

AN INVERSE APPROACH TO COEFFICIENT OF THERMAL EXPANSION
OPTIMIZATION IN OPTICAL STRUCTURES

by

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TABLE OF CONTENTS

1. INTRODUCTION AND BACKGROUND	1
Aerospace Mirrors	1
Composites of Mirrors	2
Low Coefficient of Thermal Expansion in Composites and Resins	3
Characteristics of Spherical and Cylindrical Mirrors	4
Objective	7
Scope and Approach	7
2. LITERATURE REVIEW	9
Composite Mirrors	9
Tailoring the CTE: Resin and Composite	12
3. CLOSED FORM EQUATION DEVELOPMENT	14
Infinitely Long Nested Cylinders Approach	14
Two Infinitely Long Nested Cylinders	14
Three Infinitely Long Nested Cylinders	19
N Infinitely Long Nested Cylinders	24
N Infinitely Long Nested Troughs.....	27
Concentric Cylinders Approach	28
Multi-Layer Thermostat Approach.....	40
Frequency Analysis.....	55
4. AIR FORCE RESEARCH LABORATORY BASELINE MIRROR	58
Model Development and Validation.....	58
Frequency Analysis.....	62
2D Surface Distortion Analysis.....	63
3D Surface Distortion Analysis.....	65
Surface Pressure Validation	67
Axial Load Validation.....	68
Longitudinal Load Validation	68
5. APPLICATION OF CLOSED FORM EQUATION.....	70
Layered Trough Geometry	70
Decreasing Temperature Gradient Load.....	72
Increasing Temperature Gradient Load	79
AFRL Geometry.....	83

TABLE OF CONTENTS - CONTINUED

6. CONCLUSIONS.....	90
REFERENCES CITED.....	93
APPENDICES.....	96
APPENDIX A: Closed Form Equation Development.....	97
APPENDIX B: AFRL Baseline Mirror.....	179
APPENDIX C: Application of Closed Form Equation.....	195

LIST OF TABLES

Table	Page
1. Properties of the Two Nested Cylinder System	17
2. Tangential Strains for ABAQUS vs. Theory for Two Nested Cylinders	17
3. Radial Displacements for ABAQUS vs. Theory for Two Nested Cylinders.....	18
4. Stresses for ABAQUS vs. Theory for Two Nested Cylinders.....	18
5. Properties of the Three Nested Cylinder System	21
6. Tangential Strains for ABAQUS vs. Theory for Three Nested Cylinders.....	22
7. Radial Displacements for ABAQUS vs. Theory for Three Nested Cylinders.....	22
8. Stresses for ABAQUS vs. Theory for Three Nested Cylinders.....	23
9. Properties of the Four Nested Cylinder System.....	24
10. Tangential Strains for ABAQUS vs. Theory for N Nested Cylinders	26
11. Radial Displacements for ABAQUS vs. Theory for N Nested Cylinders	26
12. Stresses for ABAQUS vs. Theory for N Nested Cylinders	26
13. Properties of the Four Nested Trough System	30
14. FE Displacement Comparison of End Conditions of Trough Model vs. Concentric Cylinders Model	30
15. CTE of the Four Nested Trough Systems.....	31

LIST OF TABLES - CONTINUED

Table	Page
16. Properties of the Four-Layer Trough.....	44
17. Comparison of RMS Surface Distortion for Closed Form and FEA Results for Uniform Temperature Loading.....	46
18. Comparison of Best-Fit Radius of Curvature for Closed Form and FEA Results for Uniform Temperature Loading.....	46
19. RMS Surface Distortion for Zero Thermal Curvature for Closed Form and FEA Results at 250 °C Temperature Loading throughout Model	47
20. Best-fit Radius of Curvature for Zero Thermal Curvature for Closed Form and FEA Results at 250 °C Temperature Loading throughout Model	47
21. Comparison of RMS Surface Distortion for Closed Form and FEA Results for Non-Uniform Temperature Loading	52
22. Comparison of Best-fit Radius of Curvature for Closed Form and FEA Results for Non-Uniform Temperature Loading	52
23. RMS Surface Distortion for Zero Thermal Curvature for Closed Form and FEA Results from a Non-Uniform Temperature Loading throughout Model	53
24. Best-fit Radius of Curvature for Zero Thermal Curvature for Closed Form and FEA Results from a Non-Uniform Temperature Loading throughout Model	54
25. ABAQUS Nat. Frequency Calculation for the Thermally Uniform Loaded Model	56
26. ABAQUS Natural Frequency Comparison for the 3D and Axisymmetric Homogenous Models.....	63
27. Properties of CTD-403 Mixed with Zr ₂ WO ₈ (wt%) at Shown Temperatures	71

LIST OF TABLES - CONTINUED

Table	Page
28. CTE (α , ppm/ $^{\circ}$ C) of CTD-403 Mixed with Zr ₂ WO ₈ (wt%) at Shown Temperatures	72
29. Material Lay-ups of Zr ₂ WO ₈ (wt%) in 4-Layer Troughs for Six Cases.....	76
30. Comparison of RMS Surface Distortion for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios	77
31. Comparison of Best-fit Radius of Curvature for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios	77
32. Stress Analysis for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios	78
33. Free End Stress Analysis for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios	79
34. Material Definitions of Zr ₂ WO ₈ (wt%) in 4-Layer Troughs for Three Cases.....	81
35. Comparison of RMS Surface Distortion for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios	82
36. Comparison of Best-fit Radius of Curvature for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios	82
37. Stress Analysis for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios	83
38. Free End Stress Analysis for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios	83
39. Material Definitions of Zr ₂ WO ₈ (wt%) in 4-Layer AFRL Axisymmetric Mirror for Five Cases	85

LIST OF TABLES - CONTINUED

Table	Page
40. Comparison of RMS Surface Distortion for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios on AFRL Mirror and MLT Trough.....	85
41. Comparison of Best-fit Radius of Curvature for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios on AFRL Cross Section and MLT Trough.....	86
42. Stress Analysis for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios with AFRL Cross Section Geometry.....	89
43. Stress Analysis for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios with AFRL Cross Section Geometry.....	89

LIST OF FIGURES

Figure	Page
1. Spherical Mirror	5
2. Paraboloidal Form and Lay-up.....	9
3. Fiber Print Through	10
4. Two Infinitely Long Nested Cylinders	14
5. Two-Cylinder FE Model w/ Quarter Symmetry BC	17
6. Three Concentric Cylinders	19
7. Three Cylinder FE Model w/ Quarter Symmetry BC.....	22
8. N Infinitely Long Nested Cylinders	25
9. Four-Cylinder FE Model w/ Quarter Symmetry BC.....	25
10. N Infinitely Long Nested Troughs	28
11. Four-Layer FE Trough Model with Horizontal Roller BC.....	28
12. Four-Layer FE Trough Model with Free BC	29
13. Four-Layer FE Trough Model with Pinned-Pinned BC	29
14. Four-Layer FE Trough Model with Vertical Roller BC	29
15. Nodal Position in Trough.....	32
16. Radial Displacement at End of Trough for 250 °C Increase.....	32
17. Tangential Displacement at End of Trough for 250 °C Increase	33
18. Resultant Displacements for Each Case	33
19. Radial Displacement at End of Trough for 250 °C Increase.....	34

LIST OF FIGURES - CONTINUED

Figure	Page
20. Tangential Displacement at End of Trough for 250 °C Increase	34
21. Resultant Displacements for Each Case	35
22. Percent Difference Between FEA and Calculated Stresses for a 4 Layer Trough with the CTE of Layer 1 Varied Compared to the System...	36
23. Percent Difference Between FEA and Calculated Stresses for a 4 Layer Trough with the CTE of Layer 2 Varied Compared to the System...	36
24. Percent Diff. Between FEA and Calculated Displacements for a 4 Layer Trough with the CTE of Layer 1 Varied Compared to the System.....	37
25. Percent Diff. Between FEA and Calculated Displacements for a 4 Layer Trough with the CTE of Layer 2 Varied Compared to the System.....	38
26. Four-Layer Strip	40
27. Initial Curvature Geometry	42
28. FEA Deformed vs. Undeformed Trough (Case 8-Uniform Loading)	45
29. Mode Shapes; a) 1 st (left), b) 2 nd (center), and c) 3 rd (right)	56
30. AFRL Mirror Side Profile.....	58
31. AFRL Axisymmetric Model	59
32. AFRL Loaded Mesh Convergence Model.....	59
33. AFRL 3D Three-Layer Model	61
34. AFRL 3D Loaded Mesh Convergence Model	61
35. Axisymmetric Mode Shapes; a) 1 st (left), b) 2 nd (center), and c) 3 rd (right) ...	62

LIST OF FIGURES - CONTINUED

Figure	Page
36. 3D Mode Shapes; a) 1 st (left), b) 2 nd (center), and c) 3 rd (right)	62
37. Deformed and Reference Surfaces Defined.....	63
38. Axial Loaded 3D AFRL Model	68
39. Longitudinally Loaded 3D AFRL Model	69
40. Layer and Decreasing Temperature Assignments.....	72
41. Case 1 CTE Lay-up	74
42. Case 1 (48%, 70%, 70%, 0%) Internal Stresses in x-direction.....	75
43. Case 1 (48%, 70%, 70%, 0%) Internal Stresses in y-direction.....	75
44. Layer and Increasing Temperature Assignments	79
45. Case 7 CTE Lay-up	80
46. AFRL Geometry Overlaid with Closed form Geometry	83
47. AFRL Mirror Layers and Temperature Loading Scenarios.....	84
48. Case 1 AFRL Axisymmetric Model x-direction Stress.....	87
49. Case 1 AFRL Axisymmetric Model y-direction Stress.....	88
50. Case 1 AFRL Axisymmetric Model Magnitude Displacements.....	90

ABSTRACT

Optical component performance increasingly demands materials with tailored properties. Optical systems see applications where ambient conditions can drastically reduce performance. Optics used in space, for example, may undergo severe changes in temperature, which results in large thermally induced stresses and distortions. To minimize these thermal effects, it was desired to manipulate the coefficient of thermal expansion (CTE) within the material. However, wholesale reductions in CTE may not be optimum since synthetic manipulation of CTE often leads to undesirable effects on other material properties, such as strength or ductility. Consequently, there was interest in distributing the CTE/material property trades over the optical structure to achieve optimum results for competing requirements. Performing such studies relied heavily on Finite Element Analysis (FEA) programs coupled with closed form solutions that give basic understanding of CTE design.

To begin fulfilling this need, it is presented here first a revisit of the problem of two infinitely long nested cylinders. From this study, design equations were developed that related the dependent variables of stress, strain, and displacement to the elastic modulus, CTE, and temperature change of the system. The equations provide insight into how the dependent variables were affected by the layer CTEs. The analysis was then extended to three nested cylinders. Extrapolation of this approach to N nested cylinders is shown. This work provided a starting point for analysis of optical cylindrical troughs.

To strengthen the analytical techniques for the cylindrical troughs, the problem of the bi-metallic strip was also revisited. Design equations were likewise developed that related the radius of curvature and surface distortion to the elastic modulus, CTE, and temperature change of the system. The analysis was extended to a tri-metallic strip and eventually a strip of N layers. Each layer in the model represents a material differing in CTE of neighboring layers. Finally, a system was observed that was traceable to an optical backing structure, namely a system of several nested cylindrical troughs. Design equations and observations of CTE distribution as well as quantifying data for this system are provided.

CHAPTER 1

INTRODUCTION AND BACKGROUND

Aerospace Mirrors

Throughout the centuries the motivation to see farther into space has been fostered by the increasing size and quality of optics. The light gathering capabilities of handheld telescopes of 16th century astronomers were good enough to begin understanding of the cosmos. The original telescopes, however, pale in size comparison to those used today. Telescopes such as the 10 m diameter Keck telescopes in Hawaii allow observers to see parts of space previously unknown to man. In addition to the large ground based telescopes, space based telescopes have increased the ability to clearly see deep space¹.

The broad use of telescopes in space began on April 24, 1990 with the deployment of the Hubble Space Telescope (HST). Championed by Lyman Spitzer Jr., the daunting creation of the HST propelled the United States of America to the forefront of space exploration. The project carried a few notable statistics including its \$500 million price tag and its size for a telescope used in space. Hubble's primary mirror, weighing one ton, is 240 cm in diameter. Compared to a telescope that is earth bound, the Hubble is small, but its viewing location in space provides advantages. The earth's atmosphere cannot distort Hubble's images like it does with ground telescopes. Although the HST has provided 16 years of reliable service, its career is nearing the end. The James

Webb Space Telescope (JWST), which is marked to replace Hubble in 2013, will see an increase in functionality due to its technological advances over the Hubble Telescope¹.

Several advances made in space-based optical technology since the creation of the HST include materials with higher strength to weight ratios (composites), optically smoother surfaces, and materials with lower coefficients of thermal expansion (CTE). The overarching goal of any mirror design used in space, however, is to create a mirror with a large light gathering capability, undistorted image surface, low weight, and the ability to react favorably to its environment.

Composites of Mirrors

In order to see deeper into space one needs a larger primary optic. As the reflection area of a mirror increases the resolution of a mirror also increases. Typically as the size of the optics increase, weight also increases. However, launch load limits and size requirements must be satisfied. A definition of areal density arises out of the desire to quantify a ratio of weight (actually mass) to the surface area of the optic. It has therefore been the goal of some researchers to create mirrors that utilize the high strength to weight ratio of composites as substrates (mirror backing support structure) for a mirror system. With composites, high areal densities can be obtained and, unlike mirrors that use traditional materials such as glass, composites have properties that can be tailored to a specific duty. For instance, the fibers used in a composite have a high strength in their longitudinal direction. If strength is needed in a system at a particular orientation, the fibers can be layered corresponding to this direction. Other properties such as CTE,

modulus of elasticity, Poisson's ratio, etc., can be tailored with composites. With these benefits, however, a few concerns arise with the use of composites.

Composites used as substrates for mirrors intrinsically contain surface errors that distort reflecting light. When cured in the matrix, the fibers have a round cross-section that creates a rough surface normal to their longitudinal direction. A reflective coating adhered to this undulating surface will take on this roughened figure. This phenomenon is called "fiber print through" and can cause an undesirable wavefront error. To counter this, several methods exist. One method is to create a form that itself maintains an optically smooth surface. When the fibers and resin mate to the form's surface, the composite substrate replicates that smooth surface. Another process is to adhere a finishing surface to the rough substrate, one that can be polished. After polishing, the reflective coating is then applied to the smooth surface². This process adds an intermediate layer in the mirror system and with it added complexity and several concerns. One concern is a mismatch of CTE between mated layers. With composites, however, comes the ability to tailor the CTE of the system. A brief overview of composite CTE proceeds in the next section.

Low Coefficient of Thermal Expansion in Composites and Resins

In space an instrument needs to be intelligently designed to accommodate a shift in temperature. From the shadows of space to the radiation of the sun, orbiting instrumentation is exposed to a minimum 30 °C range³. If a system in space is composed of several materials all with mismatched CTE, the results of a temperature shift could be

catastrophic. For instance, the stresses could potentially be high in a mirror system composed of a substrate with a near zero CTE and a high CTE in the coating. An increase in temperature would cause the coating to expand while the substrate would retain its dimension. The two bonded layers could potentially tear apart given the temperature shift. With composites, however, the CTE of the system can be tailored to meet a specific need.

Several ideas to modify the CTE of a composite system are under examination. One idea is to create a system in which all the composing materials have near zero CTE. This allows for a system to retain its shape and rigidity regardless of any temperature changes. One possible flaw exists, however, in that although the coating and substrate may have a near zero CTE, the mounting device might not. Somewhere in the orbiting structure a mismatch of CTE may occur. The answer to this dilemma is to define a coating and mounting device and then gradually tailor the CTE throughout the substrate to match the CTE of both the coating and mount to minimize undesirable effects. Many ideas exist, but the benefits of composites and resins lie within their ability to have tailored properties. The advantages of tailoring a system's material properties are not endless, however. In fact, attention to the modified material's strength is necessary because in general as the CTE of a material decreases, its strength will also decrease. A material with low CTE may satisfy thermal stability requirements, but fail in mechanical strength requirements. Attention, therefore, needs to be granted to both parameters to ensure that a material and its related material properties are appropriately assigned.

Characteristics of Spherical and Cylindrical Mirrors

Certain optics of interest to the Air Force Research Laboratory and this project are ones with spherical geometry. A spherical mirror has a constant radius of curvature, ρ , one focal point, f , and a diameter, D . The radius of curvature and the focal point are related as shown in Equation 1. Figure 1 shows the geometry of an AFRL spherical mirror.

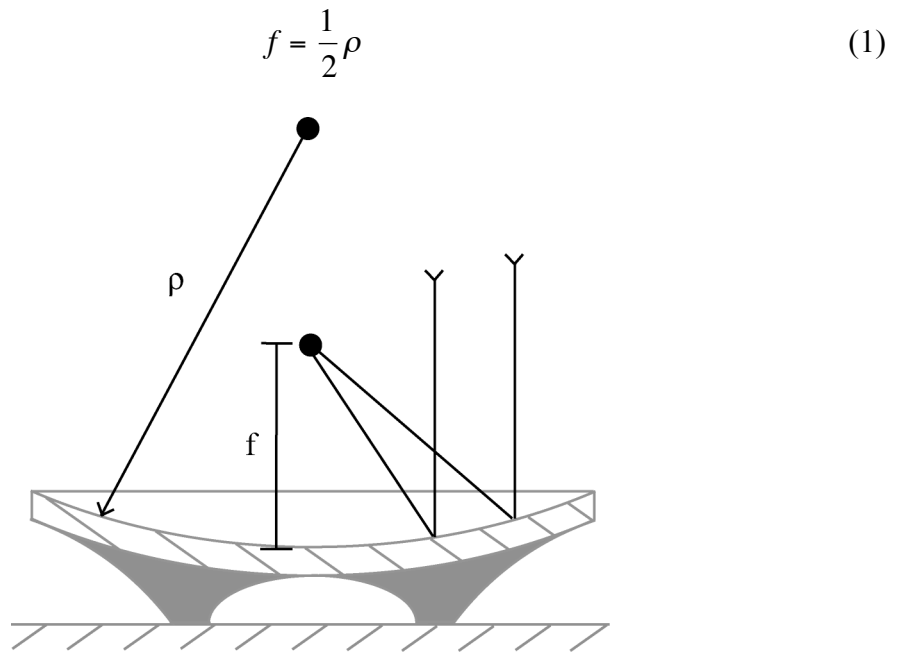


Figure 1: Spherical Mirror

A spherical mirror is an example of a concave mirror in which light rays focus on a general area, called the focal point. The area, however, is not the intersection of all reflected rays, but rather a spot to which all rays come close to intersecting. At this “point” a perfect image is therefore not produced, but in fact is a distorted image. This distorted image has what is called a spherical aberration. Although a spherical mirror has

intrinsic aberrations, it is frequently used for optics due to its ability to focus light. A surface's ability to focus light is of great interest and is often quantified to find the distortion, if any, from an ideal reflecting surface. The distortion of spherical mirrors and other mirror geometries is often presented as a fraction of the wavelength of light, e.g., $\lambda/20$ rms typical. A surface that reportedly has less than this rms value is considered to be of high optical quality.

A spherical mirror is not an ideal mirror as far as imaging is concerned, but is relatively easy in terms of fabricating and testing. Compared to a parabolic mirror surface, a constant radius of curvature is easier to machine than the variable curvature in a parabola. If the radius of curvature is large, then the difference between the sphere and associated parabola is small. The spherical aberration can also be corrected via image processing and therefore is a valid geometry for optics.

Much work has been previously completed for analysis of spherical optics, however, closed form design equations do not exist for thermally loaded systems. Such equations would provide engineers needed tools to avoid thermal and mechanical failures. These design equations also do not exist for simple configurations, such as optical troughs.

Related to the spherical mirror in geometry is the cylindrical trough optic. A cylindrical trough mirror has many of the same parameters as the spherical counterpart since it has curvature in one direction, e.g. radius of curvature related to focal point, but is easier to analyze for many variables such as stress and strain. Thermal stress and strain in a cylinder for example are easier to compute because the model can often be solved

with a plane strain assumption, thus reducing the number of dimensions. Also, the benefits of optical troughs are currently being researched at NASA and its subsidiaries⁴. It is for these reasons that the infinitely long concentric cylinder and cylindrical trough are analyzed as an intermediate step toward the analysis of more complex geometries in optics.

Objective

The primary goal of this thesis work was to observe and formulate the closed form design equations for CTE distribution in geometries related to optics. In addition to the optimization of the CTE, attention to a system's mechanical strength was also shown necessary. This two-fold approach of optimization for thermal stability and mechanical strength was first applied to simple structures and then the theory was used to consider the distribution in more complex geometries.

Scope and Approach

The first model to be considered was the classical problem of two infinitely long nested cylinders. This system, which has been widely studied, was used as the basis for observing three, four, and multiple layers of infinitely long nested cylinders. The methods of the infinitely long nested cylinders were to be extrapolated to a system of nested cylinders cut in half along the longitudinal axis (cylindrical troughs). This developmental work can be found in Chapter 3, CLOSED FORM EQUATION DEVELOPMENT. This approach was compared to other existing approaches and

ABAQUS, a Finite Element Analysis program (FEA). The finite element formulations are described in Chapter 4, FINITE ELEMENT MODELING and the results of the comparisons are shown in Chapter 5, RESULTS.

The second method for the cylindrical trough to be utilized revisited the problem of the bi-metallic strip. The analysis is extended to a tri-metallic strip and a strip and eventually a strip of N layers. Each layer in the model represents a material differing in CTE of neighboring layers. Finally, a system was observed that was traceable to an optical backing structure, namely a system of several nested cylindrical troughs. Design equations were likewise developed that related the radius of curvature and surface distortion to the elastic modulus, CTE, strength, and temperature change of the cylindrical trough system. These newly developed closed form design equations will be beneficial to optics engineering because they predict how a cylindrical trough system behaves under a temperature loading. These formulations are discussed in Chapter 3, CLOSED FORM EQUATION DEVELOPMENT. Again, FEA was used to validate this approach and is described in Chapter 4, FINITE ELEMENT MODELING. The design equation results, FEA results, quantifying data, and observations of CTE distribution for this system are provided in Chapter 5, RESULTS and all results are discussed in Chapter 6, DISCUSSION OF RESULTS. Chapter 7, CONCLUSIONS, discusses the implications of this work as well as summarizes the main details.

CHAPTER 2

LITERATURE REVIEW

Composite Mirrors

The need to see deeper into space and with increasing clarity causes telescope designers to increase the size of primary mirrors in use. With an increase in size of mirrors that use traditional materials, such as glass, weight becomes a design obstacle. On space missions the payload capacity is often the constraining parameter and it is therefore desired to minimize the weight of a mirror structure. Although the weight decreases, structural stability must not decrease. One solution that meets these requirements is to use composite materials that have a high strength to weight ratio and are also relatively inexpensive to produce.

The most common method to develop a composite mirror uses replication. For instance, in the creation of a spherical or paraboloidal mirror, replication uses a corresponding form that is optically smooth. The fibers and resin are then laid up on the form and the matrix cures (Figure 2).

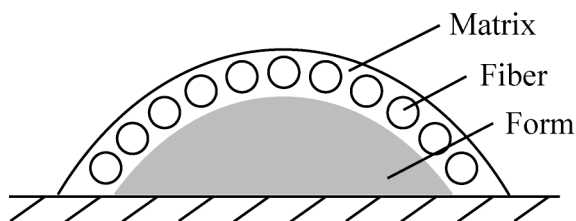


Figure 2: Paraboloidal Form and Lay-up

Many methods exist using varying temperatures and processes to inject the resin into the fibers. Once the composite cures, the matrix is removed and the desired shape is used as the substrate for a reflective coating. Although composites are viable options to use in space-based applications, several concerns need addressing.

The main dilemma with composite mirrors is the surface⁵. A mirror created using the replication process is prone to fiber print through on the reflecting surface. The surface of the composite that mates with the form is commonly used as the base for the reflective coating. As the resin cures, it often contracts and forms around the fibers. When applied to the composite the reflective surface takes the contour of the cross-sectionally round fibers and is “bumpy.” This rough surface causes light to diffract away from the intended focus. Figure 3 gives a representation of fiber print through.

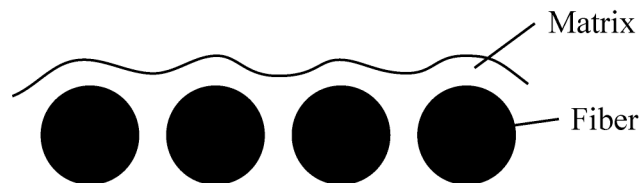


Figure 3: Fiber Print Through

To mitigate fiber print through, a group at NASA Goddard Space Flight Center has experimented with flat and spherical replication to determine the optimal process to create an optical quality mirror⁵. Varying fiber geometries, lay-up configurations, and fiber angles were catalogued. The optimized surface of 1/5 wave rms was created using “ $\pi/3$ geometry, 24 or more plies, M55J/EX1515 cyanate ester resin composite, fiber-to-resin volume ratio of 67%, and certain fine-tuning of the processing temperature and cycle times⁵.” The process eliminated print through due to the extra layer of resin added.

The resin was added to the surface of the substrate to which the reflective coating would be applied.

Although the method was optimized, the surface alone was not good enough for the optical threshold of $<1/20$ of a wavelength rms. The sources of the surface errors for the spherical mirror came not from print through, but from edge roll off, spherical aberration, and astigmatism⁵. To nullify the resultant errors it was suggested to employ ion milling or active control techniques to create a better quality surface. These techniques are common and much work has been performed in these areas, but such methods are expensive. Errors from fiber print through, however, were terminated. Other work on print through mitigation has been performed.

Another group from MER Corporation claimed a surface rms of 2 nm⁶. The flat mirror used a carbon/silicon carbide honeycomb configuration as the substrate. To the top of this substrate a slurry of graphite was applied. The slurry surface alone resulted in an rms of 1-2 μm . Once the slurry cured, a coating of silicon dioxide (SiO_2) was applied. The new surface of SiO_2 was then polished, and finished to 2 nm rms, far below the optical threshold of $<1/20$ wavelength⁶. Although the group from MER Corp. used an innovative approach to eliminate fiber print through, such a process is expensive in an industrial setting. Another method of using a combination of cutting edge composites and traditional metals was suggested by Kendrew (2006)². Kendrew proved that an addition of nickel on top of a carbon fiber substrate overcame fiber print through, because this metallic surface was ground and finished to an optically smooth surface (4 nm surface roughness)². A heterogeneous composition, however, will lead to CTE mismatch

and potentially a thermally unstable system. Regardless, the work of these three groups proved that the main obstacle of composite mirrors, fiber print through, was overcome although perhaps with great expense. Focus now turns to the work published on tailoring the CTE of the resin and composite.

Tailoring the CTE: Resin and Composite

To tailor the CTE of composites, several ideas have been approached. One method under investigation was addressed in by Kowbel (2001)⁷. Kowbel laid up a substrate of a carbon-carbon composite utilizing Cerasettm resin and heat-treated the composite at 1600 °C. On its own, a C-SiC composite exhibited a CTE of 0.1 ppm/°C. Using the chemical vapor reaction process, he converted the C-C composite and created a mirror structure that possessed a CTE of near zero for the substrate and a CTE of 4.5 ppm/°C at the surface⁷. In his other work, Kowbel created a mirror structure with a CTE of near zero throughout. Comparatively, a SiC composite alone has a greater CTE of 2.4 ppm/°C⁴.

Another method under investigation is that of particle doping, which mixes near zero or even negative CTE particles into a resin. Theoretically, a combination of one negative CTE material and a positive one could result in a resin of near zero CTE. A near zero CTE would produce a resin of dimensional stability in which a temperature shift would cause no expansion or shrinkage of the material.

A group at South Dakota School of Mines and Technology (SDSM&T) currently is using zirconium tungstate (ZrW_2O_8) to lower the CTE of a cyanate ester resin (CTD-

403)⁸. On its own, CTD-403 has a CTE of 52-61 ppm/°C. Zirconium tungstate alone, however, has a CTE of -8.7 ppm/°C, which is the lowest CTE for any known isotropic material⁸. Unlike most materials, ZrW_2O_8 will expand when cooled and contract when heated. The group at SDSM&T has shown that when CTD-403 is loaded with 81.8 wt% of zirconium tungstate, the CTE of the resultant resin was 9.8 ppm/°C. This however was not good enough to label the composition as dimensionally stable, because the material's dimensions would change with respect to changing temperature. The group concluded that this resin must be mixed with greater than 80 wt% ZrW_2O_8 , but doubts the possibility of such a composition's success due to the integrity of the composition⁸.

CHAPTER 3

CLOSED FORM EQUATION DEVELOPMENT

Infinitely Long Nested Cylinders Approach

The primary goal of the present work was to observe the closed form design equations for CTE distribution in simple structures and then use these findings to consider the distribution in more complex geometries related to optics. The first model considered was the classical problem of the two infinitely long nested cylinders. This system, which has been widely studied, was used as the basis for understanding three, four, and multiple layers of infinitely long nested cylinders.

Two Infinitely Long Nested Cylinders

The first model analyzed was that of two infinitely long nested cylinders. The cylinders were nested with no initial stresses or strains and the model was assumed to be in a state of plane strain. The layers were joined with a tie constraint, which did not allow separation at the interface. Also, following common assumptions, the walls were kept thin with respect to the radius of the cylinders (thickness/radius = 0.01). The layout of the system is shown in Figure 4.

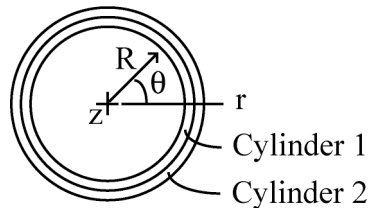


Figure 4: Two Infinitely Long Nested Cylinders

For the closed form solutions, the equations were given for an element far from either end of the cylinders, to nullify any consideration of the end boundary conditions on the cylinder. The equations for the tangential stresses of cylinder 1 and cylinder 2, respectively, are as follows¹¹:

$$\sigma_{\theta 1} = \frac{E_1 E_2}{E_1 + E_2} (\alpha_2 - \alpha_1) \Delta T \quad (2)$$

$$\sigma_{\theta 2} = -\sigma_{\theta 1} \quad (3)$$

where E = elastic modulus, α = the coefficient of thermal expansion (CTE), and ΔT = the temperature change between cylinders. Due to the thin wall assumption, the tangential strain in cylinder 1 is approximately equal to the tangential strain in cylinder 2. This means that the tangential strain of the system equals the strain of cylinder 1 which equals the strain of cylinder 2. The equations for the tangential strains of cylinder 1, cylinder 2, and the entire system are¹¹:

$$\varepsilon_{\theta 1} = \frac{\sigma_{\theta 1}}{E_1} + \alpha_1 \Delta T \quad (4)$$

$$\varepsilon_{\theta 2} = \frac{\sigma_{\theta 2}}{E_2} + \alpha_2 \Delta T \quad (5)$$

$$\varepsilon_{\theta 2} = \varepsilon_{\theta 1} = \varepsilon_{\theta} \quad (6)$$

Additionally, by substituting the stress equation of cylinder 1 (2) into the strain equation of cylinder 1 (4) and rearranging, the strain of the entire system becomes:

$$\varepsilon_{\theta} = \Delta T \left(\frac{E_1 \alpha_1 + E_2 \alpha_2}{E_1 + E_2} \right) \quad (7)$$

From the strain calculation, the radial displacement, u_r , of the cylindrical system is calculated by multiplying the strain with the initial radius, R , of the cylindrical system as given in Equation (8):

$$u_r = \varepsilon_\theta R \quad (8)$$

From the thin wall assumption, the radius of curvature is large compared to the cylinder wall's thickness. Therefore, both the cylinders share the same approximate radius of curvature (1 m). As a result, the tangential strain was also single valued for the system. Because the cylinders were tied together, each cylindrical layer was forced to displace equally throughout. As a result, the radial displacement retained a single value. Therefore, only one initial radius value was needed to approximate the radial displacement of both layers.

To test the validity of the derived equations for two infinitely long nested cylinders, a model was developed in the FEA program ABAQUS. To reduce the complexity of the finite element model, one quarter of the circular cross section was analyzed. In the FE model, symmetric boundary conditions were applied to replicate a complete cylinder and then the model was loaded with a temperature change. Figure 5 shows the FE model with the symmetric boundary conditions applied.

The inner radius of cylinder 1 was 0.99 m and the outer radius for cylinder 1 was 1.00 m., giving cylinder 1 a 0.01 m wall thickness. Additionally, cylinder 2 had an inner radius equal to the outer radius of cylinder 1 and a wall thickness 0.01 m, as well. Cylinder 1 was given an elastic modulus of 10 GPa and a CTE of 5 ppm/ °C while cylinder 2 was set at an elastic modulus of 20 GPa and a CTE of 10 ppm/ °C (properties

are summarized in Table 1). The system was subjected to several temperature changes and the resultant FEA stresses, strains, and displacements were compared to closed form solutions of these same conditions.

Table 1: Properties of the Two Nested Cylinder System

Cylinder #	Thickness (m)	Elastic Modulus (GPa)	CTE (ppm/°C)
1	0.01	10	5
2	0.01	20	10

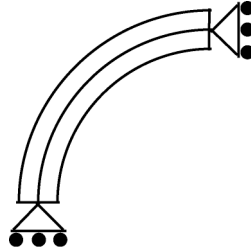


Figure 5: Two-Cylinder FE Model w/ Quarter Symmetry BC

The FE model was loaded in independent cases with temperatures changes in the range of -1000 °C to 250 °C degrees. The tangential strains reported by ABAQUS were compared to the tangential strains that were calculated by the derived Equations (4-7). Table 2 reports the results of this comparison and is provided below. Table 3 compares the radial displacements between the two approaches. The tangential hoop stresses from the FE model and the analytical equations were also compared and are reported below in Table 4.

Table 2: Tangential Strains for ABAQUS vs. Theory for Two Nested Cylinders

Temperature Change (°C)	ABAQUS Strain	Equation (4) Strain	Percent Difference
250	2.080E-03	2.083E-03	0.14
-100	-8.314E-04	-8.333E-04	0.23
-250	-2.078E-03	-2.083E-03	0.24
-500	-4.154E-03	-4.167E-03	0.31
-1000	-8.298E-03	-8.330E-03	0.38

Table 3: Radial Displacements for ABAQUS vs. Theory for Two Nested Cylinders

Temperature Change (°C)	ABAQUS Displacement (m)	Equation (8) Displacement (m)	Percent Difference
250	2.082E-03	2.083E-03	0.05
-100	-8.310E-04	-8.333E-04	0.28
-250	-2.076E-03	-2.083E-03	0.34
-500	-4.145E-03	-4.167E-03	0.53
-1000	-8.264E-03	-8.330E-03	0.79

Table 4: Stresses for ABAQUS vs. Theory for Two Nested Cylinders

Temperature Change (°C)	Cylinder Number	ABAQUS Stress (Pa)	Equation (2) and (3) Stress (Pa)	Percent Difference
250	1	8.366E+06	8.333E+06	-0.40
	2	-8.335E+06	-8.333E+06	-0.02
-100	1	-3.340E+06	-3.333E+06	-0.21
	2	3.345E+06	3.333E+06	-0.36
-250	1	-8.345E+06	-8.333E+06	-0.14
	2	8.376E+06	8.333E+06	-0.52
-500	1	-1.667E+07	-1.667E+07	0.00
	2	1.679E+07	1.667E+07	-0.72
-1000	1	-3.325E+07	-3.333E+07	0.24
	2	3.376E+07	3.333E+07	-1.29

For each temperature change, the stresses, strains, and displacements were compared. In each case, the only stress that existed was the tangential hoop stress (σ_θ) and the only strain that was considered was that of the tangential strain; the radial displacement was also the only component of displacement considered. For a concentric two-cylinder system, Tables 2 and 3 show a correlation with little percent difference between predicted displacements and strains versus corresponding values obtained through FEA. It is also shown in Tables 2 and 3 that as the magnitude of the temperature change increases, the resultant percent difference for displacements and strains also increases between predicted and FEA results. In general, the closed form solution

slightly over predicts the displacements and the strains. The corresponding stresses are reported in Table 4.

For the stress results, the equations generally under predict values compared to FEA results. No other generalizations, however, can be made concerning these values. The stress error does not appear to change with increasing temperature magnitude as the strains reacted. As shown in Table 4, however, the stresses reported for both cylinders from ABAQUS compare well with the predicted values. The displacement, strain, and stress results are sufficient to suggest that the equations derived for the two-cylinder case will predict the effect of CTE on the overall model. See Appendix A for development and example work of the two nested cylinders method. In a similar manner, the process used for this two-cylinder system was applied to a three-cylinder system.

Three Infinitely Long Nested Cylinders

The system of three infinitely long cylinders used the two cylinder system as a base. A third cylinder was tied to the outermost cylinder of the previous two-cylinder model. A schematic of the three-cylinder system is shown in Figure 6.

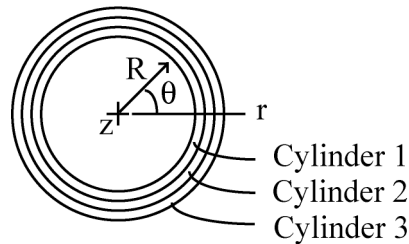


Figure 6: Three Concentric Cylinders

Again, the only stress that existed was the tangential hoop stress and the only strain that was considered was that of the tangential strain; the radial displacement was also the only

component of displacement considered. For the closed form solutions, the equations were developed using the mechanical equilibrium approach for an element far from either end of the cylinders. The results of this approach produced stress equations as follows:

$$\sigma_{\theta 1} = E_1 \left[\frac{\sigma_{\theta 3}}{E_3} + (\alpha_3 - \alpha_1) \Delta T \right] \quad (9)$$

$$\sigma_{\theta 2} = E_2 \left[\frac{\sigma_{\theta 3}}{E_3} + (\alpha_3 - \alpha_2) \Delta T \right] \quad (10)$$

$$\sigma_{\theta 3} = \frac{-E_1(\alpha_3 - \alpha_1)\Delta T - E_2(\alpha_3 - \alpha_2)\Delta T}{\frac{E_1 + E_2 + E_3}{E_3}} \quad (11)$$

Again, the model complies with the thin wall assumption as shown in Equation (12) and the tangential strain equation is extended to the third cylinder (13):

$$\varepsilon_{\theta 3} = \varepsilon_{\theta 2} = \varepsilon_{\theta 1} = \varepsilon_{\theta} \quad (12)$$

$$\varepsilon_{\theta 3} = \frac{\sigma_{\theta 3}}{E_3} + \alpha_3 \Delta T \quad (13)$$

By combining the stress Equations (9-11) and strain equations (12 and 13), the result shown in Equation (14) corresponds well with the similar formulation for two cylinders (7). Also, by similar reasoning to the two cylinder case, the radial displacement can be calculated using Equation (8). The tangential strain for a three-cylinder system however is:

$$\varepsilon_{\theta} = \Delta T \left(\frac{E_1 \alpha_1 + E_2 \alpha_2 + E_3 \alpha_3}{E_1 + E_2 + E_3} \right) \quad (14)$$

Comparing the tangential strain equation of the two cylinder case (7) and the three cylinder case (14) suggests that the tangential strain for a system of any number of

concentric cylinders is merely the temperature change multiplied by a rule of mixtures of layer CTEs and elastic modulus. This observation is the background for the theory and analysis of N concentric cylinders that is presented in the next section.

FEA models were created and tested against the closed form solution to validate Equation (14). For the models, cylinder 3 was given wall thickness of 0.01 m. The inner radius of cylinder 3 was 1.01 m and the outer radius was 1.02 m. The material properties of the inner two cylinders retained the values they had in the previous case ($\alpha_1 = 5$ ppm/ $^{\circ}\text{C}$, $\alpha_2 = 10$ ppm/ $^{\circ}\text{C}$). The third cylinder, however, had an elastic modulus of 30 GPa and a CTE of 15 ppm/ $^{\circ}\text{C}$. (These properties are summarized in Table 5). Similar to the two-cylinder model, the three-cylinder system was subjected to several temperature changes and the resultant FEA stresses, strains, and displacements were compared to closed form solutions of these same outputs.

Table 5: Properties of the Three Nested Cylinder System

Cylinder #	Thickness (m)	Elastic Modulus (GPa)	CTE (ppm/ $^{\circ}\text{C}$)
1	0.01	10	5
2	0.01	20	10
3	0.01	30	15

Similarly to the two cylinder FEA model, the three cylinder infinitely long nested cylinder system was developed in the FEA program ABAQUS. Again, to increase computational efficiency in the finite element model, one quarter of the circular cross section was analyzed. Symmetric boundary conditions were applied to the FE model to replicate a complete cylinder. Figure 7 shows the FE model with the symmetric boundary conditions applied.

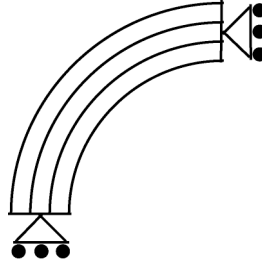


Figure 7: Three Cylinder FE Model w/ Quarter Symmetry BC

The model was then subjected to a temperature change. Five temperatures were tested: 250 °C, -100 °C, -250 °C, -500 °C, and -1000 °C and the computed tangential strains and radial displacements from the FEA ABAQUS model were compared to those calculated via the closed form solutions of Equations (14) and (8). The results of this comparison are shown below in Tables 6 and 7.

Table 6: Tangential Strains for ABAQUS vs. Theory for Three Nested Cylinders

Temperature Change (°C)	ABAQUS Strain	Equation (14) Strain	Percent Difference
250	2.916E-03	2.917E-03	0.03
-100	-1.165E-03	-1.167E-03	0.17
-250	-2.911E-03	-2.917E-03	0.21
-500	-5.816E-03	-5.833E-03	0.29
-1000	-1.169E-02	-1.167E-02	-0.17

Table 7: Radial Displacements for ABAQUS vs. Theory for Three Nested Cylinders

Temperature Change (°C)	ABAQUS Displacement (m)	Equation (8) Displacement (m)	Percent Difference
250	2.920E-03	2.917E-03	-0.10
-100	-1.164E-03	-1.167E-03	0.26
-250	-2.907E-03	-2.917E-03	0.34
-500	-5.799E-03	-5.833E-03	0.58
-1000	-1.154E-02	-1.167E-02	1.11

The tangential hoop stresses from the FE model and the Equations (9) - (11) were compared and are reported below in Table 8.

Table 8: Stresses for ABAQUS vs. Theory for Three Nested Cylinders

Temperature Change (°C)	Cylinder Number	ABAQUS Stress (Pa)	Equation (9)-(11) Stress (Pa)	Percent Difference
250	1	1.680E+07	1.667E+07	-0.75
	2	8.333E+06	8.333E+06	0.00
	3	-2.497E+07	-2.500E+07	0.11
-100	1	-6.667E+03	-6.740E+03	1.08
	2	-3.333E+03	-3.307E+03	-0.79
	3	1.002E+04	1.000E+04	-0.19
-250	1	-1.674E+06	-1.667E+06	-0.43
	2	-8.228E+06	-8.333E+06	1.26
	3	2.513E+07	2.500E+07	-0.52
-500	1	-3.343E+07	-3.333E+07	-0.30
	2	-1.635E+07	-1.667E+07	1.93
	3	5.042E+07	5.000E+07	-0.84
-1000	1	-6.664E+07	-6.667E+07	0.05
	2	-3.225E+07	-3.333E+07	3.23
	3	1.015E+08	1.000E+08	-1.49

The results of three infinitely long nested cylinders further support the results obtained from the two-cylinder model. Similarly to the two-cylinder model, the three-cylinder model's displacements and strains decrease in accuracy as the magnitude of the temperature change increases (Tables 6 and 7). Also similar to the previous model, the stress difference for the three-cylinder model shows no general trend like that seen in the displacement and strains (Table 8). All FEA reported values, however, correspond well with those predicted in the closed form solutions. The work performed on the three-infinitely long nested cylinder further proves the accuracy of the closed form solutions to predict the affect of each layer's CTE on the displacements, strains, and stresses. The low percent difference in results for the first two models formed the basis for a theory to be generalized to N infinitely long nested cylinders. Development and example work is

shown in Appendix A. A four-cylinder model ($N = 4$) was developed in ABAQUS to test the theory and its comparison to the analytical results follow.

N Infinitely Long Nested Cylinders

To formulate a model of N concentric cylinders, the results for two and three cylinder systems were extended by inference such that, e.g., Equation (14) becomes:

$$\varepsilon_{\theta} = \Delta T \frac{\sum_{i=1}^N E_i \alpha_i}{\sum_{i=1}^N E_i} \quad (15)$$

A four-cylinder model, i.e., $N = 4$ was created to test this hypothesis. As before, the inside of the fourth cylinder was tied to the outside of the third cylinder and given the same wall thickness of 0.01 m, an elastic modulus of 40 GPa and a CTE of 20 ppm/°C. The third cylinder's material properties were changed to an elastic modulus of 35 GPa and a CTE of 18 ppm/°C. (The properties of the system are summarized in Table 9).

Table 9: Properties of the Four Nested Cylinder System

Cylinder #	Thickness (m)	Elastic Modulus (GPa)	CTE (ppm/°C)
1	0.01	10	5
2	0.01	20	10
3	0.01	35	18
4	0.01	40	20

Analytical results for the four-cylinder model were compiled. This theory was tested by comparing it to the results of a FEA model built in ABAQUS. The model was constructed with four cylinders ($N = 4$). A schematic of an N cylinder model is shown in Figure 8.

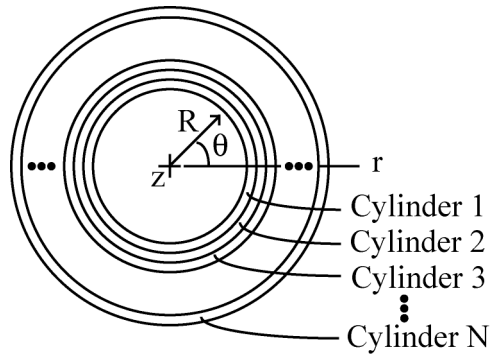


Figure 8: N Infinitely Long Nested Cylinders

To remain consistent with the previous two cases, a quarter of the circular cross section was modeled. Symmetry boundary conditions were applied to the quarter wedge to replicate the reactions of a complete circular section. This was performed to speed up computation time in the FEA program. Figure 9 shows a four-cylinder quarter model with symmetry boundary conditions applied.

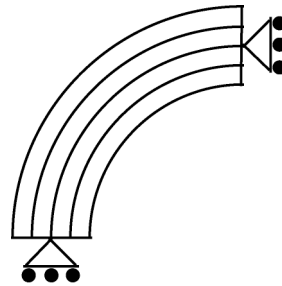


Figure 9: Four-Cylinder FE Model w/ Quarter Symmetry BC

The model was then subjected to a temperature change and the computed tangential stresses were compared to those calculated via the closed form solutions. The results of this comparison are shown below in Tables 10 and 11. The temperatures tested were in the range of -1000 °C to 250 °C.

Table 10: Tangential Strains for ABAQUS vs. Theory for N Nested Cylinders

Temperature Change (°C)	ABAQUS Strain	Equation (15) Strain	Percent Difference
250	4.001E-03	4.000E-03	-0.03
-100	-1.601E-03	-1.600E-03	-0.06
-250	-3.999E-03	-4.000E-03	0.03
-500	-7.988E-03	-8.000E-03	0.15
-1000	-1.593E-02	-1.600E-02	0.44

Table 11: Radial Displacements for ABAQUS vs. Theory for N Nested Cylinders

Temperature Change (°C)	ABAQUS Displacement (m)	Equation (8) Displacement (m)	Percent Difference
250	4.017E-03	4.000E-03	-0.42
-100	-1.600E-03	-1.600E-03	0.00
-250	-3.991E-03	-4.000E-03	0.23
-500	-7.956E-03	-8.000E-03	0.55
-1000	-1.581E-02	-1.600E-02	1.19

Also, the tangential hoop stresses from the FE model and the analytical equations were compared and are reported below in Table 12.

Table 12: Stresses for ABAQUS vs. Theory for N Nested Cylinders

Temperature Change (°C)	Cylinder Number	ABAQUS Stress (Pa)	Calculated Stress (Pa)	Percent Difference
250	1	2.78E+07	2.75E+07	-1.13
	2	3.01E+07	3.00E+07	-0.4
	3	-1.76E+07	-1.75E+07	-0.51
	4	-3.98E+07	-4.00E+07	0.43
-100	1	-1.11E+07	-1.10E+07	-0.82
	2	-1.20E+07	-1.20E+07	0.08
	3	7.13E+06	7.00E+06	-1.9
	4	1.60E+07	1.60E+07	-0.25
-250	1	-2.77E+07	-2.75E+07	-0.76
	2	-2.99E+07	-3.00E+07	0.27
	3	1.79E+07	1.75E+07	-2.51
	4	4.02E+07	4.00E+07	-0.53
-500	1	-5.53E+07	-5.50E+07	-0.58
	2	-5.96E+07	-6.00E+07	0.62
	3	3.62E+07	3.50E+07	-3.49
	4	8.08E+07	8.00E+07	-1.04

Table 12: (Continued)

-1000	1	-1.102E+08	-1.100E+08	-0.18
	2	-1.184E+08	-1.200E+08	1.33
	3	7.391E+07	7.000E+07	-5.59
	4	1.633E+08	1.600E+08	-2.06

The displacement and strain results of a model with four-cylinders ($N = 4$) is shown in Tables 10 and 11. Consistent to the two-cylinder and three-cylinder models, the percent difference in predicted displacements and strains versus the FEA results shows an increasing inaccuracy as the magnitude of the temperature change increases. Likewise, Table 12 shows no general trend in the stress differences. The low differences for all values between the FEA and closed form solutions, however, shows that the closed form equations will predict the displacements, strains, and stresses for infinitely long nested cylinders of any number. See Appendix A for development and example work. At this point, however, the accuracy of the prediction equations only fit the complete cylindrical model for which it was designed. The motivation behind this project, however, is to formulate closed form solutions for geometries that replicate ones used in optics. The following work, therefore, shows the application of the closed form equations to an optical trough (half-cylinder) of N -layers.

N Infinitely Long Nested Troughs

The concentric cylinders formulation does not model real world optics. It does however, provide for a basis from which to apply the theory to more accurate optical geometries. A step closer to a real world optic would be the optical trough. A trough has different boundary conditions than concentric cylinders, but a modified approach looked

promising. In addition to this method another approach was examined to determine which would suit the conditions of the N infinitely long nested trough. The modified infinitely long nested cylinders approach and a multi-layer thermostat (MLT) approach were both tested. The trough was set up in the following configuration (Figure 10):

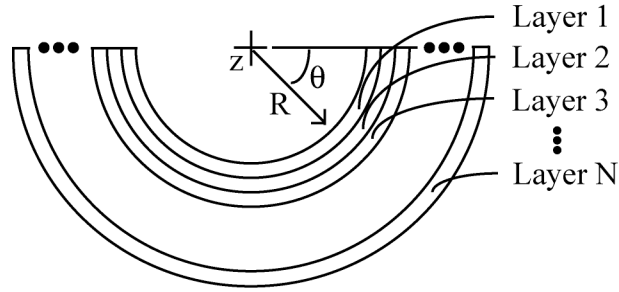


Figure 10: N Infinitely Long Nested Troughs

Concentric Cylinders Approach

The equations of the N infinitely long nested cylinders were used to determine if the stresses, strains, and displacements projected were valid for N infinitely long nested semi-circular troughs and if so, which trough end conditions suited this approach. The first boundary modeled had a horizontal roller condition on the ends of the trough (Figure 11). The second had a free end (Figure 12), the third a pinned-pinned end condition (Figure 13), and the fourth had a vertical roller on the end of the trough (Figure 14).

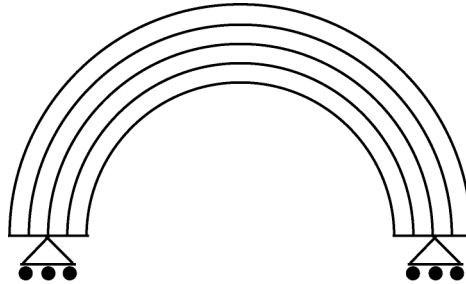


Figure 11: Four-Layer FE Trough Model with Horizontal Roller BC

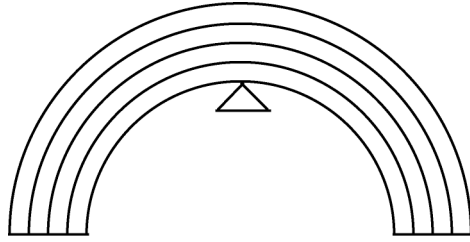


Figure 12: Four-Layer FE Trough Model with Free BC

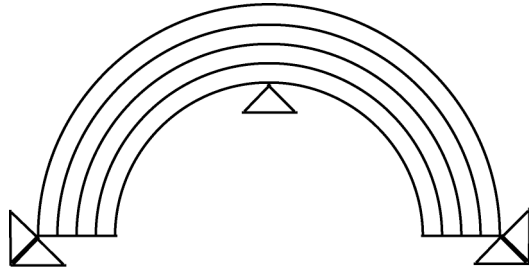


Figure 13: Four-Layer FE Trough Model with Pinned-Pinned BC

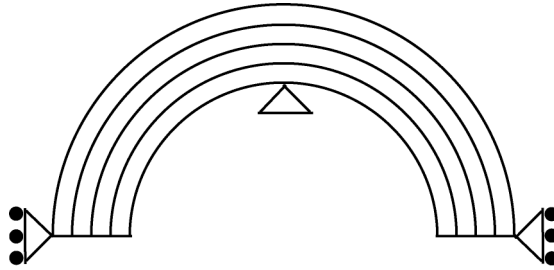


Figure 14: Four-Layer FE Trough Model with Vertical Roller BC

The cross section of the trough model is geometrically equal to half the cross section of the concentric cylinder model. Like the nested cylinders, the individual layers of the trough were tied to the neighboring cylinder layers. To compare to previous work performed on the infinitely long nested cylinders, a trough of 4 layers ($N = 4$) was developed.

All dimensions of the concentric cylinder models were maintained in the trough model. Layer 1 had an inner radius of curvature of 0.99 m and the outer radius of

curvature for layer 1 was 1.00 m, giving layer 1 a 0.01 m wall thickness. Additionally, layer 2 had an inner radius of curvature equal to the outer radius of curvature of layer 1 and a wall thickness 0.01 m, as well. Similarly, layer 3 had a wall thickness of 0.01 m. The inner radius of curvature of layer 3 was 1.01 m and the outer radius was 1.02 m. Layer 4 had an inner radius of curvature of 1.02 m and an outer radius of curvature of 1.03 m. The material properties of the trough were set uniformly with an elastic modulus of 40 GPa and a CTE of 20 ppm/°C in each layer (summarized in Table 13).

Table 13: Properties of the Four Nested Trough System

Layer #	Thickness (m)	Elastic Modulus (GPa)	CTE (ppm/°C)
1	0.01	40	20
2	0.01	40	20
3	0.01	40	20
4	0.01	40	20

Each trough with the four boundary conditions and uniform properties was subjected to a temperature change of 250 °C. This was first performed to determine which end condition more accurately modeled the trough relative to the concentric cylinders model. The results are shown in Table 14 below. Development of this test and further explanation can be found in Appendix A.

Table 14: FE Displacement Comparison of End Conditions of Trough Model vs. Concentric Cylinders Model

Trough BC Type	Cylinder Disp. (Uniform)	Trough Disp. (End u_r)	Percent Difference	Trough Disp. (End u_θ)	Percent Difference
r-direction Roller	5.013E-03	5.013E-03	0.00	5.013E-03	0.00
Free End	5.013E-03	5.096E-03	1.66	4.986E-03	-0.54
Pinned-Pinned	5.013E-03	6.050E-04	-87.93	1.432E-04	-97.14
θ -direction Roller	5.013E-03	1.494E-04	-97.02	-8.283E-03	-265.23

As is shown in Table 14, the r-direction roller end condition exactly modeled the concentric cylinder. The free end condition also closely matched the cylinder with uniform properties. A trough with free end boundary conditions, however, is more realistic in application and this condition thus became the focus of further analysis. After the end condition of Figure 11 was matched to the concentric cylinder approach, the material properties of individual layers in the FEA trough model were altered. First, layer 1 was altered to have a CTE 1.5x the CTE of the unaltered remaining three layers. In the first case, layer 1 had a CTE of 30 ppm/°C and layers 2,3, and 4 had a CTE of 20 ppm/°C. Next, layer 2 was altered to have a CTE 1.5x above the CTE of system with a CTE of 30 ppm/°C and the other layers had a CTE of 20 ppm/°C. Likewise, layer 3 and then layer 4 were altered in the same manner. The radial displacements (Figure 16) and tangential displacements (Figure 17) of the trough end for these cases were observed in comparison with each other. Similarly, each individual layer was tested with a CTE of 0.5x (10 ppm/°C) that of the adjoining layers to again compare the radial and tangential displacements of the trough end. The CTE configurations of the four-layer trough are summarized in Table 15. The radial displacements and tangential displacements of the trough end for these cases were observed in comparison with each other and are shown in Figures 19 and 20, respectively.

