laplace transform of the product of time functions, $a_o(t)y(t)$, to be found so readily. And, to complicate the situation even further, many times a well defined functional description for $a_o(t)$, such as $a_o e^{-rt}$ in the above example, is not available. The coefficient might only be described by a time plot of the function, in which case, obtaining the Laplace transform of $a_o(t)y(t)$ would be a difficult process at best.

In a case where the time-varying coefficient $a_o(t)$ is given by a time plot, a possible method of solution would be to determine a polynomial representation for $a_o(t)$ by a method such as a least squares approximation (29).

$$a_o(t) \cong \sum_{k=o}^{K} C_k t^k \tag{4.35}$$

Equation 4.35 is a very convenient representation for $a_o(t)$ because the Laplace transform of the product $a_o(t)y(t)$ is easily obtained.

$$\mathcal{L}\left[a_o(t)y(t)\right] = \mathcal{L}\left[\sum_{k=o}^{K} C_k t^k y(t)\right]$$

$$= \sum_{k=o}^{K} (-1)^k C_k Y^{(k)}(s) \qquad (4.36)$$

Equation 4.30 may now be rewritten as

$$X_1(s) = \frac{\sum_{k=o}^{K} (-1)^k C_k Y^{(k)}(s)}{\sum_{k=o}^{n} a_k s^k} \qquad (4.37)$$

Utilizing the results obtained in equation 4.37 and equation 4.29, the equation in the complex frequency domain which describes the system in Figure 4.7 may now be expressed as equation 4.38.

$$Y(s) = G(s)X(s) - W(s)\left[\sum_{k=o}^{K} (-1)^k C_k Y^{(k)}(s)\right] \qquad (4.38)$$

W(s) is as given in equation 4.10.

Figure 4.8 illustrates the block diagram model which is described by equation 4.38.

By the method of Chapter 2, the solution to equation 4.38 may be obtained as a series,

$$Y(s) = \sum_{n=0}^{\infty} Y_n(s) , \tag{4.39}$$

where

$$Y_0(s) = G(s)X(s) , \tag{4.40}$$

and

$$Y_{n+1}(s) = -W(s) \sum_{k=0}^{K} (-1)^k C_k Y_n^{(k)}(s) , \quad n = 0,1,2, \ldots . \tag{4.41}$$

Using the method described in Chapter 3, Section 3.3, the above series is found to be uniformly convergent if the following inequality is satisfied:

$$\sup_{t} \left[\int_0^t \left| w(t-z)a_0(z) \right| \, dz \right] < 1 \tag{4.42}$$

As an application of the analysis of a time-varying system with a graphically given, nonperiodic parameter, consider the practical example shown below.

