



A study of the state of stress in a two-dimensional solid containing multiple-crack systems
by Dale Edward Nottingham

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of
MASTER OF SCIENCE in Civil Engineering
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Abstract:

This study is an analysis of "the state of stress existing around a given crack in a biaxially-stressed, two-dimensional, elastic solid containing a system of cracks. Photoelasticity was used to define the stress condition around both a multiple-crack system used in the major study and around a single crack used in the minor study. This minor study was an experimental and analytical analysis of the stress conditions around a single crack. This information was used to analyze the experimental data obtained from the multiple-crack study.

Several characteristics of the specific multiple-crack systems studied were observed. The largest difference between principal stresses always occurred at the ends of the cracks. These end stresses were probably increased by decreasing the crack's radius of curvature, increasing its length, or orienting the cracks at about 60 degrees with the major applied stress. The end-stress magnitude was a function of both the sum of and the difference between the applied stresses. When the applied stresses were equal, the stress fields were symmetrical about the cracks and the end-stress magnitudes were probably proportional to the applied stresses. Changes in the stress fields due to changing the crack patterns are complex.

The results of the entire study are intended to provide part of the background information necessary for determining stress conditions in real fractured or jointed media.

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A STUDY OF THE STATE OF STRESS IN A TWO-DIMENSIONAL SOLID
CONTAINING MULTIPLE-CRACK SYSTEMS

by

DALE EDWARD NOTTINGHAM

A thesis submitted to the Graduate Faculty in partial
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of

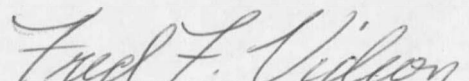
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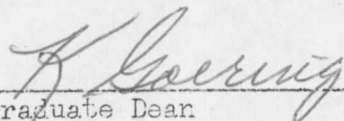
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ABSTRACT

This study is an analysis of the state of stress existing around a given crack in a biaxially-stressed, two-dimensional, elastic solid containing a system of cracks. Photoelasticity was used to define the stress condition around both a multiple-crack system used in the major study and around a single crack used in the minor study. This minor study was an experimental and analytical analysis of the stress conditions around a single crack. This information was used to analyze the experimental data obtained from the multiple-crack study.

Several characteristics of the specific multiple-crack systems studied were observed. The largest difference between principal stresses always occurred at the ends of the cracks. These end stresses were probably increased by decreasing the crack's radius of curvature, increasing its length, or orienting the cracks at about 60 degrees with the major applied stress. The end-stress magnitude was a function of both the sum of and the difference between the applied stresses. When the applied stresses were equal, the stress fields were symmetrical about the cracks and the end-stress magnitudes were probably proportional to the applied stresses. Changes in the stress fields due to changing the crack patterns are complex.

The results of the entire study are intended to provide part of the background information necessary for determining stress conditions in real fractured or jointed media.

CHAPTER I

INTRODUCTION

Background

This project was inspired by the very real and complex problems associated with determining the strength of fractured or jointed materials. These types of materials have been dealt with for many years by mining engineers who design tunnels and shafts, but recent developments in open-pit mining and nuclear excavation have greatly increased interest in the subject.

The U.S. Army Engineer Waterways Experiment Station has been conducting a major study to develop a general method for analyzing slopes formed by nuclear explosions under the study name, "Engineering Properties of Nuclear Craters" (1968). This study deals with the problems associated with both loose and intact rock slopes. The Bureau of Mines in conjunction with the Kennecott Copper Corporation made a study of a similar problem, designing an economical slope modification for the Kimbley Pit at Ely, Nevada. (Results of this study consist of four individual reports by Robert H. Merrill, Wilson Blake, Perry N. Halstead, and David W. Wiscarver, all of which were published in 1968 and are available by reprint from the Society of Mining Engineers of AIME.) These projects and others dealing with both slope stability (Terzaghi, 1962) and structural stability (Goodman, 1967) have one common problem, jointed material.

Several different methods are used for finding the stability of a slope or structure, but all require determining the strength of the materials and the mode of failure. The mode of failure commonly assumed is shear. Terzaghi (1962) uses this mode in presenting a technique for

evaluating the strength of intact rock slopes that is similar to his method for analyzing soil slopes.

In the present study, however, no mode of failure is chosen because the author is not convinced that any one failure mode controls for all applicable systems. The problem dealt with here is determining the stress conditions imposed upon a material by the presence of cracks. An extension of this information may allow finding the strength of a fractured material based on the properties of the parent material and the crack pattern.

For the case of a real material, the amount, orientation and types of joints and fractures must be known prior to analysis. Methods for determining this information on a statistical basis have been devised by Klaus (1962), Aho (1960), and others; and the parent material's physical properties can be found by conventional laboratory means. This says that methods are presently known for collecting the supporting data to solve the real problem regardless of whether or not good techniques are available for using this information.

Object and Scope

The purpose of this project is to analyze the state of stress in a two-dimensional solid containing multiple-crack systems. A real fractured media is much too complex to lend itself to analysis; so a simplified, well-controlled system must be adopted. Real problems exist in three dimensions, but here, the analysis is done in two dimensions to reduce the complexity of the experimental and analytical studies. The field of research is narrowed still further as each crack considered is entirely

surrounded by intact material, and the crack faces are not in contact with each other.

The problem that is left after all the simplifications have been made seems rather simple compared to the real problem. However, when one learns that the plate with a single discontinuity is considered to be a relatively complex problem with few, if any, satisfactory solutions, the multiple-crack problem even in the simplified form is very formidable indeed. Because so little information about the stress conditions around a single crack is available, this study has to be incorporated with the main problem.

The author chose to use a three-phase study of the problem. First, an analytical study of a single crack based on a solution presented by Durelli and Murray (1943) was carried out to define significant stress parameters and to obtain indications of how the parameters influenced the stress field. Second, a photoelastic study was conducted to give experimental indication of how the single-crack parameters behaved. The analytical and experimental conclusions were correlated and used as the single-crack characteristics. Third, a photoelastic study of multiple-crack patterns was conducted, and the results were analyzed in the light of the single-crack characteristics.

CHAPTER II

ANALYTICAL STUDY OF SINGLE CRACK

Method of Approach

An analytical approach to the problem of a single crack in a loaded plate may be made by using the solution to the problem of an elliptical discontinuity in a loaded infinite plate. By allowing the ellipse to approach the dimensions of a real crack, the ellipse solution becomes the crack solution. Several authors present solutions to the ellipse problem (Wang, 1953; Timoshenko and Lessells, 1925; etc.); however, many of these solutions exist in unclear forms or are for cases of uniaxial loading only. Durelli and Murray (1943) present a general solution which is clear and quite adaptable to this type of study; consequently, their form was chosen. The solution itself is credited to an Englishman, C.E. Inglis (1913).

The general solutions for stresses presented (Durelli and Murray, 1943) are complex infinite series that would be very difficult to evaluate. However, if several assumptions are made, a relatively simple equation is obtained that allows study of the most critical regions. Those assumptions are:

1. The largest stress concentrations in the field around a crack occur on the boundary of the crack. (This is confirmed by experimental results.)
2. The crack exists in an infinite plate and is loaded by uniform normal stresses applied on boundaries that are mutually perpendicular.
3. The larger applied stress acts in a direction that makes an angle ϕ with the major axis of the crack.