Table 15: CTE of the Four Nested Trough Systems

Case #	CTE (ppm/°C) in Layer #				Case #	CTE (ppm/°C) in Layer #			
	1	2	3	4		1	2	3	4
1	30	20	20	20	1	10	20	20	20
2	20	30	20	20	2	20	10	20	20
3	20	20	30	20	3	20	20	10	20
4	20	20	20	30	4	20	20	20	10

The results of the radial and tangential displacements were observed at nine nodal positions. Node 1 was on the inner surface of the trough and node 9 was on the outermost surface of the trough, as shown in Figure 15.

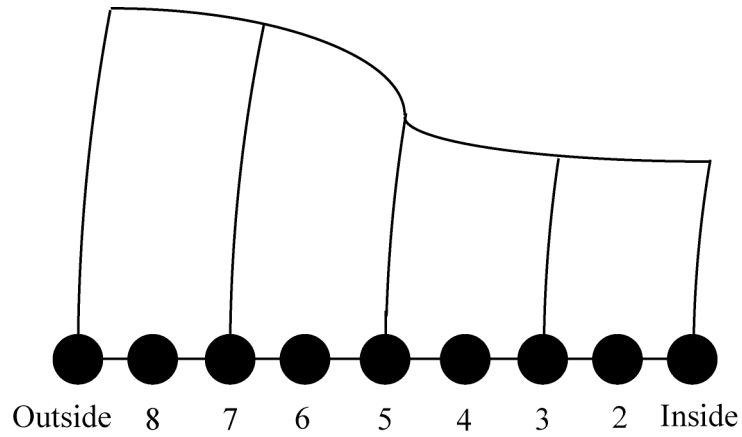


Figure 15: Nodal Position in Trough

Figure 16: Radial Displacement at End of Trough for
250 °C Increase (Free end B.C.)

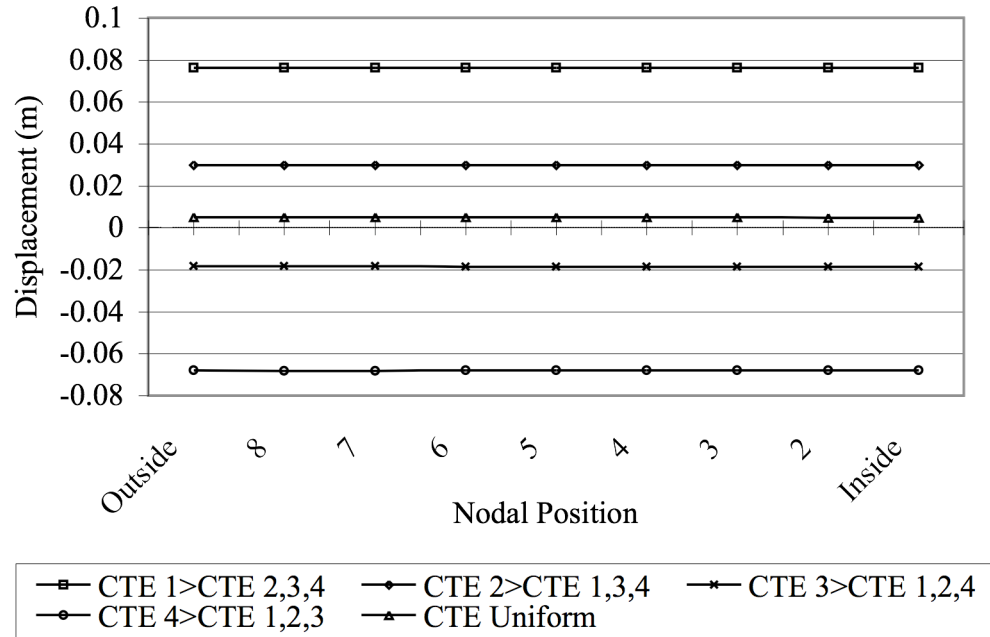
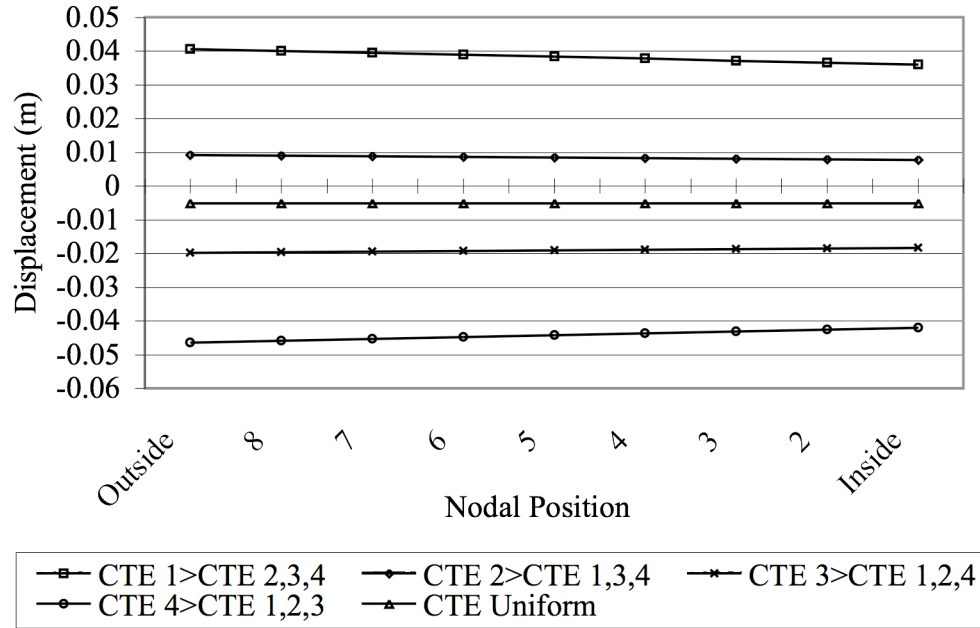


Figure 17: Tangential Displacement at End of Trough for
250 °C Increase (Free end B.C.)



To summarize the results, Figure 18 shows the resultant displacements for each case.

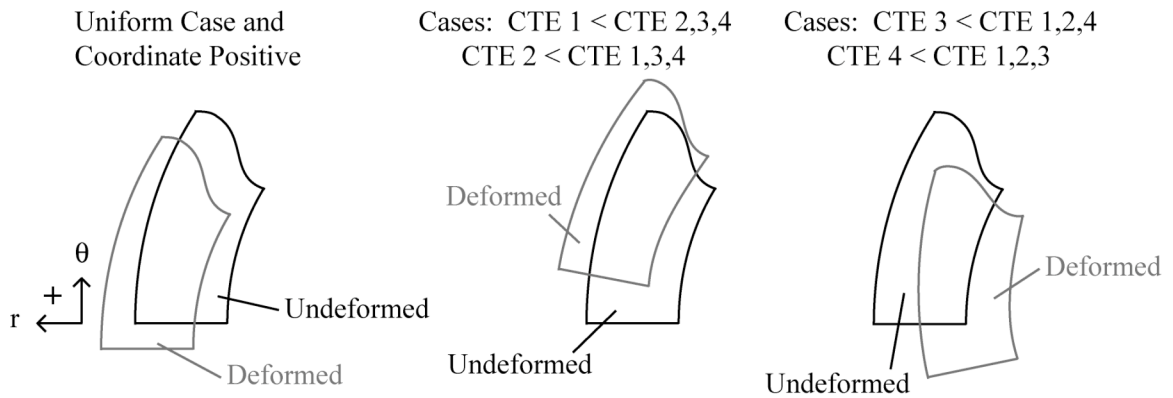


Figure 18: Resultant Displacements for each case

Similarly, the radial displacement from one layer having a lower CTE than the adjoining layers is shown in Figure 19 and the corresponding tangential displacement is shown in Figure 20.

Figure 19: Radial Displacement at Edge of Trough for
250 °C Increase (Free end B.C.)

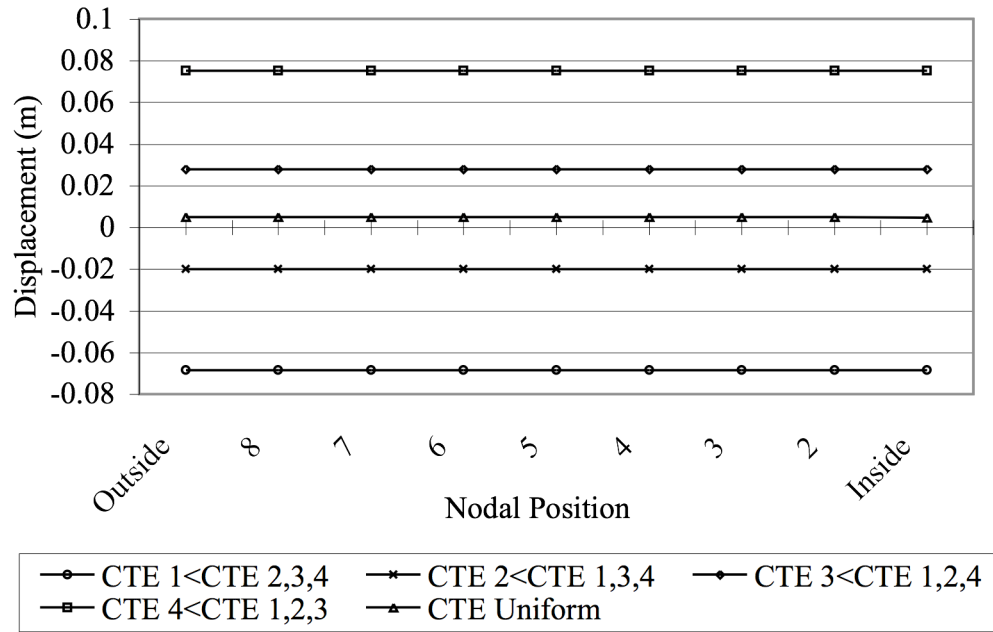
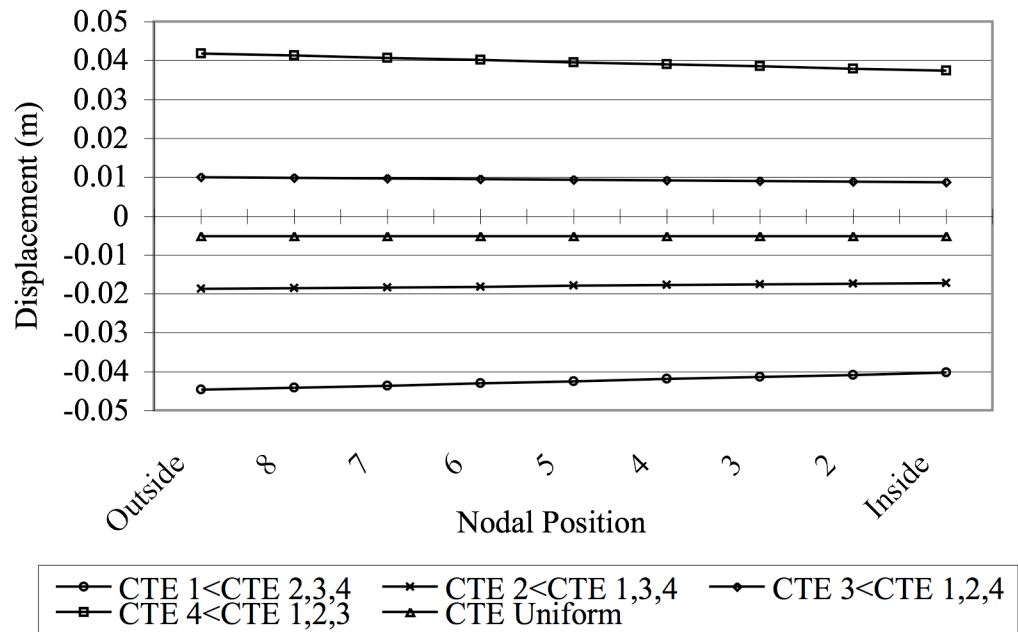


Figure 20: Tangential Displacement at Edge of Trough for
250 °C Increase (Free end B.C.)



To summarize the results, Figure 21 shows the resultant displacements for each case.

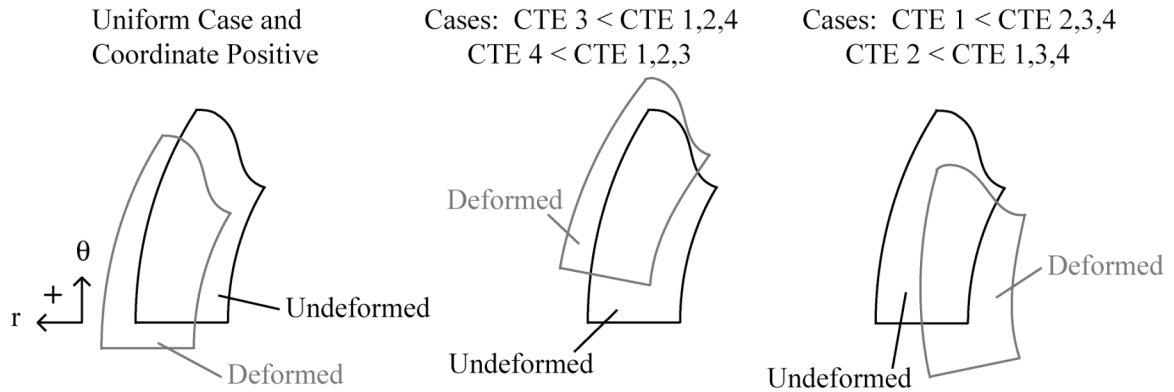


Figure 21: Resultant Displacements for each case

Further tests were performed under the same temperature change in which one layer's CTE was set to a percentage of the remaining three layer's CTE by 4x, 3x, 2x, 0.75x, and 0.25x. These were performed to determine how well the concentric cylinder approach corresponded to the FEA trough model with non-uniform layers and the free end condition. It was determined that the concentric cylinders approach correlated precisely with the r-direction roller, but an attempt was made to expand this approach to determine if other end conditions could be predicted. These tests compared the FEA resultant stresses versus the calculated stresses from the concentric cylinders approach to determine the limits of this method. Figure 22 shows the percent difference in stresses for a system in which layer 1 (see Figure 10) was the abnormal layer. Figure 23 displays the results stemming from an abnormal layer 2. Similar to the cases in Figure 22, layer two's CTE was expressed as a percentage of the equivalent CTE in layers 1, 3, and 4. Appendix A contains the figures showing similar plots from the results of an abnormal layer 3 an abnormal layer 4.

Figure 22: Percent Difference Between FEA and Calculated Stresses for a 4 Layer Trough with the CTE of Layer 1 Varied Compared to the System (Note: for the 1x case, stresses are zero and cannot be quantified)

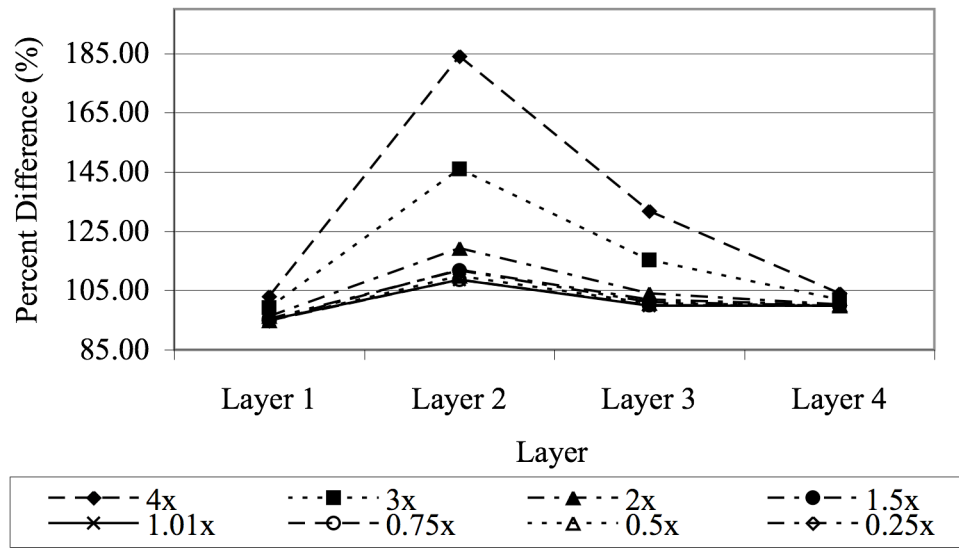
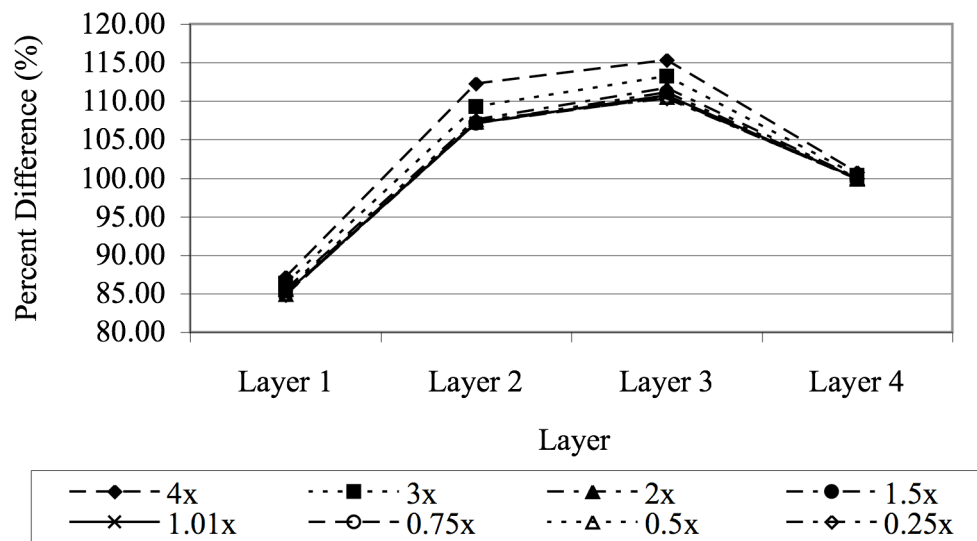


Figure 23: Percent Difference Between FEA and Calculated Stresses for a 4 Layer Trough with the CTE of Layer 2 Varied Compared to the System (Note: for the 1x case, stresses are zero and cannot be quantified)



Figures 24 and 25 show the displacements corresponding to the free end of the trough models in which layers 1 and 2 had the differing CTE, respectively. The plots show the percent difference between the FEA displacement results and the closed form calculation of the displacements. The displacement results for the remaining cases, 3 and 4, are shown in Appendix A. The stress comparison from an abnormal layer 1, as shown in Figure 22, corresponds to the displacements in Figure 24. The stress from the abnormal layer 2 was shown in Figure 23 and the corresponding displacements are shown in Figure 25.

Figure 24: Percent Diff. Between FEA and Calculated Displacement for a 4 Layer Trough with the CTE of Layer 1 Varied Compared to the System (Note: the 1.01x case is an approximation of the 1x case)

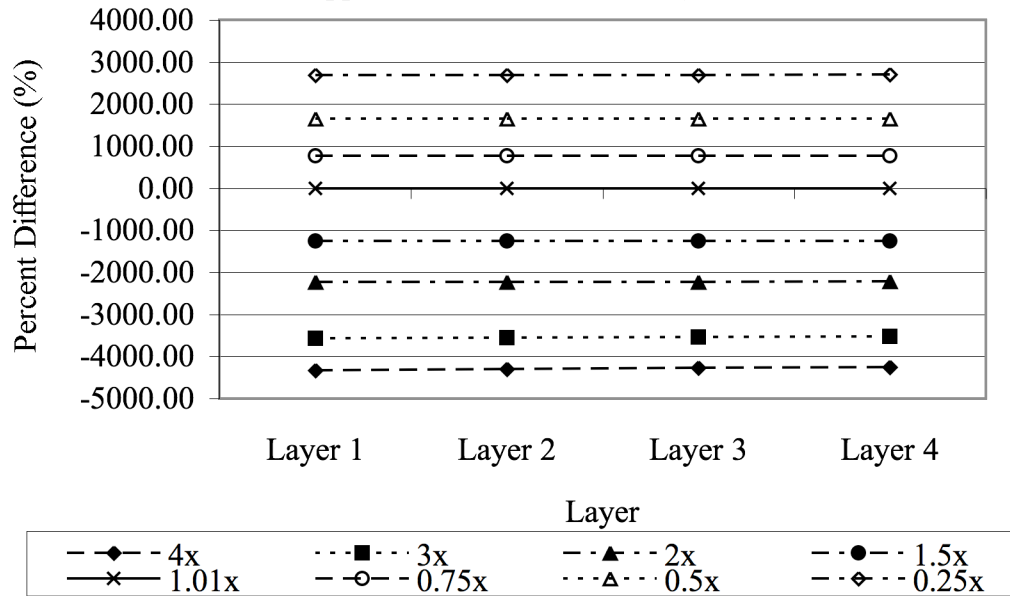
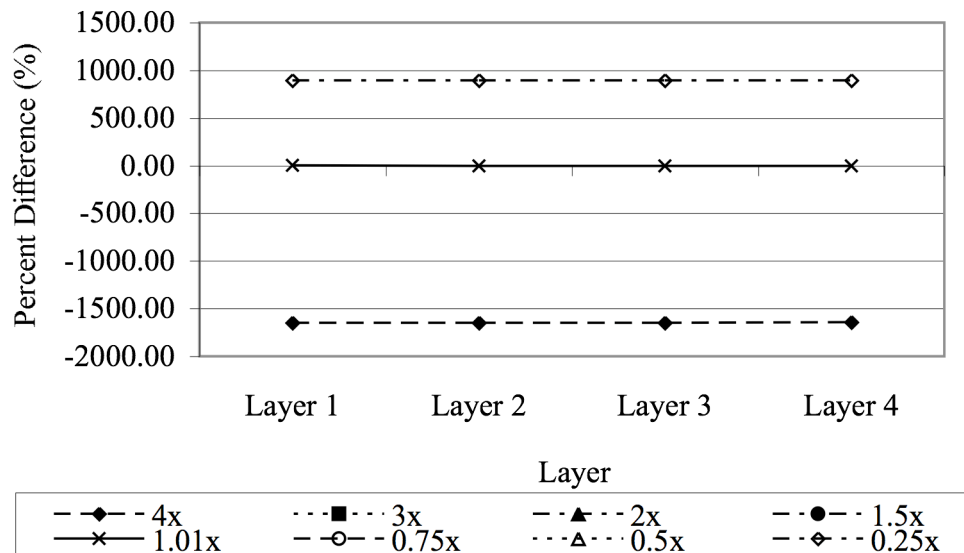


Figure 25: Percent Diff. Between FEA and Calculated Displacements for a 4 Layer Trough with the CTE of Layer 2 Varied Compared to the System (Note: the 1.01x case is an approximation of the 1x case)



To quantify the behavior, the closed form equations produced from the concentric cylinders approach will predict the stress, strain, and displacements of a trough fabricated with the end conditions of Figure 11. However, the method of the concentric cylinders was applied to the free end trough (Figure 12) with non-uniform properties and failed. Troughs with non-uniform properties in which at least one layer had properties differing from those of the system could not be modeled with the closed form equations derived through the method of concentric cylinders. Inspection of Figures 22 and 23 show that the stresses observed in the FEA models with one abnormal CTE layer were at minimum 85% higher than the stresses realized in the prediction equations. Although the differences appear visually uniform in the graphs, the results provided no concrete ground from which to develop a consistent stress prediction theory. The same was

concluded from the analysis of the displacement results shown in Figures 24 and 25. The method developed from the concentric cylinders approach does not account for moments and rotations generated when the cylinder is sectioned. It, therefore, does not predict the displacements or stresses of the free end trough, in which moments and rotations occur. To predict the conditions of the trough with more realistic boundary conditions, another method following the multi-layer thermostat (MLT) had to be examined.

A few notable observations from the previous testing were made. For a four-layer trough, the interface of the second and third layers becomes a centerline for movement. For a temperature increase in the system, a CTE that is greater in layers one or two will cause the radius of curvature to increase, while a CTE greater in layers three or four for a temperature increase will cause the radius of curvature of the system to decrease. The inverse is true for a temperature decrease. For the lowered temperature, a CTE that is greater in layers one or two will cause the radius of curvature to decrease, while a CTE greater in layers three or four for a temperature decrease will cause the radius of curvature of the system to increase. Also, a CTE difference in the layer farthest from the centerline interface will cause a greater increase in curvature in the system for equivalent CTE. For example, a temperature increase with layer 1 having a CTE of 20 ppm/°C and the remaining layers having a CTE of 10 ppm/°C will cause greater increase in the curvature than if layer 2 had the CTE of 20 ppm/°C and the remaining layers the CTE of 10 ppm/°C. This examination produced conceptual understanding on the behavior of the trough with the free end condition.

Multi-Layer Thermostat Approach

The moments and rotations generated in the trough were not captured by the concentric cylinders approach. This led to an analysis of bi-metallic, tri-metallic, and multi-layered thermostats. A thermostat detects change in temperature, often with the use of materials differing in CTE. The thermostats of interest were the ones composed of various flat strips of material welded together. The expansion or contraction of the bi-metallic strip is due to the mismatched CTE of the layers under a steady-state temperature change. This causes a strain mismatch at the interface and results in the strip bending¹². The radius of curvature of each layer, ρ_i , or its inverse the curvature, κ_i , is resultant from the thermal strains. The thermally induced curvature is calculated via the material properties of each layer. Figure 26 shows an example four-layer strip and Equation (16) shows the thermal curvature for this model¹³. Expansion of this equation to N-layers is intuitive based upon recognition of the patterns in the coefficients a, b, c, and d.

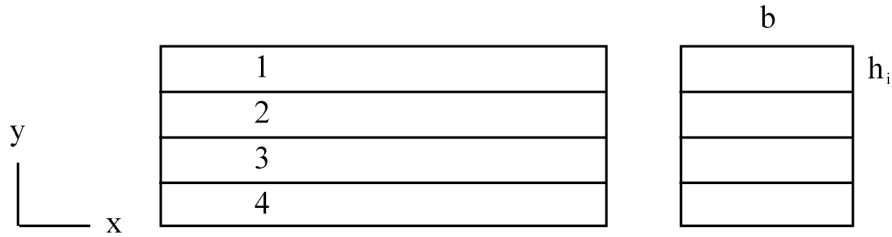


Figure 26: Four-Layer Strip

$$\kappa_{thermal} = \frac{1}{\rho} = 2\Delta T \frac{a_{12} + a_{13} + a_{14} + a_{23} + a_{24} + a_{34}}{b_{12} + b_{13} + b_{14} + b_{23} + b_{24} + b_{34} + 4cd} \quad (16)$$

where:

$$a_{12} = s_1 s_2 (h_1 + h_2) (\alpha_2 - \alpha_1) \quad (17a)$$

$$a_{23} = s_2 s_3 (h_2 + h_3) (\alpha_3 - \alpha_2) \quad (17b)$$

$$a_{34} = s_3 s_4 (h_3 + h_4)(\alpha_4 - \alpha_3) \quad (17c)$$

$$a_{13} = s_1 s_3 (h_1 + 2h_2 + h_3)(\alpha_3 - \alpha_1) \quad (17d)$$

$$a_{24} = s_2 s_4 (h_2 + 2h_3 + h_4)(\alpha_4 - \alpha_2) \quad (17e)$$

$$a_{14} = s_1 s_4 (h_1 + 2(h_2 + h_3) + h_4)(\alpha_4 - \alpha_1) \quad (17f)$$

$$b_{12} = s_1 s_2 (h_1 + h_2)^2 \quad (18a)$$

$$b_{23} = s_2 s_3 (h_2 + h_3)^2 \quad (18b)$$

$$b_{34} = s_3 s_4 (h_3 + h_4)^2 \quad (18c)$$

$$b_{13} = s_1 s_3 (h_1 + 2h_2 + h_3)^2 \quad (18d)$$

$$b_{24} = s_2 s_4 (h_2 + 2h_3 + h_4)^2 \quad (18e)$$

$$b_{14} = s_1 s_4 (h_1 + 2(h_2 + h_3) + h_4)^2 \quad (18f)$$

$$c = E_1 I_1 + E_2 I_2 + E_3 I_3 + E_4 I_4 \quad (19)$$

$$d = s_1 + s_2 + s_3 + s_4 \quad (20)$$

$$s_i = h_i E_i \quad (21)$$

where the second moment of area of the rectangular cross-section thermostat about a horizontal line through the centroid is:

$$I_i = \frac{b h_i^3}{12} \quad (22)$$

All work was performed on the unit depth basis of $b = 1$ m and h is the thickness of individual layers in the trough. The above equations predict curvature for an initially flat four layer metallic strip. The trough, however, has an initial curvature, κ_{initial} . The total curvature is conveniently additive as is shown below¹⁴:

$$\kappa_{\text{total}} = \kappa_{\text{initial}} + \kappa_{\text{thermal}} \quad (23)$$

Trigonometric relations allow the initial curvature to be calculated based on the distance between the initial coordinates of the apex and one end point (Figure 27).

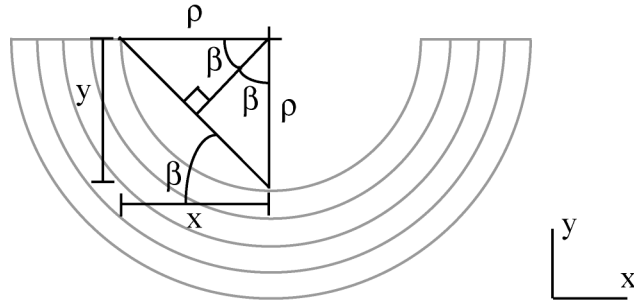


Figure 27: Initial Curvature Geometry

The equation for the initial curvature is:

$$\kappa_{i,initial} = \frac{1}{\rho_{i,initial}} = \frac{2 \sin(\beta)}{\sqrt{(x_i^2 + y_i^2)}} \quad , \quad i = 1, 2, \dots, N \quad (24)$$

Although each layer has differing initial curvatures, the primary focus is the curvature of the reflecting surface, i.e., the inner most layer (layer 1). Also, because the individual thickness of each layer is small compared to the respective radius, all layers are approximated to have equal radii. Therefore, the subscript i is dropped and the curvature of each layer is assumed to be equivalent to the curvature of the entire system. Again, the total curvature of the thermally affected N layer strip is simply the curvature determined by the initial geometry added to the thermally induced curvature.

The closed form design equation can be arranged to solve for favorable conditions, such as a mirror that does not distort during a temperature loading, i.e. no thermal curvature. To design a 4-layer mirror with no thermal curvature, Equation (16) is set to equal zero. In doing so, the following occurs:

$$0 = a_{12} + a_{13} + a_{14} + a_{23} + a_{24} + a_{34} \quad (25)$$

Solving for any a in terms of the remaining a variables and substituting/arranging the corresponding s , h , and α variables results in Equation (26):

$$0 = P_1 + P_2 + P_3 + P_4 \quad (26)$$

where:

$$P_1 = \alpha_1 [s_1 s_2 h_{12} + s_1 s_3 h_{13} + s_1 s_4 h_{14}] \quad (27a)$$

$$P_2 = \alpha_2 [s_1 s_2 (-h_{12}) + s_2 s_3 h_{23} + s_2 s_4 h_{24}] \quad (27b)$$

$$P_3 = \alpha_3 [s_1 s_3 (-h_{13}) + s_2 s_3 (-h_{23}) + s_3 s_4 h_{34}] \quad (27c)$$

$$P_4 = \alpha_4 [s_1 s_4 (-h_{14}) + s_2 s_4 (-h_{24}) + s_3 s_4 (-h_{34})] \quad (27d)$$

As has been shown, the individual coefficients of α corresponding to the layer number have been separated. The equations allow one α to be solved for in terms of the other three. For example, if layers 2 through 4 have CTE assignments, then the CTE of layer 1 can be found. The dependent CTE of layer 1 will cause the system to have zero curvature after *any* temperature loading. The resultant CTE of layer 1, however, may or may not be realistic.

The equations can be applied to a system of N-layers. For a system of N-layers, the general form of the equation that will predict a layer's CTE to produce zero curvature in the system is Equation (28) (for development and validation of this method see Appendix A):

$$0 = P_i + P_{i+1} + P_{i+2} + \cdots + P_{N-1} + P_N \quad (28)$$

$$i=1, 2, 3, \dots, N$$

where:

$$P_i = \alpha_i \left[\sum_{j=1}^N [s_i s_j (h_{ij})] - \sum_{j=1}^N [s_i s_j (h_{ij})] \cdot \delta_{ij} \right] \quad (29)$$

$$s_j s_i = -s_i s_j \text{ and } h_{ji} = h_{ij}$$

Here, δ_{ij} is the Kronecker Delta. Again, any layer CTE can be solved for in terms of the remaining layer's CTE and its value will not allow thermally induced curvature for *any* temperature loading. This is advantageous in that a mirror system of N-layers can be optimized to withstand thermal curvature under any given temperature increase. This however, does not express that the internal stresses of the system due to differing expansion rates of neighboring layers can go unaccounted. A thorough analysis of resultant stresses is necessary to avoid failure due to internal stresses that may overcome the yield or ultimate tensile strengths. Also, neighboring layers with drastically different CTE values may experience end effects that will cause aberrations in the reflected image. Regardless, the equations were tested against FEA models to determine their validity.

With equal definitions of parameters to the closed form models, the FEA models of the four layer troughs were created in ABAQUS with an initial radius of curvature of 1m. Layer one had an elastic modulus of 10 GPa and a CTE of 5 ppm/°C. Layer two had an elastic modulus of 20 GPa and a CTE of 10 ppm/°C, layer three had an elastic modulus of 35 GPa and a CTE of 18 ppm/°C, and layer four had an elastic modulus of 40 GPa and a CTE of 20 ppm/°C. Each of the four layers had a thickness of 0.023 m and was assumed to be infinitely long (properties for all layers are summarized in Table 16).

Table 16: Properties of the Four-layer Trough

Cylinder #	Thickness (m)	Elastic Modulus (GPa)	CTE (ppm/°C)
1	0.023	10	5
2	0.023	20	10
3	0.023	35	18
4	0.023	40	20

Each of the four layers was rigidly constrained at the interfaces. For the FEA model, 2816 of 8-node plane strain thermally-coupled quadrilateral 2D continuum elements were assigned to the model. To reduce computation, only half of the symmetrical trough was created and the symmetry boundary condition was employed in the x-direction. Figure 28 shows a deformed vs. undeformed FEA model with the previous defined parameters.

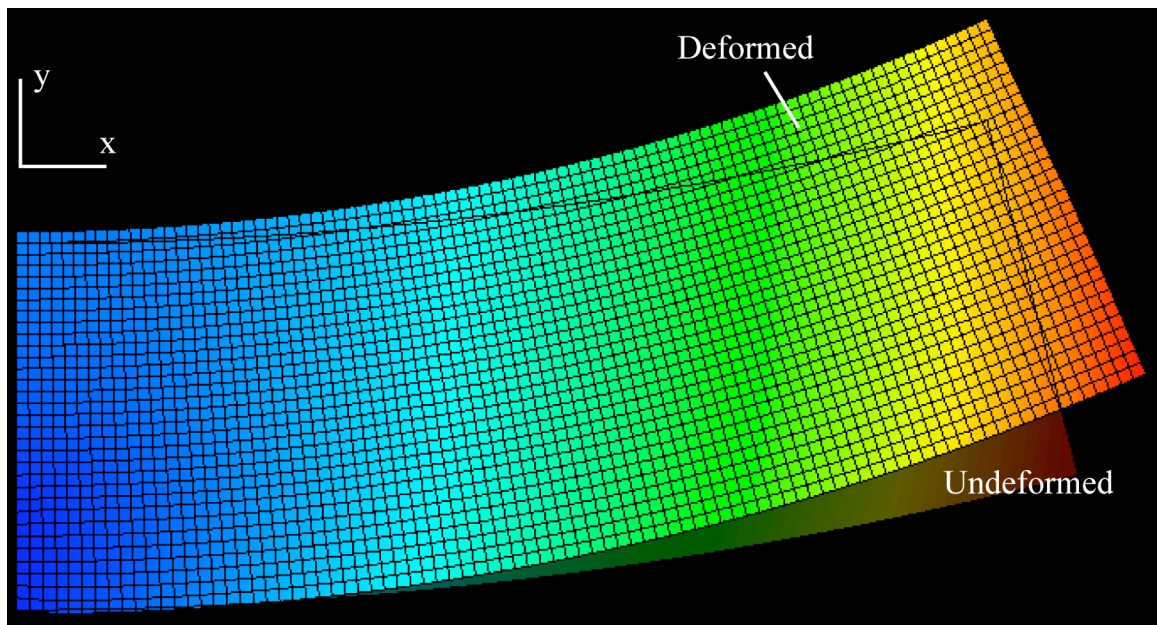


Figure 28: FEA Deformed vs. Undeformed Trough (Case 8-Uniform Loading)

A vertical boundary condition of zero displacement was also applied to the lowest point of the trough to prevent rigid body piston displacement under loading. The initial temperature over all elements in the model was set to 0 °C. Eight separate cases with uniform temperature changes of ± 250 °C, ± 100 °C, ± 50 °C, and ± 25 °C were applied to the model. The resultant RMS surface deviations (Table 17) and best-fit radius of curvature (Table 18) are reported. The FEA RMS surface distortion is calculated from the raw surface data of the FEA model and the best-fit RMS is calculated from smoothed data.

Table 17: Comparison of RMS Surface Distortion for Closed Form and FEA Results for Uniform Temperature Loading

Case #	Temperature (°C)	FEA RMS Surface Dist. (m)	Best-fit RMS Surface Dist. (m)	Percent Difference
1	-250	3.147E-06	3.134E-06	0.427
2	-100	3.148E-06	3.158E-06	-0.316
3	-50	3.135E-06	3.079E-06	1.777
4	-25	3.021E-06	3.026E-06	-0.172
5	25	3.092E-06	3.054E-06	1.216
6	50	3.079E-06	2.996E-06	2.682
7	100	3.060E-06	3.108E-06	-1.569
8	250	3.267E-06	3.224E-06	1.336

Table 18: Comparison of Best-fit Radius of Curvature for Closed Form and FEA Results for Uniform Temperature Loading

Case #	Temperature (°C)	FEA Best-fit Radius of Curvature (m)	Closed Form Best-fit Radius of Curvature (m)	Percent Difference
1	-250	1.055	1.051	0.370
2	-100	1.023	1.019	0.355
3	-50	1.012	1.009	0.336
4	-25	1.005	1.004	0.107
5	25	0.996	0.994	0.219
6	50	0.992	0.989	0.332
7	100	0.978	0.979	-0.147
8	250	0.950	0.951	-0.105

In addition to the previous cases, the closed form equation was rearranged to determine which material definitions produced a zero curvature in the system for a given temperature loading. For all of these tests, the uniform temperature change was set to +250 °C. The elastic moduli for each layer also remained the same for the individual tests. Layer one had an elastic modulus of 15 GPa, layer two 20 GPa, layer three 35 GPa, and layer four had an elastic modulus of 40 GPa. Thickness of each layer remained at 0.023m. Again, the results of the thermally loaded troughs from the closed form equation and the FEA models were compared. The results of most interest were the RMS surface deviation (Table 19) and the best-fit radius of curvature (Table 20). The best-fit

radius of curvature was a curve fit plotted through the deformed surface of the FEA model. From this curve fit, the radius of curvature of the FEA deformed surface was calculated. The bold print in Tables 19 and 20 denotes the dependent CTE predicted with Equation (26) that would produce zero thermal curvature for the entire system (the initial radius of curvature remained at 1m).

Table 19: RMS Surface Distortion for Zero Thermal Curvature for Closed Form and FEA Results at 250 °C Temperature Loading throughout Model (bold font represents dependent CTE)

Case #	CTE ppm/°C in Layer				FEA RMS Surface Dist. (m)	Best-fit RMS Surface Dist. (m)	Percent Difference
	1	2	3	4			
1	42	10	20	30	3.020E-06	3.147E-06	-4.190
2	10	35	15	20	3.100E-06	3.022E-06	2.506
3	20	-5	15	10	3.040E-06	3.154E-06	-3.756
4	-1	-5	100	-10	3.097E-06	3.050E-06	1.519
5	10	10	10	10	3.021E-06	3.216E-06	-6.473
6	10	20	30	13	3.066E-06	3.142E-06	-2.477
7	-10	-5	-1	-9	3.071E-06	3.115E-06	-1.453

Table 20: Best-fit Radius of Curvature for Zero Thermal Curvature for Closed Form and FEA Results at 250 °C Temperature Loading throughout Model (bold font represents dependent CTE)

Case #	CTE ppm/°C in Layer				FEA Best-fit Radius of Curvature (m)	Closed Form Best-fit Radius of Curvature (m)	Percent Difference
	1	2	3	4			
1	42	10	20	30	0.996	1.000	-0.393
2	10	35	15	20	1.005	1.000	0.539
3	20	-5	15	10	1.002	1.000	0.222
4	-1	-5	100	-10	1.031	1.000	3.047
5	10	10	10	10	1.003	1.000	0.268
6	10	20	30	13	1.009	1.000	0.847
7	-10	-5	-1	-9	1.002	1.000	0.153

To predict the stresses and all other variables of interest for a free end trough with non-uniform properties, the equations of multi-material thermostats proved accurate. For a system of uniform temperature change, the closed form approach and FEA results had

an absolute average of 1.187% difference between RMS surface deviations of the undeformed and deformed surfaces (Table 17) and an absolute average of 0.246% difference between the best-fit radius of curvature results (Table 18). This approach provides for a nearly exact method to predict the RMS surface distortion and radius of infinite cylindrical troughs under a thermally uniform load. The minimal difference seen, however, is attributed to the unbalance in CTE definition. The interface between the second and third layers of the four-layer system is a centerline for expansion or contraction. The greater the unbalance in the addition of the CTEs in layer 1 and 2 versus layers 3 and 4 will cause greater percent differences between the FEA and closed form models. On the other hand, if the CTEs are better balanced about this centerline, the percent differences are smaller. Observe Case 1 in Table 19 for instance. The addition of the CTEs in layers 1 and 2 is 52 and in layers 3 and 4 is 50. This case has the least percent difference of the cases reported in the table. Some of the other cases are not balanced as well and the percent difference is slightly higher. For the cases of zero curvature analysis, however, the absolute averages were 3.196% and 0.781% for RMS surface distortion (Table 19) and radius of curvature comparisons (Table 20), respectively. The multi-layer thermostat (MLT) approach proved successful in determining the reactions of a thermally uniformly loaded trough of N-layers.

The previous approach, however, only predicted the best-fit radius of curvature for N layers affected by a uniform temperature change throughout the model. It does not predict the curvature for a model with a temperature change gradient. To further the prediction capabilities of the multi-layer thermostat (MLT) approach, we have developed

the closed form equation for non-uniform temperature distribution (see Equations (30) and (31)): which has shown accuracy in comparisons to FEA results. Equation (30) is for a 4-layer trough, however, the expansion of this equation to N-layers is intuitive based upon recognition of the patterns in the coefficients a, b, c, d, and ΔT .

$$\kappa_{thermal} = \frac{1}{\rho} = 2 \frac{\Delta T_{21}a_{12} + \Delta T_{31}a_{13} + \Delta T_{41}a_{14} + \Delta T_{32}a_{23} + \Delta T_{42}a_{24} + \Delta T_{43}a_{34}}{b_{12} + b_{13} + b_{14} + b_{23} + b_{24} + b_{34} + 4cd} \quad (30)$$

where:

$$\Delta T_{ij} = \frac{T_i + T_j}{2} - T_0 \quad (31)$$

T_i or T_j are the temperature changes in each layer and $i \neq j$

$T_0 = \text{Ambient Temperature} = 0^\circ\text{C (arbitrary)}$

Equation (30) also predicts the results of the uniform thermal loading in which all layers have equivalent temperatures. In that case, Equation (31) reduces to:

$$T_i = T_j \quad (32)$$

$$\Delta T_{ii} = T_i - T_0$$

By substituting Equation (32) into Equation (30) and factoring, the resulting equation becomes that of the uniform thermal loading Equation (16).

$$\kappa_{thermal} = \frac{1}{\rho} = 2\Delta T_{ii} \frac{a_{12} + a_{13} + a_{14} + a_{23} + a_{24} + a_{34}}{b_{12} + b_{13} + b_{14} + b_{23} + b_{24} + b_{34} + 4cd}$$

As has been shown, the non-uniform thermally loading equation predicts the same results as the thermally uniform loading approach for uniform cases. The inverse, however, is not true and the results of this test are reported. The non-uniform approach is an expansion of the uniform method and works for both cases.

Similar to the uniform thermally loaded trough, the closed form design equation of the non-uniform thermal loading can be arranged to solve for favorable conditions, such as a mirror that does have an additional thermally induced curvature. The process to solve for zero curvature is similar to that of the previous method that was shown in Equation (25). Again, the thermal curvature for a 4-layer trough was set to zero and the result is shown in Equation (33):

$$0 = \Delta T_{21}a_{12} + \Delta T_{31}a_{13} + \Delta T_{41}a_{14} + \Delta T_{32}a_{23} + \Delta T_{42}a_{24} + \Delta T_{43}a_{34} \quad (33)$$

For this case, the temperature change, ΔT_{ij} , cannot be factored out due to the temperature gradient. Substitution for the a variables and rearranging of the corresponding s , h , and α variables results in Equation (34):

$$0 = P_1 + P_2 + P_3 + P_4 \quad (34)$$

where:

$$P_1 = \alpha_1 [-\Delta T_{21}s_1s_2h_{12} - \Delta T_{31}s_1s_3h_{13} - \Delta T_{41}s_1s_4h_{14}] \quad (35a)$$

$$P_2 = \alpha_2 [\Delta T_{21}s_1s_2(h_{12}) - \Delta T_{32}s_2s_3h_{23} - \Delta T_{42}s_2s_4h_{24}] \quad (35b)$$

$$P_3 = \alpha_3 [\Delta T_{31}s_1s_3(h_{13}) + \Delta T_{32}s_2s_3(h_{23}) - \Delta T_{43}s_3s_4h_{34}] \quad (35c)$$

$$P_4 = \alpha_4 [\Delta T_{41}s_1s_4(h_{14}) + \Delta T_{42}s_2s_4(h_{24}) + \Delta T_{43}s_3s_4(h_{34})] \quad (35d)$$

Equation (34) is similar to Equation (26) of the uniform loading case, but the individual equations for P_i have differing definitions; ΔT_{ij} remains the same for both cases. Again, the equations allow one α to be solved for in terms of the other three and the dependent CTE of the layer solved for will cause the system to have zero curvature for the temperature loading specified. Unlike the uniform thermally loaded case, these equations are *not* independent of the temperatures and the resultant CTE will *not* cause

zero curvature for any temperature configuration. For development work and validation of this method see Appendix A.

The equations can also be applied to a system of N-layers. For a system of N-layers, the general form of the equation that will predict a layer's CTE to produce zero curvature in the non-uniform thermally loaded system is Equation (36):

$$0 = P_i + P_{i+1} + P_{i+2} + \dots + P_{N-1} + P_N \quad (36)$$

$$i=1, 2, 3, \dots, N$$

where:

$$P_i = \alpha_i \left[\sum_{j=1}^N [\Delta T_{ji} s_i s_j (h_{ij})] - \sum_{j=1}^N [\Delta T_{ji} s_i s_j (h_{ij})] \cdot \delta_{ij} \right] \quad (37)$$

$$s_j s_i = -s_i s_j \text{ and } h_{ji} = h_{ij}$$

Again, any layer CTE can be solved for in terms of the remaining layer's CTE and its value will not allow thermally induced curvature for the temperatures initially specified. Stress analysis must be performed to determine if any violation of the material's yield or ultimate tensile strengths are violated from internal stresses caused by non-uniform layer expansion or contraction.

Similar to the uniform thermal loading cases, the FEA models of the four layer infinitely long troughs were created in ABAQUS with an initial radius of curvature of 1 m and each layer was 0.023 m thick. To better compare to the uniform cases, the material properties were defined correspondingly. Layer one had an elastic modulus of 10 GPa and a CTE of 5 ppm/°C. Layer two had an elastic modulus of 20 GPa and a CTE of 10 ppm/°C, layer three had an elastic modulus of 35 GPa and a CTE of 18 ppm/°C,

and layer four had an elastic modulus of 40 GPa and a CTE of 20 ppm/°C (refer to Table 16). Each of the four layers were rigidly constrained at their interfaces and 2816 8-node plane strain thermally-coupled quadrilateral 2D continuum elements were assigned to the model. Symmetry was again employed and the models were created with the same boundary conditions as the previous ones. In contrast to the uniformly loaded trough, each layer was assigned one temperature in the range of $\pm 25^\circ\text{C}$ to $\pm 250^\circ\text{C}$ to stepwise approximate the gradient throughout the model. These observations of the closed form equation were also compared to FEA results and the RMS surface deviations (Table 21) and the best-fit radius of curvature (Table 22) are shown.

Table 21: Comparison of RMS Surface Distortion for Closed Form and FEA Results for Non-Uniform Temperature Loading

Case #	Temperature (°C) in Layer #				FEA RMS Surface Dist. (m)	Best-fit RMS Surface Dist. (m)	Percent Diff.
	1	2	3	4			
1	250	125	62.5	25	3.110E-06	3.134E-06	-0.766
2	-250	-125	-62.5	-25	3.203E-06	3.150E-06	1.646
3	125	62.5	25	250	5.563E-06	5.019E-06	9.777
4	62.5	25	250	125	3.657E-06	3.687E-06	-0.814
5	-62.5	-25	-250	-125	3.687E-06	3.468E-06	5.923
6	25	250	125	62.5	3.127E-06	3.087E-06	1.283
7	-25	-250	-125	-62.5	2.979E-06	3.098E-06	-3.992

Table 22: Comparison of Best-fit Radius of Curvature for Closed Form and FEA Results for Non-Uniform Temperature Loading

Case #	Temperature (°C) in Layer #				FEA Best-fit Radius of Curvature (m)	Closed Form Best-fit Radius of Curvature (m)	Percent Difference
	1	2	3	4			
1	250	125	62.5	25	1.055	0.979	7.240
2	-250	-125	-62.5	-25	0.946	1.025	-8.329
3	125	62.5	25	250	0.756	0.975	-29.005
4	62.5	25	250	125	0.884	0.980	-10.805
5	-62.5	-25	-250	-125	1.158	1.024	11.571
6	25	250	125	62.5	1.018	0.981	3.640
7	-25	-250	-125	-62.5	0.988	1.023	-3.567

In the same manner as the uniform thermally loaded trough, another use of the closed form equations of this current closed form equation is to determine material definitions that provide favorable results in curvature, i.e. zero thermal curvature. For a given temperature change, this provides for a closed form equation that can be used to tailor a system's properties. The design equation solves for any one of the layer CTEs in terms of the remaining layer CTEs, the layer elastic modulus, and the thickness of each layer. Many more cases were examined to test the zero curvature prediction accuracy of the closed form equation (i.e., only the initial curvature of 1 m remaining).

Corresponding to the thermal uniformly loaded cases, layer one had an elastic modulus of 15 GPa, layer two 20 GPa, layer three 35 GPa, and layer four had an elastic modulus of 40 GPa (refer to Table 16). Table 23 shows example values for the RMS surface distortion comparison between the FEA and closed form results and Table 24 shows the comparisons of the best-fit radius of curvature. Thickness of each layer remained at 0.023 m. All tests had the temperatures in each layer set to $T_1 = 250^\circ\text{C}$, $T_2 = 125^\circ\text{C}$, $T_3 = 62.5^\circ\text{C}$, and $T_4 = 25^\circ\text{C}$ and the defined and resultant CTEs are reported. In Tables 23 and 24, the CTEs in bold are the predicted CTEs of that layer that provide a zero curvature from the closed form Equations of (34) and (35). Again, the CTE values predicted only cause zero thermal curvature at the temperatures specified.

Table 23: RMS Surface Distortion for Zero Thermal Curvature for Closed Form and FEA Results from a Non-Uniform Temperature Loading throughout Model (bold font represents dependent CTE)

Case #	CTE ppm/ $^\circ\text{C}$ in Layer				FEA RMS Surface Dist. (m)	Best-fit RMS Surface Dist. (m)	Percent Difference
	1	2	3	4			
1	10.7	1	5	10	3.148E-06	3.140E-06	0.239
2	4.5	30	10	10	4.780E-06	4.530E-06	5.232
3	10	49.7	15	20	6.776E-06	6.157E-06	9.138

Table 23: Continued

4	1	36	5	10	5.289E-06	4.978E-06	5.879
5	10	5	29.1	1	3.323E-06	3.317E-06	0.181
6	20	15	41.6	10	4.039E-06	3.685E-06	8.768
7	10	20	30	5.2	4.341E-06	4.097E-06	5.619
8	10	5	1	12	3.081E-06	3.186E-06	-3.420

Table 24: Best-fit Radius of Curvature for Zero Thermal Curvature for Closed Form and FEA Results from a Non-Uniform Temperature Loading throughout Model (bold font represents dependent CTE)

Case #	CTE ppm/°C in Layer				FEA Best-fit Radius of Curvature (m)	Closed Form Best-fit Radius of Curvature (m)	Percent Difference
	1	2	3	4			
1	10.7	1	5	10	0.997	1.000	-0.302
2	4.5	30	10	10	1.380	1.000	27.531
3	10	49.7	15	20	1.777	1.000	43.715
4	1	36	5	10	1.482	1.000	32.528
5	10	5	29.1	1	1.111	1.000	9.958
6	20	15	41.6	10	1.241	1.000	19.418
7	10	20	30	5.2	1.301	1.000	23.138
8	10	5	1	12	1.025	1.000	2.462

For the cases in which the CTEs of the non-uniform thermal loading model remained set, an absolute difference of about 3.5% was observed for the prediction of the RMS deviation between the FEA and the Best-fit values (Table 21). Also, for this loading configuration, an absolute average of about 11% difference was observed between the FEA and closed form best-fit radii of curvature (Table 22). For the cases that satisfied the zero curvature conditions, the next values apply. The results have an absolute average less than 5% difference for the prediction of the RMS deviation between the FEA deformed-undeformed and the closed form deformed-undeformed surfaces (Table 23). For the radius of curvature, the results had an absolute average of less than 20% difference between the FEA and closed form best-fit radii of curvature (Table 24).

The uniform loading of the multi-layer thermostats is an approach that has been developed for many years, but only successfully predicts the results of uniform cases. This non-uniform approach was developed during this project and has shown success for predicting troughs thermally loaded with a gradient. It has also been shown that this closed form equation also predicts the uniformly loaded trough with equal accuracy to its predecessor.

The methods predicted accurate results for all cases when the end conditions observed were concurrent with the prediction method used. As a result, the concentric cylinders approach best modeled the r-direction roller condition and the multi-material thermostat approach best modeled the free end conditions of uniform and non-uniform thermal loading.

Frequency Analysis

With each approach paired with the appropriate end conditions, frequency analysis was then performed on the multi-layered trough. This was performed to determine if a change in the geometry resulting from thermal loading affected the frequency response of the system. If thermal loading did change the frequency response of the system, this would need to be accounted for in assignment of the CTE in individual layers. Without such awareness, a system could potentially pass static loading but fail under dynamic conditions.

All observations were solely performed in the FEA ABAQUS program. Cases 1 through 8 of the thermally uniform loaded model, for which the results were reported in Tables 17 and 18, were again tested for natural frequency response. All geometric and

material definitions were retained from the previous models and a density of 3960 kg/m^3 was arbitrarily defined for all layers. A density definition was necessary for frequency calculations in the FEA program. The models were loaded with the same temperature as previously reported, but the frequency response was observed post loading. Additionally, a base case in which the model was not thermally loaded was also tested for the frequency response. All cases were then compared to the non-loaded base case to determine the thermal loading effects on the natural frequency of the system.

Under testing for the frequency response of the base case, mode 1 occurred at a frequency of 455.8 Hz. Mode 2 had a frequency of 1908.8 Hz and mode 3 had a frequency of 2575.8 Hz. The results are summarized in Table 25. For the system void of any thermal loading, Figures 29a-c show the first three mode shapes corresponding to the previous frequencies.