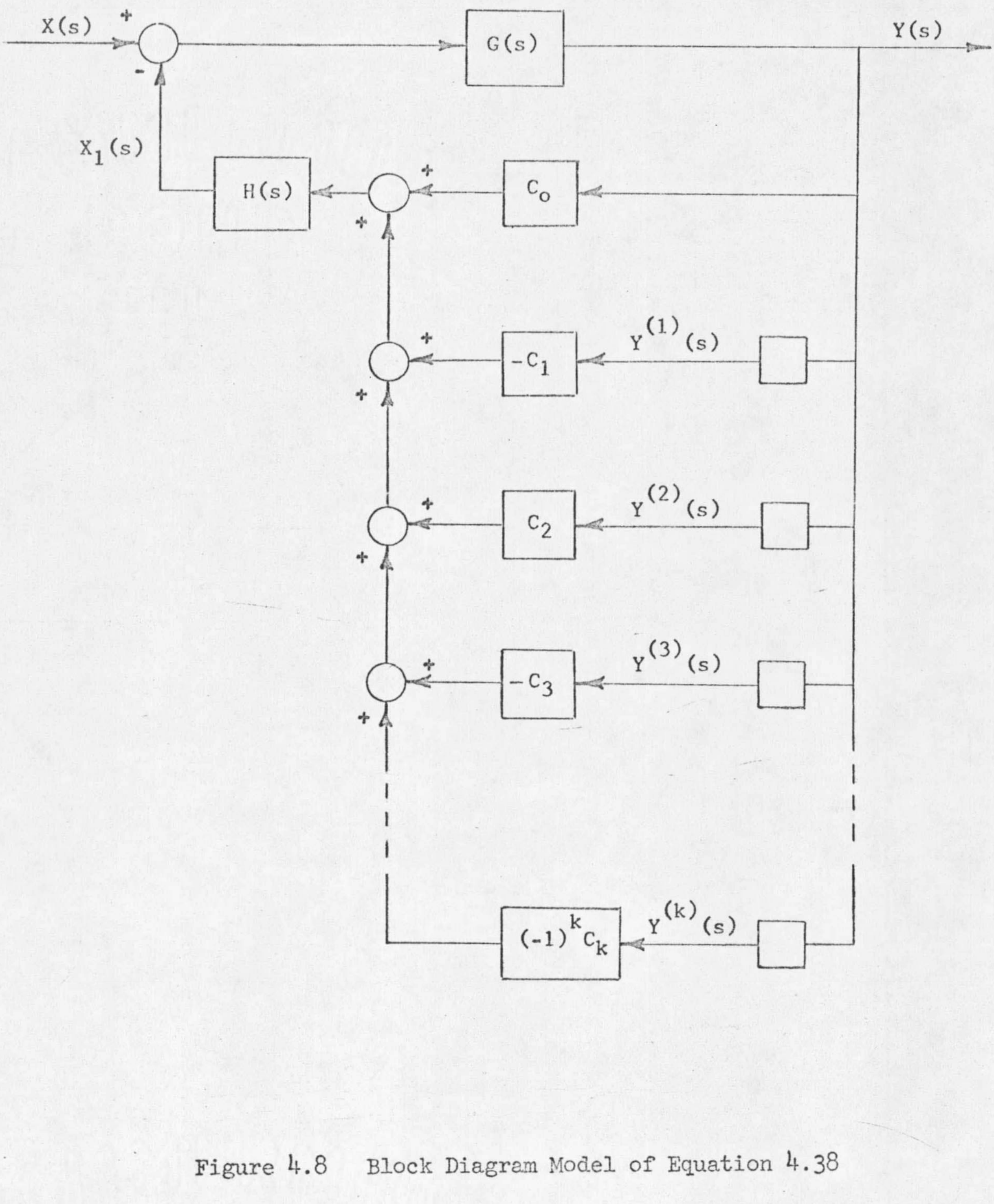

Figure 4.8 Block Diagram Model of Equation 4.38

Example 4.3.2

Transmitter oscillator stages are designed to produce carrier waveforms which are fixed in both amplitude and frequency. In operation, however, any change in the oscillator parameters, as perhaps would occur during a warm-up period, would cause changes in the waveform generated by the oscillator.

Suppose that during a warm-up period the passive portion of a certain oscillator may be described by the following time-varying differential equation.

$$v(t) = L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + S(t)\,q(t) \qquad (4.43)$$

L is the constant inductance; R the constant resistance; and S(t) is the variable elastance of the oscillator.

Suppose that the elastance variation S(t) is known beforehand to behave, during a warm-up period, as shown in Figure 4.9.

