



A characterization of the complete quotient ring of homomorphic images of Prufer domains
by John Robert Chuchel

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Abstract:

Let D be a Prufer domain and let θ be a homomorphism of D . We investigate the complete quotient ring of D/a , where the kernel $a = a_1 \cap \dots \cap a_n$ of θ is an irredundant primary decomposition. The homomorphism θ is extended in such a way that D may be taken as a semi-local Prufer domain. Then, D is the intersection of the localizations $D_{P_1}, \dots, D_{P_n}$ of D , where P_j is the radical of a_j and each D_{P_j} is a valuation ring with valuation v_j and valuation group G_j . Each a_j consists of elements in D with v_j -values greater than or equal to a fixed element in a group of which G_j is a subgroup. An approximation theorem is given which enables us to choose elements x in D with appropriate v_j -values for $j=1, \dots, n$, even though the valuations are not necessarily independent. Then, we can find the prime ideals in D whose images are dense in D/a .

Using the valuation structure of D , we obtain a characterization, in terms of a sequence of finite products, of an arbitrary element f in the complete quotient ring of D/a .

An example is given of a homomorphic image of a Prufer domain with a non-trivial complete quotient ring.

A CHARACTERIZATION OF THE COMPLETE QUOTIENT RING OF
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ABSTRACT

Let D be a Prüfer domain and let ϕ be a homomorphism of D . We investigate the complete quotient ring of $D/\mathfrak{a}$, where the kernel $\mathfrak{a} = \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_n$ of ϕ is an irredundant primary decomposition. The homomorphism ϕ is extended in such a way that D may be taken as a semi-local Prüfer domain. Then, D is the intersection of the localizations $D_{P_1}, \dots, D_{P_n}$ of D , where P_j is the radical of $\mathfrak{a}_j$ and each D_{P_j} is a valuation ring with valuation v_j and valuation group G_j . Each $\mathfrak{a}_j$ consists of elements in D with v_j -values greater than or equal to a fixed element in a group of which G_j is a subgroup. An approximation theorem is given which enables us to choose elements x in D with appropriate v_j -values for $j=1, \dots, n$, even though the valuations are not necessarily independent. Then, we can find the prime ideals in D whose images are dense in $D/\mathfrak{a}$.

Using the valuation structure of D , we obtain a characterization, in terms of a sequence of finite products, of an arbitrary element f in the complete quotient ring of $D/\mathfrak{a}$.

An example is given of a homomorphic image of a Prüfer domain with a non-trivial complete quotient ring.

CHAPTER I

In [1], Boisen and Larsen showed that the homomorphic image of a Prüfer domain is a Prüfer ring. Their work prompted an investigation of other properties possessed by homomorphic images of Prüfer domains. In this paper, we take the homomorphic image of a Prüfer domain, where the kernel of the homomorphism has a certain specified form, and look at its complete quotient ring. Before presenting the main body of material, we review the concepts mentioned above, introduce additional ones, and prove a few results needed later.

Throughout the paper, all rings are commutative and have identity 1.

Definition 1.1: A subset S of a ring R is multiplicatively closed if $s_1 \cdot s_2 \in S$ whenever both s_1 and s_2 are in S .

If P is a prime ideal of a ring R , then $R \setminus P$ is multiplicatively closed.

For S a multiplicatively closed set of a ring R , let $T = \{(x, s) | x \in R, s \in S\}$. Just as in constructing $\mathbb{Q}$, the rationals, from $\mathbb{Z}$, the integers, by taking equivalence classes in the set $\{(u, v) | u, v \in \mathbb{Z}, v \neq 0\}$, we get equivalence classes in T by defining an equivalence relation

" $\sim$ " as follows: $(x,s) \sim (x',s')$ if there is an element t in S such that $t(sx' - s'x) = 0$. The equivalence class of (x,s) in T is written as x/s , just as, in $\mathbb{Q}$, the equivalence class of $(1,3)$ is written as $1/3$. Then, $T^* = T/\sim$ becomes a ring by defining:

$$\begin{cases} x/s + y/t = tx+sy/st \\ x/s \cdot y/t = xy/st \end{cases}$$

The foregoing construction leads to three important cases:

(a) If S is the set of regular elements (non zero-divisors) in a ring R , then S is multiplicatively closed, and T^* is denoted by K and called the total quotient ring of R . If R is a domain, then K is the (classical) quotient field of R . But, R need not be a domain, so that the construction applies to an arbitrary commutative ring. Since 1 is a regular element, R is embedded in its total quotient ring by $r \rightarrow r/1$.

(b) If P is a prime ideal of a ring R , then $S = R \setminus P$ is multiplicatively closed, and T^* is denoted by R_P and called the localization of R to P . For R a domain, R is embedded in R_P . In general, there is an epimorphism from R to R_P .

(c) If $P_1, \dots, P_n$ are distinct prime ideals of R , then $R \setminus \bigcup_{i=1}^n P_i$ is a multiplicatively closed set in R . Here, T^* may be denoted by $R \setminus \bigcup_{i=1}^n P_i$.

Definition 1.2: A fractional ideal of a ring R is a subset A of the total quotient ring K of R such that:

- (i) A is an R -module.
- (ii) There is a regular element d in R such that $dA \subseteq R$.

Each ideal I of R is fractional, since $1 \cdot I = I \subseteq R$ for 1 regular in R .

Definition 1.3: A fractional ideal A of a ring R is invertible if there is a fractional ideal B of R such that $AB = R$, where

$$AB = \left\{ \sum_{i=1}^k a_i b_i \mid a_i \in A, b_i \in B, k \in \mathbb{Z}, k > 0 \right\}.$$

An invertible fractional ideal is always finitely generated, but the converse is not always true.

Definition 1.4: A domain R is a Prüfer domain if each non-zero finitely generated fractional ideal of R is invertible.

Definition 1.5: A valuation ring is an integral domain V with the property that if A and B are ideals of V , then either $A \subseteq B$ or $B \subseteq A$.

The theorems and exercises of Chapter IV in Gilmer [3] give about forty conditions equivalent to R being a Prüfer domain. One of the conditions is:

Theorem 1.1: A domain R is Prüfer if and only if for every proper prime ideal P of R , the localization R_P is a valuation ring. [Theorem 22.1, (1), p. 276, *ibid*].

Definition 1.6: An ideal I in a ring R is regular if it contains a regular element.

Definition 1.7: A ring R is a Prüfer ring if every finitely generated regular ideal in R is invertible.

Valuations are an essential part of much of the work that follows. We introduce the concept after a preliminary definition.

Definition 1.8: An ordered abelian group G is an abelian group on which there is given a total ordering " $\leq$ " such that if $\alpha, \beta, \gamma \in G$ and $\alpha \leq \beta$, then $\alpha + \gamma \leq \beta + \gamma$.

The additive group of real numbers is an ordered abelian group.

For G an ordered abelian group, let $\{\infty\}$ be a set whose sole element is not in G . Let $G^* = G \cup \{\infty\}$; for $\alpha, \beta \in G^*$, define

$$\alpha + \beta = \begin{cases} \text{their sum in } G & \text{if } \alpha, \beta \in G \\ \infty & \text{if } \alpha = \infty \text{ or } \beta = \infty \end{cases}$$

Defining $\alpha \leq \infty$ for all $\alpha \in G^*$, G^* becomes an ordered semi-group: if $\alpha, \beta, \gamma \in G^*$ and $\alpha \leq \beta$, then $\alpha + \gamma \leq \beta + \gamma$.

Definition 1.9: Let K be a field. A valuation on K is a mapping v from K onto G^* , where G is an ordered abelian group, such that:

- (i) $v(a) = \infty$ if and only if $a = 0$;
- (ii) $v(ab) = v(a) + v(b)$ for all $a, b \in K$;
- (iii) $v(a+b) \geq \min\{v(a), v(b)\}$ for all $a, b \in K$.

An important result connects valuations and valuation rings:

Theorem 1.2: Let V be a valuation ring with quotient field K . Then, there is a valuation v on K such that $V = \{a \mid a \in K, v(a) \geq 0\}$. [Proposition 5.13, p. 108, Larsen and McCarthy [5]].

We next develop the notion of the complete quotient ring of a ring R .

Definition 1.10: An ideal A in a commutative ring R is dense if $rA = 0$ implies $r = 0$ for all $r \in R$.

An immediate consequence of Definition 1.10 is that the intersection of dense ideals is dense.

Definition 1.11: A fraction is an element F of $\text{Hom}_R(A, R)$, where A is a dense ideal of R (F is a group homomorphism of A into R for which $F(ar) = F(a)r$ for all $a \in A$ and for all $r \in R$); A is called the domain of F .

To motivate the next definition, we note that in constructing $\mathbb{Q}$ from $\mathbb{Z}$, $2/4$ is a "fraction" with domain $4\mathbb{Z}$, mapping $4\mathbb{Z}$ to $\mathbb{Z}$; $3/6$ has domain $6\mathbb{Z}$ and maps $6\mathbb{Z}$ to $\mathbb{Z}$; $2/4$ and $3/6$ are equivalent in that they agree on the intersection of their domains: $12\mathbb{Z}$; both map $12\mathbb{Z}$ to $\mathbb{Z}$. Hence, $2/4$ and $3/6$ belong to the same equivalence class of "fractions", the class denoted by $1/2$.

Definition 1.12: For fractions F_1 and F_2 with domains A_1 and A_2 , respectively, we let $F_1 \theta F_2$ if and only if F_1 and F_2 agree on the intersection of their domains:

$$F_1(a) = F_2(a) \text{ for all } a \in A_1 \cap A_2.$$

Denoting the class of fractions by $\mathcal{F}$, θ is an equivalence relation and partitions $\mathcal{F}$ into equivalence classes.

Theorem 1.3: For R a commutative ring,

$(\mathcal{F}, 0, 1, -, +, \cdot,)/\theta = Q(R)$ is a commutative ring, called the complete ring of quotients of R . [Proposition 1, p. 38, Lambek [4]].

Letting $f = [F]$ be the equivalence class of the fraction F in $Q(R)$, R is embedded in $Q(R)$ by the mapping $r \rightarrow [r/1]$. The mapping is a monomorphism since $r/1$ agrees with $0/1$ on a dense ideal A only if $rA = 0$, and hence, $r = 0$. For each non zero-divisor a of R , the ideal aR is dense in R . Then, for $r \in R$, the "classical" fraction $r/a \in \text{Hom}_R(aR, R)$ is defined by:

$$(r/a)(as) = rs, \quad s \in R.$$

Theorem 1.4: The equivalence classes $[r/a]$, $r \in R$, a not a zero-divisor, comprise a subring of $Q(R)$, called the classical ring of quotients of R : $Q_{cl}(R)$. [Proposition 2, p. 39, Lambek [4]].

As expected, the classical ring of quotients of a ring R is identical to the total quotient ring of R .

Collecting results, we have for an arbitrary commutative ring R that

$$R \subseteq Q_{cl}(R) \subseteq Q(R).$$

To better illustrate the concept of a complete quotient ring, we look at two examples:

(a) Since all non-zero elements of $\mathbb{Z}$ are non zero-divisors, $\mathbb{Z} \not\subseteq Q_{cl}(\mathbb{Z}) = Q(\mathbb{Z}) = \mathbb{Q}$, $\mathbb{Q}$ being the rational numbers.

(b) Let R be the collection of all finite and cofinite subsets of some infinite set, say $\mathbb{Z}$. Then, R is a Boolean ring with operations of symmetric difference and intersection, where $\mathbb{Z} = 1$ and $\emptyset = 0$. Let $\mathcal{a}$ be the collection of all finite subsets of $\mathbb{Z}$; $\mathcal{a}$ is a dense maximal ideal of R . Further, R itself is dense. We define fractions for R as follows:

If L is a finite or cofinite subset of $\mathbb{Z}$, then $L \in \text{Hom}_R(R, R)$, where $L(V) = L \cap V$ for $V \in R$.

However, if L is neither a finite nor a cofinite subset of $\mathbb{Z}$, then $L(V') = L \cap V'$ need not be in R for V' cofinite in R . For example, let $L = 2\mathbb{Z}$ (the even integers), and let $V' = \mathbb{Z} \setminus \{0, 1, 2\}$. Then, $L \cap V' = 2\mathbb{Z} \setminus \{0, 2\} \notin R$.

Therefore, for L in the power set of $\mathbb{Z}$ but not in R , we must restrict the domain of L to $\mathcal{a}$, thus ensuring that $L \in \text{Hom}_R(\mathcal{a}, R)$. Identifying the subset L with the "fraction" L , we get $Q(R) \cong \mathcal{P}(\mathbb{Z})$, the power set of $\mathbb{Z}$.

All elements of R , except $\mathbb{Z}$ itself, are zero-divisors: for $V \in R$, $V \cap c(V) = \emptyset = 0$, where $c(V)$, the complement of V , is in R by definition. Since R has only one regular element, all elements of $Q_{cl}(R)$ have the form $E/\mathbb{Z} = E/1$ for $E \in R$. Then, $Q_{cl}(R) \cong R$, $E \mapsto E/1$ being an isomorphism. By construction, $R \not\cong \mathcal{P}(\mathbb{Z})$, and it follows that $R \cong Q_{cl}(R) \subsetneq Q(R)$. [R. Gilmer [3a]].

We thus have one example each of a trivial and a non-trivial complete quotient ring. In the work that follows, we show that certain homomorphic images of Prüfer domains have non-trivial complete quotient rings.

For $f \in Q(R)$, let the domain of f be $\text{dom}_R f = \text{dom } f = \{x \in R \mid fx \in R\}$, and let the range of f be $\text{ran}_R f = \text{ran } f = f(\text{dom}_R f)$.

Lemma 1.5: If $f \in Q(R)$ and $\text{dom } f \subseteq P$, where P is a minimal prime ideal of a ring R , then $\text{dom } f + \text{ran } f \subseteq P$.

Proof. Using Proposition 12, p. 73, Bourbaki [2], if P is a minimal prime and x is in P , then there is an element s in $R \setminus P$ and a positive integer z such that $sx^z = 0$. Therefore, for $x \in \text{dom } f \subseteq P$, we have $sx^z = 0$, where s and z are as stated above. Since $f \in \text{Hom}_R(A, R)$ for some dense ideal A of R , $fsx^z = 0 = s(fx)^z \in P$. Then, P a prime ideal and

$s \notin P$ imply that $(fx)^Z \in P$. Continuing this reasoning, $fx \in P$, and $\text{ran } f \subseteq P$ since x is an arbitrary element in $\text{dom } f$. Hence, $\text{dom } f + \text{ran } f \subseteq P$, for P a minimal prime ideal of R .

(End of Lemma 1.5)

If ϕ is a homomorphism of a ring R and I is an ideal in R , we write $\phi(I)$, the image or extension of I in $\phi(R)$, as I^e . For J an ideal in $\phi(R)$, the contraction of J in R is $J^c = \{a \in R \mid \phi(a) \in J\}$. Throughout the paper, we use Zariski and Samuel [6] as a source for properties of extensions and contractions.

Definition 1.13: For A and B ideals in a ring R , let $A:B = \{x \mid x \in R \wedge Bx \subseteq A\}$.

The following lemma provides a useful relationship between the kernel of a homomorphism of a domain D and ideals in D and their extensions in $\phi(D)$.

Lemma 1.6: Let ϕ be a homomorphism from a domain D onto a ring R , with K the kernel of ϕ . For an ideal A of D , $K:A = K$ if and only if A^e is dense in R .

Proof: $\Rightarrow$: Let $K:A = K$ and suppose that A^e is not dense in R . Then, for some $\bar{r} \neq \bar{0}$ in R , we have that $\bar{r}A^e = \bar{0}$.

This is equivalent to $\phi(r)\phi(A) = \bar{0} = \phi(rA)$ for $r \notin K$ in D , where $\phi(r) = \bar{r}$. Hence, $rA \subseteq K$ and $r \in K:A$. But, $r \notin K$ implies that $K \subsetneq K:A$, a contradiction. Thus, A^e is dense in R .

$\Leftarrow$: Let A^e be dense in R and assume that $K \subsetneq K:A$ ($K \subseteq K:A$ always holds). Therefore, there is an element $s \in K:A \setminus K$ for which $sA \subseteq K$. Since $s \notin K$, we have that $\phi(s) \neq \bar{0}$ in R . But, $sA \subseteq K$ implies that $\phi(sA) = \phi(s) \cdot \phi(A) = \phi(s) \cdot A^e = \bar{0}$ in R . So, A^e is not dense in R , a contradiction. We conclude that $K = K:A$.

(End of Lemma 1.6)

Definition 1.14: Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be primary ideals with associated prime ideals $P_i = \sqrt{\mathfrak{a}_i}$, $i = 1, \dots, n$. The intersection $\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_n$ of the $\mathfrak{a}_i$'s is a primary representation or a primary decomposition and is said to be irredundant if:

- (i) No $\mathfrak{a}_i$ contains the intersection of the others:
 $\mathfrak{a}_j \not\supseteq \bigcap_{i \neq j} \mathfrak{a}_i$.
- (ii) The $\mathfrak{a}_i$ have distinct associated prime ideals:
 $P_i \neq P_j$ for $i \neq j$.

Definition 1.15: The prime ideals $P_1, \dots, P_n$ are isolated if $P_i \not\subseteq P_j$ for $i \neq j$.

We will study the homomorphic image of a Prüfer domain where the kernel of the homomorphism has an irredundant primary decomposition. The Prüfer domain property insures that the prime ideals associated with the kernel are isolated.

Lemma 1.7: Let $\bigcap_{i=1}^n \mathfrak{a}_i$ be an irredundant primary decomposition in a Prüfer domain D . Let $P_i = \sqrt{\mathfrak{a}_i}$, $i = 1, \dots, n$. Then, $P_1, \dots, P_n$ are isolated prime ideals.

Proof: Consider the primary ideals $\mathfrak{a}_i, \mathfrak{a}_j$ in D and the associated prime ideals P_i, P_j . Without loss of generality, assume that $P_i \subseteq P_j$. Then, $\mathfrak{a}_i + \mathfrak{a}_j \subseteq P_j$. Since D is a Prüfer domain and $\mathfrak{a}_i + \mathfrak{a}_j \neq D$, by pp. 130-1, Larsen and McCarthy [5], $\mathfrak{a}_i \subseteq \mathfrak{a}_j$ or $\mathfrak{a}_j \subseteq \mathfrak{a}_i$. But, this is a contradiction to $\bigcap_{i=1}^n \mathfrak{a}_i$ being an irredundant primary decomposition. Therefore, $P_i \not\subseteq P_j$.