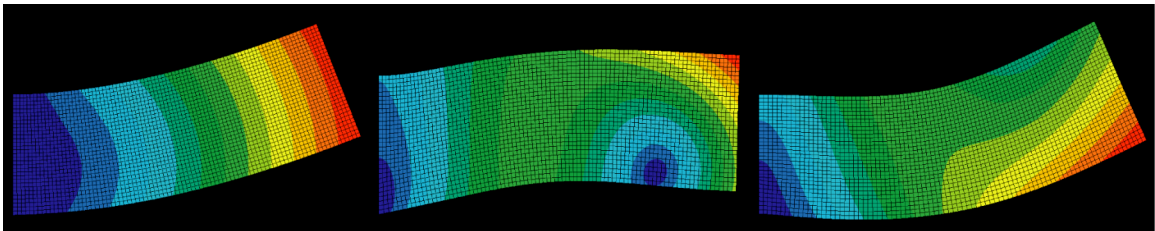


Figure 29: Mode Shapes; a) 1st (left), b) 2nd (center), and c) 3rd (right).

Table 25: ABAQUS Nat. Frequency Calculation for the Thermally Uniform Loaded Model

Mode Number	ABAQUS Frequency (Hz)
1	455.8
2	1908.8
3	2575.8

For the thermally loaded models, all eight cases reported a frequency of 455.8 Hz for the 1st mode, 1908.8 Hz for the 2nd mode, and 2575.8 Hz for the 3rd mode. This

showed that within the range of temperatures tested, -250 °C to 250 °C, no affect by thermal loading on the natural frequency was found. Thus, it was determined that frequency analysis on a trough system with tailored CTE will provide ample results in preloading tests. Determination of the natural frequencies on a post-thermally loaded model will not be necessary to design for dynamic requirements.

CHAPTER 4

AIR FORCE RESEARCH LABORATORY BASELINE MIRROR

Model Development and Validation

The Air Force Research Laboratory (AFRL) provided specifics of a mirror geometry of interest. The spherical mirror was designed with a 50 cm diameter and an f-number ($f/\#$) of 1.0. These characteristics resulted in a 1 m radius of curvature. Shown in Figure 30 is a diametral section profile of this spherical mirror.

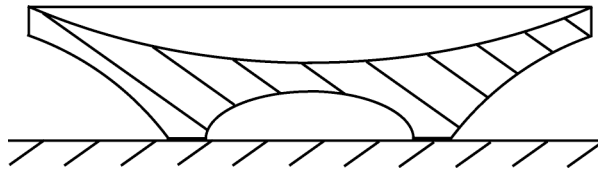


Figure 30: AFRL Mirror Side profile

The model was created axisymmetrically in ABAQUS. This model served as an analytical basis for work performed later on the AFRL model with the same properties. Results from the computationally simpler axisymmetric model were compared to the results of the more complex 3D model to determine the accuracy of the 3D model. Formation of the axisymmetric model is shown in Figure 31, which was composed of solid continuum elements in the substrate and shell elements for the mirror surface coating.

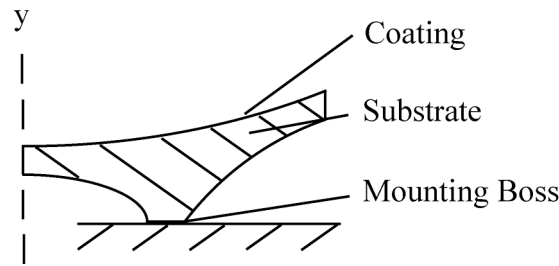


Figure 31: AFRL Axisymmetric Model

To ease validation purposes, the model first underwent analysis in a homogeneous configuration for which Zerodur was used as the material. With properties similar to that of glass (e.g., $CTE = -0.09 \text{ ppm}/^{\circ}\text{C}$), Zerodur is commonly used in mirrors with space based applications and was used here as a reference material. (Future work will include other materials of interest to this project.) With the homogeneous model created, a mesh convergence study was performed to create a meshed model with the highest efficiency in calculation time and accuracy in reported outputs. In the study, the axisymmetric model was loaded with a 100 Pa internal pressure and the displacement of outermost node on the coating was observed (Figure 32).

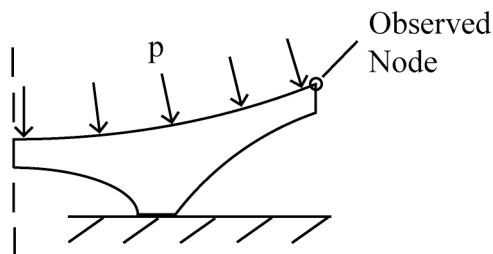


Figure 32: AFRL Loaded Mesh Convergence Model

The first mesh studied contained 286 elements and the observed displacement of the node was -5.034 nm . As the mesh increased in size, the observed displacement converged upon -4.278 nm . To get this value, however, a mesh of 20934 elements was

needed. To obtain the increased accuracy from the initial mesh to the last, a magnitude of 100 times more elements were needed. For all FEA models presented previously and hereafter, a similar mesh convergence study was performed to determine appropriate mesh sizing. For this model, it was determined that 12925 elements obtained a reasonable value of -4.281 nm for displacement at the end (percent difference = 0.07%). With almost half the number of elements, this mesh size of 12925 elements was used for future work performed on this model. Such work performed included frequency analysis and loads applied in the axial and longitudinal directions. The corresponding work and results are discussed in the sections labeled appropriately. With completion of the axisymmetric model, the 3D model was created.

The creation of the 3D model included much of the same characteristics as that of the axisymmetric model. Zerodur, for instance, was used for all layers of the homogeneous model. The geometry as well was kept the same between the two models. Although all defining parameters were held constant, the construction and placement of the shell and solid layers differed. Due to convergence errors in the first 3D model, it was determined to create the model with three layers. The first layer consisted of solid continuum elements as the mirror backing structure. The second layer was the substrate layer between the backing and the coated surface. The substrate consisted of solid continuum elements, but contained a more refined mesh than that of the backing. Each part was meshed independently of the others and this configuration allowed an optimized mesh based on the geometry of each layer. This allowed the model to retain accuracy where needed and reduced computation compared to a model meshed as one layer. The

top layer, at a thickness of 1.0×10^{-5} m, was composed of shell continuum elements and meshed independently of the others. Figure 33 shows this 3D model.

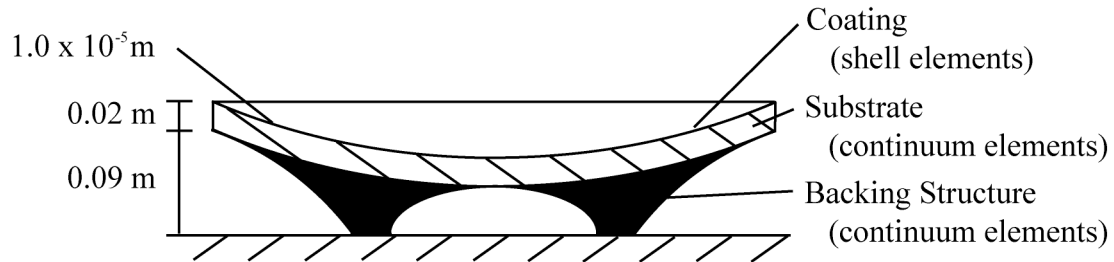


Figure 33: AFRL 3D Three Layer Model

Similarly to the axisymmetric model, the 3D model was studied to determine how refined the mesh needed to be to return accurate results. The first meshed model contained 12318 elements. Like that of the axisymmetric model, a 100 Pa internal pressure was applied the top surface and the outermost node's displacement was observed (Figure 34).

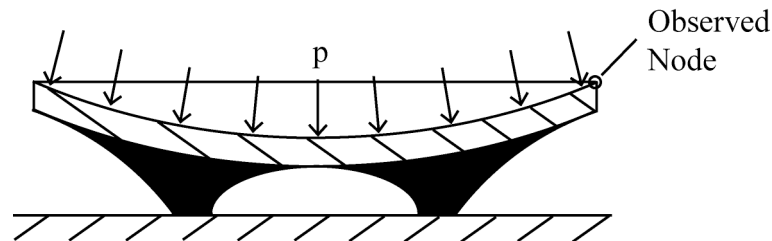


Figure 34: AFRL 3D Loaded Mesh Convergence Model

The mesh of 12318 elements returned a nodal displacement of -4.002 nm. As the meshed was refined, the displacement value converged to -3.986 nm. The final mesh chosen was a mesh of 19820 elements. When the displacements of the outermost nodes were compared between the two final meshed models (axisymmetric and 3D), a difference of 6.9% resulted. This low value gave confidence in the two models and

allowed for further testing to be performed on the 3D model. All work corresponding to the mesh convergence analysis can be found in Appendix B.

Frequency Analysis

With completed axisymmetric and 3D models, the first study performed was a frequency analysis on the FEA models. Built as an axisymmetric model, the analysis of this model only returned the symmetric natural frequency modes of interest. For the rigidly supported model shown in Figure 31, the first three symmetric modes were: 4913.1 Hz, 7626.1 Hz, and 9529.4 Hz. Their respective mode shapes are shown in Figures 35a-c.

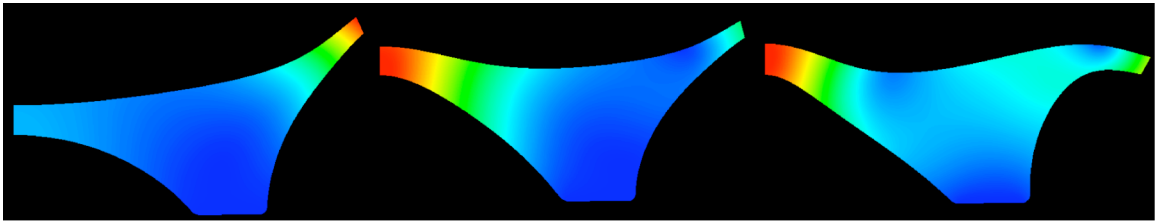


Figure 35: Axisymmetric Mode Shapes; a) 1st (left), b) 2nd (center), and c) 3rd (right).

The 3D model returned all mode shapes (not solely the symmetric ones). To compare, however, only the symmetric modes and shapes will be discussed here. Figure 36a-c show the respective mode shapes for the first three modes.

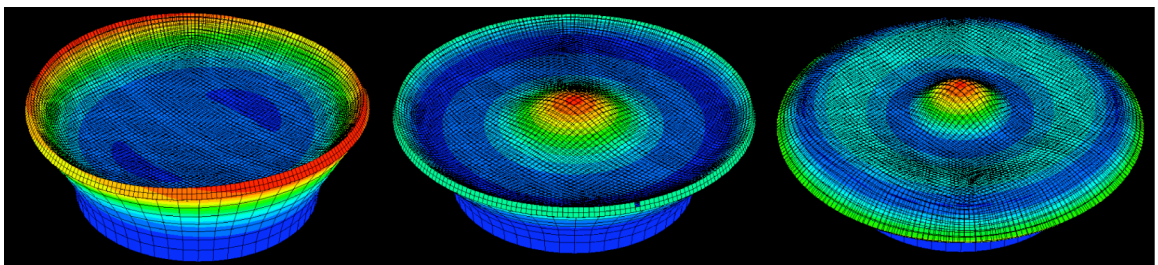


Figure 36: 3D Mode Shapes; a) 1st (left), b) 2nd (center), and c) 3rd (right).

The first mode occurred at 4879.8 Hz, the second at 7643.2 Hz, and the third occurred at 9582.8 Hz. The comparative results between the two models are summarized in Table 26 (a list of the first ten natural frequencies not limited to the symmetric modes and natural frequencies of flat and spherical discs are shown in Appendix B).

Table 26: ABAQUS Natural Frequency Comparison for the 3D and Axisymmetric Homogenous Models

Symmetric Mode Number	ABAQUS Axisymmetric Frequency (Hz)	ABAQUS 3D Frequency (Hz)	Percent Difference
1	4913.1	4879.8	0.68
2	7626.1	7643.2	-0.22
3	9529.4	9582.8	-0.56

2D Surface Distortion Analysis

With successful validation of the static and dynamic solutions, the next step performed was analysis of the surface distortion. The concept of surface distortion considered how a loaded surface deviated from that of an unloaded ideal surface. In 2D Cartesian coordinates, surface deviation was calculated by a comparison of FE coordinates from the deformed (x' , $f(x')$) and initial (x , $f(x)$) coordinates (Figure 37).

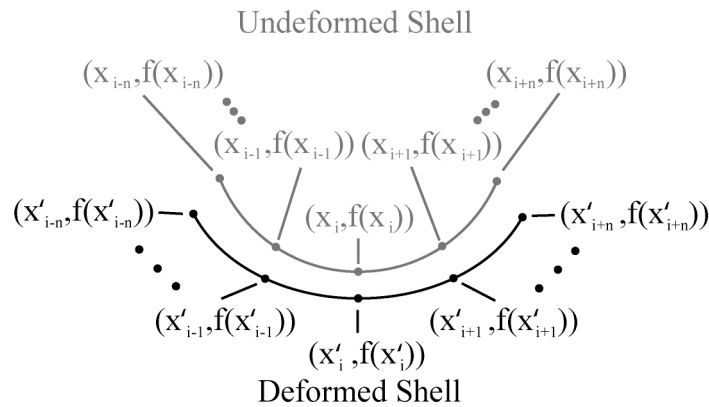


Figure 37: Deformed and Reference Surfaces Defined

To determine the surface distortion, two Root-Mean-Square methods were employed. The first method takes the Δx and $\Delta f(x)$ values of the current node and subtracts the previous node's values. The values are summed up over the entire set of nodes (N) and then the square root is taken over the final answer to get the Root-Mean-Square. The following Equation (38) was used:

$$RMS = \sqrt{\frac{\sum_{i=1}^N ((f(x_i') - f(x_i)) - (f(x_{i-1}') - f(x_{i-1})))^2 (x_i - x_{i-1})}{N \left[\sum_{i=1}^N (x_i - x_{i-1}) \right]}} \quad (38)$$

It is noted that $\Delta x = x_i - x_{i-1}$ in Equation (38) can be calculated with either the deformed or the undeformed coordinates. Also, if the reference surface used is flat, the unprimed $f(x)$ terms drop out and Equation (38) simplifies.

A second method, the approach of summing over one less than the entire set of nodes, was employed. Intrinsically, this second method causes discrepancy on which end of the node set to begin, the left or right. To cancel the arbitrary decision both the Left Hand (LH) and the Right Hand (RH) methods were used and then an average was taken of both. This method takes the x and $f(x)$ values of the current node and subtracts the previous node's values as in the first method, but includes one less node than the entire set as was stated. This method uses all the nodes from the first to the N-1 as the Left Hand calculation and the Right Hand calculation uses all the nodes from the second to the last, Nth, node. Again, the two RMS values are then averaged to get a final RMS. The equations used for the second method are:

$$RMS_{RH} = \sqrt{\frac{\sum_{i=1}^{N-1} ((f(x_i') - f(x_i)) - (f(x_{i-1}') - f(x_{i-1})))^2 (x_i - x_{i-1})}{N \left[\sum_{i=1}^{N-1} (x_i - x_{i-1}) \right]}} \quad (39a)$$

$$RMS_{LH} = \sqrt{\frac{\sum_{i=2}^N ((f(x_i') - f(x_i)) - (f(x_{i-1}') - f(x_{i-1})))^2 (x_i - x_{i-1})}{N \left[\sum_{i=2}^N (x_i - x_{i-1}) \right]}} \quad (39b)$$

$$RMS_{AVE} = \frac{1}{2} (RMS_{RH} + RMS_{LH}) \quad (40)$$

For a mirror loaded with a 400 Pa surface pressure, the surface distortion using Equation (38) was calculated to be $RMS = 7.412 \text{ nm}$. For the same mirror loaded with the same internal pressure, the surface distortion given by the second method (Equations 39 and 40) was calculated to be $RMS = 7.361 \text{ nm}$. This resulted in a difference of 0.13% between the two methods. (See Appendix B for full details).

3D Surface Distortion Analysis

To determine the surface distortion of the axisymmetric model, the methods of the 2D Surface Distortion section were employed. For the 3D model, however, two 3D methods were needed. Ku, (1966), provided a method for determining how the coordinates of a deformed sphere deviated from a perfect sphere⁹. The displacements of each node were incorporated into the standard deviation for the three spatial variables and an unweighted RMS was calculated¹⁰. By unweighted, it is meant that the deviations of the points only are considered. (A weighted method, which is not presented here,

incorporates the displacements of individual surface areas that the nodes contain. This deformed surface area is then used to calculate the deviance from the perfect surface's area contained by the same nodes). In the unweighted method, the standard deviation, S , is mathematically equivalent to a RMS calculation. The equation for a sphere with origin of $(x = 0, y = 0, z = 0)$ is:

$$f(x_i, y_j, z_k) = \sqrt{x_i^2 + y_j^2 + z_k^2} \quad (41)$$

Equation (41) was substituted into the equation for the standard deviation of the function f as:

$$S_f = \sqrt{\left(\frac{df}{dx_i}\right)^2 S_x^2 + \left(\frac{df}{dy_j}\right)^2 S_y^2 + \left(\frac{df}{dz_k}\right)^2 S_z^2} \quad (42)$$

where the standard deviations of x, y, z were given by:

$$S_x = \sqrt{\frac{1}{N} \sum_{i=1}^N (x'_i - x_i)^2} \quad S_y = \sqrt{\frac{1}{N} \sum_{j=1}^N (y'_j - y_j)^2} \quad S_z = \sqrt{\frac{1}{N} \sum_{k=1}^N (z'_k - z_k)^2} \quad (43)$$

After all the partial derivatives were taken, the final equation became:

$$RMS = \left(\frac{x_i^2}{x_i^2 + y_j^2 + z_k^2} S_x^2 + \frac{y_j^2}{x_i^2 + y_j^2 + z_k^2} S_y^2 + \frac{z_k^2}{x_i^2 + y_j^2 + z_k^2} S_z^2 \right)^{1/2} \quad (44)$$

Because the partial differentiation in Equation (44) is not summed over the entire set of N nodes, the RMS is calculated relative to one point. For instance, an RMS can be calculated relative to the origin of $(0,0,0)$ or relative to the focal point of a mirror. A second unweighted method was employed to validate calculated values of the previous method. For each node i , the second method uses the distance between the unloaded ideal sphere of coordinates (x, y, z) and the deformed coordinates of the loaded sphere $(x',$

$y', z')$. The 3D Pythagorean distance formula, below, was then used to determine this distance.

$$d^2 = (x'_i - x_i)^2 + (y'_j - y_j)^2 + (z'_k - k_k)^2 \quad (45)$$

Once the distance was determined, all distances for all nodes were summed and then divided by the number of nodes N . As is shown below the square root of this value was then taken to determine the RMS surface distortion¹⁰.

$$RMS = \sqrt{\sum_{i=1}^N \frac{d^2}{N}} \quad (46)$$

For both methods the initial and final coordinates of the mirror surface were reported from ABAQUS. A spreadsheet was set-up that used coordinates to determine the surface distortions based on the above equations.

Surface Pressure Validation

Both models, the axisymmetric and the 3D models were analyzed in ABAQUS with a pressure of 100 Pa on the mirror surface (Figures 32 and 34, respectively). The surface of the shell coating was examined to determine the surface distortion. For the axisymmetric case, Method 1 produced a RMS of 0.036 nm and Method 2 produced a RMS of 0.035 nm, which resulted in a difference of 3.06%. The 3D model resulted in RMS values of 2.805 nm and 2.338 nm respective to Methods 1 and 2, a difference of 16.7%. Between the two models for the same loading conditions and geometries, the RMS values were expected to be closer. This, however, could be due to the larger

number of nodes surveyed for the 3D model. With a larger sample more accurate results are expected. The models were compared under one more load condition.

Axial Load Validation

For this test, a gravitational load of 1g was applied to the entire model in the vertical (axial) direction (y-direction shown in Figure 38). From ABAQUS the initial and final coordinates of the coating were reported. For the axisymmetric model, Methods 1 and 2, respectively, calculated a RMS of 1.189×10^{-11} m and 1.186×10^{-11} m, a difference of 0.27%. For the 3D case RMS values obtained were 7.051 nm for Method 1 and 6.233 nm for Method 2. This resulted in a difference of 11.6%. Again, the discrepancies between the two models most likely occurred due to sampling size.

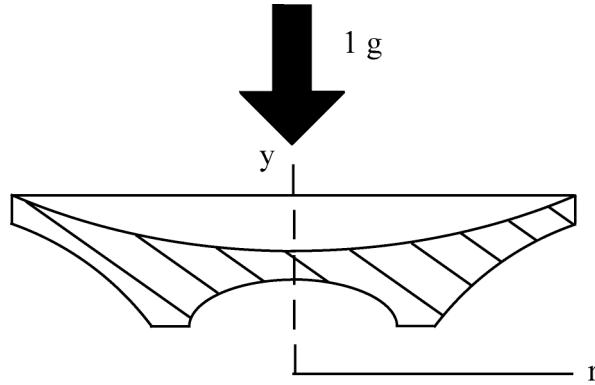


Figure 38: Axial Loaded 3D AFRL Model

Longitudinal Load Validation

For this test a gravitational load of 1g was applied to the entire model in the horizontal longitudinal direction (r-direction shown in Figure 39).

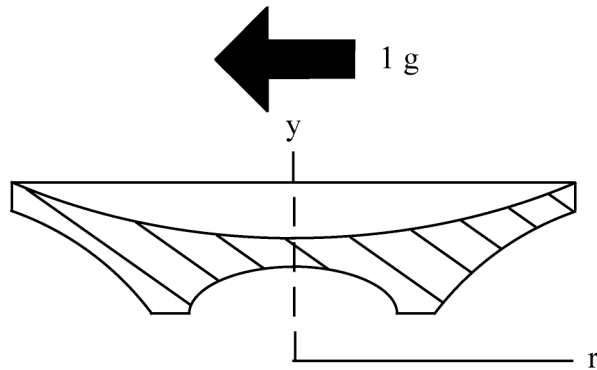


Figure 39: Longitudinally Loaded 3D AFRL Model

The axisymmetric model, however, was not tested because an axisymmetric model could not be loaded in an anti-symmetric manner. Therefore, only the 3D model was observed and the RMS values were as follows: 12.230 nm and 5.627 nm for Methods 1 and 2, a difference of 53.99%. (For the further detail on this preliminary work for surface distortion analysis covered in these sections, please see Appendix B).

CHAPTER 5

APPLICATION OF CLOSED FORM EQUATION

Layered Trough Geometry

The non-uniform thermal loading closed form Equation (30) proved reliable as a tool for tailoring the CTE of a multi-layered trough mirror system. The previous work was performed on rudimentary models necessary to validate the method. The models were given arbitrary material definitions and basic design parameters. The following work, however, was completed to show the method's usefulness in real world applications.

The model tested was that of the 4-layer infinitely long nested troughs loaded with a temperature gradient. Two loading configurations were examined; the reflecting surface as the hot side (back side as the cold side) and the reflecting surface as the cold side (back side as the hot side). Both of these scenarios are environments that an optical structure may be exposed to in space. The first scenario of the reflecting surface as the hot side would occur if the mirror faced the sun. The backside would thus be directed away from the sun's radiation and would be exposed to the cold temperatures of space. The second scenario is the reverse. The reflecting surface directed away from the sun's radiation would cause the backside to be the hotter side and the reflecting surface to be the colder side. In addition to the sun as a source of heat generation, the backside would most commonly be the attachment point to the structure of the telescope and may pick up

heat from the internal sources such as electronics. These two loading scenarios with different temperature gradients were tested on a mirror system with tailored CTE.

The mirror system contained four layers of filled composites. The resin of the composite was a cyanate ester resin, CTD-403. The filler was a zirconium tungstate, Zr_2WO_8 , which when added to a composite lowers the bulk CTE, as stated in Chapter 1⁸. The neat resin only, CTD-403, has a CTE of 52-61 ppm/ $^{\circ}\text{C}$ in the temperature range of 0 $^{\circ}\text{C}$ -100 $^{\circ}\text{C}$. In published and unpublished data from partners in this work, CTD-403 mixed with 30 wt% Zr_2WO_8 drops the CTE of the composite to roughly 39-45 ppm/ $^{\circ}\text{C}$ for the same temperature range¹⁵. For 48 wt% Zr_2WO_8 added to the composite, the CTE drops to approximately 30-40 ppm/ $^{\circ}\text{C}$ and for 70 wt% Zr_2WO_8 added, the CTE drops to roughly 19-23 ppm/ $^{\circ}\text{C}$. With an increase in weight percent of Zr_2WO_8 , the bulk CTE of the composite decreases, however, a trade off exists. With a decrease in bulk CTE, the yield strength of the material also decreases. This necessitates attention to the mechanical strength of a composite when assigned to a system. For instance, a system may be thermally stable (appropriately defined CTE), but fail mechanically. The material properties of zirconium tungstate mixes are summarized in Table 27. In this table, the interaction of CTE, yield strength, ultimate tensile strength, and elastic modulus is shown. Table 28 displays the CTE range for each weight percent zirconium tungstate at each temperature used in the two loading conditions.

Table 27: Properties of CTD-403 Mixed with Zr_2WO_8 (wt%) at Shown Temperatures

Material Property	0 wt%	30 wt%	48 wt%	70 wt%
CTE, α (ppm/ $^{\circ}\text{C}$) between 0 $^{\circ}\text{C}$ -100 $^{\circ}\text{C}$	52-61	39-45	30-40	19-23
Elastic Modulus, E (GPa) at room temp.	4.727	6.000	7.250	8.571
Yield Strength, S_y (MPa) at room temp.	52.0	43.0	33.5	24.0
Ultimate Strength, S_u (MPa) at room temp.	90.0	71.0	53.0	34.0

Table 28: CTE (α , ppm/ $^{\circ}\text{C}$) of CTD-403 Mixed with Zr_2WO_8 (wt%) at Shown Temperatures

Temperature ($^{\circ}\text{C}$)	0 wt%	30 wt%	48 wt%	70 wt%
25	52-54.25	39-40.5	30-32.5	19-20
50	54.25-56.5	40.5-42	32.5-35	20-21
75	56.5-58.75	42-43.5	35-37.5	21-22
100	58.75-61	43.5-45	37.5-40	22-23

Decreasing Temperature Gradient Load

The first model tested had a temperature gradient of 100°C - 25°C from the reflecting surface to the backside. To approximate this gradient each layer of the 4-layer model was assigned a temperature corresponding linearly to the drop in temperature. Layer 1 was set to 100°C , layer 2 was set to 75°C , layer 3 was set to 50°C , and layer 4 was set to 25°C . Figure 40 shows the assignment of temperatures to each layer, as well as a definition of layer numbers. Layer 1 is the top reflecting surface and layer 4 is the backside from which the system would be mounted.

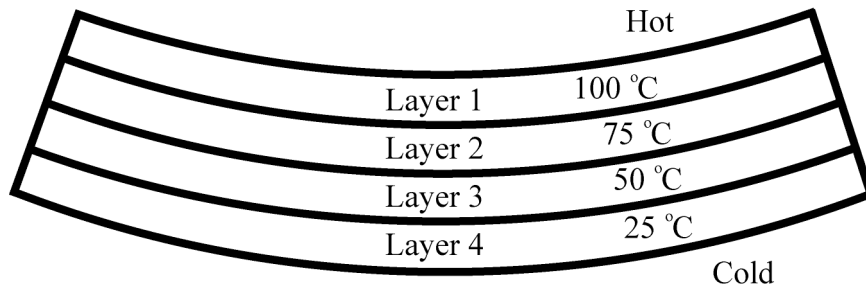


Figure 40: Layer and Decreasing Temperature Assignments

As stated, layer 4 was the layer in contact with the system mounts. These mounts distribute mechanical loads from the main body of the system to the optic. The highest mechanical stresses are usually seen during liftoff into space and are distributed into the base layer 4. Layer 4 therefore needed the assignment of the composite with the highest

yield strength (S_y) and ultimate tensile strength (S_u). The highest yield strength and ultimate tensile strength of the four materials in consideration existed in the neat CTD-403 resin (but also the highest CTE). Layer 4 was thus assigned the neat resin for all models tested.

The assignment of the material for layers 2 and 3 were arbitrary. Layer 1 was then iteratively solved for to determine the necessary CTE and elastic modulus that would cause the system to have no thermally induced curvature. This was done by plugging in the material properties of layers 2-4 into Equation (34). The first scenario investigated was a trough that had the following definitions: layer 4 – 0 wt% Zr_2WO_8 (neat CTD-403), layer 3 – 70 wt% Zr_2WO_8 , and layer 2 – 70 wt% Zr_2WO_8 . To begin iteratively solving for the CTE of layer 1, it was initially assigned the elastic modulus of 70 wt% Zr_2WO_8 . Each material had a range of CTE values that were input into Equation (34). From the individual ranges, Equation (34) predicted a range of CTE values necessary in layer 1 to create zero thermal curvature. The calculated range, however, was above the tested CTE range of 70 wt% Zr_2WO_8 , but was within the range of 48 wt% Zr_2WO_8 . The next step reran the model with layer 1 given the elastic modulus of 48 wt% Zr_2WO_8 . This time, the output CTE range fit the material definition it was given. The closed form Equation (34) determined that a system with the following wt% Zr_2WO_8 definitions in layer 1 through 4 would produce zero thermal curvature: 48%, 70%, 70%, and 0%, respectively. Figure 41 shows the initial lay-up of the CTE in each layer and the resultant lay-up required.

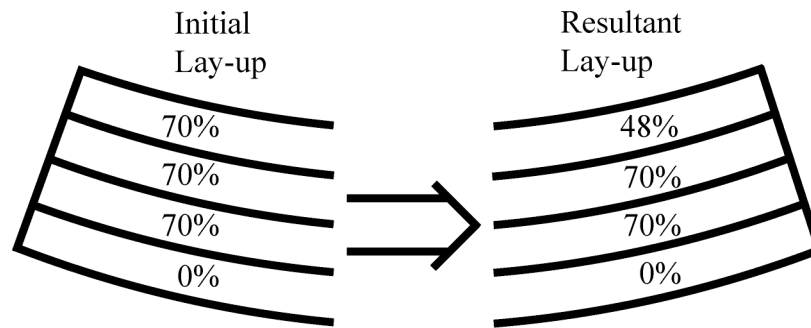


Figure 41: Case 1 CTE Lay-up

Finite element analysis was performed on this model to validate the zero thermal curvature prediction, test the RMS surface deformation and to analyze the internal stresses induced in each layer. The stresses were observed to determine if the yield strength limits in any of the four layers was breached. First, the percent error between the RMS surface deformation of FEA deformed to undeformed and a best-fit deformed to FEA undeformed was an absolute value of 1.06%. The best-fit radius of curvature from the FEA data reported a value of 1.040 m. This was a 3.86% difference from the expected 1.000 m radius of curvature (only initial radius of curvature remained, no thermal curvature existed). This showed that the closed form Equation (34) predicted CTE values that worked. The next step examined the internal stresses reported in the FEA model.

The internal stresses in the FEA model are shown in Figures 42 and 43. Figure 42 shows the internal stresses in the x-direction and Figure 43 shows the y-direction internal stresses. The higher stresses are denoted with a red-orange hue and the lower stresses are green in tint. For the x-direction (Figure 42), stresses were examined away from the free end because they were fully developed void of the end condition affects.

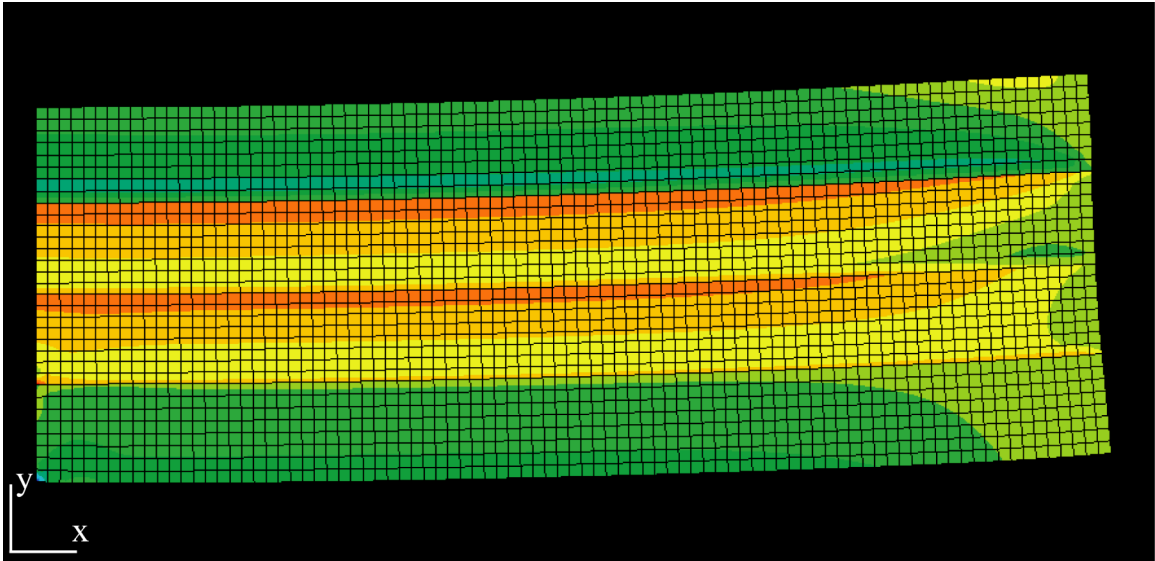


Figure 42: Case 1 (48%, 70%, 70%, 0%) Internal Stresses in x-direction.
Maximum Tensile Stress is 10.97 MPa

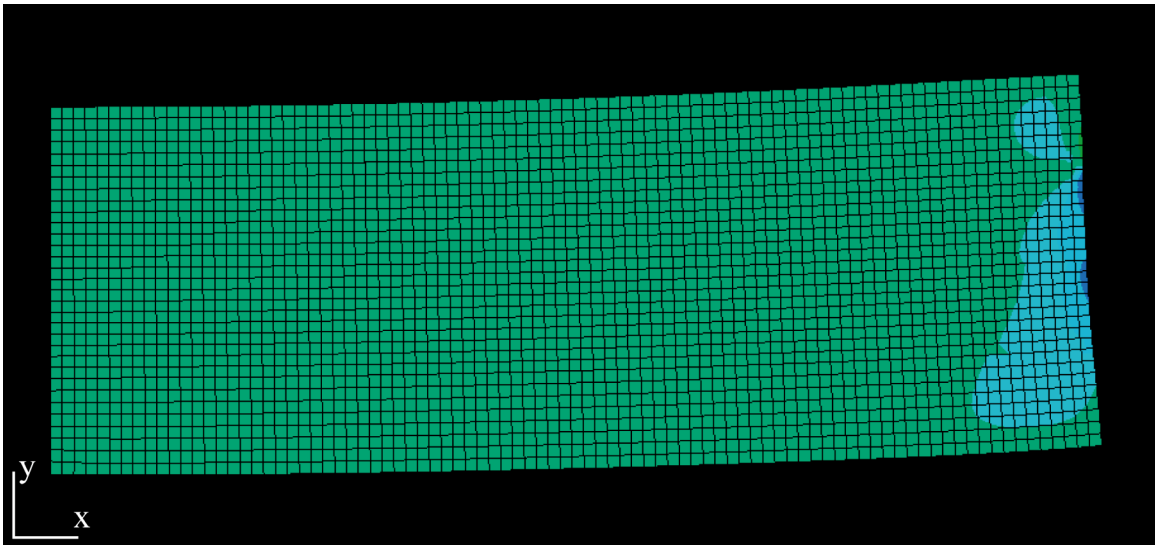


Figure 43: Case 1 (48%, 70%, 70%, 0%) Internal Stresses in y-direction.
Maximum Compressive Stress is 20.44 MPa

The largest stress experienced in the x-direction occurred in both layers 2 and 3. This maximum x-direction stress was a tensile 10.97 MPa, which is 45.7% of the yield strength of 70 wt% Zr_2WO_8 as assigned to layers 2 and 3. This low load allows for

additional loading, gravity for instance. A compressive stress was also observed in both layers 2 and 3 on the free end of the trough as shown in Figure 43. This end was compressed with 20.44 MPa in the radial direction of the trough. Figure 43 shows that no other stress existed in the radial direction except for the stress caused from end conditions. This matched the assumptions of the infinitely long nested troughs.

Five additional tests were performed with the same temperature gradient and similar steps as the previous test. First, layer 1 was always defined with the properties of 0 wt% Zr_2WO_8 and the next two layer's properties were arbitrarily defined. The elastic modulus of layer 1 was defined and the corresponding CTE was determined via the closed form Equation (34) and all of the previously defined properties. If possible, the material of layer 1 was then adjusted to the CTE range outputted by the closed form equation. The CTE ranges were limited by the materials reported in Table 28. For a CTE required by the closed form equation, but not achievable with the material properties reported, it was noted that a hypothetical material with a wt% Zr_2WO_8 between known values could work. The resultant material lay-ups of all six cases are shown in Table 29 with the untested but probable cases shown in bold.

Table 29: Material Lay-ups of Zr_2WO_8 (wt%) in 4-Layer Troughs for Six Cases (Temps Shown for Each Layer)

Case #	Layer 1 (100 °C)	Layer 2 (75 °C)	Layer 3 (50 °C)	Layer 4 (25 °C)
1	48	70	70	0
2	0	70	0	0
3	30-48	0	70	0
4	0-30	0	0	0
5	0-30	48	30	0
6	30	30	48	0

The final 4-layer material definitions that worked for the known materials resulted in a mirror with zero thermal curvature. As shown in Table 29, three scenarios were tested (non-bold cases). All the conditions of these three tests produced favorable results. For the RMS surface distortion resultant from the three tests, the percent difference between FEA deformed to undeformed and a best-fit deformed to FEA undeformed was under an absolute value of 2%. Less than 6% difference was calculated between predicted radii of curvature and FEA resultant radii of curvature. These results together show the accuracy of the closed form equation. The RMS comparison results of each case are reported in Table 30 and the radius of curvature comparison is shown in Table 31. For detailed work on the closed form calculation of CTE values and other related work for each case including RMS surface calculations, see Appendix C.

Table 30: Comparison of RMS Surface Distortion for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios

Case #	FEA RMS Surface Distortion (m)	Best-fit RMS Surface Distortion (m)	Percent Difference
1	3.100E-06	3.051E-06	1.581
2	3.080E-06	3.097E-06	-0.552
6	3.154E-06	3.118E-06	1.141

Table 31: Comparison of Best-fit Radius of Curvature for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios

Case #	FEA Best-fit Radius of Curvature (m)	Closed Form Best-fit Radius of Curvature (m)	Percent Difference
1	1.039	1.000	3.754
2	1.053	1.000	5.033
6	1.058	1.000	5.482

The RMS surface distortions for all three cases are within acceptable values for optics. None of the surfaces distorted beyond a reflective quality surface. Also, the percent difference between the predicted radius of curvature (1 m) and the FEA radius of

curvature shows that the FEA models do not deviate far from the prediction theory. The three cases reported validate the zero curvature prediction equations and show that the equations provide for thermal stability design. However, attention to mechanical stability has yet to be discussed.

The internal stresses of the three cases were examined. Observation was given to the highest x-direction stresses and the layer to which it corresponded. Also, y-direction stresses were examined to determine the end affects. Although the predicted CTE of the system indicates a favorable zero thermal curvature, a test would fail if the maximum stress in any layer was higher than its respective yield strength. Table 32 shows the maximum stresses and which layer it occurred in for each test. It also compares the maximum stress against the yield strength of the material assigned to the respective layer.

Table 32: Stress Analysis for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios (x-direction stress)

Case #	Layer Number of Max. Stress	Yield strength of Material in Layer	Max. Stress in Respective Layer	Percent of Yield strength
1	2	24 MPa	10.97 MPa	45.7
2	2	24 MPa	16.87 MPa	70.3
6	2	33.5 MPa	8.71 MPa	26.0

None of the three tests reported values that violated the yield strength of the materials for which the highest stressed layer was assigned. In fact, the worst case (case 2), had a safety factor of about 1.4. The FEA analysis showed that the material properties defined for the scenarios provide for both thermal stability and mechanical stability.

The end affects of the loaded trough are reported in Table 33. For all three cases, the stresses observed were compressive. This is beneficial with a hypothetical crack situation. A compressive load will close any crack that could form at the interface or in

any of the layers. Also, compressive yield strengths are often higher than tensile strengths and therefore are typically not the limiting factor in design. However, a non-uniform stress like those seen in the end of the optic will distort images. Regardless, this thesis was not concerned with image quality, but with CTE design and manipulation. The end affects are therefore reported, but not analyzed in depth (Table 33).

Table 33: Free End Stress Analysis for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios (y-direction stress)

Case #	Layer Number of Maximum Stress	Maximum Stress in Respective Layer
1	2	-20.44 MPa
2	2	-20.88 MPa
6	2	-10.33 MPa

Increasing Temperature Gradient Load

The next loading conditions had a temperature gradient of 25 °C - 100 °C from the reflecting surface to the backside. The gradient load was again approximated by assigning a linear increase in temperature to each layer of the 4-layer model. Layer 1 was set to 25 °C, layer 2 was set to 50 °C, layer 3 was set to 75 °C, and layer 4 was set to 100 °C. Figure 44 shows the assignment of temperatures to each layer. Layer 1 is the reflecting surface and layer 4 is the backside from which the system would be mounted.

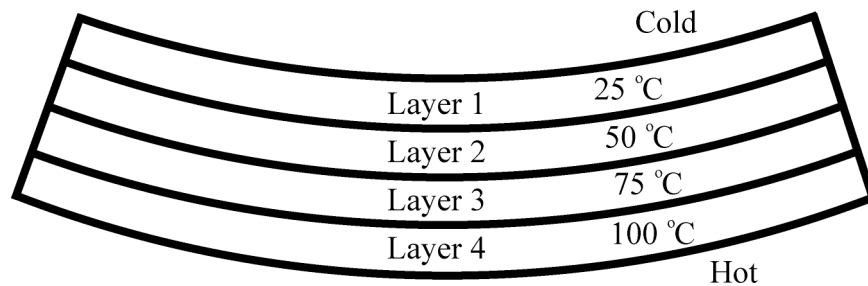


Figure 44: Layer and Increasing Temperature Assignments

Layer 4 was again assigned the neat resin for all models tested and layers 2 and 3 were arbitrarily assigned. Although arbitrary, the increasing temperature gradient did not allow for many different material assignments in these layers, because layer 4 was assigned the neat CTD-403. This was the material with the highest CTE in this study, and it was assigned the highest temperature. Recall that the interface of layers 2 and 3 acted as a centerline for expansion or contraction. Due to this, layer 3 needed a material assignment with lower CTE than layer 4 and layers 1 and 2 needed high CTE values.

The first model had layer 3 set to 70 wt% Zr_2WO_8 , the material with the lowest CTE range in this study. Layer 2 was assigned the properties of the neat resin and layer 1 was given the elastic modulus of the neat resin, as well. After plugging in the material properties into the closed form Equation (34), it was determined that layer 1 at 25 °C in fact needed a CTE in the range of the neat resin to produce an overall zero thermal curvature. Unlike the models of the decreasing temperature gradient, the assignment of CTE to individual layers for the increasing gradient was easier to perform by inspection. No extra iterations were needed after the initial lay-up. The first model with the layer assignments of wt% Zr_2WO_8 as 0, 0, 70, and 0 is shown in Figure 45 (case 7).

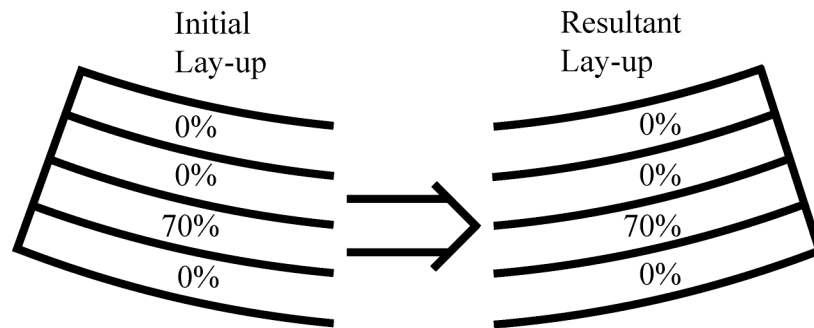


Figure 45: Case 7 CTE Lay-up

Two other models were analyzed with the closed form Equation (34) to determine the arrangement's validity as a zero thermal curvature configuration. The second lay-up (case 8) had the following wt% Zr_2WO_8 : 0, 0, 48, and 0. The third lay-up (case 9) was: 0, 0, 30, and 0. Of these two, only the former configuration provided a system that would produce zero thermal curvature. The 0, 0, 30, and 0 did not work, but from this test it was determined that any arrangement with layer 3 possessing a CTE in the range of 48-70 wt% Zr_2WO_8 would work. Other assignments slightly outside of either end of the range might also work, but were not tested. For a summary of the possible material layer assignments for the increasing temperature gradient loading, see Table 34 (bold print indicates untested conditions).

Table 34: Material Definitions of Zr_2WO_8 (wt%) in 4-Layer Troughs for Three Cases (Temps shown for each layer)

Case #	Layer 1 (25 °C)	Layer 2 (50 °C)	Layer 3 (75 °C)	Layer 4 (100 °C)
7	0	0	70	0
8	0	0	48	0
9	0	0	48-70	0

Like cases 1, 2, and 6 of the previous temperature loading, finite element analysis was performed on these models to validate the zero thermal curvature prediction, test the RMS surface deformation and to analyze the internal stresses induced in each layer. For case 7, the percent error between the RMS surface deformation of FEA deformed to undeformed and a best-fit deformed to FEA undeformed was 4.221%. The best-fit radius of curvature from the FEA data reported a value of 0.948 m. This was a -5.485% difference from the expected 1.000 m radius of curvature. Table 35 displays the RMS

surface distortion comparison for both cases 7 and 8. Table 36 shows the comparison between FEA and closed form best-fit radii of curvature for both cases.

Table 35: Comparison of RMS Surface Distortion for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios

Case #	FEA RMS Surface Distortion (m)	Best-fit RMS Surface Distortion (m)	Percent Difference
7	3.292E-06	3.154E-06	4.221
8	3.192E-06	3.213E-05	-0.658

Table 36: Comparison of Best-fit Radius of Curvature for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios

Case #	FEA Best-fit Radius of Curvature (m)	Closed Form Best-fit Radius of Curvature (m)	Percent Difference
7	0.948	1.000	-5.485
8	0.946	1.000	-5.708

Again, the RMS distortion values and the radius of curvature comparison show that the zero thermal curvature predicted material properties create a thermally stable optic.

Attention is next given to the mechanical stability of the system.

The internal stresses reported in ABAQUS were examined. The corresponding figures are omitted, but the values are reported. The largest stress experienced in the x-direction for case 7 occurred in layer 3. This maximum x-direction stress was a tensile 18.6 MPa, which is 77.5% of the yield strength of 70 wt% Zr_2WO_8 as assigned to layer 3. A compressive stress was also observed in layer 3 on the free end of the trough. This end was compressed with 20.67 MPa in the y-direction of the trough. No other stress existed in the y-direction direction except for the stress caused from end conditions. The results of case 8 are similar and Table 37 summarized x-direction stresses for both tests. Again, none of the observed x-direction stresses violated the yield strength in that layer and the worst case had a 1.3 safety factor. Table 38 shows y-direction stresses in cases 7 and 8.

Table 37: Stress Analysis for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios (x-direction stress)

Case #	Layer Number of Max. Stress	Yield strength of Material in Layer	Max. Stress in Respective Layer	Percent of Yield strength
7	3	24.0 MPa	18.60 MPa	45.7
8	3	33.5 MPa	10.28 MPa	30.7

Table 38: Free End Stress Analysis for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios (y-direction stress)

Case #	Layer Number of Maximum Stress	Maximum Stress in Respective Layer
7	3	-20.67 MPa
8	3	-10.98 MPa

AFRL Geometry

The previous models compared zero thermal curvature prediction of the closed form method to FEA models of similar uniform geometry. The following work, however, compares the closed form method to the spherical AFRL geometry. The multi-layer thermostat (MLT) geometry was overlaid onto the AFRL geometry (Figure 46). The FEA results from this model were then compared to the results of the closed form equations that have been validated for the geometry with uniform layers.

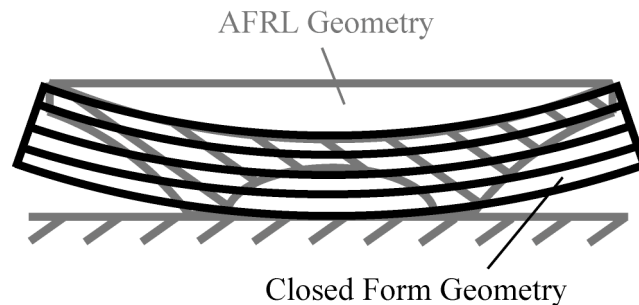


Figure 46: AFRL Geometry Overlaid with MLT Closed Form Geometry

This exercise was performed to determine how well the closed form equations work for more realistic geometries. The closed form method has been validated in this work for the uniform layers geometry, but its range of efficacy was of interest. Such work could provide a base for modification of the closed form equations to compensate for specific geometries.

The FEA models with the spherical AFRL cross section were given the same properties of zirconium tungstate as the previous FEA infinitely long trough models; all elastic modulus, CTE, and layer thicknesses were maintained. The AFRL model was given similar layer definitions and the gradient thermal loading scenarios were also maintained (Figure 47).

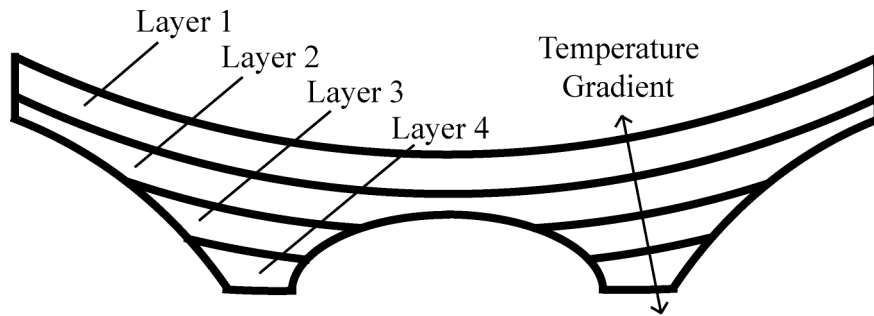


Figure 47: AFRL Mirror Layers and Temperature Loading Scenarios

Similar to the model with nested infinitely long trough layers, layer 1 was defined as the top surface of the optic and layer 4 was assigned to the bottom. Layers 2 and 3 were the mid layers. Each layer was assigned a thickness of 0.023 m, as defined previously. Both temperature gradient loading scenarios were applied, the decreasing gradient (layer 1 the hot side) and the increasing gradient (layer 1 the cold side). To create zero thermal curvature in the closed form method, the CTE of layer 1 was determined in terms of the

material properties of the remaining layers. Again, using zirconium tungstate's material properties and the prediction results of the previous section, five scenarios of layer definitions provide for zero thermal curvature via the closed form method (Table 39).

Table 39: Material Definitions of Zr_2WO_8 (wt%) in 4-Layer AFRL Axisymmetric Mirror for Five Cases (Temps shown for each layer)

Case	Layer 1 (100 °C)	Layer 2 (75 °C)	Layer 3 (50 °C)	Layer 4 (25 °C)
1	48	70	70	0
2	0	70	0	0
6	30	30	48	0
7	0	0	70	0
8	0	0	48	0

The closed form results have previously been discussed, however, the material definitions of Table 39 were input into the AFRL FEA axisymmetric model. The results were then compared to the closed form method and are shown in Tables 40 and 41. Table 40 shows the comparison for RMS surface distortion and Table 41 shows the curvature comparison between the AFRL cross section FEA model and the closed form model (see Appendix C for detailed work on these cases including RMS surface distortion calculations).

Table 40: Comparison of RMS Surface Distortion for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios on AFRL Mirror and MLT Trough

Case #	AFRL FEA RMS Surface Distortion (m)	MLT FEA RMS Surface Distortion (m)	Percent Difference
1	3.992E-06	3.051E-06	23.585
2	4.714E-06	3.097E-06	34.296
6	4.196E-06	3.118E-06	25.699
7	4.174E-06	3.154E-06	24.444
8	4.182E-06	3.213E-06	23.163

Table 41: Comparison of Best-fit Radius of Curvature for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios on AFRL Cross Section and MLT Trough

Case #	FEA Best-fit Radius of Curvature (m)	MLT Best-fit Radius of Curvature (m)	Percent Difference
1	1.038	1.000	3.690
2	1.051	1.000	4.825
6	1.051	1.000	4.825
7	0.969	1.000	-3.213
8	0.955	1.000	-4.675

The increase in the RMS surface distortion values from the equivalent cases for the uniform layered models shows that the optical surface of the AFRL cross-section model was not as smoothly deformed. The best-fit RMS surface distortion is from the uniform layered models. An increase in the percent difference between this values and the FEA RMS shows that the AFRL axisymmetric model did not distort to a constant radius. Although the FEA RMS values showed potential for optical quality surfaces, an increase in surface roughness shows that the FEA models deviated from the closed form theory. The radius of curvature comparison, however, suggests that the theory might work. The radius of curvature is calculated from a best-fit approximation of the deformed surface. It does not account for the magnitudes of the perturbations between the undeformed and deformed surfaces. However, the RMS values do provide for these magnitudes and, although the percent differences are low for the radius of curvature comparison, the increase RMS calculations showed that the surface has increased in distortion when compared to the equivalent cases for the uniform layered model. Regardless, the closed form equation produced approximate values for necessary CTE that achieved zero thermal curvature in the optic. This showed promise that the equations

can be used as a tools to find rough estimates of required CTE definitions or that the equation might be modifiable to a specific geometry.

With thermal stability considered, mechanical strength was next examined.

ABAQUS was used to report maximum x-direction stresses in the AFRL trough. Case 1 x-direction stresses are shown in Figure 48.

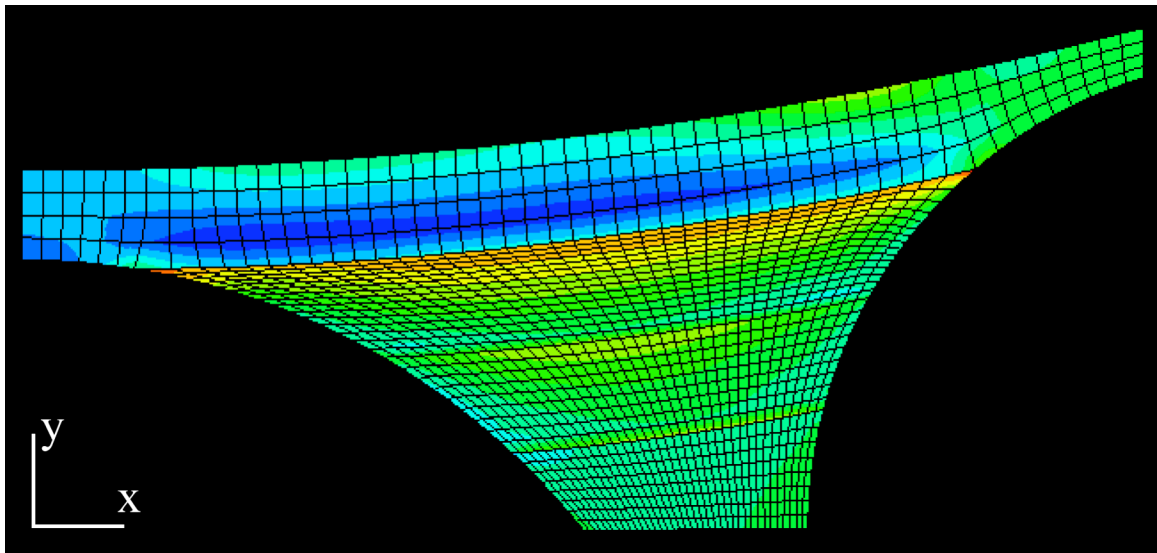


Figure 48: Case 1 AFRL Axisymmetric Model x-direction Stress

For case 1 (Figure 48), the maximum stress observed in the x-direction was 14.66 MPa. This was a shearing stress that occurred at the interface of layers 1 and 2. In Figure 48, the darkest of a gray scale (dark red for color photos) displays the maximum stress and it is shown on the left of the trough. Of layers 1 and 2, layer 2 had the zirconium tungstate mix with the lowest yield strength. At 24 MPa for a yield strength, the observed stress of 14.66 MPa was about 160% below. For the y-direction stresses reported in case 1, the highest tensile stress was 7.33 MPa. This maximum y-direction tensile stress also occurred at the interface of layers 1 and 2 and was far below the corresponding yield

strength of 24 MPa (layer 2). Figure 49 shows the ABAQUS plot for y-direction stresses and the maximum value is again shown by the darkest of a gray scale (dark red for color photos) on the right side of the figure (highlighted for convenience).