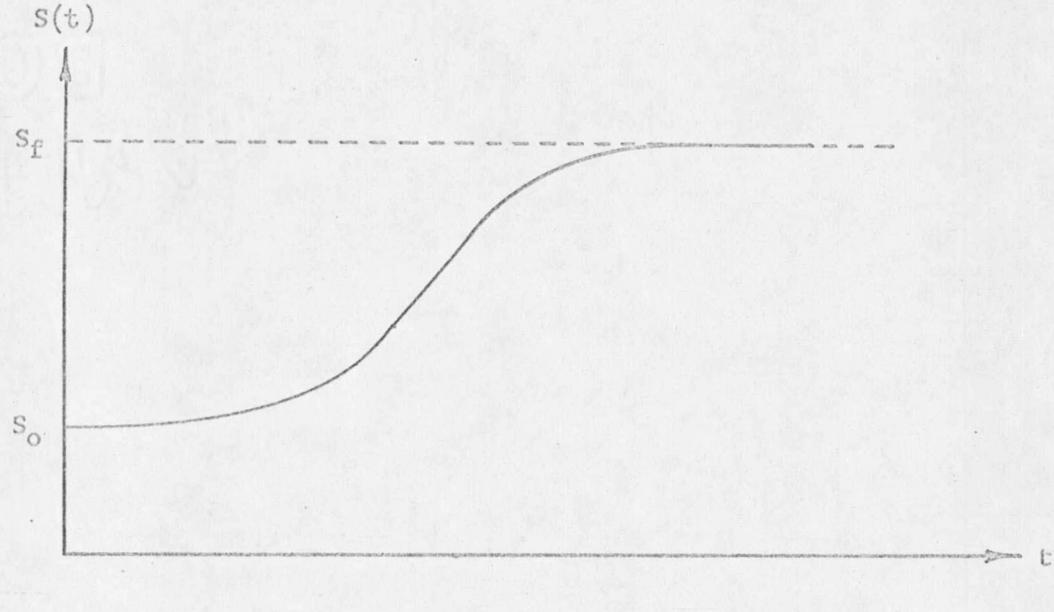

Figure 4.9 Time-Plot of Elastance Variation S(t)

An appropriate expression which describes S(t) analytically for the variation shown in Figure 4.9 may be found by the least squares method.

$$S(t) = S_o + S_1 t + S_2 t^2 \tag{4.44}$$

Substituting equation 4.44 into equation 4.43 and transposing the time-varying terms to the left hand side results in equation 4.45.

$$v(t) - S_1 t q(t) - S_2 t^2 q(t)$$

$$= L \frac{d^2 q(t)}{dt^2} + R \frac{dq(t)}{dt} + S_o q(t) \qquad (4.45)$$

Laplace transforming equation 4.45 yields:

$$V(s) - S_1 \frac{d}{ds} \left[Q(s) \right] - S_2 \frac{d^2}{ds^2} \left[Q(s) \right]$$

$$= \left[s^2 L + sR + S_o \right] Q(s) \qquad (4.46)$$

The block diagram model of the system described by equation 4.46 is illustrated in Figure 4.10.

Using the method of Chapter 2, it is found that the solution to equation 4.46 is a series,

$$Q(s) = \sum_{n=0}^{\infty} Q(s) \qquad (4.47)$$

where

$$Q_o(s) = \frac{V(s)}{s^2 L + sR + S_o} \qquad (4.48)$$

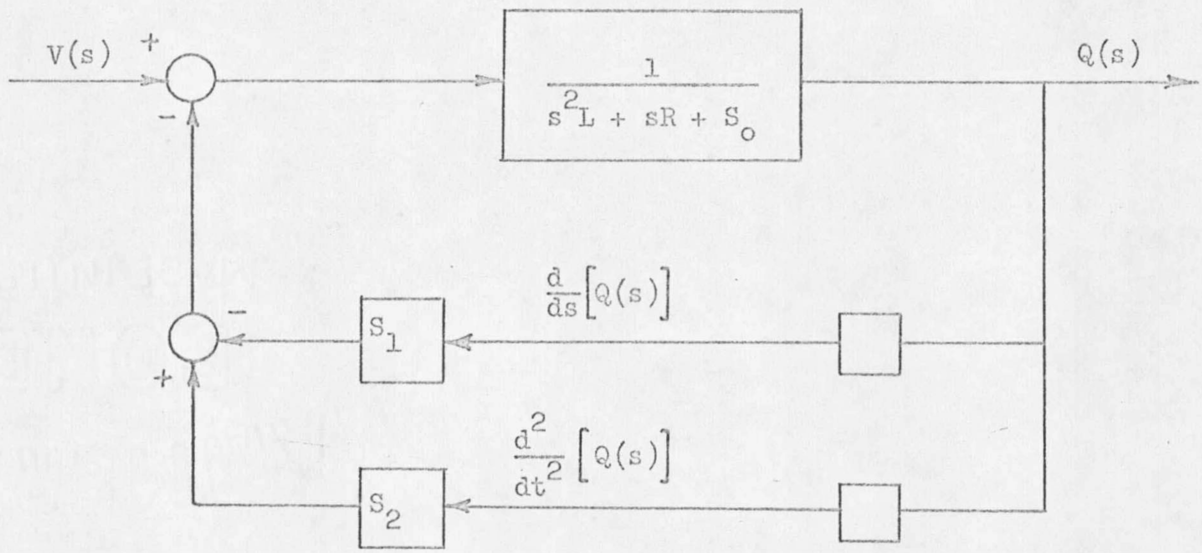

Figure 4.10 Block Diagram of System Described by Equation 4.46.

