Robust Optimization Model for a Dynamic Network Design Problem Under Demand Uncertainty

Authors: Byung Do Chung, Tao Yao, Chi Xie, and Andreas Thorsen

The final publication is available at Springer via http://dx.doi.org/10.1007/s11067-010-9147-2.

Chung, Byung Do, Tao Yao, Chi Xie, and Andreas Thorsen. "Robust Optimization Model for a Dynamic Network Design Problem Under Demand Uncertainty." Networks and Spatial Economics 11, no. 2 (September 4, 2010): 371–389. doi:10.1007/s11067-010-9147-2.

Robust Optimization Model for a Dynamic Network Design Problem Under Demand Uncertainty

Byung Do Chung · Tao Yao · Chi Xie ·
Andreas Thorsen

Abstract This paper describes a robust optimization approach for a network design problem explicitly incorporating traffic dynamics and demand uncertainty. In particular, we consider a cell transmission model based network design problem of the linear programming type and use box uncertainty sets to characterize the demand uncertainty. The major contribution of this paper is to formulate such a robust network design problem as a tractable linear programming model and demonstrate the model robustness by comparing its solution performance with the nominal solution from the corresponding deterministic model. The results of the numerical experiments justify the modeling advantage of the robust optimization approach and provide useful managerial insights for enacting capacity expansion policies under demand uncertainty.

1 Introduction

Network design consists of a broad spectrum of problems, each corresponding to different sets of objectives, decision variables and resource constraints, implying different behavioral and system assumptions, and possessing varying data requirements and capabilities in terms of representing network supplies and demands. Network design models have been extensively used as various types of strategic, tactical and operational decision-making tools and spanned over a variety of applications in, for example, transportation, production, distribution, and communication fields. In a transportation network, traffic congestion has long been a major concern of the network operator, which occurs when traffic volumes exceed the road capacity. Network design problems (NDP) for transportation networks in general aim at minimizing network traffic congestions (or minimizing some general network-wide traveler costs) through implementing an optimal capacity expansion policy in the network.

An optimal capacity expansion policy, however, may not be reached without properly considering the behavioral nature of travel demands, which are inherently time-variant and uncertain. Travel demands are an aggregate result of individual travel activities, which are determined by various observed and unobserved socioeconomic factors and subject to geographical, technological and temporal constraints. The vast body of the literature has focused on static deterministic NDPs (see, for example, Magnanti and Wong 1984; Minoux 1989; Yang and Bell 1998). A major limitation of static network design models is the inability to capture traffic dynamics, such as traffic shockwave propagation and the build-up and avoidance of queues. Dynamic models, on the other hand, allow us to model the time-dependent variation of traffic flows and travel behaviors and hence better describe traffic evolution and interaction phenomena over the network (Peeta and Ziliaskopoulos 2001). Travel demand uncertainty is not only the underlying characteristic of travel activities but also a likely result of our inaccurate or inconsistent travel demand estimation procedures. Without explicit and rigorous recognition of uncertainty in travel demands, any transportation network development plans and policies may take on unnecessary risk and even result in misleading outcomes (Zhao and Kockelman 2002).

In terms of their mathematical functional forms, dynamic traffic assignment (DTA) based NDPs can be classified into two major groups: single-level models and bi-level models (see the discussion in Lin et al. 2008). The focus of this paper is on an application of robust optimization (RO) for dynamic NDPs under demand uncertainty, or more succinctly, a robust dynamic NDP (RDNDP), which has a single-level strcture. The single-level structure provides an easier way to manipulate robust counterpart and make RDNDP to be computationally tractable.

The research community has observed a number of recent network design studies that explicitly incorporate demand uncertainty into NDPs with time-varying flows (see Waller and Ziliaskopoulos 2001; Karoonsoontawong and Waller 2007; Ukkusuri and Waller 2008; Karoonsoontawong and Waller 2008). The common feature of these problems is that time-varying flows are described by the cell transmission model (CTM) (Daganzo 1994, 1995) and the network flow pattern is then characterized by CTM-based DTA methods, under either the system-optimal (Ziliaskopoulos 2000) or user-optimal assignment mechanism (Ukkusuri and Waller

2008). The demand uncertainty of these problems is accommodated by a chance constraint setting, a two-stage recourse model, or a scenario-based simulation method. These techniques, however, suffer from deficiencies related to lack of data availability and problem tractability, which limit their applicability to a broad range of applications. Resulting models from these stochastic modeling methods are often computationally intractable and require known probability distributions.

We follow a similar fashion to form our RDNDP using the CTM-based system-optimal DTA model, but employ the RO approach to account for demand uncertainty. Given the fact that the CTM-based DTA model has a linear programming (LP) formulation, we use the set-based RO method (Ben-Tal and Nemirovski 1998, 1999, 2000, 2002) to form a tractable LP model for the RDNDP, which overcomes the limitations of previous stochastic optimization methods. Specifically, in our RDNDP, no probability distribution is presumed; instead, we only need to simply specify an uncertain set, which is readily available in most applications. The solution feasibility is guaranteed by the RO method through the use of the prescribed uncertainty set and can be readily made computationally tractable through an appropriate reformulation.

We highlight the main contributions of our work at a glance below:

- We develop an RO framework for the SO-DTA based RDNDP. For simplicity, we present our RO model only for single-destination, system-optimal networks. However, the basic RO counterpart formulation method can be readily transferred to the multi-destination problem case. This work adds to the body of knowledge in the dynamic network design by presenting an emerging method related to the solution robustness.
- An appealing feature of our robust counterpart problem is that it still has an LP formulation, so it is in general computationally tractable and can be solved in polynomial time by a few well-known solution algorithms.
- Our numerical experiments demonstrate the value of RO in the context of dynamic traffic assignment and network design problems. The computational viability is illustrated for the proposed modeling framework. The numerical analysis for the impact of the investment budget bound and the demand uncertainty level on network design solutions justifies the solution robustness.

The remaining part of this paper is structured as follows. Section 2 provides a discussion of the relevant literature. In Section 3, we generalize the formulation given by Ukkusuri and Waller (2008) as a CTM-based deterministic dynamic NDP (DDNDP). We then in Section 4 propose a robust counterpart formulation of the DDNDP to account for demand uncertainty, which we name the RDNDP. Computational experiments and result analyses from applying the RDNDP model to a few numerical examples are elaborated in Section 5. Finally, Section 6 concludes the paper and indicates some future research directions.