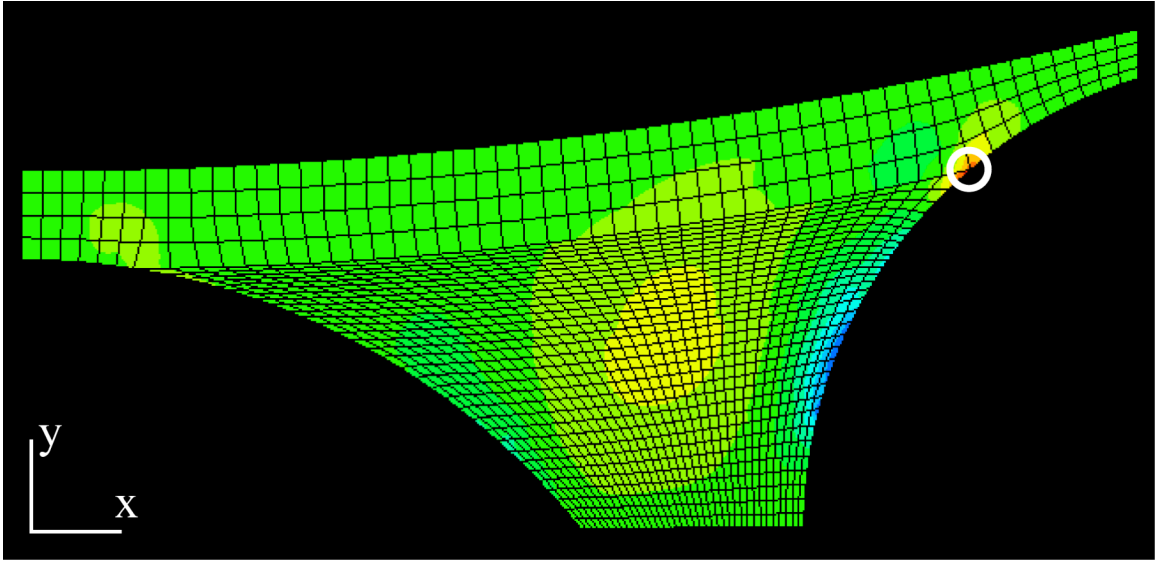


Figure 49: Case 1 AFRL Axisymmetric Model y-direction Stress.
Maximum stress indicated by O.

Like the previous uniform layered models, however, case 1 with the AFRL cross-section showed maximum absolute y-direction stresses that were compressive. For the layered geometries of these models, a tensile stress has more potential for failure due to delamination. However, the tensile stress observed in case 1 with the AFRL axisymmetric model was far below the yield strength of the 70 wt% zirconium tungstate (24 MPa) and therefore will not cause catastrophic failure.

From observations of stresses in case 1, it was determined that the properties required to provide for a thermally stable mirror with zero induced thermal curvature also allowed the mirror to be mechanically reliable. None of the maximum stresses observed

in case 1 violated the yield strength of the corresponding layer definitions. This exercise was performed for the remaining four cases and is presented in Table 42.

Table 42: Stress Analysis for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios with AFRL Cross Section Geometry (x-direction stress)

Case #	Layer Number of Max. Stress	Yield Strength of Material in Layer	Max. Stress in Respective Layer	Percent of Yield Strength
1	2	24 MPa	14.66 MPa	61.1
2	2	24 MPa	37.28 MPa	Failure
6	3	33.5 MPa	10.57 MPa	31.6
7	3	24 MPa	23.73 MPa	98.9
8	3	33.5 MPa	13.76 MPa	41.1

One of the cases, case 2 failed and case 7 was close to failure. The maximum x-direction stresses observed in both of these cases were higher or close to the yield strength of the corresponding maximum stressed layer. Although the predicted properties produced a thermally stable optic with zero thermal curvature, the required material definitions failed for mechanical reliability. This study provided evidence that attention to mechanical stresses was necessary while design of a system's CTE is under consideration. The other three cases (cases 1, 6, and 8), however, produced both a thermally and mechanically desirable optic. Additionally, the largest magnitude y-direction stresses are reported in Table 43. Again, these values are reported because they are not negligible. However, this thesis was not concerned with their affects.

Table 43: Stress Analysis for Non-Uniform Temperature Loading Zirconium Tungstate Scenarios with AFRL Cross Section Geometry (y-direction stress)

Case #	Layer Number of Max. Stress	Max. Stress in Respective Layer
1	2	-9.44 MPa
2	2	-19.74 MPa
6	3	7.72 MPa
7	3	-41.49 MPa
8	3	27.07 MPa

The displacements of the mirror were of interest, because these values directly affect the RMS calculation of the optic's surface. Figure 50 shows the magnitude displacements for case 1. The largest displacements were observed on the mirror ends and at the mounting boss. For the remaining decreasing temperature gradient cases (2 and 6), the magnitude displacements are similar in distribution as shown in Figure 50. For the increasing temperature gradient cases (7 and 8), the largest displacements and thus the greatest RMS surface distortions were centered about the axisymmetric condition (left side of mirror as is displayed in Figure 50) and at the mounting boss.

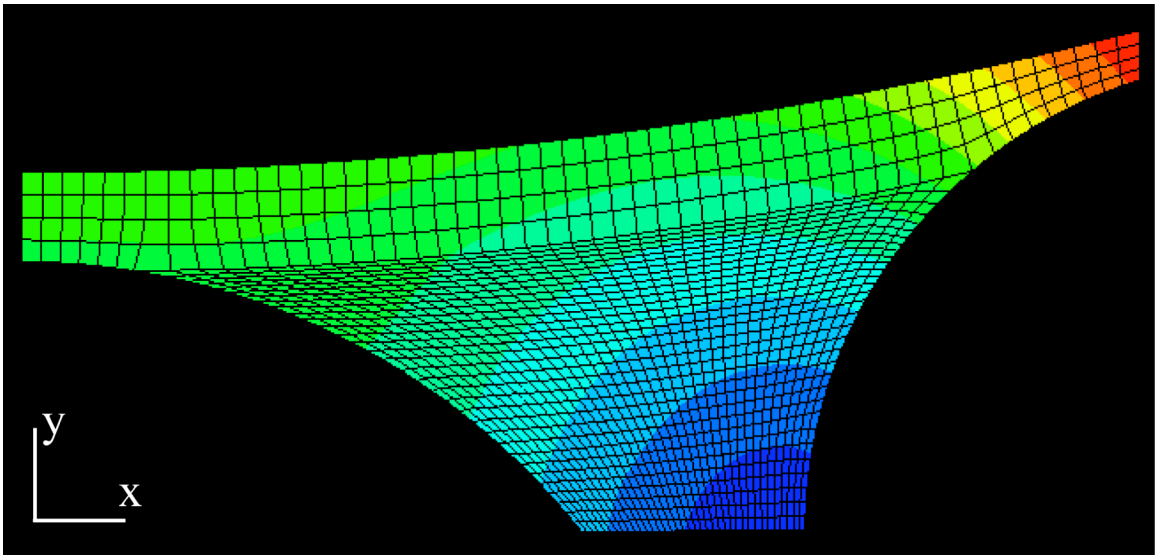


Figure 50: Case 1 AFRL Axisymmetric Model Magnitude Displacements

CHAPTER 6

CONCLUSIONS

Two closed form methods were developed to aid in the design of CTE distribution in optical structures. Each method produced favorable results for the assumptions and conditions specified. The two methods were the infinitely long nested trough approach and the multi-layer thermostat (MLT) approach. The former method was derived from the infinitely long concentric cylinders method and the latter was derived from the metallic strip method.

The first approach explored was the infinitely long nested concentric cylinders method applied to nested troughs. The well-known system of two embedded cylinders was reexamined. The equation of strain from the observations of two and three rings was expanded and applied to N-rings. In this thesis, the strain equation for N-rings was presented as a new observation and tested with less than 0.5% difference between the closed form equation and FEA results. The equations and assumptions were then applied to a semi-circular cross section trough with horizontal roller boundary conditions. This trough was equivalent to replacing half of the circular cross section with symmetrical boundary conditions represented by the horizontal rollers. This allowed the assumptions of the concentric cylinders approach to match a trough with these end conditions. As tested, the closed form equations for this approach produced less than 0.5% difference between the closed form equation and FEA results for stress and strain in the trough. This method is presented here as a viable tool in determining the stresses, strains, and

displacements for a trough with horizontal roller end conditions. For a trough with free end conditions, the metallic thermostat strip approach was expanded.

The metallic thermostat approach of N-layers was used as a model for the trough of N-layers with free end conditions and uniform thermal loading. For these trough models, the predictions between the closed form method and FEA resulted in less than an absolute difference of 7% for RMS surface distortion. The equations were also easily modifiable to solve for one layer's CTE in terms of remaining layers' CTE and other material properties. This method, however, was limited to uniform thermal loads. Developed in the thesis was the method of the multi-layer thermostat applied to thermal gradients. The thermal gradient was approximated by assigning one temperature to each layer in the trough corresponding to the increase or decrease in the temperature. This method achieved less than 10% difference against comparable results from ABAQUS. This non-uniform thermal loading multi-layer thermostat (MLT) approach also predicted the results of the previous uniform equation. It also was easily modifiable in solving for one layer's CTE in terms of the remaining layers' material properties. This allowed for definition of an optical system, which prevented thermally induced curvature for specified temperatures and material properties. This method is presented here as a viable tool in determining the curvature and surface distortion for a trough with free end conditions and a non-uniform thermal loading.

Application of the non-uniform thermally loaded multi-layer thermostat (MLT) approach to non-uniform layer geometries was also presented. Although the closed form equations are well suited for cross sections with uniform layers, the approach was

expanded to other geometries. This exercise demonstrated the closed form equation's usefulness in approximating CTE distribution for specific geometries like that of the AFRL's cross-section. A decrease in accuracy was found for RMS surface distortion of thermally loaded AFRL optics containing the required CTE values to produce zero thermally induced curvature. Although inexact, the method nonetheless produced estimated CTE values that produced distorted surfaces of potential optical quality. The equations honed in required CTE values and with further refinement or equation modification the exact CTE of the system could be found for non-uniform cross-sections.

The methods presented in conjunction with modern age materials provide useful tools for the optical engineer. For the respective assumptions and end conditions, the two discussed closed form methods allow an infinitely long optical trough's CTE to be tailored to favorable conditions. One such condition is design of the optic that produces zero thermal curvature. These methods, however, would only exist as theoretical tools if such composites like those of CTD-403 and zirconium tungstate did not exist. The mixing of zirconium tungstate to the resin of CTE-403 allows for control of the CTE and other material properties of the composite. The modifiable properties of materials in existence today, in conjunction with these closed form methods that provide optimization of their respective properties, provide tools necessary to meet current and future design requirements.

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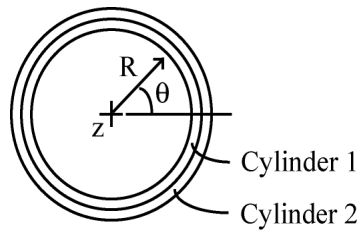
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APPENDICES

APPENDIX A

CLOSED FORM EQUATION DEVELOPMENT

The following work has been performed to determine the stresses in a two nested cylinder system. The two cylinders are fabricated to be in contact with one another, but there exists no resultant pressure in any of the interfaces from manufacturing. A temperature change is then applied simultaneously to all two cylinders and the steady-state analysis of the resultant stresses is determined. Cylinders 1 and 2 are labeled as shown below.



Cylinder 1

Inner Radius = 0.99m

Outer Radius = 1.00m = R

Cylinder 2

Inner Radius = 1.00m

Outer Radius = 1.01m

Material Properties of the two cylinders:

$$E_1 := 10 \cdot 10^9 \text{ Pa}$$

$$E_2 := 20 \cdot 10^9 \text{ Pa}$$

$$R := 1 \text{ m}$$

$$\alpha_1 := 5 \cdot 10^{-6} \cdot \frac{1}{\text{C}}$$

$$\alpha_2 := 10 \cdot 10^{-6} \cdot \frac{1}{\text{C}}$$

Steady-State temperature change:

$$\Delta T := -100 \text{ C}$$

Stress equations of the system (from equilibrium equations):

$$\sigma_{\theta 1} = -\sigma_{\theta 2}$$

$$\sigma_{\theta 1} := \frac{E_1 \cdot E_2}{E_1 + E_2} \cdot (\alpha_2 - \alpha_1) \cdot \Delta T$$

Strains of the two cylinders

Because the walls are thin, the assumption of this problem is:

$$\varepsilon_{\theta 1} = \varepsilon_{\theta 2} = \varepsilon_{\theta}$$

Total strain of cylinder 1 is:

Total strain of cylinder 2 is:

$$\varepsilon_{\theta 1} = \frac{\sigma_{\theta 1}}{E_1} + \alpha_1 \cdot \Delta T$$

$$\varepsilon_{\theta 2} = \frac{\sigma_{\theta 2}}{E_2} + \alpha_2 \cdot \Delta T$$

Substitue $\sigma_{\theta 1}$ into $\epsilon_{\theta 1}$

$$\epsilon_{\theta 1} = \frac{\frac{E_1 \cdot E_2}{E_1 + E_2} \cdot (\alpha_2 - \alpha_1) \cdot \Delta T}{E_1} + \alpha_1 \cdot \Delta T$$

$$(E_1 + E_2) \cdot \epsilon_{\theta 1} = E_2 \cdot (\alpha_2 - \alpha_1) \cdot \Delta T + (E_1 + E_2) \cdot \alpha_1 \cdot \Delta T$$

$$(E_1 + E_2) \cdot \epsilon_{\theta 1} = \alpha_2 \cdot E_2 \cdot \Delta T - \alpha_1 \cdot E_2 \cdot \Delta T + E_1 \cdot \alpha_1 \cdot \Delta T + E_2 \cdot \alpha_1 \cdot \Delta T$$

$$(E_1 + E_2) \cdot \epsilon_{\theta 1} = \alpha_2 \cdot E_2 \cdot \Delta T + \alpha_1 \cdot E_1 \cdot \Delta T$$

$$\epsilon_{\theta} := \Delta T \cdot \frac{E_1 \cdot \alpha_1 + E_2 \cdot \alpha_2}{E_1 + E_2} \quad \epsilon_{\theta} = -8.333 \times 10^{-4}$$

Displacements of the two Cylinders

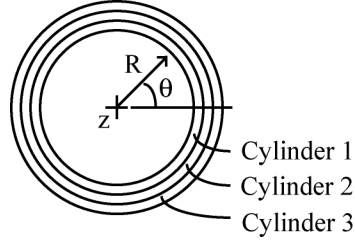
$$\epsilon_{\theta} = \frac{r}{R} \quad r = \text{radial displacement} \quad R = \text{initial radius}$$

$$r := \epsilon_{\theta} \cdot R$$

$$r = -8.333 \times 10^{-4} \text{ m}$$

Three Nested Cylinders

The following work has been performed to determine the stresses in a three nested cylindrical system. The three cylinders are fabricated to be in contact with one another, but there exists no resultant pressure in any of the interfaces from manufacturing. A temperature change is then applied simultaneously to all three cylinders and the steady-state analysis of the resultant stresses is determined. Cylinders 1,2 and 3 are labeled as shown below.



Cylinder 1

Inner Radius = 0.99m

Outer Radius = 1.00m = R

Cylinder 2

Inner Radius = 1.00m

Outer Radius = 1.01m

Cylinder 3

Inner Radius = 1.01m

Outer Radius = 1.02m

Material Properties of the three cylinders:

$$E_1 := 10 \cdot 10^9 \text{ Pa}$$

$$E_2 := 20 \cdot 10^9 \text{ Pa}$$

$$E_3 := 30 \cdot 10^9 \text{ Pa}$$

$$R := 1 \text{ m}$$

$$\alpha_1 := 5 \cdot 10^{-6} \cdot \frac{1}{^\circ\text{C}}$$

$$\alpha_2 := 10 \cdot 10^{-6} \cdot \frac{1}{^\circ\text{C}}$$

$$\alpha_3 := 15 \cdot 10^{-6} \cdot \frac{1}{^\circ\text{C}}$$

Steady-State temperature change:

$$\Delta T := -100^\circ\text{C}$$

Stress equations of the system:

$$\sigma_{\theta 3} := \frac{-E_1 \cdot (\alpha_3 - \alpha_1) \cdot \Delta T - E_2 \cdot (\alpha_3 - \alpha_2) \cdot \Delta T}{\frac{E_2}{E_3} + \frac{E_1}{E_3} + 1}$$

$$\sigma_{\theta 3} = 1 \times 10^7 \text{ Pa}$$

$$\sigma_{\theta 2} := E_2 \cdot \left[\frac{\sigma_{\theta 3}}{E_3} + (\alpha_3 - \alpha_2) \cdot \Delta T \right]$$

$$\sigma_{\theta 2} = -3.333 \times 10^6 \text{ Pa}$$

$$\sigma_{\theta 1} := E_1 \cdot \left[\frac{\sigma_{\theta 3}}{E_3} + (\alpha_3 - \alpha_1) \cdot \Delta T \right]$$

$$\sigma_{\theta 1} = -6.667 \times 10^6 \text{ Pa}$$

Strains of the three cylinders

Because the walls are thin, the assumption of this problem is:

$$\epsilon_{\theta 1} = \epsilon_{\theta 2} = \epsilon_{\theta 3} = \epsilon_{\theta}$$

Total strain of cylinder 1 is:

Total strain of cylinder 2 is:

Total strain of cylinder 3 is:

$$\varepsilon_{\theta 1} = \frac{\sigma_{\theta 1}}{E_1} + \alpha_1 \cdot \Delta T$$

$$\varepsilon_{\theta 2} = \frac{\sigma_{\theta 2}}{E_2} + \alpha_2 \cdot \Delta T$$

$$\varepsilon_{\theta 3} = \frac{\sigma_{\theta 3}}{E_3} + \alpha_3 \cdot \Delta T$$

Substitute $\sigma_{\theta 1}$ into $\varepsilon_{\theta 1}$

$$\varepsilon_{\theta 1} = \left[\frac{\sigma_{\theta 3}}{E_3} + (\alpha_3 - \alpha_1) \cdot \Delta T \right] + \alpha_1 \cdot \Delta T$$

Substitute $\sigma_{\theta 3}$ into $\varepsilon_{\theta 1}$

$$\varepsilon_{\theta 1} = \left[\frac{-E_1 \cdot (\alpha_3 - \alpha_1) \cdot \Delta T - E_2 \cdot (\alpha_3 - \alpha_2) \cdot \Delta T}{\frac{E_2}{E_3} + \frac{E_1}{E_3} + 1} + (\alpha_3 - \alpha_1) \cdot \Delta T \right] + \alpha_1 \cdot \Delta T$$

$$\varepsilon_{\theta 1} = \frac{-E_1 \cdot (\alpha_3 - \alpha_1) \cdot \Delta T - E_2 \cdot (\alpha_3 - \alpha_2) \cdot \Delta T}{E_1 + E_2 + E_3} + \alpha_3 \cdot \Delta T$$

$$(E_1 + E_2 + E_3) \cdot \varepsilon_{\theta 1} = -E_1 \cdot \alpha_3 \cdot \Delta T + E_1 \cdot \alpha_1 \cdot \Delta T - E_2 \cdot \alpha_3 \cdot \Delta T + E_2 \cdot \alpha_2 \cdot \Delta T + (E_1 + E_2 + E_3) \cdot \alpha_3 \cdot \Delta T$$

$$(E_1 + E_2 + E_3) \cdot \varepsilon_{\theta 1} = E_1 \cdot \alpha_1 \cdot \Delta T + E_2 \cdot \alpha_2 \cdot \Delta T + E_3 \cdot \alpha_3 \cdot \Delta T$$

$$\varepsilon_{\theta} := \Delta T \cdot \frac{E_1 \cdot \alpha_1 + E_2 \cdot \alpha_2 + E_3 \cdot \alpha_3}{E_1 + E_2 + E_3}$$

$$\varepsilon_{\theta} = -1.167 \times 10^{-3}$$

Displacement of Three Cylinders

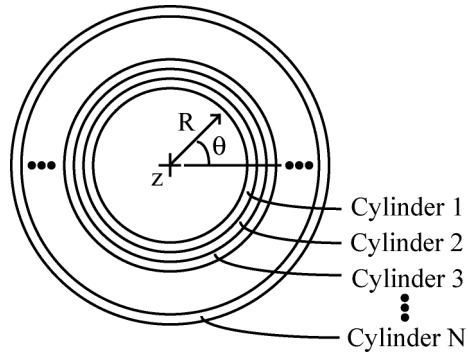
$$\varepsilon_{\theta} = \frac{r}{R} \quad r = \text{radial displacement}$$

$$r := \varepsilon_{\theta} \cdot R \quad R = \text{initial radius}$$

$$r = -1.167 \times 10^{-3} \text{ m}$$

Multiple Layer Nested Cylinders

The following work has been performed to determine the stresses in a multiple layer nested cylindrical system. The layers are fabricated to be in contact with one another, but there exists no resultant pressure in any of the interfaces from manufacturing. A temperature change is then applied simultaneously to all cylinders and the steady-state analysis of the resultant stresses is determined. Cylinders 1, 2, 3, and the nth layer are labeled as shown below.



Cylinder 1

Inner Radius = 0.99m
Outer Radius = 1.00m = R

Cylinder 2

Inner Radius = 1.00m
Outer Radius = 1.01m

Cylinder 3

Inner Radius = 1.01m
Outer Radius = 1.02m

Cylinder 4 n = 4

Inner Radius = 1.02m
Outer Radius = 1.03m

Material Properties of the three cylinders:

$$E_1 := 10 \cdot 10^9 \text{ Pa} \quad E_2 := 20 \cdot 10^9 \text{ Pa} \quad E_3 := 30 \cdot 10^9 \text{ Pa} \quad E_4 := 40 \cdot 10^9 \text{ Pa} \quad R := 1 \text{ m}$$

$$\alpha_1 := 5 \cdot 10^{-6} \cdot \frac{1}{\text{C}} \quad \alpha_2 := 10 \cdot 10^{-6} \cdot \frac{1}{\text{C}} \quad \alpha_3 := 18 \cdot 10^{-6} \cdot \frac{1}{\text{C}} \quad \alpha_4 := 20 \cdot 10^{-6} \cdot \frac{1}{\text{C}}$$

Steady-State temperature change:

$$\Delta T := -100 \text{ C}$$

Strains of the multiple layer cylinders

Because the walls are thin, the assumption of this problem is:

$$\epsilon_{\theta 1} = \epsilon_{\theta 2} = \epsilon_{\theta 3} = \epsilon_{\theta n} = \epsilon_{\theta}$$

Total strain of cylinder 1:

$$\varepsilon_{\theta 1} = \frac{\sigma_{\theta 1}}{E_1} + \alpha_1 \cdot \Delta T$$

Total strain of cylinder 2 is:

$$\varepsilon_{\theta 2} = \frac{\sigma_{\theta 2}}{E_2} + \alpha_2 \cdot \Delta T$$

Total strain of cylinder 3 is:

$$\varepsilon_{\theta 3} = \frac{\sigma_{\theta 3}}{E_3} + \alpha_3 \cdot \Delta T$$

Total strain of cylinder n is:

$$\varepsilon_{\theta n} = \frac{\sigma_{\theta n}}{E_n} + \alpha_n \cdot \Delta T$$

By observations from a 2 layer and a 3 layer cylinder configuration, a system with n cylinders is hypothesized to have a strain equation of:

$$\varepsilon_{\theta} = \Delta T \cdot \frac{\sum_{i=1}^n (E_i \cdot \alpha_i)}{\sum_{i=1}^n E_i}$$

With a multilayered system where n = 4 the strain equation becomes:

$$\varepsilon_{\theta} := \Delta T \cdot \frac{E_1 \cdot \alpha_1 + E_2 \cdot \alpha_2 + E_3 \cdot \alpha_3 + E_4 \cdot \alpha_4}{E_1 + E_2 + E_3 + E_4}$$

$$\varepsilon_{\theta} = -1.59 \times 10^{-3}$$

Displacement of Four Cylinders

$$\varepsilon_{\theta} = \frac{r}{R} \quad r = \text{radial displacement}$$

$$r := \varepsilon_{\theta} \cdot R \quad R = \text{initial radius}$$

$$r = -1.59 \times 10^{-3} \text{ m}$$

Stresses of the System

$$\varepsilon_{\theta} = \frac{\sigma_{\theta n}}{E_n} + \alpha_n \cdot \Delta T$$

$$\sigma_{\theta n} = E_n \cdot (\varepsilon_{\theta} - \alpha_n \cdot \Delta T)$$

Stress in Cylinder 1

$$\sigma_{\theta 1} := E_1 \cdot (\epsilon_{\theta} - \alpha_1 \cdot \Delta T)$$

$$\sigma_{\theta 1} = -1.09 \times 10^7 \text{ Pa}$$

Stress in Cylinder 2

$$\sigma_{\theta 2} := E_2 \cdot (\epsilon_{\theta} - \alpha_2 \cdot \Delta T)$$

$$\sigma_{\theta 2} = -1.18 \times 10^7 \text{ Pa}$$

Stress in Cylinder 3

$$\sigma_{\theta 3} := E_3 \cdot (\epsilon_{\theta} - \alpha_3 \cdot \Delta T)$$

$$\sigma_{\theta 3} = 6.3 \times 10^6 \text{ Pa}$$

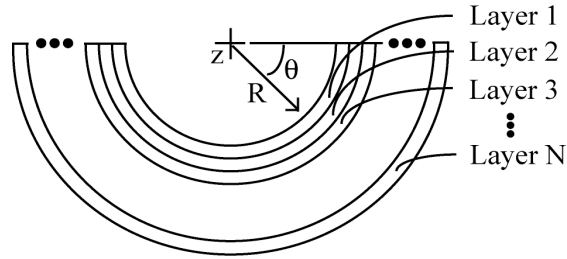
Stress in Cylinder 4

$$\sigma_{\theta 4} := E_4 \cdot (\epsilon_{\theta} - \alpha_4 \cdot \Delta T)$$

$$\sigma_{\theta 4} = 1.64 \times 10^7 \text{ Pa}$$

Multiple Layer Nested Half-Cylinders (Trough) with Three Layers having the Same Material Properties and One Layer Different

The following work has been performed to determine which boundary conditions give accurate results compared to the complete cylindrical system method. Using the design equations of nested complete cylinders, this work was performed to determine if there exists a correlation between the equations of the complete cylinders and the trough system. The layers are fabricated to be in contact with one another, but there exists no resultant pressure in any of the interfaces from manufacturing. A temperature change is then applied simultaneously to all layers and the steady-state analysis of the resultant stresses is determined. Layers 1, 2, 3, and the nth layer are labeled as shown below.



Layer 1

Inner Radius = 0.99m
Outer Radius = 1.00m = R

Layer 2

Inner Radius = 1.00m
Outer Radius = 1.01m

Layer 3

Inner Radius = 1.01m
Outer Radius = 1.02m

Layer 4

n = 4

Inner Radius = 1.02m
Outer Radius = 1.03m

Material Properties of the four Layers:

The Elastic Modulus and geometric definitions of the four layers are the same, but one layer's CTE differs from the other three. The material properties of the layers that share the same CTE and other properties have the following:

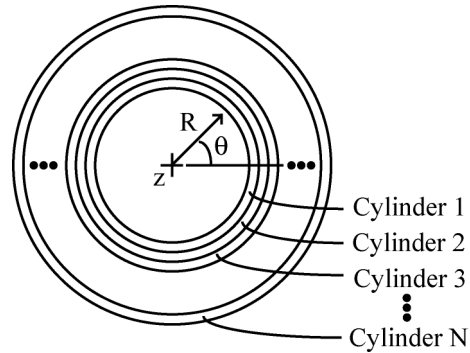
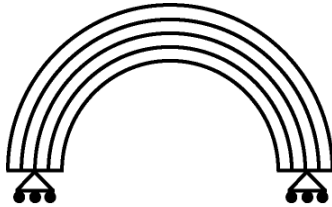
$$E := 40 \cdot 10^9 \text{ Pa} \quad \alpha := 20 \cdot 10^{-6} \cdot \frac{1}{\text{C}}$$

The layer that differs has material properties of:

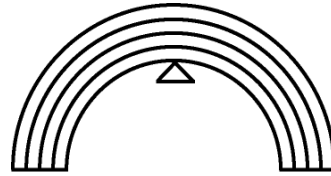
$$E := 40 \cdot 10^9 \text{ Pa} \quad \alpha := 10 \cdot 10^{-6} \cdot \frac{1}{\text{C}}$$

The equations and processes of the complete four layer nested cylinder system can be found on previous pages.

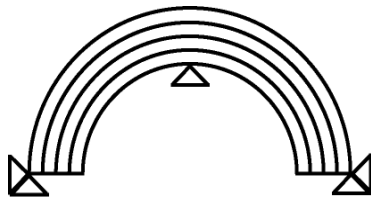
Displacements and Strains for Several Boudary Condition Types

Standard Boundary Condition (Complete Cylinder)**Boundary Condition 1** (Half Trough with Horizontal Roller End):

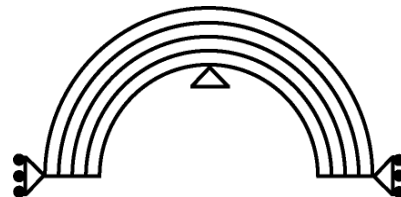
The trough end, at the bottom of the diagram is on horizontal rollers.

Boundary Condition 2 (Half Trough with Free End):

The trough end, at the bottom of the diagram is free.

Boundary Condition 3 (Half Trough with Pinned End):

The trough end, at the bottom of the diagram is pinned.

Boundary Condition 4 (Half Trough with Vertical Roller End):

The trough end, at the bottom of the diagram has a vertical roller.

Displacements from ABAQUS under a Temperature Change of +250C Comparing Different BC's in a Trough System with Uniform Properties to a Complete Cylinder System

Trough BC Type	Cylinder Disp. (Uniform)	Trough Disp. (End Horizontal)	Percent Difference	Trough Disp. (End Vertical)	Percent Difference
1	5.013E-03	5.013E-03	0.00	5.013E-03	0.00
2	5.013E-03	5.096E-03	1.66	4.986E-03	-0.54
3	5.013E-03	6.050E-04	-87.93	1.432E-04	-97.14
4	5.013E-03	1.494E-04	-97.02	-8.283E-03	-265.23

Figure 37: Percent Diff. Between FEA and Calculated Displacements for a 4 Layer Trough with the CTE of Layer 3 Varied Compared to the System (Note: the 1.01x case is an approximation of the 1x case)

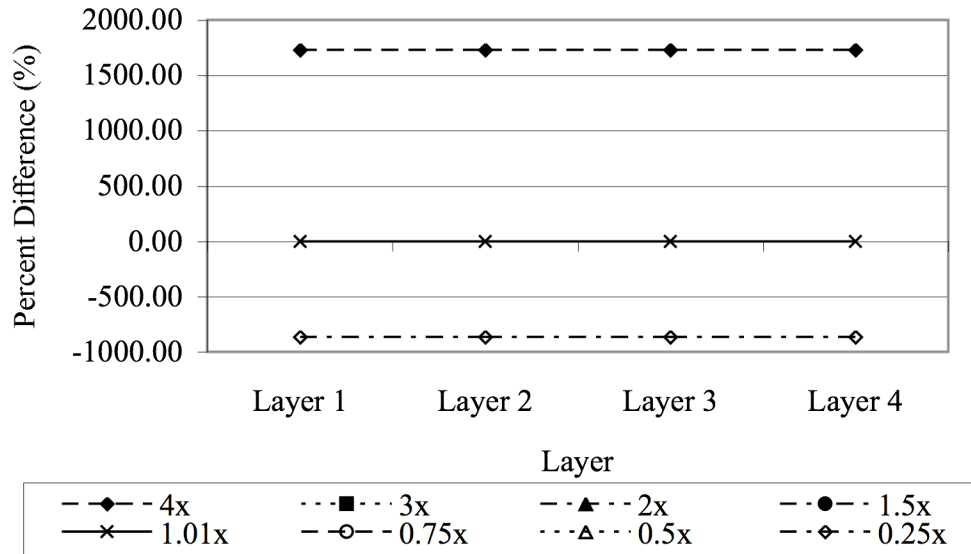
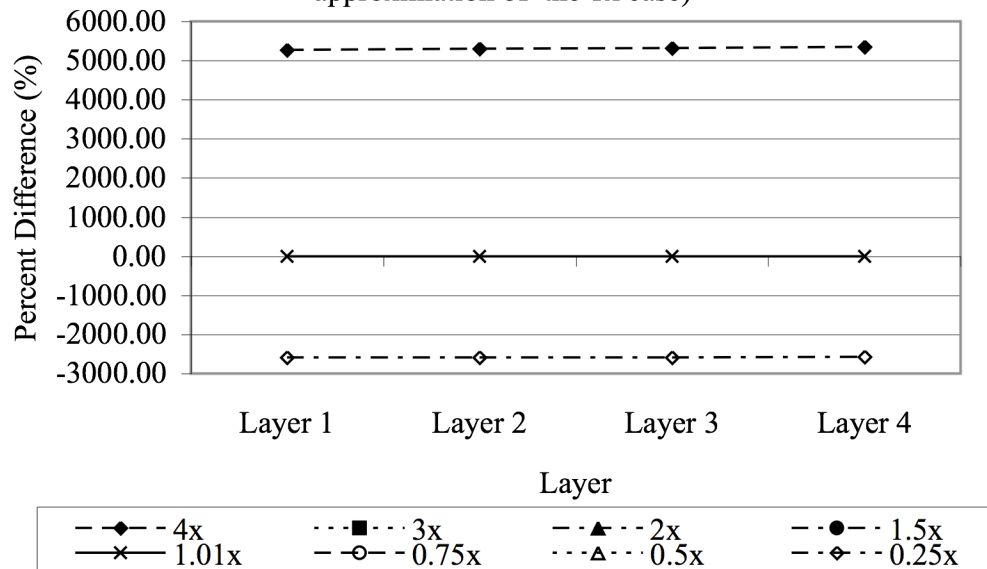


Figure 38: Percent Diff. Between FEA and Calculated Displacements for a 4 Layer Trough with the CTE of Layer 4 Varied Compared to the System (Note: the 1.01x case is an approximation of the 1x case)



Curvature of Two Layer Trough

Method of a Bimetallic strip

Curvature of a bimetallic strip under a uniform temperature change ΔT .

$$\kappa = \frac{1}{R} = \frac{2 \cdot \sin(\theta)}{\sqrt{x^2 + y^2}}$$

Case 1

The trough has an initial offset curvature before the temperature change ΔT

$$R := 1\text{m}$$

$$\kappa_{\text{offset}} := \frac{1}{R} \quad \kappa_{\text{offset}} = 1 \frac{1}{\text{m}}$$

The curvature of the trough from the thermal strain after the temperature is:

Material properties of the trough layers

$$E_1 := 40 \cdot 10^9 \text{Pa} \quad E_2 := 40 \cdot 10^9 \text{Pa} \quad \alpha_1 := 20 \cdot 10^{-6} \frac{1}{\text{C}} \quad \alpha_2 := 40 \cdot 10^{-6} \frac{1}{\text{C}}$$

Material geometries of the trough layers

$$\text{layer thickness:} \quad h_1 := 0.02\text{m} \quad h_2 := 0.02\text{m}$$

Temperature change

$$\Delta T := 1000\text{C}$$

$$\kappa_{\text{thermal}} := \frac{6 \cdot E_1 \cdot E_2 \cdot (h_1 + h_2) \cdot h_1 \cdot h_2 \cdot (\alpha_2 - \alpha_1) \cdot \Delta T}{E_1^2 \cdot h_1^4 + 4 \cdot E_1 \cdot E_2 \cdot h_1^3 \cdot h_2 + 6 \cdot E_1 \cdot E_2 \cdot h_1^2 \cdot h_2^2 + 4 \cdot E_1 \cdot E_2 \cdot h_1 \cdot h_2^3 + E_2^2 \cdot h_2^4}$$

$$\kappa_{\text{thermal}} = 0.75 \frac{1}{\text{m}}$$

The total curvature of the trough after the temperature change is:

$$\kappa_{\text{total}} := \kappa_{\text{offset}} + \kappa_{\text{thermal}}$$

$$\kappa_{\text{total}} = 1.75 \frac{1}{\text{m}}$$

Radius of thermally affected trough

$$R_{\text{predicted}} := \frac{1}{\kappa_{\text{total}}} \quad R_{\text{predicted}} = 0.571 \text{ m}$$

Curvature from finite element analysis

The following are the coordinates of the deformed trough $\pi/2$ radians ccw from the apex:

$$x := 0.5716\text{m} \quad y := 0.5738\text{m}$$

$$\theta := \text{atan}\left(\frac{y}{x}\right) \quad \theta = 0.787$$

$$\kappa_{\text{FEA}} := \frac{2 \cdot \sin(\theta)}{\sqrt{x^2 + y^2}}$$

$$\kappa_{\text{FEA}} = 1.749 \frac{1}{\text{m}}$$

Radius of thermally affected trough

$$R_{\text{FEA}} := \frac{1}{\kappa_{\text{FEA}}} \quad R_{\text{FEA}} = 0.572 \text{ m}$$

Percent difference between FEA and closed form calculation of radius of curvature:

$$\text{Difference} := \frac{R_{\text{FEA}} - R_{\text{predicted}}}{R_{\text{FEA}}}$$

$$\text{Difference} = 0.031 \%$$

Stresses in each layer of the trough^{A1}:

$$K_1 := 4 + 6 \cdot \frac{h_1}{h_2} + 4 \cdot \left(\frac{h_1}{h_2}\right)^2 + \frac{E_1}{E_2} \cdot \left(\frac{h_1}{h_2}\right)^3 + \frac{E_2}{E_1} \cdot \frac{h_2}{h_1}$$

$$\sigma_1 := \frac{-(\alpha_2 - \alpha_1) \cdot \Delta T \cdot E_1}{K_1} \cdot \left[3 \cdot \frac{h_1}{h_2} + 2 \cdot \left(\frac{h_1}{h_2}\right)^2 - \frac{E_2 \cdot h_2}{E_1 \cdot h_1} \right]$$

$$\sigma_1 = -2 \times 10^8 \text{ Pa}$$

$$\sigma_2 := \frac{(\alpha_2 - \alpha_1) \cdot \Delta T \cdot E_2}{K_1} \cdot \left[3 \cdot \frac{h_1}{h_2} + 2 - \frac{E_1}{E_2} \cdot \left(\frac{h_1}{h_2}\right)^3 \right]$$

$$\sigma_2 = 2 \times 10^8 \text{ Pa}$$

Case 2

Temperature change

$$\Delta T := -1000\text{C}$$

The trough has an initial offset curvature before the temperature change ΔT

$$R := 1\text{m}$$

$$\kappa_{\text{offset}} := \frac{1}{R} \quad \kappa_{\text{offset}} = 1 \frac{1}{\text{m}}$$

The curvature of the trough from the thermal strain after the temperature is:

Material properties of the trough layers

$$E_1 := 40 \cdot 10^9 \text{Pa} \quad E_2 := 40 \cdot 10^9 \text{Pa} \quad \alpha_1 := 20 \cdot 10^{-6} \frac{1}{\text{C}} \quad \alpha_2 := 40 \cdot 10^{-6} \frac{1}{\text{C}}$$

Material geometries of the trough layers

$$\text{layer thickness:} \quad h_1 := 0.02\text{m} \quad h_2 := 0.02\text{m}$$

$$\kappa_{\text{thermal}} := \frac{6 \cdot E_1 \cdot E_2 \cdot (h_1 + h_2) \cdot h_1 \cdot h_2 \cdot (\alpha_2 - \alpha_1) \cdot \Delta T}{E_1^2 \cdot h_1^4 + 4 \cdot E_1 \cdot E_2 \cdot h_1^3 \cdot h_2 + 6 \cdot E_1 \cdot E_2 \cdot h_1^2 \cdot h_2^2 + 4 \cdot E_1 \cdot E_2 \cdot h_1 \cdot h_2^3 + E_2^2 \cdot h_2^4}$$

$$\kappa_{\text{thermal}} = -0.75 \frac{1}{\text{m}}$$

The total curvature of the trough after the temperature change is:

$$\kappa_{\text{total}} := \kappa_{\text{offset}} + \kappa_{\text{thermal}}$$

$$\kappa_{\text{total}} = 0.25 \frac{1}{\text{m}}$$

Radius of thermally affected trough

$$R_{\text{predicted}} := \frac{1}{\kappa_{\text{total}}} \quad R_{\text{predicted}} = 4 \text{ m}$$

Curvature from finite element analysis

The following are the coordinates of the deformed trough $\pi/2$ radians ccw from the apex:

$$x := 1.49843\text{m} \quad y := 0.291123\text{m}$$

$$\theta := \text{atan}\left(\frac{y}{x}\right) \quad \theta = 0.192$$

$$\kappa_{\text{FEA}} := \frac{2 \cdot \sin(\theta)}{\sqrt{x^2 + y^2}}$$

$$\kappa_{\text{FEA}} = 0.25 \frac{1}{\text{m}}$$

Radius of thermally affected trough

$$R_{\text{FEA}} := \frac{1}{\kappa_{\text{FEA}}} \quad R_{\text{FEA}} = 4.002 \text{ m}$$

Percent difference between FEA and closed form calculation of radius of curvature:

$$\text{Difference} := \frac{R_{\text{FEA}} - R_{\text{predicted}}}{R_{\text{FEA}}}$$

$$\text{Difference} = 0.046 \%$$

Stresses in each layer of the trough^{B1}:

$$K_1 := 4 + 6 \cdot \frac{h_1}{h_2} + 4 \cdot \left(\frac{h_1}{h_2} \right)^2 + \frac{E_1}{E_2} \cdot \left(\frac{h_1}{h_2} \right)^3 + \frac{E_2}{E_1} \cdot \frac{h_2}{h_1}$$

$$\sigma_1 := \frac{-(\alpha_2 - \alpha_1) \cdot \Delta T \cdot E_1}{K_1} \cdot \left[3 \cdot \frac{h_1}{h_2} + 2 \cdot \left(\frac{h_1}{h_2} \right)^2 - \frac{E_2 \cdot h_2}{E_1 \cdot h_1} \right]$$

$$\sigma_1 = 2 \times 10^8 \text{ Pa}$$

$$\sigma_2 := \frac{(\alpha_2 - \alpha_1) \cdot \Delta T \cdot E_2}{K_1} \cdot \left[3 \cdot \frac{h_1}{h_2} + 2 - \frac{E_1}{E_2} \cdot \left(\frac{h_1}{h_2} \right)^3 \right]$$

$$\sigma_2 = -2 \times 10^8 \text{ Pa}$$

Case 3

Temperature change

$$\Delta T := 250^\circ\text{C}$$

The trough has an initial offset curvature before the temperature change ΔT

$$R := 1 \text{ m}$$

$$\kappa_{\text{offset}} := \frac{1}{R} \quad \kappa_{\text{offset}} = 1 \frac{1}{\text{m}}$$

The curvature of the trough from the thermal strain after the temperature is:

Material properties of the trough layers

$$E_1 := 40 \cdot 10^9 \text{ Pa} \quad E_2 := 40 \cdot 10^9 \text{ Pa} \quad \alpha_1 := 20 \cdot 10^{-6} \frac{1}{\text{C}} \quad \alpha_2 := 40 \cdot 10^{-6} \frac{1}{\text{C}}$$

Material geometries of the trough layers

$$\text{layer thickness:} \quad h_1 := 0.02 \text{ m} \quad h_2 := 0.02 \text{ m}$$

$$\kappa_{\text{thermal}} := \frac{6 \cdot E_1 \cdot E_2 \cdot (h_1 + h_2) \cdot h_1 \cdot h_2 \cdot (\alpha_2 - \alpha_1) \cdot \Delta T}{E_1^2 \cdot h_1^4 + 4 \cdot E_1 \cdot E_2 \cdot h_1^3 \cdot h_2 + 6 \cdot E_1 \cdot E_2 \cdot h_1^2 \cdot h_2^2 + 4 \cdot E_1 \cdot E_2 \cdot h_1 \cdot h_2^3 + E_2^2 \cdot h_2^4}$$

$$\kappa_{\text{thermal}} = 0.187 \frac{1}{\text{m}}$$

The total curvature of the trough after the temperature change is:

$$\kappa_{\text{total}} := \kappa_{\text{offset}} + \kappa_{\text{thermal}}$$

$$\kappa_{\text{total}} = 1.188 \frac{1}{\text{m}}$$

Radius of thermally affected trough

$$R_{\text{predicted}} := \frac{1}{\kappa_{\text{total}}} \quad R_{\text{predicted}} = 0.842 \text{ m}$$

Curvature from finite element analysis

The following are the coordinates of the deformed trough $\pi/2$ radians ccw from the apex:

$$x := .845736 \text{ m} \quad y := .846882 \text{ m}$$

$$\theta := \text{atan}\left(\frac{y}{x}\right) \quad \theta = 0.786$$

$$\kappa_{\text{FEA}} := \frac{2 \cdot \sin(\theta)}{\sqrt{x^2 + y^2}}$$

$$\kappa_{\text{FEA}} = 1.182 \frac{1}{\text{m}}$$

Radius of thermally affected trough

$$R_{\text{FEA}} := \frac{1}{\kappa_{\text{FEA}}} \quad R_{\text{FEA}} = 0.846 \text{ m}$$

Percent difference between FEA and closed form calculation of radius of curvature:

$$\text{Difference} := \frac{R_{\text{FEA}} - R_{\text{predicted}}}{R_{\text{FEA}}}$$

$$\text{Difference} = 0.429 \%$$

Stresses in each layer of the trough^{B1}:

$$K_1 := 4 + 6 \cdot \frac{h_1}{h_2} + 4 \cdot \left(\frac{h_1}{h_2} \right)^2 + \frac{E_1}{E_2} \cdot \left(\frac{h_1}{h_2} \right)^3 + \frac{E_2}{E_1} \cdot \frac{h_2}{h_1}$$

$$\sigma_1 := \frac{-(\alpha_2 - \alpha_1) \cdot \Delta T \cdot E_1}{K_1} \cdot \left[3 \cdot \frac{h_1}{h_2} + 2 \cdot \left(\frac{h_1}{h_2} \right)^2 - \frac{E_2 \cdot h_2}{E_1 \cdot h_1} \right]$$

$$\sigma_1 = -5 \times 10^7 \text{ Pa}$$

$$\sigma_2 := \frac{(\alpha_2 - \alpha_1) \cdot \Delta T \cdot E_2}{K_1} \cdot \left[3 \cdot \frac{h_1}{h_2} + 2 - \frac{E_1}{E_2} \cdot \left(\frac{h_1}{h_2} \right)^3 \right]$$

$$\sigma_2 = 5 \times 10^7 \text{ Pa}$$

Case 4

Temperature change

$$\Delta T := -250^\circ\text{C}$$

The trough has an initial offset curvature before the temperature change ΔT

$$R := 1\text{m}$$

$$\kappa_{\text{offset}} := \frac{1}{R} \quad \kappa_{\text{offset}} = 1 \frac{1}{\text{m}}$$

The curvature of the trough from the thermal strain after the temperature is:

Material properties of the trough layers

$$E_1 := 40 \cdot 10^9 \text{ Pa} \quad E_2 := 40 \cdot 10^9 \text{ Pa} \quad \alpha_1 := 20 \cdot 10^{-6} \frac{1}{^\circ\text{C}} \quad \alpha_2 := 40 \cdot 10^{-6} \frac{1}{^\circ\text{C}}$$

Material geometries of the trough layers

$$\text{layer thickness:} \quad h_1 := 0.02\text{m} \quad h_2 := 0.02\text{m}$$

$$\kappa_{\text{thermal}} := \frac{6 \cdot E_1 \cdot E_2 \cdot (h_1 + h_2) \cdot h_1 \cdot h_2 \cdot (\alpha_2 - \alpha_1) \cdot \Delta T}{E_1^2 \cdot h_1^4 + 4 \cdot E_1 \cdot E_2 \cdot h_1^3 \cdot h_2 + 6 \cdot E_1 \cdot E_2 \cdot h_1^2 \cdot h_2^2 + 4 \cdot E_1 \cdot E_2 \cdot h_1 \cdot h_2^3 + E_2^2 \cdot h_2^4}$$

$$\kappa_{\text{thermal}} = -0.187 \frac{1}{\text{m}}$$

The total curvature of the trough after the temperature change is:

$$\kappa_{\text{total}} := \kappa_{\text{offset}} + \kappa_{\text{thermal}}$$

$$\kappa_{\text{total}} = 0.813 \frac{1}{\text{m}}$$

Radius of thermally affected trough

$$R_{\text{predicted}} := \frac{1}{\kappa_{\text{total}}} \quad R_{\text{predicted}} = 1.231 \text{ m}$$

Curvature from finite element analysis

The following are the coordinates of the deformed trough $\pi/2$ radians ccw from the apex:

$$x := 1.17230 \text{ m} \quad y := .867333 \text{ m}$$

$$\theta := \text{atan}\left(\frac{y}{x}\right) \quad \theta = 0.637$$

$$\kappa_{\text{FEA}} := \frac{2 \cdot \sin(\theta)}{\sqrt{x^2 + y^2}}$$

$$\kappa_{\text{FEA}} = 0.816 \frac{1}{\text{m}}$$

Radius of thermally affected trough

$$R_{\text{FEA}} := \frac{1}{\kappa_{\text{FEA}}} \quad R_{\text{FEA}} = 1.226 \text{ m}$$

Percent difference between FEA and closed form calculation of radius of curvature:

$$\text{Difference} := \frac{R_{\text{FEA}} - R_{\text{predicted}}}{R_{\text{FEA}}}$$

$$\text{Difference} = -0.396 \%$$

Stresses in each layer of the trough^{B1}:

$$K_1 := 4 + 6 \cdot \frac{h_1}{h_2} + 4 \cdot \left(\frac{h_1}{h_2}\right)^2 + \frac{E_1}{E_2} \cdot \left(\frac{h_1}{h_2}\right)^3 + \frac{E_2}{E_1} \cdot \frac{h_2}{h_1}$$

$$\sigma_1 := \frac{-(\alpha_2 - \alpha_1) \cdot \Delta T \cdot E_1}{K_1} \cdot \left[3 \cdot \frac{h_1}{h_2} + 2 \cdot \left(\frac{h_1}{h_2} \right)^2 - \frac{E_2 \cdot h_2}{E_1 \cdot h_1} \right]$$

$$\sigma_1 = 5 \times 10^7 \text{ Pa}$$

$$\sigma_2 := \frac{(\alpha_2 - \alpha_1) \cdot \Delta T \cdot E_2}{K_1} \cdot \left[3 \cdot \frac{h_1}{h_2} + 2 - \frac{E_1}{E_2} \cdot \left(\frac{h_1}{h_2} \right)^3 \right]$$

$$\sigma_2 = -5 \times 10^7 \text{ Pa}$$

Curvature of Three Layer Trough

Method of a Tri-metallic strip

Curvature of a tri-metallic strip before temperature change ΔT

$$\kappa = \frac{1}{R} = \frac{2 \cdot \sin(\theta)}{\sqrt{x^2 + y^2}}$$

Case 1

The trough has an initial offset curvature before the temperature change ΔT

$$R := 1\text{m}$$

$$\kappa_{\text{offset}} := \frac{1}{R} \quad \kappa_{\text{offset}} = 1 \frac{1}{\text{m}}$$

The curvature of the trough from the thermal strain after the temperature change is:

Material properties of the trough layers

$$E_1 := 40 \cdot 10^9 \text{Pa} \quad E_2 := 40 \cdot 10^9 \text{Pa} \quad E_3 := 40 \cdot 10^9 \text{Pa}$$

$$\alpha_1 := 20 \cdot 10^{-6} \frac{1}{\text{C}} \quad \alpha_2 := 30 \cdot 10^{-6} \frac{1}{\text{C}} \quad \alpha_3 := 40 \cdot 10^{-6} \frac{1}{\text{C}}$$

Material geometries of the trough layers

$$\text{layer thickness:} \quad h_1 := 0.01\text{m} \quad h_2 := 0.01\text{m} \quad h_3 := 0.02\text{m}$$

$$\text{Unit depth:} \quad \text{unit} := 1$$

Temperature change

$$\Delta T := -1000\text{C}$$

$$s_1 := h_1 \cdot E_1 \quad s_2 := h_2 \cdot E_2 \quad s_3 := h_3 \cdot E_3$$

$$I_1 := \text{unit} \cdot \frac{h_1^3}{12} \quad I_2 := \text{unit} \cdot \frac{h_2^3}{12} \quad I_3 := \text{unit} \cdot \frac{h_3^3}{12}$$

$$a_{12} := s_1 \cdot s_2 \cdot (h_1 + h_2) \cdot (\alpha_2 - \alpha_1)$$

$$a_{23} := s_2 \cdot s_3 \cdot (h_2 + h_3) \cdot (\alpha_3 - \alpha_2)$$

$$a_{13} := s_1 \cdot s_3 \cdot (h_1 + 2 \cdot h_2 + h_3) \cdot (\alpha_3 - \alpha_1)$$

$$b_{12} := s_1 \cdot s_2 \cdot (h_1 + h_2)^2$$

$$b_{23} := s_2 \cdot s_3 \cdot (h_2 + h_3)^2$$

$$b_{13} := s_1 \cdot s_3 \cdot (h_1 + 2 \cdot h_2 + h_3)^2$$

$$c := E_1 \cdot I_1 + E_2 \cdot I_2 + E_3 \cdot I_3$$

$$d := s_1 + s_2 + s_3$$

$$\kappa_{\text{thermal}} := 2 \cdot \Delta T \cdot \frac{a_{12} + a_{13} + a_{23}}{b_{12} + b_{13} + b_{23} + 4 \cdot c \cdot d}$$

$$\kappa_{\text{thermal}} = -0.656 \frac{1}{\text{m}}$$

The total curvature of the trough after the temperature change is:

$$\kappa_{\text{total}} := \kappa_{\text{offset}} + \kappa_{\text{thermal}}$$

$$\kappa_{\text{total}} = 0.344 \frac{1}{\text{m}}$$

Radius of thermally affected trough

$$R_{\text{predicted}} := \frac{1}{\kappa_{\text{total}}} \quad R_{\text{predicted}} = 2.909 \text{ m}$$

Curvature from finite element analysis

The following are the coordinates of the deformed trough $\pi/2$ radians ccw from the apex:

$$x := 1.46068 \text{ m} \quad y := 0.393884 \text{ m}$$

$$\theta := \text{atan}\left(\frac{y}{x}\right) \quad \theta = 0.263$$

$$\kappa_{\text{FEA}} := \frac{2 \cdot \sin(\theta)}{\sqrt{x^2 + y^2}}$$

$$\kappa_{\text{FEA}} = 0.344 \frac{1}{\text{m}}$$

Radius of thermally affected trough

$$R_{\text{FEA}} := \frac{1}{\kappa_{\text{FEA}}} \quad R_{\text{FEA}} = 2.905 \text{ m}$$

Percent difference between FEA and closed form calculation of radius of curvature:

$$\text{Difference} := \frac{R_{\text{FEA}} - R_{\text{predicted}}}{R_{\text{FEA}}}$$

$$\text{Difference} = -0.129 \%$$

Closed Form and FEA Curvature Comparison of Four Layer Thermostat Trough for Uniform Temperatures

Predicted Curvature

The following are the coordinates of the undeformed trough in both the thermostat and real geometry FEA models:

$$\begin{aligned} x_{\text{apex}} &:= 0\text{m} & y_{\text{apex}} &:= -1\text{m} \\ x_{\text{edgetrough}} &:= .2496\text{m} & y_{\text{edgetrough}} &:= -.9683\text{m} \end{aligned}$$

The trough has an initial offset curvature before the temperature change ΔT . The distance between the above coordinates gives the desired x and y values below which are used in determining the initial curvature.

$$\begin{aligned} x &:= x_{\text{edgetrough}} - x_{\text{apex}} & y &:= y_{\text{edgetrough}} - y_{\text{apex}} \\ \theta &:= \text{atan}\left(\frac{y}{x}\right) & \theta &= 0.126 \\ \kappa_{\text{offset}} &:= \frac{2 \cdot \sin(\theta)}{\sqrt{x^2 + y^2}} & \kappa_{\text{offset}} &= 1.001 \frac{1}{\text{m}} \end{aligned}$$

The curvature of the trough from the thermal strain after the temperature change is determined by the following:

Material properties of the trough layers

$$\begin{aligned} E_1 &:= 10 \cdot 10^9 \text{Pa} & E_2 &:= 20 \cdot 10^9 \text{Pa} & E_3 &:= 35 \cdot 10^9 \text{Pa} & E_4 &:= 40 \cdot 10^9 \text{Pa} \\ \alpha_1 &:= 5 \cdot 10^{-6} \frac{1}{\text{C}} & \alpha_2 &:= 10 \cdot 10^{-6} \frac{1}{\text{C}} & \alpha_3 &:= 18 \cdot 10^{-6} \frac{1}{\text{C}} & \alpha_4 &:= 20 \cdot 10^{-6} \frac{1}{\text{C}} \end{aligned}$$