and $$Q_{n+1}(s) = -\frac{1}{\left[s^2L + sR + S_o\right]} \sum_{k=1}^{2} (-1)^k S_k \frac{d^k}{ds^k}\left[Q_n(s)\right]$$

$$n = 0,1,2, \ldots . \qquad (4.49)$$

The series solution 4.47 will converge uniformly in $0 \leq t \leq T$, where T is any positive number if the following inequality is satisfied:

$$S_f \sup_t \left[\int_0^t |g(x)|\, dx\right] < 1 \qquad (4.50)$$

where

$$\mathcal{L}\left[g(t)\right] = \frac{1}{s^2L + sR + S_o} \qquad (4.51)$$

4.4 EQUATIONS WITH MORE THAN ONE TIME-VARYING COEFFICIENT

In the two preceding sections, systems were examined which were described by time-varying differential equations with a single time-varying coefficient. This, of course, is a special case. More often the differential equation will contain more than one time-varying coefficient, and in fact, a single time-varying parameter in a system may sometimes result in a differential equation containing more than one time-varying coefficient. Solutions to equations of this sort are still obtained by the method of Chapter 2. All of the time-varying terms must be transposed to the left hand side of the equation and considered as a modified input to a system containing only constant parameters.

Example 4.4.1

Consider the case of an air-to-air missile. The missile has a mass $M(t)$ which changes considerably due to the rapid consumption of propellant and has an air friction coefficient B which is practically constant due to the fact that the missile trajectory is nearly horizontal. The missile is propelled by a constant thrust F. The horizontal component $x(t)$ of the missile trajectory is described by the following differential equation.

$$\begin{aligned} F &= \frac{d}{dt}\left[M(t)\frac{dx(t)}{dt}\right] + B\,\frac{dx(t)}{dt} \\ &= M(t)\,\frac{d^2x(t)}{dt^2} + \left[B + \frac{dM(t)}{dt}\right]\frac{dx(t)}{dt} \end{aligned} \tag{4.52}$$

Thus a system with a single time-varying parameter is described, in this case, by a differential equation containing two time-varying coefficients.

The fuel consumption of the missile results in a change in the mass of the missile from an initial high of M_o to a low of M_1 at the time of detonation, T_d. The description of the mass $M(t)$ as a function of time is shown in equation 4.53.

$$M(t) = M_o + M_1 t + M_2 t^2 \tag{4.53}$$

Substituting the value for M(t) above into the differential equation and transposing the time-varying terms to the left hand side results in equation 4.54.

$$F - M_1 t \frac{d^2x(t)}{dt^2} - M_2 t^2 \cdot \frac{d^2x(t)}{dt^2} - 2M_2 t \cdot \frac{dx(t)}{dt}$$

$$= M_o \frac{d^2x(t)}{dt^2} + \left[B + M_1 \right] \cdot \frac{dx(t)}{dt} \tag{4.54}$$

Laplace transforming the above equation results in equation 4.55.

$$\frac{F}{s} + M_1 \frac{d}{ds} \left[s^2 x(s) \right] - M_2 \frac{d^2}{ds^2} \left[s^2 x(s) \right] + 2M_2 \frac{d}{ds} \left[sx(s) \right]$$

$$= M_o s^2 + \left[B + M_1 \right] s \tag{4.55}$$

The above equation suggests the block diagram model of the system shown in Figure 4.11, where G(s) is defined by equation 4.56.