4. The length of the crack is very large compared to the width.
5. The plate containing the crack acts as a true elastic material.
6. The edges of the crack are not in contact with each other.
7. An assumption that is implied by the choice of the coordinate system is that the stress pattern is symmetrical about the major axis of the ellipse.

No restriction is placed on the type of failure because, at the crack boundary, only normal stresses parallel to the edge exist on a boundary element. The stresses on all other planes may be found by simply multiplying this stress by the appropriate value from a Mohr's circle relationship.

Definition of the Solution

Prior to presentation of the solution to be studied, an explanation of the coordinate system is in order. The following long quotation taken directly from the paper by Durelli and Murray (1943) best describes the coordinate system used:

The mathematical analysis for the elliptical discontinuity is simplified by the use of elliptical coordinates. According to this system each point in a plane can be represented in terms of the coordinates, α and ν , whose values in the cartesian system correspond to,

$$\begin{aligned}x &= c \cosh \alpha \cos \nu \\y &= c \sinh \alpha \sin \nu\end{aligned}\tag{1}$$

For conditions such that α equals a constant these may

be expressed as

$$\frac{x^2}{c^2 \cosh^2 \alpha} + \frac{y^2}{c^2 \sinh^2 \alpha} = 1 \quad (2)$$

which represents a family of confocal ellipses with a distance of $2c$ between the foci.

For conditions such that ν equals a constant there results

$$\frac{x^2}{c^2 \cos^2 \nu} - \frac{y^2}{c^2 \sin^2 \nu} = 1 \quad (3)$$

which corresponds to a family of confocal hyperbolae, orthogonal to the previous ellipses.

Every point represented by "x" and "y" in the cartesian system is the intersection between an ellipse and hyperbola of the elliptical system whose corresponding coordinates will be α and ν (Figure 1). At first this system may seem unduly complicated, however, it has the advantage that the boundary of an elliptical discontinuity can be given by holding one of the parameters, α , constant.

The parameter ν is also known as the eccentric angle of the ellipse. It can be determined for each ellipse (α equals a constant) in a simple manner by using the major and minor auxiliary circles of the ellipse as indicated in Figure 2.

If one thinks of the detail of the stress distribution and considers conditions at some point in the field then it is an easy matter to visualize a small rectangular particle whose center is on the intersection of a confocal ellipse and hyperbolae and whose sides are parallel to the tangents at the point of intersection. Such an element is shown in Figure 3.

Since this study is concerned with the region bordering the crack, Inglis' general solution takes the particular form shown by Durelli and Murray (1943)

$$\sigma_{\nu} = \frac{(\sigma_1 + \sigma_2) \sinh 2\alpha_0 + (\sigma_1 - \sigma_2) [\cos 2\phi - e^{2\alpha_0} \cos 2(\phi - \nu)]}{\cosh 2\alpha_0 - \cos 2\nu} \quad (4)$$

where

$$\sigma_{\alpha} = \tau_{\alpha\nu} = 0,$$

σ_1 = major applied principal stress,

σ_2 = minor applied principal stress,

$\sigma_v, \sigma_a, \tau_{av}$ = stresses on the element defined in Figure 3,

a, v = the elliptical coordinates of any point,

a_0 = the coordinate that is constant along the edge of the crack,

ϕ = angle between the crack and direction of the major applied stress.

Having the solution completely defined, one can now look at the modifications for the problem at hand. First, the ranges of some of the terms in the σ_v equation may be defined by applying the assumption that the crack's length is large compared to its width. The ellipse information derived in Appendix A provides the relationship between crack length-to-width ratio and a_0 .

$$a_0 = \sinh^{-1} \frac{b}{\sqrt{a^2 - b^2}} \quad (5)$$

If "a" is much larger than "b", then a_0 is approximately equal to $\sinh^{-1} b/a$, where b/a is very small compared to one. Table I shows that for small a_0 , a_0 is approximately equal to $\sinh a_0$ from which one concludes that a_0 is approximately equal to b/a , where b/a is very small compared to one.

Study of the Crack Boundary Solution

As the equation for σ_v is simplified by applicable assumptions and approximations, one is allowed to look at how σ_v varies with location on the crack edge, orientation of applied stresses, crack size and crack

TABLE I. VALUES OF FUNCTIONS
FROM HANDBOOK OF MATHEMATICS BY ABRAMOWITZ (1964)

$2a_0$	$\sinh 2a_0$	$\cosh 2a_0$	e^{2a_0}
0.0000	0.000000000	1.000000000	1.000000000
0.0001	0.000100000	1.000000005	1.000100005
0.0010	0.001000000	1.000000500	1.001000500
0.0100	0.010000167	1.000050000	1.010050167
0.1000	0.100166756	1.005004168	1.105170918
1.0000	1.175201194	1.543080635	2.718281828

shape. These variables will be studied in the following three phases:

1. Let σ_1, σ_2 and ϕ vary while other variables are treated as constants.
2. Let the ratio a/b vary while all other variables are held constant (vary crack shape).
3. Let "a" vary while the ratio, a/b as well as all other variables, are held constant (vary crack size).

Variation of Crack Orientation and Loads' Magnitudes

Consider the two terms comprising σ_v in the form

$$\sigma_v = \frac{(\sigma_1 + \sigma_2) \sinh 2a_0}{\cosh 2a_0 - \cos 2v} + \frac{(\sigma_1 - \sigma_2) [\cos 2\phi - e^{2a_0} \cos 2(\phi - v)]}{\cosh 2a_0 - \cos 2v}$$

where, from Table I for very small $2a_0$

$$\cosh 2a_0 \approx 1 \quad (\text{slightly larger}),$$

$$e^{2a_0} \approx 1 \quad (\text{slightly larger}),$$

$$\sinh 2a_0 \approx 0 \quad (\text{slightly larger}),$$

such that the first term becomes

$$\frac{(\sigma_1 + \sigma_2) (0+)}{(1+) - \cos 2v}, \quad (\text{Term One})$$

where ν is arbitrary so that $\cos 2\nu$ can have any value between plus one and minus one. In general, this term will be small (excluding the case when $\cos 2\nu$ approaches one). The second term becomes

$$\frac{(\sigma_1 - \sigma_2) [\cos 2\phi - (1+) \cos 2(\phi - \nu)]}{(1+) - \cos 2\nu} \quad (\text{Term Two})$$

Immediately, two cases have become apparent: when σ_1 equals σ_2 , and when σ_1 does not equal σ_2 .

If σ_1 equals σ_2 then

$$\sigma_\nu = \frac{(\sigma_1 + \sigma_2) (0+)}{(1+) - \cos 2\nu},$$

which says σ_ν will have significant values only if $(1+) - \cos 2\nu$ approaches zero and where, in turn, 2ν approaches zero or 2π . This says σ_ν is independent of ϕ and is maximum at the ends of the crack where ν equals zero or π . Also, σ_ν is directly proportional to $(\sigma_1 + \sigma_2)$ at all points. (Consider only the range of ν from zero to π , because that is the total range of ν according to this coordinate system.)

If σ_1 is not equal to σ_2 , the first term behaves as it did above; however, the total σ_ν value includes the second term which does not act as nicely. Looking at Term Two independently from Term One may permit evaluation of the effect of the value of ϕ . Variation of Term Two's value with respect to $(\sigma_1 - \sigma_2)$ is obviously a direct proportionality, but the ϕ and ν effects are not immediately apparent. Consequently, Figure 4 is used to show the nature of these variations.