2 Literature review

Numerous NDPs for transportation applications have been presented in the past three decades (see Magnanti and Wong 1984; Minoux 1989; Yang and Bell 1998). These

NDPs are distinguished by a variety of problem settings and supply and demand assumptions. The literature review presented below by no means provides a comprehensive survey to general network design problems or to network design applications in the transportation field; instead, our discussion is focused on those network design models and solution methods with data uncertainty, particularly network design problems with time-varying flows.

A great amount of attention has been paid to NDPs with data uncertainty in past years and various modeling techniques are used for dealing with uncertain input data and parameters. The main approaches can be classified into two groups: stochastic programming (SP) and robust optimization (RO). The SP approach requires known probability distributions of the uncertain data and includes techniques such as the Monte Carlo sampling approach and chance-constrained programming. For example, Waller and Ziliaskopoulos (2001) solved a NDP under uncertain demands where the probability distributions of demand rates are known a priori. They used a CTM-based system-optimal NDP formulation with chance constraints. Ukkusuri and Waller (2008) extended the CTM to model both the system-optimal and user-optimal NDPs and presented the formulations of a chance-constrained NDP model and a two-stage resource NDP model to account for demand uncertainty.

Mulvey et al. (1995) proposed a scenario-based RO approach for general LP problems. Karoonsoontawong and Waller (2007) applied this approach to a CTM-based dynamic NDP with stochastic demands under both the system-optimal and user-optimal conditions. A similar RO model formulation approach was employed by Ukkusuri et al. (2007), in which a scenario-based robust NDP with discrete decision variables was tackled by a genetic algorithm. The limitations of scenario-based RO approach are similar to stochastic programming in that we must know the probability of each scenario in advance and it is computationally expensive when there are a large number of scenarios.

Recently, a variety of papers have used the set-based RO technique to characterize optimization models with data uncertainty. Interested readers are referred to Ben-Tal and Nemirovski (2002) and Bertsimas et al. (2007) for reviews of the set-based RO methods. For NDPs with uncertain demands, Yin and Lawpongpanich (2007) considered a static continuous equilibrium NDP under demand uncertainty. Ordonez and Zhao (2007) formulated and solved a static multicommodity NDP with demand and travel time uncertainties bounded by polyhedral sets. Mudchanatongsuk et al. (2008) extended the work by Ordonez and Zhao by considering some generalized assumptions on demand uncertainty, in which they discussed a path-constrained NDP and introduced a column generation method to solve the robust NDP with polyhedral uncertainty sets. Ban et al. (2009) considered a robust road pricing problem (which is an NDP in the broader definition) that contains multiple traffic assignment solutions. Atamturk and Zhang (2007) formulated and solved a NDP by using the two-stage RO method and taking advantage of the network structure for its solutions. To characterize their uncertainty sets, they used a budget of uncertainty which limits the number of observed demand values that can differ from nominal values. They also discussed the numerical results for a simple location-transportation problem and compared the two-stage robust approach with the single-stage robust approach as well as two-stage scenario-based stochastic programming.

There have also been approaches where the set-based RO approach is used to construct discrete network design models. For example, Lou et al. (2009) described a discrete NDP with user-equilibrium flows based on the concept of uncertainty budget and proposed a cutting-plane method for problem solutions; Lu (2007) addressed a discrete user-equilibrium NDP with polyhedral uncertainty sets using the RO approach and used an iterative solution algorithm to solve the problem.

To the best of our knowledge, no work has been done in applying the set-based RO technique to investigate a NDP with dynamic flows and uncertain demands. In this paper, our effort is given to analytically developing and numerically analyzing the robust counterpart model of such an NDP in the context of transportation network design.

3 Deterministic model

This section presents the deterministic version of the dynamic NDP model we have discussed, or the DDNDP model in abbreviation, which provides the basic modeling platform and functional form for the RDNDP model we will introduce in the next section. For discussion convenience, let us first present the notation used throughout these models (see Table 1).

Table 1 The notation

Sets	Description
$\mathcal{T}$	Set of discrete time intervals, $\{1,...,T\}$
C	Set of cells, $\{1,...,I\}$, including the set of sink cells (C_S) and the set of source cells (C_R)
C_E	Set of cells that can be expanded, $C_E \subset C \backslash C_S$
A	Adjacency matrix, $A=\{a_{ij}\}$, where each (i, j) component, a_{ij}, equals 1 if cell i is connected to cell j, and equals 0 otherwise
Parameters	Description
d_i^t	Demand generated in cell i at time t
c_i^t	Travel cost in cell i at time t
N_i^t	Capacity in cell i at time t
Q_i^t	Inflow/outflow capacity of cell i at time t
δ_i^t	Ratio of the free-flow speed over the backward propagation speed of cell i at time t
$\widehat{x}_i$	Initial number of vehicles of cell i
B	Total investment budget available for capacity expansion
f_i	Conversion coefficient of investment cost of cell i for a unit increase of b_i
χ_i	Increase in capacity of cell i for a unit increase of b_i
φ_i	Increase in inflow/outflow capacity of cell i for a unit increase of b_i
Variables	Description
b_i	Investment cost spent on cell i
x_i^t	Number of vehicles staying in cell i at time t
y_i^t	Number of vehicles moving from cell i to cell j at time t

The network design problem aims at minimizing the sum of the total system travel cost and the capacity expansion cost. To penalize the unmet demand by the end of planning horizon, the travel cost in cell i at time t, c_i^t, is set as follows:

$$c_i^t = \begin{cases} 1 & i \in C \backslash C_S, t \neq T \\ M & i \in C \backslash C_S, t = T \end{cases}$$

where M is a sufficiently large, positive number. The big-M value can also simply serve as a penalty cost, for example, in emergency evacuation networks, representing the potential loss of life and property caused by vehicles that do not arrive at the destination by the end of the time horizon. Use of the penalty cost has the effect of minimizing the number of vehicles staying in an evacuation network.