$$G(s) = \frac{1}{M_o s^2 + \left[B + M_1 \right] \cdot s} \tag{4.56}$$

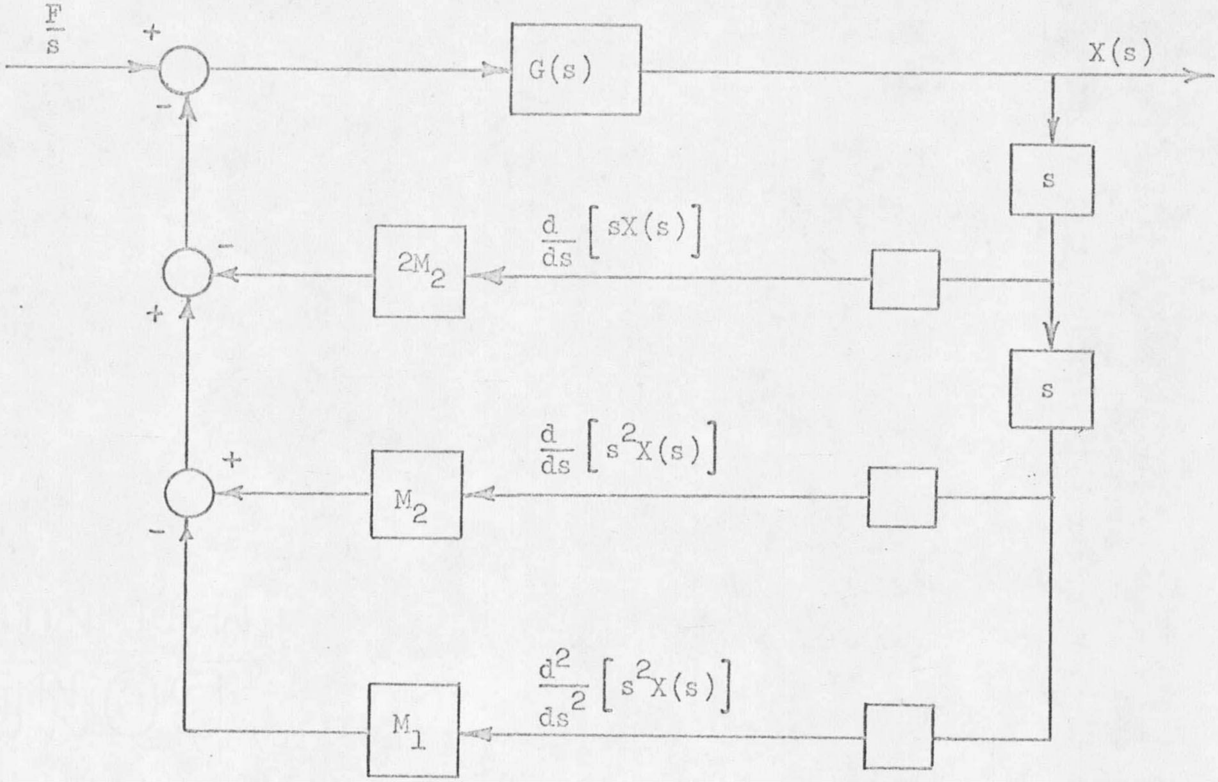

Figure 4.11 Block Diagram Model of Equation 4.55

Utilizing Figure 4.11, the following equation may be written which describes the system defined by equation 4.55.

$$X(s) = G(s)\frac{F}{s} + G(s)\left[2M_2 \frac{d}{ds}\left[sX(s)\right] - M_2 \frac{d}{ds}\left[s^2X(s)\right] + M_1 \frac{d^2}{ds^2}\left[s^2X(s)\right]\right] \tag{4.57}$$

The solution to equation 4.57 is the series

$$X(s) = \sum_{n=o}^{\infty} X_n(s) , \tag{4.58}$$

where
$$X_o(s) = G(s)\frac{F}{s} , \tag{4.59}$$

and
$$X_{n+1}(s) = G(s)\left[2M_2 \frac{d}{ds}\left[sX_n(s)\right] - M_2 \frac{d}{ds}\left[s^2X_n(s)\right] + M_1 \frac{d^2}{ds^2}\left[s^2X_n(s)\right]\right]$$

$$n = 0,1,2, \ldots . \tag{4.60}$$

The series 4.58 is uniformly convergent in the time interval $0 \leq t \leq T$ if the requirement of inequality 4.61 is satisfied.