The curve connecting maximum M (M is defined by Figure 4) values

for constant ν 's allows prediction of the ϕ value that gives the absolute maximum stress contribution from Term Two. By fitting a curve through calculated points, the equation for this curve was found to be M equals $\sec 2.00\phi$. The ϕ angle between zero and $\pi/2$ that makes M maximum is 45 degrees. A second conclusion drawn from Figure 4 is that for Term Two, maximum stresses occur at the ends of the crack (ν equals zero and π) for all values of ϕ .

The effects of Terms One and Two can be combined only in the light of certain qualifying remarks. First, compression stresses are considered to be positive and tension stresses negative. The largest compressive stress or most compressive stress is σ_1 ; the smaller compressive stress or most tensile stress is σ_2 . Therefore, by this convention $(\sigma_1 - \sigma_2)$ will always be positive, but $(\sigma_1 + \sigma_2)$ may be of either sign. The Term One coefficient of $(\sigma_1 + \sigma_2)$ is $\sinh 2\alpha_0 / (1 - \cos 2\nu)$ which is always positive; and the coefficient of $(\sigma_1 - \sigma_2)$ in Term Two is M which varies in sign. For arbitrary ν , nothing can be concluded about the combination of Term One with Term Two because of the sign variation. However, both these terms independently indicate that the largest stress concentrations occur at the ends of the crack regardless of the value of ϕ . For this reason, the points where ν equals zero and π will be studied in more detail.

For ν equals zero, the equation for σ_ν (Equation (4)) becomes

$$\sigma_\nu = \frac{(\sigma_1 + \sigma_2) \sinh 2\alpha_0 + (\sigma_1 - \sigma_2) \cos 2\phi (1 - e^{2\alpha_0})}{\cosh 2\alpha_0 - 1} \quad (6)$$

From Table I it is seen that for $2a_0$ in the range of 0.001, $\sinh 2a_0$ is very nearly equal to $(e^{2a_0} - 1)$. This relationship may be shown more analytically by an algebraic manipulation. By the definition of the hyperbolic sine

$$\sinh 2a_0 = \frac{e^{2a_0} - e^{-2a_0}}{2}.$$

Forming a least common denominator gives

$$\sinh 2a_0 = \frac{e^{4a_0} - 1}{2e^{2a_0}},$$

and factoring leaves

$$\sinh 2a_0 = \frac{(e^{2a_0} - 1)(e^{2a_0} + 1)}{2e^{2a_0}}. \quad (7)$$

When $2a_0$ becomes very small, Table I shows that e^{2a_0} approaches one. This says the remaining term in Equation (7), $(e^{2a_0} + 1)/2e^{2a_0}$, also approaches one. That leaves

$$\sinh 2a_0 = e^{2a_0} - 1. \quad (8)$$

(See Figure 5 for a graphical comparison of $\sinh 2a_0$ and $(e^{2a_0} - 1)$ over the range of $2a_0$ shown in Table I.) Making the substitution of $\sinh 2a_0$ for $(1 - e^{-2a_0})$ in Equation (6) gives

$$\sigma_v = \frac{(\sigma_1 + \sigma_2) \sinh 2a_0 + (\sigma_1 - \sigma_2) \cos 2\phi (-\sinh 2a_0)}{\cosh 2a_0 - 1}$$

or

$$\sigma_v = \left[(\sigma_1 + \sigma_2) - \cos 2\phi (\sigma_1 - \sigma_2) \right] \left[\frac{\sinh 2a_0}{\cosh 2a_0 - 1} \right]. \quad (9)$$

The variation of σ_v with respect to ϕ is expressed by

$$(\sigma_1 + \sigma_2) - \cos 2\phi (\sigma_1 - \sigma_2).$$

The remaining term will be explored later to show σ_v variation with respect to change in crack size and shape.

It has already been shown that for any given σ_1 and σ_2 , Terms One and Two, independently, give maximum stress contribution when v equals zero. Therefore, if their contributions are added with like sign when v equals zero, the absolute maximum stress magnitude occurs. The sign will be that of $(\sigma_1 + \sigma_2)$. The Term One contribution is represented by $(\sigma_1 + \sigma_2)$ and the Term Two contribution is represented by $(\sigma_1 - \sigma_2) \cos 2\phi$, as all other terms are factorable. The minimum value of stress at v equals zero for given σ_1 and σ_2 values will occur when $(\sigma_1 + \sigma_2)$ and $(\sigma_1 - \sigma_2)$ are added with opposite signs. The value for the combination will have the sign of the larger (that is, larger in absolute value). The only way that the sign between $(\sigma_1 + \sigma_2)$ and $(\sigma_1 - \sigma_2)$ can be varied is by picking ϕ such that $\cos 2\phi$ changes sign. Any values of $|\cos 2\phi|$ less than one will not give maximum or minimum values of σ_v ; consequently, for these conditions ϕ must be either zero or $\pi/2$.

Perhaps one of the most important observations is that, if σ_2 is a tensile stress, then it is also possible for σ_v to be a tensile stress. A stress condition of this type would be particularly important if it occurred in a material such as rock, where the tensile strength is much smaller than the compressive strength. This may mean that in rock any of three types of failure are possible: compressive, tensile or shear.

The other point of interest on the crack is where v equals π . Equation (4) becomes

$$\sigma_v = \frac{(\sigma_1 + \sigma_2) \sinh 2\alpha_0 + (\sigma_1 - \sigma_2) \cos 2\phi - e^{2\alpha_0} \cos 2(\phi - \pi)}{\cosh 2\alpha_0 - \cos 2\pi},$$

but $\cos 2\pi$ equals one and $\cos(2\phi - 2\pi)$ is equal in sign and magnitude to $\cos 2\phi$; consequently, the argument is reduced to that of v equals zero. This says the stress condition described by v equals zero applies to both ends of the crack.

Variation of Crack Shape

When the length of a crack is changed with respect to its width, the shape undergoes a change that may be related by the elliptical parameters, "a" and "b". Consider what will happen to the stress at v equals zero if the crack becomes longer with respect to its width, but all other factors are held constant. This says, although "b" is already small with respect to "a", it becomes even smaller. Since

$$\alpha_0 = \sinh^{-1} b/a,$$

α_0 will decrease as this change occurs.

The expression developed previously that isolated the α_0 effect upon σ_v is

$$\sigma_v = K \left[\frac{\sinh 2\alpha_0}{\cosh 2\alpha_0 - 1} \right] \quad (10)$$

where K (K is equal to $[(\sigma_1 + \sigma_2) - \cos 2\phi(\sigma_1 - \sigma_2)]$) is now held constant. Substituting the definition for $\cosh 2\alpha_0$ and $\sinh 2\alpha_0$ into this equation and simplifying, gives an expression that makes obvious the effect of changing α_0 .

$$\frac{\sinh 2\alpha_0}{\cosh 2\alpha_0 - 1} = \frac{\frac{1}{2}(e^{2\alpha_0} - e^{-2\alpha_0})}{\frac{1}{2}(e^{2\alpha_0} + e^{-2\alpha_0} - 2)}$$

$$\frac{\sinh 2\alpha_0}{\cosh 2\alpha_0 - 1} = \frac{e^{4\alpha_0} - 1}{e^{4\alpha_0} + 1 - 2e^{2\alpha_0}}$$

$$\frac{\sinh 2\alpha_0}{\cosh 2\alpha_0 - 1} = \frac{(e^{2\alpha_0} + 1)(e^{2\alpha_0} - 1)}{(e^{2\alpha_0} - 1)(e^{2\alpha_0} - 1)} \quad (11)$$

As α_0 decreases, $e^{2\alpha_0}$ decreases toward one and the value $\sinh 2\alpha_0 / (\cosh 2\alpha_0 - 1)$ increases (See Figure 6); σ_v increases.