By assuming the system-optimal principle and the linear relationship between investment and capacity increase, the DDNDP model can be written as a LP program with the notation listed in Table 1:

$$\min_{x,y,b} \sum_{t \in \tau} \sum_{i \in C \backslash C_S} c_i^t x_i^t + \sum_{i \in C_E} f_i b_i$$

subject to

$$x_i^t - x_i^{t-1} - \sum_{k \in C} a_{ki} y_{ki}^{t-1} + \sum_{j \in C} a_{ij} y_{ij}^{t-1} = d_i^{t-1} \quad \forall i \in C, \forall t \in \tau \tag{1}$$

$$\sum_{k \in C} a_{ki} y_{ki}^t \leq Q_i^t \; \forall i \in C \backslash C_E, \forall t \in \tau \tag{2 – 1}$$

$$\sum_{k \in C} a_{ki} y_{ki}^t \leq Q_i^t + b_i \varphi_i \; \forall i \in C_E, \forall t \in \tau \tag{2 – 2}$$

$$\sum_{k \in C} a_{ki} y_{ki}^t + \delta_i^t x_i^t \leq \delta_i^t N_i^t \; \forall i \in C \backslash C_E, \forall t \in \tau \tag{3 – 1}$$

$$\sum_{k \in C} a_{ki} y_{ki}^t + \delta_i^t x_i^t \leq \delta_i^t (N_i^t + b_i \chi_i) \; \forall i \in C_E, \forall t \in \tau \tag{3 – 2}$$

$$\sum_{j \in C} a_{ij} y_{ij}^t \leq Q_i^t \; \forall i \in C \backslash C_E, \forall t \in \tau \tag{4 – 1}$$

$$\sum_{j \in C} a_{ij} y_{ij}^t \leq Q_i^t + b_i \varphi_i \forall i \in C_E, \forall t \in \tau \tag{4 – 2}$$

$$\sum_{j \in C} a_{ij} y_{ij}^t - x_i^t \leq 0 \; \forall i \in C, \forall t \in \tau \tag{5}$$

$$\sum_{i \in C_E} b_i \leq B \tag{6}$$

$$x_i^0 = \widehat{x}_i \forall i \in C \tag{7}$$

$$y_{ij}^0 = 0 \quad \forall (i,j) \in C \times C \tag{8}$$

$$x_i^t \geq 0 \quad \forall i \in C, \forall t \in \tau \tag{9}$$

$$y_{ij}^t \geq 0 \quad \forall (i,j) \in C \times C, \forall t \in \tau \tag{10}$$

$$b_i \geq 0 \quad \forall i \in C_E \tag{11}$$

The objective function includes both the travel cost and expansion cost.[1] The coefficient f_i converts the investment cost (money measure) to the travel cost (time measure). In fact, such coefficient is the reciprocal of value of time, which can be measured by empirical methods (Wardman (1998)). Note here that the expansion cost appears in the objective function and is subject to the investment budget constraint, which makes this formulation different from the traditional charge design problem (where the expansion cost term is only included in the objective function) and budget design problem (where the expansion cost terms only appears in the investment budget constraint).

The constraint set of the DDNDP model specifies the capacity expansion limit, flow conservation and propagation relationships, initial network conditions and flow non-negativity conditions. The flow conservation constraint (i.e., Eq. (1)) for cell i at time t can be generalized by setting d_i^t to be zero in ordinary and sink cells. Constraints (2-1) and (2-2) are the bounds for the total inflow rate of non-expandable and expandable cell i at time t, respectively. Similarly, the total outflow rate of cell i at time t is restricted by constraints (4-1) and (4-2). Constraints (3-1) and (3-2) bounds the total inflow rate into a cell by its remaining space. Constraint (5) bounds the total outflow rate of a cell by its current occupancy, and constraint (6) sets the upper bound on the sum of capacity investments over all cells. The remaining constraints from Eq. (7) to Eq. (11) set initial network conditions and flow non-negativity conditions.

4 Robust formulation

Now we develop the robust counterpart of the DDNDP model, which incorporates the demand uncertainty into a LP program via the RO approach. In the deterministic

[1] Costs in general do not vary linearly with respect to the transportation facility capacity or size. Typically, scale economies or diseconomies exist. Abdulaal and LeBlanc (1979) discussed the cases of linear relationship, scale economies and scale diseconomies in the context of transportation network design problems. If the average investment cost per unit of capacity is declining, then scale economies exist. Empirical data are needed to establish the economies of scale for road construction. This paper assumes a linear relationship between the investment cost and the capacity, for the reasons of simplicity and the requirement of the linear model. The linear case can be regarded as an approximation to the case of scale economies in an expected capacity-increasing range.

version, Eq. (1) is the only set of constraints related to the demand generation. This equality constraint can be rewritten as an inequality constraint (Waller and Ziliaskopoulos 2006),

$$x_i^t - x_i^{t-1} - \sum_{k \in C} a_{ki} y_{ki}^{t-1} + \sum_{j \in C} a_{ij} y_{ij}^{t-1} \geq d_i^{t-1}. \tag{12}$$

It is assumed that all possible demand instances of d_i^t belong to a box uncertainty set,

$$U_{d_i^t} = \left[\overline{d}_i^t \left(1 - \theta_i^t\right), \overline{d}_i^t \left(1 + \theta_i^t\right)\right] \tag{13}$$

where $\overline{d}_i^t$ is the nominal demand level and θ_i^t is the demand uncertainty level. Then, the robust counterpart of the Eq. (12) with demand uncertainty becomes

$$x_i^t - x_i^{t-1} - \sum_{k \in C} a_{ki} y_{ki}^{t-1} + \sum_{j \in C} a_{ij} y_{ij}^{t-1} \geq d_i^{t-1}, \quad d_i^{t-1} \in U_{d_i^{t-1}}. \tag{14}$$