$$M_o \sup_t \left[\int_0^t \left| g(x) \right| dx \right] < 1 \tag{4.61}$$

4.5 SYSTEMS IN WHICH EVERY PARAMETER IS TIME-VARYING

Sometimes systems are encountered in which every parameter of the system is time dependent. It would seem that the solution of such a system would require very little more effort than the solution of systems, such as those in the previous section, where all but one of the system parameters were time-varying. The solution is, however, somewhat more involved because, if the same procedure is followed, and all of the time-varying terms considered as modified inputs to a time-invariant system, there are essentially no time-invariant terms from which to form the invariant part. In the method used previously, the first term $Y_o(s)$ of the series solution was the response of the time-invariant network to the external excitation, and if all the parameters are time-dependent, then $Y_o(s)$ must equal zero, which in turn would cause all the remaining terms of the series solution to also equal zero, because of the recursive relationships. On account of this situation it is obvious that some modification of the method of solution is necessary.

Let us examine the procedure as it was used before. The solution in all cases was found to be a series.

$$Y(s) = \sum_{n=o}^{\infty} Y_n(s) \tag{4.62}$$

The first term in the series, $Y_o(s)$, is the first approximation to the solution Y(s). The second approximation to the solution is $Y_o(s) + Y_1(s)$ where $Y_1(s)$ is determined from $Y_o(s)$ in a recursion relationship. The third approximation to the solution is $Y_o(s) + Y_1(s) + Y_2(s)$ where $Y_2(s)$ is determined from $Y_1(s)$. Obviously, then, each successive term in the series is dependent upon the first approximation $Y_o(s)$. Previously, the first approximation has always been the response of the time-invariant part of the system to the external input X(s). It is not necessary, however, that the first approximation be found in this manner. Any real function which is continuous in the time interval of interest may be used as the first approximation. The reason is that even though each term in the series is dependent upon $Y_o(s)$, the limit of the series

$$Y(s) = \underset{N \to \infty}{\text{Limit}} \sum_{n=o}^{N} Y_n(s) \tag{4.63}$$

is independent of the choice of $Y_o(s)$ (48,56,76).

In cases where all the system parameters are time dependent and there is no fixed part of the system whose response may be used as the first approximation to the solution, it is necessary to choose a first approximation $Y_o(s)$ to the series solution. As has been stated above, the choice of the first approximation does not affect the limit function of the series $Y(s)$ provided the function used is real and continuous in the time interval of interest (48,56,76). The choice of $Y_o(s)$ may, however, have an influence on how rapidly the series solution converges. A judicious choice of $Y_o(s)$ in such a case might therefore lead to considerable savings in computation time required to yield an adequate solution. After the choice of a suitable first approximation $Y_o(s)$, the procedure to obtain the total solution is the same as was applied in the previous section.

Example 4.5.1

A manned space module trying to maintain a given attitude may be considered as a simple second order system.

$$T_A(t) = \frac{d}{dt} \; J(t) \, \frac{d\Theta(t)}{dt} \qquad (4.64)$$

T_A is the reactive torque developed by small thrusters, $\Theta(t)$ is the angle deviation from a given attitude position, and $J(t)$ is the moment of inertia of the vehicle which is constantly changing due to such factors as fuel consumption and the ever changing locations of both men and material aboard the craft.

Since all the terms of the equation are time-varying, it is necessary according to the procedure described above, to choose a $\Theta_o(s)$ as a first approximation to the solution. The choice, of course, is no problem because any real function continuous in the time interval of interest will eventually yield the solution. However, a judicious choice of $\Theta_o(s)$ would lead to a more rapid convergence of the series solution. If the craft had an average value of $J(t)$, perhaps J_o, then a good first approximation would be that given by equation 4.65.

$$\Theta_o(s) = \frac{1}{s^2 J_o} \cdot T_A(s) \tag{4.65}$$

Figure 4.12 illustrates the block diagram model which describes the operation by which the series solution is obtained.