(End of Lemma 1.7)

CHAPTER II

Let D be a Prüfer domain and let ϕ be a homomorphism of D . Let K , the kernel of ϕ , have an irredundant primary decomposition $\bigcap_{i=1}^n \mathfrak{a}_i$, where $P_i = \sqrt{\mathfrak{a}_i}$. Each $D \setminus P_i$ is multiplicatively closed, and, hence, $\bigcap_{i=1}^n (D \setminus P_i)$ is multiplicatively closed. But, $\bigcap_{i=1}^n (D \setminus P_i) = D \setminus \bigcup_{i=1}^n P_i$. Therefore, we form the intersection of local rings $\bigcap_{P_i} D_{P_i}$, which, by Proposition 17(c), p. 93, Bourbaki [2], is equal to $D_{\bigcup P_i}$. We investigate a homomorphism of $D_{\bigcup P_i}$ which is related to the homomorphism ϕ of D .

Lemma 2.1: The homomorphism ϕ from D onto D/K can be extended to a homomorphism μ from $D_{\bigcup P_i}$ onto $Q_{cl}(D/K)$, where, for $a \in D$ and $b \in D \setminus \bigcup_{i=1}^n P_i$, we define $\mu(a/b) = \phi(a)/\phi(b)$:

$$\begin{array}{ccc}
 D & \xrightarrow{\quad} & D_{\bigcup P_i} \\
 \phi \downarrow & & \downarrow \mu \\
 D/K & \xrightarrow{\quad} & Q_{cl}(D/K).
 \end{array}$$

Proof: We must show that if $b \in D \setminus \bigcup_{i=1}^n P_i$, then $\phi(b)$ is regular in D/K . But, clearly, $K:b = K$ for $b \notin \bigcup_{i=1}^n P_i$. Therefore, $(\phi(b))$ is dense in D/K by Lemma 1.6, and $\phi(b)$ is regular in D/K .

The natural mapping μ is onto: let $b \in D$ such that $\phi(b)$ is regular in D/K . Then, $(\phi(b))$ is dense in D/K and $K:b = K$ by Lemma 1.6. Equivalently,

$\bigcap_{i=1}^n \mathfrak{a}_i : b = \bigcap_{i=1}^n \mathfrak{a}_i$. Assume that $b \in \bigcup_{i=1}^n P_i$. Since $b \in \bigcap_{i=1}^n \mathfrak{a}_i$ implies that $\phi(b) = 0$, which is certainly not regular, $b \in \bigcap_{i=1}^n \mathfrak{a}_i$ is impossible. Since each $\mathfrak{a}_j$ is P_j -primary, there exists minimal $k_j \in \mathbb{Z}^+$ such that $b^{k_j} \in \mathfrak{a}_j$. Letting $N = \max\{k_1, \dots, k_n\} \geq 1$, we have $b^N \in \mathfrak{a}_i$ for all i , but $b^{N-1} \notin \mathfrak{a}_j$ for at least one j , by definition of N . Therefore, $b^{N-1} \notin \bigcap_{i=1}^n \mathfrak{a}_i$. But, $b^{N-1} \in \bigcap_{i=1}^n \mathfrak{a}_i : b$, since $b^{N-1} \cdot b = b^N \in \bigcap_{i=1}^n \mathfrak{a}_i$. Therefore, $\bigcap_{i=1}^n \mathfrak{a}_i : b \not\supseteq \bigcap_{i=1}^n \mathfrak{a}_i$, a contradiction. Thus, if $\phi(b)$ is regular, then $b \in D \setminus \bigcup P_i$.

Hence, for $a, b \in D$ and $\phi(b)$ regular, $a/b \in D_{\bigcup P_i}$, $\mu(a/b) = \phi(a)/\phi(b)$, and μ is onto.

(End of Lemma 2.1)

Now since D is a Prüfer domain, each D_{P_i} , $i = 1, \dots, n$, is a valuation domain. Since $K = \bigcap_{i=1}^n \mathfrak{a}_i$ is an irredundant decomposition, each $P_j = \sqrt{\mathfrak{a}_j}$, $j = 1, \dots, n$, is isolated, by Lemma 1.7. Then, by Proposition 17(a), p. 93, Bourbaki [2], the isolated prime ideals P_j of D extend to

maximal ideals $P_j \cdot D_{\bigcup P_i}$ of $D_{\bigcup P_i}$. This leads us to the following:

Lemma 2.2: Let D be a domain with isolated prime ideals $P_1, \dots, P_n$; $E = D_{\bigcup P_i} = \bigcap D_{P_i}$ is the domain with maximal ideals $P_1 E = P_1^*, \dots, P_n E = P_n^*$. Let $P_i^!$ be the unique maximal ideal in D_{P_i} , $i = 1, \dots, n$. Then, $P_i^* = P_i^! \cap E$, $i = 1, \dots, n$. Further, if $\mathfrak{a}_i^*$ is a P_i^* -primary ideal in E , then there exists a $P_i^!$ -primary ideal $\mathfrak{a}_i^!$ in D_{P_i} such that $\mathfrak{a}_i^* = \mathfrak{a}_i^! \cap E$. In fact, this relation is a one-to-one order preserving correspondence between P_i^* -primary ideals in E and $P_i^!$ -primary ideals in D_{P_i} .

Proof: We have the following diagram:

$$\begin{array}{ccccccc}
 P_j & \xrightarrow{\quad} & P_j^* & \xrightarrow{\quad} & P_j^! & & \\
 \downarrow & & \searrow & & \searrow & & \\
 D & \xrightarrow{\iota} & \bigcap D_{P_i} & = & D_{\bigcup P_i} & \xrightarrow{e_j} & D_{P_j} \cong (D_{\bigcup P_i})_{P_j^*},
 \end{array}$$

where ι extends D to the localization $D_{\bigcup P_i}$ and e_j extends $D_{\bigcup P_i}$ to the localization $(D_{\bigcup P_i})_{P_j^*}$. By Proposition 17(b), p. 93, Bourbaki [2], D_{P_j} is canonically isomorphic to $(D_{\bigcup P_i})_{P_j^*}$ (as indicated in the diagram). Then,

$(P_j^*)^{e_j} = P_j^!$; letting c_j be the contraction from D_{P_j} to $D_{\bigcup P_i}$, we get:

$$(P_j^!)^{c_j} = (P_j^*)^{e_j c_j} = (P_j^*)^{e_j c_j} = P_j^*.$$

So, $P_j^* = (P_j^!)^{c_j} = P_j^! \cap E$. Since j is an arbitrary subscript from 1 to n , this gives us the first result in Lemma 2.2.

To show the one-to-one order preserving correspondence between primary ideals, we cite Zariski and Samuel [6] and its material on extensions and contractions of ideals.

Hence, for j between 1 and n , if $\mathfrak{a}_j^*$ is a P_j^* -primary ideal in E , then there is a $P_j^!$ -primary ideal $\mathfrak{a}_j^!$ in D_{P_j} for which $\mathfrak{a}_j^* = \mathfrak{a}_j^! \cap E$, and the correspondence between the $\mathfrak{a}_j^*$'s and the $\mathfrak{a}_j^!$'s is one-to-one.

(End of Lemma 2.2)

In light of the aforementioned lemmas and in order to study the complete quotient ring of $D/\mathfrak{a}$, we may assume the extension:

$$\begin{array}{ccc} D & \longrightarrow & D_{\bigcup P_i} \\ \downarrow & & \downarrow \\ D/\mathfrak{a} & \longrightarrow & Q_{cl}(D/\mathfrak{a}). \end{array}$$

In the following, $D = D_{\bigcup P_i} = \bigcap D_{P_i}$ will be a semi-local Prüfer domain with maximal ideals $P_1, \dots, P_n$, and the homomorphism ϕ of D has kernel $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{a}_i$, an irredundant primary decomposition, where $P_j = \sqrt{\mathfrak{a}_j}$. Since $D = D_{\bigcup P_i}$, we have $D/\mathfrak{a} = Q_{cl}(D/\mathfrak{a})$. Each D_{P_j} has unique maximal ideal P'_j , where $P_j = P'_j \cap D$ by Lemma 2. Denote arbitrary P_j -primary ideals in D by $\mathfrak{P}_j$ and arbitrary P'_j -primary ideals in D_{P_j} by $\mathfrak{P}'_j$. Since D_{P_j} is a valuation domain, we take v_j to be the valuation from D_{P_j} to the valuation group G_j (by Definition 1.9). Let B'_j be the intersection of all P'_j -primary ideals in D_{P_j} : $\bigcap \mathfrak{P}'_j = B'_j$.

Before proceeding to another lemma, we need two definitions.

Definition 2.1: Let G be an ordered abelian group. A subgroup H of G is isolated if for each $\alpha \geq 0$ in H , $0 \leq \beta \leq \alpha$ forces β to be in H .

Definition 2.2: An ordered group is of rank one if its only isolated subgroups are itself and 0.

Lemma 2.3: Let B'_j and G_j be given as above, where $B'_j \neq P'_j$ (that is, there are P'_j -primary ideals $\mathfrak{P}'_j$ not equal

to P_j'). Then, there is a rank one ordered group $H_j \leq G_j$ which corresponds to B_j' in a natural way, where H_j is isomorphic to an additive subgroup of the reals.

Proof: By Theorem 5.11 (3), p. 106, Larsen and McCarthy [5], the intersection B_j' of all P_j' -primary ideals of D_{P_j} is a prime ideal in D_{P_j} and there are no prime ideals of D_{P_j} properly between B_j' and P_j' . Thus, we have

$$B_j' \subsetneq \mathfrak{B}_j' \subseteq P_j' \subsetneq D_{P_j}$$

for all P_j' -primary ideals $\mathfrak{B}_j'$ in D_{P_j} , with v_j the valuation on D_{P_j} . By Theorem 5.17, p. 111, *ibid.*, there exists a one-to-one order reversing correspondence between isolated subgroups of G_j and proper prime ideals of D_{P_j} :

$$\begin{array}{ccccc} B_j' & \subsetneq & \mathfrak{B}_j' & \subseteq & P_j' & \subsetneq & D_{P_j} \\ \updownarrow & & & & \updownarrow & & \\ H_j & & \supsetneq & & 0 & & \end{array}$$

By Definition 2.2, since there are no prime ideals of D_{P_j} properly between B_j' and P_j' , we have that H_j is a rank one ordered group. Then, using Proposition 5.15, p. 110, *ibid.*, H_j is isomorphic to an additive subgroup of the reals.

Hence, we take $H_j \leq \mathbb{R}$. By the construction in the proof of Theorem 5.17, pp. 111-2, *ibid.*,

$B_j' = \{x \in D_{P_j} \mid v_j(x) \notin H_j\}$. This gives us the exact nature of the correspondence between B_j' and H_j .

(End of Lemma 2.3)

Note: If P_j' is the only P_j' -primary ideal in D_{P_j} , then

$B_j' = P_j'$ and $H_j = 0$. We exclude this case from the following discussion.

For $B_j' \neq P_j'$, since H_j is an additive subgroup of the reals, we can find a concise representation for each P_j' -primary ideal $\mathfrak{B}_j'$ in D_{P_j} . Since $B_j' \subsetneq \mathfrak{B}_j'$ and B_j' is given as above, there are elements y in $\mathfrak{B}_j' \setminus B_j'$ such that $v_j(y) \in H_j$. For $x \in B_j'$, suppose there is an element y in $\mathfrak{B}_j' \setminus B_j'$ such that $v_j(x) \leq v_j(y)$. Since H_j is an isolated subgroup of G_j (see Definition 2.1), $v_j(x) \in H_j$, a contradiction. Therefore, for arbitrary x in B_j' , $v_j(x) > v_j(y)$ for all $y \in \mathfrak{B}_j' \setminus B_j'$. Because of the total ordering for v_j , $\mathfrak{B}_j'$ must have one of two forms:

either
$$\mathfrak{B}_j' = \bigcup_{y \in \mathfrak{B}_j'} \{x \in D_{P_j} \mid v_j(x) \geq v_j(y)\},$$

or
$$\mathfrak{B}_j' = \bigcup_{y \in \mathfrak{B}_j'} \{x \in D_{P_j} \mid v_j(x) > v_j(y)\}.$$

Since H_j is an additive subgroup of the reals, let

$r_j = \inf\{v_j(y) | y \in \mathfrak{B}_j^!\}$. Then, either

$$\mathfrak{B}_j^! = \{x \in D_{P_j} | v_j(x) \geq r_j\} \quad \text{or} \quad \mathfrak{B}_j^! = \{x \in D_{P_j} | v_j(x) > r_j\}.$$

If $r_j \notin H_j$, then $\{x \in D_{P_j} | v_j(x) \geq r_j\} = \{x \in D_{P_j} | v_j(x) > r_j\}$;

it is when $r_j \in H_j$ that the two sets differ.

Further, anything of the form $\mathfrak{a}_j^! = \{x \in D_{P_j} | v_j(x) \geq r_j\}$

or $\mathfrak{a}_j^! = \{x \in D_{P_j} | v_j(x) > r_j\}$ must be $P_j^!$ -primary. Take an

arbitrary element z_j in $P_j^!$, where $v_j(z_j) = g_j > 0$. If

$z_j \in \mathfrak{a}_j^!$ already, we have nothing to show. If $z_j \in P_j^! \setminus \mathfrak{a}_j^!$,

then $g_j \in H_j$, and, by the Archimedean property of the reals,

there is a positive integer n_j such that $n_j g_j > r_j$. Therefore,

$$v_j(z_j^{n_j}) = n_j \cdot v_j(z_j) = n_j \cdot g_j > r_j \quad \text{and}$$

$$z_j^{n_j} \in \mathfrak{a}_j^!.$$

Hence, $\mathfrak{a}_j^!$ is $P_j^!$ -primary.

We are now able to characterize the primary ideals

$\mathfrak{a}_j$ in the irredundant primary decomposition

$$\bigcap_{i=1}^n \mathfrak{a}_i = \text{Ker } \phi. \quad \text{Since } \mathfrak{a}_j \text{ is } P_j\text{-primary in } D, \text{ by Lemma}$$

2.2 there is a $P_j^!$ -primary ideal $\mathfrak{a}_j^!$ in D_{P_j} such that

$\mathfrak{a}_j = \mathfrak{a}'_j \cap D$. From above, there exists a real number $r_j \in H_j$ for which either $\mathfrak{a}'_j = \{x \in D_{P_j} \mid v_j(x) \geq r_j\}$ or $\mathfrak{a}'_j = \{x \in D_{P_j} \mid v_j(x) > r_j\}$. Hence, either $\mathfrak{a}_j = \{x \in D \mid v_j(x) \geq r_j\}$ or $\mathfrak{a}_j = \{x \in D \mid v_j(x) > r_j\}$. If $\mathfrak{a}_j \subsetneq P_j$, then $r_j \neq 0$; if $\mathfrak{a}_j = P_j$, then $r_j = 0$ and $\mathfrak{a}_j = \{x \in D \mid v_j(x) > 0\}$, since $v_j(x) = 0$ only if x is a unit in D_{P_j} .

In subsequent work, we will investigate $Q(D/\mathfrak{a})$, the complete quotient ring of $D/\mathfrak{a}$, where $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{a}_i$. For this work, we will need a theorem which will enable us to choose elements in D with appropriate valuations. Some preliminary material is needed.

Definition 2.3: The rank of the order group of a valuation is called the rank of that valuation.

For each ordered group G_j associated with the local domain D_{P_i} , one of four cases holds:

Case 1a: G_j has no rank one subgroup since P_j has no proper primary ideals; that is, $\mathfrak{a}_j$ is forced to be P_j .

Case 1b: G_j has a rank one subgroup H_j (P_j has proper primary ideals), but $\mathfrak{a}_j$ is chosen to be equal to P_j .

Case 2: G_j has a rank one subgroup H_j and $H_j \cong \mathbb{Z}$.

Case 3: G_j has a rank one subgroup H_j and $H_j \not\cong \mathbb{Z}$, where $\mathfrak{a}_j = \{x \in D \mid v_j(x) > r_j\}$.

Case 4: G_j has a rank one subgroup H_j and $H_j \not\cong \mathbb{Z}$, where $\mathfrak{a}_j = \{x \in D \mid v_j(x) \geq r_j\}$.

In Cases 1a and 1b, $\mathfrak{a}_j = \{x \in D \mid v_j(x) > 0\}$. In Cases 2, 3, and 4, $\mathfrak{a}_j$ is properly contained in P_j .

Let Cases 2, 3, and 4 hold for $i = 1, \dots, \gamma$ and let Cases 1a and 1b hold for $i = \gamma+1, \dots, n$. With this agreement, H_i is a rank one subgroup of G_i for $i = 1, \dots, \gamma$, while G_i may or may not have a rank one subgroup for $i = \gamma+1, \dots, n$. Recall that $D = D_{\bigcup P_i} = \bigcap D_{P_i}$, where $D_{P_i} = \{x \in Q_{cl}(D) \mid v_i(x) \geq 0\}$, and $P_i = \{x \in Q_{cl}(D) \mid v_i(x) > 0\}$. We now state and prove the theorem on valuations, a variant of Theorem 18 (an approximation theorem), pp. 45-6, Zariski and Samuel [6].

Theorem 2.4 (The Approximation Theorem): Let D be a semi-local domain with maximal ideals $P_1, \dots, P_n$. Let v_i , $i = 1, \dots, n$, be valuations of the field $Q_{cl}(D)$ with valuation groups G_i corresponding to D_{P_i} . Let H_i be rank one subgroups of G_i , $i = 1, \dots, \gamma$. Let $u_i \in D$ for $i = 1, \dots, n$ and $\alpha_i \in H_i$, $\alpha_i \geq 0$, for $i = 1, \dots, \gamma$. Then, there is an element $u \in Q_{cl}(D)$ such that

- (a) $v_i(u - u_i) = \alpha_i$, $i = 1, \dots, \gamma$
- (b) $v_i(u - u_i) = 0$, $i = \gamma + 1, \dots, \rho$
- (c) $v_i(u - u_i) > 0$, $i = \rho + 1, \dots, n$.