This is equivalent to the following inequality,

$$x_i^t - x_i^{t-1} - \sum_{k \in C} a_{ki} y_{ki}^{t-1} + \sum_{j \in C} a_{ij} y_{ij}^{t-1} \geq \max_{d_i^{t-1} \in U_{d_i^{t-1}}} d_i^{t-1}, \tag{15}$$

which becomes the flow conservation constraint for the RDNDP model. The above conversion of the flow conservation constraint leads the RDNDP to be in a deterministic functional form with the maximum possible demand in the box uncertainty set. Given that other constraints can be directly transferred from the DDNDP model to the RDNDP model, the RDNDP formulation can be written into the following LP form:

$$\min_{x,y,b} \sum_{t \in \tau} \sum_{i \in C \setminus C_S} c_i^t x_i^t + \sum_{i \in C \setminus C_S} f_i b_i$$

subject to

$$x_i^t - x_i^{t-1} - \sum_{k \in C} a_{ki} y_{ki}^{t-1} + \sum_{j \in C} a_{ij} y_{ij}^{t-1} = \overline{d}_i^{t-1} (1 + \theta_i^{t-1}) \forall i \in C, \forall t \in \tau \tag{16}$$

$$\sum_{k \in C} a_{ki} y_{ki}^t \leq Q_i^t \; \forall i \in C \setminus C_E, \forall t \in \tau$$

$$\sum_{k \in C} a_{ki} y_{ki}^t \leq Q_i^t + b_i \varphi_i \; \forall i \in C_E, \forall t \in \tau$$

$$\sum_{k \in C} a_{ki} y_{ki}^t + \delta_i^t x_i^t \leq \delta_i^t N_i^t \; \forall i \in C \setminus C_E, \forall t \in \tau$$

$$\sum_{k\in C} a_{ki}y_{ki}^t + \delta_i^t x_i^t \le \delta_i^t(N_i^t + b_i\chi_i)\ \forall i\in C_E, \forall t\in\tau$$

$$\sum_{j\in C} a_{ij}y_{ij}^t \le Q_i^t\ \forall i\in C\backslash C_E, \forall t\in\tau$$

$$\sum_{j\in C} a_{ij}y_{ij}^t \le Q_i^t + b_i\varphi_i\ \forall i\in C_E, \forall t\in\tau$$

$$\sum_{j\in C} a_{ij}y_{ij}^t - x_i^t \le 0\ \forall i\in C, \forall t\in\tau$$

$$\sum_{i\in C_E} b_i \le B$$

$$y_{ij}^0 = 0\ \forall (i,j)\in C\times C$$

$$x_i^t \ge 0\ \forall i\in C, \forall t\in\tau$$

$$y_{ij}^t \ge 0\ \forall (i,j)\in C\times C, \forall t\in\tau$$

$$b_i \ge 0\ \forall i\in C_E$$

In Eq. (16), the value of $\overline{d}_i^{t-1}\left(1+\theta_i^{t-1}\right)$ is the maximum possible demand in cell i at time t–1, according to the uncertainty set $U_{d_i^{t-1}}$, which represents the worst-case scenario. Therefore, the optimal solution will remain feasible for all instances of demand. In other words, we will obtain an optimal solution with the cell capacity values that are adequate for any realized demand scenarios within the uncertainty set $U_{d_i^{t-1}}$.

We make the following observation between the optimal objective value and the total budget level B from the RDNDP model. The implication of this property is that the network designers should consider the budget level as large as possible even if the objective function minimize the money used for network expansion together with the travel cost.

Property 1 The optimal objective function value of the RDNDP monotonically decreases with respect to the investment budget level.

Proof Let the objective function of RDNDP be $z_r^*(B)$, given the total budget level B. Without loss of generality, we assume that two budget levels B_1 and B_2 are given as $B_1 < B_2$. Since the RDNDP with B_2 has a larger feasible region than the RDNDP with B_1, $z_r^*(B_1)$ is smaller than or equal to $z_r^*(B_2)$, i.e. $z_r^*(B_1) \le z_r^*(B_2)$. ■

When other types of uncertainty sets such as an ellipsoidal uncertainty set or a polyhedral uncertainty set are assumed, different deterministic formulations are derived. For example, the equivalent tractable robust counterpart with an ellipsoidal uncertainty set is a conic quadratic problem; if a polyhedral uncertainty set is assumed, it becomes a linear problem (see Yao et al. 2009 for details).

5 Numerical analysis

The purpose of presenting computational experiments in this section is twofold: 1) to demonstrate the difference between robust network design solutions and corresponding nominal solutions from DDNPP; and 2) to illustrate the advantage of the RO approach for network design under demand uncertainty. Two numerical examples are selected from the literature for the experiments: 1) a smaller network with 16 cells and 15 time intervals; and 2) a larger network with 167 cells and 300 time intervals. For each example, under the assumption that all cells except destination cells can be invested, we derived the optimal capacity investment solutions and the objective function values from the DDNPP and RDNDP models with various demand uncertainty levels. To evaluate the solution robustness, we also conducted a parallel simulation experiment to randomly generate 100 demand instances within the given box uncertainty set. The objective function values from the simulation experiment are also evaluated by solving the embedded SO DTA problem based on the same capacity expansion scheme as the one derived by the RDNDP model.

5.1 A toy network

The first experiment uses the test network shown in Fig. 1 and the data set in Table 2, which are from Ukkusuri and Waller (2008). Since they considered a set of deterministic demands, it is assumed that the demand data in their paper are nominal values of the network design problem under uncertainty. Let us assume that uncertain demands from source cell 1 and 14 are $[2(1-\theta), 2(1+\theta)]$ at time 0 and 1, and $[1(1-\theta), 1(1+\theta)]$ at time 3. Note that when θ is equal to 0, the uncertainty sets become the nominal values. The investment cost coefficient (f_i) and penalty cost (M) for this example are set to 0.1 and 10, respectively. The resulting RDNDP model has 748 constraints and 344 variables, which has been solved within 3 seconds on a PC with an Intel 1.87 GHz CPU and 2 GB RAM using GAMS/CPLEX.

5.1.1 Optimal solutions under different uncertain levels

The objective function value is calculated and plotted as the total budget level is varied from 0 to 80 in the interval of 1 unit. Figure 2 shows the change of the objective function values of the DDNDP and RDNDP models with three different uncertainty levels (including, θ=0.1, 0.2 and 0.3). As the budget level increases, the objective function value of the RDNDP model decreases and it converges to a certain value. (see Property 1) Robust solutions are the best worst-case solutions and

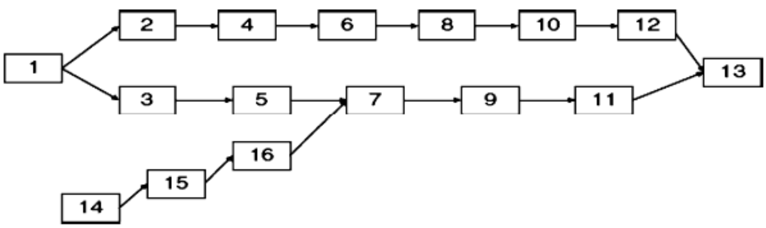


Fig. 1 Cell representation of the Toy network (Ukkusuri and Waller 2008)

Table 2 Cell characteristics of the Toy network (Ukkusuri and Waller 2008)

Cell	2	3	4	5	6	7	8	9	10	11	12	15	16
N_i^t	4	4	4	4	4	2	4	4	4	4	4	4	4
Q_i^t	1	2	2	2	1	2	1	2	1	2	1	1	1
$\hat{x}_i$	0	0	0	0	0	0	0	0	0	0	0	0	0

thus their objective function values are greater than those of the corresponding deterministic cases. Note that any nominal solution is equivalent to its robust solution with the zero uncertainty level ($\theta = 0$).