Material geometries of the trough layers

$$\begin{aligned} \text{layer thickness:} & & h_1 &:= 0.023\text{m} & h_2 &:= 0.023\text{m} & h_3 &:= 0.023\text{m} & h_4 &:= 0.023\text{m} \\ \text{Unit depth:} & & \text{unit} &:= 1 \end{aligned}$$

Temperature change

$$\Delta T := \begin{pmatrix} -250 \\ -100 \\ -50 \\ -25 \\ 25 \\ 50 \\ 100 \\ 250 \end{pmatrix} \text{C}$$

The following definitions are used in the constitutive equation for the thermally induced curvature:

$$\begin{aligned}
 s_1 &:= h_1 \cdot E_1 & s_2 &:= h_2 \cdot E_2 & s_3 &:= h_3 \cdot E_3 & s_4 &:= h_4 \cdot E_4 \\
 I_1 &:= \text{unit} \cdot \frac{h_1^3}{12} & I_2 &:= \text{unit} \cdot \frac{h_2^3}{12} & I_3 &:= \text{unit} \cdot \frac{h_3^3}{12} & I_4 &:= \text{unit} \cdot \frac{h_4^3}{12} \\
 a_{12} &:= s_1 \cdot s_2 \cdot (h_1 + h_2) \cdot (\alpha_2 - \alpha_1) & a_{23} &:= s_2 \cdot s_3 \cdot (h_2 + h_3) \cdot (\alpha_3 - \alpha_2) \\
 a_{34} &:= s_3 \cdot s_4 \cdot (h_3 + h_4) \cdot (\alpha_4 - \alpha_3) \\
 a_{13} &:= s_1 \cdot s_3 \cdot (h_1 + 2 \cdot h_2 + h_3) \cdot (\alpha_3 - \alpha_1) & a_{24} &:= s_2 \cdot s_4 \cdot (h_2 + 2 \cdot h_3 + h_4) \cdot (\alpha_4 - \alpha_2) \\
 a_{14} &:= s_1 \cdot s_4 \cdot [h_1 + 2 \cdot (h_2 + h_3) + h_4] \cdot (\alpha_4 - \alpha_1) \\
 b_{12} &:= s_1 \cdot s_2 \cdot (h_1 + h_2)^2 & b_{23} &:= s_2 \cdot s_3 \cdot (h_2 + h_3)^2 & b_{34} &:= s_3 \cdot s_4 \cdot (h_3 + h_4)^2 \\
 b_{13} &:= s_1 \cdot s_3 \cdot (h_1 + 2 \cdot h_2 + h_3)^2 & b_{24} &:= s_2 \cdot s_4 \cdot (h_2 + 2 \cdot h_3 + h_4)^2 \\
 b_{14} &:= s_1 \cdot s_4 \cdot [h_1 + 2 \cdot (h_2 + h_3) + h_4]^2 \\
 c &:= E_1 \cdot I_1 + E_2 \cdot I_2 + E_3 \cdot I_3 + E_4 \cdot I_4 & d &:= s_1 + s_2 + s_3 + s_4
 \end{aligned}$$

The thermally induced curvature is:

$$\kappa_{\text{thermal}} := 2 \cdot \Delta T \cdot \frac{a_{12} + a_{13} + a_{14} + a_{23} + a_{24} + a_{34}}{b_{12} + b_{13} + b_{14} + b_{23} + b_{24} + b_{34} + 4 \cdot c \cdot d} \quad \kappa_{\text{thermal}} = \begin{pmatrix} -0.05 \\ -0.02 \\ -9.995 \times 10^{-3} \\ -4.998 \times 10^{-3} \\ 4.998 \times 10^{-3} \\ 9.995 \times 10^{-3} \\ 0.02 \\ 0.05 \end{pmatrix} \frac{1}{\text{m}}$$

The total curvature of the trough after the temperature change is:

$$\kappa_{\text{total}} := \kappa_{\text{offset}} + \kappa_{\text{thermal}} \quad \kappa_{\text{total}} = \begin{pmatrix} 0.952 \\ 0.982 \\ 0.992 \\ 0.997 \\ 1.006 \\ 1.011 \\ 1.021 \\ 1.051 \end{pmatrix} \frac{1}{\text{m}}$$

The radius of thermally affected trough is:

$$R_{\text{predicted}} := \frac{1}{\kappa_{\text{total}}}$$

$$R_{\text{predicted}} = \begin{pmatrix} 1.051 \\ 1.019 \\ 1.009 \\ 1.004 \\ 0.994 \\ 0.989 \\ 0.979 \\ 0.951 \end{pmatrix} \text{ m}$$

These values are plugged into the following tables and compared to the resultant FEA radius values

RMS Surface Distortion and Radius of Curvature Comparison for FEA vs. Closed Form Equation Results for Case 1 (-250°C)

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.147E-06
RMS of best fit of FEA deformed	3.134E-06
Percent Difference between FEA and Closed Form	0.427

Radius of Curvature Results:

FEA Radius of Curvature	1.055
Closed Form Radius of Curvature	1.051
Percent Difference between FEA and Closed Form	0.370

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4317E-15	-0.001432
2	0.00143	0.00143	-1.00000	-1.00000	-1.2886E-14	-0.001432
3	0.00287	0.00286	-1.00000	-1.00000	-3.5793E-14	-0.001432
4	0.00430	0.00430	-0.99999	-1.00000	-7.0154E-14	-0.001432
5	0.00573	0.00573	-0.99998	-1.00000	-1.4317E-13	-0.001432
6	0.00717	0.00716	-0.99997	-1.00000	-1.7323E-13	-0.001432
7	0.00860	0.00859	-0.99996	-1.00000	-2.4195E-13	-0.001432

Performed for all surface nodes and RMS is calculated according to the above equation

RMS Surface Distortion and Radius of Curvature Comparison for FEA vs. Closed Form Equation Results for Case 2 (-100°C)

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.148E-06
RMS of best fit of FEA deformed	3.158E-06
Percent Difference between FEA and Closed Form	-0.316

Radius of Curvature Results:

FEA Radius of Curvature	1.023
Closed Form Radius of Curvature	1.019
Percent Difference between FEA and Closed Form	0.355

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4326E-15	-0.001433
2	0.00143	0.00143	-1.00000	-1.00000	-1.2893E-14	-0.001433
3	0.00287	0.00287	-1.00000	-1.00000	-3.5815E-14	-0.001433
4	0.00430	0.00430	-0.99999	-1.00000	-7.0197E-14	-0.001433
5	0.00573	0.00573	-0.99998	-1.00000	-1.4326E-13	-0.001433
6	0.00717	0.00716	-0.99997	-1.00000	-1.7334E-13	-0.001433
7	0.00860	0.00860	-0.99996	-1.00000	-2.4210E-13	-0.001433

Performed for all surface nodes and RMS is calculated according to the above equation

RMS Surface Distortion and Radius of Curvature Comparison for FEA vs. Closed Form Equation Results for Case 3 (-50°C)

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.135E-06
RMS of best fit of FEA deformed	3.079E-06
Percent Difference between FEA and Closed Form	1.777

Radius of Curvature Results:

FEA Radius of Curvature	1.012
Closed Form Radius of Curvature	1.009
Percent Difference between FEA and Closed Form	0.336

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4329E-15	-0.001433
2	0.00143	0.00143	-1.00000	-1.00000	-1.2896E-14	-0.001433
3	0.00287	0.00287	-1.00000	-1.00000	-3.5822E-14	-0.001433
4	0.00430	0.00430	-0.99999	-1.00000	-7.0212E-14	-0.001433
5	0.00573	0.00573	-0.99998	-1.00000	-1.4329E-13	-0.001433
6	0.00717	0.00716	-0.99997	-1.00000	-1.7338E-13	-0.001433
7	0.00860	0.00860	-0.99996	-1.00000	-2.4215E-13	-0.001433

Performed for all surface nodes and RMS is calculated according to the above equation

RMS Surface Distortion and Radius of Curvature Comparison for FEA vs. Closed Form Equation Results for Case 4 (-25°C)

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.021E-06
RMS of best fit of FEA deformed	3.026E-06
Percent Difference between FEA and Closed Form	-0.172

Radius of Curvature Results:

FEA Radius of Curvature	1.005
Closed Form Radius of Curvature	1.004
Percent Difference between FEA and Closed Form	0.107

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4331E-15	-0.001433
2	0.00143	0.00143	-1.00000	-1.00000	-1.2897E-14	-0.001433
3	0.00287	0.00287	-1.00000	-1.00000	-3.5826E-14	-0.001433
4	0.00430	0.00430	-0.99999	-1.00000	-7.0219E-14	-0.001433
5	0.00573	0.00573	-0.99998	-1.00000	-1.4330E-13	-0.001433
6	0.00717	0.00717	-0.99997	-1.00000	-1.7339E-13	-0.001433
7	0.00860	0.00860	-0.99996	-1.00000	-2.4217E-13	-0.001433

Performed for all surface nodes and RMS is calculated according to the above equation

RMS Surface Distortion and Radius of Curvature Comparison for FEA vs. Closed Form Equation Results for Case 5 (25°C)

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.092E-06
RMS of best fit of FEA deformed	3.054E-06
Percent Difference between FEA and Closed Form	1.216

Radius of Curvature Results:

FEA Radius of Curvature	0.996
Closed Form Radius of Curvature	0.994
Percent Difference between FEA and Closed Form	0.219

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4333E-15	-0.001433
2	0.00143	0.00143	-1.00000	-1.00000	-1.2900E-14	-0.001433
3	0.00287	0.00287	-1.00000	-1.00000	-3.5833E-14	-0.001433
4	0.00430	0.00430	-0.99999	-1.00000	-7.0233E-14	-0.001433
5	0.00573	0.00573	-0.99998	-1.00000	-1.4333E-13	-0.001433
6	0.00717	0.00717	-0.99997	-1.00000	-1.7343E-13	-0.001433
7	0.00860	0.00860	-0.99996	-1.00000	-2.4222E-13	-0.001433

Performed for all surface nodes and RMS is calculated according to the above equation

RMS Surface Distortion and Radius of Curvature Comparison for FEA vs. Closed Form Equation Results for Case 6 (50°C)

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.079E-06
RMS of best fit of FEA deformed	2.996E-06
Percent Difference between FEA and Closed Form	2.682

Radius of Curvature Results:

FEA Radius of Curvature	0.992
Closed Form Radius of Curvature	0.989
Percent Difference between FEA and Closed Form	0.332

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4335E-15	-0.001433
2	0.00143	0.00143	-1.00000	-1.00000	-1.2901E-14	-0.001433
3	0.00287	0.00287	-1.00000	-1.00000	-3.5837E-14	-0.001433
4	0.00430	0.00430	-0.99999	-1.00000	-7.0240E-14	-0.001433
5	0.00573	0.00573	-0.99998	-1.00000	-1.4335E-13	-0.001433
6	0.00717	0.00717	-0.99997	-1.00000	-1.7345E-13	-0.001433
7	0.00860	0.00860	-0.99996	-1.00000	-1.0850E-11	-0.001433

Performed for all surface nodes and RMS is calculated according to the above equation

RMS Surface Distortion and Radius of Curvature Comparison for FEA vs. Closed Form Equation Results for Case 7 (100°C)

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.060E-06
RMS of best fit of FEA deformed	3.108E-06
Percent Difference between FEA and Closed Form	-1.569

Radius of Curvature Results:

FEA Radius of Curvature	0.978
Closed Form Radius of Curvature	0.979
Percent Difference between FEA and Closed Form	-0.147

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4338E-15	-0.001434
2	0.00143	0.00143	-1.00000	-1.00000	-1.2904E-14	-0.001434
3	0.00287	0.00287	-1.00000	-1.00000	-3.5844E-14	-0.001434
4	0.00430	0.00430	-0.99999	-1.00000	-7.0254E-14	-0.001434
5	0.00573	0.00574	-0.99998	-1.00000	-1.4338E-13	-0.001434
6	0.00717	0.00717	-0.99997	-1.00000	-1.7348E-13	-0.001434
7	0.00860	0.00860	-0.99996	-1.00000	-2.4230E-13	-0.001434

Performed for all surface nodes and RMS is calculated according to the above equation

RMS Surface Distortion and Radius of Curvature Comparison for FEA vs. Closed Form Equation Results for Case 8 (250°C)

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.267E-06
RMS of best fit of FEA deformed	3.224E-06
Percent Difference between FEA and Closed Form	1.336

Radius of Curvature Results:

FEA Radius of Curvature	0.950
Closed Form Radius of Curvature	0.951
Percent Difference between FEA and Closed Form	-0.105

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4347E-15	-0.001435
2	0.00143	0.00143	-1.00000	-1.00000	-1.2912E-14	-0.001435
3	0.00287	0.00287	-1.00000	-1.00000	-3.5866E-14	-0.001435
4	0.00430	0.00430	-0.99999	-1.00000	-7.0297E-14	-0.001435
5	0.00573	0.00574	-0.99998	-1.00000	-1.4346E-13	-0.001435
6	0.00717	0.00717	-0.99997	-1.00000	-1.7359E-13	-0.001435
7	0.00860	0.00861	-0.99996	-1.00000	-1.0858E-11	-0.001435

Performed for all surface nodes and RMS is calculated according to the above equation

Zero Curvature Equation Development for Uniform Temperature Loading

This case determines the definitions of the material properties to make the curvature resultant from the temperature change to remain at 0 curvature. The trough has an initial offset curvature and there will be no resultant curvature from the temperature change ΔT .

$$R := 1\text{m} \quad \kappa_{\text{offset}} := \frac{1}{R} \quad \kappa_{\text{offset}} = 1 \frac{1}{\text{m}}$$

The curvature of the trough from the thermal strain is to remain at 0 after the temperature change is:

$$\kappa_{\text{thermal}} := 0$$

The constitutive equation for the thermal curvature is:

$$\kappa_{\text{thermal}} = 2 \cdot \Delta T \cdot \frac{a_{12} + a_{13} + a_{14} + a_{23} + a_{24} + a_{34}}{b_{12} + b_{13} + b_{14} + b_{23} + b_{24} + b_{34} + 4 \cdot c \cdot d}$$

The top is set equal to 0, because $\Delta T = 0$ is not desired and the bottom cannot be 0 (undefined).

$$0 = a_{12} + a_{13} + a_{14} + a_{23} + a_{24} + a_{34}$$

Where:

$$a_{12} = s_1 \cdot s_2 \cdot (h_1 + h_2) \cdot (\alpha_2 - \alpha_1) \quad s_1 = h_1 \cdot E_1$$

$$a_{34} = s_3 \cdot s_4 \cdot (h_3 + h_4) \cdot (\alpha_4 - \alpha_3) \quad s_3 = h_3 \cdot E_3$$

$$a_{13} = s_1 \cdot s_3 \cdot (h_1 + 2 \cdot h_2 + h_3) \cdot (\alpha_3 - \alpha_1) \quad s_2 = h_2 \cdot E_2$$

$$a_{14} = s_1 \cdot s_4 \cdot [h_1 + 2 \cdot (h_2 + h_3) + h_4] \cdot (\alpha_4 - \alpha_1) \quad s_4 = h_4 \cdot E_4$$

$$a_{23} = s_2 \cdot s_3 \cdot (h_2 + h_3) \cdot (\alpha_3 - \alpha_2)$$

$$a_{24} = s_2 \cdot s_4 \cdot (h_2 + 2 \cdot h_3 + h_4) \cdot (\alpha_4 - \alpha_2)$$

This creates five possibilities for the equation above to equal 0. From this, all CTE's can be written in terms of the other material properties.

$$a_{12} = -(a_{13} + a_{14} + a_{23} + a_{24} + a_{34})$$

By substitution and rearranging the result is:

$$h_{12} = h_1 + h_2 \quad h_{13} = h_1 + 2 \cdot h_2 + h_3 \quad h_{14} = h_1 + 2 \cdot h_2 + 2 \cdot h_3 + h_4$$

$$h_{23} = h_2 + h_3 \quad h_{24} = h_2 + 2 \cdot h_3 + h_4 \quad h_{34} = h_3 + h_4$$

$$P_1 = \alpha_2 \cdot [s_1 \cdot s_2 \cdot (-h_{12}) + s_2 \cdot s_3 \cdot (h_{23}) + s_2 \cdot s_4 \cdot (h_{24})]$$

$$P_2 = \alpha_3 [s_1 \cdot s_3 \cdot (-h_{13}) + s_2 \cdot s_3 \cdot (-h_{23}) + s_3 \cdot s_4 \cdot (h_{34})]$$

$$P_3 = \alpha_4 [[s_1 \cdot s_4 \cdot (-h_{14}) + s_2 \cdot s_4 \cdot (-h_{24})] + s_3 \cdot s_4 \cdot (-h_{34})]$$

$$\alpha_1 = -\frac{P_1 + P_2 + P_3}{s_1 \cdot s_3 \cdot (h_{13}) + s_1 \cdot s_2 \cdot (h_{12}) + s_1 \cdot s_4 \cdot (h_{14})}$$

Test of above equation:

Material properties of the trough layers

$$E_1 := 10 \cdot 10^9 \text{ Pa} \quad E_2 := 20 \cdot 10^9 \text{ Pa} \quad E_3 := 35 \cdot 10^9 \text{ Pa} \quad E_4 := 40 \cdot 10^9 \text{ Pa}$$

$$\alpha_2 := 10 \cdot 10^{-6} \frac{1}{\text{C}} \quad \alpha_3 := 18 \cdot 10^{-6} \frac{1}{\text{C}} \quad \alpha_4 := 20 \cdot 10^{-6} \frac{1}{\text{C}}$$

Material geometries of the trough layers

$$\text{layer thickness:} \quad h_1 := 0.023 \text{ m} \quad h_2 := 0.023 \text{ m} \quad h_3 := 0.023 \text{ m} \quad h_4 := 0.023 \text{ m}$$

$$\text{Unit depth:} \quad \text{unit} := 1$$

Temperature change

$$\Delta T := 250 \text{ C}$$

Material Property of Layer 1 to obtain 0 curvature in the trough:

$$s_1 := h_1 \cdot E_1 \quad s_3 := h_3 \cdot E_3 \quad s_2 := h_2 \cdot E_2 \quad s_4 := h_4 \cdot E_4$$

$$h_{12} := h_1 + h_2 \quad h_{13} := h_1 + 2 \cdot h_2 + h_3 \quad h_{14} := h_1 + 2 \cdot h_2 + 2 \cdot h_3 + h_4$$

$$h_{23} := h_2 + h_3 \quad h_{24} := h_2 + 2 \cdot h_3 + h_4 \quad h_{34} := h_3 + h_4$$

$$P_2 := \alpha_2 [s_1 \cdot s_2 \cdot (-h_{12}) + s_2 \cdot s_3 \cdot (h_{23}) + s_2 \cdot s_4 \cdot (h_{24})]$$

$$P_3 := \alpha_3 [s_1 \cdot s_3 \cdot (-h_{13}) + s_2 \cdot s_3 \cdot (-h_{23}) + s_3 \cdot s_4 \cdot (h_{34})]$$

$$P_4 := \alpha_4 [s_1 \cdot s_4 \cdot (-h_{14}) + s_2 \cdot s_4 \cdot (-h_{24}) + s_3 \cdot s_4 \cdot (-h_{34})]$$

$$\alpha_1 := -\frac{P_2 + P_3 + P_4}{s_1 \cdot s_3 \cdot (h_{13}) + s_1 \cdot s_2 \cdot (h_{12}) + s_1 \cdot s_4 \cdot (h_{14})}$$

$$\alpha_1 = 3 \times 10^{-5} \frac{1}{\text{K}}$$

The next possibility to produce 0 curvature is:

$$a_{13} = -(a_{12} + a_{14} + a_{23} + a_{24} + a_{34})$$

By substitution and rearranging the result is:

$$h_{12} := h_1 + h_2 \quad h_{13} := h_1 + 2 \cdot h_2 + h_3 \quad h_{14} := h_1 + 2 \cdot h_2 + 2 \cdot h_3 + h_4$$

$$h_{23} := h_2 + h_3 \quad h_{24} := h_2 + 2 \cdot h_3 + h_4 \quad h_{34} := h_3 + h_4$$

$$P_2 := \alpha_2 [s_1 \cdot s_2 \cdot (-h_{12}) + s_2 \cdot s_3 \cdot (h_{23}) + s_2 \cdot s_4 \cdot (h_{24})]$$

$$P_3 := \alpha_3 \cdot [s_1 \cdot s_3 \cdot (-h_{13}) + s_2 \cdot s_3 \cdot (-h_{23}) + s_3 \cdot s_4 \cdot (h_{34})]$$

$$P_4 := \alpha_4 \cdot [s_1 \cdot s_4 \cdot (-h_{14}) + s_2 \cdot s_4 \cdot (-h_{24}) + s_3 \cdot s_4 \cdot (-h_{34})]$$

$$\alpha_1 := -\frac{P_2 + P_3 + P_4}{s_1 \cdot s_3 \cdot (h_{13}) + s_1 \cdot s_2 \cdot (h_{12}) + s_1 \cdot s_4 \cdot (h_{14})}$$

This is an identical result to the first derivation. The next possibilities were not performed, because the derivation would have turned out the same.

Coefficient of Thermal expansion design equations that produce a 0 curvature written in terms of the four CTE's of a four layer trough.

$$P_1 := \alpha_1 \cdot [s_1 \cdot s_2 \cdot h_{12} + s_1 \cdot s_3 \cdot (h_{13}) + s_1 \cdot s_4 \cdot (h_{14})]$$

$$P_2 := \alpha_2 \cdot [s_1 \cdot s_2 \cdot (-h_{12}) + s_2 \cdot s_3 \cdot (h_{23}) + s_2 \cdot s_4 \cdot (h_{24})]$$

$$P_3 := \alpha_3 \cdot [s_1 \cdot s_3 \cdot (-h_{13}) + s_2 \cdot s_3 \cdot (-h_{23}) + s_3 \cdot s_4 \cdot (h_{34})]$$

$$P_4 := \alpha_4 \cdot [s_1 \cdot s_4 \cdot (-h_{14}) + s_2 \cdot s_4 \cdot (-h_{24}) + s_3 \cdot s_4 \cdot (-h_{34})]$$

$$0 = P_1 + P_2 + P_3 + P_4$$

$$\alpha_1 = -\frac{P_2 + P_3 + P_4}{s_1 \cdot s_3 \cdot (h_{13}) + s_1 \cdot s_2 \cdot (h_{12}) + s_1 \cdot s_4 \cdot (h_{14})}$$

$$\alpha_2 = -\frac{P_1 + P_3 + P_4}{s_1 \cdot s_2 \cdot (-h_{12}) + s_2 \cdot s_3 \cdot (h_{23}) + s_2 \cdot s_4 \cdot (h_{24})}$$

$$\alpha_3 = -\frac{P_1 + P_2 + P_4}{s_1 \cdot s_3 \cdot (-h_{13}) + s_2 \cdot s_3 \cdot (-h_{23}) + s_3 \cdot s_4 \cdot (h_{34})}$$

$$\alpha_4 = -\frac{P_1 + P_2 + P_3}{s_1 \cdot s_4 \cdot (-h_{14}) + s_2 \cdot s_4 \cdot (-h_{24}) + s_3 \cdot s_4 \cdot (-h_{34})}$$

General Form of 0 curvature equation derivation:

$$P_i = \alpha_i \cdot \left[\sum_{j=1}^N [s_i \cdot s_j \cdot (h_{ij})] - \sum_{j=1}^N [s_i \cdot s_j \cdot (h_{ij})] \cdot \delta(i, j) \right]$$

$$\text{for } i < j \quad s_i \cdot s_j > 0$$

$$\text{for } i > j \quad s_i \cdot s_j < 0$$

$$0 = P_i + P_{i+1} + P_{i+2} \dots P_{N-1} + P_N$$

Where

$$s_j \cdot s_i = -s_i \cdot s_j$$

$$s = \begin{bmatrix} 0 & s_1 \cdot s_2 & s_1 \cdot s_3 & s_1 \cdot s_{N-1} & s_1 \cdot s_N \\ -(s_1 \cdot s_2) & 0 & s_2 \cdot s_3 & s_2 \cdot s_{N-1} & s_2 \cdot s_N \\ -(s_1 \cdot s_3) & -(s_2 \cdot s_3) & 0 & s_3 \cdot s_{N-1} & s_3 \cdot s_N \\ -(s_1 \cdot s_{N-1}) & -(s_2 \cdot s_{N-1}) & -(s_3 \cdot s_{N-1}) & 0 & s_{N-1} \cdot s_N \\ -(s_1 \cdot s_N) & -(s_2 \cdot s_N) & -(s_3 \cdot s_N) & -(s_{N-1} \cdot s_N) & 0 \end{bmatrix}$$

$$h_{ij} = h_{ji}$$

$$h_{21} = h_{12} = h_1 + h_2$$

$$h_{31} = h_{13} = h_1 + 2 \cdot h_2 + h_3$$

$$h_{41} = h_{14} = h_1 + 2 \cdot h_2 + 2 \cdot h_3 + h_4$$

$$h_{N1} = h_{1N} = h_1 + 2 \cdot h_2 + 2 \cdot h_3 \dots 2 \cdot h_{N-1} + h_N$$

$$h_{Ni} = h_{iN} = h_i + 2 \cdot h_{i+1} + 2 \cdot h_{i+2} \dots 2 \cdot h_{N-1} + h_N$$

Zero Curvature of Four Layer Trough Scenarios for Uniform Temperature Loading

These scenarios explore the range of dependent CTE that is required to keep the system at 0 thermal curvature by defining 3 of the 4 CTE's.

Scenarios for Layer One

Material properties of the trough layers

$$E_1 := 15 \cdot 10^9 \text{ Pa} \quad E_2 := 20 \cdot 10^9 \text{ Pa} \quad E_3 := 35 \cdot 10^9 \text{ Pa} \quad E_4 := 40 \cdot 10^9 \text{ Pa}$$

Case 1

$$\alpha_2 := \begin{pmatrix} 10 \\ 10 \\ 1 \\ -1 \\ -10 \\ -10 \\ 10 \\ 10 \\ 10 \\ 30 \\ 30 \\ 30 \\ 30 \\ -30 \\ -20 \\ -10 \\ 10 \\ 20 \\ 30 \\ 1 \\ 40 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}} \quad \alpha_3 := \begin{pmatrix} 20 \\ 15 \\ 5 \\ -5 \\ -15 \\ -20 \\ 10 \\ 10 \\ 30 \\ 30 \\ 30 \\ 30 \\ 10 \\ -20 \\ -15 \\ -5 \\ 5 \\ 15 \\ 20 \\ 40 \\ 40 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}} \quad \alpha_4 := \begin{pmatrix} 30 \\ 20 \\ 10 \\ -10 \\ -20 \\ -30 \\ 10 \\ 30 \\ 30 \\ 30 \\ 10 \\ 10 \\ 10 \\ -10 \\ -10 \\ -1 \\ 1 \\ 10 \\ 10 \\ 40 \\ 1 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$$

Case 13

Material geometries of the trough layers

$$\text{layer thickness:} \quad h_1 := 0.023 \text{ m} \quad h_2 := 0.023 \text{ m} \quad h_3 := 0.023 \text{ m} \quad h_4 := 0.023 \text{ m}$$

$$\text{Unit depth:} \quad \text{unit} := 1$$

Temperature change

$$\Delta T := 250 \text{ C}$$

Material Property of Layer 1 to obtain 0 curvature in the trough:

$$s_1 := h_1 \cdot E_1 \quad s_2 := h_2 \cdot E_2 \quad s_3 := h_3 \cdot E_3 \quad s_4 := h_4 \cdot E_4$$

$$\begin{aligned}
h_{12} &:= h_1 + h_2 & h_{13} &:= h_1 + 2 \cdot h_2 + h_3 & h_{14} &:= h_1 + 2 \cdot h_2 + 2 \cdot h_3 + h_4 \\
h_{23} &:= h_2 + h_3 & h_{24} &:= h_2 + 2 \cdot h_3 + h_4 & h_{34} &:= h_3 + h_4
\end{aligned}$$

$$P_2 := \alpha_2 \cdot [s_1 \cdot s_2 \cdot (-h_{12}) + s_2 \cdot s_3 \cdot (h_{23}) + s_2 \cdot s_4 \cdot (h_{24})]$$

$$P_3 := \alpha_3 \cdot [s_1 \cdot s_3 \cdot (-h_{13}) + s_2 \cdot s_3 \cdot (-h_{23}) + s_3 \cdot s_4 \cdot (h_{34})]$$

$$P_4 := \alpha_4 \cdot [s_1 \cdot s_4 \cdot (-h_{14}) + s_2 \cdot s_4 \cdot (-h_{24}) + s_3 \cdot s_4 \cdot (-h_{34})]$$

$$\alpha_1 := -\frac{P_2 + P_3 + P_4}{s_1 \cdot s_3 \cdot (h_{13}) + s_1 \cdot s_2 \cdot (h_{12}) + s_1 \cdot s_4 \cdot (h_{14})}$$

$$\alpha_1 = \frac{1}{C} \begin{pmatrix} 4.159 \times 10^{-5} \\ 2.579 \times 10^{-5} \\ 1.516 \times 10^{-5} \\ -1.516 \times 10^{-5} \\ -2.579 \times 10^{-5} \\ -4.159 \times 10^{-5} \\ 1 \times 10^{-5} \\ 4.048 \times 10^{-5} \\ 4.27 \times 10^{-5} \\ 3 \times 10^{-5} \\ -4.762 \times 10^{-7} \\ -2.698 \times 10^{-6} \\ 1.587 \times 10^{-6} \\ -4.206 \times 10^{-6} \\ 4.27 \times 10^{-6} \\ -4.27 \times 10^{-6} \\ 4.206 \times 10^{-6} \\ -1.587 \times 10^{-6} \\ 6.476 \times 10^{-5} \\ -1.943 \times 10^{-5} \end{pmatrix} \quad \begin{array}{l} \text{Case 1} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \text{Case 13} \\ \\ \\ \\ \\ \\ \\ \\ \end{array}$$

Scenarios for Layer Two

Material properties of the trough layers

$$E_1 := 15 \cdot 10^9 \text{ Pa} \quad E_2 := 20 \cdot 10^9 \text{ Pa} \quad E_3 := 35 \cdot 10^9 \text{ Pa} \quad E_4 := 40 \cdot 10^9 \text{ Pa}$$

Case 22	$\alpha_1 := \begin{pmatrix} 10 \\ 10 \\ 1 \\ -1 \\ -10 \\ -10 \\ 10 \\ 10 \\ 10 \\ 30 \\ 30 \\ 30 \\ -30 \\ -20 \\ -10 \\ 10 \\ 20 \\ 30 \\ 1 \\ 40 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$	$\alpha_3 := \begin{pmatrix} 20 \\ 15 \\ 5 \\ -5 \\ -15 \\ -20 \\ 10 \\ 10 \\ 30 \\ 30 \\ 30 \\ 10 \\ -20 \\ -15 \\ -5 \\ 5 \\ 15 \\ 20 \\ 40 \\ 40 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$	$\alpha_4 := \begin{pmatrix} 30 \\ 20 \\ 10 \\ -10 \\ -20 \\ -30 \\ 10 \\ 30 \\ 30 \\ 30 \\ 10 \\ 10 \\ -10 \\ -10 \\ -1 \\ 1 \\ 10 \\ 10 \\ 40 \\ 1 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$
Case 37			

Material geometries of the trough layers

$$\text{layer thickness:} \quad h_1 := 0.023 \text{ m} \quad h_2 := 0.023 \text{ m} \quad h_3 := 0.023 \text{ m} \quad h_4 := 0.023 \text{ m}$$

$$\text{Unit depth:} \quad \text{unit} := 1$$

Temperature change

$$\Delta T := 250 \text{ C}$$

Material Property of Layer 1 to obtain 0 curvature in the trough:

$$s_1 := h_1 \cdot E_1 \quad s_2 := h_2 \cdot E_2 \quad s_3 := h_3 \cdot E_3 \quad s_4 := h_4 \cdot E_4$$

$$h_{12} := h_1 + h_2 \quad h_{13} := h_1 + 2 \cdot h_2 + h_3 \quad h_{14} := h_1 + 2 \cdot h_2 + 2 \cdot h_3 + h_4$$

$$h_{23} := h_2 + h_3 \quad h_{24} := h_2 + 2 \cdot h_3 + h_4 \quad h_{34} := h_3 + h_4$$

$$P_1 := \alpha_1 \cdot [s_1 \cdot s_2 \cdot h_{12} + s_1 \cdot s_3 \cdot (h_{13}) + s_1 \cdot s_4 \cdot (h_{14})]$$

$$P_3 := \alpha_3 \cdot [s_1 \cdot s_3 \cdot (-h_{13}) + s_2 \cdot s_3 \cdot (-h_{23}) + s_3 \cdot s_4 \cdot (h_{34})]$$

$$P_4 := \alpha_4 \cdot [s_1 \cdot s_4 \cdot (-h_{14}) + s_2 \cdot s_4 \cdot (-h_{24}) + s_3 \cdot s_4 \cdot (-h_{34})]$$

$$\alpha_2 := -\frac{P_1 + P_3 + P_4}{s_1 \cdot s_2 \cdot (-h_{12}) + s_2 \cdot s_3 \cdot (h_{23}) + s_2 \cdot s_4 \cdot (h_{24})}$$

$$\alpha_2 = \begin{pmatrix} 5.975 \times 10^{-5} \\ 3.487 \times 10^{-5} \\ 2.33 \times 10^{-5} \\ -2.33 \times 10^{-5} \\ -3.487 \times 10^{-5} \\ -5.975 \times 10^{-5} \\ 1 \times 10^{-5} \\ 5.8 \times 10^{-5} \\ 6.15 \times 10^{-5} \\ 3 \times 10^{-5} \\ -1.8 \times 10^{-5} \\ -2.15 \times 10^{-5} \\ 1.975 \times 10^{-5} \\ 4.875 \times 10^{-6} \\ 1.247 \times 10^{-5} \\ -1.247 \times 10^{-5} \\ -4.875 \times 10^{-6} \\ -1.975 \times 10^{-5} \\ 1.014 \times 10^{-4} \\ -5.36 \times 10^{-5} \end{pmatrix} \frac{1}{C}$$

Case 22

Case 37

Scenarios for Layer Three

Material properties of the trough layers

$$E_1 := 15 \cdot 10^9 \text{ Pa} \quad E_2 := 20 \cdot 10^9 \text{ Pa} \quad E_3 := 35 \cdot 10^9 \text{ Pa} \quad E_4 := 40 \cdot 10^9 \text{ Pa}$$

	$\begin{pmatrix} 10 \\ 10 \\ 1 \\ -1 \\ -10 \\ -10 \\ 10 \\ 10 \\ 10 \\ 30 \\ 30 \\ 30 \\ -30 \\ -20 \\ -10 \\ 10 \\ 20 \\ 30 \\ 1 \\ 40 \end{pmatrix}$		$\begin{pmatrix} 20 \\ 15 \\ 5 \\ -5 \\ -15 \\ -20 \\ 10 \\ 10 \\ 30 \\ 30 \\ 30 \\ 10 \\ -20 \\ -15 \\ -5 \\ 5 \\ 15 \\ 20 \\ 40 \\ 40 \end{pmatrix}$		$\begin{pmatrix} 30 \\ 20 \\ 10 \\ -10 \\ -20 \\ -30 \\ 10 \\ 30 \\ 30 \\ 30 \\ 10 \\ 10 \\ -10 \\ -10 \\ -1 \\ 1 \\ 10 \\ 10 \\ 40 \\ 1 \end{pmatrix}$
Case 44					
Case 47					
	$\alpha_1 := 10^{-6} \cdot \frac{1}{\text{C}}$		$\alpha_2 := 10^{-6} \cdot \frac{1}{\text{C}}$		$\alpha_4 := 10^{-6} \cdot \frac{1}{\text{C}}$

Material geometries of the trough layers

$$\text{layer thickness:} \quad h_1 := 0.023\text{m} \quad h_2 := 0.023\text{m} \quad h_3 := 0.023\text{m} \quad h_4 := 0.023\text{m}$$

$$\text{Unit depth:} \quad \text{unit} := 1$$

Temperature change

$$\Delta T := 250\text{C}$$

Material Property of Layer 1 to obtain 0 curvature in the trough:

$$s_1 := h_1 \cdot E_1 \quad s_2 := h_2 \cdot E_2 \quad s_3 := h_3 \cdot E_3 \quad s_4 := h_4 \cdot E_4$$

$$h_{12} := h_1 + h_2 \quad h_{13} := h_1 + 2 \cdot h_2 + h_3 \quad h_{14} := h_1 + 2 \cdot h_2 + 2 \cdot h_3 + h_4$$

$$h_{23} := h_2 + h_3 \quad h_{24} := h_2 + 2 \cdot h_3 + h_4 \quad h_{34} := h_3 + h_4$$

$$P_1 := \alpha_1 \cdot [s_1 \cdot s_2 \cdot (h_{12}) + s_1 \cdot s_3 \cdot (h_{13}) + s_1 \cdot s_4 \cdot (h_{14})]$$

$$P_2 := \alpha_2 \cdot [s_1 \cdot s_2 \cdot (-h_{12}) + s_2 \cdot s_3 \cdot (h_{23}) + s_2 \cdot s_4 \cdot (h_{24})]$$

$$P_4 := \alpha_4 \cdot [s_1 \cdot s_4 \cdot (-h_{14}) + s_2 \cdot s_4 \cdot (-h_{24}) + s_3 \cdot s_4 \cdot (-h_{34})]$$

$$\alpha_3 := -\frac{P_1 + P_2 + P_4}{s_1 \cdot s_3 \cdot (-h_{13}) + s_2 \cdot s_3 \cdot (-h_{23}) + s_3 \cdot s_4 \cdot (h_{34})}$$

$$\alpha_3 = \frac{1}{C} \begin{pmatrix} -2.071 \times 10^{-4} \\ -9.857 \times 10^{-5} \\ -9.957 \times 10^{-5} \\ 9.957 \times 10^{-5} \\ 9.857 \times 10^{-5} \\ 2.071 \times 10^{-4} \\ 10 \times 10^{-6} \\ -2.643 \times 10^{-4} \\ -1.5 \times 10^{-4} \\ 3 \times 10^{-5} \\ 3.043 \times 10^{-4} \\ 1.9 \times 10^{-4} \\ -2.471 \times 10^{-4} \\ -1.286 \times 10^{-4} \\ -1.049 \times 10^{-4} \\ 1.049 \times 10^{-4} \\ 1.286 \times 10^{-4} \\ 2.471 \times 10^{-4} \\ -3.11 \times 10^{-4} \\ 5.749 \times 10^{-4} \end{pmatrix}$$

Case 44

Case 47

Scenarios for Layer Four

Material properties of the trough layers

$$E_1 := 15 \cdot 10^9 \text{ Pa} \quad E_2 := 20 \cdot 10^9 \text{ Pa} \quad E_3 := 35 \cdot 10^9 \text{ Pa} \quad E_4 := 40 \cdot 10^9 \text{ Pa}$$

Case 61	$\alpha_1 := \begin{pmatrix} 10 \\ 10 \\ 1 \\ -1 \\ -10 \\ -10 \\ 10 \\ 10 \\ 10 \\ 30 \\ 30 \\ 30 \\ -30 \\ -20 \\ -10 \\ 10 \\ 20 \\ 30 \\ 1 \\ 40 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$	$\alpha_2 := \begin{pmatrix} 20 \\ 15 \\ 5 \\ -5 \\ -15 \\ -20 \\ 10 \\ 10 \\ 30 \\ 30 \\ 30 \\ 10 \\ -20 \\ -15 \\ -5 \\ 5 \\ 15 \\ 20 \\ 40 \\ 40 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$	$\alpha_3 := \begin{pmatrix} 30 \\ 20 \\ 10 \\ -10 \\ -20 \\ -30 \\ 10 \\ 30 \\ 30 \\ 30 \\ 10 \\ 10 \\ -10 \\ -10 \\ -1 \\ 1 \\ 10 \\ 10 \\ 40 \\ 1 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$
Case 75			

Material geometries of the trough layers

$$\text{layer thickness:} \quad h_1 := 0.023\text{m} \quad h_2 := 0.023\text{m} \quad h_3 := 0.023\text{m} \quad h_4 := 0.023\text{m}$$

$$\text{Unit depth:} \quad \text{unit} := 1$$

Temperature change

$$\Delta T := 250\text{C}$$

Material Property of Layer 1 to obtain 0 curvature in the trough:

$$s_1 := h_1 \cdot E_1 \quad s_2 := h_2 \cdot E_2 \quad s_3 := h_3 \cdot E_3 \quad s_4 := h_4 \cdot E_4$$

$$h_{12} := h_1 + h_2 \quad h_{13} := h_1 + 2 \cdot h_2 + h_3 \quad h_{14} := h_1 + 2 \cdot h_2 + 2 \cdot h_3 + h_4$$

$$h_{23} := h_2 + h_3 \quad h_{24} := h_2 + 2 \cdot h_3 + h_4 \quad h_{34} := h_3 + h_4$$

$$P_1 := \alpha_1 \cdot [s_1 \cdot s_2 \cdot (h_{12}) + s_1 \cdot s_3 \cdot (h_{13}) + s_1 \cdot s_4 \cdot (h_{14})]$$

$$P_2 := \alpha_2 \cdot [s_1 \cdot s_2 \cdot (-h_{12}) + s_2 \cdot s_3 \cdot (h_{23}) + s_2 \cdot s_4 \cdot (h_{24})]$$

$$P_3 := \alpha_3 \cdot [s_1 \cdot s_3 \cdot (-h_{13}) + s_2 \cdot s_3 \cdot (-h_{23}) + s_3 \cdot s_4 \cdot (h_{34})]$$

$$\alpha_4 := -\frac{P_1 + P_2 + P_3}{s_1 \cdot s_4 \cdot (-h_{14}) + s_2 \cdot s_4 \cdot (-h_{24}) + s_3 \cdot s_4 \cdot (-h_{34})}$$

$\alpha_4 =$	1.271×10^{-5}	$\frac{1}{C}$	Case 61
	1.135×10^{-5}		
	2.01×10^{-6}		
	-2.01×10^{-6}		
	-1.135×10^{-5}		
	-1.271×10^{-5}		
	1×10^{-5}		
	8.542×10^{-6}		
	1.687×10^{-5}		
	3×10^{-5}		
	3.146×10^{-5}	Case 75	
	2.312×10^{-5}		
	-2.729×10^{-5}		
	-1.865×10^{-5}		
	-8.573×10^{-6}		
	8.573×10^{-6}		
	1.865×10^{-5}		
	2.729×10^{-5}		
	1.441×10^{-5}		
	4.284×10^{-5}		

**RMS Surface Distortion and Radius of Curvature Comparison for FEA vs.
Zero Curvature Closed Form Equation Results for Case 1 (250°C)**

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.020E-06
RMS of best fit of FEA deformed	3.147E-06
Percent Difference between FEA and Closed Form	-4.190

Radius of Curvature Results:

FEA Radius of Curvature	0.996
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	-0.393

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4416E-15	-0.001442
2	0.00143	0.00144	-1.00000	-1.00000	-1.2974E-14	-0.001442
3	0.00287	0.00288	-1.00000	-1.00000	-3.6039E-14	-0.001442
4	0.00430	0.00432	-0.99999	-1.00000	-7.0637E-14	-0.001442
5	0.00573	0.00577	-0.99998	-1.00000	-1.4416E-13	-0.001442
6	0.00717	0.00721	-0.99997	-1.00000	-1.7443E-13	-0.001442
7	0.00860	0.00865	-0.99996	-1.00000	-2.4362E-13	-0.001442

Performed for all surface nodes and RMS is calculated according to the above equation

**RMS Surface Distortion and Radius of Curvature Comparison for FEA vs.
Zero Curvature Closed Form Equation Results for Case 2 (250°C)**

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.100E-06
RMS of best fit of FEA deformed	3.022E-06
Percent Difference between FEA and Closed Form	2.506

Radius of Curvature Results:

FEA Radius of Curvature	1.005
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	0.539

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2 (x'_i - x'_{i-1})$	
1	0.00000	0.00000	-1.00000	-1.00000	-1.4401E-15	-0.001440
2	0.00143	0.00144	-1.00000	-1.00000	-1.2961E-14	-0.001440
3	0.00287	0.00288	-1.00000	-1.00000	-3.6003E-14	-0.001440
4	0.00430	0.00432	-0.99999	-1.00000	-7.0566E-14	-0.001440
5	0.00573	0.00576	-0.99998	-1.00000	-1.4401E-13	-0.001440
6	0.00717	0.00720	-0.99997	-1.00000	-1.7425E-13	-0.001440
7	0.00860	0.00864	-0.99996	-1.00000	-2.4337E-13	-0.001440

Performed for all surface nodes and RMS is calculated according to the above equation

**RMS Surface Distortion and Radius of Curvature Comparison for FEA vs.
Zero Curvature Closed Form Equation Results for Case 3 (250°C)**

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.040E-06
RMS of best fit of FEA deformed	3.154E-06
Percent Difference between FEA and Closed Form	-3.756

Radius of Curvature Results:

FEA Radius of Curvature	1.002
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	0.222

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4368E-15	-0.001437
2	0.00143	0.00144	-1.00000	-1.00000	-1.2931E-14	-0.001437
3	0.00287	0.00287	-1.00000	-1.00000	-3.5919E-14	-0.001437
4	0.00430	0.00431	-0.99999	-1.00000	-7.0400E-14	-0.001437
5	0.00573	0.00575	-0.99998	-1.00000	-1.4367E-13	-0.001437
6	0.00717	0.00718	-0.99997	-1.00000	-1.7384E-13	-0.001437
7	0.00860	0.00862	-0.99996	-1.00000	-2.4280E-13	-0.001437

Performed for all surface nodes and RMS is calculated according to the above equation

**RMS Surface Distortion and Radius of Curvature Comparison for FEA vs.
Zero Curvature Closed Form Equation Results for Case 4 (250°C)**

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.097E-06
RMS of best fit of FEA deformed	3.050E-06
Percent Difference between FEA and Closed Form	1.519

Radius of Curvature Results:

FEA Radius of Curvature	1.031
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	3.047

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4443E-15	-0.001444
2	0.00143	0.00144	-1.00000	-1.00000	-1.2999E-14	-0.001444
3	0.00287	0.00289	-1.00000	-1.00000	-3.6107E-14	-0.001444
4	0.00430	0.00433	-0.99999	-1.00000	-7.0770E-14	-0.001444
5	0.00573	0.00578	-0.99998	-1.00000	-1.4443E-13	-0.001444
6	0.00717	0.00722	-0.99997	-1.00000	-1.7475E-13	-0.001444
7	0.00860	0.00867	-0.99996	-1.00000	-2.4408E-13	-0.001444

Performed for all surface nodes and RMS is calculated according to the above equation

**RMS Surface Distortion and Radius of Curvature Comparison for FEA vs.
Zero Curvature Closed Form Equation Results for Case 5 (250°C)**

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.021E-06
RMS of best fit of FEA deformed	3.216E-06
Percent Difference between FEA and Closed Form	-6.473

Radius of Curvature Results:

FEA Radius of Curvature	1.003
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	0.268

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4367E-15	-0.001437
2	0.00143	0.00144	-1.00000	-1.00000	-1.2930E-14	-0.001437
3	0.00287	0.00287	-1.00000	-1.00000	-3.5917E-14	-0.001437
4	0.00430	0.00431	-0.99999	-1.00000	-7.0397E-14	-0.001437
5	0.00573	0.00575	-0.99998	-1.00000	-1.4367E-13	-0.001437
6	0.00717	0.00718	-0.99997	-1.00000	-1.7384E-13	-0.001437
7	0.00860	0.00862	-0.99996	-1.00000	-2.4279E-13	-0.001437

Performed for all surface nodes and RMS is calculated according to the above equation

**RMS Surface Distortion and Radius of Curvature Comparison for FEA vs.
Zero Curvature Closed Form Equation Results for Case 6 (250°C)**

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.066E-06
RMS of best fit of FEA deformed	3.142E-06
Percent Difference between FEA and Closed Form	-2.477

Radius of Curvature Results:

FEA Radius of Curvature	1.009
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	0.847

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4402E-15	-0.001440
2	0.00143	0.00144	-1.00000	-1.00000	-1.2962E-14	-0.001440
3	0.00287	0.00288	-1.00000	-1.00000	-3.6005E-14	-0.001440
4	0.00430	0.00432	-0.99999	-1.00000	-7.0569E-14	-0.001440
5	0.00573	0.00576	-0.99998	-1.00000	-1.4402E-13	-0.001440
6	0.00717	0.00720	-0.99997	-1.00000	-1.7426E-13	-0.001440
7	0.00860	0.00864	-0.99996	-1.00000	-2.4339E-13	-0.001440

Performed for all surface nodes and RMS is calculated according to the above equation

**RMS Surface Distortion and Radius of Curvature Comparison for FEA vs.
Zero Curvature Closed Form Equation Results for Case 7 (250°C)**

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.071E-06
RMS of best fit of FEA deformed	3.115E-06
Percent Difference between FEA and Closed Form	-1.453

Radius of Curvature Results:

FEA Radius of Curvature	1.002
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	0.153

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4313E-15	-0.001431
2	0.00143	0.00143	-1.00000	-1.00000	-1.2882E-14	-0.001431
3	0.00287	0.00286	-1.00000	-1.00000	-3.5783E-14	-0.001431
4	0.00430	0.00429	-0.99999	-1.00000	-7.0135E-14	-0.001431
5	0.00573	0.00573	-0.99998	-1.00000	-1.4313E-13	-0.001431
6	0.00717	0.00716	-0.99997	-1.00000	-1.7319E-13	-0.001431
7	0.00860	0.00859	-0.99996	-1.00000	-2.4188E-13	-0.001431

Performed for all surface nodes and RMS is calculated according to the above equation

Closed Form and FEA Curvature Comparison of Four Layer Thermostat Trough for Non-Uniform Temperatures

Predicted Curvature by method of Individual Temperatures in each layer

The following are the coordinates of the undeformed trough in both the thermostat and real geometry FEA models:

$$\begin{aligned} x_{\text{apex}} &:= 0\text{m} & y_{\text{apex}} &:= -1\text{m} \\ x_{\text{edgetrough}} &:= .2496\text{m} & y_{\text{edgetrough}} &:= -.9684\text{m} \end{aligned}$$

The trough has an initial offset curvature before the temperature change ΔT . The distance between the above coordinates gives the desired x and y values below which are used in determining the initial curvature.

$$\begin{aligned} x &:= x_{\text{edgetrough}} - x_{\text{apex}} & y &:= y_{\text{edgetrough}} - y_{\text{apex}} \\ \theta &:= \text{atan}\left(\frac{y}{x}\right) & \theta &= 0.126 \\ \kappa_{\text{offset}} &:= \frac{2 \cdot \sin(\theta)}{\sqrt{x^2 + y^2}} & \kappa_{\text{offset}} &= 0.998 \frac{1}{\text{m}} \end{aligned}$$

The curvature of the trough from the thermal strain after the temperature change is determined by the following:

Material properties of the trough layers

$$\begin{aligned} E_1 &:= 10 \cdot 10^9 \text{Pa} & E_2 &:= 20 \cdot 10^9 \text{Pa} & E_3 &:= 35 \cdot 10^9 \text{Pa} & E_4 &:= 40 \cdot 10^9 \text{Pa} \\ \alpha_1 &:= 5 \cdot 10^{-6} \frac{1}{\text{C}} & \alpha_2 &:= 10 \cdot 10^{-6} \frac{1}{\text{C}} & \alpha_3 &:= 18 \cdot 10^{-6} \frac{1}{\text{C}} & \alpha_4 &:= 20 \cdot 10^{-6} \frac{1}{\text{C}} \end{aligned}$$

Material geometries of the trough layers

$$\text{layer thickness:} \quad h_1 := 0.023\text{m} \quad h_2 := 0.023\text{m} \quad h_3 := 0.023\text{m} \quad h_4 := 0.023\text{m}$$

$$\text{Unit depth:} \quad \text{unit} := 1$$

Temperature change

$$\begin{aligned} T_0 &:= 0\text{C} \\ T_1 &:= \begin{pmatrix} 250 \\ -250 \\ 125 \\ -125 \\ 62.5 \\ -62.5 \\ 25 \\ -25 \end{pmatrix} \text{C} & T_2 &:= \begin{pmatrix} 125 \\ -125 \\ 62.5 \\ -62.5 \\ 25 \\ -25 \\ 250 \\ -250 \end{pmatrix} \text{C} & T_3 &:= \begin{pmatrix} 62.5 \\ -62.5 \\ 25 \\ -25 \\ 250 \\ -250 \\ 125 \\ -125 \end{pmatrix} \text{C} & T_4 &:= \begin{pmatrix} 25 \\ -25 \\ 250 \\ -250 \\ 125 \\ -125 \\ 62.5 \\ -62.5 \end{pmatrix} \text{C} \end{aligned}$$

The following definitions are used in the constitutive equation for the thermally induced curvature:

$$\begin{aligned}
 s_1 &:= h_1 \cdot E_1 & s_2 &:= h_2 \cdot E_2 & s_3 &:= h_3 \cdot E_3 & s_4 &:= h_4 \cdot E_4 \\
 I_1 &:= \text{unit} \cdot \frac{h_1^3}{12} & I_2 &:= \text{unit} \cdot \frac{h_2^3}{12} & I_3 &:= \text{unit} \cdot \frac{h_3^3}{12} & I_4 &:= \text{unit} \cdot \frac{h_4^3}{12} \\
 a_{12} &:= s_1 \cdot s_2 \cdot (h_1 + h_2) \cdot (\alpha_2 - \alpha_1) & a_{23} &:= s_2 \cdot s_3 \cdot (h_2 + h_3) \cdot (\alpha_3 - \alpha_2) \\
 a_{34} &:= s_3 \cdot s_4 \cdot (h_3 + h_4) \cdot (\alpha_4 - \alpha_3) \\
 a_{13} &:= s_1 \cdot s_3 \cdot (h_1 + 2 \cdot h_2 + h_3) \cdot (\alpha_3 - \alpha_1) & a_{24} &:= s_2 \cdot s_4 \cdot (h_2 + 2 \cdot h_3 + h_4) \cdot (\alpha_4 - \alpha_2) \\
 a_{14} &:= s_1 \cdot s_4 \cdot [h_1 + 2 \cdot (h_2 + h_3) + h_4] \cdot (\alpha_4 - \alpha_1) \\
 b_{12} &:= s_1 \cdot s_2 \cdot (h_1 + h_2)^2 & b_{23} &:= s_2 \cdot s_3 \cdot (h_2 + h_3)^2 & b_{34} &:= s_3 \cdot s_4 \cdot (h_3 + h_4)^2 \\
 b_{13} &:= s_1 \cdot s_3 \cdot (h_1 + 2 \cdot h_2 + h_3)^2 & b_{24} &:= s_2 \cdot s_4 \cdot (h_2 + 2 \cdot h_3 + h_4)^2 \\
 b_{14} &:= s_1 \cdot s_4 \cdot [h_1 + 2 \cdot (h_2 + h_3) + h_4]^2 \\
 c &:= E_1 \cdot I_1 + E_2 \cdot I_2 + E_3 \cdot I_3 + E_4 \cdot I_4 & d &:= s_1 + s_2 + s_3 + s_4
 \end{aligned}$$

The thermally induced curvature is theorized as:

$$T_0 := 0\text{C}$$

$$\begin{aligned}
 \Delta T_{21} &:= \frac{T_2 + T_1}{2} - T_0 & \Delta T_{31} &:= \frac{T_3 + T_1}{2} - T_0 & \Delta T_{41} &:= \frac{T_4 + T_1}{2} - T_0 \\
 \Delta T_{32} &:= \frac{T_3 + T_2}{2} - T_0 & \Delta T_{42} &:= \frac{T_4 + T_2}{2} - T_0 & \Delta T_{43} &:= \frac{T_4 + T_3}{2} - T_0
 \end{aligned}$$

$$\kappa_{\text{thermal3}} := 2 \cdot \frac{\Delta T_{21} \cdot a_{12} + \Delta T_{31} \cdot a_{13} + \Delta T_{41} \cdot a_{14} + \Delta T_{32} \cdot a_{23} + \Delta T_{42} \cdot a_{24} + \Delta T_{43} \cdot a_{34}}{b_{12} + b_{13} + b_{14} + b_{23} + b_{24} + b_{34} + 4 \cdot c \cdot d}$$

$$\kappa_{\text{thermal3}} = \begin{pmatrix} 0.023 \\ -0.023 \\ 0.028 \\ -0.028 \\ 0.022 \\ -0.022 \\ 0.021 \\ -0.021 \end{pmatrix} \frac{1}{\text{m}} \quad \kappa_{\text{total}} := \kappa_{\text{offset}} + \kappa_{\text{thermal3}}$$