Proof: It is sufficient to prove the following: given any positive integer m there is an x in $Q_{cl}(D)$ such that:

$$(1) \quad \begin{cases} v_i(x - u_i) \geq m, & i = 1, \dots, \gamma \\ v_i(x - u_i) > 0, & i = \gamma + 1, \dots, n. \end{cases}$$

To see this, assume that (1) has already been shown. Then, since the α_i , $i = 1, \dots, \gamma$, have been chosen, and since H_i is an additive subgroup of the reals, there exists a positive integer m for which $m > \alpha_i$ for each $i = 1, \dots, \gamma$. Recalling that valuations are onto maps, there are elements x_i in $Q_{cl}(D)$ such that:

$$\begin{cases} v_i(x_i) = \alpha_i, & i = 1, \dots, \gamma \\ v_i(x_i) = 0, & i = \gamma+1, \dots, \rho \\ v_i(x_i) > 0, & i = \rho+1, \dots, n. \end{cases}$$

By assumption, there is an element y in $Q_{cl}(D)$ such that:

$$\begin{cases} v_i(y-x_i) \geq m, & i = 1, \dots, \gamma \\ v_i(y-x_i) > 0, & i = \gamma+1, \dots, n. \end{cases}$$

Since $y = (y-x_i) + x_i$ and $v_i(y-x_i) \geq m > \alpha_i = v_i(x_i)$, we get that $v_i(y) = v_i(x_i) = \alpha_i$, $i = 1, \dots, \gamma$. For $i = \gamma+1, \dots, \rho$, $v_i(x_i) = 0$ and $v_i(y-x_i) > 0$ together imply that $v_i(y) = 0$. Similarly, for $i = \rho+1, \dots, n$, $v_i(x_i) > 0$ and $v_i(y-x_i) > 0$ give us that $v_i(y) > 0$.

Let x be an element satisfying inequalities (1) and let $u = x + y$. Then, $u - u_i = (x-u_i) + y$. For $i = 1, \dots, \gamma$, $v_i(y) = \alpha_i < m \leq v_i(x-u_i)$; therefore, $v_i(u-u_i) = v_i(y) = \alpha_i$, and u satisfies relations (a). For $i = \gamma+1, \dots, \rho$, $v_i(x-u_i) > 0$ and $v_i(y) = 0$ together imply that $v_i(u-u_i) = 0$, and u satisfies relations (b). For $i = \rho+1, \dots, n$, $v_i(u-u_i) > 0$ follows from $v_i(x-u_i) > 0$ and $v_i(y) > 0$. Hence, u satisfies relations (c).

We now prove (1). Since D is a semi-local domain with maximal ideals $P_1, \dots, P_n$, there is an element $\theta_i \in P_i$

such that $v_i(\theta_i) > 0$ and $v_j(\theta_i) = 0$ for $j \neq i$. Letting $\eta_i = \theta_1 \dots \theta_{i-1} \theta_{i+1} \dots \theta_n$, we get that $v_i(\eta_i) = 0$ and $v_i(\eta_j) > 0$ for $i \neq j$, $i, j = 1, \dots, n$. Replace the elements η_i by the following elements λ_i :

$$\lambda_i = \frac{\eta_i}{\eta_1 + \dots + \eta_n}, \quad i = 1, \dots, n.$$

Then, it is still true that $v_i(\lambda_i) = 0$ and $v_i(\lambda_j) > 0$ if $i \neq j$, but, in addition, the image of λ_i under the canonical mapping $D_{P_i} \rightarrow D_{P_i}/P_i'$ is $1 + P_i'$. Hence, $v_i(\lambda_i - 1) > 0$, 1 the identity in $Q_{cl}(D)$. Recalling that $m \in H_i \subseteq G_i$ and $v_k(u_i) \geq 0$, fix a positive integer L for which:

$$(2) \quad \begin{cases} L \cdot v_i(\lambda_i - 1) + v_i(u_i) \geq m, & i = 1, \dots, \gamma \\ L \cdot v_i(\lambda_i - 1) + v_i(u_i) > 0, & i = \gamma + 1, \dots, n \end{cases};$$

and,

$$(3) \quad \begin{cases} L \cdot v_j(\lambda_i) + v_j(u_i) \geq m, & i \neq j; i = 1, \dots, n; \\ & j = 1, \dots, \gamma \\ L \cdot v_j(\lambda_i) + v_j(u_i) > 0, & i \neq j, i = 1, \dots, n; \\ & j = \gamma + 1, \dots, n. \end{cases}$$

Consider the elements ξ_i of $Q_{cl}(D)$:

$$\xi_i = 1 - (1 - \lambda_i^L)^L, \quad i = 1, \dots, n.$$

Then, $v_i(\xi_i - 1) = L \cdot v_i(1 - \lambda_i^L) \geq L \cdot v_i(1 - \lambda_i)$ for $i = 1, \dots, n$. So, by (2):

$$\begin{aligned} (4) \quad & v_i[u_i(\xi_i - 1)] \\ &= v_i(u_i) + v_i(\xi_i - 1) \\ &\geq m - L \cdot v_i(\lambda_i - 1) + L \cdot v_i(1 - \lambda_i) = m, \quad i = 1, \dots, \gamma; \end{aligned}$$

and,

$$\begin{aligned} (5) \quad & v_i[u_i(\xi_i - 1)] \\ &> -L \cdot v_i(\lambda_i - 1) + L \cdot v_i(1 - \lambda_i) = 0, \quad i = \gamma + 1, \dots, n. \end{aligned}$$

Now, $\xi_i = \lambda_i^L \cdot g(\lambda_i)$, where g is a polynomial with coefficients in D . Hence, if $i \neq j$ for $i, j = 1, \dots, n$, we have

$$\begin{aligned} v_j(\xi_i) &= v_j(\lambda_i^L \cdot g(\lambda_i)) \\ &= v_j(\lambda_i^L) + v_j(g(\lambda_i)) \\ &= L \cdot v_j(\lambda_i) + v_j(g(\lambda_i)) \\ &\geq L \cdot v_j(\lambda_i). \end{aligned}$$

Therefore, by (3):

$$\begin{aligned}
 (6) \quad v_j(u_i \xi_i) &= v_j(u_i) + v_j(\xi_i) \\
 &\geq m - L \cdot v_j(\lambda_i) + L \cdot v_j(\lambda_i) \\
 &= m, \text{ for } i = 1, \dots, n \text{ and } j = 1, \dots, \gamma;
 \end{aligned}$$

and,

$$\begin{aligned}
 (7) \quad v_j(u_i \xi_i) &> -L \cdot v_j(\lambda_i) + L \cdot v_j(\lambda_i) = 0, \\
 &\text{for } i = 1, \dots, n \text{ and } j = \gamma+1, \dots, n.
 \end{aligned}$$

Set $x = u_1 \xi_1 + \dots + u_n \xi_n$; then, it follows immediately from (4), (5), (6), and (7) that x satisfies inequalities (1).

(End of Theorem 2.4)

Note: If $u_i = 0$ for $i = 1, \dots, n$ and $\alpha_i \in H_i$ is given, then the approximation theorem guarantees the existence of an element u in $Q_{c\ell}(D)$ for which:

- (a) $v_i(u) = \alpha_i, \quad i = 1, \dots, \gamma$
- (b) $v_i(u) = 0, \quad i = \gamma+1, \dots, p$
- (c) $v_i(u) > 0, \quad i = p+1, \dots, n.$

All applications of the approximation theorem will be of this form.

For Case 4, we get the following important result.

Theorem 2.5: Let D , $\mathfrak{a}$, v_j , and H_j be given as above, where $P_j = \sqrt{\mathfrak{a}_j}$. Then, P_j^e is dense in $D/\mathfrak{a}$ if and only if there is a real number r_j such that

$\mathfrak{a}_j = \{x \in D \mid v_j(x) \geq r_j, r_j \in H_j\}$, and H_j is not discrete.

Proof: $\Rightarrow$: Let P_j^e be dense in $D/\mathfrak{a}$ and suppose that $\mathfrak{a}_j = \{x \in D \mid v_j(x) > r_j\}$, $r_j \geq 0$, $r_j \in H_j$. By the approximation theorem there is an element a in $Q_{cl}(D)$ such that:

$$\begin{cases} v_j(a) = r_j \\ v_i(a) = \alpha_i > r_i, i \neq j; i = 1, \dots, \gamma \\ v_i(a) > 0, i \neq j; i = \gamma+1, \dots, n. \end{cases}$$

Then, $a \in D_{P_i}$ for all $i = 1, \dots, n$ and $a \in D$, but $a \notin \mathfrak{a}$ since $a \notin \mathfrak{a}_j$.

Let p be an arbitrary element of the prime ideal P_j . Then, $v_j(p) > 0$ and $v_i(p) \geq 0$, $i \neq j$, since $p \in D \subseteq D_{P_i}$.

So:
$$\begin{cases} v_j(ap) = v_j(a) + v_j(p) > r_j \\ v_i(ap) = v_i(a) + v_i(p) > r_i, i \neq j. \end{cases}$$

Note: For $i = \gamma+1, \dots, n$, $r_i = 0$; in the above and subsequent work, we will use this shortened form.

Therefore, $ap \in \mathfrak{a}$ and $a \in \mathfrak{a}:P_j$. But, $\mathfrak{a}:P_j \not\supseteq \mathfrak{a}$ since $a \notin \mathfrak{a}$. Hence, by Lemma 1.6, P_j^e is not dense in $D/\mathfrak{a}$, a contradiction. With P_j^e dense in $D/\mathfrak{a}$, we must conclude that $\mathfrak{a}_j = \{x \in D \mid v_j(x) \geq r_j\}$.

We then ask: Can $\mathfrak{a}_j$ have this form and H_j be discrete? If H_j is discrete, then $\mathfrak{a}_j = \{x \in D \mid v_j(x) \geq r_j, r_j \geq 1\}$. Use the approximation theorem to pick $a \in D \setminus \mathfrak{a}$ such that

$$\begin{cases} v_j(a) = r_j - 1 \\ v_i(a) > r_i, i \neq j; i = 1, \dots, n. \end{cases}$$

Let p be an arbitrary element in P_j . Then, $v_j(p) \geq 1$ and $v_i(p) \geq 0, i \neq j$, the latter holding since $p \in D$.

Then:
$$\begin{cases} v_j(ap) = v_j(a) + v_j(p) \geq r_j - 1 + 1 = r_j \\ v_i(ap) = v_i(a) + v_i(p) > r_i, i \neq j, \end{cases}$$

and we get $ap \in \mathfrak{a}$. Therefore, $aP_j \subseteq \mathfrak{a}$ and $a \in \mathfrak{a}:P_j \setminus \mathfrak{a}$.

So, $\mathfrak{a}:P_j \not\supseteq \mathfrak{a}$ and P_j^e is not dense in $D/\mathfrak{a}$. Equivalently, if P_j^e is dense in $D/\mathfrak{a}$, then H_j is not discrete.

$\Leftarrow$: Assume that $\mathfrak{a}_j = \{x \in D \mid v_j(x) \geq r_j, r_j \in H_j\}$, where H_j is not discrete. To prove that P_j^e is dense in $D/\mathfrak{a}$, we take an arbitrary element $a \in D \setminus \mathfrak{a}$ and show that $a \notin \mathfrak{a}:P_j$.

Two cases arise:

$$(i) \quad v_j(a) < r_j$$

$$(ii) \quad v_j(a) \geq r_j.$$

Case (i): Let $v_j(a) = s \in H_j$; since H_j is not discrete there is an element $t \in H_j$ for which $s < t < r_j$. By the approximation theorem, choose an element $z \in P_j$ such that

$$\begin{cases} v_j(z) = t - s > 0 \\ v_i(z) = 0, i \neq j. \end{cases}$$

Then,

$$\begin{cases} v_j(az) = v_j(a) + v_j(z) = s + (t-s) = t < r_j \\ v_i(az) = v_i(a) + v_i(z) = v_i(a), i \neq j. \end{cases}$$

Since $v_j(az) < r_j$, we get that $az \in D \setminus \mathfrak{a}$, and, hence, $a \notin \mathfrak{a} : P_j$.

Case (ii): Here, $a \in \mathfrak{a}_j$, and hence, since $a \in D \setminus \mathfrak{a}$, there is a k , $1 \leq k \leq n$, such that $a \notin \mathfrak{a}_k$. If

$\mathfrak{a}_k = \{x \in D \mid v_k(x) \geq r_k\}$, then $v_k(a) < r_k$. If

$\mathfrak{a}_k = \{x \in D \mid v_k(x) > r_k\}$, then $v_k(a) \leq r_k$. We will work

with the first form, the argument for the second form

being similar. By the approximation theorem there is an

element z in P_j such that:

$$\begin{cases} v_j(z) > 0 \\ v_k(z) = 0 \\ v_i(z) = 0, i \neq j, k. \end{cases}$$

Then, $v_k(az) = v_k(a) + v_k(z) = v_k(a) < r_k$. Now, in D_{P_k} , if $x \in B$, an ideal, and $v_k(y) \geq v_k(x)$, then $y \in B$. Therefore, since $v_k(az) = v_k(a)$, $az \in \mathfrak{a}'_k$ if and only if $a \in \mathfrak{a}'_k$, where, by Lemma 2.2, $\mathfrak{a}'_k$ is the P'_k -primary ideal in D_{P_k} corresponding to $\mathfrak{a}_k$ in D ($\mathfrak{a}_k = \mathfrak{a}'_k \cap D$). But, $a \notin \mathfrak{a}'_k$, so that $az \notin \mathfrak{a}'_k$ and $az \notin \mathfrak{a}_k$. Hence, for $a \notin \mathfrak{a}$, we have $az \notin \mathfrak{a}_k$. Since $\mathfrak{a} = \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_n$, we then get $az \notin \mathfrak{a}$. With $z \in P_j$, this means that $a \notin \mathfrak{a}:P_j$.

Thus, by cases (i) and (ii), $\mathfrak{a}:P_j = \mathfrak{a}$ and P_j^e is dense in $D/\mathfrak{a}$.

(End of Theorem 2.5)

Since we can reassign subscripts if necessary, without loss of generality, we will work with the following classification of the $\mathfrak{a}_k$'s, $k = 1, \dots, n$.

(i) For $k = 1, \dots, \tau$, $\mathfrak{a}_k = \{x \in D \mid v_k(x) \geq r_k$,

$r_k \neq 0, r_k \in \mathbb{R}\}$:

case 4 primaries.

- (ii) For $k = \tau+1, \dots, \theta$, $\mathfrak{a}_k = \{x \in D \mid v_k(x) > r_k, r_k \neq 0, r_k \in \mathbb{R}\}$:

case 3 primaries.

- (iii) For $k = \theta+1, \dots, \gamma$, $\mathfrak{a}_k = \{x \in D \mid v_k(x) \geq r_k, r_k \geq 2, v_k(x), r_k \in \mathbb{Z}\}$:

case 2 primaries.

- (iv) For $k = \gamma+1, \dots, n$, $\mathfrak{a}_k = \{x \in D \mid v_k(x) > 0 = r_k\}$
 $= P_k$:

case 1 primaries.

In particular, the Case 4 rank one subgroups

$H_1, \dots, H_\tau$ are not discrete.

Corollary to Theorem 2.5: Since $H_1, \dots, H_\tau$ are not discrete, $D/\mathfrak{a}$ is dense and any ideal of the form

$P_{i_1}^e \cap \dots \cap P_{i_\ell}^e$, $1 \leq i_1, \dots, i_\ell \leq \tau$, is dense in $D/\mathfrak{a}$.

Furthermore, no dense ideals are properly contained in

$P_1^e \cap \dots \cap P_\tau^e$, and $P_1^e \cap \dots \cap P_\tau^e$ is the unique minimal dense ideal in $D/\mathfrak{a}$.

Proof: Since $D/\mathfrak{a}$ has an identity, it must be dense in itself. As just shown in Theorem 2.5, since $H_1, \dots, H_\tau$ are

not discrete, $P_1^e, \dots, P_\tau^e$ are all dense in $D/\mathfrak{a}$. Then by Lambek [4], p. 37, each ideal of the form

$P_{i_1}^e \cap \dots \cap P_{i_\ell}^e$, $1 \leq i_1, \dots, i_\ell \leq n$, including

$P_1^e \cap \dots \cap P_\tau^e$, is dense in $D/\mathfrak{a}$.

Now, it is true that

$(P_1 \cap \dots \cap P_\tau)^e = P_1^e \cap \dots \cap P_\tau^e$. Then, we take an ideal

I in D such that

$$\mathfrak{a} \subsetneq I \subsetneq P_1 \cap \dots \cap P_\tau = \{x \in D \mid v_k(x) > 0, k = 1, \dots, \tau; \\ v_\ell(x) \geq 0, \ell = \tau+1, \dots, n\},$$

and show that $\mathfrak{a}:I \not\supseteq \mathfrak{a}$ (that is, I^e is not dense in $D/\mathfrak{a}$).

To accomplish this, we must know how the valuations on elements in I differ from the valuations on elements in

$$P_1 \cap \dots \cap P_\tau.$$

Lemma 2.6: Suppose there exist elements x_i in I such that $v_i(x_i)$ is arbitrarily small for $i \leq \tau$ and $v_i(x_i) = 0$ for $i > \tau$. Then, $I = P_1 \cap \dots \cap P_\tau$.