In all the above cases, the same cells (including cells 7, 9, 11, 15 and 16) are chosen for capacity expansion, which indicates that they are bottleneck cells in the network. However, the proportions of the investment on the cells are dependent on the investment budget level and the demand uncertainty level. Figure 3 shows the investment distribution over the cells. The implication behind these distribution curves is that the investment strategy should be changed depending on the budget bound we set and the demand uncertainty degree we expect to face.

It is readily observed that there is a critical/maximum investment point associated with the investment budget level, beyond which a higher investment does not reduce the travel cost, or a higher investment even increases the objective function value if it is used for capacity expansion in the network. For example, this maximum investment point is between 30 and 40 monetary units in the DDNDP case, and the point is about 70 monetary units in the RDNDP case with θ=0.3. The critical investment point can be interpreted as the threshold for investment: when the budget is less than this threshold, the marginal travel cost (reduction) is greater than the marginal construction cost (increase); when the budget is greater than the threshold, the marginal travel cost (reduction) is less than the marginal construction cost (increase).

5.1.2 Worst-case analysis

After obtaining the investment solutions from the DDNDP and RDNDP models, we then evaluated the relative improvement of robust solutions from their corresponding

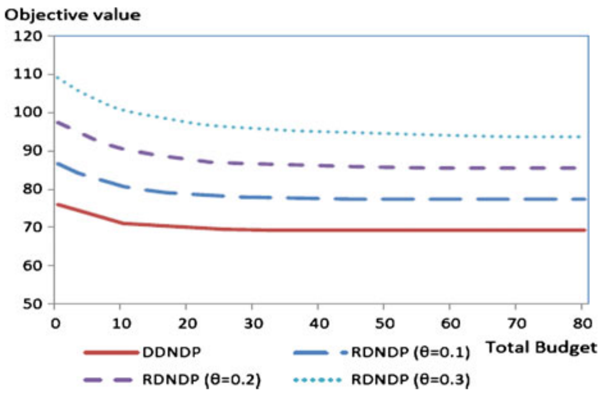


Fig. 2 The objective-budget relationship under different demand uncertainty levels

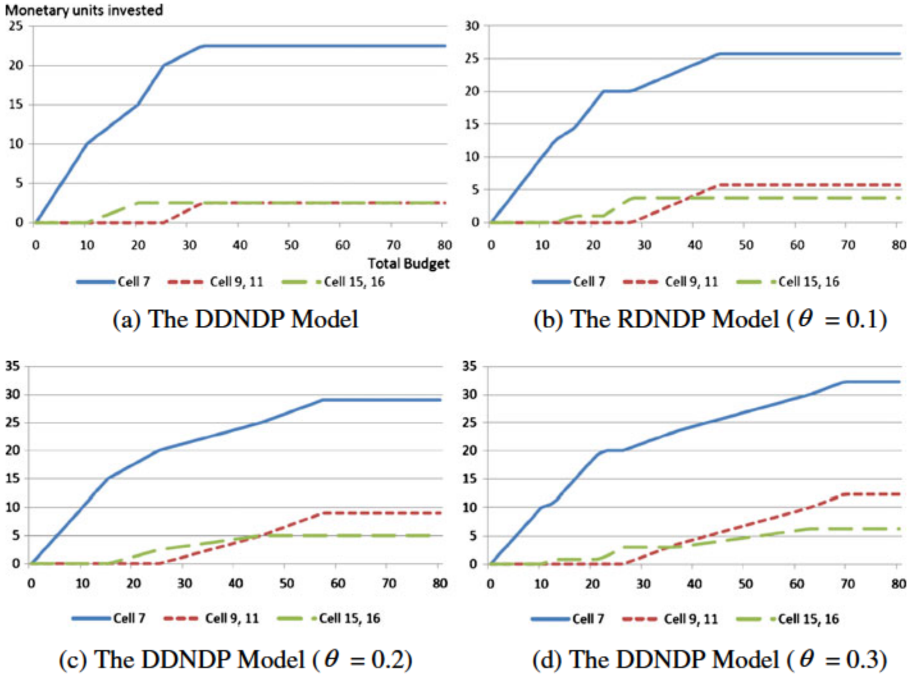


Fig. 3 Optimal investment distributions over the network

nominal solutions under the worst-case scenario. The relative improvement (*RI*) in this study is defined as:

$$RI = \frac{TC_d - TC_r}{TC_r}$$

where TC_d is the total travel cost from the nominal solution and TC_r is the total travel cost from the robust solution.

Table 3 Total travel cost of robust and nominal solutions in worst-case scenarios

Budget	θ=0.1		θ=0.2		θ=0.3	
	DDNDP	RDNDP	DDNDP	RDNDP	DDNDP	RDNDP
0	86.7	86.7	97.4	97.4	109.1	109.1
10	79.7	79.7	89.4	89.4	99.6	99.6
20	77.2	76.7	86.4	85.77	95.7	95.35
30	75.03	74.87	83.73	83.65	92.85	92.8
40	74.7	73.53	83.4	82.15	92.35	90.98
50	74.7	72.9	83.4	80.73	92.35	89.48
60	74.7	72.9	83.4	79.8	92.35	87.98
70	74.7	72.9	83.4	79.8	92.35	86.7
80	74.7	72.9	83.4	79.8	92.35	86.7