The shape of this crack may also be defined in terms of the length and the radius of curvature at the ends. William Gerberich (1962) states that for an internal crack the principal stress difference is approximately proportional to the minus one-fourth root of the radius of curvature. A corresponding relationship based upon the assumptions previously stated in this paper will now be derived.

The expression for radius of curvature at the ends of the crack, ρ , as derived in Appendix B, is

$$\rho = b^2/a.$$

Substituting ρ into $\sinh \alpha_0$ equals b/a gives

$$\sinh \alpha_0 = \rho/b,$$

and

$$b = \sqrt{\rho a},$$

so that

$$\sinh \alpha_0 = \sqrt{\rho a}/a \approx \alpha_0.$$

The expression for σ_v can now be written

$$\sigma_v = K \left[\frac{e^{2\alpha_0} + 1}{e^{2\alpha_0} - 1} \right]$$

where $(e^{2\alpha_0} - 1)$, as shown in Equation (8), is approximately equal to $\sinh 2\alpha_0$ which is in turn approximately equal to $2\alpha_0$. Also, that same

section shows that $(e^{2\alpha_0} + 1)$ is approximately equal to two. If the substitution of α_0 equals $\sqrt{\rho a}/a$ is carried out, the σ_v expression may then be written as

$$\sigma_v = (K\sqrt{a})\rho^{-\frac{1}{2}}, \quad (12)$$

which says σ_v is proportional to the minus one-half power of radius of curvature at the ends of the crack. This expression agrees with Gerberich's (1962) conclusion about how the principal stress differences vary, with respect to radius-of-curvature change, around a single-edged crack (external crack).

This last expression for σ_v (Equation (12)) not only shows how σ_v varies with respect to radius of curvature (ρ) when length (a) is held constant, but also shows that σ_v increases in direct proportion to the square root of length as length increases.

Variation of Crack Size

When the crack size is varied without changing its shape, the ratio a/b remains constant while "a" varies. The only term in σ_v that reflects this type of change is α_0 . Since α_0 equals $\sinh^{-1} b/a$ and b/a remains constant, α_0 remains constant and the stress σ_v will be unchanged by varying "a". This means that the magnitude of σ_v is unchanged for all points in the plate, if the crack changes size but not shape and if all other variables are held constant.

Summary of the Analytical Observations

The following observations can be made concerning the variation of σ_v with variation of σ_1 , σ_2 and ϕ :

1. The maximum σ_v values always occur at the ends of the

crack. (Here, maximum means largest tensile or largest compressive stresses occurring in the region of the crack.)

2. Stress, σ_v , is directly proportional to $(\sigma_1 + \sigma_2) - \cos 2\phi(\sigma_1 - \sigma_2)$.
3. When the applied stresses are equal, σ_v is independent of ϕ and is directly proportional to twice the applied stress.
4. The absolute maximum value of σ_v occurs when $(\sigma_1 - \sigma_2)$ is added with like sign to $(\sigma_1 + \sigma_2)$, and the sign of σ_v is the same as that of $(\sigma_1 + \sigma_2)$.
5. The minimum values of σ_v at the ends of the crack occur when $(\sigma_1 - \sigma_2)$ is added with unlike sign to $(\sigma_1 + \sigma_2)$, and the sign of σ_v will be the same as that of the term having the larger absolute value.
6. Maximum and minimum stresses will occur at the ends of the crack when the crack is either parallel or perpendicular ($\phi = 0, \pi/2$) to the direction of the largest applied stress.
7. If one of the applied stresses is a tensile stress, σ_v may be a tensile stress, depending on the value of ϕ .

The following observations can be made concerning σ_v as the crack size and shape are varied:

1. Changing the crack's size without changing its shape does not alter the stress magnitude or distribution.
2. If the length of the crack is held constant, σ_v is directly proportional to the minus one-half power of the radius of

curvature at the ends of the crack.

3. If the radius of curvature of the crack is held constant, σ_v is directly proportional to the square root of the length.

CHAPTER III

LABORATORY RESEARCH

The laboratory approach to this problem was dictated by the available analytical approaches and the assumptions made therein. Consequently, a plane model of linearly elastic material loaded by uniform stresses on mutually perpendicular edges was chosen. Since high stress gradients were expected in some portions of the model and the shape of stress patterns were of interest, a photoelastic study using birefringent plastic was made. This provided continuous stress patterns throughout the model and stress magnitudes at the crack boundary.

Brief Explanation of Photoelasticity

A detailed explanation of photoelasticity and the physical principles involved was found in several sources. Coker and Filon (1957) gave what was probably the most rigorous treatment of the subject, while Frocht (1948) also gave quite a complete explanation. The author found that, for his purposes, Durelli and Riley's Introduction to Photomechanics (1965) presented the simplest, most direct explanation of photoelasticity.

When polarized light passes through a stressed birefringent material and is viewed through an analyzer, fringe patterns may be seen. If the light is circularly polarized and monochromatic, the patterns that appear (called isochromatics) are a function of the difference between the principal stresses. In other words, each line represents a continuous locus of points where the maximum shear stress at each point on the locus is constant. The number of fringes existing between a point of zero maximum shear stress and a point of non-zero maximum shear stress is called the fringe order. If the fringe order is known (found by simply counting the

fringes), then the maximum shear stress at that point may be found by applying the stress-optic law which, in two-dimensional form (Durelli and Riley, 1965), is

$$n = \frac{Ct}{\lambda} (\sigma_1 - \sigma_2) = \frac{2Ct}{\lambda} \tau_{\max}$$

where

n = fringe order,

C = relative stress-optic coefficient which is a characteristic of the birefringent material and is found experimentally,

λ = wavelength of the light,

t = thickness of the birefringent material,

$\sigma_1, \sigma_2, \tau_{\max}$ = secondary principal stresses and secondary maximum shear stress associated with the direction of light propagation.

In this study only the magnitudes of stresses acting along the crack boundary are of interest, because stresses at other points can not be correlated with analytical results. The boundary-element stresses may be directly related to fringe order, because the shear stresses and the principal stress acting normal to the boundary are zero. This says that the principal stress acting parallel to the boundary equals twice the maximum shear stress at that point.

The Models and Model Material

The model selected was a 6 by 6 by $\frac{1}{4}$ in. plastic, plane-simulating model of "Hysol 4290 Epoxy Resin." Limitations of the optical equipment available controlled the model size, and the analytical solution dictated the plane shape. "Hysol 4290 Epoxy Resin" was chosen as the model material.

because it had a relatively high proportional limit (7000 psi), low stress fringe value (30 psi-in./fringe), and good machinability.

Crack Patterns

A standard crack (See Figure 7) was used for the major study so that the effects of crack orientation and influences of multiple cracks on a given crack's stresses could be isolated. Basically, two categories of cracks were tested: single cracks and multiple-crack systems. The single-crack models consisted of the standard crack cut in the center of the plate and oriented at various angles with the plate edges (See Figure 8a). The multiple-crack systems and their variations are shown in Figure 8, b, c and d. The central crack was considered to be representative of the standard crack surrounded by an infinite field of cracks, or at least to indicate how the presence of other cracks influenced the stress of a given crack.