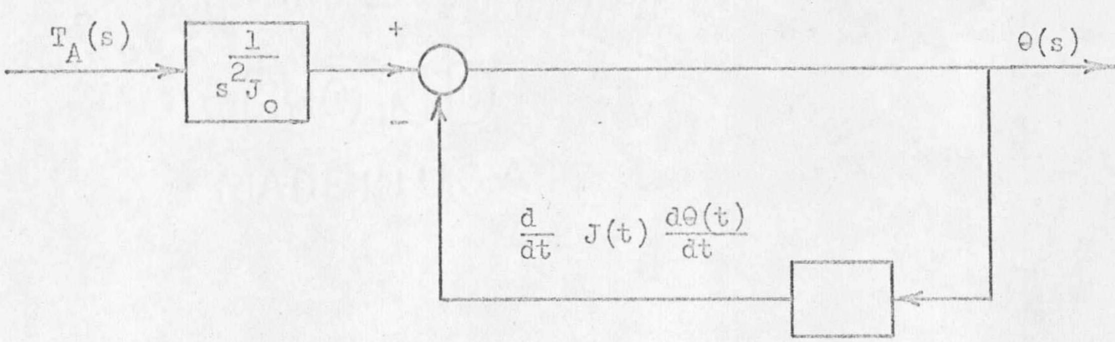

Figure 4.12 Block Diagram Model Describing Method of Solution to Equation 4.64

The solution for the above system is the series

$$\Theta(s) = \sum_{n=0}^{\infty} \Theta_n(s) , \qquad (4.66)$$

where $\Theta_o(s)$ is as given by equation 4.65, and the succeeding terms are found from the recursion relationship

$$\Theta_{n+1}(s) = -\mathcal{L}\left[\frac{d}{dt}\left[J(t)\,\frac{d\Theta(t)}{dt}\right]\right] \qquad n = 0,1,2, \ldots . \qquad (4.67)$$

4.6 TIME-VARYING SYSTEMS EXCITED BY MODULATED SIGNALS

A very large percentage of all the electrical circuits used in transmitting and receiving information are exposed to modulated signals of one sort or another. In the past, most all of the network elements were fixed, but in recent years more and more interest has developed in circuits whose elements may be changed in some desirable fashion. It is appropriate, then, that the method of solution developed thus far be extended to include time-varying systems excited by modulated signals.

4.6.1 AMPLITUDE MODULATED INPUT SIGNALS

About the simplest case of amplitude modulation is the situation in which a carrier waveform with maximum amplitude A_c and natural frequency w_c is amplitude modulated by a sinusoidal single frequency signal which has a frequency of w_m. This amplitude modulated signal may be described by equation 4.68, where M is the modulation index (70).

$$i(t) = A_c \left[1 + M \cos(w_m t) \right] \cos(w_c t)$$

$$= A_c \cos(w_c t) + MA_c \cos(w_c + w_m)$$

$$+ MA_c \cos(w_c - w_m) \qquad (4.68)$$

As can be seen from the above equation, the effect of the modulation process is to produce two new frequencies, the sum and the difference of the carrier and modulating frequencies. In other words, the amplitude modulated signal can be considered as three separate signals as shown in equation 4.68. By the use of the principle of superposition, each of the three signals may be applied separately to a linear system and the output of the system would simply be the sum of the three responses.

The procedure to be followed to solve for the response of time-varying systems excited by amplitude modulated signals is to first reduce the signal to the three frequency components as shown in equation 4.68, and second to apply each signal separately to the time-varying system and solve for the output of the particular signal by the procedure of Chapter 2. The total response to the modulated signal is the sum of the responses to the three separate signals.

If the modulating signal is not sinusoidal but is still periodic, it should be expanded in its Fourier series representation and each term of the Fourier series considered as a separate modulating signal. The total response would then be obtained by performing the procedure of the preceding paragraph for each significant term of the Fourier series. The computations could easily get out of hand unless the Fourier series can be truncated after a very few terms.