Radius of thermally affected trough

$$R_{\text{total}} := \frac{1}{\kappa_{\text{total}}}$$

$$R_{\text{total}} = \begin{pmatrix} 0.979 \\ 1.025 \\ 0.975 \\ 1.03 \\ 0.98 \\ 1.024 \\ 0.981 \\ 1.023 \end{pmatrix} \text{ m}$$

RMS Surface Distortion and Radius of Curvature Comparison for FEA vs. Closed Form Equation Results for Case 1 (Non-Uniform Temperature Change)

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.110E-06
RMS of best fit of FEA deformed	3.134E-06
Percent Difference between FEA and Closed Form	-0.766

Radius of Curvature Results:

FEA Radius of Curvature	1.055
Closed Form Radius of Curvature	0.979
Percent Difference between FEA and Closed Form	7.240

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4423E-15	-0.001442
2	0.00143	0.00144	-1.00000	-1.00000	-1.2980E-14	-0.001442
3	0.00287	0.00288	-1.00000	-1.00000	-3.6056E-14	-0.001442
4	0.00430	0.00433	-0.99999	-1.00000	-7.0670E-14	-0.001442
5	0.00573	0.00577	-0.99998	-1.00000	-1.4422E-13	-0.001442
6	0.00717	0.00721	-0.99997	-1.00000	-1.7451E-13	-0.001442
7	0.00860	0.00865	-0.99996	-1.00000	-2.4373E-13	-0.001442

Performed for all surface nodes and RMS is calculated according to the above equation

RMS Surface Distortion and Radius of Curvature Comparison for FEA vs. Closed Form Equation Results for Case 2 (Non-Uniform Temperature Change)

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.203E-06
RMS of best fit of FEA deformed	3.150E-06
Percent Difference between FEA and Closed Form	1.646

Radius of Curvature Results:

FEA Radius of Curvature	0.946
Closed Form Radius of Curvature	1.025
Percent Difference between FEA and Closed Form	-8.329

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x'_{i-1}))^2 (x'_i - x'_{i-1})$	
1	0.00000	0.00000	-1.00000	-1.00000	-1.4241E-15	-0.001424
2	0.00143	0.00142	-1.00000	-1.00000	-1.2817E-14	-0.001424
3	0.00287	0.00285	-1.00000	-1.00000	-1.2853E-11	-0.001424
4	0.00430	0.00427	-0.99999	-0.99990	-6.9781E-14	-0.001424
5	0.00573	0.00570	-0.99998	-0.99990	-1.4241E-13	-0.001424
6	0.00717	0.00712	-0.99997	-0.99990	-1.7231E-13	-0.001424
7	0.00860	0.00854	-0.99996	-0.99990	-2.4066E-13	-0.001424

Performed for all surface nodes and RMS is calculated according to the above equation

RMS Surface Distortion and Radius of Curvature Comparison for FEA vs. Closed Form Equation Results for Case 3 (Non-Uniform Temperature Change)

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	5.563E-06
RMS of best fit of FEA deformed	5.019E-06
Percent Difference between FEA and Closed Form	9.777

Radius of Curvature Results:

FEA Radius of Curvature	0.756
Closed Form Radius of Curvature	0.975
Percent Difference between FEA and Closed Form	-29.005

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x'_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.3929E-11	-0.001421
2	0.00143	0.00142	-1.00000	-0.99990	-1.2791E-14	-0.001421
3	0.00287	0.00284	-1.00000	-0.99990	-1.2826E-11	-0.001421
4	0.00430	0.00426	-0.99999	-0.99980	-6.9640E-14	-0.001421
5	0.00573	0.00568	-0.99998	-0.99980	-1.1512E-11	-0.001421
6	0.00717	0.00711	-0.99997	-0.99970	-1.7197E-13	-0.001421
7	0.00860	0.00853	-0.99996	-0.99970	-1.0757E-11	-0.001421

Performed for all surface nodes and RMS is calculated according to the above equation

RMS Surface Distortion and Radius of Curvature Comparison for FEA vs. Closed Form Equation Results for Case 4 (Non-Uniform Temperature Change)

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.657E-06
RMS of best fit of FEA deformed	3.687E-06
Percent Difference between FEA and Closed Form	-0.814

Radius of Curvature Results:

FEA Radius of Curvature	0.884
Closed Form Radius of Curvature	0.980
Percent Difference between FEA and Closed Form	-10.805

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4366E-15	-0.001437
2	0.00143	0.00144	-1.00000	-1.00000	-1.2929E-14	-0.001437
3	0.00287	0.00287	-1.00000	-1.00000	-1.2965E-11	-0.001437
4	0.00430	0.00431	-0.99999	-0.99990	-7.0390E-14	-0.001437
5	0.00573	0.00575	-0.99998	-0.99990	-1.4365E-13	-0.001437
6	0.00717	0.00718	-0.99997	-0.99990	-1.1379E-11	-0.001437
7	0.00860	0.00862	-0.99996	-0.99980	-2.4277E-13	-0.001436

Performed for all surface nodes and RMS is calculated according to the above equation

RMS Surface Distortion and Radius of Curvature Comparison for FEA vs. Closed Form Equation Results for Case 5 (Non-Uniform Temperature Change)

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.687E-06
RMS of best fit of FEA deformed	3.468E-06
Percent Difference between FEA and Closed Form	5.923

Radius of Curvature Results:

FEA Radius of Curvature	1.158
Closed Form Radius of Curvature	1.024
Percent Difference between FEA and Closed Form	11.571

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x'_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4298E-15	-0.001430
2	0.00143	0.00143	-1.00000	-1.00000	-1.2869E-14	-0.001430
3	0.00287	0.00286	-1.00000	-1.00000	-3.5746E-14	-0.001430
4	0.00430	0.00429	-0.99999	-1.00000	-1.6370E-11	-0.001430
5	0.00573	0.00572	-0.99998	-1.00010	-1.4298E-13	-0.001430
6	0.00717	0.00715	-0.99997	-1.00010	-1.7300E-13	-0.001430
7	0.00860	0.00858	-0.99996	-1.00010	-2.4163E-13	-0.001430

Performed for all surface nodes and RMS is calculated according to the above equation

RMS Surface Distortion and Radius of Curvature Comparison for FEA vs. Closed Form Equation Results for Case 6 (Non-Uniform Temperature Change)

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.127E-06
RMS of best fit of FEA deformed	3.087E-06
Percent Difference between FEA and Closed Form	1.283

Radius of Curvature Results:

FEA Radius of Curvature	1.018
Closed Form Radius of Curvature	0.981
Percent Difference between FEA and Closed Form	3.640

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4435E-15	-0.001444
2	0.00143	0.00144	-1.00000	-1.00000	-1.2992E-14	-0.001444
3	0.00287	0.00289	-1.00000	-1.00000	-3.6088E-14	-0.001444
4	0.00430	0.00433	-0.99999	-1.00000	-7.0732E-14	-0.001444
5	0.00573	0.00577	-0.99998	-1.00000	-1.4435E-13	-0.001444
6	0.00717	0.00722	-0.99997	-1.00000	-1.7466E-13	-0.001443
7	0.00860	0.00866	-0.99996	-1.00000	-2.4395E-13	-0.001443

Performed for all surface nodes and RMS is calculated according to the above equation

RMS Surface Distortion and Radius of Curvature Comparison for FEA vs. Closed Form Equation Results for Case 7 (Non-Uniform Temperature Change)

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	2.979E-06
RMS of best fit of FEA deformed	3.098E-06
Percent Difference between FEA and Closed Form	-3.992

Radius of Curvature Results:

FEA Radius of Curvature	0.988
Closed Form Radius of Curvature	1.023
Percent Difference between FEA and Closed Form	-3.567

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2 (x'_i - x'_{i-1})$	
1	0.00000	0.00000	-1.00000	-1.00000	-1.4229E-15	-0.001423
2	0.00143	0.00142	-1.00000	-1.00000	-1.2806E-14	-0.001423
3	0.00287	0.00285	-1.00000	-1.00000	-3.5571E-14	-0.001423
4	0.00430	0.00427	-0.99999	-1.00000	-6.9719E-14	-0.001423
5	0.00573	0.00569	-0.99998	-1.00000	-1.1525E-11	-0.001423
6	0.00717	0.00711	-0.99997	-0.99990	-1.7216E-13	-0.001423
7	0.00860	0.00854	-0.99996	-0.99990	-2.4045E-13	-0.001423

Performed for all surface nodes and RMS is calculated according to the above equation

Zero Curvature Equation Development for Non-Uniform Temperature Loading

Material properties of the trough layers

$$E_1 := 10 \cdot 10^9 \text{ Pa} \quad E_2 := 20 \cdot 10^9 \text{ Pa} \quad E_3 := 35 \cdot 10^9 \text{ Pa} \quad E_4 := 40 \cdot 10^9 \text{ Pa}$$

$$\alpha_2 := 10 \cdot 10^{-6} \frac{1}{\text{C}} \quad \alpha_3 := 18 \cdot 10^{-6} \frac{1}{\text{C}} \quad \alpha_4 := 20 \cdot 10^{-6} \frac{1}{\text{C}}$$

Material geometries of the trough layers

$$\text{layer thickness:} \quad h_1 := 0.023 \text{ m} \quad h_2 := 0.023 \text{ m} \quad h_3 := 0.023 \text{ m} \quad h_4 := 0.023 \text{ m}$$

$$\text{Unit depth:} \quad \text{unit} := 1$$

Temperature Changes of each layer:

$$T_0 := 0 \text{ C}$$

$$T_1 := 1000 \text{ C} \quad T_2 := 500 \text{ C} \quad T_3 := 250 \text{ C} \quad T_4 := 100 \text{ C}$$

$$\Delta T_{21} := \frac{T_2 + T_1}{2} - T_0 \quad \Delta T_{31} := \frac{T_3 + T_1}{2} - T_0 \quad \Delta T_{41} := \frac{T_4 + T_1}{2} - T_0$$

$$\Delta T_{32} := \frac{T_3 + T_2}{2} - T_0 \quad \Delta T_{42} := \frac{T_4 + T_2}{2} - T_0 \quad \Delta T_{43} := \frac{T_4 + T_3}{2} - T_0$$

The curvature of the trough from the thermal strain is to remain at 0 after the temperature change is:

$$\kappa_{\text{thermal}} := 0$$

The constitutive equation for the thermal curvature is:

$$\kappa_{\text{thermal}} = 2 \cdot \frac{\Delta T_{21} \cdot a_{12} + \Delta T_{31} \cdot a_{13} + \Delta T_{41} \cdot a_{14} + \Delta T_{32} \cdot a_{23} + \Delta T_{42} \cdot a_{24} + \Delta T_{43} \cdot a_{34}}{b_{12} + b_{13} + b_{14} + b_{23} + b_{24} + b_{34} + 4 \cdot c \cdot d}$$

The top is set equal to 0, because $\Delta T = 0$ is not desired and the bottom cannot be 0 (undefined).

$$0 = \Delta T_{21} \cdot a_{12} + \Delta T_{31} \cdot a_{13} + \Delta T_{41} \cdot a_{14} + \Delta T_{32} \cdot a_{23} + \Delta T_{42} \cdot a_{24} + \Delta T_{43} \cdot a_{34}$$

Where:

$$a_{12} = s_1 \cdot s_2 \cdot (h_1 + h_2) \cdot (\alpha_2 - \alpha_1) \quad s_1 := h_1 \cdot E_1$$

$$a_{34} = s_3 \cdot s_4 \cdot (h_3 + h_4) \cdot (\alpha_4 - \alpha_3) \quad s_2 := h_2 \cdot E_2$$

$$a_{13} = s_1 \cdot s_3 \cdot (h_1 + 2 \cdot h_2 + h_3) \cdot (\alpha_3 - \alpha_1) \quad s_3 := h_3 \cdot E_3$$

$$a_{14} = s_1 \cdot s_4 \cdot [h_1 + 2 \cdot (h_2 + h_3) + h_4] \cdot (\alpha_4 - \alpha_1) \quad s_4 := h_4 \cdot E_4$$

$$a_{23} = s_2 \cdot s_3 \cdot (h_2 + h_3) \cdot (\alpha_3 - \alpha_2)$$

$$a_{24} = s_2 \cdot s_4 \cdot (h_2 + 2 \cdot h_3 + h_4) \cdot (\alpha_4 - \alpha_2)$$

By substitution and rearranging the above equations:

$$h_{12} := h_1 + h_2 \quad h_{13} := h_1 + 2 \cdot h_2 + h_3 \quad h_{14} := h_1 + 2 \cdot h_2 + 2 \cdot h_3 + h_4$$

$$h_{23} := h_2 + h_3 \quad h_{24} := h_2 + 2 \cdot h_3 + h_4 \quad h_{34} := h_3 + h_4$$

$$I_1 := \text{unit} \cdot \frac{h_1^3}{12} \quad I_2 := \text{unit} \cdot \frac{h_2^3}{12} \quad I_3 := \text{unit} \cdot \frac{h_3^3}{12} \quad I_4 := \text{unit} \cdot \frac{h_4^3}{12}$$

$$P_1 = \alpha_1 \cdot \left[-\Delta T_{21} \cdot s_1 \cdot s_2 \cdot (h_{12}) - \Delta T_{31} \cdot s_1 \cdot s_3 \cdot (h_{13}) - \Delta T_{41} \cdot s_1 \cdot s_4 \cdot (h_{14}) \right]$$

$$P_2 := \alpha_2 \cdot \left[\Delta T_{21} \cdot s_1 \cdot s_2 \cdot (h_{12}) - \Delta T_{32} \cdot s_2 \cdot s_3 \cdot (h_{23}) - \Delta T_{42} \cdot s_2 \cdot s_4 \cdot (h_{24}) \right]$$

$$P_3 := \alpha_3 \cdot \left[\Delta T_{31} \cdot s_1 \cdot s_3 \cdot (h_{13}) + \Delta T_{32} \cdot s_2 \cdot s_3 \cdot (h_{23}) - \Delta T_{43} \cdot s_3 \cdot s_4 \cdot (h_{34}) \right]$$

$$P_4 := \alpha_4 \cdot \left[\Delta T_{41} \cdot s_1 \cdot s_4 \cdot (h_{14}) + \Delta T_{42} \cdot s_2 \cdot s_4 \cdot (h_{24}) + \Delta T_{43} \cdot s_3 \cdot s_4 \cdot (h_{34}) \right]$$

$$0 = P_1 + P_2 + P_3 + P_4$$

Test of above equation:

$$\alpha_1 := \frac{-(P_2 + P_3 + P_4)}{-\Delta T_{21} \cdot s_1 \cdot s_2 \cdot (h_{12}) - \Delta T_{31} \cdot s_1 \cdot s_3 \cdot (h_{13}) - \Delta T_{41} \cdot s_1 \cdot s_4 \cdot (h_{14})}$$

$$\alpha_1 = 2.402 \times 10^{-5} \frac{1}{\text{K}}$$

$$a_{12} := s_1 \cdot s_2 \cdot (h_1 + h_2) \cdot (\alpha_2 - \alpha_1) \quad a_{23} := s_2 \cdot s_3 \cdot (h_2 + h_3) \cdot (\alpha_3 - \alpha_2)$$

$$a_{34} := s_3 \cdot s_4 \cdot (h_3 + h_4) \cdot (\alpha_4 - \alpha_3)$$

$$a_{13} := s_1 \cdot s_3 \cdot (h_1 + 2 \cdot h_2 + h_3) \cdot (\alpha_3 - \alpha_1) \quad a_{24} := s_2 \cdot s_4 \cdot (h_2 + 2 \cdot h_3 + h_4) \cdot (\alpha_4 - \alpha_2)$$

$$a_{14} := s_1 \cdot s_4 \cdot [h_1 + 2 \cdot (h_2 + h_3) + h_4] \cdot (\alpha_4 - \alpha_1)$$

$$b_{12} := s_1 \cdot s_2 \cdot (h_1 + h_2)^2 \quad b_{23} := s_2 \cdot s_3 \cdot (h_2 + h_3)^2 \quad b_{34} := s_3 \cdot s_4 \cdot (h_3 + h_4)^2$$

$$b_{13} := s_1 \cdot s_3 \cdot (h_1 + 2 \cdot h_2 + h_3)^2 \quad b_{24} := s_2 \cdot s_4 \cdot (h_2 + 2 \cdot h_3 + h_4)^2$$

$$b_{14} := s_1 \cdot s_4 \cdot [h_1 + 2 \cdot (h_2 + h_3) + h_4]^2$$

$$c := E_1 \cdot I_1 + E_2 \cdot I_2 + E_3 \cdot I_3 + E_4 \cdot I_4 \quad d := s_1 + s_2 + s_3 + s_4$$

The resultant thermally induced curvature is theorized as:

$$\kappa_{\text{thermal}} := 2 \cdot \frac{\Delta T_{21} \cdot a_{12} + \Delta T_{31} \cdot a_{13} + \Delta T_{41} \cdot a_{14} + \Delta T_{31} \cdot a_{23} + \Delta T_{42} \cdot a_{24} + \Delta T_{43} \cdot a_{34}}{b_{12} + b_{13} + b_{14} + b_{23} + b_{24} + b_{34} + 4 \cdot c \cdot d}$$

$$\kappa_{\text{thermal}} = 5.331 \times 10^{-3} \frac{1}{\text{m}}$$

Coefficient of Thermal expansion design equations that produce a 0 curvature written in terms of the four CTE's of a four layer trough.

$$\alpha_1 = \frac{-(P_2 + P_3 + P_4)}{-\Delta T_{21} \cdot s_1 \cdot s_2 \cdot (h_{12}) - \Delta T_{31} \cdot s_1 \cdot s_3 \cdot (h_{13}) - \Delta T_{41} \cdot s_1 \cdot s_4 \cdot (h_{14})}$$

$$\alpha_2 = \frac{-(P_1 + P_3 + P_4)}{\Delta T_{21} \cdot s_1 \cdot s_2 \cdot (h_{12}) - \Delta T_{32} \cdot s_2 \cdot s_3 \cdot (h_{23}) - \Delta T_{42} \cdot s_2 \cdot s_4 \cdot (h_{24})}$$

$$\alpha_3 = \frac{-(P_1 + P_2 + P_4)}{\Delta T_{31} \cdot s_1 \cdot s_3 \cdot (h_{13}) + \Delta T_{32} \cdot s_2 \cdot s_3 \cdot (h_{23}) - \Delta T_{43} \cdot s_3 \cdot s_4 \cdot (h_{34})}$$

$$\alpha_4 = \frac{-(P_1 + P_2 + P_3)}{\Delta T_{41} \cdot s_1 \cdot s_4 \cdot (h_{14}) + \Delta T_{42} \cdot s_2 \cdot s_4 \cdot (h_{24}) + \Delta T_{43} \cdot s_3 \cdot s_4 \cdot (h_{34})}$$

General Form of 0 curvature equation derivation:

$$\Delta T_{21} := \frac{T_2 + T_1}{2} - T_0 \quad \Delta T_{31} := \frac{T_3 + T_1}{2} - T_0 \quad \Delta T_{41} := \frac{T_4 + T_1}{2} - T_0$$

$$\Delta T_{32} := \frac{T_3 + T_2}{2} - T_0 \quad \Delta T_{42} := \frac{T_4 + T_2}{2} - T_0 \quad \Delta T_{43} := \frac{T_4 + T_3}{2} - T_0$$

$$P_1 := -\alpha_1 \cdot [\Delta T_{21} \cdot s_1 \cdot s_2 \cdot (h_{12}) + \Delta T_{31} \cdot s_1 \cdot s_3 \cdot (h_{13}) + \Delta T_{41} \cdot s_1 \cdot s_4 \cdot (h_{14})]$$

$$P_2 := -\alpha_2 \cdot [-\Delta T_{21} \cdot s_1 \cdot s_2 \cdot (h_{12}) + \Delta T_{32} \cdot s_2 \cdot s_3 \cdot (h_{23}) + \Delta T_{42} \cdot s_2 \cdot s_4 \cdot (h_{24})]$$

$$P_3 := -\alpha_3 \cdot [-\Delta T_{31} \cdot s_1 \cdot s_3 \cdot (h_{13}) - \Delta T_{32} \cdot s_2 \cdot s_3 \cdot (h_{23}) + \Delta T_{43} \cdot s_3 \cdot s_4 \cdot (h_{34})]$$

$$P_4 := -\alpha_4 \cdot [-\Delta T_{41} \cdot s_1 \cdot s_4 \cdot (h_{14}) - \Delta T_{42} \cdot s_2 \cdot s_4 \cdot (h_{24}) - \Delta T_{43} \cdot s_3 \cdot s_4 \cdot (h_{34})]$$

$$P_i = -\alpha_i \cdot \left[\sum_{j=1}^N [\Delta T_{ji} \cdot (s_i \cdot s_j) \cdot (h_{ij})] - \sum_{j=1}^N [\Delta T_{ji} \cdot (s_i \cdot s_j) \cdot (h_{ij})] \cdot \delta(i, j) \right]$$

$$\text{for } i < j \quad s_i \cdot s_j > 0$$

$$\text{for } i > j \quad s_i \cdot s_j < 0$$

$$0 = P_i + P_{i+1} + P_{i+2} \dots P_{N-1} + P_N$$

This translates to:

$$s_j \cdot s_i = -s_i \cdot s_j$$

$$s = \begin{bmatrix} 0 & s_1 \cdot s_2 & s_1 \cdot s_3 & s_1 \cdot s_{N-1} & s_1 \cdot s_N \\ -(s_1 \cdot s_2) & 0 & s_2 \cdot s_3 & s_2 \cdot s_{N-1} & s_2 \cdot s_N \\ -(s_1 \cdot s_3) & -(s_2 \cdot s_3) & 0 & s_3 \cdot s_{N-1} & s_3 \cdot s_N \\ -(s_1 \cdot s_{N-1}) & -(s_2 \cdot s_{N-1}) & -(s_3 \cdot s_{N-1}) & 0 & s_{N-1} \cdot s_N \\ -(s_1 \cdot s_N) & -(s_2 \cdot s_N) & -(s_3 \cdot s_N) & -(s_{N-1} \cdot s_N) & 0 \end{bmatrix}$$

$$h_{ij} = h_{ji}$$

$$h_{21} = h_{12} = h_1 + h_2$$

$$h_{31} = h_{13} = h_1 + 2 \cdot h_2 + h_3$$

$$h_{41} = h_{14} = h_1 + 2 \cdot h_2 + 2 \cdot h_3 + h_4$$

$$h_{N1} = h_{1N} = h_1 + 2 \cdot h_2 + 2 \cdot h_3 \dots 2 \cdot h_{N-1} + h_N$$

$$h_{Ni} = h_{iN} = h_i + 2 \cdot h_{i+1} + 2 \cdot h_{i+2} \dots 2 \cdot h_{N-1} + h_N$$

Zero Curvature of Four Layer Trough Scenarios for Uniform Temperature Loading

These scenarios explore the range of dependent CTE that is required to keep the system at 0 thermal curvature by defining 3 of the 4 CTE's.

Scenarios for Layer One

Material properties of the trough layers

$$E_1 := 15 \cdot 10^9 \text{ Pa} \quad E_2 := 20 \cdot 10^9 \text{ Pa} \quad E_3 := 35 \cdot 10^9 \text{ Pa} \quad E_4 := 40 \cdot 10^9 \text{ Pa}$$

Case 1

$$\alpha_2 := \begin{pmatrix} 10 \\ 10 \\ 1 \\ -1 \\ -10 \\ -10 \\ 10 \\ 10 \\ 10 \\ 30 \\ 30 \\ 30 \\ 30 \\ -30 \\ -20 \\ -10 \\ 10 \\ 20 \\ 30 \\ 1 \\ 40 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$$

$$\alpha_3 := \begin{pmatrix} 20 \\ 15 \\ 5 \\ -5 \\ -15 \\ -20 \\ 10 \\ 10 \\ 30 \\ 30 \\ 30 \\ 30 \\ 10 \\ -20 \\ -15 \\ -5 \\ 5 \\ 15 \\ 20 \\ 40 \\ 40 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$$

$$\alpha_4 := \begin{pmatrix} 30 \\ 20 \\ 10 \\ -10 \\ -20 \\ -30 \\ 10 \\ 30 \\ 30 \\ 30 \\ 10 \\ 10 \\ 10 \\ -10 \\ -10 \\ -1 \\ 1 \\ 10 \\ 10 \\ 40 \\ 1 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$$

Case 13

Material geometries of the trough layers

$$\text{layer thickness:} \quad h_1 := 0.023 \text{ m} \quad h_2 := 0.023 \text{ m} \quad h_3 := 0.023 \text{ m} \quad h_4 := 0.023 \text{ m}$$

$$\text{Unit depth:} \quad \text{unit} := 1$$

Temperature change

$$\Delta T := 250 \text{ C}$$

Material Property of Layer 1 to obtain 0 curvature in the trough:

$$s_1 := h_1 \cdot E_1 \quad s_2 := h_2 \cdot E_2 \quad s_3 := h_3 \cdot E_3 \quad s_4 := h_4 \cdot E_4$$

$$\begin{aligned}
h_{12} &:= h_1 + h_2 & h_{13} &:= h_1 + 2 \cdot h_2 + h_3 & h_{14} &:= h_1 + 2 \cdot h_2 + 2 \cdot h_3 + h_4 \\
h_{23} &:= h_2 + h_3 & h_{24} &:= h_2 + 2 \cdot h_3 + h_4 & h_{34} &:= h_3 + h_4
\end{aligned}$$

$$P_2 := \alpha_2 \cdot [s_1 \cdot s_2 \cdot (-h_{12}) + s_2 \cdot s_3 \cdot (h_{23}) + s_2 \cdot s_4 \cdot (h_{24})]$$

$$P_3 := \alpha_3 \cdot [s_1 \cdot s_3 \cdot (-h_{13}) + s_2 \cdot s_3 \cdot (-h_{23}) + s_3 \cdot s_4 \cdot (h_{34})]$$

$$P_4 := \alpha_4 \cdot [s_1 \cdot s_4 \cdot (-h_{14}) + s_2 \cdot s_4 \cdot (-h_{24}) + s_3 \cdot s_4 \cdot (-h_{34})]$$

$$\alpha_1 := -\frac{P_2 + P_3 + P_4}{s_1 \cdot s_3 \cdot (h_{13}) + s_1 \cdot s_2 \cdot (h_{12}) + s_1 \cdot s_4 \cdot (h_{14})}$$

$$\alpha_1 = \begin{pmatrix} 4.159 \times 10^{-5} \\ 2.579 \times 10^{-5} \\ 1.516 \times 10^{-5} \\ -1.516 \times 10^{-5} \\ -2.579 \times 10^{-5} \\ -4.159 \times 10^{-5} \\ 1 \times 10^{-5} \\ 4.048 \times 10^{-5} \\ 4.27 \times 10^{-5} \\ 3 \times 10^{-5} \\ -4.762 \times 10^{-7} \\ -2.698 \times 10^{-6} \\ 1.587 \times 10^{-6} \\ -4.206 \times 10^{-6} \\ 4.27 \times 10^{-6} \\ -4.27 \times 10^{-6} \\ 4.206 \times 10^{-6} \\ -1.587 \times 10^{-6} \\ 6.476 \times 10^{-5} \\ -1.943 \times 10^{-5} \end{pmatrix} \quad \text{Case 1}$$

$$\alpha_1 = \begin{pmatrix} 4.159 \times 10^{-5} \\ 2.579 \times 10^{-5} \\ 1.516 \times 10^{-5} \\ -1.516 \times 10^{-5} \\ -2.579 \times 10^{-5} \\ -4.159 \times 10^{-5} \\ 1 \times 10^{-5} \\ 4.048 \times 10^{-5} \\ 4.27 \times 10^{-5} \\ 3 \times 10^{-5} \\ -4.762 \times 10^{-7} \\ -2.698 \times 10^{-6} \\ 1.587 \times 10^{-6} \\ -4.206 \times 10^{-6} \\ 4.27 \times 10^{-6} \\ -4.27 \times 10^{-6} \\ 4.206 \times 10^{-6} \\ -1.587 \times 10^{-6} \\ 6.476 \times 10^{-5} \\ -1.943 \times 10^{-5} \end{pmatrix} \quad \frac{1}{C} \quad \text{Case 13}$$

Scenarios for Layer Two

Material properties of the trough layers

$$E_1 := 15 \cdot 10^9 \text{ Pa} \quad E_2 := 20 \cdot 10^9 \text{ Pa} \quad E_3 := 35 \cdot 10^9 \text{ Pa} \quad E_4 := 40 \cdot 10^9 \text{ Pa}$$

Case 22	$\alpha_1 := \begin{pmatrix} 10 \\ 10 \\ 1 \\ -1 \\ -10 \\ -10 \\ 10 \\ 10 \\ 10 \\ 30 \\ 30 \\ 30 \\ -30 \\ -20 \\ -10 \\ 10 \\ 20 \\ 30 \\ 1 \\ 40 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$	$\alpha_3 := \begin{pmatrix} 20 \\ 15 \\ 5 \\ -5 \\ -15 \\ -20 \\ 10 \\ 10 \\ 30 \\ 30 \\ 30 \\ 10 \\ -20 \\ -15 \\ -5 \\ 5 \\ 15 \\ 20 \\ 40 \\ 40 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$	$\alpha_4 := \begin{pmatrix} 30 \\ 20 \\ 10 \\ -10 \\ -20 \\ -30 \\ 10 \\ 30 \\ 30 \\ 30 \\ 10 \\ 10 \\ -10 \\ -10 \\ -1 \\ 1 \\ 10 \\ 10 \\ 40 \\ 1 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$
Case 37			

Material geometries of the trough layers

$$\text{layer thickness:} \quad h_1 := 0.023 \text{ m} \quad h_2 := 0.023 \text{ m} \quad h_3 := 0.023 \text{ m} \quad h_4 := 0.023 \text{ m}$$

$$\text{Unit depth:} \quad \text{unit} := 1$$

Temperature change

$$\Delta T := 250 \text{ C}$$

Material Property of Layer 1 to obtain 0 curvature in the trough:

$$s_1 := h_1 \cdot E_1 \quad s_2 := h_2 \cdot E_2 \quad s_3 := h_3 \cdot E_3 \quad s_4 := h_4 \cdot E_4$$

$$h_{12} := h_1 + h_2 \quad h_{13} := h_1 + 2 \cdot h_2 + h_3 \quad h_{14} := h_1 + 2 \cdot h_2 + 2 \cdot h_3 + h_4$$

$$h_{23} := h_2 + h_3 \quad h_{24} := h_2 + 2 \cdot h_3 + h_4 \quad h_{34} := h_3 + h_4$$

$$P_1 := \alpha_1 \cdot [s_1 \cdot s_2 \cdot h_{12} + s_1 \cdot s_3 \cdot (h_{13}) + s_1 \cdot s_4 \cdot (h_{14})]$$

$$P_3 := \alpha_3 \cdot [s_1 \cdot s_3 \cdot (-h_{13}) + s_2 \cdot s_3 \cdot (-h_{23}) + s_3 \cdot s_4 \cdot (h_{34})]$$

$$P_4 := \alpha_4 \cdot [s_1 \cdot s_4 \cdot (-h_{14}) + s_2 \cdot s_4 \cdot (-h_{24}) + s_3 \cdot s_4 \cdot (-h_{34})]$$

$$\alpha_2 := -\frac{P_1 + P_3 + P_4}{s_1 \cdot s_2 \cdot (-h_{12}) + s_2 \cdot s_3 \cdot (h_{23}) + s_2 \cdot s_4 \cdot (h_{24})}$$

$$\alpha_2 = \begin{pmatrix} 5.975 \times 10^{-5} \\ 3.487 \times 10^{-5} \\ 2.33 \times 10^{-5} \\ -2.33 \times 10^{-5} \\ -3.487 \times 10^{-5} \\ -5.975 \times 10^{-5} \\ 1 \times 10^{-5} \\ 5.8 \times 10^{-5} \\ 6.15 \times 10^{-5} \\ 3 \times 10^{-5} \\ -1.8 \times 10^{-5} \\ -2.15 \times 10^{-5} \\ 1.975 \times 10^{-5} \\ 4.875 \times 10^{-6} \\ 1.247 \times 10^{-5} \\ -1.247 \times 10^{-5} \\ -4.875 \times 10^{-6} \\ -1.975 \times 10^{-5} \\ 1.014 \times 10^{-4} \\ -5.36 \times 10^{-5} \end{pmatrix} \frac{1}{C}$$

Case 22

Case 37

Scenarios for Layer Three

Material properties of the trough layers

$$E_1 := 15 \cdot 10^9 \text{ Pa} \quad E_2 := 20 \cdot 10^9 \text{ Pa} \quad E_3 := 35 \cdot 10^9 \text{ Pa} \quad E_4 := 40 \cdot 10^9 \text{ Pa}$$

	$\begin{pmatrix} 10 \\ 10 \\ 1 \\ -1 \\ -10 \\ -10 \\ 10 \\ 10 \\ 10 \\ 30 \\ 30 \\ 30 \\ -30 \\ -20 \\ -10 \\ 10 \\ 20 \\ 30 \\ 1 \\ 40 \end{pmatrix}$		$\begin{pmatrix} 20 \\ 15 \\ 5 \\ -5 \\ -15 \\ -20 \\ 10 \\ 10 \\ 30 \\ 30 \\ 30 \\ 10 \\ -20 \\ -15 \\ -5 \\ 5 \\ 15 \\ 20 \\ 40 \\ 40 \end{pmatrix}$		$\begin{pmatrix} 30 \\ 20 \\ 10 \\ -10 \\ -20 \\ -30 \\ 10 \\ 30 \\ 30 \\ 30 \\ 10 \\ 10 \\ -10 \\ -10 \\ -1 \\ 1 \\ 10 \\ 10 \\ 40 \\ 1 \end{pmatrix}$
Case 44					
Case 47					
	$\alpha_1 := 10^{-6} \cdot \frac{1}{\text{C}}$		$\alpha_2 := 10^{-6} \cdot \frac{1}{\text{C}}$		$\alpha_4 := 10^{-6} \cdot \frac{1}{\text{C}}$

Material geometries of the trough layers

$$\text{layer thickness:} \quad h_1 := 0.023\text{m} \quad h_2 := 0.023\text{m} \quad h_3 := 0.023\text{m} \quad h_4 := 0.023\text{m}$$

$$\text{Unit depth:} \quad \text{unit} := 1$$

Temperature change

$$\Delta T := 250\text{C}$$

Material Property of Layer 1 to obtain 0 curvature in the trough:

$$s_1 := h_1 \cdot E_1 \quad s_2 := h_2 \cdot E_2 \quad s_3 := h_3 \cdot E_3 \quad s_4 := h_4 \cdot E_4$$

$$h_{12} := h_1 + h_2 \quad h_{13} := h_1 + 2 \cdot h_2 + h_3 \quad h_{14} := h_1 + 2 \cdot h_2 + 2 \cdot h_3 + h_4$$

$$h_{23} := h_2 + h_3 \quad h_{24} := h_2 + 2 \cdot h_3 + h_4 \quad h_{34} := h_3 + h_4$$

$$P_1 := \alpha_1 \cdot [s_1 \cdot s_2 \cdot (h_{12}) + s_1 \cdot s_3 \cdot (h_{13}) + s_1 \cdot s_4 \cdot (h_{14})]$$

$$P_2 := \alpha_2 \cdot [s_1 \cdot s_2 \cdot (-h_{12}) + s_2 \cdot s_3 \cdot (h_{23}) + s_2 \cdot s_4 \cdot (h_{24})]$$

$$P_4 := \alpha_4 \cdot [s_1 \cdot s_4 \cdot (-h_{14}) + s_2 \cdot s_4 \cdot (-h_{24}) + s_3 \cdot s_4 \cdot (-h_{34})]$$

$$\alpha_3 := -\frac{P_1 + P_2 + P_4}{s_1 \cdot s_3 \cdot (-h_{13}) + s_2 \cdot s_3 \cdot (-h_{23}) + s_3 \cdot s_4 \cdot (h_{34})}$$

$$\alpha_3 = \frac{1}{C} \begin{pmatrix} -2.071 \times 10^{-4} \\ -9.857 \times 10^{-5} \\ -9.957 \times 10^{-5} \\ 9.957 \times 10^{-5} \\ 9.857 \times 10^{-5} \\ 2.071 \times 10^{-4} \\ 10 \times 10^{-6} \\ -2.643 \times 10^{-4} \\ -1.5 \times 10^{-4} \\ 3 \times 10^{-5} \\ 3.043 \times 10^{-4} \\ 1.9 \times 10^{-4} \\ -2.471 \times 10^{-4} \\ -1.286 \times 10^{-4} \\ -1.049 \times 10^{-4} \\ 1.049 \times 10^{-4} \\ 1.286 \times 10^{-4} \\ 2.471 \times 10^{-4} \\ -3.11 \times 10^{-4} \\ 5.749 \times 10^{-4} \end{pmatrix}$$

Case 44

Case 47

Scenarios for Layer Four

Material properties of the trough layers

$$E_1 := 15 \cdot 10^9 \text{ Pa} \quad E_2 := 20 \cdot 10^9 \text{ Pa} \quad E_3 := 35 \cdot 10^9 \text{ Pa} \quad E_4 := 40 \cdot 10^9 \text{ Pa}$$

Case 61

$$\alpha_1 := \begin{pmatrix} 10 \\ 10 \\ 1 \\ -1 \\ -10 \\ -10 \\ 10 \\ 10 \\ 10 \\ 30 \\ 30 \\ 30 \\ -30 \\ -20 \\ -10 \\ -10 \\ 10 \\ 20 \\ 30 \\ 1 \\ 40 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}} \quad \alpha_2 := \begin{pmatrix} 20 \\ 15 \\ 5 \\ -5 \\ -15 \\ -20 \\ 10 \\ 10 \\ 30 \\ 30 \\ 30 \\ 10 \\ -20 \\ -15 \\ -5 \\ 5 \\ 15 \\ 20 \\ 40 \\ 40 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}} \quad \alpha_3 := \begin{pmatrix} 30 \\ 20 \\ 10 \\ -10 \\ -20 \\ -30 \\ 10 \\ 30 \\ 30 \\ 30 \\ 10 \\ 10 \\ -10 \\ -10 \\ -1 \\ 1 \\ 10 \\ 10 \\ 40 \\ 1 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$$

Case 75

Material geometries of the trough layers

$$\text{layer thickness:} \quad h_1 := 0.023 \text{ m} \quad h_2 := 0.023 \text{ m} \quad h_3 := 0.023 \text{ m} \quad h_4 := 0.023 \text{ m}$$

$$\text{Unit depth:} \quad \text{unit} := 1$$

Temperature change

$$\Delta T := 250 \text{ C}$$

Material Property of Layer 1 to obtain 0 curvature in the trough:

$$s_1 := h_1 \cdot E_1 \quad s_2 := h_2 \cdot E_2 \quad s_3 := h_3 \cdot E_3 \quad s_4 := h_4 \cdot E_4$$

$$h_{12} := h_1 + h_2 \quad h_{13} := h_1 + 2 \cdot h_2 + h_3 \quad h_{14} := h_1 + 2 \cdot h_2 + 2 \cdot h_3 + h_4$$

$$h_{23} := h_2 + h_3 \quad h_{24} := h_2 + 2 \cdot h_3 + h_4 \quad h_{34} := h_3 + h_4$$

$$P_1 := \alpha_1 \cdot [s_1 \cdot s_2 \cdot (h_{12}) + s_1 \cdot s_3 \cdot (h_{13}) + s_1 \cdot s_4 \cdot (h_{14})]$$

$$P_2 := \alpha_2 \cdot [s_1 \cdot s_2 \cdot (-h_{12}) + s_2 \cdot s_3 \cdot (h_{23}) + s_2 \cdot s_4 \cdot (h_{24})]$$

$$P_3 := \alpha_3 \cdot [s_1 \cdot s_3 \cdot (-h_{13}) + s_2 \cdot s_3 \cdot (-h_{23}) + s_3 \cdot s_4 \cdot (h_{34})]$$

$$\alpha_4 := -\frac{P_1 + P_2 + P_3}{s_1 \cdot s_4 \cdot (-h_{14}) + s_2 \cdot s_4 \cdot (-h_{24}) + s_3 \cdot s_4 \cdot (-h_{34})}$$

$$\alpha_4 = \frac{1}{C} \begin{pmatrix} 1.271 \times 10^{-5} \\ 1.135 \times 10^{-5} \\ 2.01 \times 10^{-6} \\ -2.01 \times 10^{-6} \\ -1.135 \times 10^{-5} \\ -1.271 \times 10^{-5} \\ 1 \times 10^{-5} \\ 8.542 \times 10^{-6} \\ 1.687 \times 10^{-5} \\ 3 \times 10^{-5} \\ 3.146 \times 10^{-5} \\ 2.312 \times 10^{-5} \\ -2.729 \times 10^{-5} \\ -1.865 \times 10^{-5} \\ -8.573 \times 10^{-6} \\ 8.573 \times 10^{-6} \\ 1.865 \times 10^{-5} \\ 2.729 \times 10^{-5} \\ 1.441 \times 10^{-5} \\ 4.284 \times 10^{-5} \end{pmatrix}$$

Case 61

Case 75

**RMS Surface Distortion and Radius of Curvature Comparison for FEA vs.
Zero Curvature Closed Form Equation Results for Case 1
(Non-Uniform Temperature Change)**

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N ((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})))^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.148E-06
RMS of best fit of FEA deformed	3.140E-06
Percent Difference between FEA and Closed Form	0.239

Radius of Curvature Results:

FEA Radius of Curvature	0.997
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	-0.302

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4335E-15	-0.001434
2	0.00143	0.00143	-1.00000	-1.00000	-1.2902E-14	-0.001434
3	0.00287	0.00287	-1.00000	-1.00000	-3.5837E-14	-0.001433
4	0.00430	0.00430	-0.99999	-1.00000	-7.0241E-14	-0.001433
5	0.00573	0.00573	-0.99998	-1.00000	-1.4335E-13	-0.001433
6	0.00717	0.00717	-0.99997	-1.00000	-1.7345E-13	-0.001433
7	0.00860	0.00860	-0.99996	-1.00000	-2.4225E-13	-0.001433

Performed for all surface nodes and RMS is calculated according to the above equation

**RMS Surface Distortion and Radius of Curvature Comparison for FEA vs.
Zero Curvature Closed Form Equation Results for Case 2
(Non-Uniform Temperature Change)**

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	4.780E-06
RMS of best fit of FEA deformed	4.530E-06
Percent Difference between FEA and Closed Form	5.232

Radius of Curvature Results:

FEA Radius of Curvature	1.380
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	27.531

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4625E-15	-0.001462
2	0.00143	0.00146	-1.00000	-1.00000	-1.5515E-11	-0.001462
3	0.00287	0.00292	-1.00000	-1.00010	-3.6562E-14	-0.001462
4	0.00430	0.00439	-0.99999	-1.00010	-1.6744E-11	-0.001462
5	0.00573	0.00585	-0.99998	-1.00020	-1.4625E-13	-0.001462
6	0.00717	0.00731	-0.99997	-1.00020	-1.8019E-11	-0.001462
7	0.00860	0.00877	-0.99996	-1.00030	-2.4715E-13	-0.001462

Performed for all surface nodes and RMS is calculated according to the above equation

**RMS Surface Distortion and Radius of Curvature Comparison for FEA vs.
Zero Curvature Closed Form Equation Results for Case 3
(Non-Uniform Temperature Change)**

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N ((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})))^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	6.776E-06
RMS of best fit of FEA deformed	6.157E-06
Percent Difference between FEA and Closed Form	9.138

Radius of Curvature Results:

FEA Radius of Curvature	1.777
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	43.715

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2 (x'_i - x'_{i-1})$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.5103E-11	-0.001481
2	0.00143	0.00148	-1.00000	-1.00010	-1.5707E-11	-0.001481
3	0.00287	0.00296	-1.00000	-1.00020	-3.7015E-14	-0.001481
4	0.00430	0.00444	-0.99999	-1.00020	-1.6951E-11	-0.001481
5	0.00573	0.00592	-0.99998	-1.00030	-1.7915E-11	-0.001481
6	0.00717	0.00740	-0.99997	-1.00040	-1.7915E-13	-0.001481
7	0.00860	0.00888	-0.99996	-1.00040	-1.8906E-11	-0.001481

Performed for all surface nodes and RMS is calculated according to the above equation

**RMS Surface Distortion and Radius of Curvature Comparison for FEA vs.
Zero Curvature Closed Form Equation Results for Case 4
(Non-Uniform Temperature Change)**

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N ((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})))^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	5.289E-06
RMS of best fit of FEA deformed	4.978E-06
Percent Difference between FEA and Closed Form	5.879

Radius of Curvature Results:

FEA Radius of Curvature	1.482
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	32.528

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4964E-11	-0.001467
2	0.00143	0.00147	-1.00000	-1.00010	-1.3202E-14	-0.001467
3	0.00287	0.00293	-1.00000	-1.00010	-1.6173E-11	-0.001467
4	0.00430	0.00440	-0.99999	-1.00020	-7.1880E-14	-0.001467
5	0.00573	0.00587	-0.99998	-1.00020	-1.7750E-11	-0.001467
6	0.00717	0.00733	-0.99997	-1.00030	-1.7750E-13	-0.001467
7	0.00860	0.00880	-0.99996	-1.00030	-1.8731E-11	-0.001467

Performed for all surface nodes and RMS is calculated according to the above equation

**RMS Surface Distortion and Radius of Curvature Comparison for FEA vs.
Zero Curvature Closed Form Equation Results for Case 5
(Non-Uniform Temperature Change)**

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N ((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})))^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.323E-06
RMS of best fit of FEA deformed	3.317E-06
Percent Difference between FEA and Closed Form	0.181

Radius of Curvature Results:

FEA Radius of Curvature	1.111
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	9.958

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4453E-15	-0.001445
2	0.00143	0.00145	-1.00000	-1.00000	-1.3007E-14	-0.001445
3	0.00287	0.00289	-1.00000	-1.00000	-3.6131E-14	-0.001445
4	0.00430	0.00434	-0.99999	-1.00000	-1.6547E-11	-0.001445
5	0.00573	0.00578	-0.99998	-1.00010	-1.4452E-13	-0.001445
6	0.00717	0.00723	-0.99997	-1.00010	-1.7487E-13	-0.001445
7	0.00860	0.00867	-0.99996	-1.00010	-2.4424E-13	-0.001445

Performed for all surface nodes and RMS is calculated according to the above equation

**RMS Surface Distortion and Radius of Curvature Comparison for FEA vs.
Zero Curvature Closed Form Equation Results for Case 6
(Non-Uniform Temperature Change)**

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	4.039E-06
RMS of best fit of FEA deformed	3.685E-06
Percent Difference between FEA and Closed Form	8.768

Radius of Curvature Results:

FEA Radius of Curvature	1.241
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	19.418

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4563E-15	-0.001456
2	0.00143	0.00146	-1.00000	-1.00000	-1.5450E-11	-0.001456
3	0.00287	0.00291	-1.00000	-1.00010	-3.6407E-14	-0.001456
4	0.00430	0.00437	-0.99999	-1.00010	-7.1358E-14	-0.001456
5	0.00573	0.00583	-0.99998	-1.00010	-1.4563E-13	-0.001456
6	0.00717	0.00728	-0.99997	-1.00010	-1.7943E-11	-0.001456
7	0.00860	0.00874	-0.99996	-1.00020	-2.4611E-13	-0.001456

Performed for all surface nodes and RMS is calculated according to the above equation

**RMS Surface Distortion and Radius of Curvature Comparison for FEA vs.
Zero Curvature Closed Form Equation Results for Case 7
(Non-Uniform Temperature Change)**

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N ((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})))^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	4.341E-06
RMS of best fit of FEA deformed	4.097E-06
Percent Difference between FEA and Closed Form	5.619

Radius of Curvature Results:

FEA Radius of Curvature	1.301
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	23.138

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4590E-15	-0.001459
2	0.00143	0.00146	-1.00000	-1.00000	-1.5478E-11	-0.001459
3	0.00287	0.00292	-1.00000	-1.00010	-3.6475E-14	-0.001459
4	0.00430	0.00438	-0.99999	-1.00010	-7.1491E-14	-0.001459
5	0.00573	0.00584	-0.99998	-1.00010	-1.7654E-11	-0.001459
6	0.00717	0.00730	-0.99997	-1.00020	-1.7654E-13	-0.001459
7	0.00860	0.00875	-0.99996	-1.00020	-2.4657E-13	-0.001459

Performed for all surface nodes and RMS is calculated according to the above equation

**RMS Surface Distortion and Radius of Curvature Comparison for FEA vs.
Zero Curvature Closed Form Equation Results for Case 8
(Non-Uniform Temperature Change)**

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.081E-06
RMS of best fit of FEA deformed	3.186E-06
Percent Difference between FEA and Closed Form	-3.420

Radius of Curvature Results:

FEA Radius of Curvature	1.025
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	2.462

Closed Form Equation Sample Results

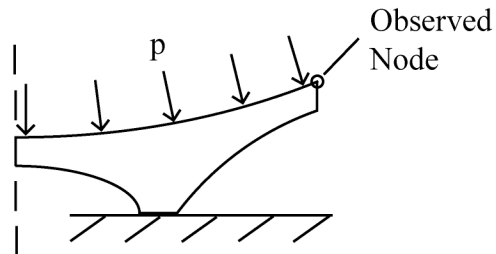
Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4359E-15	-0.001436
2	0.00143	0.00144	-1.00000	-1.00000	-1.2923E-14	-0.001436
3	0.00287	0.00287	-1.00000	-1.00000	-3.5897E-14	-0.001436
4	0.00430	0.00431	-0.99999	-1.00000	-7.0359E-14	-0.001436
5	0.00573	0.00574	-0.99998	-1.00000	-1.4359E-13	-0.001436
6	0.00717	0.00718	-0.99997	-1.00000	-1.7374E-13	-0.001436
7	0.00860	0.00862	-0.99996	-1.00000	-2.4266E-13	-0.001436

Performed for all surface nodes and RMS is calculated according to the above equation

APPENDIX B

AIR FORCE RESEARCH LABORATORY BASELINE MIRROR

AFRL Axisymmetric Geometry Mirror Mesh Convergence Testing of Abaqus for Internal Pressure vs. Displacement of the Homogenous Case



Configuration:

The spherical shell is composed of a solid layer of elements with a thin layer of shell elements as its surface.

100 Pa (p) was applied to the top surface of the mirror. Nodal Displacements are taken from a node on the farthest edge away from the axisymmetric axis (apex of mirror).

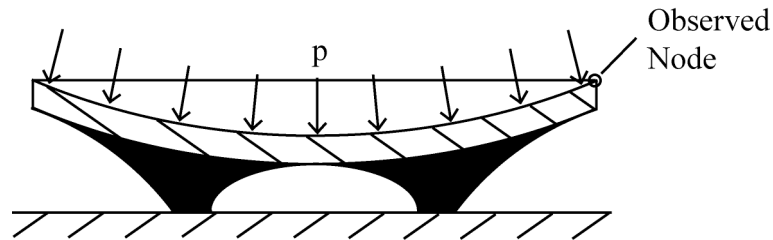
Shell Layer:	Zerodur	Solid Layer:	Zerodur
Thickness	1.00E-05	Elastic Modulus	9.20E+10
Elastic Modulus	9.20E+10	(Homogenous Case)	
(Homogenous Case)		Possion's Ratio	0.25
Possion's Ratio	0.25	Density	2530.00
Density	2530.00		

Abaqus Results (Non-Linear Geometry)

Axisymmetric Homogenous Case

Top Edge	Number of Total	Nodal
Elements	Elements	Displacement
40	286	-5.034E-09
80	866	-4.422E-09
120	2086	-4.333E-09
160	4038	-4.301E-09
200	6196	-4.292E-09
240	9102	-4.286E-09
280	12925	-4.281E-09
320	16205	-4.279E-09
360	20934	-4.278E-09

AFRL 3D Geometry Mirror Mesh Convergence Testing of Abaqus for Internal Pressure vs. Displacement of the Homogenous Case



Configuration:

The spherical shell is composed of a solid layer of elements with a thin layer of shell elements as its surface.

100 Pa (p) was applied to the top surface of the mirror. Nodal Displacements are taken from a node on the farthest edge away from the axisymmetric axis (apex of mirror)

Shell Layer:	Zerodur	Solid Layer:	Zerodur
Thickness	1.00E-05	Elastic Modulus	9.20E+10
Elastic Modulus	9.20E+10	(Homogenous Case)	
(Homogenous Case)		Possion's Ratio	0.25
Possion's Ratio	0.25	Density	2530.00
Density	2530.00		

Abaqus Results (Non-Linear Geometry)*

3D Homo. Case with substrate and coating

Top Outer Ring Edge Elements	Number of Total Elements	Nodal Displacement
40	5286	-4.844E-09
80	7798	-5.187E-09
120	8040	-5.080E-09
160	10638	-5.055E-09
200	41668	-5.148E-09

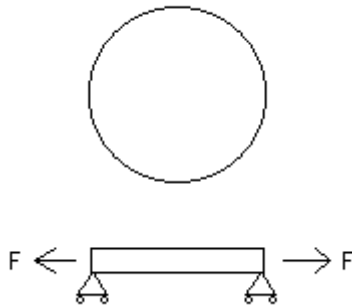
*This model did not converge and therefore the model was recreated to the one used for the table below which included a different model formulation and a different mesh configuration.

Abaqus Results (Non-Linear Geometry)

3D Homo. Case with substrate, plate, and coating

Top Outer Ring Edge (Plate and Coating)	Number of Total Elements	Nodal Displacement
200	12318	-4.002E-09
205	18340	-3.983E+03
220	19820	-3.986E-09
240	21720	-3.986E-09
280	25418	-3.986E-09

Natural Frequency Calculations



This model (taken from Blevins, Robert D. PhD. Formulas for Natural Frequency and Mode Shape) calculates the natural frequency of a flat round disc and then calculates the natural frequencies of a shallow spherical shell^{B1}. The axisymmetric mirror disc will assume the conditions as shown on the right (simply supported with load F action outwardly from the centroid).