Minor studies were carried out to show variation of stress magnitudes as a function of crack shape. The models for these studies consisted of the standard 6 by 6 by $\frac{1}{4}$ in. plate with a single crack in the center. The crack shape was modified by increasing the length while holding the end radius constant for one case, and changing the end radius while holding the length constant for the other case.

A model of a prismatic beam was used as a calibration device to determine the physical properties of the plastic.

Construction of Models

The "Hysol 4290" was purchased in sheets 24 by 24 by $\frac{1}{4}$ in. and covered with masking tape to prevent scratching of the surfaces during sawing.

These large sheets were marked and sawed with a band saw into six-inch squares. Circular cuts were made at each corner so that the final configuration was that shown in Figure 9. These last cuts were unwanted modifications that were made necessary by the geometry of the loading device. After cutting was complete and the saw marks were removed by filing, the models were annealed to remove residual stresses.

The second phase of preparation involved cutting the cracks. The crack pattern was scribed on the model surface and holes were drilled at the center of each crack. A jeweler's sawblade was inserted into the hole and the crack was sawed in the proper orientation radiating outward from the hole. Then holes were drilled at each end to provide a controlled radius of curvature. The model was annealed again to remove most of the machining stresses created around the crack.

Because of "edge effect," the second phase of model construction, cutting the crack, was performed one day before the test on that model was to be run. When a freshly cut surface was exposed to air for a period of time, fringes appeared at the edge and progressed slowly into the plastic. These fringes, called "edge effect," could not be removed by annealing, and they distorted the temporary patterns due to loading. The crack provided free edges that were important points of study and therefore had to be free from "edge effect." However, the model boundaries, which were also free surfaces, were outside the field of study; consequently, there, the "edge effects" did no harm.

The Apparatus

The apparatus used in this study consisted of commercially-manufactured

optical and photographic equipment, and "home-made" model-loading devices. Other standard laboratory equipment was used to calibrate the loading devices.

Loading Machine

The initial phase of this study involved designing and constructing a machine that would apply a uniform compressive stress along the edges of a 6 by 6 by $\frac{1}{4}$ in. plate. Dr. R.L. Sanks, a Montana State University faculty member, suggested the means of loading which was finally adopted. A flexible hose containing a fluid under pressure was confined on three sides by a loading frame, and on the fourth side by the edge to be loaded. The result was that the fluid pressure was applied to the model edge by way of the flexible rubber hose.

Since a condition of biaxial loading was desired, two independent sources of fluid pressure were necessary. These sources were the pressure chambers of two hydraulic jacks (See Figure 10). The jacks were jacked against large springs to create a load on the jack piston. This load forced the piston into the pressure chamber, inducing a fluid pressure that was proportional to the load. Because of the linearity of the spring force with respect to displacement, the force and consequently the fluid pressures were directly proportional to the piston displacement. The springs were included so that when jacking occurred, the change in chamber pressure would be gradual instead of instantaneous.

Copper tubes were used to tap the fluid pressure from the pressure chamber and conduct it to a standard gage where the pressure was measured. From there, two rubber tubes, confined in an outer copper tube, ran from each gage to the loading frame where the tubes were then confined by the

frame and model (See Figure 11). When fluid was forced into the hoses, they tried to expand, but no space was available for expansion so pressure was created and applied to the surfaces bounding the hoses.

Each pair of hoses from the same gage applied an equal stress on opposite edges of the model. The result was a loaded model that was always in static equilibrium and did not move during the loading process. A biaxial condition was obtained by pressurizing both sets of hoses, and since the hoses were flexible, negligible shear stresses were applied as loads.

The corners were cut out of the models to provide space for the hose end-caps and spacers. Spacers were required to prevent hose collapse at intersections and to keep the hose confined at all points. Whenever gaps occurred around a pressurized hose, the rubber would bulge and eventually rupture.

A second loading machine was devised for calibrating the birefringent plastic. This machine consisted of a frame that supported a six-inch plastic beam on two fulcrums, and had guides for moveable load-applying fulcrums. Weights were placed on the moveable fulcrums so that a condition of pure bending was induced in the beam (See Figure 12). Calibration was accomplished by counting the fringes between two points of known stress (the neutral axis and the extreme outer fiber) and applying the stress-optic law (See Appendix C).

Polariscope and Camera

The polariscope and accessories were standard laboratory equipment (See Figure 13). A mercury vapor lamp with green and yellow filters

produced a monochromatic light that radiated from a vertical filament. The light was collected and concentrated at a point by a short-focal-length lens. A light diffuser was placed at the point of light concentration so that the filament image was obliterated, and a field of uniformly intense light was produced. This uniform field acted as a light source that was collimated by a second lens and projected through polaroid lenses.

The first lens was the polarizer which produced plane-polarized light. Then the light was circularly polarized by a quarter-wave plate and passed through the model to a second quarter-wave plate. Here the light was restored to the plane-polarized state and rotated ninety degrees from the original polarization. All the light that was retarded (by the model) an integral multiple of the wave length was passed through the analyzer to produce an image of green light. Light that was retarded an integral multiple of one-half the wave length (0.5, 1.5, 2.5, etc.) was blocked by the analyzer to produce an image that was black. All other degrees of retardation produced images that were some shade between green and black. This combination of polaroid lenses produced a light field.

Two long-focal-length lenses were used to magnify the model's image; then it was projected into a commercial portrait camera for viewing and photographing.

Laboratory Procedures

Two of the procedures used in this study require a certain amount of explanation because their accuracy had a direct bearing on the quality of the experimental results. Since calibration of the loading machine was continuous throughout the model-testing operation, these two procedures

were tied together.

Calibration of Equipment

Calibration of the loading machine consisted of relating the stress applied along a model's boundaries to a reading on the pressure gage. This correlation was done indirectly by relating loads applied to the steel coupon shown in Figure 14 to strains experienced by its strain gages as recorded by a strain indicator. Then the coupon was placed in the model-loading machine and loaded to give a relationship between strains and pressure-gage readings (See Figure 15). The load-strain and the strain-gage data were equated to give load versus gage-readings data which was then reduced to applied stress versus pressure-gage readings. (The applied stress was based on the 6 by $\frac{1}{4}$ in. area, not the 4.85 by $\frac{1}{4}$ in. area.)

Part of the applied stress was due to compression of the pressure hoses by the loading frame. This portion of the stress simply added to the stress due to the fluid pressure; however, the "hose-compression stress" varied with the age of the rubber hose. For this reason, the graphical representation of applied stress versus pressure-gage readings consisted of a family of parallel curves (See Figure 16). Each curve was labeled with a strain reading which showed the amount of strain induced in the coupon by bolting it into the loading frame. Prior to each test, the coupon was bolted into the frame and the strain was read. With this information the applicable curves were selected and the test was ready to begin.

Testing Procedure

The gage readings corresponding to applied stresses of 200, 400 and

600 psi were read from the curves for both gages, and the model was bolted into the loading frame. Positioning of the optical equipment was checked and a name tag indicating the model number and loading was stuck to the model. The desired loading was applied, the camera was focused, and a picture was taken. Another name tag was then stuck to the model and the procedure was repeated.