Proof of Lemma 2.6: Let z be an arbitrary element in

$P_1 \cap \dots \cap P_\tau$. Then, $v_i(z) = s_i > 0$ for $i \leq \tau$, and

$v_i(z) \geq 0$ for $i > \tau$. By assumption, there are elements x_i in I such that

$$\begin{cases} 0 < v_i(x_i) < s_i, & i \leq \tau \\ v_i(x_i) = 0, & i > \tau. \end{cases}$$

By the approximation theorem, there exist elements

$y_1, \dots, y_n \in D$ such that:

$$\begin{aligned} \text{(a)} \quad & \text{For } i = 1, \dots, \tau, \quad \begin{cases} v_i(y_i) = 0 \\ v_i(y_j) > \max\{s_1, \dots, s_\tau\}, & j \neq i; \end{cases} \\ \text{(b)} \quad & \text{For } i = \tau+1, \dots, n, \quad \begin{cases} v_i(y_i) = 0 \\ v_i(y_j) > r_j, & j \neq i. \end{cases} \end{aligned}$$

Now, we look at $x_1 y_1 + \dots + x_n y_n = \sum_1^n x_j y_j$. For i from 1 to τ :

$$\begin{cases} v_i(x_j y_j) = v_i(x_j) + v_i(y_j) > 0 + \max\{s_1, \dots, s_\tau\} \geq s_i, \\ \qquad \qquad \qquad j \neq i \\ v_i(x_i y_i) = v_i(x_i) + v_i(y_i) < s_i + 0 = s_i. \end{cases}$$

Therefore, $v_i(\sum_1^n x_j y_j) = \min_{1 \leq j \leq n} \{v_i(x_j y_j)\} < s_i$,

since $v_i(x_i y_i) < v_i(x_j y_j)$ for $j \neq i$ and $i \leq \tau$. For i from $\tau+1$ to n :

$$\begin{cases} v_i(x_j y_j) = v_i(x_j) + v_i(y_j) > r_j \geq 0, & j \neq i \\ v_i(x_i y_i) = v_i(x_i) + v_i(y_i) = 0. \end{cases}$$

Therefore, $v_i(\sum_1^n x_j y_j) = \min_{1 \leq j \leq n} \{v_i(x_j y_j)\} = 0$,
 since $v_i(x_i y_i) < v_i(x_j y_j)$ for $j \neq i$ and $i > \tau$. Hence,
 by construction, we have that $v_i(\sum_1^n x_j y_j) \leq v_i(z)$ for
 $i = 1, \dots, n$.

Let B be the ideal generated by $\sum_1^n x_j y_j$. Then $B \subseteq I$
 since $\sum_1^n x_j y_j \in I$. Since $v_i(\sum_1^n x_j y_j) \leq v_i(z)$ for
 $i = 1, \dots, n$, we have $z \in B \subseteq I$. Thus, $P_1 \cap \dots \cap P_\tau = I$,
 since z is an arbitrary element in $P_1 \cap \dots \cap P_\tau$.

(End of Lemma 2.6)

Continuing the proof of the corollary, by Lemma 2.6,
 since I is properly contained in $P_1 \cap \dots \cap P_\tau$, either:

- (i) for some j between 1 and τ , $v_j(x)$ must be
 bounded away from 0 for all x in I ; or:
- (ii) for some j between $\tau+1$ and n , $v_j(x)$ must be
 greater than 0 for all x in I .

That is, if (i) holds, there is a positive s_j in $\mathbb{R}$
 such that $v_j(x) \geq s_j$ for all x in D ; if (ii) holds, then
 $v_j(x) > 0$ for all x in D .

For (i), two cases arise for $s_j > 0$, j between 1
 and τ :

$$(ia) \quad r_j \leq s_j$$

Since H_j is not discrete for j between 1 and τ , we use the approximation theorem to get an element $z \notin \mathfrak{a}$ such that

$$\begin{cases} 0 < v_j(z) < r_j \\ v_i(z) > r_i, i \neq j. \end{cases}$$

Then, for arbitrary $x \in I$:

$$\begin{cases} v_j(zx) = v_j(z) + v_j(x) > s_j \geq r_j \\ v_j(zx) > r_i, i \neq j. \end{cases}$$

Therefore, $zx \in \mathfrak{a}$ and $z \in \mathfrak{a} : I \setminus \mathfrak{a}$.

$$(ib) \quad 0 < s_j < r_j$$

Apply the approximation theorem to get an element $z \notin \mathfrak{a}$ such that

$$\begin{cases} r_j - s_j < v_j(z) < r_j \\ v_i(z) > r_i, i \neq j. \end{cases}$$

So, for arbitrary $x \in I$:

$$\begin{cases} v_j(zx) = v_j(z) + v_j(x) > (r_j - s_j) + s_j = r_j \\ v_i(zx) > r_i, i \neq j. \end{cases}$$

Hence, $zx \in \mathfrak{a}$ and $z \in \mathfrak{a} : I \setminus \mathfrak{a}$.

Now, note that for $k = \theta+1, \dots, \gamma$,

$\mathfrak{a}_k = \{x \in D \mid v_k(x) \geq r_k, v_k(x), r_k \in \mathbb{Z}\}$ can be written as $\{x \in D \mid v_k(x) > r_k - 1, v_k(x), r_k \in \mathbb{Z}\}$. Then, for each k from $\tau+1, \dots, n$, $\mathfrak{a}_k$ has the form $\{x \in D \mid v_k(x) > r_k\}$, where $r_k = 0$ for $k = \gamma+1, \dots, n$.

For (ii), use the approximation theorem to get an element $z \notin \mathfrak{a}$ such that

$$\begin{cases} v_j(z) = r_j \\ v_i(z) > r_i, i \neq j. \end{cases}$$

Then, for arbitrary $x \in I$:

$$\begin{cases} v_j(zx) = v_j(z) + v_j(x) > r_j \\ v_i(zx) > r_i, i \neq j. \end{cases}$$

Therefore, $zx \in \mathfrak{a}$ and $z \in \mathfrak{a} : I \setminus \mathfrak{a}$.

From each case we conclude that $\mathfrak{a} : I \not\supseteq \mathfrak{a}$ and I^e is not dense in $D/\mathfrak{a}$ by Lemma 1.6.

Suppose there is an ideal B in $D/\mathfrak{a}$ such that

$0 \subsetneq B \subsetneq (P_1 \cap \dots \cap P_\tau)^e = P_1^e \cap \dots \cap P_\tau^e$. Then, there is an ideal I in D such that $\mathfrak{a} \subsetneq I \subsetneq (P_1^e \cap \dots \cap P_\tau^e)^c = P_1 \cap \dots \cap P_\tau$, and $I^e = B$. By what we have above, $I^e = B$.

is not dense in $D/\mathfrak{a}$. Therefore, $P_1^e \cap \dots \cap P_\tau^e$ is a minimal dense ideal in $D/\mathfrak{a}$.

Finally, we will show that $P_1^e \cap \dots \cap P_\tau^e$ is the minimal dense ideal in $D/\mathfrak{a}$. Assume that J is an ideal in D such that J^e is dense in $D/\mathfrak{a}$. Since D is semi-local with maximal ideals $P_1, \dots, P_n$, J must be contained in one or more of the P_i 's. But, if $J \subseteq P_i$ for i between $\tau+1$ and n , then P_i^e is dense, a contradiction (by Definition 1.10, if $A \subseteq B$ and A is dense, then B must be dense). Hence, J can only be contained in some or all of $P_1, \dots, P_\tau$. Without loss of generality, let $J \subseteq P_1, \dots, P_\mu$, while $J \not\subseteq P_{\mu+1}, \dots, P_\tau$. If $J \not\subseteq P_1 \cap \dots \cap P_\mu$, then either:

- (i) for some j between 1 and μ , $v_j(x)$ is bounded away from 0 for all x in J ; or:
- (ii) for some j between $\tau+1$ and n , $v_j(x)$ must be greater than 0 for all x in J .

We then proceed as above to show that J is not dense in $D/\mathfrak{a}$. If $J = P_1 \cap \dots \cap P_\mu$, then $P_1 \cap \dots \cap P_\tau \subseteq J$ and $P_1^e \cap \dots \cap P_\tau^e$ is the minimal dense ideal in $D/\mathfrak{a}$.

(End of the Corollary to
Theorem 2.5)

CHAPTER III

As in the previous chapter, D is a semi-local Prüfer domain with maximal ideals $P_1, \dots, P_n$, ϕ is a homomorphism of D with kernel $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{a}_i$, an irredundant primary decomposition where $P_j = \sqrt{\mathfrak{a}_j}$, and $D/\mathfrak{a} = Q_{cl}(D/\mathfrak{a})$.

Earlier we considered four cases for G_j , as to whether or not it possesses a rank one subgroup. In light of Theorem 2.5, we return to these cases, relating them to dense ideals for the purpose of investigating $Q(D/\mathfrak{a})$.

Cases 1a and 1b: $\mathfrak{a}_j = P_j = \{x \in D \mid v_j(x) > 0\}$; by Theorem 2.5, P_j^e is not dense in $D/\mathfrak{a}$.

Case 2: $H_j \cong \mathbb{Z}$; by Theorem 2.5, P_j^e is not dense in $D/\mathfrak{a}$.

Case 3: $H_j \not\cong \mathbb{Z}$ and $\mathfrak{a}_j = \{x \in D \mid v_j(x) > r_j\}$; by Theorem 2.5, P_j^e is not dense in $D/\mathfrak{a}$.

Case 4: $H_j \not\cong \mathbb{Z}$ and $\mathfrak{a}_j = \{x \in D \mid v_j(x) \geq r_j\}$; by Theorem 2.5, P_j^e is dense in $D/\mathfrak{a}$.

In what will follow concerning the complete quotient ring of $D/\mathfrak{a}$, we are interested in the order group G_j if P_j^e is dense in $D/\mathfrak{a}$.

By the Corollary to Theorem 2.5, $P_1^e \cap \dots \cap P_\tau^e$ is the minimal dense ideal in $D/\mathfrak{a}$. Letting $A = P_1 \cap \dots \cap P_\tau$, we have $A^e = P_1^e \cap \dots \cap P_\tau^e$. Then, recall the following classification of the $\mathfrak{a}_k$'s, $k = 1, \dots, n$:

- (a) For $k = 1, \dots, \tau$, $\mathfrak{a}_k = \{x \in D \mid v_k(x) \geq r_k, r_k \neq 0, r_k \in \mathbb{R}\}$: case 4 primaries corresponding to prime ideals in the dense ideal A^e .
- (b) For $k = \tau+1, \dots, \theta$, $\mathfrak{a}_k = \{x \in D \mid v_k(x) > r_k, r_k \neq 0, r_k \in \mathbb{R}\}$: case 3 primaries.
- (c) For $k = \theta+1, \dots, \gamma$, $\mathfrak{a}_k = \{x \in D \mid v_k(x) \geq r_k, r_k \geq 2, v_k(x), r_k \in \mathbb{Z}\}$: case 2 primaries where $H_k \cong \mathbb{Z}$.
- (d) For $k = \gamma+1, \dots, n$, $\mathfrak{a}_k = \{x \in D \mid v_k(x) > 0 = r_k\} = P_k$: case 1 primaries.

If $x \in D$, let $\bar{x}$ be the image of x under the canonical map $D \rightarrow D/\mathfrak{a}$.

Lemma 3.1: Let $\mathfrak{a} = \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_n$, with D given as before. If $x, x' \notin \mathfrak{a}_k$ and if $\bar{x} = \bar{x}'$ in $D/\mathfrak{a}$, then $v_k(x) = v_k(x')$, k fixed.

Proof: If $x, x' \notin \mathfrak{a}_k$, then $\bar{x}, \bar{x}' \neq \bar{0}$ in $D/\mathfrak{a}$. Then, for

$\overline{x} = \overline{x'}$, we have $x + \mathfrak{a} = x' + \mathfrak{a}$, or

$x - x' \in \mathfrak{a} = \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_n \subseteq \mathfrak{a}_k$. So:

If k is one of $1, \dots, \tau$ or $\theta+1, \dots, \gamma$, then $x - x' \in \mathfrak{a} \subseteq \mathfrak{a}_k$ implies that $v_k(x-x') \geq r_k$.

If k is one of $\tau+1, \dots, \theta$ or $\gamma+1, \dots, n$, then $x - x' \in \mathfrak{a} \subseteq \mathfrak{a}_k$ implies that $v_k(x-x') > r_k$.

Hence, $v_k(x-x') \geq r_k$ for all k .

If $v_k(x) \neq v_k(x')$, then both $v_k(x) \geq r_k$ and $v_k(x') \geq r_k$ since $v_k(x-x') = \min\{v_k(x), v_k(x')\} \geq r_k$.

But then, both x and x' are in $\mathfrak{a}_k$, a contradiction. Therefore, $v_k(x) = v_k(x')$ for fixed k .

(End of Lemma 3.1)

Now, if $f \in Q(D/\mathfrak{a})$, then $\text{dom} f$ is a dense ideal in $D/\mathfrak{a}$. Since $A^e = P_1^e \cap \dots \cap P_\tau^e$ is the minimal dense ideal in $D/\mathfrak{a}$, $\text{dom} f \supseteq A^e$ for arbitrary f in $D/\mathfrak{a}$.

Theorem 3.2: Let $f \in Q(D/\mathfrak{a})$ be given. For

$\overline{x} \in \text{dom} f \supseteq A^e$, let $\overline{y} = f\overline{x}$, where y is a preimage of $\overline{y}$.

Then, if there is a k , $1 \leq k \leq n$, such that $v_k(y) < v_k(x)$, we must have $x \in \mathfrak{a}_k$.

Proof: First, if k is one of $\gamma+1, \dots, n$, then

$\mathfrak{a}_k = \{z \in D \mid v_k(z) > 0\}$. Let $v_k(y) < v_k(x)$ for k one of $\gamma+1, \dots, n$. Then, since $y \in D$, $v_k(y) \geq 0$, and $v_k(y) < v_k(x)$ implies that $v_k(x) > 0$. But, this means that $x \in \mathfrak{a}_k$.

Next, suppose that k is one of $1, \dots, \gamma$. Then, if $v_k(y) < v_k(x)$, we have that $x \in (y)$ in D_{P_k} , and, hence, there is an element $b_k \in D_{P_k}$ such that $x = b_k y$, since D_{P_k} is a valuation domain. And, $v_k(x) = v_k(b_k y) = v_k(b_k) + v_k(y)$, so that $v_k(y) < v_k(x)$ implies that $v_k(b_k) > 0$. By the approximation theorem, Theorem 2.4, there is an element b' in $Q_{cl}(D)$ such that

$$\begin{cases} v_k(b') = v_k(b_k) > 0 \\ v_j(b') > 0, j = 1, \dots, n, j \neq k; \end{cases}$$

and, $b' \in D$. Now, for $j = 1, \dots, \gamma$ (including the fixed k), $\mathfrak{a}_j \neq P_j$, and there exists a positive integer q_j such that $q_j \cdot v_j(b') > r_j$, since $v_j(b') > 0$. Therefore, $v_j(b'^{q_j}) > r_j$, and $b'^{q_j} \in \mathfrak{a}_j$. For $j = \gamma+1, \dots, n$, $\mathfrak{a}_j = P_j$, and $v_j(b') > 0$ implies that $b' \in \mathfrak{a}_j$. Let $M = \max(q_1, \dots, q_\gamma)$. Then, $b'^M \in \mathfrak{a}_j$ for all $j = 1, \dots, n$,

or $b'^M \in \mathfrak{a} = \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_n$. So, $\overline{b'^M} = \overline{0}$ in $D/\mathfrak{a}$. Since $v_j(b') > 0$ for $j = 1, \dots, r$, $b' \in A$ and $\overline{b'} \in A^e \subseteq \text{dom} f$. Therefore, $\overline{b'^M} \in \text{dom} f$ and

$$\begin{aligned} \overline{0} &= f^M \overline{0} \\ &= f^M \overline{b'^M} \\ &= (f \overline{b'})^M; \end{aligned}$$

hence, $f \overline{b'}$ is nilpotent in $D/\mathfrak{a}$.

Since $v_k(b') = v_k(b_k)$, there is a unit u_k in D_{P_k} such that $b_k = u_k b'$, where $u_k = \alpha_k / \beta_k$, $\alpha_k, \beta_k \in D \setminus P_k$. Then, $x/y = b_k = (\alpha_k / \beta_k) \cdot b'$ in D_{P_k} , or: $\beta_k x = \alpha_k b' y$ in D . Therefore, in $D/\mathfrak{a}$,

$$\begin{aligned} \overline{\beta_k x} &= \overline{\alpha_k b' y} \\ &= \overline{\alpha_k b' f x} \\ &= \overline{x} \overline{\alpha_k f b'}, \end{aligned}$$

since $\overline{b'} \in A^e \subseteq \text{dom} f$.

$$\begin{aligned} \text{Hence, } \overline{\beta_k^M x} &= \overline{\beta_k^{M-1} x \overline{\alpha_k f b'}} \\ &\quad \vdots \\ &= \overline{x} \overline{\alpha_k^M (f b')^M} = \overline{0}, \end{aligned}$$

since $(f b')^M = \overline{0}$ and $\overline{\beta_k x} = \overline{x} \overline{\alpha_k f b'}$.

Therefore, we have shown $\overline{\beta_k^M} \overline{x} = \overline{0}$ and, hence,

$\overline{\beta_k^M} \overline{x} \in \mathfrak{a}_k^e$. Since $\beta_k \in D \setminus P_k$, we have that $\overline{\beta_k} \notin P_k^e$ and

$\overline{\beta_k^M} \notin P_k^e$ (P_k^e being prime in $D/\mathfrak{a}$). But, then, $\overline{x} \in \mathfrak{a}_k^e$,

since $\mathfrak{a}_k^e$ is P_k^e -primary. Therefore, $x \in \mathfrak{a}_k$, as was to be shown.

(End of Theorem 3.2)

Equivalently, if there is a k , $1 \leq k \leq n$, and there is an element $x \notin \mathfrak{a}_k$, where $\overline{x} \in \text{dom} f$, then

$v_k(y) \geq v_k(x)$, where $\overline{y} = f\overline{x}$.

Corollary to Theorem 3.2: Let $f \in Q(D/\mathfrak{a})$ be given. Then, for each k from 1 to γ , there exists an element $\overline{x}_k \in \text{dom} f$ such that $v_k(y_k) \geq v_k(x_k)$, where $\overline{y}_k = f\overline{x}_k$.