4.6.2 FREQUENCY MODULATED INPUT SIGNALS

As was true in the case of amplitude modulated signals, the frequency modulated waveform is not a sinusoid, but it can be expanded in a series of sinusoids. The equation describing a carrier of frequency w_c frequency modulated by a sinusoidal signal of frequency w_m is shown in equation 4.69.

$$i(t) = \cos\left[w_c t + M \sin(w_m t) \right] \tag{4.69}$$

M is the modulation index. Equation 4.69 may be expanded in a Fourier series whose coefficients $J_k(M)$ are Bessel functions of the first kind (70).

$$\begin{aligned} i(t) = {} & J_o(M) \cos(w_c t) \\ & + J_1(M) \left[\cos(w_c + w_m)t - \cos(w_c - w_m)t \right] \\ & + J_2(M) \left[\cos(w_c + 2w_m)t + \cos(w_c - 2w_m)t \right] \\ & + J_3(M) \left[\cos(w_c + 3w_m)t - \cos(w_c - 3w_m)t \right] \\ & + \ldots \end{aligned} \tag{4.70}$$

The frequency modulation process, therefore, produces a series of side-band frequency pairs. The number of significant pairs depends on the modulation index M.

In order to find the response of a time-varying system which is excited by a frequency modulated signal, it is necessary to apply the method of Chapter 2 to each of the significant terms of the series 4.70 and consider each term as a separate input to the system.

CHAPTER 5

SUMMARY AND SUGGESTED FUTURE RESEARCH

5.1 SUMMARY OF ESSENTIAL VALUES OF THE THESIS RESEARCH

In this thesis a general method of solution has been developed for single input, single output, time-varying, linear systems. The method yields the total solution, transient as well as steady-state solution and is independent of both the order of the system and the manner of the time dependence of the system parameters. The parameter variations, however, are restricted in magnitude.

In Chapter 2, the method was used to obtain the solution of three specific time-varying systems in which a parameter was a sinusoidal function of time. In each case it is found that the solution is an infinite series. In the complex frequency domain, each successive term of the series contributes two additional frequency components to the solution. The series solution for two of the systems converged. However, the series for the third system diverged.

In Chapter 3, this problem of the divergence of the series solution was examined and criteria for the uniform convergence of the series solutions were developed in terms of the physical parameters of the systems. Also, a series for steady-state solutions was developed which had more lenient convergence requirements than the series for the total solution.

In Chapter 4, the series method of solution was applied to several problems which are representative of the most encountered types of time-varying systems.

5.2 ASPECTS OF THE THESIS MERITING ADDITIONAL STUDY

Several topics worthy of further investigation are evident as a result of this dissertation.

1. Development of necessary as well as sufficient conditions for convergence of the series solutions would facilitate application of this method of solution.

2. Investigation into the realm of frequency domain characteristics of time-varying filters would contribute greatly to the understanding of such filters. As an example, consider a frequency tracking filter whose input is a frequency modulated signal (27,36). The filter has a very narrow bandwidth whose passband can be continuously tuned to the instantaneous frequency of the incoming FM signal. The noise rejection capability of the frequency tracking filter is much superior to the ordinarily used wideband FM filter. The reasons for this are not fully understood. A lucid explanation of the frequency rejection and pass characteristics of such time-varying filters would be a welcomed contribution. The techniques developed in the frequency domain for the series method of solution provide an avenue well worth investigating.

3. The results of an analysis such as described in topic 2 above could be put to good use in the design of special time-varying filters to perform specific functions upon incoming signals that cannot be performed by time-invariant networks. Again the tracking filter provides an appropriate example. The problem of specifying the properties of such

a time-varying filter so as to remove in some optimal fashion as much of the incoming noise as possible is a problem that needs to be solved if optimal design procedures for time-varying filters are going to become available to electronic circuit designers.

4. Also of interest is the performance of time-varying plants or compensators within control loops. In topic 2 above, it was shown that time-varying systems can be used to advantage in particular applications. In such cases, the question arises as to how the stability of the overall system is affected by the inclusion of time-varying subsystems within the control loop. A stability investigation might be pursued using the Popov Circle criterion in light of the frequency domain method developed in this thesis (99).

5. Since most linear time-varying parameter systems encountered are actually simplified models of nonlinear systems, it would be logical to extend the methods developed herein to an analysis of nonlinear time-varying systems. This extension should not be as drastic as it may initially appear because the series solution method is related to an approximation technique for solving Fredholm and Volterra integral equations in the time domain (1,29,48,76). This approximation technique applies also to nonlinear time-varying integral equations with but minor changes.

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