After all the loading combinations for a model were photographed, the negatives were developed and pictures were made. These pictures constituted the experimental data that was to be analyzed.

Experimental Source of Errors

The major obstacle in production of experimental data that would correlate with the analytical predictions made was the crack shape. For reasons of practical model production, the crack was not even a good approximation of an ellipse; consequently, experimental results were assumed to be good qualitatively, but not especially good quantitatively.

The assumption that the cracks existed in an infinite plate undoubtedly affected some of the results. This effect was apparent where the stress patterns extended outside the test region, but no quantitative estimate of the error could be made.

The loading-machine calibration procedure was a multiple source of error. Most unpredictable was the effect of the pressure hoses on applied stresses. A slight variation in the sizes and shapes of the models meant that the amount they compressed the pressure hoses would vary; yet this variation had to be neglected because hose pressure was always measured by the steel coupon. Even though the hoses probably did not apply a perfectly

uniform stress, stress patterns of loaded plates with no cracks showed a uniform field of light over the test area; so uniformity was no problem. The machine calibration also allowed for a possible accumulation of error due to the following: the strain gages not being on the neutral axis of the coupon; not using the exact gage factor for the strain gages; making errors in readings; and using an uncalibrated "Soil-Test" loading machine.

Load creep, a consequence of the loading-machine system, required that the load applied be slightly higher than the desired load and that the picture be taken when the desired load was passed. Another problem arose when a hose broke during a test. A new hose was installed, but time did not always allow recalibration of the loading frame.

The models were not all the same thickness and the plastic calibration varied with time. These variables were evaluated in Appendix C and taken into account in the analysis.

All the errors mentioned, other than crack shape, undoubtedly had certain influences on the experimental results. However, these errors were judged to be small relative to those errors due to crack shape and to the inability to count fringes precisely; for these reasons, the errors are not stated in numerical terms.

Interpretation of Experimental Single-Crack Results

The experimental data interpreted in this section is the set of isochromatic photographs dealing with single cracks. Each of the four models was loaded with the combinations listed in columns one and two of Table II. A photograph of each model under each loading combination was

TABLE II. LOADING COMBINATIONS FOR MODELS TESTED

(1) Model Top psi	(2) Model Side psi	(3) Major Applied Stress psi	(4) Minor Applied Stress psi
400	0	400	0
400	200	400	200
400	400	400	400
400	600	600	400
0	400	---	---
200	400	---	---
600	400	---	---

taken so that the array of results represented cracks oriented at angles of 0, 15, 30, 45, 60, 75, and 90 degrees with the major applied stress, and under the loading combinations listed in columns three and four of Table II. The presentation of experimental results is made in the light of the analytical assumptions and conclusions previously stated and the crack used is assumed to behave similarly to a real crack.

Observations about Assumptions

Only one of the assumptions made in the analytical presentation seems to be in error. The stress patterns are not always symmetrical about the crack. The only conditions that produce axially symmetrical stress patterns are: (1) where the crack is parallel or perpendicular to the applied stress, and (2) where the applied stresses are equal.

Observations about Loading and Crack Inclination

All stress patterns show that the highest fringe gradient occurs at the end of the crack. In this study only the isochromatics were obtained, and they do not indicate whether the gradient is toward increasing maximum shear stress or decreasing maximum shear stress. However, intuition and

experience tell one that a crack propagates from itself, which means the failure-causing stress at the point of propagation is larger than the stresses in the field around the crack. In addition, a crack propagates from its end, which means the highest stress around a crack occurs at or near that end. Using this reasoning, one can conclude that the fringe order is highest at the end of the crack; therefore, since σ_v on the crack is directly proportional to the fringe order, σ_v is maximum at the ends of a crack. This experimental data confirms only the case when σ_1 and σ_2 are compressive stresses.

Enlarged photographs of the single-crack models were analyzed to obtain the variations in stress due to crack orientation and applied stress. These results are shown as the solid lines on the graphs in Figures 17 and 18. The long-dashed lines in these same figures are the corresponding analytical results based on Equation (9), and the short-dashed lines are based on Equation (4). The crack properties were calculated on the basis of the derivations appearing in Appendices A and B.

A graphical comparison of experimental data and analytical data (See Figure 17) shows that for all sets of data, σ_v increases as ϕ increases from zero toward ninety degrees. Experimental results indicate that the maximum σ_v occurs when ϕ is around 75 degrees, while the equations predict maximum stresses when ϕ is ninety degrees. This difference in maximums may be due to experimental errors, or it may be an indication that the analytical solutions do not predict the correct stress conditions. Regardless of which observation is accepted, one can conclude that when the applied stresses are compressive, the minimum σ_v

occurs when ϕ is zero. Figure 17c illustrates another relationship between applied and induced stresses. For both experimental and analytical data, when σ_1 equals σ_2 , the stress σ_v is independent of ϕ and is directly proportional to σ_1 .

Figure 18 is used to show why the analytical curves and the experimental curves in Figure 17 are not more closely related quantitatively. Graphs a, c, e, g, and i illustrate that for all values of ϕ , the agreement between experimental and analytical results is poor when σ_2 is much smaller than σ_1 or nearly equal to σ_1 . Furthermore, experimental σ_v 's decrease or remain constant as σ_2 approaches σ_1 , while analytical σ_v 's always increase for the same change. Graphs b, d, f, h, and j indicate that when $(\sigma_1 - \sigma_2)$ is held constant, the experimental and analytical solutions diverge for all values of ϕ as $(\sigma_1 + \sigma_2)$ increases; but they both indicate that σ_v increases. These variations may mean that the relationships containing σ_1 , σ_2 and ϕ are not good in general, but give good solutions for the case when σ_1/σ_2 is one-half.

Observations about Crack Shape

A test was conducted to study the variation of σ_v at the end of a crack with respect to radius of curvature at that point. This test consisted of uniaxially loading a model containing a one-half inch crack and changing the radius of curvature while the applied stress was held constant. The variation in radius was 0.010, 0.014, and 0.020 inches.

The results were assumed to be inconclusive because such a small variation in curvature resulted in a small change in fringe order, and the fringes could not be counted accurately. Also, the crack shape or deviation

from the proper crack shape probably changed the proportionality somewhat. However, the results as plotted in Figure 19 indicate that σ_v is inversely proportional to the one-fifth power of ρ . This agrees quite well with Gerberich (1962) who stated σ_v is inversely proportional to the one-fourth power of ρ .

The proportionality between crack length and σ_v was studied by holding the applied stress and radius of curvature constant while changing the length. Only one such test was conducted, but since the variation was quite large ($\frac{1}{2}$, 1, and $1\frac{1}{2}$ in.) and the length was easily controlled, the results were assumed to have been fair. According to the analytically-derived relationship, σ_v should be proportional to the square root of length. The data plotted in Figure 20 agrees with this proportionality, but more supporting data is required before the conclusion can be verified.

CHAPTER IV

STUDY OF MULTIPLE-CRACK SYSTEMS

The graduation in study from a single-crack to a multiple-crack system is not a simple one. For instance, to the author's knowledge, no analytical solutions of this specific problem exist today and probably few, if any, numerical solutions exist. No attempt at obtaining an analytical solution is made in this study; only suggestions of how to approach the problem are presented.