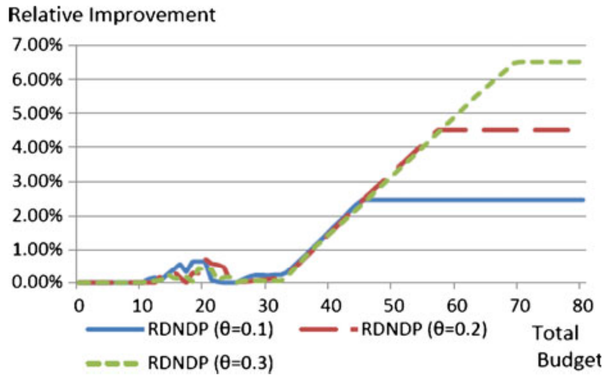


Fig. 4 Relative improvement of travel cost in worst-case scenarios under different demand uncertainty levels

The following worst-case analysis consists of two parts. First, we fixed the demand uncertain level θ and increased the investment budget level B. The computation results are shown in Table 3 and Fig. 4. When the budget level is low, it is natural that there is little difference between the nominal and robust solutions. Moreover, when the investment budget is less than 10 monetary units, the model always selects cell 7 as the site for capacity expansion, in that it is a merging cell and the bottleneck of the network. The total travel cost associated from the robust design solutions is slightly lower than that of the corresponding nominal solutions when the total budget is between 10 and 35 units. We can also see that the robust solutions significantly outperform the nominal solutions when the budget is large enough and the demand uncertainty is on a sufficiently high level. However, the relative improvement of the robust solution against the nominal solution shown in Fig. 4 is not necessarily a monotonically increasing function with respect to the investment budget level. Though TC_r and TC_d both decrease as the investment budget level increases, the travel cost reduction rates of the two terms change over the budget level, which is a result of the tradeoff between marginal investment costs and marginal travel costs in the two different problem cases.

Next, we fixed the total budget level B at four different levels (including 30, 40, 50 or 60 monetary units) with the demand uncertainty level θ ranging from 0 to 0.5. The computation result is depicted in Fig. 5. We can see that with a lower budget level, the demand uncertainty has a weaker affect on the performance of the RDNDP model. However, the solution of the RDNDP model may be largely different from

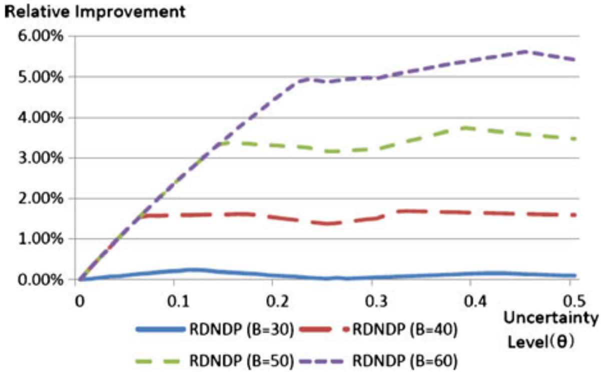


Fig. 5 Relative improvement of travel cost in worst-case scenarios under different investment budget levels

Table 4 Comparison of simulation results

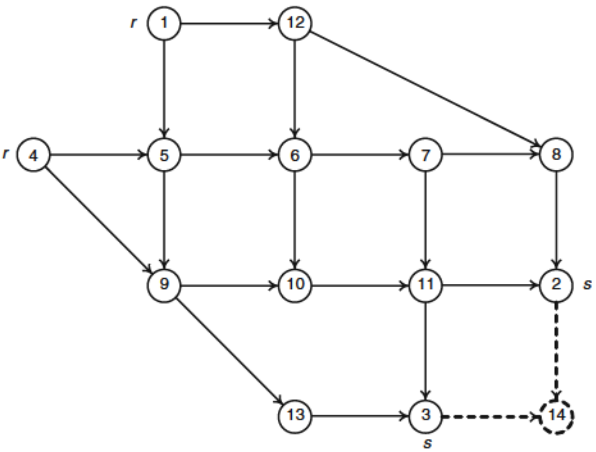
Budget	Mean		Standard deviation		Maximum	
	DDNDP	RDNDP	DDNDP	RDNDP	DDNDP	RDNDP
(a) θ=0.1						
0	80.66	80.66	1.52	1.52	83.55	83.55
10	74.36	74.36	1.27	1.27	77.10	77.10
20	72.01	71.78	1.35	1.29	75.16	74.81
30	70.06	70.18	1.38	1.30	72.78	72.72
40	69.82	68.95	1.07	1.01	72.55	71.55
50	69.62	68.34	1.20	1.07	72.14	70.56
60	69.64	68.35	1.16	1.04	71.88	70.39
70	69.62	68.34	1.09	0.98	71.61	70.02
80	69.64	68.36	1.32	1.18	72.79	71.18
(b) θ=0.2						
0	85.27	85.27	2.96	2.96	93.02	93.02
10	78.22	78.22	2.51	2.51	83.35	83.35
20	76.16	75.69	2.50	2.41	82.14	81.51
30	73.90	73.83	2.47	2.44	78.84	78.76
40	73.56	72.62	2.49	2.34	78.58	77.33
50	73.51	71.49	2.35	2.18	78.82	76.48
60	72.92	70.39	2.68	2.44	77.92	74.89
70	73.41	70.83	2.20	1.96	78.75	75.57
80	73.65	71.05	2.63	2.35	80.03	76.78
(c) θ=0.3						
0	90.21	90.21	4.24	4.24	99.54	99.54
10	82.88	82.88	4.20	4.20	90.37	90.37
20	79.86	79.44	3.64	3.61	87.50	86.97
30	77.30	77.24	3.99	3.98	86.49	86.58
40	77.42	76.34	3.57	3.52	84.98	83.85
50	76.83	74.57	3.45	3.23	84.96	82.33
60	76.88	73.72	3.92	3.57	85.75	81.93
70	76.88	73.72	3.92	3.57	85.75	81.93
80	77.57	73.64	3.52	3.19	84.35	79.85

the solution of the corresponding DDNDP model when the budget level is relatively high. Similar to Fig. 4, we can also observe that the relative improvement of the total travel cost of the robust solution against the nominal solution is not always a monotonically increasing function with respect to the demand uncertainty level.