Proof: Fix k between 1 and γ . Suppose that for all elements $\overline{x}$ in $\text{dom} f$, we have that $v_k(y) < v_k(x)$, where $\overline{y} = f\overline{x}$. Then, by Theorem 3.2, $\overline{x} \in \mathfrak{a}_k^e$ for all $\overline{x}$ in $\text{dom} f$. Therefore, $A^e = P_1^e \cap \dots \cap P_\tau^e \subseteq \text{dom} f \subseteq \mathfrak{a}_k^e$, or, equivalently, $A = P_1 \cap \dots \cap P_\tau \subseteq \mathfrak{a}_k$. We will now show that $A \subseteq \mathfrak{a}_k$ implies that $\mathfrak{a}_k$ must equal P_k . Assume the contrary: $\mathfrak{a}_k \subsetneq P_k$. Then, by the approximation theorem,

there is an element $c \in Q_{cl}(D)$ such that:

$$\begin{cases} 0 < v_k(c) < r_k \\ v_j(c) > 0, \quad j \neq k. \end{cases}$$

Hence, $c \in P_1 \cap \dots \cap P_\tau$, but $c \notin \mathfrak{a}_k$, a contradiction to $P_1 \cap \dots \cap P_\tau \subseteq \mathfrak{a}_k$. So, $\mathfrak{a}_k = P_k$. But this, in turn, contradicts the condition that for $k = 1, \dots, \gamma$ we have $\mathfrak{a}_k \subsetneq P_k$. Thus, for our fixed k , there must exist an element $\bar{x}_k \in \text{dom} f$ such that $v_k(y_k) \geq v_k(x_k)$, where $\bar{y}_k = f\bar{x}_k$.

(End of the Corollary)

Note: For $k = \gamma+1, \dots, n$, if $v_k(x_k) = 0$, then, automatically, $v_k(y_k) \geq v_k(x_k)$, where $\bar{y}_k = f\bar{x}_k$.

Theorem 3.3: Let $\bar{x}, \bar{x}^* \in \text{dom} f \supseteq A^e$, where $\bar{y} = f\bar{x}$ and $\bar{y}^* = f\bar{x}^*$, and where y and y^* are respective preimages of $\bar{y}$ and $\bar{y}^*$ under the map $D \rightarrow D/\mathfrak{a}$. Then, if there exists a k , $1 \leq k \leq n$, such that $x, x^*, y, y^* \notin \mathfrak{a}_k$, we must have $v_k(y^*) - v_k(x^*) = v_k(y) - v_k(x)$. That is, the difference $v_k(y) - v_k(x)$ is invariant for all pairs $\langle \bar{x}, \bar{y} \rangle$ associated with f , where $x, y \notin \mathfrak{a}_k$.

Proof: For $x, x^* \notin \mathfrak{a}_k$, Theorem 3.2 implies that

$v_k(y) \geq v_k(x)$ and $v_k(y^*) \geq v_k(x^*)$ for fixed k , $1 \leq k \leq n$.

Therefore, $v_k(y) - v_k(x) \geq 0$ and $v_k(y^*) - v_k(x^*) \geq 0$ for the fixed k . If k lies between 1 and γ , we may assume without loss of generality that $v_k(x) \leq v_k(x^*)$ and $x^* \in x D_{P_k}$. Therefore, there exists $c_k \in D_{P_k}$ such that $x^* = c_k x$. And: $v_k(c_k) \geq 0$. By the approximation theorem there exists $c' \in Q_{c\ell}(D)$ such that:

$$\begin{cases} v_k(c') = v_k(c_k) \geq 0 \\ v_j(c') \geq 0, j = 1, \dots, n, j \neq k. \end{cases}$$

And, $c' \in D$. Now, since $v_k(c') = v_k(c_k)$, there exists a unit u_k in D_{P_k} such that $c_k = u_k c'$, where $u_k = \alpha_k / \beta_k$ for $\alpha_k, \beta_k \in D \setminus P_k$.

So, $x^*/x = c_k = (\alpha_k / \beta_k) c'$ in D_{P_k} , and $\beta_k x^* = \alpha_k x c'$ in D . Then, since $\bar{\beta}_k \in D/\mathfrak{a}$ and $\bar{x}^* \in \text{domf}$, we have that $\bar{\beta}_k \bar{x}^* \in \text{domf}$. Similarly, since $\bar{\alpha}_k \bar{c}' \in D/\mathfrak{a}$ and $\bar{x} \in \text{domf}$, we have that $\bar{\alpha}_k \bar{x} \bar{c}' \in \text{domf}$. But, $\beta_k x^* = \alpha_k x c'$. Hence, $f\bar{\beta}_k \bar{x}^* = f\bar{\alpha}_k \bar{x} \bar{c}'$, and $\bar{\beta}_k \bar{y}^* = \bar{\alpha}_k \bar{c}' \bar{y}$.

Since, for the fixed k , $\beta_k y^* \notin \mathfrak{a}_k$ and $\alpha_k c' y \notin \mathfrak{a}_k$, we have $v_k(\beta_k y^*) = v_k(\alpha_k c' y)$ by Lemma 3.1; hence

$$v_k(\beta_k) + v_k(y^*) = v_k(\alpha_k) + v_k(c') + v_k(y).$$

But, $v_k(\beta_k) = 0$ and $v_k(\alpha_k) = 0$. Therefore,

$$\begin{aligned}
v_k(y^*) &= v_k(c') + v_k(y) \\
&= v_k(c_k) + v_k(y) \\
&= v_k(c_k y).
\end{aligned}$$

So,

$$\begin{aligned}
v_k(x^*) + v_k(y) &= v_k(x^* y) \\
&= v_k(c_k x y) \\
&= v_k(x) + v_k(c_k y) \\
&= v_k(x) + v_k(y^*).
\end{aligned}$$

Hence, $v_k(y^*) - v_k(x^*) = v_k(y) - v_k(x)$, k between 1 and γ , and the v_k -difference for f is invariant for all $\bar{x}, \bar{x}^* \in \text{dom} f \setminus \mathfrak{a}_k^e$ for which we also have $\bar{y}, \bar{y}^* \notin \mathfrak{a}_k^e$, k between 1 and γ .

For k between $\gamma+1$ and n , if $x, x^* \notin \mathfrak{a}_k$, then $v_k(x) = 0$ and $v_k(x^*) = 0$. But, by hypothesis, $y, y^* \notin \mathfrak{a}_k$ and, therefore, $v_k(y) = 0$ and $v_k(y^*) = 0$. Hence, $v_k(y^*) - v_k(x^*) = 0 = v_k(y) - v_k(x)$ for k between $\gamma+1$ and n .

Therefore, letting the v_k -difference $v_k(y) - v_k(x)$ be represented by $v_k[f]$, we have that $v_k[f]$ is invariant for all pairs $\langle \bar{x}, \bar{y} \rangle$ associated with f , where $x, y \notin \mathfrak{a}_k$.

(End of Theorem 3.3)

We come now to a fundamental theorem characterizing all elements in $Q(D/\mathfrak{a})$. Later, we will get a sharper result, one stating a necessary and sufficient condition for an element to be in $Q(D/\mathfrak{a})$.

Theorem 3.4: Let $f \in Q(D/\mathfrak{a})$, where $\mathfrak{a}_k$, $\mathfrak{a}$, and D are given as before. Then, there is a positive integer N , elements a and c_N in D , and a sequence of elements

$\langle b_j \rangle_{j=N}^{\infty}$ in D such that:

$$f\bar{x} = \left(\prod_{j=N}^{i-1} \bar{b}_j \right) \bar{c}_N \bar{a} \bar{x}, \text{ where } v_k(x) > 1/i$$

for $k = 1, \dots, \tau$.

Proof: Let $f \neq \bar{0}$ be an element in $Q(D/\mathfrak{a})$. Pick a positive integer W such that $1/W < \min_{k=1, \dots, \tau} \{r_k\}$. Pick a sequence

$\langle \bar{x}_\ell \rangle_{\ell \geq W}$ in $D/\mathfrak{a}$ such that $0 < v_k(x_\ell) < 1/\ell$ and

$v_k(x_{\ell+1}) < v_k(x_\ell)$ for $k = 1, \dots, \tau$, and $v_k(x_\ell) = 0$ for

$k = \tau+1, \dots, n$. This is possible, using the approximation theorem, since $H_1, \dots, H_\tau$ are not discrete (the latter by Theorem 2.5). Further, for $\ell \geq W$, let $\bar{y}_\ell = f\bar{x}_\ell$, where y_ℓ is a preimage of $\bar{y}_\ell$ under the map $D \rightarrow D/\mathfrak{a}$. Note that by our choice of valuations, $x_\ell \notin \mathfrak{a}_k$ for $\ell \geq W$ and $k = 1, \dots, n$.

Lemma 3.5: Let c and d be elements of D , where $\bar{c} \neq \bar{0}$ in $D/\mathfrak{a}$, such that $v_k(c) \leq v_k(d)$ for $k = 1, \dots, n$. Then, there is an element s in D for which $d = sc$.

Proof of Lemma 3.5: Since $v_k(c) \leq v_k(d)$ for $k = 1, \dots, n$, $d \in (c)$ in D_{P_k} . Therefore, there exists $s_k \in D_{P_k}$ such that $d = s_k c$, where $v_k(s_k) \geq 0$, $k = 1, \dots, n$. Let $g \neq h$ be two arbitrary subscripts taken from 1 through n . Since $s_g c = d = s_h c$ in $Q_{c\ell}(D)$, we have that $(s_g - s_h)c = 0$. Then, $\bar{c} \neq \bar{0}$ in $D/\mathfrak{a}$ implies that $c \neq 0$ in D . Therefore, $s_g - s_h$ must be 0, or $s_g = s_h$. Since g and h are arbitrary, $s_1 = \dots = s_n$ for $s_k \in D_{P_k}$; call this common element s . Then, $s \in D_{P_1} \cap \dots \cap D_{P_n} = D$ and $d = sc$ in D .

(End of Lemma 3.5)

Lemma 3.6: Let $\langle \bar{x}_\ell \rangle_{\ell \geq W}$ be given as above. Then, since $f \neq \bar{0}$, there is a positive integer M such that for each $\ell \geq M$, there must exist a k , dependent on ℓ , $1 \leq k \leq n$, such that $f\bar{x}_\ell \notin \mathfrak{a}_k^e$.

Proof of Lemma 3.6: Assume the contrary: for all positive integers M there are an infinite number of subscripts $\ell \geq M$ such that $f\bar{x}_\ell = \bar{0}$ for all $k = 1, \dots, n$. Then for an arbitrary positive integer M and for arbitrary

$\bar{x} \in \text{dom} f$, $v_k(x) > 0$ for $k = 1, \dots, \tau$. Therefore, pick $i \geq M$ such that $v_k(x) > 1/i$ for $k = 1, \dots, \tau$, and where $f\bar{x}_i = \bar{0}$ (the latter by hypothesis). Then, by choice of the elements in $\langle \bar{x}_\ell \rangle_{\ell \geq W}$, we have $v_k(x) > 1/i \geq 1/j > v_k(x_j)$ for $k = 1, \dots, \tau$ and $j \geq i$. For $k = \tau+1, \dots, n$, $v_k(x) \geq 0 = v_k(x_j)$. Applying Lemma 3.5, there is an element s_j in D such that $x = s_j x_j$.

Since $f\bar{x}_j = \bar{0}$ for some $j \geq i$, by assumption, $f\bar{x} = f\bar{s}_j \bar{x}_j = \bar{s}_j \cdot f\bar{x}_j = \bar{0}$. Then, since $\bar{x}$ is arbitrary, $f \equiv \bar{0}$, a contradiction.

(End of Lemma 3.6)

We are now able to show a relationship between f and successive elements in the sequence $\langle \bar{x}_\ell \rangle_{\ell \geq W}$.

Lemma 3.7: If $y_\ell \notin \mathfrak{a}_k$ for a fixed k , $1 \leq k \leq n$, then we must have $y_{\ell+1} \notin \mathfrak{a}_k$.

Proof of Lemma 3.7: By construction of $\langle \bar{x}_\ell \rangle_{\ell \geq W}$,

$v_k(x_{\ell+1}) < v_k(x_\ell)$ for $k = 1, \dots, \tau$, and

$v_k(x_{\ell+1}) = 0 = v_k(x_\ell)$ for $k = \tau+1, \dots, n$. Therefore, by

Lemma 3.5, there exists s_ℓ in D such that

$x_\ell = s_\ell \cdot x_{\ell+1}$ in D . Suppose that $y_{\ell+1} \in \mathfrak{a}_k$ for a fixed

k. Then:

$$f\bar{x}_\ell = f\bar{s}_\ell \bar{x}_{\ell+1} = \bar{s}_\ell \cdot f\bar{x}_{\ell+1} \in \mathfrak{a}_k^e,$$

since $f\bar{x}_{\ell+1} = \bar{y}_{\ell+1} \in \mathfrak{a}_k^e$ by assumption. So, $y_\ell \in \mathfrak{a}_k$, a contradiction. Hence, if $y_\ell \notin \mathfrak{a}_k$, we must have that $y_{\ell+1} \notin \mathfrak{a}_k$, k fixed.

(End of Lemma 3.7)

Corollary to Lemma 3.7: If $y_\ell \notin \mathfrak{a}_k$ for a fixed k , then $y_m \notin \mathfrak{a}_k$ for all $m \geq \ell$.

For a given k , $1 \leq k \leq n$, we may have the following conditions arise, where $f\bar{x}_\ell = \bar{y}_\ell$, still referring to the sequence $\langle \bar{x}_\ell \rangle_{\ell \geq W}$:

- (ia) $y_\ell \notin \mathfrak{a}_k$ for all $\ell \geq W$.
- (ib) There exists a positive integer M_k , $M_k > W$, such that $y_1, \dots, y_{M_k-1} \in \mathfrak{a}_k$, but $y_{M_k} \notin \mathfrak{a}_k$.
- (ii) $y_\ell \in \mathfrak{a}_k$ for all $\ell \geq W$.

For condition (ib), the Corollary to Lemma 3.7 guarantees that if there is a positive integer M_k for which $y_{M_k} \notin \mathfrak{a}_k$, then $y_m \notin \mathfrak{a}_k$ for all $m \geq M_k$, k fixed.

Let $N^* = \max_{1 \leq k \leq n} \{M_k, W\}$.

Then, for a fixed k , $1 \leq k \leq n$, one of two conditions must hold regarding $\langle \bar{x}_\ell \rangle_{\ell \geq N^*}$, where $\bar{y}_\ell = f\bar{x}_\ell$; either:

(i) $y_\ell \notin \mathfrak{a}_k$, and thus $\bar{y}_\ell \notin \mathfrak{a}_k^e$, for all $\ell \geq N^*$;

or

(ii) $y_\ell \in \mathfrak{a}_k$, and thus $\bar{y}_\ell \in \mathfrak{a}_k^e$, for all $\ell \geq N^*$.

Since $f \neq \bar{0}$, condition (i) must hold for at least one k .

Further, for each $\ell \geq N^*$, $x_\ell \notin \mathfrak{a}_k$, $k = 1, \dots, n$, since $N^* \geq W$.

If k lies between 1 and τ , without loss of generality we will let condition (i) hold for $k = 1, \dots, \delta$, and we will let condition (ii) hold for $k = \delta+1, \dots, \tau$.

Then, let $\mathfrak{L} = \{k | 1 \leq k \leq n, k \text{ satisfies condition (i)}\}$, and let $\mathfrak{M} = \{k | 1 \leq k \leq n, k \text{ satisfies condition (ii)}\}$. Hence, $\{1, \dots, \delta\} \subseteq \mathfrak{L}$ and $\{\delta+1, \dots, \tau\} \subseteq \mathfrak{M}$.

By Theorem 3.3, for $k \in \mathfrak{L}$ we have that

$v_k(y_{\ell+1}) - v_k(x_{\ell+1}) = v_k(y_\ell) - v_k(x_\ell) = v_k[f] < r_k$ for all $\ell \geq N^*$; that is, the v_k -difference $v_k[f]$ is invariant for all $\ell \geq N^*$.

We are now able to say how large we want N to be (see the statement of Theorem 3.4). Let Y be a positive integer such that

$$1/Y < \min_{1 \leq k \leq \tau} \left\{ \min_{k \in \mathcal{L}} \{r_k - v_k[f]\}, \min_{k \in \mathcal{M}} \{r_k\} \right\}$$

and then let $N = \max_{1 \leq k \leq n} \{N^*, Y\}$.

Upon picking the sequence $\langle \bar{x}_\ell \rangle_{\ell \geq N \geq W}$, we fixed the elements x_ℓ in D , $\bar{x}_\ell$ in $D/\mathfrak{a}$, and $\bar{y}_\ell$ in $D/\mathfrak{a}$. For $k \in \mathcal{L}$, the y_ℓ 's are bound by Theorem 3.3 to satisfy the v_k -difference property. But, for $k \in \mathcal{M}$ the y_ℓ 's are not fixed at this point. That is, for $k \in \mathcal{M}$, we may replace y_ℓ by y'_ℓ provided that $\bar{y}'_\ell = \bar{y}_\ell = f\bar{x}_\ell$. So, for $k \in \mathcal{M}$, we will use an inductive argument to show that the y_ℓ 's may be chosen such that both

$$(1) \quad v_k(y_{\ell+1}) - v_k(x_{\ell+1}) > v_k(y_\ell) - v_k(x_\ell),$$

and

$$(2) \quad v_k(y_{\ell+2}) - v_k(x_{\ell+2}) - (v_k(y_{\ell+1}) - v_k(x_{\ell+1})) \\ > v_k(y_{\ell+1}) - v_k(x_{\ell+1}) - (v_k(y_\ell) - v_k(x_\ell)),$$

where $\bar{y}_\ell = f\bar{x}_\ell$ for all $\ell \geq N$. That is, not only do we want the successive v_k -differences to be larger, but we also want each difference to exceed the preceding one by

by an ever increasing amount. Thus, letting

$v_k(y_\ell) - v_k(x_\ell) = d_\ell$, we want both $d_{\ell+1} > d_\ell$ and

$d_{\ell+2} - d_{\ell+1} > d_{\ell+1} - d_\ell$ for $\ell \geq N$. Note: (2) may be

rewritten in the form:

$$(2') \quad v_k(y_{\ell+2}) > 2v_k(y_{\ell+1}) - v_k(y_\ell) + v_k(x_{\ell+2}) \\ - 2v_k(x_{\ell+1}) + v_k(x_\ell).$$

Now, let $y'_N = y_N$. Suppose for $N < \ell < L$ and for $k \in \mathfrak{M}$ we have chosen y'_ℓ such that:

$$(3) \quad (i) \quad v_k(y'_\ell) > v_k(y'_{\ell-1}); \\ (ii) \quad v_k(y'_\ell) > 2v_k(y'_{\ell-1}) - v_k(y'_{\ell-2}) + v_k(x_\ell) \\ - 2v_k(x_{\ell-1}) + v_k(x_{\ell-2}); \\ (iii) \quad f\bar{x}_\ell = \bar{y}'_\ell.$$

Note: The second part of the assumption only applies when $\ell \geq N + 2$.