Analytical Approach

At first glance one may assume that superposition of several single-crack solutions would give a good solution. However, these solutions are only good for a single crack in an infinite plate. When a second crack is introduced, the boundary conditions, from which both these solutions were obtained, are changed so that the individual solutions are no longer valid for the individual cracks. This means that when one tries to superimpose two such solutions, he invalidates both solutions just by the act of superposition. How much in error these solutions become is a question that must be answered before this method can be completely dismissed. Before this error can be evaluated, a general solution for the stresses at all points in the plate must be known. This brings one back to the problem of first finding a good general solution to the single-crack problem.

The single-crack solution studied in this paper is not general enough to be used in the evaluation of superposition, and obtaining the constants required to complete Inglis' (Durelli and Murray, 1943) general solution is a problem too large to be solved in a study such as this one. Furthermore, it has been shown that Inglis' solution applies only for specific

loading conditions. For this reason one may wish to choose some other solution for evaluation. Limitations in the scope of this study prevent extensive study of other solutions, and therefore do not permit evaluation of the superposition technique.

A numerical method that may be quite applicable to the solution of this problem is the finite element approach. Zienkiewicz (1967) states that the method has an almost limitless scope of applicability in dealing with practical problems. Furthermore, problems involving anisotropic materials, thermal stresses, or body forces may be solved with relative ease.

The presence of discontinuities, such as cracks which cause high stress gradients, requires that a large number of elements be used for good results in those regions. The complexity of the multiple-crack problem would probably require such a large number of elements that the accuracy of the solution would be dictated by the computing facilities used. Because of the flexibility of the finite element approach, it may be one of the better methods of solving this problem, especially if anisotropic materials are being analyzed.

Another method for solving this problem is the electrical analogue. Redshaw and Rushton (1958) used this method to solve the problem of a single crack in a uniaxially-loaded thin plate. According to their article, results obtained by the electrical analogue appeared to be reasonable and agreed with the available analytical results. They considered the major limitation of this technique to be its inability to represent the infinite stress at the crack ends. This problem would

probably be partially eliminated in the case of a real crack because no infinite stresses could exist. Use of a continuous grid may also eliminate this shortcoming of the electrical analogue.

Experimental Approach and Results

The experimental approach to the multiple-crack problem was exactly the same as the approach to the single-crack problem, except, here, systems of the standard crack were tested. All the equipment and procedures used in this portion of the study were the same as those described in Chapter III.

The systems of cracks shown in Figure 8b are studied to show the effects of ϕ , σ_1 and σ_2 variations on the isochromatic patterns. The other crack systems (See Figure 8c and d) are used to give some indication of how pattern variation causes changes in fringe patterns and gradients. This indication is obviously superficial because complete treatment of this aspect is a large project in itself. The effect of crack-shape change is not studied and neither is the effect of crack size, although both conditions are discussed later.

Obtaining fringe orders anywhere on the pictures is impossible because, due to the limited field of illumination, the outer regions of the test field which contain a known fringe order fall outside the pictures. For this reason, trends in stress changes must be studied by way of fringe gradients.

Since no analytical solutions have been found for this problem, the experimental results constitute the only solutions. No comparison of experimental and analytical results (as is done for the single-crack

problem) is possible for the multiple-crack problem. Furthermore, just interpreting the isochromatic pictures requires a certain amount of insight into the behavior of stresses around the individual cracks. Because no other results are available, this insight must be gleaned from the single-crack observations stated previously.

Certain conditions of loading produce isochromatic patterns that are axially symmetrical with respect to the central crack. Data observation shows that these conditions occur when the crack axis is parallel or perpendicular to the major applied stress for all combinations of loading, and for all crack orientations when σ_1 equals σ_2 . (These are the same conditions that produced symmetrical isochromatic patterns in the single-crack tests.) All other loadings and crack orientations produced unsymmetrical patterns.

As in the single-crack fringe patterns, high fringe gradients occur at the crack ends. Based on single-crack observations, these gradients probably indicate that the highest fringe orders, and consequently the largest stresses, occur near the ends of the cracks. For any given set of compressive loads, the highest gradients occur when ϕ is in the range of 45 to 75 degrees (See Figure 21). For ϕ 's decreasing from 45 to zero degrees, the gradients drop to nearly zero, while for ϕ 's approaching 90 degrees, the gradients drop to about 60 per cent of the maximum. This indicates that for this crack pattern and the range of loads used, the maximum stresses occur when ϕ is about 60 degrees.

The observations about loading variation as related to stress conditions are inconclusive and may apply only to the specific models and loadings used. Stress gradients, and probably stresses, seem to be a

function of both the sum of and the difference between the applied stresses. When the major applied stress is held constant and the minor applied stress is increased, the crack end-stress gradients decrease (See Figure 22). When the difference between the applied stresses is held constant and both are increased, the stress gradients increase. This relationship is analogous to the same conditions for the single-crack situation. In addition, when σ_1 equals σ_2 the stress gradients appear to be independent of ϕ just as was the case for the single crack.

From the brief study conducted here concerning the change of stress field with the change of the crack pattern, one can only conclude that the change is complex. The complexity seems to increase when two systems of cracks are superpositioned. The effects of using unequal-length cracks in a system was not studied.

Analysis of Results

Consideration of an individual crack's parameters when the crack is only one of a system is made without the aide of experimental data. The influence of crack shape on the crack end-stress is probably somewhat similar to the corresponding influences in the single-crack case. For instance, under a condition of constant applied loads to the system, decreasing the radius of curvature at the ends would tend to increase the stress at those points.

The effect of size change, however, is probably much different than that same influence on the single crack. For a single crack, changing the crack size does nothing to change the stress field; but for a system of cracks where the crack center points are fixed, length change undoubtedly

is an important factor. Here the distance between cracks and the percentage of continuous material are functions of the crack length. Two factors probably tend to increase the crack end-stress as the lengths are increased:

1) for a given system of loading applied to each crack, increasing the length without changing the radius of curvature at the ends increases the end stress in some proportion to the length; 2) for a given system of crack centers, increasing the crack lengths decreases the amount of material supporting the load and increases the stresses.

Summary of Observations

The observed and implied factors that influence the end stresses acting on a given crack in a system of cracks may now be summarized. In a system of cracks subjected to a compressive loading field, the maximum stresses occur near the ends of the cracks. These stresses are increased by decreasing the radius of curvature, lengthening the cracks, or orienting the cracks at about 60 degrees with the major applied stress. The stress magnitude is a function of both the sum of and the difference between the applied stresses. When σ_1 equals σ_2 , the stress fields are symmetrical, independent of ϕ , and proportional to some function of σ_1 . Changes in stress fields due to changing crack patterns are complex and not easily analyzed.

The multiple-crack problem may be approached analytically, numerically, or experimentally. Due to the complexity of some crack systems, an experimental approach similar to the one presented here may be the best way to obtain results, while relatively simple systems can probably be solved numerically.

CHAPTER V

SUGGESTIONS FOR FURTHER STUDY

Complex crack systems cannot be analyzed directly by using the results of this study, but the information obtained should prove valuable as a background for extended study. In addition several specific areas requiring further study have been pointed out.

With the advent and common use of digital computers, numerical problem solutions which were formerly unobtainable are now becoming commonplace. This procedure in the finite element form appears to be a promising method of attack. Another numerical type of solution can be found with the electrical analogue. These and probably other numerical techniques unknown to the author may be some of the better ways to solve the specific problem studied in this project.

The method of superposition of single-crack solutions may warrant extended study for problems where the crack patterns are quite open and the intercrack influences are small. Validation or invalidation of this technique can come only through further study.