5.1.3 Simulation results

Finally, we evaluated the objective function by implementing the robust network design solutions and nominal solution with random demands generated by the given

Fig. 6 The node-link topology of the Nguyen-Dupis network



box uncertainty sets. Specifically, 100 sets of random data generated from a beta distribution (i.e., beta(5, 2)) are used for this evaluation. Note that we only know the support of the primitive uncertain data and accordingly use box uncertainty sets to characterize the bounded uncertain demand. The beta distribution is a reasonable choice for simulating bounded uncertain data. The mean, standard deviation, and maximum values of the objective function values generated from the simulation experiment are shown and compared in Table 4. It can be seen that, in almost every case, the mean objective function value of the robust solutions is better than that of the nominal solutions; in all cases, the standard deviation and maximum values of the robust solutions are less than or equal to those of the nominal solutions.

5.2 The nguyen-dupis network

Now we present a second numerical example to show the computational tractability and the performance consistency of with the RDNDP model in larger networks. The Nguyen-Dupis network with 13 nodes in total (including 2 source nodes and 1 super sink node) is considered here (see Fig. 6). An equivalent cell network with 167 cells is created from the original node-link version. The resulting RDNDP model from the

Fig. 7 The objective-budget relationship under different demand uncertainty levels

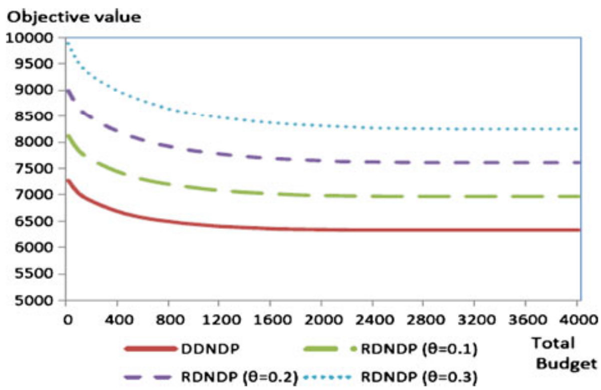
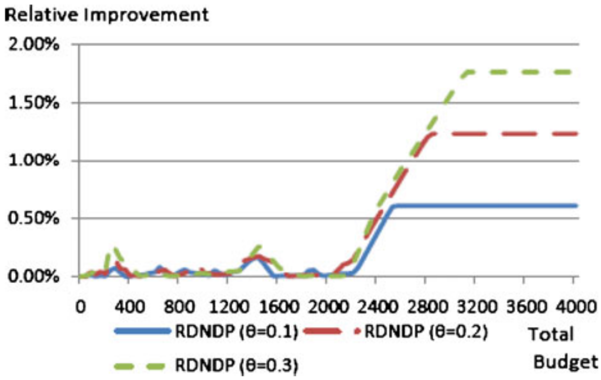


Fig. 8 Relative improvement of travel cost in worst-case scenarios under different demand uncertainty levels



Nguyen-Dupis network has 96,794 constraints and 190,694 variables, which has been solved in about 60 s on a PC with an Intel 1.87 GHz CPU and 2 GB RAM using GAMS/CPLEX.

Figure 7 shows the optimal objective function values of the DDNDP and RDNDP models with three different demand uncertainty levels. As similar to the previous example, there is a set of cells that are chosen for capacity expansion (where, in this case, there are 36 cells in total) in the Nguyen-Dupis network, which delivers a similar objective-budget relationship to the previous toy example. Investment decisions vary with different demand uncertainty levels.

The relative improvement of the robust solutions from the corresponding nominal solutions in the worst-case scenarios is aggregated in Figs. 8 and 9. It is shown from Fig. 8 that the robust solution significantly improves the nominal solution when the investment budget level is greater than 2,200 monetary units, in particular when the demand uncertainty level is high. A similar phenomenon can be observed from Fig. 9.

Finally, the simulation results are compared in Table 5. It is found that the simulated objective function values from DDNDP and RDNPD are comparable when the investment budget level is less than 1,500 monetary units. However, the robust solutions provide a lower travel cost when the investment budget goes higher. Our computational results show that the robust solution is more attractive than the

Fig. 9 Relative improvement of travel cost in worst-case scenarios under different investment budget levels

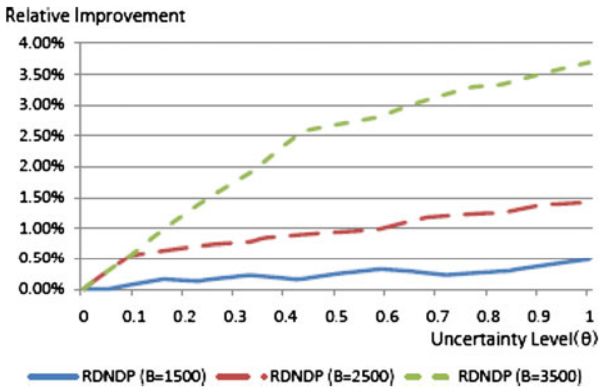


Table 5 Comparison of the robust optimization results and simulation results

Budget	Mean		Standard deviation		Maximum	
	DDNDP	RDNDP	DDNDP	RDNDP	DDNDP	RDNDP
(a) θ=0.1						
0	7,631.63	7,631.63	111.42	111.42	7,846.08	7,846.08
1,500	6,488.78	6,488.61	91.84	91.42	6,685.74	6,684.25
2,500	6,391.82	6,366.79	78.54	75.08	6,549.64	6,517.81
3,500	6,380.82	6,353.72	78.88	75.25	6,530.00	6,497.45
(b) θ=0.2						
0	7,976.02	7,976.02	204.57	204.57	8,395.07	8,395.07
1,500	6,788.14	6,784.49	182.44	179.68	7,192.98	7,185.69
2,500	6,671.00	6,645.71	149.01	143.96	7,030.38	6,990.74
3,500	6,666.09	6,614.52	154.87	146.25	7,006.21	6,937.27
(c) θ=0.3						
0	8,389.44	8,389.44	369.82	369.82	9,046.43	9,046.43
1,500	7,096.40	7,089.95	272.83	269.47	7,847.98	7,838.24
2,500	6,963.85	6,931.39	251.54	241.73	7,494.99	7,441.00
3,500	6,957.23	6,878.45	297.76	278.49	7,520.06	7,414.78

nominal solution from the simulation experiment. We note that the total travel cost is affected by 1) the network capacity expansion policy and 2) the underlying traffic flow pattern. Since our focus is the network design problem, we only tested the impact of robust network capacity expansion solution in simulation with the deterministic traffic assignment solutions. We expect the improvement be more significant when both a robust capacity expansion policy and a robust traffic assignment procedure are used.