To show that there is an element y'_L which satisfies conditions (3), we will need to use the approximation theorem to guarantee the existence of certain elements z and z' such that $z + z'$ is a unit. However, without additional work this cannot be done now, since it is possible for $j, j' \in \mathfrak{M}$ that some proper prime ideal of

D_{P_j} coincides with a proper prime ideal of $D_{P_{j'}}$. That is, v_j and $v_{j'}$ are dependent valuations.

Let $y'_{L-1} = y$ for convenience of notation. Then, by definition of $\mathfrak{M}$, $y \in P_j$ for all $j \in \mathfrak{M}$. Define:

$$\begin{cases} P_{[i]} = P_i & \text{for } i \in \mathcal{L}; \\ P_{[j]} = \bigcap_{\substack{P \subseteq P_j \\ y \in P}} P & \text{for } j \in \mathfrak{M}, \text{ where } P \text{ is a prime ideal in } D. \end{cases}$$

Since D is a Prüfer domain, D_{P_j} is a valuation domain for each $j \in \mathfrak{M}$ by Theorem 1.1. Hence, by Definition 1.5, the ideals in D_{P_j} are linearly ordered, and, by the one-to-one order preserving correspondence of Lemma 2.2, the prime ideals P defining $P_{[j]}$ are linearly ordered. Then, it follows immediately that $P_{[j]}$ is a prime ideal of D containing y . We then form the localization $D_{P_{[j]}}$ for

$j \in \mathfrak{M}$, and $D_{P_j} \subseteq D_{P_{[j]}}$ since $P_{[j]} \subseteq P_j$. For $i \in \mathcal{L}$,

$D_{P_i} = D_{P_{[i]}}$. Note that if some proper prime ideal of D_{P_j}

containing y is also a prime ideal of $D_{P_{j'}}$, $j \neq j'$, then

$$P_{[j]} = P_{[j']} \text{ and } D_{P_{[j]}} = D_{P_{[j]'}}.$$

By Theorem 1.2, let $v_{[j]}$ be the valuation on $Q_{cl}(D)$ determined by $D_{P_{[j]}}$. Therefore, if $D_{P_{[j]}} = D_{P_{[j'()]}}$ for $j, j' \in \mathfrak{M}$, the valuation $v_{[j]}$ is identical to the valuation $v_{[j'()]}$. As in Chapter II, we let $G_{[j]}$ be the valuation group corresponding to $v_{[j]}$. Next, let $D' = \bigcap_{k=1, \dots, n} D_{P_{[k]}}$. Since $D \subseteq D'$, D' is a Prüfer domain. By Lemma 2.2, D' is a semi-local domain with maximal ideals $P_{[k]}^{D'}$, $k = 1, \dots, n$. Further, $P_{[k]}^{D'} = P_{[k'()] }^{D'}$ if and only if $P_{[k]} = P_{[k'()]}$. Hence, for $j, j' \in \mathfrak{M}$, $P_{[j]}^{D'} = P_{[j'()] }^{D'}$ if and only if $P_{[j]} = P_{[j'()]}$, but, for $i, i' \in \mathfrak{L}$, $P_{[i]}^{D'} \neq P_{[i'()] }^{D'}$ since $P_{[i]} \neq P_{[i'()]}$. Since $y \in P_{[j]} \subseteq P_{[j]}^{D_{P_{[j]}}}$, we have that $v_{[j]}(y) > 0$ for all $j \in \mathfrak{M}$.

We then claim that, for each k from 1 to n , either $v_{[k]}(y) = 0$ or there is a rank one ordered subgroup $H_{[k]}$ of $G_{[k]}$ such that $v_{[k]}(y) \in H_{[k]}$. Let

$$H_{[k]} = \{h \in G_{[k]} \mid \text{there is a positive integer } q \text{ such that } h < q \cdot v_{[k]}(y)\},$$

where $v_{[k]}(y) > 0$. Then, $H_{[k]}$ is an isolated subgroup of

$G_{[k]}$ containing $v_{[k]}(y)$, and $H_{[k]} \neq 0$ for $k \in \mathfrak{M}$ since $v_{[k]}(y) > 0$ for $k \in \mathfrak{M}$. Assume that $H_{[k]}$ is not of rank one. Then, there is an isolated subgroup $H_{[k]}^*$ of $G_{[k]}$, where $0 \subsetneq H_{[k]}^* \subsetneq H_{[k]}$. And, $v_{[k]}(y) \notin H_{[k]}^*$. By Theorem 5.17, p. 111, Larsen and McCarthy [5], there is a one-to-one order reversing correspondence between isolated subgroups of $G_{[k]}$ and proper prime ideals of $D_{P[k]}$. Therefore, we have

$$\begin{array}{ccccc} B' & \subsetneq & P_{[k]} & D_{P[k]} & \subsetneq & D_{P[k]} \\ \updownarrow & & & \updownarrow & & \\ H_{[k]}^* & \supsetneq & 0, & & & \end{array}$$

where $B' = \{c \in D_{P[k]} \mid v_{[k]}(c) \notin H_{[k]}^*\}$ is a prime ideal in $D_{P[k]}$. Then, $y \in B'$ since $v_{[k]}(y) \notin H_{[k]}^*$. By Lemma 2.2, there is a prime ideal B of D such that $B = B' \cap D$. And $B \subsetneq P_{[k]}$. But then, since $y \in D$ and $y \in B'$, it follows that $y \in B \subsetneq P_{[k]}$, a contradiction to the definition of $P_{[k]}$. Therefore, $H_{[k]}$ is a rank one ordered subgroup of $G_{[k]}$. As in Chapter II, $H_{[k]}$ is isomorphic to an additive subgroup of the real numbers.

Now, since we are seeking an element y_L' that satisfies conditions (3), it suffices to show that such an element satisfies the inequality

$v_k(y_L') > 2 \cdot v_k(y_{L-1}')$ for each $j \in \mathfrak{M}$. This is true because

$$\begin{aligned}
 & 2v_k(y_{L-1}') - v_k(y_{L-2}') + v_k(x_L) - 2v_k(x_{L-1}) \\
 & \quad + v_k(x_{L-2}) \\
 & = 2v_k(y_{L-1}') - (v_k(y_{L-2}') - v_k(x_{L-2})) \\
 & \quad - (v_k(x_{L-1}) - v_k(x_L)) - v_k(x_{L-1}) \\
 & < 2v_k(y_{L-1}').
 \end{aligned}$$

For $v_{[k]}(y) > 0$, by definition of $H_{[k]}$, there are elements z in $D_{P_{[k]}}$ such that $v_{[k]}(z) > 2v_{[k]}(y)$. In particular, this is true for $k \in \mathfrak{M}$. So, define

$$C_{[k]} = \begin{cases} a_k D' & \text{if } k \in \mathfrak{L}; \\ \{z \in D' \mid v_{[k]}(z) > 2v_{[k]}(y) \\ \quad = 2v_{[k]}(y_{L-1}')\} & \text{if } k \in \mathfrak{M}. \end{cases}$$

If $P_{[j]} = P_{[j']}$ for $j, j' \in \mathfrak{M}$, then $v_{[j]}$ is identical to $v_{[j']}$ and $C_{[j]} = C_{[j']}$. So, let $C_{[k_1]}, \dots, C_{[k_m]}$ be the

distinct sets defined above: $C_{[k_i]} \neq C_{[k_j]}$ if and only if $k_i \neq k_j$. Then, for $k_i \neq k_j$ we will show that

$$C_{[k_i]} + C_{[k_j]} = D'.$$

By definition, either

$$C_{[k_i]} = \{z \in D' \mid v_{[k_i]}(z) \geq \omega_{k_i}\}$$

or

$$C_{[k_i]} = \{z \in D' \mid v_{[k_i]}(z) > \omega_{k_i}\},$$

ω_{k_i} not infinite, and either

$$C_{[k_j]} = \{z \in D' \mid v_{[k_j]}(z) \geq \omega_{k_j}\}$$

or

$$C_{[k_j]} = \{z \in D' \mid v_{[k_j]}(z) > \omega_{k_j}\},$$

ω_{k_j} not infinite. We will consider

$$C_{[k_i]} = \{z \in D' \mid v_{[k_i]}(z) \geq \omega_{k_i}\}$$

and

$$C_{[k_j]} = \{z \in D' \mid v_{[k_j]}(z) \geq \omega_{k_j}\}$$

(proofs for the other cases are analogous).

Since $H_{[k]}$ is a rank one subgroup of $G_{[k]}$ for all $k = 1, \dots, n$, by the approximation theorem there are elements z, z' in D' such that

$$\left\{ \begin{array}{l} v_{[k_i]}(z) > \omega_{k_i}, \quad v_{[k_j]}(z) = 0, \\ \quad \quad \quad v_{[k_w]}(z) > 0 \\ v_{[k_i]}(z') = 0, \quad v_{[k_j]}(z') > \omega_{k_j}, \quad k_w \neq k_i, k_j. \\ \quad \quad \quad v_{[k_w]}(z') = 0 \end{array} \right. \quad \text{for}$$

Then, $z \in C_{[k_i]}$, $z' \in C_{[k_j]}$, and:

$$\left\{ \begin{array}{l} v_{[k_i]}(z+z') = \min\{v_{[k_i]}(z), v_{[k_i]}(z')\} = 0, \\ \quad \quad \quad \text{since } 0 = v_{[k_i]}(z') < v_{[k_i]}(z); \\ v_{[k_j]}(z+z') = \min\{v_{[k_j]}(z), v_{[k_j]}(z')\} = 0, \\ \quad \quad \quad \text{since } 0 = v_{[k_j]}(z) < v_{[k_j]}(z'); \\ v_{[k_w]}(z+z') = \min\{v_{[k_w]}(z), v_{[k_w]}(z')\} = 0, \\ \quad \quad \quad \text{since } 0 = v_{[k_w]}(z') < v_{[k_w]}(z) \\ \quad \quad \quad \text{for } k_w \neq k_i, k_j. \end{array} \right.$$

Therefore, $z + z'$ is a unit in D' . But,

$z + z' \in C_{[k_i]} + C_{[k_j]}$; hence, $C_{[k_i]} + C_{[k_j]} = D'$ for $k_i \neq k_j$.

Next, let $f\bar{x}_L = \bar{y}_L$ for $y_L \in D$. We will show that y_L may be replaced by y'_L , where y'_L satisfies conditions (3). Define z_{k_i} for $k_i = k_1, \dots, k_m$ as follows:

$$z_{k_i} = \begin{cases} y_L & \text{if } k_i \in \mathcal{L} \\ 0 & \text{if } k_i \in \mathcal{M}. \end{cases}$$

Since $C_{[k_i]} + C_{[k_j]} = D'$ for $k_i \neq k_j$, and since the Chinese Remainder Theorem is equivalent to D' being a Prüfer domain (pp. 307-10, Gilmer [3]), in D' we can solve the system of congruences

$$\begin{cases} y'_L \equiv z_{k_1} \pmod{C_{[k_1]}} \\ \vdots \\ y'_L \equiv z_{k_m} \pmod{C_{[k_m]}} \end{cases} \quad \text{for } y'_L.$$

For $k_i \in \mathcal{M}$, $z_{k_i} = 0$ and $v_{[k_i]}(z_{k_i}) = \infty$. Hence,

$z_{k_i} \in C_{[k_i]}$ and $y'_L \equiv z_{k_i} \pmod{C_{[k_i]}}$ imply that $y'_L \in C_{[k_i]}$.

Therefore,

$$(4) \quad v_{[k_i]}(y_L') > 2 \cdot v_{[k_i]}(y_{L-1}') \text{ for } k_i \in \mathfrak{M},$$

by definition of $C_{[k_i]}$.

Now, for $j \in \mathfrak{M}$, $v_{[j]}$ can be written as a composite of valuations, since we have the following:

$$\begin{array}{ccccc} Q_{cl}(D) & \xrightarrow{v_j} & G_j^+ & \xrightarrow{g_j} & G_{[j]}^+ \\ & \searrow & & \nearrow & \\ & & v_{[j]} & & \end{array}$$

Here, g_j is a group homomorphism from G_j^+ to $G_{[j]}^+$. Hence,

$$v_{[j]} = g_j \circ v_j \text{ for each } j \in \mathfrak{M}.$$

By (4), $v_{[k_i]}(y_L') > 2v_{[k_i]}(y_{L-1}')$ for $k_i \in \mathfrak{M}$. We would like to show that $v_{k_i}(y_L') > 2v_{k_i}(y_{L-1}')$. Suppose not:

$$v_{k_i}(y_L') \leq 2v_{k_i}(y_{L-1}') = v_{k_i}(y_{L-1}'^2).$$

Then,

$$g_{k_i}(v_{k_i}(y_L')) \leq g_{k_i}(v_{k_i}(y_{L-1}'^2)),$$

or

$$v_{[k_i]}(y_L') \leq v_{[k_i]}(y_{L-1}'^2) = 2 \cdot v_{[k_i]}(y_{L-1}'),$$

a contradiction. Therefore,

$$v_{k_i}(y_L^!) > 2v_{k_i}(y_{L-1}^!) \text{ for } k_i \in \mathfrak{M},$$

$$k_1 \leq k_i \leq k_m.$$

If $k \in \mathfrak{M}$, where $k \neq k_i$, such that $P_{[k]} = P_{[k_i]}$, then $v_{[k]}$ is identical to $v_{[k_i]}$ and $v_{[k_i]}(y_L^!) > 2v_{[k_i]}(y_{L-1}^!)$ implies that

$$v_{[k]}(y_L^!) > 2v_{[k]}(y_{L-1}^!)$$

or

$$v_k(y_L^!) > 2v_k(y_{L-1}^!).$$

Hence, $v_k(y_L^!) > 2v_k(y_{L-1}^!)$ for all $k \in \mathfrak{M}$. Now we must show that $y_L^!$ is in D . For $k \in \mathfrak{M}$, $v_k(y_L^!) > 2v_k(y_{L-1}^!) > 0$, and, for $k \in \mathfrak{L}$, $v_k(y_L^!) = v_{[k]}(y_L^!) \geq 0$. Thus, $v_k(y_L^!) \geq 0$ for all $k = 1, \dots, n$, and $y_L^! \in D$. Then, for all $k \in \mathfrak{M}$, since $v_k(y_L^!) > 2v_k(y_{L-1}^!)$, we have shown that there is an element $y_L^!$ in D such that

$$(5) \quad \begin{cases} v_k(y_L^!) > v_k(y_{L-1}^!) \\ v_k(y_L^!) > 2v_k(y_{L-1}^!) - v_k(y_{L-2}^!) + v_k(x_L) \\ \quad - 2v_k(x_{L-1}) + v_k(x_{L-2}). \end{cases}$$

To complete the induction step, we must show that $\overline{y_L'} = f\overline{x_L} = \overline{y_L}$. For $k \in \mathfrak{L}$, $y_L' \equiv y_L(C_{[k]})$, where $y_L = z_k$ and $C_{[k]} = \mathfrak{a}_k D'$. Hence, $y_L' - y_L \in \mathfrak{a}_k D'$ for $k \in \mathfrak{L}$. For $k \in \mathfrak{M}$, $y_L' \equiv 0(C_{[k]})$, where $\mathfrak{a}_k D' \supseteq C_{[k]}$. Therefore, $y_L' \in C_{[k]} \subseteq \mathfrak{a}_k D'$; but, $y_L \in \mathfrak{a}_k \subseteq \mathfrak{a}_k D'$ by the definition of $\mathfrak{M}$. So, $y_L' - y_L \in \mathfrak{a}_k D'$ for $k \in \mathfrak{M}$. We thus have $y_L' - y_L \in \mathfrak{a}_1 D' \cap \dots \cap \mathfrak{a}_n D' = \mathfrak{a} D'$. But, $y_L' - y_L \in D$ since $y_L', y_L \in D$; hence $y_L' - y_L \in \mathfrak{a} D' \cap D = \mathfrak{a}$. Therefore, $\overline{y_L'} - \overline{y_L} = \overline{0}$ in $D/\mathfrak{a}$, or $\overline{y_L'} = \overline{y_L}$.

Since each y_ℓ for $\ell \geq N$ may be replaced by y_ℓ' such that conditions (3) are satisfied, to simplify subsequent work we shall assume that the y_ℓ 's already satisfy conditions (3).

So, for $k \in \mathfrak{M}$ and for all $\ell \geq N$, we have

$$v_k(y_{\ell+1}) > v_k(y_\ell)$$

and

$$\begin{aligned} v_k(y_{\ell+2}) &> 2v_k(y_{\ell+1}) - v_k(y_\ell) + v_k(x_{\ell+2}) \\ &\quad - 2v_k(x_{\ell+1}) + v_k(x_\ell). \end{aligned}$$

But, $v_k(x_{\ell+1}) \leq v_k(x_\ell)$ for all $k \in \mathfrak{M}$, so that

$$-v_k(x_{\ell+1}) \geq -v_k(x_\ell). \text{ Hence,}$$

$$(6) \quad v_k(y_\ell) - v_k(x_\ell) < v_k(y_{\ell+1}) - v_k(x_{\ell+1})$$

for $k \in \mathfrak{M}$ and for all $\ell \geq N$. This is the first of two results which we wished to show for $k \in \mathfrak{M}$.

Combining (6) with what we have previously shown for $k \in \mathfrak{L}$, we get:

$$(7) \quad v_k(y_\ell) - v_k(x_\ell) \leq v_k(y_{\ell+1}) - v_k(x_{\ell+1})$$

for all $k = 1, \dots, n$.

Next, if $k \in \mathfrak{M}$, then

$$\begin{aligned} v_k(y_{\ell+2}) &> 2v_k(y_{\ell+1}) - v_k(y_\ell) + v_k(x_{\ell+2}) \\ &\quad - 2v_k(x_{\ell+1}) + v_k(x_\ell) \end{aligned}$$

implies that

$$\begin{aligned} (8) \quad v_k(y_{\ell+2}) - v_k(x_{\ell+2}) - (v_k(y_{\ell+1}) - v_k(x_{\ell+1})) &> \\ v_k(y_{\ell+1}) - v_k(x_{\ell+1}) - (v_k(y_\ell) - v_k(x_\ell)) \end{aligned}$$

for all $\ell \geq N$. This is the second of the two desired results for $k \in \mathfrak{M}$.