Homogenous Case

$$E := 2.08 \cdot 10^8 \text{ Pa}$$

$$h := .01001 \text{ m} \quad \text{Thickness of the disc}$$

$$a := .5 \text{ m} \quad \text{Radius of the disc}$$

$$\gamma := .10214 \frac{\text{kg}}{\text{m}^2} \quad \text{mass per unit area of the disc}$$

$$F := 0 \text{ N} \quad \text{The load will be 0 Newtons}$$

$$\nu := .3$$

$$A := \frac{12 \cdot (1 - \nu)^2 \cdot F \cdot a^2}{4.2 \cdot E \cdot h^3} \quad A = 0$$

From page 240 in the reference at A=0:

Where λ_{ij} $i := 0$ $j := 0$ For the first mode of vibration

$i := 1$ $j := 0$ For the second mode

$$\lambda_{00} := 4.977^{\frac{1}{2}} \quad \lambda_{10} := 13.94^{\frac{1}{2}}$$

$$f_{ij} = \frac{\lambda_{ij}^2}{2 \cdot \pi \cdot a^2} \cdot \left[\frac{E \cdot h^3}{12 \cdot \gamma \cdot (1 - \nu^2)} \right]^{\frac{1}{2}}$$

Mode 1 (1st axisymmetric mode)

$$f_{00} := \frac{\lambda_{00}^2}{2 \cdot \pi \cdot a^2} \cdot \left[\frac{E \cdot h^3}{12 \cdot \gamma \cdot (1 - \nu^2)} \right]^{\frac{1}{2}}$$

$$f_{00} = 43.333 \frac{1}{s}$$

Mode 2

$$f_{10} := \frac{\lambda_{10}^2}{2 \cdot \pi \cdot a^2} \cdot \left[\frac{E \cdot h^3}{12 \cdot \gamma \cdot (1 - \nu^2)} \right]^{\frac{1}{2}}$$

$$f_{10} = 121.371 \frac{1}{s}$$

Mode 4 (2nd axisymmetric mode)

$$\lambda_{01} := 29.76^{\frac{1}{2}}$$

$$f_{01} := \frac{\lambda_{01}^2}{2 \cdot \pi \cdot a^2} \cdot \left[\frac{E \cdot h^3}{12 \cdot \gamma \cdot (1 - \nu^2)} \right]^{\frac{1}{2}}$$

$$f_{01} = 259.111 \frac{1}{s}$$

Mode 7 (3rd axisymmetric mode)

$$\lambda_{02} := 74.20^{\frac{1}{2}}$$

$$f_{02} := \frac{\lambda_{02}^2}{2 \cdot \pi \cdot a^2} \cdot \left[\frac{E \cdot h^3}{12 \cdot \gamma \cdot (1 - \nu^2)} \right]^{\frac{1}{2}}$$

$$f_{02} = 646.037 \frac{1}{s}$$

To calculate the natural frequency of a shallow spherical shell, the natural frequency of a flat circular disc was calculated and then a correction factor will be added in for the curvature. This formulation is based upon the shadow area of the curved mirror that is equivalent to the area of the flat disc. From p.332 of reference:

ω_{ijss} = frequency of the shallow spherical shell

ω_{ijfp} = frequency of the flat plate

$$\omega_{00fp} := f_{00} \cdot 2 \cdot \pi \quad \omega_{00fp} = 272.271 \frac{\text{rad}}{s}$$

$$\omega_{10fp} := f_{10} \cdot 2 \cdot \pi \quad \omega_{10fp} = 762.599 \frac{\text{rad}}{s}$$

$$\mu := 10.204 \frac{\text{kg}}{\text{m}^3} \quad R := 1.0675 \text{m}$$

Mode 1 (1st axisymmetric mode)

$$f_{00ss} := \frac{1}{2 \cdot \pi} \cdot \left(\omega_{00fp}^2 + \frac{E}{\mu \cdot R^2} \right)^{\frac{1}{2}}$$

Mode 2

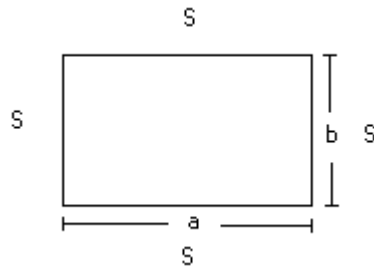
$$f_{10ss} := \frac{1}{2 \cdot \pi} \cdot \left(\omega_{10fp}^2 + \frac{E}{\mu \cdot R^2} \right)^{\frac{1}{2}}$$

$$f_{00\text{sss}} = 674.523 \frac{1}{\text{s}}$$

$$f_{10\text{sss}} = 683.984 \frac{1}{\text{s}}$$

Square plate- The preceding calculations are to determine the natural frequency of the disc if it were modeled as a square plate. This is done to later calculate a shallow spherical shell using the shadow area of the first determined square plate. From the reference on p.258:

Simply supported on all edges



$$\text{Area}_{\text{circulardisc}} := \pi \cdot a^2$$

$$\text{Area}_{\text{circulardisc}} = 0.785 \text{ m}^2$$

The dimensions used will be ones that cause equivalent areas between the circular disc and the square plate. Because a square plate is being used, the ratio of the two sides is:

$$a = b$$

$$\text{Area}_{\text{squareplate}} := \text{Area}_{\text{circulardisc}}$$

$$\text{Area}_{\text{squareplate}} = 0.785 \text{ m}^2$$

$$a := \sqrt{\text{Area}_{\text{squareplate}}}$$

$$a = 0.886 \text{ m}$$

With a ratio as stated above From p.258:

Mode 1

$$\lambda_{11} := 19.74^{\frac{1}{2}}$$

$$f_{11} := \frac{\lambda_{11}^2}{2 \cdot \pi \cdot a^2} \cdot \left[\frac{E \cdot h^3}{12 \cdot \gamma \cdot (1 - \nu^2)} \right]^{\frac{1}{2}}$$

$$f_{11} = 54.708 \frac{1}{\text{s}}$$

Mode 2

$$\lambda_{21} := 49.35^{\frac{1}{2}}$$

$$f_{21} := \frac{\lambda_{21}^2}{2 \cdot \pi \cdot a^2} \cdot \left[\frac{E \cdot h^3}{12 \cdot \gamma \cdot (1 - \nu^2)} \right]^{\frac{1}{2}}$$

$$f_{21} = 136.77 \frac{1}{\text{s}}$$

To again calculate the natural frequency of a shallow spherical shell, the natural frequency of the flat square disc was calculated and then a correction factor will be added in for the curvature. This formulation is based upon the shadow area of the curved mirror that is equivalent to the area of the flat square disc. From p.332 of reference:

$\omega_{ij\text{sss}}$ = frequency of the shallow spherical shell

$\omega_{ij\text{fp}}$ = frequency of the flat plate

$\omega_{11fp} := f_{11} \cdot 2 \cdot \pi$

$\omega_{21fp} := f_{21} \cdot 2 \cdot \pi$

$\mu := 10.204 \frac{\text{kg}}{\text{m}^3}$

$\omega_{11fp} = 343.741 \frac{\text{rad}}{\text{s}}$

$\omega_{21fp} = 859.351 \frac{\text{rad}}{\text{s}}$

$R := 1.0675\text{m}$

Mode 1

$\omega_{11ss} := \frac{1}{2 \cdot \pi} \cdot \left(\omega_{11fp}^2 + \frac{E}{\mu \cdot R^2} \right)^{\frac{1}{2}}$

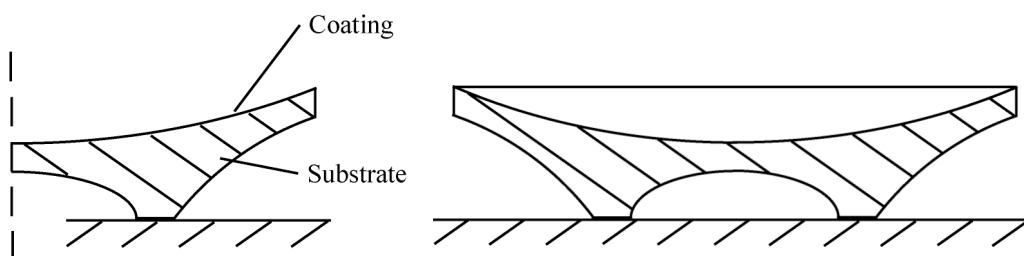
$\omega_{11ss} = 675.349 \frac{1}{\text{s}}$

Mode 2

$\omega_{21ss} := \frac{1}{2 \cdot \pi} \cdot \left(\omega_{21fp}^2 + \frac{E}{\mu \cdot R^2} \right)^{\frac{1}{2}}$

$\omega_{21ss} = 686.884 \frac{1}{\text{s}}$

AFRL Axisymmetric and 3D Spherical Shell Natural Frequency Analysis from Abaqus for the Homogenous and Heterogeneous Cases



Configuration:

The spherical shell is composed of a solid layer of elements with a thin layer of shell elements as its surface.

Shell Layer:	Zerodur
Thickness	1.00E-05
Elastic Modulus	9.20E+10
(Homogenous case)	
Possion's Ratio	0.25
Density	2530.00

Solid Layer:	Zerodur
Elastic Modulus	9.20E+10
(Homogenous case)	
Possion's Ratio	0.25
Density	2530.00

Abaqus Results

3D Model

Spherical Homogenous Case

Non-Symmetric Mode Numbers (1st 10)	Frequency (Hz)
1	4356.1
2	4358
3	4838
4	4848.3
5	4879.8
6	5222.9
7	5223.4
8	5669.5
9	5670.5
10	6145.5

Abaqus Results

Axisymmetric Model

Spherical Homogenous Case

Symmetric Mode Number	Frequency (Hz)
1	4913.1
2	7626.1
3	9529.4

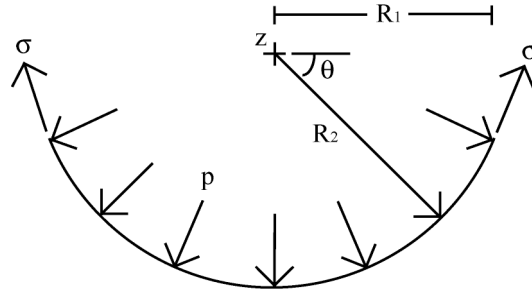
Abaqus Results

3D Model

Spherical Homogenous Case

Symmetric Mode Number	Frequency (Hz)
1	4879.8
2	7643.2
3	9582.8

Axisymmetric Flat and Spherical Shell Natural Frequency Calculations from Abaqus for the Homogenous and Heterogeneous Cases



Configuration:

The spherical shell is composed of a solid layer of elements with a thin layer of shell elements as its surface. Theta opening approximately = 26 degrees to end of shell
Reference layer of the shell element was offset to bottom surface (sneg).

Shell Layer:

Thickness .00001 m
Elastic Modulus 2.08×10^8 Pa
(Homogenous case)
Elastic Modulus 2.08×10^{10} Pa
(Heterogeneous Case)
Poisson's Ratio 0.3
Density 10.204 kg/m^3

Solid Layer:

Thickness .01 m

Analytical Calculations

Flat Homogenous Case

Symmetric Mode Number	Frequency (Hz)
1	43.011
2	259.11
3	646.037

Abaqus Results

Flat Homogenous Case

Symmetric Mode Number	Frequency (Hz)
1	42.903
2	257.82
3	642.64

Flat Heterogeneous Case

Symmetric Mode Number	Frequency (Hz)
1	48.19
2	289.58
3	719.73

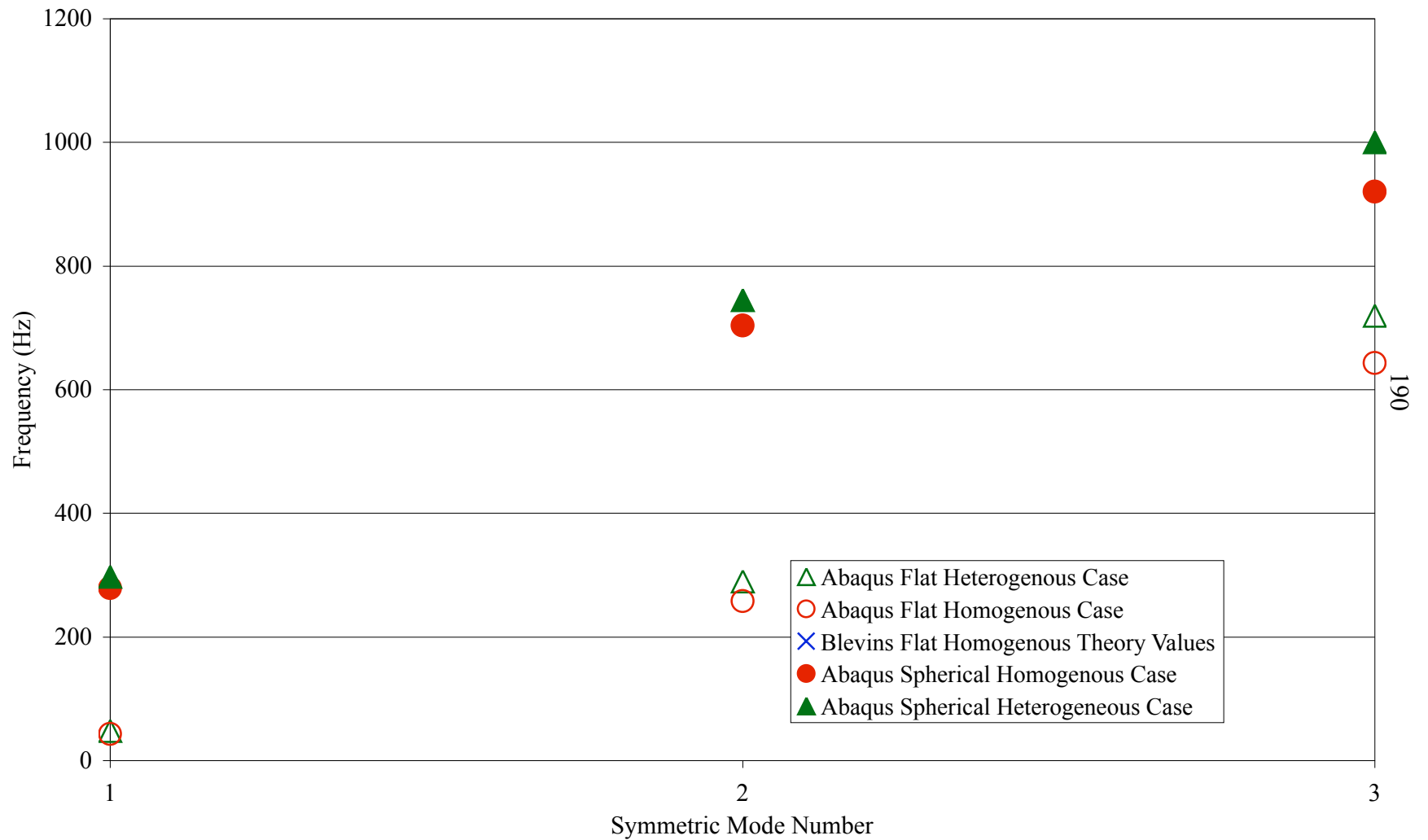
Spherical Homogenous Case

Symmetric Mode Number	Frequency (Hz)
1	278.9
2	703.86
3	920.11

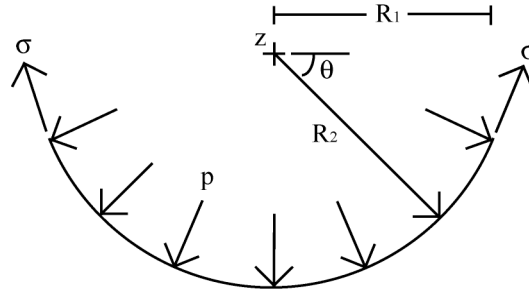
Spherical Heterogeneous Case

Symmetric Mode Number	Frequency (Hz)
1	297.18
2	745.3
3	1000.7

Axisymmetric Flat and Spherical Mirror Comparison of Natural Frequencies



Axisymmetric Spherical Shell Natural Frequency Calculations Comparing to Published Values for the Homogenous Case



Configuration:

The spherical shell is composed of a shell layer of elements

Theta opening = 90 degrees to the end of shell (hemisphere)

The published calculations come from *Gould, Finite Element Analysis of Shells of Revolution*, p. 93^{B2}

The published values were obtained by another FEA program (Shore III) and analytical values published in the material.

Shell Layer:

Thickness .25 in

Elastic Modulus 30×10^6 psi

(Homogenous case)

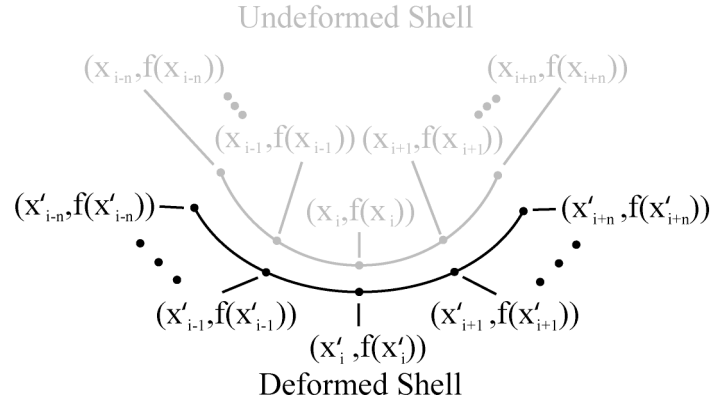
Possion's Ratio 0.3

Density $0.000736 \text{ lb} \cdot \text{s}^2/\text{in}^4$

ABAQUS and Published Calculations

Symmetric Mode Number	ABAQUS Frequency (Hz)	Published Frequency (Hz)	Percent Difference Between Values
1	944.68	970	2.61
2	1192	1212	1.65
3	1265.6	1304	2.94
4	1343.7	1421	5.44

**Surface Distortion Analysis for an Axisymmetric Spherical Shell using
Abaqus to Load the Shell with an Internal Pressure to Distort the Homogenous Cas**



Configuration:

The spherical shell is composed of a solid layer of elements with a thin layer of shell elements as its surface.

Displacement values are in the vertical U2 direction, and the horizontal U1 direction shown in the diagram

The internal pressure used was 400 Pa

Shell Layer:

Thickness	.00001 m	Possion's Ratio	0.3
Elastic Mod. (Homogenous)	2.08×10^8 Pa	Density	10.204 kg/m ³

RMS Calculation:

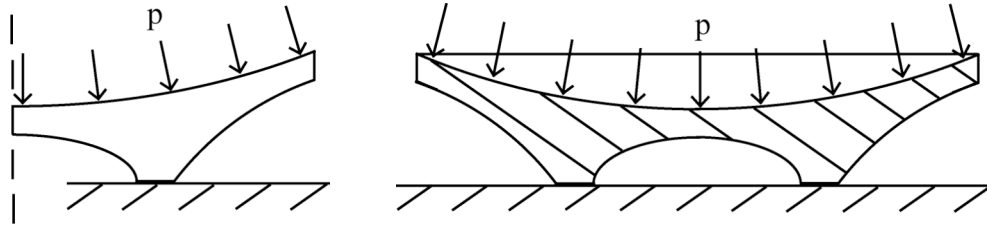
$$RMS = \sqrt{\frac{\sum_{i=1}^N ((f'(x_i) - f(x_i)) - (f'(x_{i-1}) - f(x_{i-1})))^2 (x_i - x_{i-1}))}{N \left[\sum_{i=1}^N (x_i - x_{i-1}) \right]}}$$

$$RMS_{RH} = \sqrt{\frac{\sum_{i=1}^{N-1} ((f'(x_i) - f(x_i)) - (f'(x_{i-1}) - f(x_{i-1})))^2 (x_i - x_{i-1}))}{N \left[\sum_{i=1}^{N-1} (x_i - x_{i-1}) \right]}}$$

$$RMS_{AVE} = \frac{1}{2} (RMS_{RH} + RMS_{LH})$$

RMS (m)	RMS-RH (m)	RMS-LH (m)	RMS-AVE (m)	% Difference
7.412E-09	7.361E-09	7.445E-09	7.403E-09	0.13

**RMS Surface Analysis for the 3D and Axisymmetric Spherical Shell Model
Loaded with an Internal Pressure to Distort the Homogenous Case in ABAQUS**



Configuration:

The spherical shell is composed of a solid layer of elements with a thin layer of shell elements as its mirror surface.

Displacement values are in the vertical U2 direction, and the horizontal U1 and U3 (Perpendicular to U1 in the 3D model) directions

The loading is a 100Pa Internal Load

Shell Layer: Zerodur

Thickness	1.00E-05	Possion's Ratio	0.25
Elastic Mod. (Homogenous)	9.20E+10	Density	2530.00

Axisymmetric RMS Calculation:

Method 1 Method 2

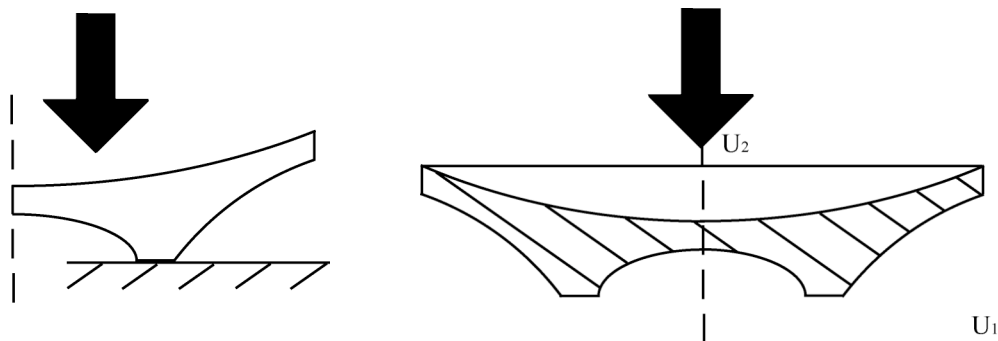
RMS (m)	RMS-RH (m)	RMS-LH (m)	RMS-AVE (m)	% Difference
3.614E-11	3.659E-11	3.346E-11	3.503E-11	3.06

3D RMS Calculation:

Method 1 Method 2

RMS (m)	RMS (m)	% Difference
2.805E-09	2.338E-09	16.67

**RMS Surface Analysis for the 3D and Axisymmetric Spherical Shell Model
Loaded with a 1 g Axial Load to Distort the Homogenous Case in ABAQUS**



Configuration:

The spherical shell is composed of a solid layer of elements with a thin layer of shell elements as its mirror surface.

Displacement values are in the vertical U_2 direction, and the horizontal U_1 and U_3 (Perpendicular to U_1 in the 3D model) directions

The loading is a 1 g Axial Load in the U_2 direction

Shell Layer: Zerodur

Thickness	1.00E-05	Possion's Ratio	0.25
Elastic Mod. (Homogenous)	9.20E+10	Density	2530.00

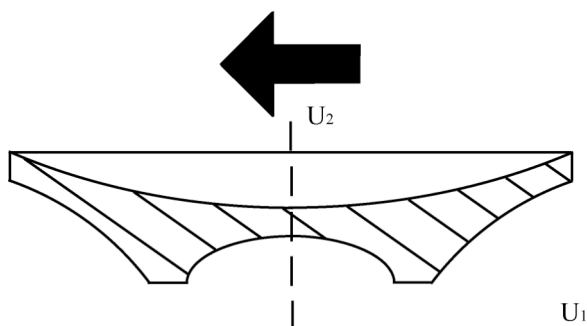
Axisymmetric RMS Calculation:

Method 1	Method 2			
RMS (m)	RMS-RH (m)	RMS-LH (m)	RMS-AVE (m)	% Difference
1.189E-11	1.192E-11	1.181E-11	1.186E-11	0.27

3D RMS Calculation:

Method 1	Method 2	
RMS (m)	RMS (m)	% Difference
7.051E-09	6.233E-09	11.60

**RMS Surface Analysis for the 3D Spherical Shell Model Loaded
with a 1 g Longitudinal Load to Distort the Homogenous Case in ABAQUS**



Configuration:

The spherical shell is composed of a solid layer of elements with a thin layer of shell elements as its mirror surface.

Displacement values are in the vertical U2 direction, and the horizontal U1 and U3 (Perpendicular to U1 in the 3D model) directions

The loading is a 1 g Longitudinal Load in the U2 direction

Shell Layer:	Zerodur		
Thickness	1.00E-05	Possion's	0.25
Elastic Mod.	9.20E+10	Ratio	
(Homogenous)		Density	2530.00

3D RMS Calculation:

Method 1

Method 2

RMS (m)	RMS (m)	% Difference
1.223E-08	5.627E-09	53.99

APPENDIX C

APPLICATION OF CLOSED FORM EQUATION

Case 1 AFRL Approximating Geometry to Predict Zero Curvature of Four Layer Trough Scenarios for Individual Temperatures per each Layer

These scenarios explore the range of dependent CTE and wt% of Zirconium Tungstate that is required to keep the system at 0 thermal curvature by defining 3 of the 4 CTE's.

Material properties of the trough layers

48 wt% approx.

70 wt% approx.

Neat CTD-403

$$E_1 := 7.25 \cdot 10^9 \text{ Pa}$$

$$E_2 := 8.571 \cdot 10^9 \text{ Pa} \quad E_3 := 8.571 \cdot 10^9 \text{ Pa}$$

$$E_4 := 4.727 \cdot 10^9 \text{ Pa}$$

$$\alpha_4 := \begin{pmatrix} 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$$

$$\alpha_3 := \begin{pmatrix} 21 & 20 & 21 & 20 & 21 & 20 & 21 & 20 \\ 20.95 & 20.05 & 20.95 & 20.05 & 20.95 & 20.05 & 20.95 & 20.05 \\ 20.9 & 20.1 & 20.9 & 20.1 & 20.9 & 20.1 & 20.9 & 20.1 \\ 20.85 & 20.15 & 20.85 & 20.15 & 20.85 & 20.15 & 20.85 & 20.15 \\ 20.8 & 20.2 & 20.8 & 20.2 & 20.8 & 20.2 & 20.8 & 20.2 \\ 20.75 & 20.25 & 20.75 & 20.25 & 20.75 & 20.25 & 20.75 & 20.25 \\ 20.7 & 20.3 & 20.7 & 20.3 & 20.7 & 20.3 & 20.7 & 20.3 \\ 20.65 & 20.35 & 20.65 & 20.35 & 20.65 & 20.35 & 20.65 & 20.35 \\ 20.6 & 20.4 & 20.6 & 20.4 & 20.6 & 20.4 & 20.6 & 20.4 \\ 20.55 & 20.45 & 20.55 & 20.45 & 20.55 & 20.45 & 20.55 & 20.45 \\ 20.5 & 20.5 & 20.5 & 20.5 & 20.5 & 20.5 & 20.5 & 20.5 \\ 20.45 & 20.55 & 20.45 & 20.55 & 20.45 & 20.55 & 20.45 & 20.55 \\ 20.4 & 20.6 & 20.4 & 20.6 & 20.4 & 20.6 & 20.4 & 20.6 \\ 20.35 & 20.65 & 20.35 & 20.65 & 20.35 & 20.65 & 20.35 & 20.65 \\ 20.3 & 20.7 & 20.3 & 20.7 & 20.3 & 20.7 & 20.3 & 20.7 \\ 20.25 & 20.75 & 20.25 & 20.75 & 20.25 & 20.75 & 20.25 & 20.75 \\ 20.2 & 20.8 & 20.2 & 20.8 & 20.2 & 20.8 & 20.2 & 20.8 \\ 20.15 & 20.85 & 20.15 & 20.85 & 20.15 & 20.85 & 20.18 & 20.85 \\ 20.1 & 20.9 & 20.1 & 20.9 & 20.1 & 20.9 & 20.1 & 20.9 \\ 20.05 & 20.95 & 20.05 & 20.95 & 20.05 & 20.95 & 20.05 & 20.95 \end{pmatrix} 10^{-6} \frac{1}{\text{C}}$$

$$\alpha_2 := \begin{pmatrix} 22 & 22 & 21 & 21 & 22 & 22 & 21 & 21 \\ 21.95 & 21.95 & 21.05 & 21.05 & 21.95 & 21.95 & 21.05 & 21.05 \\ 21.9 & 21.9 & 21.1 & 21.1 & 21.9 & 21.9 & 21.1 & 21.1 \\ 21.85 & 21.85 & 21.15 & 21.15 & 21.85 & 21.85 & 21.15 & 21.15 \\ 21.8 & 21.8 & 21.2 & 21.2 & 21.8 & 21.8 & 21.2 & 21.2 \\ 21.75 & 21.75 & 21.25 & 21.25 & 21.75 & 21.75 & 21.25 & 21.25 \\ 21.7 & 21.7 & 21.3 & 21.3 & 21.7 & 21.7 & 21.3 & 21.3 \\ 21.65 & 21.65 & 21.35 & 21.35 & 21.65 & 21.65 & 21.35 & 21.35 \\ 21.6 & 21.6 & 21.4 & 21.4 & 21.6 & 21.6 & 21.4 & 21.4 \\ 21.55 & 21.55 & 21.45 & 21.45 & 21.55 & 21.55 & 21.45 & 21.45 \\ 21.5 & 21.5 & 21.5 & 21.5 & 21.5 & 21.5 & 21.5 & 21.5 \\ 21.45 & 21.45 & 21.55 & 21.55 & 21.45 & 21.45 & 21.55 & 21.55 \\ 21.4 & 21.4 & 21.6 & 21.6 & 21.4 & 21.4 & 21.6 & 21.6 \\ 21.35 & 21.35 & 21.65 & 21.65 & 21.35 & 21.35 & 21.65 & 21.65 \\ 21.3 & 21.3 & 21.7 & 21.7 & 21.3 & 21.3 & 21.7 & 21.7 \\ 21.25 & 21.25 & 21.75 & 21.75 & 21.25 & 21.25 & 21.75 & 21.75 \\ 21.2 & 21.2 & 21.8 & 21.8 & 21.2 & 21.2 & 21.8 & 21.8 \\ 21.15 & 21.15 & 21.85 & 21.85 & 21.15 & 21.15 & 21.85 & 21.85 \\ 21.1 & 21.1 & 21.9 & 21.9 & 21.1 & 21.1 & 21.9 & 21.9 \\ 21.05 & 21.05 & 21.95 & 21.95 & 21.05 & 21.05 & 21.95 & 21.95 \end{pmatrix} 10^{-6} \cdot \frac{1}{C}$$

Material geometries of the trough layers

layer thickness: $h_1 := 0.023m$ $h_2 := 0.023m$ $h_3 := 0.023m$ $h_4 := 0.023m$

Unit depth: $unit := 1$

Temperature Changes of each layer:

$T_0 := 0C$ $T_1 := 100C$ $T_2 := 75C$ $T_3 := 50C$ $T_4 := 25C$

$$\Delta T_{21} := \frac{T_2 + T_1}{2} - T_0 \quad \Delta T_{31} := \frac{T_3 + T_1}{2} - T_0 \quad \Delta T_{41} := \frac{T_4 + T_1}{2} - T_0$$

$$\Delta T_{32} := \frac{T_3 + T_2}{2} - T_0 \quad \Delta T_{42} := \frac{T_4 + T_2}{2} - T_0 \quad \Delta T_{43} := \frac{T_4 + T_3}{2} - T_0$$

To find the CTE in layer one:

$s_1 := h_1 \cdot E_1$ $s_2 := h_2 \cdot E_2$ $s_3 := h_3 \cdot E_3$ $s_4 := h_4 \cdot E_4$

$$I_1 := unit \cdot \frac{h_1^3}{12} \quad I_2 := unit \cdot \frac{h_2^3}{12} \quad I_3 := unit \cdot \frac{h_3^3}{12} \quad I_4 := unit \cdot \frac{h_4^3}{12}$$

$h_{12} := h_1 + h_2$ $h_{13} := h_1 + 2 \cdot h_2 + h_3$ $h_{14} := h_1 + 2 \cdot h_2 + 2 \cdot h_3 + h_4$

$h_{23} := h_2 + h_3$ $h_{24} := h_2 + 2 \cdot h_3 + h_4$ $h_{34} := h_3 + h_4$

$$P_2 := \alpha_2 [\Delta T_{21} \cdot s_1 \cdot s_2 (h_{12}) - \Delta T_{32} \cdot s_2 \cdot s_3 (h_{23}) - \Delta T_{42} \cdot s_2 \cdot s_4 (h_{24})]$$

$$P_3 := \alpha_3 [\Delta T_{31} \cdot s_1 \cdot s_3 (h_{13}) + \Delta T_{32} \cdot s_2 \cdot s_3 (h_{23}) - \Delta T_{43} \cdot s_3 \cdot s_4 (h_{34})]$$

$$P_4 := \alpha_4 [\Delta T_{41} \cdot s_1 \cdot s_4 (h_{14}) + \Delta T_{42} \cdot s_2 \cdot s_4 (h_{24}) + \Delta T_{43} \cdot s_3 \cdot s_4 (h_{34})]$$

CTE of layer one:

$$\alpha_1 := \frac{-(P_2 + P_3 + P_4)}{-\Delta T_{21} \cdot s_1 \cdot s_2 (h_{12}) - \Delta T_{31} \cdot s_1 \cdot s_3 (h_{13}) - \Delta T_{41} \cdot s_1 \cdot s_4 (h_{14})}$$

$$\alpha_1 = \begin{pmatrix} 38.4 & 37.82 & 38.56 & 37.97 & 39.68 & 39.09 & 39.83 & 39.24 \\ 38.38 & 37.86 & 38.52 & 37.99 & 39.66 & 39.13 & 39.79 & 39.27 \\ 38.36 & 37.89 & 38.48 & 38.01 & 39.63 & 39.17 & 39.76 & 39.29 \\ 38.34 & 37.93 & 38.45 & 38.04 & 39.61 & 39.2 & 39.72 & 39.31 \\ 38.32 & 37.97 & 38.41 & 38.06 & 39.59 & 39.24 & 39.68 & 39.33 \\ 38.3 & 38 & 38.37 & 38.08 & 39.57 & 39.28 & 39.65 & 39.35 \\ 38.27 & 38.04 & 38.33 & 38.1 & 39.55 & 39.31 & 39.61 & 39.37 \\ 38.25 & 38.08 & 38.3 & 38.12 & 39.53 & 39.35 & 39.57 & 39.4 \\ 38.23 & 38.11 & 38.26 & 38.14 & 39.5 & 39.39 & 39.54 & 39.42 \\ 38.21 & 38.15 & 38.22 & 38.17 & 39.48 & 39.42 & 39.5 & 39.44 \\ 38.19 & 38.19 & 38.19 & 38.19 & 39.46 & 39.46 & 39.46 & 39.46 \\ 38.17 & 38.22 & 38.15 & 38.21 & 39.44 & 39.5 & 39.42 & 39.48 \\ 38.14 & 38.26 & 38.11 & 38.23 & 39.42 & 39.54 & 39.39 & 39.5 \\ 38.12 & 38.3 & 38.08 & 38.25 & 39.4 & 39.57 & 39.35 & 39.53 \\ 38.1 & 38.33 & 38.04 & 38.27 & 39.37 & 39.61 & 39.31 & 39.55 \\ 38.08 & 38.37 & 38 & 38.3 & 39.35 & 39.65 & 39.28 & 39.57 \\ 38.06 & 38.41 & 37.97 & 38.32 & 39.33 & 39.68 & 39.24 & 39.59 \\ 38.04 & 38.45 & 37.93 & 38.34 & 39.31 & 39.72 & 39.22 & 39.61 \\ 38.01 & 38.48 & 37.89 & 38.36 & 39.29 & 39.76 & 39.17 & 39.63 \\ 37.99 & 38.52 & 37.86 & 38.38 & 39.27 & 39.79 & 39.13 & 39.66 \end{pmatrix} 10^{-6} \cdot \frac{1}{C}$$

Note: Most of the scenarios produce CTE values for layer 1 that are in the range of the neat 48 wt% for the temperature of 100C as desired.

**RMS Surface Distortion and Radius of Curvature Comparison for
Zirconium Tungstate Thermostat Application in Trough Configuration
FEA vs. Zero Curvature Closed Form Equation Results for Case 1**

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.100E-06
RMS of best fit of FEA deformed	3.051E-06
Percent Difference between FEA and Closed Form	1.580

Radius of Curvature Results:

FEA Radius of Curvature	1.039
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	3.781

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4380E-15	-0.001438
2	0.00143	0.00144	-1.00000	-1.00000	-1.2942E-14	-0.001438
3	0.00287	0.00288	-1.00000	-1.00000	-3.5950E-14	-0.001438
4	0.00430	0.00431	-0.99999	-1.00000	-7.0462E-14	-0.001438
5	0.00573	0.00575	-0.99998	-1.00000	-1.4380E-13	-0.001438
6	0.00717	0.00719	-0.99997	-1.00000	-1.7399E-13	-0.001438
7	0.00860	0.00863	-0.99996	-1.00000	-2.4301E-13	-0.001438

Performed for all surface nodes and RMS is calculated according to the above equation

**RMS Surface Distortion and Radius of Curvature Comparison for
Zirconium Tungstate AFRL Axisymmetric Spherical Mirror Geometry
FEA vs. Zero Curvature Closed Form Equation Results for Case 1**

Thermal Loading Condition:

Decreasing Temperature Gradient

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

AFRL Axisymmetric FEA RMS Surface Distortion	3.992E-06
MLT FEA RMS Surface Distortion	3.051E-06
Percent Difference between FEA and Closed Form	23.585

Radius of Curvature Results:

FEA Radius of Curvature	1.038
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	3.690

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4380E-15	-0.001438
2	0.00143	0.00144	-1.00000	-1.00000	-1.2942E-14	-0.001438
3	0.00287	0.00288	-1.00000	-1.00000	-3.5950E-14	-0.001438
4	0.00430	0.00431	-0.99999	-1.00000	-7.0462E-14	-0.001438
5	0.00573	0.00575	-0.99998	-1.00000	-1.4380E-13	-0.001438
6	0.00717	0.00719	-0.99997	-1.00000	-1.7399E-13	-0.001438
7	0.00860	0.00863	-0.99996	-1.00000	-2.4301E-13	-0.001438

Performed for all surface nodes and RMS is calculated according to the above equation

Case 2 AFRL Approximating Geometry to Predict Zero Curvature of Four Layer Trough Scenarios for Individual Temperatures per each Layer

These scenarios explore the range of dependent CTE and wt% of Zirconium Tungstate that is required to keep the system at 0 thermal curvature by defining 3 of the 4 CTE's. Material properties of the trough layers

	70 wt% approx.	Neat CTD-403	
	$E_2 := 8.571 \cdot 10^9 \text{ Pa}$	$E_1 := 4.727 \cdot 10^9 \text{ Pa}$	$E_3 := 4.727 \cdot 10^9 \text{ Pa} \quad E_4 := 4.727 \cdot 10^9 \text{ Pa}$
$\alpha_4 :=$	$\left(\begin{array}{cccccccc} 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \end{array} \right) 10^{-6} \cdot \frac{1}{\text{C}}$		
$\alpha_3 :=$	$\left(\begin{array}{cccccccc} 54.25 & 56.5 & 54.25 & 56.5 & 54.25 & 56.5 & 54.25 & 56.5 \\ 54.5 & 56.25 & 54.5 & 56.25 & 54.5 & 56.25 & 54.5 & 56.25 \\ 54.75 & 56 & 54.75 & 56 & 54.75 & 56 & 54.75 & 56 \\ 55 & 55.75 & 55 & 55.75 & 55 & 55.75 & 55 & 55.75 \\ 55.25 & 55.5 & 55.25 & 55.5 & 55.25 & 55.5 & 55.25 & 55.5 \\ 55.5 & 55.25 & 55.5 & 55.25 & 55.5 & 55.25 & 55.5 & 55.25 \\ 55.75 & 55 & 55.75 & 55 & 55.75 & 55 & 55.75 & 55 \\ 56 & 54.75 & 56 & 54.75 & 56 & 54.75 & 56 & 54.75 \\ 56.25 & 54.5 & 56.25 & 54.5 & 56.25 & 54.5 & 56.25 & 54.5 \\ 56.5 & 54.25 & 56.5 & 54.25 & 56.5 & 54.25 & 56.5 & 54.25 \end{array} \right) 10^{-6} \cdot \frac{1}{\text{C}}$		
$\alpha_2 :=$	$\left(\begin{array}{cccccccc} 21 & 21 & 22 & 22 & 21 & 21 & 22 & 22 \\ 21.1 & 21.1 & 21.9 & 21.9 & 21.1 & 21.1 & 21.9 & 21.9 \\ 21.2 & 21.2 & 21.8 & 21.8 & 21.2 & 21.2 & 21.8 & 21.8 \\ 21.3 & 21.3 & 21.7 & 21.7 & 21.3 & 21.3 & 21.7 & 21.7 \\ 21.4 & 21.4 & 21.6 & 21.6 & 21.4 & 21.4 & 21.6 & 21.6 \\ 21.5 & 21.5 & 21.5 & 21.5 & 21.5 & 21.5 & 21.5 & 21.5 \\ 21.6 & 21.6 & 21.4 & 21.4 & 21.6 & 21.6 & 21.4 & 21.4 \\ 21.7 & 21.7 & 21.3 & 21.3 & 21.7 & 21.7 & 21.3 & 21.3 \\ 21.8 & 21.8 & 21.2 & 21.2 & 21.8 & 21.8 & 21.2 & 21.2 \\ 21.9 & 21.9 & 21.1 & 21.1 & 21.9 & 21.9 & 21.1 & 21.1 \end{array} \right) 10^{-6} \cdot \frac{1}{\text{C}}$		

Material geometries of the trough layers

layer thickness: $h_1 := 0.023\text{m}$ $h_2 := 0.023\text{m}$ $h_3 := 0.023\text{m}$ $h_4 := 0.023\text{m}$

Unit depth: $\text{unit} := 1$

Temperature Changes of each layer:

$$\begin{aligned} T_0 &:= 0\text{C} & T_1 &:= 100\text{C} & T_2 &:= 75\text{C} & T_3 &:= 50\text{C} & T_4 &:= 25\text{C} \\ \Delta T_{21} &:= \frac{T_2 + T_1}{2} - T_0 & \Delta T_{31} &:= \frac{T_3 + T_1}{2} - T_0 & \Delta T_{41} &:= \frac{T_4 + T_1}{2} - T_0 \\ \Delta T_{32} &:= \frac{T_3 + T_2}{2} - T_0 & \Delta T_{42} &:= \frac{T_4 + T_2}{2} - T_0 & \Delta T_{43} &:= \frac{T_4 + T_3}{2} - T_0 \end{aligned}$$

To find the CTE in layer one:

$$\begin{aligned} s_1 &:= h_1 \cdot E_1 & s_2 &:= h_2 \cdot E_2 & s_3 &:= h_3 \cdot E_3 & s_4 &:= h_4 \cdot E_4 \\ I_1 &:= \text{unit} \cdot \frac{h_1^3}{12} & I_2 &:= \text{unit} \cdot \frac{h_2^3}{12} & I_3 &:= \text{unit} \cdot \frac{h_3^3}{12} & I_4 &:= \text{unit} \cdot \frac{h_4^3}{12} \\ h_{12} &:= h_1 + h_2 & h_{13} &:= h_1 + 2 \cdot h_2 + h_3 & h_{14} &:= h_1 + 2 \cdot h_2 + 2 \cdot h_3 + h_4 \\ h_{23} &:= h_2 + h_3 & h_{24} &:= h_2 + 2 \cdot h_3 + h_4 & h_{34} &:= h_3 + h_4 \\ P_2 &:= \alpha_2 \cdot [\Delta T_{21} \cdot s_1 \cdot s_2 \cdot (h_{12}) - \Delta T_{32} \cdot s_2 \cdot s_3 \cdot (h_{23}) - \Delta T_{42} \cdot s_2 \cdot s_4 \cdot (h_{24})] \\ P_3 &:= \alpha_3 \cdot [\Delta T_{31} \cdot s_1 \cdot s_3 \cdot (h_{13}) + \Delta T_{32} \cdot s_2 \cdot s_3 \cdot (h_{23}) - \Delta T_{43} \cdot s_3 \cdot s_4 \cdot (h_{34})] \\ P_4 &:= \alpha_4 \cdot [\Delta T_{41} \cdot s_1 \cdot s_4 \cdot (h_{14}) + \Delta T_{42} \cdot s_2 \cdot s_4 \cdot (h_{24}) + \Delta T_{43} \cdot s_3 \cdot s_4 \cdot (h_{34})] \end{aligned}$$

CTE of layer one:

$$\alpha_1 := \frac{-(P_2 + P_3 + P_4)}{-\Delta T_{21} \cdot s_1 \cdot s_2 \cdot (h_{12}) - \Delta T_{31} \cdot s_1 \cdot s_3 \cdot (h_{13}) - \Delta T_{41} \cdot s_1 \cdot s_4 \cdot (h_{14})}$$

$$\alpha_1 = \begin{pmatrix} 61.521 & 62.545 & 61.247 & 62.271 & 63.363 & 64.388 & 63.089 & 64.113 \\ 61.607 & 62.404 & 61.388 & 62.184 & 63.45 & 64.246 & 63.231 & 64.027 \\ 61.694 & 62.263 & 61.529 & 62.098 & 63.536 & 64.105 & 63.372 & 63.941 \\ 61.78 & 62.121 & 61.67 & 62.012 & 63.623 & 63.964 & 63.513 & 63.854 \\ 61.866 & 61.98 & 61.812 & 61.925 & 63.709 & 63.823 & 63.654 & 63.768 \\ 61.953 & 61.839 & 61.953 & 61.839 & 63.795 & 63.682 & 63.795 & 63.682 \\ 62.039 & 61.698 & 62.094 & 61.753 & 63.882 & 63.54 & 63.937 & 63.595 \\ 62.125 & 61.557 & 62.235 & 61.666 & 63.968 & 63.399 & 64.078 & 63.509 \\ 62.212 & 61.415 & 62.376 & 61.58 & 64.054 & 63.258 & 64.219 & 63.422 \\ 62.298 & 61.274 & 62.517 & 61.493 & 64.141 & 63.117 & 64.36 & 63.336 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$$

Note: A few of the scenarios produce CTE values for layer 1 close the range of the neat 0 wt% for the temperature of 100C as desired.

**RMS Surface Distortion and Radius of Curvature Comparison for
Zirconium Tungstate Thermostat Application in Trough Configuration
FEA vs. Zero Curvature Closed Form Equation Results for Case 2**

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.080E-06
RMS of best fit of FEA deformed	3.097E-06
Percent Difference between FEA and Closed Form	-0.566

Radius of Curvature Results:

FEA Radius of Curvature	1.053
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	5.046

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4400E-15	-0.001440
2	0.00143	0.00144	-1.00000	-1.00000	-1.2960E-14	-0.001440
3	0.00287	0.00288	-1.00000	-1.00000	-3.5999E-14	-0.001440
4	0.00430	0.00432	-0.99999	-1.00000	-7.0558E-14	-0.001440
5	0.00573	0.00576	-0.99998	-1.00000	-1.4400E-13	-0.001440
6	0.00717	0.00720	-0.99997	-1.00000	-1.7423E-13	-0.001440
7	0.00860	0.00864	-0.99996	-1.00000	-2.4335E-13	-0.001440

Performed for all surface nodes and RMS is calculated according to the above equation

**RMS Surface Distortion and Radius of Curvature Comparison for
Zirconium Tungstate AFRL Axisymmetric Spherical Mirror Geometry
FEA vs. Zero Curvature Closed Form Equation Results for Case 2**

Thermal Loading Condition:

Decreasing Temperature Gradient

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

AFRL Axisymmetric FEA RMS Surface Distortion	4.714E-06
MLT FEA RMS Surface Distortion	3.097E-06
Percent Difference between FEA and Closed Form	34.296

Radius of Curvature Results:

FEA Radius of Curvature	1.051
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	4.825

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4380E-15	-0.001438
2	0.00143	0.00144	-1.00000	-1.00000	-1.2942E-14	-0.001438
3	0.00287	0.00288	-1.00000	-1.00000	-3.5950E-14	-0.001438
4	0.00430	0.00431	-0.99999	-1.00000	-7.0462E-14	-0.001438
5	0.00573	0.00575	-0.99998	-1.00000	-1.4380E-13	-0.001438
6	0.00717	0.00719	-0.99997	-1.00000	-1.7399E-13	-0.001438
7	0.00860	0.00863	-0.99996	-1.00000	-2.4301E-13	-0.001438

Performed for all surface nodes and RMS is calculated according to the above equation

Case 3 AFRL Approximating Geometry to Predict Zero Curvature of Four Layer Trough Scenarios for Individual Temperatures per each Layer

These scenarios explore the range of dependent CTE and wt% of Zirconium Tungstate that is required to keep the system at 0 thermal curvature by defining 3 of the 4 CTE's.

Material properties of the trough layers

48 wt% approx.

$$E_1 := 7.25 \cdot 10^9 \text{ Pa}$$

70 wt% approx.

$$E_3 := 8.571 \cdot 10^9 \text{ Pa}$$

Neat CTD-403

$$E_2 := 4.727 \cdot 10^9 \text{ Pa}$$

$$E_4 := 4.727 \cdot 10^9 \text{ Pa}$$

$$\alpha_4 := \begin{pmatrix} 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$$

$$\alpha_3 := \begin{pmatrix} 21 & 20 & 21 & 20 & 21 & 20 & 21 & 20 \\ 20.95 & 20.05 & 20.95 & 20.05 & 20.95 & 20.05 & 20.95 & 20.05 \\ 20.9 & 20.1 & 20.9 & 20.1 & 20.9 & 20.1 & 20.9 & 20.1 \\ 20.85 & 20.15 & 20.85 & 20.15 & 20.85 & 20.15 & 20.85 & 20.15 \\ 20.8 & 20.2 & 20.8 & 20.2 & 20.8 & 20.2 & 20.8 & 20.2 \\ 20.75 & 20.25 & 20.75 & 20.25 & 20.75 & 20.25 & 20.75 & 20.25 \\ 20.7 & 20.3 & 20.7 & 20.3 & 20.7 & 20.3 & 20.7 & 20.3 \\ 20.65 & 20.35 & 20.65 & 20.35 & 20.65 & 20.35 & 20.65 & 20.35 \\ 20.6 & 20.4 & 20.6 & 20.4 & 20.6 & 20.4 & 20.6 & 20.4 \\ 20.55 & 20.45 & 20.55 & 20.45 & 20.55 & 20.45 & 20.55 & 20.45 \\ 20.5 & 20.5 & 20.5 & 20.5 & 20.5 & 20.5 & 20.5 & 20.5 \\ 20.45 & 20.55 & 20.45 & 20.55 & 20.45 & 20.55 & 20.45 & 20.55 \\ 20.4 & 20.6 & 20.4 & 20.6 & 20.4 & 20.6 & 20.4 & 20.6 \\ 20.35 & 20.65 & 20.35 & 20.65 & 20.35 & 20.65 & 20.35 & 20.65 \\ 20.3 & 20.7 & 20.3 & 20.7 & 20.3 & 20.7 & 20.3 & 20.7 \\ 20.25 & 20.75 & 20.25 & 20.75 & 20.25 & 20.75 & 20.25 & 20.75 \\ 20.2 & 20.8 & 20.2 & 20.8 & 20.2 & 20.8 & 20.2 & 20.8 \\ 20.15 & 20.85 & 20.15 & 20.85 & 20.15 & 20.85 & 20.18 & 20.85 \\ 20.1 & 20.9 & 20.1 & 20.9 & 20.1 & 20.9 & 20.1 & 20.9 \\ 20.05 & 20.95 & 20.05 & 20.95 & 20.05 & 20.95 & 20.05 & 20.95 \end{pmatrix} 10^{-6} \frac{1}{C}$$

$$\alpha_2 := \begin{pmatrix} 56.5 & 56.5 & 58.5 & 58.5 & 56.5 & 56.5 & 58.5 & 58.5 \\ 56.6 & 56.6 & 58.4 & 58.4 & 56.6 & 56.6 & 58.4 & 58.4 \\ 56.7 & 56.7 & 58.3 & 58.3 & 56.7 & 56.7 & 58.3 & 58.3 \\ 56.8 & 56.8 & 58.2 & 58.2 & 56.8 & 56.8 & 58.2 & 58.2 \\ 56.9 & 56.9 & 58.1 & 58.1 & 56.9 & 56.9 & 58.1 & 58.1 \\ 57 & 57 & 58 & 58 & 57 & 57 & 58 & 58 \\ 57.1 & 57.1 & 57.9 & 57.9 & 57.1 & 57.1 & 57.9 & 57.9 \\ 57.2 & 57.2 & 57.8 & 57.8 & 57.2 & 57.2 & 57.8 & 57.8 \\ 57.3 & 57.3 & 57.7 & 57.7 & 57.3 & 57.3 & 57.7 & 57.7 \\ 57.4 & 57.4 & 57.6 & 57.6 & 57.4 & 57.4 & 57.6 & 57.6 \\ 57.5 & 57.5 & 57.5 & 57.5 & 57.5 & 57.5 & 57.5 & 57.5 \\ 57.6 & 57.6 & 57.4 & 57.4 & 57.6 & 57.6 & 57.4 & 57.4 \\ 57.7 & 57.7 & 57.3 & 57.3 & 57.7 & 57.7 & 57.3 & 57.3 \\ 57.8 & 57.8 & 57.2 & 57.2 & 57.8 & 57.8 & 57.2 & 57.2 \\ 57.9 & 57.9 & 57.1 & 57.1 & 57.9 & 57.9 & 57.1 & 57.1 \\ 58 & 58 & 57 & 57 & 58 & 58 & 57 & 57 \\ 58.1 & 58.1 & 56.9 & 56.9 & 58.1 & 58.1 & 56.9 & 56.9 \\ 58.2 & 58.2 & 56.8 & 56.8 & 58.2 & 58.2 & 56.8 & 56.8 \\ 58.3 & 58.3 & 56.7 & 56.7 & 58.3 & 58.3 & 56.7 & 56.7 \\ 58.4 & 58.4 & 56.6 & 56.6 & 58.4 & 58.4 & 56.6 & 56.6 \end{pmatrix} 10^{-6} \cdot \frac{1}{C}$$

Material geometries of the trough layers

layer thickness: $h_1 := 0.023m$ $h_2 := 0.023m$ $h_3 := 0.023m$ $h_4 := 0.023m$

Unit depth: $unit := 1$

Temperature Changes of each layer:

$T_0 := 0C$ $T_1 := 100C$ $T_2 := 75C$ $T_3 := 50C$ $T_4 := 25C$

$$\Delta T_{21} := \frac{T_2 + T_1}{2} - T_0 \quad \Delta T_{31} := \frac{T_3 + T_1}{2} - T_0 \quad \Delta T_{41} := \frac{T_4 + T_1}{2} - T_0$$

$$\Delta T_{32} := \frac{T_3 + T_2}{2} - T_0 \quad \Delta T_{42} := \frac{T_4 + T_2}{2} - T_0 \quad \Delta T_{43} := \frac{T_4 + T_3}{2} - T_0$$

To find the CTE in layer one:

$s_1 := h_1 \cdot E_1$ $s_2 := h_2 \cdot E_2$ $s_3 := h_3 \cdot E_3$ $s_4 := h_4 \cdot E_4$

$$I_1 := unit \cdot \frac{h_1^3}{12} \quad I_2 := unit \cdot \frac{h_2^3}{12} \quad I_3 := unit \cdot \frac{h_3^3}{12} \quad I_4 := unit \cdot \frac{h_4^3}{12}$$

$h_{12} := h_1 + h_2$ $h_{13} := h_1 + 2 \cdot h_2 + h_3$ $h_{14} := h_1 + 2 \cdot h_2 + 2 \cdot h_3 + h_4$

$$h_{23} := h_2 + h_3 \quad h_{24} := h_2 + 2 \cdot h_3 + h_4 \quad h_{34} := h_3 + h_4$$

$$P_2 := \alpha_2 \cdot [\Delta T_{21} \cdot s_1 \cdot s_2 \cdot (h_{12}) - \Delta T_{32} \cdot s_2 \cdot s_3 \cdot (h_{23}) - \Delta T_{42} \cdot s_2 \cdot s_4 \cdot (h_{24})]$$

$$P_3 := \alpha_3 \cdot [\Delta T_{31} \cdot s_1 \cdot s_3 \cdot (h_{13}) + \Delta T_{32} \cdot s_2 \cdot s_3 \cdot (h_{23}) - \Delta T_{43} \cdot s_3 \cdot s_4 \cdot (h_{34})]$$

$$P_4 := \alpha_4 \cdot [\Delta T_{41} \cdot s_1 \cdot s_4 \cdot (h_{14}) + \Delta T_{42} \cdot s_2 \cdot s_4 \cdot (h_{24}) + \Delta T_{43} \cdot s_3 \cdot s_4 \cdot (h_{34})]$$

CTE of layer one:

$$\alpha_1 := \frac{-(P_2 + P_3 + P_4)}{-\Delta T_{21} \cdot s_1 \cdot s_2 \cdot (h_{12}) - \Delta T_{31} \cdot s_1 \cdot s_3 \cdot (h_{13}) - \Delta T_{41} \cdot s_1 \cdot s_4 \cdot (h_{14})}$$

$$\alpha_1 = \begin{pmatrix} 34.486 & 33.935 & 34.298 & 33.746 & 35.708 & 35.157 & 35.519 & 34.968 \\ 34.449 & 33.953 & 34.279 & 33.783 & 35.671 & 35.175 & 35.501 & 35.005 \\ 34.412 & 33.971 & 34.261 & 33.82 & 35.634 & 35.193 & 35.483 & 35.042 \\ 34.375 & 33.989 & 34.243 & 33.857 & 35.597 & 35.211 & 35.465 & 35.079 \\ 34.338 & 34.007 & 34.225 & 33.894 & 35.56 & 35.229 & 35.447 & 35.116 \\ 34.301 & 34.026 & 34.207 & 33.931 & 35.523 & 35.247 & 35.429 & 35.153 \\ 34.264 & 34.044 & 34.189 & 33.968 & 35.486 & 35.265 & 35.411 & 35.19 \\ 34.227 & 34.062 & 34.171 & 34.005 & 35.449 & 35.284 & 35.392 & 35.227 \\ 34.19 & 34.08 & 34.152 & 34.042 & 35.412 & 35.302 & 35.374 & 35.264 \\ 34.153 & 34.098 & 34.134 & 34.079 & 35.375 & 35.32 & 35.356 & 35.301 \\ 34.116 & 34.116 & 34.116 & 34.116 & 35.338 & 35.338 & 35.338 & 35.338 \\ 34.079 & 34.134 & 34.098 & 34.153 & 35.301 & 35.356 & 35.32 & 35.375 \\ 34.042 & 34.152 & 34.08 & 34.19 & 35.264 & 35.374 & 35.302 & 35.412 \\ 34.005 & 34.171 & 34.062 & 34.227 & 35.227 & 35.392 & 35.284 & 35.449 \\ 33.968 & 34.189 & 34.044 & 34.264 & 35.19 & 35.411 & 35.265 & 35.486 \\ 33.931 & 34.207 & 34.026 & 34.301 & 35.153 & 35.429 & 35.247 & 35.523 \\ 33.894 & 34.225 & 34.007 & 34.338 & 35.116 & 35.447 & 35.229 & 35.56 \\ 33.857 & 34.243 & 33.989 & 34.375 & 35.079 & 35.465 & 35.228 & 35.597 \\ 33.82 & 34.261 & 33.971 & 34.412 & 35.042 & 35.483 & 35.193 & 35.634 \\ 33.783 & 34.279 & 33.953 & 34.449 & 35.005 & 35.501 & 35.175 & 35.671 \end{pmatrix} 10^{-6} \cdot \frac{1}{C}$$

Note: None of the scenarios produce CTE values for layer 1 that are in the range of the 48 wt% Zirconium Tungstate for the temperature range between 0C and 100C as desired. If layer 1 was set as some %wt in the range of 48-70 wt% Zirconium Tungstate composite, this scenario might produce zero curvature.