6 Conclusion and further research

In this paper, we formulated and solved the RDNDP, a robust network design problem for dynamic and uncertain demands, and numerically evaluated its solution performance. The appealing LP formulation of the RDNDP model is rooted from the underlying LP-based DTA model—CTM. A box uncertainty set is assumed for modeling uncertain demands. Through this NDP example, we demonstrate how the constraints affected by uncertain parameters can be manipulated to derive a tractable mathematical program.

Since it becomes particularly important to provide a solution which is robust to extreme events and reduce the variance of cost after the realization of uncertain parameters (Waller and Ziliaskopoulos 2006), we choose a beta distribution (which is an asymmetric distribution) to model random demands and conduct a worst-case analysis. The RO approach can provide better network design solutions that produce lower objective function values than the corresponding deterministic approach, especially at a high demand uncertainty level and a high investment budget level.

Numerous future research directions remain. First, the RDNDP model with various types of uncertainty set including a polyhedral uncertainty set or an ellipsoidal uncertainty set should be investigated to find a less conservative solution. Second, the ambiguous chance-constrained programming can be applied to the model when we have more information about the uncertain data. For example, this approach may be particularly interesting when we only know the support and mean of uncertain parameters or when we know that demand can arise from a set of distributions. Third, while we dealt with the RDNDP with a single-destination, system-optimum network setting, which has potential applications in emergency evacuation planning, optimal traffic detouring, lower-bound evaluation of traffic systems, etc., the user-optimal and multi-destination versions of the same problem are worth further investigation and evaluation along the track of RO.

Acknowledgment This work was partially supported by the grant awards CMMI-0824640 and CMMI-0900040 from the National Science Foundation.

References

Abdulaal M, LeBlanc LJ (1979) Continuous equilibrium network design models. Transp Res B 13:19–32

Atamturk A, Zhang M (2007) Two-stage robust network flow and design under demand uncertainty. Oper Res 55:662–673

Ban X, Lu S, Ferris M et al (2009) Risk-averse second-best toll pricing. Proc 18th Int Sympo Transp Traffic Theory 197–218

Ben-Tal A, Nemirovski A (1998) Robust convex optimization. Math Oper Res 23:769–805

Ben-Tal A, Nemirovski A (1999) Robust solutions of uncertain linear programs. Oper Res Lett 25:1–13

Ben-Tal A, Nemirovski A (2000) Robust solutions of linear programming problems contaminated with uncertain data. Math Program 88:411–424

Ben-Tal A, Nemirovski A (2002) Robust optimization—methodology and applications. Math Program 92:453–480

Bertsimas D, Brown DB, Caramanis C (2007) Theory and applications of robust optimization. Available via http://users.ece.utexas.edu/~cmcaram/pubs/RobustOptimizationSV.pdf. Accessed 15 Mar, 2010

Daganzo CF (1994) The cell transmission model part I: a simple dynamic representation of highway traffic. Transp Res B 28:269–287

Daganzo CF (1995) The cell transmission model part II: network traffic. Transp Res B 29:79–93

Karoonsoontawong A, Waller ST (2007) Robust dynamic continuous network design problem. J Transp Res Board 2029:58–71

Karoonsoontawong A, Waller ST (2008) Integrated network capacity expansion and traffic signal optimization: robust bi-level dynamic formulation. Network Spatial Econ. doi:10.1007/s11067-008-9071-x

Lin D, Karoonsoontawong A, Waller ST (2008) A Dantzig-Wolfe decomposition based heuristic scheme for bi-level dynamic network design problem (2008). Network Spatial Econ. doi:10.1007/s11067-008-9093-4

Lou Y, Yin Y, Lawpongpanich S (2009) A robust approach to discrete network designs with demand uncertainty. J Transp Res Board 2090:86–94

Lu Y (2007) Robust transportation network design under user equilibrium, Master's Thesis, MIT, Cambridge, MA.

Magnanti TL, Wong RT (1984) Network design and transportation planning: models and algorithms. Transp Sci 18:1–55

Minoux M (1989) Network synthesis and optimum network design problems: models, solution methods and applications. Netw 19:313–360

Mudchanatongsuk S, Ordonez F, Liu J (2008) Robust solutions for network design under transportation cost and demand uncertainty. J Oper Res Soc 59:652–662

Mulvey JM, Vanderbei RJ, Zenios SA (1995) Robust optimization of large-scale systems. Oper Res 43:264–281

Ordonez F, Zhao J (2007) Robust capacity expansion of network flows. Netw 50:136–145

Peeta S, Ziliaskopoulos AK (2001) Foundations of dynamic traffic assignment: the past, the present and the future. Network Spatial Econ 1:233–265

Ukkusuri SV, Waller ST (2008) Linear programming models for the user and system optimal dynamic network design problem: formulations, comparisons and extensions. NetwOrk Spatial Econ 8:383–406

Ukkusuri SV, Mathew T, Waller T (2007) Robust transportation network design under demand uncertainty. Comput Aided Civ Infrastruct Eng 22:6–18

Waller ST, Ziliaskopoulos AK (2001) Stochastic dynamic network design problem. J Transp Res Board 1771:106–113

Waller ST, Ziliaskopoulos AK (2006) A chance-constrained based stochastic dynamic traffic assignment model: analysis, formulation and solution algorithms. Transp Res C 14:418–427

Wardman M (1998) The value of travel time: a review of British evidence. J Transp Econ Policy 32:285–316

Yang H, Bell MGH (1998) Models and algorithms for road network design: a review and some new developments. Transp Rev 18:257–278

Yao T, Mandala SR, Chung BD (2009) Evacuation transportation planning under uncertainty: a robust optimization approach. Network Spatial Econ 9:171–189

Yin Y, Lawpongpanich S (2007) A robust approach to continuous network design with demand uncertainty. Proc 17th Int Sympo Transp Traffic Theory 111–126

Zhao Y, Kockelman K (2002) The propagation of uncertainty through travel demand models: an exploratory analysis. Ann Reg Sci 36:145–163

Ziliaskopoulos AK (2000) A linear programming model for the single destination system optimum dynamic traffic assignment problem. Transp Sci 34:37–49