For each $k \in \mathfrak{L}$, the difference

$v_k(y_\ell) - v_k(x_\ell) = v_k[f]$ is independent of ℓ , $\ell \geq N$, by Theorem 3.3. By the approximation theorem, there is an

element a in D such that:

$$\begin{cases} v_k(a) = v_k[f], & k \in \mathcal{L} \\ v_k(a) = 0, & k \in \mathcal{M}. \end{cases}$$

Let x_ℓ be a fixed element in the sequence $\langle x_\ell \rangle_{\ell \geq N}$, and let $\bar{y}_\ell = f\bar{x}_\ell$ in $D/\mathfrak{a}$, as above.

For $k \in \mathcal{L}$, $v_k(y_\ell) - v_k(x_\ell) = v_k[f] = v_k(a)$, and $v_k(y_\ell) = v_k(a) + v_k(x_\ell) = v_k(ax_\ell)$. For $k \in \{\delta+1, \dots, \tau\} \subseteq \mathcal{M}$, $v_k(x_\ell) < 1/\ell$ and $v_k(y_\ell) - v_k(x_\ell) > r_k - 1/\ell > 0 = v_k(a)$ (by choice of N). For all other $k \in \mathcal{M}$, since $v_k(a) = 0$, $v_k(x_\ell) = 0$, and $v_k(y_\ell) > 0$, we have $0 < v_k(y_\ell) - v_k(x_\ell) = v_k(y_\ell) - v_k(a) - v_k(x_\ell) = v_k(y_\ell) - v_k(ax_\ell)$.

Hence, $v_k(y_\ell) > v_k(ax_\ell)$ for all $k \in \mathcal{M}$.

Then, since $v_k(y_\ell) \geq v_k(ax_\ell)$ for all $k = 1, \dots, n$, by Lemma 3.5 there is an element c_ℓ in D such that $y_\ell = c_\ell ax_\ell$. Since $y_\ell \in \mathfrak{a}_k$ and $v_k(a) = 0$ for $k \in \{\delta+1, \dots, \tau\} \subseteq \mathcal{M}$, we have

$$(9) \quad v_k(y_\ell) = v_k(c_\ell) + v_k(x_\ell) \geq r_k \text{ for all } \ell \geq N.$$

For all other $k \in \mathcal{M}$, $v_k(a) = 0$ and $v_k(x_\ell) = 0$, so that

$v_k(y_\ell) = v_k(c_\ell)$, and hence $y_\ell \in \mathfrak{A}_k$ implies that

$$(10) \begin{cases} v_k(y_\ell) = v_k(c_\ell) \geq r_k \text{ for } k = \theta+1, \dots, \gamma \text{ and} \\ \quad \text{for all } \ell \geq N; \\ v_k(y_\ell) = v_k(c_\ell) > r_k \text{ for } k = \tau+1, \dots, \theta \text{ or} \\ \quad k = \gamma+1, \dots, n \text{ and for all } \ell \geq N. \end{cases}$$

Note that $v_k(c_\ell) = 0$ for $k \in \mathfrak{L}$ since $v_k(y_\ell) = v_k(ax_\ell)$, and that $v_k(c_\ell) > 0$ for $k \in \mathfrak{M}$ since $v_k(y_\ell) > v_k(ax_\ell)$.

Now, for each $\ell \geq N$, we have:

$$(11) \quad y_\ell = c_\ell ax_\ell \text{ and } y_{\ell+1} = c_{\ell+1} ax_{\ell+1}.$$

Since $v_k(y_\ell) - v_k(x_\ell) \leq v_k(y_{\ell+1}) - v_k(x_{\ell+1})$ for all k , $1 \leq k \leq n$, we have

$$\begin{aligned} v_k(c_\ell) &= v_k(y_\ell) - v_k(a) - v_k(x_\ell) \\ &\leq v_k(y_{\ell+1}) - v_k(a) - v_k(x_{\ell+1}) = v_k(c_{\ell+1}). \end{aligned}$$

Therefore, by Lemma 3.5, there is an element b_ℓ in D such that:

$$(12) \quad b_\ell c_\ell = c_{\ell+1}.$$

For $k \in \mathfrak{L}$, $v_k(c_\ell) = v_k(c_{\ell+1}) = 0$ and $v_k(b_\ell) = 0$. For $k \in \mathfrak{M}$, by line (6), $v_k(c_\ell) < v_k(c_{\ell+1})$ and $v_k(b_\ell) > 0$; and, for $k \in \mathfrak{N}$, by line (8), $v_k(b_{\ell+1}) > v_k(b_\ell)$ for all $\ell \geq N$.

Then, $c_{N+1} = b_N c_N$, $c_{N+2} = b_{N+1} c_{N+1} = b_{N+1} b_N c_N$, and, in general,

$$c_\ell = (\prod_{j=N}^{\ell-1} b_j) c_N, \quad \ell > N.$$

For arbitrary $\bar{x} \in \text{dom} f \supseteq A^e$, we are now able to obtain an expression for $f\bar{x}$. Pick $i \geq N$ such that $v_k(x) > 1/i$ for $k = 1, \dots, \tau$. Then $v_k(x) > 1/i > v_k(x_i)$ for $k = 1, \dots, \tau$ and $v_k(x) \geq v_k(x_i) = 0$ for $k = \tau+1, \dots, n$. Hence, by Lemma 3.5, there is an element s_i in D such that $x = s_i x_i$.

$$\begin{aligned} \text{So, } f\bar{x} &= f\bar{s}_i \bar{x}_i = \bar{s}_i \cdot f\bar{x}_i \\ &= \bar{s}_i \cdot \bar{y}_i \\ &= \bar{s}_i \cdot \bar{c}_i \bar{a} \bar{x}_i: \text{ line (11)} \\ &= \bar{s}_i (\prod_{j=N}^{i-1} \bar{b}_j) \bar{c}_N \bar{a} \bar{x}_i \end{aligned}$$

$$(13) \quad f\bar{x} = (\prod_{j=N}^{i-1} \bar{b}_j) \bar{c}_N \bar{a} \bar{x}$$

since $\bar{x} = \bar{s}_i \bar{x}_i$. Note: If $i = N$, then $f\bar{x} = \bar{c}_N \bar{a} \bar{x}$.

(End of Theorem 3.4)

The foregoing proof contains some results which are not claimed in the statement of Theorem 3.4. Thus, an attempt at a converse of Theorem 3.4 will require a

refinement of the statement of the theorem.

In Lemma 3.7, we have shown that for all $l \geq N$ there exists $s_l \in D$ such that $x_l = s_l \cdot x_{l+1}$. Therefore, for the fixed f in $Q(D/\mathfrak{a})$, we have for all $l \geq N$:

$$\begin{aligned}
 \overline{c}_l \overline{a} \overline{x}_l &= \overline{y}_l: \text{ line (11)} \\
 &= f\overline{x}_l \\
 &= f\overline{s}_l \overline{x}_{l+1} \\
 &= \overline{s}_l \cdot f\overline{x}_{l+1} \\
 &= \overline{s}_l \cdot \overline{y}_{l+1} \\
 &= \overline{s}_l \cdot \overline{c}_{l+1} \overline{a} \overline{x}_{l+1}: \text{ line (11)} \\
 &= \overline{c}_{l+1} \overline{a} \overline{x}_l: x_l = s_l \cdot x_{l+1}.
 \end{aligned}$$

Hence, $\overline{c}_l \overline{a} \overline{x}_l = \overline{c}_{l+1} \overline{a} \overline{x}_l$ for $l \geq N$, or $(\overline{c}_{l+1} - \overline{c}_l) \overline{a} \overline{x}_l = \overline{0}$.

Thus, $(c_{l+1} - c_l)ax_l \in \mathfrak{a} = \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_n$, and:

$$\left\{ \begin{array}{l} v_k(c_{l+1} - c_l) \geq r_k - v_k(a) - v_k(x_l) \\ \quad \text{for } k = 1, \dots, \tau \text{ or } k = \theta+1, \dots, \gamma; \\ v_k(c_{l+1} - c_l) > r_k - v_k(a) - v_k(x_l) \\ \quad \text{for } k = \tau+1, \dots, \theta \text{ or } k = \gamma+1, \dots, n. \end{array} \right.$$

Now, for $k \in \{1, \dots, \delta\} \subseteq \mathfrak{L}$, $v_k(x_\ell) < 1/\ell$ and
 $v_k(c_{\ell+1} - c_\ell) \geq r_k - v_k(a) - v_k(x_\ell) > r_k - v_k[f] - 1/\ell > 0$
for all $\ell \geq N$, by definition of N . For
 $k \in \{\delta+1, \dots, \tau\} \subseteq \mathfrak{M}$, $v_k(x_\ell) < 1/\ell$, $v_k(a) = 0$, and
 $v_k(c_{\ell+1} - c_\ell) \geq r_k - 1/\ell > 0$ for all $\ell \geq N$, by definition of
 N . For $k \in \{\tau+1, \dots, \gamma\} \cap \mathfrak{L}$, $v_k(x_\ell) = 0$ and
 $v_k(a) = v_k[f] < r_k$, so that
 $v_k(c_{\ell+1} - c_\ell) \geq r_k - v_k(a) - v_k(x_\ell) = r_k - v_k(a) > 0$ for all
 $\ell \geq N$. For $k \in \{\tau+1, \dots, \gamma\} \cap \mathfrak{M}$, $v_k(x_\ell) = 0$ and $v_k(a) = 0$;
hence, $v_k(c_{\ell+1} - c_\ell) \geq r_k > 0$ for all $\ell \geq N$. Finally, for
 $k \in \{\gamma+1, \dots, n\}$, $v_k(x_\ell) = 0$, $v_k(a) = 0$, and $r_k = 0$, so that
 $v_k(c_{\ell+1} - c_\ell) > r_k - v_k(a) - v_k(x_\ell) = 0$ for all $\ell \geq N$.

Therefore:

$$(14) \quad v_k(c_{\ell+1} - c_\ell) > 0 \text{ for all } \ell \geq N \text{ and } k = 1, \dots, n.$$

Now, for all $\ell \geq N$, let $a_\ell = b_\ell - 1$. By (12), we
have that $b_\ell c_\ell = c_{\ell+1}$. Therefore,

$$b_\ell c_\ell - c_\ell = c_{\ell+1} - c_\ell, \text{ and } c_\ell a_\ell = c_\ell (b_\ell - 1) = c_{\ell+1} - c_\ell.$$

For $k \in \mathfrak{L}$, $v_k(c_\ell) = 0$ and $v_k(c_{\ell+1} - c_\ell) = v_k(c_\ell (b_\ell - 1))$
 $= v_k(a_\ell)$. Thus, for $k \in \mathfrak{L}$, by (14), we have $v_k(a_\ell) > 0$
for all $\ell \geq N$.

For $k \in \mathfrak{M}$, $v_k(b_\ell) > 0$ and $v_k(a_\ell) = v_k(b_\ell - 1) = 0$, since $v_k(-1) = 0$.

Let $f \in Q(D/\mathfrak{a})$. Then, for arbitrary $\bar{x} \in \text{dom} f \supseteq A^e$ such that $v_k(x) > 1/i$ for $i \geq N$ and $k = 1, \dots, \tau$, where $\mathfrak{L} \neq \emptyset$ and $\mathfrak{M}$ are partitions of $\{1, \dots, n\}$, where $1 \leq \delta \leq \tau$, where $\tau < \theta < \gamma < n$ if θ and γ exist, where $a, c_N \in D$, and using (9) and (10), we have the following alternate representation for (13):

$$(15) \quad f\bar{x} = \prod_{j=N}^{i-1} (\overline{1+a_j}) \bar{c}_N \bar{a} \bar{x},$$

where, for $k \in \mathfrak{L}$,

$$v_k(c_N) = 0,$$

for $k \in \{1, \dots, \delta\} \subseteq \mathfrak{L}$,

$$v_k(a_j) > r_k - v_k(a) - 1/j > 0,$$

for $k \in \mathfrak{L} \setminus \{1, \dots, \delta\}$,

$$v_k(a_j) > r_k - v_k(a),$$

for $k \in \mathfrak{M}$,

$$\begin{cases} 0 < v_k(1+a_j) < v_k(1+a_{j+1}) \\ v_k(a) = 0, \end{cases}$$

for $k \in \{\delta+1, \dots, \tau\} \subseteq \mathfrak{M}$,

$$v_k(c_N \cdot \prod_{j=N}^{\ell-1} (1+a_j)) > r_k - 1/\ell \text{ for all } \ell \geq N,$$

for $k \in \{\theta+1, \dots, \gamma\} \subseteq \mathfrak{M}$,

$$v_k(c_N \cdot \prod_{j=N}^{\ell-1} (1+a_j)) \geq r_k \text{ for all } \ell \geq N,$$

and for $k \in \{\tau+1, \dots, \theta\} \cup \{\gamma+1, \dots, n\} \subseteq \mathfrak{M}$,

$$v_k(c_N \cdot \prod_{j=N}^{\ell-1} (1+a_j)) > r_k \text{ for all } \ell \geq N.$$

We now wish to show that anything of form (15) is an element of $Q(D/a)$.

(i) f is well-defined:

For s a positive integer, if $v_k(x) > 1/i$ for $i \geq N$ and $k = 1, \dots, \tau$, then $v_k(x) > 1/i+s$. Hence, we must show that

$$\prod_{j=N}^{i-1} (\overline{1+a_j}) \overline{c_N} \overline{a} \overline{x} = \prod_{j=N}^{i+s-1} (\overline{1+a_j}) \overline{c_N} \overline{a} \overline{x}.$$

$$\text{Let } e = \prod_{j=N}^{i-1} (1+a_j) c_N a x - \prod_{j=N}^{i+s-1} (1+a_j) c_N a x.$$

(a) For $k \in \mathfrak{L}$,

$$\begin{aligned} v_k(e) &= v_k(c_N) + v_k(a) + v_k(x) + \\ &\quad v_k \left\{ \prod_{j=N}^{i-1} (1+a_j) [1 - \prod_{j=i}^{i+s-1} (1+a_j)] \right\} \\ (16) \quad &= v_k(c_N) + v_k(a) + v_k(x) + \\ &\quad v_k \left(\prod_{j=N}^{i-1} (1+a_j) \right) + v_k [1 - \prod_{j=i}^{i+s-1} (1+a_j)]. \end{aligned}$$

For $k \in \{1, \dots, \delta\} \subseteq \mathfrak{L}$, $v_k(c_N) = 0$ and $v_k(a_j) >$

$r_k - v_k(a) - 1/j > 0$ for $j = N, \dots, i+s-1$. Hence,

$$v_k(\prod_{j=N}^{i-1}(1+a_j)) = v_k(1) = 0. \text{ Then, } g = 1 - \prod_{j=i}^{i+s-1}(1+a_j) = \\ a_i + \dots + a_{i+s-1} + a_i \cdot a_{i+1} + \dots + a_i \cdot \dots \cdot a_{i+s-1}.$$

Since $v_k(a_j) > 0$ for $j = 1, \dots, i+s-1$,

$$v_k(g) \geq \min_{j=i, \dots, i+s-1} \{v_k(a_j)\}. \text{ Therefore, there is an}$$

integer w_k , $0 \leq w_k \leq s-1$, such that

$$v_k(g) \geq v_k(a_{i+w_k}) > r_k - v_k(a) - 1/i + w_k.$$

Continuing from (16), for $k \in \{1, \dots, \delta\} \subseteq \mathfrak{L}$,

$$\begin{aligned} v_k(e) &= v_k(a) + v_k(x) + v_k[1 - \prod_{j=i}^{i+s-1}(1+a_j)] \\ &\geq v_k(a) + v_k(x) + v_k(a_{i+w_k}) \\ &> v_k(a) + 1/i + r_k - v_k(a) - 1/i + w_k \\ &\geq r_k, \text{ since } w_k \geq 0. \end{aligned}$$

Similarly, for $k \in \mathfrak{L} \setminus \{1, \dots, \delta\}$, $v_k(e) > r_k$. Therefore,

$$e \in \bigcap_{k \in \mathfrak{L}} \mathfrak{a}_k.$$

(b) For $k \in \mathfrak{M}$, $v_k(\prod_{j=i}^{i+s-1}(1+a_j)) > 0$, and hence

$$v_k[1 - \prod_{j=i}^{i+s-1}(1+a_j)] = v_k(1) = 0. \text{ Then,}$$

$$\begin{aligned} v_k(e) &= v_k(c_N) + v_k(a) + v_k(x) \\ &\quad + v_k(\prod_{j=N}^{i-1}(1+a_j)) + v_k[1 - \prod_{j=i}^{i+s-1}(1+a_j)] \end{aligned}$$

$$\begin{aligned}
&= v_k(c_N) + 0 + v_k(x) + v_k(\prod_{j=N}^{i-1}(1+a_j)) + 0 \\
&= v_k(x) + v_k(c_N \cdot \prod_{j=N}^{i-1}(1+a_j)).
\end{aligned}$$

Then, for $k \in \{\delta+1, \dots, \tau\} \subseteq \mathfrak{M}$,

$$v_k(x) + v_k(c_N \cdot \prod_{j=N}^{i-1}(1+a_j)) > 1/i + r_k - i/i = r_k,$$

since we are assuming the representation in (15). For $k \in \{\theta+1, \dots, \gamma\} \subseteq \mathfrak{M}$,

$$v_k(x) + v_k(c_N \cdot \prod_{j=N}^{i-1}(1+a_j)) \geq r_k.$$

For $k \in \{\tau+1, \dots, \theta\} \cup \{\gamma+1, \dots, n\} \subseteq \mathfrak{M}$,

$$v_k(x) + v_k(c_N \cdot \prod_{j=N}^{i-1}(1+a_j)) > r_k.$$

Therefore, $e \in \bigcap_{k \in \mathfrak{M}} \mathfrak{a}_k$.

Combining (a) and (b), we get $e \in \bigcap_{k=1, \dots, n} \mathfrak{a}_k = \mathfrak{a}$.