**Case 4 AFRL Approximating Geometry to Predict Zero Curvature of Four
Layer Trough Scenarios for Individual Temperatures per each Layer**

These scenarios explore the range of dependent CTE and wt% of Zirconium Tungstate that is required to keep the system at 0 thermal curvature by defining 3 of the 4 CTE's.

Material properties of the trough layers

$$\text{Neat CTD-403} \quad E_1 := 4.727 \cdot 10^9 \text{ Pa} \quad E_2 := 4.727 \cdot 10^9 \text{ Pa} \quad E_3 := 4.727 \cdot 10^9 \text{ Pa} \quad E_4 := 4.727 \cdot 10^9 \text{ Pa}$$

$$\alpha_4 := \begin{pmatrix} 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$$

$$\alpha_3 := \begin{pmatrix} 54.25 & 56.5 & 54.25 & 56.5 & 54.25 & 56.5 & 54.25 & 56.5 \\ 54.5 & 56.25 & 54.5 & 56.25 & 54.5 & 56.25 & 54.5 & 56.25 \\ 54.75 & 56 & 54.75 & 56 & 54.75 & 56 & 54.75 & 56 \\ 55 & 55.75 & 55 & 55.75 & 55 & 55.75 & 55 & 55.75 \\ 55.25 & 55.5 & 55.25 & 55.5 & 55.25 & 55.5 & 55.25 & 55.5 \\ 55.5 & 55.25 & 55.5 & 55.25 & 55.5 & 55.25 & 55.5 & 55.25 \\ 55.75 & 55 & 55.75 & 55 & 55.75 & 55 & 55.75 & 55 \\ 56 & 54.75 & 56 & 54.75 & 56 & 54.75 & 56 & 54.75 \\ 56.25 & 54.5 & 56.25 & 54.5 & 56.25 & 54.5 & 56.25 & 54.5 \\ 56.5 & 54.25 & 56.5 & 54.25 & 56.5 & 54.25 & 56.5 & 54.25 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$$

$$\alpha_2 := \begin{pmatrix} 56.5 & 56.5 & 58.75 & 58.75 & 56.5 & 56.5 & 58.75 & 58.75 \\ 56.75 & 56.75 & 58.5 & 58.5 & 56.75 & 56.75 & 58.5 & 58.5 \\ 56.5 & 56.5 & 58.25 & 58.25 & 56.5 & 56.5 & 58.25 & 58.25 \\ 56.25 & 56.25 & 58 & 58 & 56.25 & 56.25 & 58 & 58 \\ 56 & 56 & 57.75 & 57.75 & 56 & 56 & 57.75 & 57.75 \\ 58.75 & 58.75 & 57.5 & 57.5 & 58.75 & 58.75 & 57.5 & 57.5 \\ 58.5 & 58.5 & 57.25 & 57.25 & 58.5 & 58.5 & 57.25 & 57.25 \\ 58.25 & 58.25 & 57 & 57 & 58.25 & 58.25 & 57 & 57 \\ 58 & 58 & 56.75 & 56.75 & 58 & 58 & 56.75 & 56.75 \\ 58.25 & 58.25 & 56.5 & 56.5 & 58.25 & 58.25 & 56.5 & 56.5 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$$

Material geometries of the trough layers

layer thickness: $h_1 := 0.023\text{m}$ $h_2 := 0.023\text{m}$ $h_3 := 0.023\text{m}$ $h_4 := 0.023\text{m}$

Unit depth: $\text{unit} := 1$

Temperature Changes of each layer:

$$\begin{aligned} T_0 &:= 0\text{C} & T_1 &:= 100\text{C} & T_2 &:= 75\text{C} & T_3 &:= 50\text{C} & T_4 &:= 25\text{C} \\ \Delta T_{21} &:= \frac{T_2 + T_1}{2} - T_0 & \Delta T_{31} &:= \frac{T_3 + T_1}{2} - T_0 & \Delta T_{41} &:= \frac{T_4 + T_1}{2} - T_0 \\ \Delta T_{32} &:= \frac{T_3 + T_2}{2} - T_0 & \Delta T_{42} &:= \frac{T_4 + T_2}{2} - T_0 & \Delta T_{43} &:= \frac{T_4 + T_3}{2} - T_0 \end{aligned}$$

To find the CTE in layer one:

$$\begin{aligned} s_1 &:= h_1 \cdot E_1 & s_2 &:= h_2 \cdot E_2 & s_3 &:= h_3 \cdot E_3 & s_4 &:= h_4 \cdot E_4 \\ I_1 &:= \text{unit} \cdot \frac{h_1^3}{12} & I_2 &:= \text{unit} \cdot \frac{h_2^3}{12} & I_3 &:= \text{unit} \cdot \frac{h_3^3}{12} & I_4 &:= \text{unit} \cdot \frac{h_4^3}{12} \\ h_{12} &:= h_1 + h_2 & h_{13} &:= h_1 + 2 \cdot h_2 + h_3 & h_{14} &:= h_1 + 2 \cdot h_2 + 2 \cdot h_3 + h_4 \\ h_{23} &:= h_2 + h_3 & h_{24} &:= h_2 + 2 \cdot h_3 + h_4 & h_{34} &:= h_3 + h_4 \\ P_2 &:= \alpha_2 \cdot [\Delta T_{21} \cdot s_1 \cdot s_2 \cdot (h_{12}) - \Delta T_{32} \cdot s_2 \cdot s_3 \cdot (h_{23}) - \Delta T_{42} \cdot s_2 \cdot s_4 \cdot (h_{24})] \\ P_3 &:= \alpha_3 \cdot [\Delta T_{31} \cdot s_1 \cdot s_3 \cdot (h_{13}) + \Delta T_{32} \cdot s_2 \cdot s_3 \cdot (h_{23}) - \Delta T_{43} \cdot s_3 \cdot s_4 \cdot (h_{34})] \\ P_4 &:= \alpha_4 \cdot [\Delta T_{41} \cdot s_1 \cdot s_4 \cdot (h_{14}) + \Delta T_{42} \cdot s_2 \cdot s_4 \cdot (h_{24}) + \Delta T_{43} \cdot s_3 \cdot s_4 \cdot (h_{34})] \end{aligned}$$

CTE of layer one:

$$\alpha_1 := \frac{-(P_2 + P_3 + P_4)}{-\Delta T_{21} \cdot s_1 \cdot s_2 \cdot (h_{12}) - \Delta T_{31} \cdot s_1 \cdot s_3 \cdot (h_{13}) - \Delta T_{41} \cdot s_1 \cdot s_4 \cdot (h_{14})}$$

$$\alpha_1 = \begin{pmatrix} 52.132 & 53.059 & 51.735 & 52.662 & 53.853 & 54.779 & 53.456 & 54.382 \\ 52.191 & 52.912 & 51.882 & 52.603 & 53.912 & 54.632 & 53.603 & 54.324 \\ 52.338 & 52.853 & 52.029 & 52.544 & 54.059 & 54.574 & 53.75 & 54.265 \\ 52.485 & 52.794 & 52.176 & 52.485 & 54.206 & 54.515 & 53.897 & 54.206 \\ 52.632 & 52.735 & 52.324 & 52.426 & 54.353 & 54.456 & 54.044 & 54.147 \\ 52.25 & 52.147 & 52.471 & 52.368 & 53.971 & 53.868 & 54.191 & 54.088 \\ 52.397 & 52.088 & 52.618 & 52.309 & 54.118 & 53.809 & 54.338 & 54.029 \\ 52.544 & 52.029 & 52.765 & 52.25 & 54.265 & 53.75 & 54.485 & 53.971 \\ 52.691 & 51.971 & 52.912 & 52.191 & 54.412 & 53.691 & 54.632 & 53.912 \\ 52.75 & 51.818 & 53.059 & 52.132 & 54.471 & 53.544 & 54.779 & 53.853 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$$

Note: None of the scenarios produce CTE values for layer 1 that are in the range of the 0 wt% Zirconium Tungstate for the temperature of 100C as desired, but a material with 0-30 wt% Zirconium Tungstate might work.

Case 5 AFRL Approximating Geometry to Predict Zero Curvature of Four Layer Trough Scenarios for Individual Temperatures per each Layer

These scenarios explore the range of dependent CTE and wt% of Zirconium Tungstate that is required to keep the system at 0 thermal curvature by defining 3 of the 4 CTE's.

Material properties of the trough layers

48 wt% approx. $E_3 := 7.250 \cdot 10^9 \text{ Pa}$	30 wt% approx. $E_2 := 6.00 \cdot 10^9 \text{ Pa}$	Neat CTD-403 $E_1 := 4.727 \cdot 10^9 \text{ Pa} \quad E_4 := 4.727 \cdot 10^9 \text{ Pa}$
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$\alpha_4 :=$

$$\begin{pmatrix} 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \\ 52 & 52 & 52 & 52 & 54.25 & 54.25 & 54.25 & 54.25 \end{pmatrix}$$

$10^{-6} \cdot \frac{1}{\text{C}}$

$\alpha_3 :=$

$$\begin{pmatrix} 32.5 & 35 & 32.5 & 35 & 32.5 & 35 & 32.5 & 35 \\ 32.75 & 34.5 & 32.75 & 34.5 & 32.75 & 34.5 & 32.75 & 34.5 \\ 33 & 34.25 & 33 & 34.25 & 33 & 34.25 & 33 & 34.25 \\ 33.25 & 34 & 33.25 & 34 & 33.25 & 34 & 33.25 & 34 \\ 33.5 & 33.75 & 33.5 & 33.75 & 33.5 & 33.75 & 33.5 & 33.75 \\ 33.75 & 33.5 & 33.75 & 33.5 & 33.75 & 33.5 & 33.75 & 33.5 \\ 34 & 33.25 & 34 & 33.25 & 34 & 33.25 & 34 & 33.25 \\ 34.25 & 33 & 34.25 & 33 & 34.25 & 33 & 34.25 & 33 \\ 34.5 & 32.75 & 34.5 & 32.75 & 34.5 & 32.75 & 34.5 & 32.75 \\ 34.75 & 32.5 & 34.75 & 32.5 & 34.75 & 32.5 & 34.75 & 32.5 \end{pmatrix}$$

$10^{-6} \cdot \frac{1}{\text{C}}$

$\alpha_2 :=$

$$\begin{pmatrix} 42 & 42 & 43.5 & 43.5 & 42 & 42 & 43.5 & 43.5 \\ 42.125 & 42.125 & 43.25 & 43.25 & 42.125 & 42.125 & 43.25 & 43.25 \\ 42.25 & 42.25 & 43 & 43 & 42.25 & 42.25 & 43 & 43 \\ 42.375 & 42.375 & 42.75 & 42.75 & 42.375 & 42.375 & 42.75 & 42.75 \\ 42.5 & 42.5 & 42.625 & 42.625 & 42.5 & 42.5 & 42.625 & 42.625 \\ 42.625 & 42.625 & 42.5 & 42.5 & 42.625 & 42.625 & 42.5 & 42.5 \\ 42.75 & 42.75 & 42.375 & 42.375 & 42.75 & 42.75 & 42.375 & 42.375 \\ 43 & 43 & 42.25 & 42.25 & 43 & 43 & 42.25 & 42.25 \\ 43.25 & 43.25 & 42.125 & 42.125 & 43.25 & 43.25 & 42.125 & 42.125 \\ 43.5 & 43.5 & 42 & 42 & 43.5 & 43.5 & 42 & 42 \end{pmatrix}$$

$10^{-6} \cdot \frac{1}{\text{C}}$

Material geometries of the trough layers

layer thickness: $h_1 := 0.023\text{m}$ $h_2 := 0.023\text{m}$ $h_3 := 0.023\text{m}$ $h_4 := 0.023\text{m}$

Unit depth: $\text{unit} := 1$

Temperature Changes of each layer:

$$\begin{aligned} T_0 &:= 0\text{C} & T_1 &:= 100\text{C} & T_2 &:= 75\text{C} & T_3 &:= 50\text{C} & T_4 &:= 25\text{C} \\ \Delta T_{21} &:= \frac{T_2 + T_1}{2} - T_0 & \Delta T_{31} &:= \frac{T_3 + T_1}{2} - T_0 & \Delta T_{41} &:= \frac{T_4 + T_1}{2} - T_0 \\ \Delta T_{32} &:= \frac{T_3 + T_2}{2} - T_0 & \Delta T_{42} &:= \frac{T_4 + T_2}{2} - T_0 & \Delta T_{43} &:= \frac{T_4 + T_3}{2} - T_0 \end{aligned}$$

To find the CTE in layer one:

$$\begin{aligned} s_1 &:= h_1 \cdot E_1 & s_2 &:= h_2 \cdot E_2 & s_3 &:= h_3 \cdot E_3 & s_4 &:= h_4 \cdot E_4 \\ I_1 &:= \text{unit} \cdot \frac{h_1^3}{12} & I_2 &:= \text{unit} \cdot \frac{h_2^3}{12} & I_3 &:= \text{unit} \cdot \frac{h_3^3}{12} & I_4 &:= \text{unit} \cdot \frac{h_4^3}{12} \\ h_{12} &:= h_1 + h_2 & h_{13} &:= h_1 + 2 \cdot h_2 + h_3 & h_{14} &:= h_1 + 2 \cdot h_2 + 2 \cdot h_3 + h_4 \\ h_{23} &:= h_2 + h_3 & h_{24} &:= h_2 + 2 \cdot h_3 + h_4 & h_{34} &:= h_3 + h_4 \\ P_2 &:= \alpha_2 \cdot [\Delta T_{21} \cdot s_1 \cdot s_2 \cdot (h_{12}) - \Delta T_{32} \cdot s_2 \cdot s_3 \cdot (h_{23}) - \Delta T_{42} \cdot s_2 \cdot s_4 \cdot (h_{24})] \\ P_3 &:= \alpha_3 \cdot [\Delta T_{31} \cdot s_1 \cdot s_3 \cdot (h_{13}) + \Delta T_{32} \cdot s_2 \cdot s_3 \cdot (h_{23}) - \Delta T_{43} \cdot s_3 \cdot s_4 \cdot (h_{34})] \\ P_4 &:= \alpha_4 \cdot [\Delta T_{41} \cdot s_1 \cdot s_4 \cdot (h_{14}) + \Delta T_{42} \cdot s_2 \cdot s_4 \cdot (h_{24}) + \Delta T_{43} \cdot s_3 \cdot s_4 \cdot (h_{34})] \end{aligned}$$

CTE of layer one:

$$\alpha_1 := \frac{-(P_2 + P_3 + P_4)}{-\Delta T_{21} \cdot s_1 \cdot s_2 \cdot (h_{12}) - \Delta T_{31} \cdot s_1 \cdot s_3 \cdot (h_{13}) - \Delta T_{41} \cdot s_1 \cdot s_4 \cdot (h_{14})}$$

$$\alpha_1 = \begin{pmatrix} 43.749 & 45.14 & 43.358 & 44.75 & 45.332 & 46.723 & 44.941 & 46.333 \\ 43.855 & 44.829 & 43.563 & 44.537 & 45.438 & 46.412 & 45.146 & 46.12 \\ 43.962 & 44.658 & 43.767 & 44.462 & 45.545 & 46.241 & 45.35 & 46.046 \\ 44.068 & 44.486 & 43.971 & 44.388 & 45.652 & 46.069 & 45.554 & 45.971 \\ 44.175 & 44.314 & 44.143 & 44.282 & 45.758 & 45.897 & 45.726 & 45.865 \\ 44.282 & 44.143 & 44.314 & 44.175 & 45.865 & 45.726 & 45.897 & 45.758 \\ 44.388 & 43.971 & 44.486 & 44.068 & 45.971 & 45.554 & 46.069 & 45.652 \\ 44.462 & 43.767 & 44.658 & 43.962 & 46.046 & 45.35 & 46.241 & 45.545 \\ 44.537 & 43.563 & 44.829 & 43.855 & 46.12 & 45.146 & 46.412 & 45.438 \\ 44.611 & 43.358 & 45.001 & 43.749 & 46.194 & 44.941 & 46.584 & 45.332 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$$

Note: None of the scenarios produce CTE values for layer 1 close the range of the neat 0 wt% for the temperature of 100C as desired.

Case 6 AFRL Approximating Geometry to Predict Zero Curvature of Four Layer Trough Scenarios for Individual Temperatures per each Layer

These scenarios explore the range of dependent CTE and wt% of Zirconium Tungstate that is required to keep the system at 0 thermal curvature by defining 3 of the 4 CTE's.

Material properties of the trough layers

	48 wt% approx.	30 wt% approx.	Neat CTD-403
	$E_3 := 7.250 \cdot 10^9 \text{ Pa}$	$E_2 := 6.00 \cdot 10^9 \text{ Pa}$	$E_1 := 6.00 \cdot 10^9 \text{ Pa}$
			$E_4 := 4.727 \cdot 10^9 \text{ Pa}$

$\alpha_4 :=$

{

52	52	52	52	54.25	54.25	54.25	54.25
52	52	52	52	54.25	54.25	54.25	54.25
52	52	52	52	54.25	54.25	54.25	54.25
52	52	52	52	54.25	54.25	54.25	54.25
52	52	52	52	54.25	54.25	54.25	54.25
52	52	52	52	54.25	54.25	54.25	54.25
52	52	52	52	54.25	54.25	54.25	54.25
52	52	52	52	54.25	54.25	54.25	54.25
52	52	52	52	54.25	54.25	54.25	54.25
52	52	52	52	54.25	54.25	54.25	54.25

}

$10^{-6} \cdot \frac{1}{^\circ\text{C}}$

$\alpha_3 :=$

{

32.5	35	32.5	35	32.5	35	32.5	35
32.75	34.5	32.75	34.5	32.75	34.5	32.75	34.5
33	34.25	33	34.25	33	34.25	33	34.25
33.25	34	33.25	34	33.25	34	33.25	34
33.5	33.75	33.5	33.75	33.5	33.75	33.5	33.75
33.75	33.5	33.75	33.5	33.75	33.5	33.75	33.5
34	33.25	34	33.25	34	33.25	34	33.25
34.25	33	34.25	33	34.25	33	34.25	33
34.5	32.75	34.5	32.75	34.5	32.75	34.5	32.75
34.75	32.5	34.75	32.5	34.75	32.5	34.75	32.5

}

$10^{-6} \cdot \frac{1}{^\circ\text{C}}$

$\alpha_2 :=$

{

42	42	43.5	43.5	42	42	43.5	43.5
42.125	42.125	43.25	43.25	42.125	42.125	43.25	43.25
42.25	42.25	43	43	42.25	42.25	43	43
42.375	42.375	42.75	42.75	42.375	42.375	42.75	42.75
42.5	42.5	42.625	42.625	42.5	42.5	42.625	42.625
42.625	42.625	42.5	42.5	42.625	42.625	42.5	42.5
42.75	42.75	42.375	42.375	42.75	42.75	42.375	42.375
43	43	42.25	42.25	43	43	42.25	42.25
43.25	43.25	42.125	42.125	43.25	43.25	42.125	42.125
43.5	43.5	42	42	43.5	43.5	42	42

}

$10^{-6} \cdot \frac{1}{^\circ\text{C}}$

Material geometries of the trough layers

layer thickness: $h_1 := 0.023\text{m}$ $h_2 := 0.023\text{m}$ $h_3 := 0.023\text{m}$ $h_4 := 0.023\text{m}$

Unit depth: $\text{unit} := 1$

Temperature Changes of each layer:

$$\begin{aligned} T_0 &:= 0\text{C} & T_1 &:= 100\text{C} & T_2 &:= 75\text{C} & T_3 &:= 50\text{C} & T_4 &:= 25\text{C} \\ \Delta T_{21} &:= \frac{T_2 + T_1}{2} - T_0 & \Delta T_{31} &:= \frac{T_3 + T_1}{2} - T_0 & \Delta T_{41} &:= \frac{T_4 + T_1}{2} - T_0 \\ \Delta T_{32} &:= \frac{T_3 + T_2}{2} - T_0 & \Delta T_{42} &:= \frac{T_4 + T_2}{2} - T_0 & \Delta T_{43} &:= \frac{T_4 + T_3}{2} - T_0 \end{aligned}$$

To find the CTE in layer one:

$$\begin{aligned} s_1 &:= h_1 \cdot E_1 & s_2 &:= h_2 \cdot E_2 & s_3 &:= h_3 \cdot E_3 & s_4 &:= h_4 \cdot E_4 \\ I_1 &:= \text{unit} \cdot \frac{h_1^3}{12} & I_2 &:= \text{unit} \cdot \frac{h_2^3}{12} & I_3 &:= \text{unit} \cdot \frac{h_3^3}{12} & I_4 &:= \text{unit} \cdot \frac{h_4^3}{12} \\ h_{12} &:= h_1 + h_2 & h_{13} &:= h_1 + 2 \cdot h_2 + h_3 & h_{14} &:= h_1 + 2 \cdot h_2 + 2 \cdot h_3 + h_4 \\ h_{23} &:= h_2 + h_3 & h_{24} &:= h_2 + 2 \cdot h_3 + h_4 & h_{34} &:= h_3 + h_4 \\ P_2 &:= \alpha_2 \cdot [\Delta T_{21} \cdot s_1 \cdot s_2 \cdot (h_{12}) - \Delta T_{32} \cdot s_2 \cdot s_3 \cdot (h_{23}) - \Delta T_{42} \cdot s_2 \cdot s_4 \cdot (h_{24})] \\ P_3 &:= \alpha_3 \cdot [\Delta T_{31} \cdot s_1 \cdot s_3 \cdot (h_{13}) + \Delta T_{32} \cdot s_2 \cdot s_3 \cdot (h_{23}) - \Delta T_{43} \cdot s_3 \cdot s_4 \cdot (h_{34})] \\ P_4 &:= \alpha_4 \cdot [\Delta T_{41} \cdot s_1 \cdot s_4 \cdot (h_{14}) + \Delta T_{42} \cdot s_2 \cdot s_4 \cdot (h_{24}) + \Delta T_{43} \cdot s_3 \cdot s_4 \cdot (h_{34})] \end{aligned}$$

CTE of layer one:

$$\alpha_1 := \frac{-(P_2 + P_3 + P_4)}{-\Delta T_{21} \cdot s_1 \cdot s_2 \cdot (h_{12}) - \Delta T_{31} \cdot s_1 \cdot s_3 \cdot (h_{13}) - \Delta T_{41} \cdot s_1 \cdot s_4 \cdot (h_{14})}$$

$$\alpha_1 = \begin{pmatrix} 43.253 & 44.58 & 43.012 & 44.339 & 44.67 & 45.997 & 44.429 & 45.756 \\ 43.366 & 44.295 & 43.185 & 44.114 & 44.782 & 45.711 & 44.602 & 45.531 \\ 43.478 & 44.142 & 43.358 & 44.022 & 44.895 & 45.558 & 44.775 & 45.438 \\ 43.591 & 43.989 & 43.531 & 43.929 & 45.007 & 45.406 & 44.947 & 45.345 \\ 43.704 & 43.836 & 43.684 & 43.816 & 45.12 & 45.253 & 45.1 & 45.233 \\ 43.816 & 43.684 & 43.836 & 43.704 & 45.233 & 45.1 & 45.253 & 45.12 \\ 43.929 & 43.531 & 43.989 & 43.591 & 45.345 & 44.947 & 45.406 & 45.007 \\ 44.022 & 43.358 & 44.142 & 43.478 & 45.438 & 44.775 & 45.558 & 44.895 \\ 44.114 & 43.185 & 44.295 & 43.366 & 45.531 & 44.602 & 45.711 & 44.782 \\ 44.207 & 43.012 & 44.447 & 43.253 & 45.623 & 44.429 & 45.864 & 44.67 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$$

Note: Most of the scenarios produce CTE values for layer 1 in the range of the 30 wt% for the temperature of 100C as desired.

**RMS Surface Distortion and Radius of Curvature Comparison for
Zirconium Tungstate Thermostat Application in Trough Configuration
FEA vs. Zero Curvature Closed Form Equation Results for Case 6**

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N ((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})))^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.154E-06
RMS of best fit of FEA deformed	3.118E-06
Percent Difference between FEA and Closed Form	1.153

Radius of Curvature Results:

FEA Radius of Curvature	1.058
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	5.517

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4408E-15	-0.001441
2	0.00143	0.00144	-1.00000	-1.00000	-1.2967E-14	-0.001441
3	0.00287	0.00288	-1.00000	-1.00000	-3.6020E-14	-0.001441
4	0.00430	0.00432	-0.99999	-1.00000	-7.0599E-14	-0.001441
5	0.00573	0.00576	-0.99998	-1.00000	-1.4408E-13	-0.001441
6	0.00717	0.00720	-0.99997	-1.00000	-1.7433E-13	-0.001441
7	0.00860	0.00864	-0.99996	-1.00000	-2.4349E-13	-0.001441

Performed for all surface nodes and RMS is calculated according to the above equation

**RMS Surface Distortion and Radius of Curvature Comparison for
Zirconium Tungstate AFRL Axisymmetric Spherical Mirror Geometry
FEA vs. Zero Curvature Closed Form Equation Results for Case 6**

Thermal Loading Condition:

Decreasing Temperature Gradient

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

AFRL Axisymmetric FEA RMS Surface Distortion	4.196E-06
MLT FEA RMS Surface Distortion	3.118E-06
Percent Difference between FEA and Closed Form	25.699

Radius of Curvature Results:

FEA Radius of Curvature	1.051
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	4.875

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4380E-15	-0.001438
2	0.00143	0.00144	-1.00000	-1.00000	-1.2942E-14	-0.001438
3	0.00287	0.00288	-1.00000	-1.00000	-3.5950E-14	-0.001438
4	0.00430	0.00431	-0.99999	-1.00000	-7.0462E-14	-0.001438
5	0.00573	0.00575	-0.99998	-1.00000	-1.4380E-13	-0.001438
6	0.00717	0.00719	-0.99997	-1.00000	-1.7399E-13	-0.001438
7	0.00860	0.00863	-0.99996	-1.00000	-2.4301E-13	-0.001438

Performed for all surface nodes and RMS is calculated according to the above equation

Case 7 AFRL Approximating Geometry to Predict Zero Curvature of Four Layer Trough Scenarios for Individual Temperatures per each Layer

These scenarios explore the range of dependent CTE and wt% of Zirconium Tungstate that is required to keep the system at 0 thermal curvature by defining 3 of the 4 CTE's.

Material properties of the trough layers

70 wt%

Neat CTD-403

$$E_3 := 8.574 \cdot 10^9 \text{ Pa}$$

$$E_2 := 4.727 \cdot 10^9 \text{ Pa}$$

$$E_1 := 4.727 \cdot 10^9 \text{ Pa}$$

$$E_4 := 4.727 \cdot 10^9 \text{ Pa}$$

$$\alpha_4 := \begin{pmatrix} 58.75 & 58.75 & 58.75 & 58.75 & 61 & 61 & 61 & 61 \\ 58.75 & 58.75 & 58.75 & 58.75 & 61 & 61 & 61 & 61 \\ 58.75 & 58.75 & 58.75 & 58.75 & 61 & 61 & 61 & 61 \\ 58.75 & 58.75 & 58.75 & 58.75 & 61 & 61 & 61 & 61 \\ 58.75 & 58.75 & 58.75 & 58.75 & 61 & 61 & 61 & 61 \\ 58.75 & 58.75 & 58.75 & 58.75 & 61 & 61 & 61 & 61 \\ 58.75 & 58.75 & 58.75 & 58.75 & 61 & 61 & 61 & 61 \\ 58.75 & 58.75 & 58.75 & 58.75 & 61 & 61 & 61 & 61 \\ 58.75 & 58.75 & 58.75 & 58.75 & 61 & 61 & 61 & 61 \\ 58.75 & 58.75 & 58.75 & 58.75 & 61 & 61 & 61 & 61 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$$

$$\alpha_3 := \begin{pmatrix} 21 & 21 & 22 & 22 & 21 & 21 & 22 & 22 \\ 21.1 & 21.1 & 21.9 & 21.9 & 21.1 & 21.1 & 21.9 & 21.9 \\ 21.2 & 21.2 & 21.8 & 21.8 & 21.2 & 21.2 & 21.8 & 21.8 \\ 21.3 & 21.3 & 21.7 & 21.7 & 21.3 & 21.3 & 21.7 & 21.7 \\ 21.4 & 21.4 & 21.6 & 21.6 & 21.4 & 21.4 & 21.6 & 21.6 \\ 21.5 & 21.5 & 21.5 & 21.5 & 21.5 & 21.5 & 21.5 & 21.5 \\ 21.6 & 21.6 & 21.4 & 21.4 & 21.6 & 21.6 & 21.4 & 21.4 \\ 21.7 & 21.7 & 21.3 & 21.3 & 21.7 & 21.7 & 21.3 & 21.3 \\ 21.8 & 21.8 & 21.2 & 21.2 & 21.8 & 21.8 & 21.2 & 21.2 \\ 21.9 & 21.9 & 21.1 & 21.1 & 21.9 & 21.9 & 21.1 & 21.1 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$$

$$\alpha_2 := \begin{pmatrix} 54.25 & 56.5 & 54.25 & 56.5 & 54.25 & 56.5 & 54.25 & 56.5 \\ 54.5 & 56.25 & 54.5 & 56.25 & 54.5 & 56.25 & 54.5 & 56.25 \\ 54.75 & 56 & 54.75 & 56 & 54.75 & 56 & 54.75 & 56 \\ 55 & 55.75 & 55 & 55.75 & 55 & 55.75 & 55 & 55.75 \\ 55.25 & 55.5 & 55.25 & 55.5 & 55.25 & 55.5 & 55.25 & 55.5 \\ 55.5 & 55.25 & 55.5 & 55.25 & 55.5 & 55.25 & 55.5 & 55.25 \\ 55.75 & 55 & 55.75 & 55 & 55.75 & 55 & 55.75 & 55 \\ 56 & 54.75 & 56 & 54.75 & 56 & 54.75 & 56 & 54.75 \\ 56.25 & 54.5 & 56.25 & 54.5 & 56.25 & 54.5 & 56.25 & 54.5 \\ 56.5 & 54.25 & 56.5 & 54.25 & 56.5 & 54.25 & 56.5 & 54.25 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$$

Material geometries of the trough layers

layer thickness: $h_1 := 0.023\text{m}$ $h_2 := 0.023\text{m}$ $h_3 := 0.023\text{m}$ $h_4 := 0.023\text{m}$

Unit depth: $\text{unit} := 1$

Temperature Changes of each layer:

$$\begin{aligned} T_0 &:= 0\text{C} & T_1 &:= 25\text{C} & T_2 &:= 50\text{C} & T_3 &:= 75\text{C} & T_4 &:= 100\text{C} \\ \Delta T_{21} &:= \frac{T_2 + T_1}{2} - T_0 & \Delta T_{31} &:= \frac{T_3 + T_1}{2} - T_0 & \Delta T_{41} &:= \frac{T_4 + T_1}{2} - T_0 \\ \Delta T_{32} &:= \frac{T_3 + T_2}{2} - T_0 & \Delta T_{42} &:= \frac{T_4 + T_2}{2} - T_0 & \Delta T_{43} &:= \frac{T_4 + T_3}{2} - T_0 \end{aligned}$$

To find the CTE in layer one:

$$\begin{aligned} s_1 &:= h_1 \cdot E_1 & s_2 &:= h_2 \cdot E_2 & s_3 &:= h_3 \cdot E_3 & s_4 &:= h_4 \cdot E_4 \\ I_1 &:= \text{unit} \cdot \frac{h_1^3}{12} & I_2 &:= \text{unit} \cdot \frac{h_2^3}{12} & I_3 &:= \text{unit} \cdot \frac{h_3^3}{12} & I_4 &:= \text{unit} \cdot \frac{h_4^3}{12} \\ h_{12} &:= h_1 + h_2 & h_{13} &:= h_1 + 2 \cdot h_2 + h_3 & h_{14} &:= h_1 + 2 \cdot h_2 + 2 \cdot h_3 + h_4 \\ h_{23} &:= h_2 + h_3 & h_{24} &:= h_2 + 2 \cdot h_3 + h_4 & h_{34} &:= h_3 + h_4 \\ P_2 &:= \alpha_2 \cdot [\Delta T_{21} \cdot s_1 \cdot s_2 \cdot (h_{12}) - \Delta T_{32} \cdot s_2 \cdot s_3 \cdot (h_{23}) - \Delta T_{42} \cdot s_2 \cdot s_4 \cdot (h_{24})] \\ P_3 &:= \alpha_3 \cdot [\Delta T_{31} \cdot s_1 \cdot s_3 \cdot (h_{13}) + \Delta T_{32} \cdot s_2 \cdot s_3 \cdot (h_{23}) - \Delta T_{43} \cdot s_3 \cdot s_4 \cdot (h_{34})] \\ P_4 &:= \alpha_4 \cdot [\Delta T_{41} \cdot s_1 \cdot s_4 \cdot (h_{14}) + \Delta T_{42} \cdot s_2 \cdot s_4 \cdot (h_{24}) + \Delta T_{43} \cdot s_3 \cdot s_4 \cdot (h_{34})] \end{aligned}$$

CTE of layer one:

$$\alpha_1 := \frac{-(P_2 + P_3 + P_4)}{-\Delta T_{21} \cdot s_1 \cdot s_2 \cdot (h_{12}) - \Delta T_{31} \cdot s_1 \cdot s_3 \cdot (h_{13}) - \Delta T_{41} \cdot s_1 \cdot s_4 \cdot (h_{14})}$$

$$\alpha_1 = \begin{pmatrix} 48.614 & 47.364 & 48.949 & 47.698 & 51.362 & 50.111 & 51.696 & 50.446 \\ 48.509 & 47.536 & 48.777 & 47.804 & 51.256 & 50.283 & 51.524 & 50.551 \\ 48.403 & 47.708 & 48.604 & 47.909 & 51.151 & 50.456 & 51.351 & 50.657 \\ 48.298 & 47.881 & 48.432 & 48.015 & 51.045 & 50.628 & 51.179 & 50.762 \\ 48.192 & 48.053 & 48.259 & 48.12 & 50.94 & 50.801 & 51.007 & 50.868 \\ 48.087 & 48.226 & 48.087 & 48.226 & 50.834 & 50.973 & 50.834 & 50.973 \\ 47.981 & 48.398 & 47.914 & 48.331 & 50.729 & 51.146 & 50.662 & 51.079 \\ 47.876 & 48.571 & 47.742 & 48.437 & 50.623 & 51.318 & 50.489 & 51.184 \\ 47.77 & 48.743 & 47.57 & 48.542 & 50.518 & 51.49 & 50.317 & 51.29 \\ 47.665 & 48.915 & 47.397 & 48.648 & 50.412 & 51.663 & 50.144 & 51.395 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$$

Note: A few of the scenarios produce CTE values for layer 1 that are in the range of the neat 0 wt% for the temperature of 100C as desired.

**RMS Surface Distortion and Radius of Curvature Comparison for
Zirconium Tungstate Thermostat Application in Trough Configuration
FEA vs. Zero Curvature Closed Form Equation Results for Case 7**

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.293E-06
RMS of best fit of FEA deformed	3.154E-06
Percent Difference between FEA and Closed Form	4.203

Radius of Curvature Results:

FEA Radius of Curvature	0.948
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	-5.474

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4335E-15	-0.001434
2	0.00143	0.00143	-1.00000	-1.00000	-1.2902E-14	-0.001434
3	0.00287	0.00287	-1.00000	-1.00000	-3.5838E-14	-0.001434
4	0.00430	0.00430	-0.99999	-1.00000	-1.2398E-11	-0.001434
5	0.00573	0.00573	-0.99998	-0.99990	-1.4335E-13	-0.001434
6	0.00717	0.00717	-0.99997	-0.99990	-1.7345E-13	-0.001433
7	0.00860	0.00860	-0.99996	-0.99990	-2.4225E-13	-0.001433

Performed for all surface nodes and RMS is calculated according to the above equation

**RMS Surface Distortion and Radius of Curvature Comparison for
Zirconium Tungstate AFRL Axisymmetric Spherical Mirror Geometry
FEA vs. Zero Curvature Closed Form Equation Results for Case 7**

Thermal Loading Condition:

Increasing Temperature Gradient

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

AFRL Axisymmetric FEA RMS Surface Distortion	4.174E-06
MLT FEA RMS Surface Distortion	3.154E-06
Percent Difference between FEA and Closed Form	24.444

Radius of Curvature Results:

FEA Radius of Curvature	0.969
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	-3.213

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4380E-15	-0.001438
2	0.00143	0.00144	-1.00000	-1.00000	-1.2942E-14	-0.001438
3	0.00287	0.00288	-1.00000	-1.00000	-3.5950E-14	-0.001438
4	0.00430	0.00431	-0.99999	-1.00000	-7.0462E-14	-0.001438
5	0.00573	0.00575	-0.99998	-1.00000	-1.4380E-13	-0.001438
6	0.00717	0.00719	-0.99997	-1.00000	-1.7399E-13	-0.001438
7	0.00860	0.00863	-0.99996	-1.00000	-2.4301E-13	-0.001438

Performed for all surface nodes and RMS is calculated according to the above equation

Case 8 AFRL Approximating Geometry to Predict Zero Curvature of Four Layer Trough Scenarios for Individual Temperatures per each Layer

These scenarios explore the range of dependent CTE and wt% of Zirconium Tungstate that is required to keep the system at 0 thermal curvature by defining 3 of the 4 CTE's.

Material properties of the trough layers

	70 wt%	Neat CTD-403
	$E_3 := 8.574 \cdot 10^9 \text{ Pa}$	$E_2 := 4.727 \cdot 10^9 \text{ Pa}$
	$E_1 := 4.727 \cdot 10^9 \text{ Pa}$	$E_4 := 4.727 \cdot 10^9 \text{ Pa}$

$\alpha_4 :=$

58.75	58.75	58.75	58.75	61	61	61	61
58.75	58.75	58.75	58.75	61	61	61	61
58.75	58.75	58.75	58.75	61	61	61	61
58.75	58.75	58.75	58.75	61	61	61	61
58.75	58.75	58.75	58.75	61	61	61	61
58.75	58.75	58.75	58.75	61	61	61	61
58.75	58.75	58.75	58.75	61	61	61	61
58.75	58.75	58.75	58.75	61	61	61	61
58.75	58.75	58.75	58.75	61	61	61	61

$10^{-6} \cdot \frac{1}{\text{C}}$

$\alpha_3 :=$

21	21	22	22	21	21	22	22
21.1	21.1	21.9	21.9	21.1	21.1	21.9	21.9
21.2	21.2	21.8	21.8	21.2	21.2	21.8	21.8
21.3	21.3	21.7	21.7	21.3	21.3	21.7	21.7
21.4	21.4	21.6	21.6	21.4	21.4	21.6	21.6
21.5	21.5	21.5	21.5	21.5	21.5	21.5	21.5
21.6	21.6	21.4	21.4	21.6	21.6	21.4	21.4
21.7	21.7	21.3	21.3	21.7	21.7	21.3	21.3
21.8	21.8	21.2	21.2	21.8	21.8	21.2	21.2
21.9	21.9	21.1	21.1	21.9	21.9	21.1	21.1

$10^{-6} \cdot \frac{1}{\text{C}}$

$\alpha_2 :=$

54.25	56.5	54.25	56.5	54.25	56.5	54.25	56.5
54.5	56.25	54.5	56.25	54.5	56.25	54.5	56.25
54.75	56	54.75	56	54.75	56	54.75	56
55	55.75	55	55.75	55	55.75	55	55.75
55.25	55.5	55.25	55.5	55.25	55.5	55.25	55.5
55.5	55.25	55.5	55.25	55.5	55.25	55.5	55.25
55.75	55	55.75	55	55.75	55	55.75	55
56	54.75	56	54.75	56	54.75	56	54.75
56.25	54.5	56.25	54.5	56.25	54.5	56.25	54.5
56.5	54.25	56.5	54.25	56.5	54.25	56.5	54.25

$10^{-6} \cdot \frac{1}{\text{C}}$

Material geometries of the trough layers

layer thickness: $h_1 := 0.023\text{m}$ $h_2 := 0.023\text{m}$ $h_3 := 0.023\text{m}$ $h_4 := 0.023\text{m}$

Unit depth: $\text{unit} := 1$

Temperature Changes of each layer:

$$\begin{aligned} T_0 &:= 0\text{C} & T_1 &:= 25\text{C} & T_2 &:= 50\text{C} & T_3 &:= 75\text{C} & T_4 &:= 100\text{C} \\ \Delta T_{21} &:= \frac{T_2 + T_1}{2} - T_0 & \Delta T_{31} &:= \frac{T_3 + T_1}{2} - T_0 & \Delta T_{41} &:= \frac{T_4 + T_1}{2} - T_0 \\ \Delta T_{32} &:= \frac{T_3 + T_2}{2} - T_0 & \Delta T_{42} &:= \frac{T_4 + T_2}{2} - T_0 & \Delta T_{43} &:= \frac{T_4 + T_3}{2} - T_0 \end{aligned}$$

To find the CTE in layer one:

$$\begin{aligned} s_1 &:= h_1 \cdot E_1 & s_2 &:= h_2 \cdot E_2 & s_3 &:= h_3 \cdot E_3 & s_4 &:= h_4 \cdot E_4 \\ I_1 &:= \text{unit} \cdot \frac{h_1^3}{12} & I_2 &:= \text{unit} \cdot \frac{h_2^3}{12} & I_3 &:= \text{unit} \cdot \frac{h_3^3}{12} & I_4 &:= \text{unit} \cdot \frac{h_4^3}{12} \\ h_{12} &:= h_1 + h_2 & h_{13} &:= h_1 + 2 \cdot h_2 + h_3 & h_{14} &:= h_1 + 2 \cdot h_2 + 2 \cdot h_3 + h_4 \\ h_{23} &:= h_2 + h_3 & h_{24} &:= h_2 + 2 \cdot h_3 + h_4 & h_{34} &:= h_3 + h_4 \\ P_2 &:= \alpha_2 \cdot [\Delta T_{21} \cdot s_1 \cdot s_2 \cdot (h_{12}) - \Delta T_{32} \cdot s_2 \cdot s_3 \cdot (h_{23}) - \Delta T_{42} \cdot s_2 \cdot s_4 \cdot (h_{24})] \\ P_3 &:= \alpha_3 \cdot [\Delta T_{31} \cdot s_1 \cdot s_3 \cdot (h_{13}) + \Delta T_{32} \cdot s_2 \cdot s_3 \cdot (h_{23}) - \Delta T_{43} \cdot s_3 \cdot s_4 \cdot (h_{34})] \\ P_4 &:= \alpha_4 \cdot [\Delta T_{41} \cdot s_1 \cdot s_4 \cdot (h_{14}) + \Delta T_{42} \cdot s_2 \cdot s_4 \cdot (h_{24}) + \Delta T_{43} \cdot s_3 \cdot s_4 \cdot (h_{34})] \end{aligned}$$

CTE of layer one:

$$\alpha_1 := \frac{-(P_2 + P_3 + P_4)}{-\Delta T_{21} \cdot s_1 \cdot s_2 \cdot (h_{12}) - \Delta T_{31} \cdot s_1 \cdot s_3 \cdot (h_{13}) - \Delta T_{41} \cdot s_1 \cdot s_4 \cdot (h_{14})}$$

$$\alpha_1 = \begin{pmatrix} 48.614 & 47.364 & 48.949 & 47.698 & 51.362 & 50.111 & 51.696 & 50.446 \\ 48.509 & 47.536 & 48.777 & 47.804 & 51.256 & 50.283 & 51.524 & 50.551 \\ 48.403 & 47.708 & 48.604 & 47.909 & 51.151 & 50.456 & 51.351 & 50.657 \\ 48.298 & 47.881 & 48.432 & 48.015 & 51.045 & 50.628 & 51.179 & 50.762 \\ 48.192 & 48.053 & 48.259 & 48.12 & 50.94 & 50.801 & 51.007 & 50.868 \\ 48.087 & 48.226 & 48.087 & 48.226 & 50.834 & 50.973 & 50.834 & 50.973 \\ 47.981 & 48.398 & 47.914 & 48.331 & 50.729 & 51.146 & 50.662 & 51.079 \\ 47.876 & 48.571 & 47.742 & 48.437 & 50.623 & 51.318 & 50.489 & 51.184 \\ 47.77 & 48.743 & 47.57 & 48.542 & 50.518 & 51.49 & 50.317 & 51.29 \\ 47.665 & 48.915 & 47.397 & 48.648 & 50.412 & 51.663 & 50.144 & 51.395 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$$

Note: A few of the scenarios produce CTE values for layer 1 that are in the range of the neat 0 wt% for the temperature of 100C as desired.

**RMS Surface Distortion and Radius of Curvature Comparison for
Zirconium Tungstate Thermostat Application in Trough Configuration
FEA vs. Zero Curvature Closed Form Equation Results for Case 8**

Thermal Loading Condition:

Uniform Temperature throughout model

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

RMS of FEA Deformed to Undeformed Surface	3.192E-06
RMS of best fit of FEA deformed	3.213E-06
Percent Difference between FEA and Closed Form	-0.662

Radius of Curvature Results:

FEA Radius of Curvature	0.946
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	-5.684

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4339E-15	-0.001434
2	0.00143	0.00143	-1.00000	-1.00000	-1.2905E-14	-0.001434
3	0.00287	0.00287	-1.00000	-1.00000	-3.5847E-14	-0.001434
4	0.00430	0.00430	-0.99999	-1.00000	-1.2402E-11	-0.001434
5	0.00573	0.00574	-0.99998	-0.99990	-1.4339E-13	-0.001434
6	0.00717	0.00717	-0.99997	-0.99990	-1.7350E-13	-0.001434
7	0.00860	0.00860	-0.99996	-0.99990	-2.4232E-13	-0.001434

Performed for all surface nodes and RMS is calculated according to the above equation

**RMS Surface Distortion and Radius of Curvature Comparison for
Zirconium Tungstate AFRL Axisymmetric Spherical Mirror Geometry
FEA vs. Zero Curvature Closed Form Equation Results for Case 8**

Thermal Loading Condition:

Increasing Temperature Gradient

RMS Calculation:

$$RMS = \sqrt{\frac{\sum_{i=1}^N \left((f(x'_i) - f(x_i)) - (f(x'_{i-1}) - f(x_{i-1})) \right)^2 (x'_i - x'_{i-1})}{N \left[\sum_{i=1}^N (x'_i - x'_{i-1}) \right]}}$$

Notes: Piston Shifting was necessary due to the boundary conditions placed on the FEA model. Vertical displacement was set to 0 at the bottom, therefore contraction or expansion of the individual layers caused the top surface to distort in the y-direction.

RMS Results:

AFRL Axisymmetric FEA RMS Surface Distortion	4.182E-06
MLT FEA RMS Surface Distortion	3.213E-06
Percent Difference between FEA and Closed Form	23.163

Radius of Curvature Results:

FEA Radius of Curvature	0.955
Closed Form Radius of Curvature	1.000
Percent Difference between FEA and Closed Form	-4.675

Closed Form Equation Sample Results

Node Number	Position x_i	Position x'_i	Reference $f(x_i)$	Predicted $f(x'_i)$ w/ Piston Shift	$(\Delta f(x_i) - \Delta f(x_{i-1}))^2$	$(x'_i - x'_{i-1})$
1	0.00000	0.00000	-1.00000	-1.00000	-1.4380E-15	-0.001438
2	0.00143	0.00144	-1.00000	-1.00000	-1.2942E-14	-0.001438
3	0.00287	0.00288	-1.00000	-1.00000	-3.5950E-14	-0.001438
4	0.00430	0.00431	-0.99999	-1.00000	-7.0462E-14	-0.001438
5	0.00573	0.00575	-0.99998	-1.00000	-1.4380E-13	-0.001438
6	0.00717	0.00719	-0.99997	-1.00000	-1.7399E-13	-0.001438
7	0.00860	0.00863	-0.99996	-1.00000	-2.4301E-13	-0.001438

Performed for all surface nodes and RMS is calculated according to the above equation

Case 9 AFRL Approximating Geometry to Predict Zero Curvature of Four Layer Trough Scenarios for Individual Temperatures per each Layer

These scenarios explore the range of dependent CTE and wt% of Zirconium Tungstate that is required to keep the system at 0 thermal curvature by defining 3 of the 4 CTE's.

Material properties of the trough layers

48 wt%

Neat CTD-403

$$E_3 := 7.25 \cdot 10^9 \text{ Pa}$$

$$E_2 := 4.727 \cdot 10^9 \text{ Pa}$$

$$E_1 := 4.727 \cdot 10^9 \text{ Pa}$$

$$E_4 := 4.727 \cdot 10^9 \text{ Pa}$$

$$\alpha_4 := \begin{pmatrix} 58.75 & 58.75 & 58.75 & 58.75 & 61 & 61 & 61 & 61 \\ 58.75 & 58.75 & 58.75 & 58.75 & 61 & 61 & 61 & 61 \\ 58.75 & 58.75 & 58.75 & 58.75 & 61 & 61 & 61 & 61 \\ 58.75 & 58.75 & 58.75 & 58.75 & 61 & 61 & 61 & 61 \\ 58.75 & 58.75 & 58.75 & 58.75 & 61 & 61 & 61 & 61 \\ 58.75 & 58.75 & 58.75 & 58.75 & 61 & 61 & 61 & 61 \\ 58.75 & 58.75 & 58.75 & 58.75 & 61 & 61 & 61 & 61 \\ 58.75 & 58.75 & 58.75 & 58.75 & 61 & 61 & 61 & 61 \\ 58.75 & 58.75 & 58.75 & 58.75 & 61 & 61 & 61 & 61 \\ 58.75 & 58.75 & 58.75 & 58.75 & 61 & 61 & 61 & 61 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$$

$$\alpha_3 := \begin{pmatrix} 35 & 35 & 37.5 & 37.5 & 35 & 35 & 37.5 & 37.5 \\ 35.25 & 35.25 & 37 & 37 & 35.25 & 35.25 & 37 & 37 \\ 35.5 & 35.5 & 36.75 & 36.75 & 35.5 & 35.5 & 36.75 & 36.75 \\ 35.75 & 35.75 & 36.5 & 36.5 & 35.75 & 35.75 & 36.5 & 36.5 \\ 36 & 36 & 36.25 & 36.25 & 36 & 36 & 36.25 & 36.25 \\ 36.25 & 36.25 & 36 & 36 & 36.25 & 36.25 & 36 & 36 \\ 36.5 & 36.5 & 35.75 & 35.75 & 36.5 & 36.5 & 35.75 & 35.75 \\ 36.75 & 36.75 & 35.5 & 35.5 & 36.75 & 36.75 & 35.5 & 35.5 \\ 37 & 37 & 35.25 & 35.25 & 37 & 37 & 35.25 & 35.25 \\ 37.5 & 37.5 & 35 & 35 & 37.5 & 37.5 & 35 & 35 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$$

$$\alpha_2 := \begin{pmatrix} 54.25 & 56.5 & 54.25 & 56.5 & 54.25 & 56.5 & 54.25 & 56.5 \\ 54.5 & 56.25 & 54.5 & 56.25 & 54.5 & 56.25 & 54.5 & 56.25 \\ 54.75 & 56 & 54.75 & 56 & 54.75 & 56 & 54.75 & 56 \\ 55 & 55.75 & 55 & 55.75 & 55 & 55.75 & 55 & 55.75 \\ 55.25 & 55.5 & 55.25 & 55.5 & 55.25 & 55.5 & 55.25 & 55.5 \\ 55.5 & 55.25 & 55.5 & 55.25 & 55.5 & 55.25 & 55.5 & 55.25 \\ 55.75 & 55 & 55.75 & 55 & 55.75 & 55 & 55.75 & 55 \\ 56 & 54.75 & 56 & 54.75 & 56 & 54.75 & 56 & 54.75 \\ 56.25 & 54.5 & 56.25 & 54.5 & 56.25 & 54.5 & 56.25 & 54.5 \\ 56.5 & 54.25 & 56.5 & 54.25 & 56.5 & 54.25 & 56.5 & 54.25 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$$

Material geometries of the trough layers

layer thickness: $h_1 := 0.023\text{m}$ $h_2 := 0.023\text{m}$ $h_3 := 0.023\text{m}$ $h_4 := 0.023\text{m}$

Unit depth: $\text{unit} := 1$

Temperature Changes of each layer:

$$\begin{aligned} T_0 &:= 0\text{C} & T_1 &:= 25\text{C} & T_2 &:= 50\text{C} & T_3 &:= 75\text{C} & T_4 &:= 100\text{C} \\ \Delta T_{21} &:= \frac{T_2 + T_1}{2} - T_0 & \Delta T_{31} &:= \frac{T_3 + T_1}{2} - T_0 & \Delta T_{41} &:= \frac{T_4 + T_1}{2} - T_0 \\ \Delta T_{32} &:= \frac{T_3 + T_2}{2} - T_0 & \Delta T_{42} &:= \frac{T_4 + T_2}{2} - T_0 & \Delta T_{43} &:= \frac{T_4 + T_3}{2} - T_0 \end{aligned}$$

To find the CTE in layer one:

$$\begin{aligned} s_1 &:= h_1 \cdot E_1 & s_2 &:= h_2 \cdot E_2 & s_3 &:= h_3 \cdot E_3 & s_4 &:= h_4 \cdot E_4 \\ I_1 &:= \text{unit} \cdot \frac{h_1^3}{12} & I_2 &:= \text{unit} \cdot \frac{h_2^3}{12} & I_3 &:= \text{unit} \cdot \frac{h_3^3}{12} & I_4 &:= \text{unit} \cdot \frac{h_4^3}{12} \\ h_{12} &:= h_1 + h_2 & h_{13} &:= h_1 + 2 \cdot h_2 + h_3 & h_{14} &:= h_1 + 2 \cdot h_2 + 2 \cdot h_3 + h_4 \\ h_{23} &:= h_2 + h_3 & h_{24} &:= h_2 + 2 \cdot h_3 + h_4 & h_{34} &:= h_3 + h_4 \\ P_2 &:= \alpha_2 \cdot [\Delta T_{21} \cdot s_1 \cdot s_2 \cdot (h_{12}) - \Delta T_{32} \cdot s_2 \cdot s_3 \cdot (h_{23}) - \Delta T_{42} \cdot s_2 \cdot s_4 \cdot (h_{24})] \\ P_3 &:= \alpha_3 \cdot [\Delta T_{31} \cdot s_1 \cdot s_3 \cdot (h_{13}) + \Delta T_{32} \cdot s_2 \cdot s_3 \cdot (h_{23}) - \Delta T_{43} \cdot s_3 \cdot s_4 \cdot (h_{34})] \\ P_4 &:= \alpha_4 \cdot [\Delta T_{41} \cdot s_1 \cdot s_4 \cdot (h_{14}) + \Delta T_{42} \cdot s_2 \cdot s_4 \cdot (h_{24}) + \Delta T_{43} \cdot s_3 \cdot s_4 \cdot (h_{34})] \end{aligned}$$

CTE of layer one:

$$\alpha_1 := \frac{-(P_2 + P_3 + P_4)}{-\Delta T_{21} \cdot s_1 \cdot s_2 \cdot (h_{12}) - \Delta T_{31} \cdot s_1 \cdot s_3 \cdot (h_{13}) - \Delta T_{41} \cdot s_1 \cdot s_4 \cdot (h_{14})}$$

$$\alpha_1 = \begin{pmatrix} 54.008 & 52.769 & 54.768 & 53.529 & 56.813 & 55.574 & 57.573 & 56.334 \\ 53.946 & 52.982 & 54.478 & 53.514 & 56.751 & 55.787 & 57.283 & 56.319 \\ 53.884 & 53.196 & 54.264 & 53.576 & 56.689 & 56.001 & 57.069 & 56.381 \\ 53.823 & 53.41 & 54.051 & 53.638 & 56.628 & 56.215 & 56.856 & 56.443 \\ 53.761 & 53.623 & 53.837 & 53.699 & 56.566 & 56.428 & 56.642 & 56.504 \\ 53.699 & 53.837 & 53.623 & 53.761 & 56.504 & 56.642 & 56.428 & 56.566 \\ 53.638 & 54.051 & 53.41 & 53.823 & 56.443 & 56.856 & 56.215 & 56.628 \\ 53.576 & 54.264 & 53.196 & 53.884 & 56.381 & 57.069 & 56.001 & 56.689 \\ 53.514 & 54.478 & 52.982 & 53.946 & 56.319 & 57.283 & 55.787 & 56.751 \\ 53.529 & 54.768 & 52.769 & 54.008 & 56.334 & 57.573 & 55.574 & 56.813 \end{pmatrix} 10^{-6} \cdot \frac{1}{\text{C}}$$

Note: None of the scenarios produce CTE values for layer 1 that are in the range of the neat 0 wt% for the temperature of 100C as desired. A system between 48 wt% and 70 wt% Zirconium Tungstate in Layer 1 would potentially create a zero thermal curvature scenario.