Hence, $\bar{e} = \bar{0}$ in $D/\mathfrak{a}$, or

$$\prod_{j=N}^{i-1}(\overline{1+a_j}) \bar{c}_N \bar{a} \bar{x} - \prod_{j=N}^{i+s-1}(\overline{1+a_j}) \bar{c}_N \bar{a} \bar{x} \text{ in } D/\mathfrak{a}, \text{ or}$$

$$\prod_{j=N}^{i-1}(\overline{1+a_j}) \bar{c}_N \bar{a} \bar{x} = \prod_{j=N}^{i+s-1}(\overline{1+a_j}) \bar{c}_N \bar{a} \bar{x}, \text{ as was to be}$$

shown. Therefore, f is well-defined.

(ii) Let $v_k(x_1) > 1/i_1$ and $v_k(x_2) > 1/i_2$ for all $k = 1, \dots, \tau$, where i_1, i_2 are positive integers, $i_1, i_2 \geq N$, and where $\bar{x}_1, \bar{x}_2 \in \text{dom} f \supseteq A^e$. Let $i = \max\{i_1, i_2\}$. Then:

$$\begin{aligned}
f(\bar{x}_1 + \bar{x}_2) &= \prod_{j=N}^{i'-1} (\overline{1+a_j}) \bar{c}_N \bar{a} (\bar{x}_1 + \bar{x}_2) \\
&= \prod_{j=N}^{i'-1} (\overline{1+a_j}) \bar{c}_N \bar{a} \bar{x}_1 + \prod_{j=N}^{i'-1} (\overline{1+a_j}) \bar{c}_N \bar{a} \bar{x}_2 \\
&= f\bar{x}_1 + f\bar{x}_2.
\end{aligned}$$

Let $v_k(x) > 1/i'$ for all $k = 1, \dots, \tau$, where i' is a positive integer, $i' \geq N$, and where $\bar{x} \in \text{dom} f \supseteq A^e$. Let $z \in D$ be arbitrary. Then, $v_k(z) \geq 0$ for $k = 1, \dots, \tau$, and $v_k(zx) = v_k(z) + v_k(x) > 1/i'$ for $k = 1, \dots, \tau$.

$$\begin{aligned}
\text{So: } f\bar{z} \bar{x} &= \prod_{j=N}^{i'-1} (\overline{1+a_j}) \bar{c}_N \bar{a} \bar{z} \bar{x} \\
&= \bar{z} \cdot \prod_{j=N}^{i'-1} (\overline{1+a_j}) \bar{c}_N \bar{a} \bar{x} \\
&= \bar{z} \cdot f\bar{x}.
\end{aligned}$$

By (i) and (ii), $f \in Q(D/\mathfrak{a})$.

Expanding upon Theorem 3.4, using the representation in (15), and letting $d = a \cdot c_N$, we may now state the main theorem with a bit more clarity.

Theorem 3.8: Let D be a semi-local Prüfer domain, let $\mathfrak{a}$, τ , P_k , $\mathfrak{a}_k$, and r_k , $k = 1, \dots, n$, be given as in Chapter II, and let $A^e = P_1^e \cap \dots \cap P_\tau^e$. Then, f is an element of $Q(D/\mathfrak{a})$ if and only if there is a positive integer N , sub-

sets $\mathcal{L} \neq \emptyset$ and $\mathcal{M}$ of $\{1, \dots, n\}$, an integer δ where $1 \leq \delta \leq \tau$, integers θ and γ (if they exist) where $\tau < \theta < \gamma < n$, an element d in D , and a sequence of elements $\langle a_j \rangle_{j=N}^{\infty}$ in D such that

$$f\bar{x} = \prod_{j=N}^{i-1} (\overline{1+a_j}) \bar{d} \bar{x},$$

where $\bar{x} \in \text{dom} f \supseteq A^e$, $v_k(x) > 1/i$

for $k = 1, \dots, \tau$, and where,

for $k \in \{1, \dots, \delta\} \subseteq \mathcal{L}$,

$$v_k(a_j) > r_k - v_k(d) - 1/j > 0,$$

for $k \in \mathcal{L} \setminus \{1, \dots, \delta\}$,

$$v_k(a_j) > r_k - v_k(d),$$

for $k \in \mathcal{M}$,

$$0 < v_k(1+a_j) < v_k(1+a_{j+1}),$$

for $k \in \{\delta+1, \dots, \tau\} \subseteq \mathcal{M}$,

$$v_k(c_N \cdot \prod_{j=N}^{\ell-1} (1+a_j)) > r_k - 1/\ell \text{ for all } \ell \geq N,$$

for $k \in \{\theta+1, \dots, \gamma\} \subseteq \mathcal{M}$,

$$v_k(c_N \cdot \prod_{j=N}^{\ell-1} (1+a_j)) \geq r_k \text{ for all } \ell \geq N,$$

and for $k \in \{\tau+1, \dots, \theta\} \cup \{\gamma+1, \dots, n\} \subseteq \mathcal{M}$,

$$v_k(c_N \cdot \prod_{j=N}^{\ell-1} (1+a_j)) > r_k \text{ for all } \ell \geq N.$$

Little is known about conditions under which $\text{dom}f + \text{ran}f = \text{dom}f$ for $f \in Q(R)$, where R is an arbitrary commutative ring. Lemma 1.5 shows that if $\text{dom}f \subseteq P$, where P is a minimal prime ideal in R , then $\text{dom}f + \text{ran}f \subseteq P$. By the nature of the expression for $f\bar{x}$ in Theorem 3.8, however, we can state the following:

Corollary to Theorem 3.8: For $f \in Q(D/\mathfrak{a})$, we have

$$\text{dom}f + \text{ran}f = \text{dom}f \subseteq D/\mathfrak{a}.$$

Proof: Let $\bar{x}$ be an arbitrary element of $\text{dom}f$, where $v_k(x) > 1/i$ for $k = 1, \dots, \tau$. Then since

$\prod_{j=N}^{i-1} (\overline{1+a_j}) \bar{d} \in D/\mathfrak{a}$ and since $\text{dom}f$ is an ideal in $D/\mathfrak{a}$, we have that

$$[\prod_{j=N}^{i-1} (\overline{1+a_j}) \bar{d}] \bar{x} = f\bar{x} \in \text{dom}f.$$

Hence,

$$\text{ran}f \subseteq \text{dom}f$$

and

$$\text{dom}f + \text{ran}f = \text{dom}f \subseteq D/\mathfrak{a}.$$

(End of the Corollary)

CHAPTER IV

In Chapter III, we have given a necessary and sufficient condition for an element f to be in the complete quotient ring of $D/\mathfrak{a}$, where D is a Prüfer domain, provided that $\mathfrak{a}$ is an irredundant primary decomposition. We now give an example of a homomorphic image of a Prüfer domain R which has a non-trivial complete quotient ring. That is, with $\mathfrak{a}$ the kernel of the homomorphism, we will show that

$$R/\mathfrak{a} = Q_{cl}(R/\mathfrak{a}) \subsetneq Q(R/\mathfrak{a}).$$

The following is a version of Example (6), pp. 390-391, Bourbaki [2]:

Let $\Gamma = \mathbf{R}$ and let F be an arbitrary field. Let $\Gamma_+ = \mathbf{R}^+ \cup \{0\}$ and let C be the semi-group algebra of Γ_+ over F . By definition, C is an F -algebra with basis $\langle x_\alpha \rangle_{\alpha \in \Gamma_+}$ and multiplication given by $x_\alpha x_\beta = x_{\alpha+\beta}$; and, C is an integral domain. An arbitrary element of C has the form $\sum_\alpha a_\alpha x_\alpha$, a finite sum. Let K be the quotient field of C : $K = Q_{cl}(C)$.

Define $v: K \rightarrow \Gamma \cup \{\infty\}$ by

$$v(\sum_\alpha a_\alpha x_\alpha / \sum_\beta b_\beta x_\beta) = \min_{a_\alpha \neq 0} (\alpha) - \min_{b_\beta \neq 0} (\beta),$$

where $v(0) = \infty$. By Definition 1.9, v is a valuation on K . Let $R = \{p \mid p \in K \text{ and } v(p) \geq 0\}$, R being the valuation ring of v . Then, R is a Prüfer domain. Further, R is a local domain with its unique maximal ideal being

$$M = \{p \in R \mid v(p) > 0\}.$$

Let $\mathfrak{a} = \{p \in R \mid v(p) \geq 1\}$; $\mathfrak{a}$ is an M -primary ideal of R , and, by Theorem 2.5, M^e is dense in $R/\mathfrak{a}$. By the Corollary to Theorem 2.5, M^e is the only dense ideal in $R/\mathfrak{a}$. Since units are the only regular elements in $R/\mathfrak{a}$, $R/\mathfrak{a} = Q_{cl}(R/\mathfrak{a})$.

Using Theorem 3.8, let $d=1$, $N=2$, and $\mathcal{L}=\{1\}$, and choose $\langle a_j \rangle_{j=N}^{\infty}$ in R such that $a_j = x_{1-1/j}/1 = x_{1-1/j}$. Note that $v(a_j) = 1 - 1/j$. Then, since $\mathcal{L}=\{1\}$, an element $f \neq \bar{0}$ in $Q(R/\mathfrak{a})$ is determined such that

$$f\bar{x} = \prod_{j=2}^{n-1} (1+a_j)\bar{x} = \bar{y} \text{ for } v(x) > 1/n, \bar{x} \in \text{dom} f = M^e.$$

Now, $\prod_{j=2}^{n-1} (1+a_j) = 1 + a_2 + \dots + a_{n-1} + a_2 a_3 + a_2 a_4 + \dots + a_2 \dots a_{n-1}$. But,

$$\begin{aligned} v(a_2 a_3) &= v(a_2) + v(a_3) \\ &= v(x_{1-1/2}/1) + v(x_{1-1/3}/1) \\ &= 1 - 1/2 + 1 - 1/3 \\ &= 2 - 5/6 > 1, \end{aligned}$$

and $\overline{a_2 a_3} = \overline{0}$. Similarly, $\overline{a_{i_1}} \dots \overline{a_{i_k}} = \overline{0}$ for $2 \leq i_1 < \dots < i_k \leq n-1$, $k \geq 2$. Therefore,

$$\begin{aligned} \prod_{j=2}^{n-1} (\overline{1+a_j}) &= \overline{1} + \overline{a_2} + \dots + \overline{a_{n-1}} \\ &= \overline{1} + \overline{x_{1/2}} + \dots + \overline{x_{n-2/n-1}} \\ &= \sum_{j=0}^{n-2} \overline{x_{j/j+1}}. \end{aligned}$$

Hence, for $\overline{x} \in \text{dom } f$, $v(x) > 1/n$, we get

$$f\overline{x} = \prod_{j=2}^{n-1} (\overline{1+a_j})\overline{x} = (\sum_{j=0}^{n-2} \overline{x_{j/j+1}})\overline{x}.$$

We will show that the element f in $Q(R/\mathfrak{a})$ cannot be in $R/\mathfrak{a}$, and hence $Q(R/\mathfrak{a})$ properly contains $R/\mathfrak{a}$.

If $f \in R/\mathfrak{a}$, then there is an element $e \in R$ such that $f = \overline{e}$. Hence, $f\overline{x} = \overline{e} \overline{x}$ for x an arbitrary element in $\text{dom } f = M^e$. Then, $e = \sum_{\delta} d_{\delta} x_{\delta} / \sum_{\sigma} s_{\sigma} x_{\sigma}$, where $0 \leq v(e) < 1$. Since $f \neq \overline{0}$, $v(e) \geq 1$ ($\overline{e} = \overline{0}$) is impossible. If $v(x) > 1/n$, then

$$f\overline{x} = (\sum_{j=0}^{n-2} \overline{x_{j/j+1}})\overline{x} = \overline{e} \overline{x},$$

and

$$(1) \quad (\sum_{j=0}^{n-2} \overline{x_{j/j+1}} - \overline{e})\overline{x} = \overline{0},$$

or

$$(\sum_{j=0}^{n-2} \overline{x_{j/j+1}} - [\sum_{\delta} d_{\delta} x_{\delta} / \sum_{\sigma} s_{\sigma} x_{\sigma}])\overline{x} = \overline{0}.$$

$$\begin{aligned} \text{Let } q_n &= \sum_{j=0}^{n-2} x_j/j+1 - [\sum_{\delta} d_{\delta} x_{\delta} / \sum_{\sigma} s_{\sigma} x_{\sigma}] \\ &= [(\sum_{j=0}^{n-2} x_j/j+1)(\sum_{\sigma} s_{\sigma} x_{\sigma}) - \sum_{\delta} d_{\delta} x_{\delta}] / \sum_{\sigma} s_{\sigma} x_{\sigma}. \end{aligned}$$

Two cases arise for e:

$$(a) \quad 0 < v(e) < 1$$

$$\text{Here, } \min_{d_{\delta} \neq 0} (\delta) > \min_{s_{\sigma} \neq 0} (\sigma),$$

and,

$$\min [(\sum_{j=0}^{n-2} x_j/j+1)(\sum_{\sigma} s_{\sigma} x_{\sigma}) - \sum_{\delta} d_{\delta} x_{\delta}] = \min_{s_{\sigma} \neq 0} (\sigma),$$

since $v(\sum_{j=0}^{n-2} x_j/j+1) = 1$. Therefore,

$$v(q_n) = \min_{s_{\sigma} \neq 0} (\sigma) - \min_{s_{\sigma} \neq 0} (\sigma) = 0 \text{ for arbitrary } n. \text{ So, for}$$

$\bar{x} \in \text{dom } f$ such that $1/n < v(x) < 1$, we get

$$v(q_n \cdot x) = v(q_n) + v(x) = v(x) < 1. \text{ Therefore,}$$

$\bar{q}_n \cdot \bar{x} \neq \bar{0}$, a contradiction to (1).

$$(b) \quad v(e) = 0$$

Here, $\min_{d_{\delta} \neq 0} (\delta) = \min_{s_{\sigma} \neq 0} (\sigma)$. We look at the expression

$$(\sum_{j=0}^{n-2} \bar{x}_j/j+1 - \bar{e})\bar{x} = \bar{0} \text{ for arbitrary } n. \text{ There are two}$$

possibilities:

$$(i) \quad \sum_{j=0}^{n-2} \bar{x}_{j/j+1} - \bar{e} = \bar{0} \text{ for some fixed } n;$$

$$\text{that is, } \bar{e} = \sum_{j=0}^{n-2} \bar{x}_{j/j+1}.$$

Let $x = x_\alpha$ where $1/n+1 < \alpha < 1/n$. Then $\bar{x} \in \text{dom } f$ and $f\bar{x} = \bar{e} \bar{x}$ implies that

$$(\sum_{j=0}^{n-1} \bar{x}_{j/j+1})\bar{x} = (\sum_{j=0}^{n-2} \bar{x}_{j/j+1})\bar{x},$$

or

$$\begin{aligned} (\bar{1} + \bar{x}_{1/2} + \dots + \bar{x}_{n-1/n})\bar{x} \\ = (\bar{1} + \bar{x}_{1/2} + \dots + \bar{x}_{n-2/n-1})\bar{x}, \end{aligned}$$

or

$$\begin{aligned} (\bar{1} + \bar{x}_{1/2} + \dots + \bar{x}_{n-1/n} - \bar{1} - \bar{x}_{1/2} - \dots \\ - \bar{x}_{n-2/n-1})\bar{x} = \bar{0}, \end{aligned}$$

or

$$\bar{x}_{n-1/n} \cdot \bar{x} = \bar{0}.$$

Therefore,

$$(2) \quad v(x_{n-1/n} \cdot x) \geq 1.$$

But, $v(x_{n-1/n}) = n-1/n = 1 - 1/n$ and $v(x) < 1/n$. So,

$v(x_{n-1/n}) + v(x) < 1 - 1/n + 1/n = 1$, a contradiction to

(2).

$$(ii) \quad \sum_{j=0}^{n-2} \bar{x}_{j/j+1} - \bar{e} \neq \bar{0} \text{ for all } n; \text{ that is,}$$

$$\bar{e} \neq \sum_{j=0}^{n-2} \bar{x}_{j/j+1} \text{ for all } n.$$

First, (ii) implies that $v(q_n) < 1$ for all n . But, we need $v(q_n)$ bounded away from 1 for all n . We may choose n sufficiently large, say $n \geq M$, so that no term in $(x_{n-2/n-1}) \cdot (\sum_{\sigma} s_{\sigma} x_{\sigma})$ appears in $\sum_{\delta} d_{\delta} x_{\delta}$. Then, $v(q_n)$ remains fixed for $n > M$ and equals $v(q_M)$. Therefore, there is a positive integer $s > M$ such that $v(q_n) < 1 - 1/s$, $n \geq M$. Let $x = x_{\alpha}$ where $1/s+1 < \alpha < 1/s$. Then, $\bar{x} \in \text{dom } f$ and

$$(3) \quad f\bar{x} = (\sum_{j=0}^{s-1} \bar{x}_{j/j+1})\bar{x} = \bar{e} \bar{x}$$

implies that $v(q_{s+1} x) = v(q_{s+1}) + v(x)$

$$< 1 - 1/s + 1/s = 1,$$

a contradiction to (3).

Therefore, there is no element $e \in R$ for which $\bar{e} = f$. We must conclude that $f \in Q(R/\mathfrak{a}) \setminus R/\mathfrak{a}$, or, since $R/\mathfrak{a} = Q_{cl}(R/\mathfrak{a})$, $f \in Q(R/\mathfrak{a}) \setminus Q_{cl}(R/\mathfrak{a})$. So, $R/\mathfrak{a} = Q_{cl}(R/\mathfrak{a}) \subsetneq Q(R/\mathfrak{a})$. Letting $\mathfrak{a}$ be the kernel of a homomorphism of R , we have an example of a non-trivial complete quotient ring of a homomorphic image of a

Prüfer domain.

A natural extension of the work in Chapters II and III would be a characterization of the complete quotient ring of $D/\mathfrak{a}$ for an arbitrary kernel ϕ of a homomorphism of a Prüfer domain D . A uniformity can be defined on a valuation ring. Then, it appears that the complete quotient ring of a homomorphic image of a valuation ring can be characterized using the uniformity. A reduction from an arbitrary Prüfer domain to a semi-local Prüfer domain is accomplished in Chapter II. The techniques of Chapter III might be applied in the characterization of the complete quotient ring of a homomorphic image of a semi-local Prüfer domain D , without the condition that the kernel be an irredundant primary decomposition. However, the characterization of the elements would no longer be in terms of products, but would most likely be in the form of convergence with respect to the uniformity on D determined by the uniformities on the individual valuation rings.

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