Areas requiring further study that stem from the limitations of this project can be divided into those related to the single-crack system and those related to the multiple-crack system. For the single crack more experimental data concerning the influences of radius-of-curvature change and crack-length change on the stress field is needed.

Much more experimental information about the parameters of multiple-crack systems is needed. An expanded form of the study described earlier could be used to find the variation in stress magnitudes with respect to loading and crack orientation. Expansion in another form could provide

data about the stress changes in a stress field caused by changing the crack shape and size while holding the system constant. Combinations of various lengths of cracks and various stress patterns need evaluation.

The general problem created by cracks and joints is so complex and covers such a large scope that many more areas than those presented here could be covered in research projects. The suggested areas of study described pertain only to the specific problem encountered in this project.

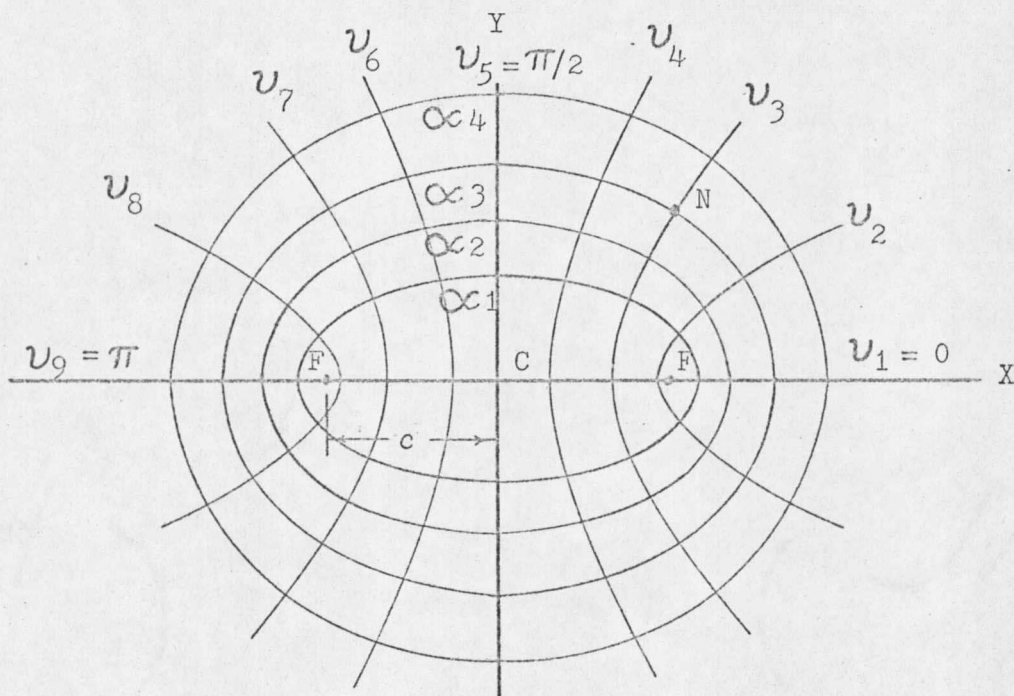


Figure 1. Elliptical coordinates. Point N is defined by (α_3, v_3) .

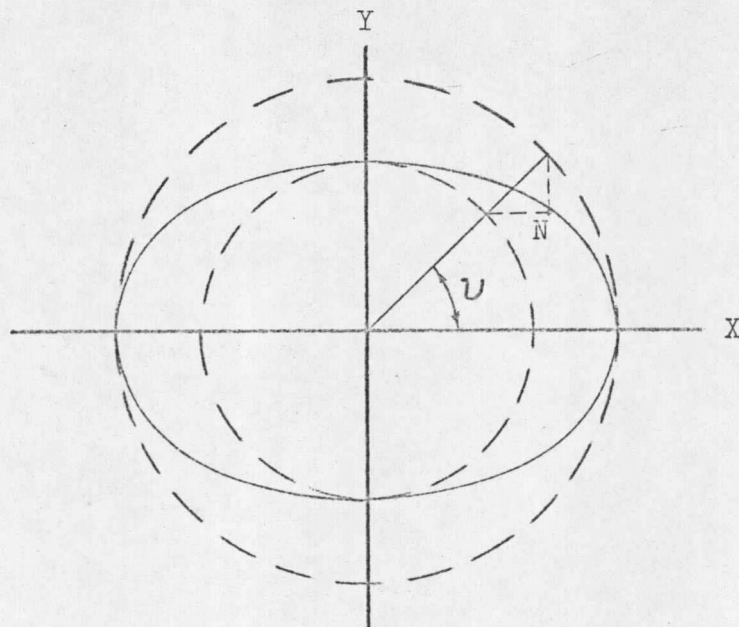


Figure 2. Eccentric angle of an ellipse: v is the eccentric angle for the point N. The two dotted circles are the major and minor auxiliaries of the ellipse.

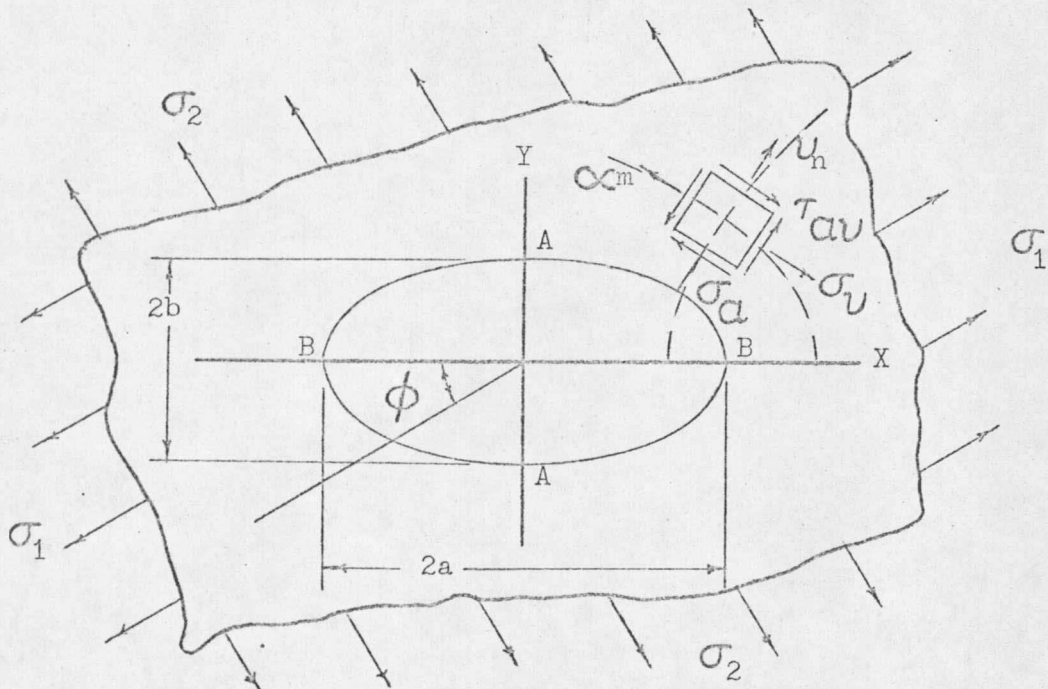


Figure 3. Stresses around a hole in a plate subjected to two perpendicular uniformly distributed loads.

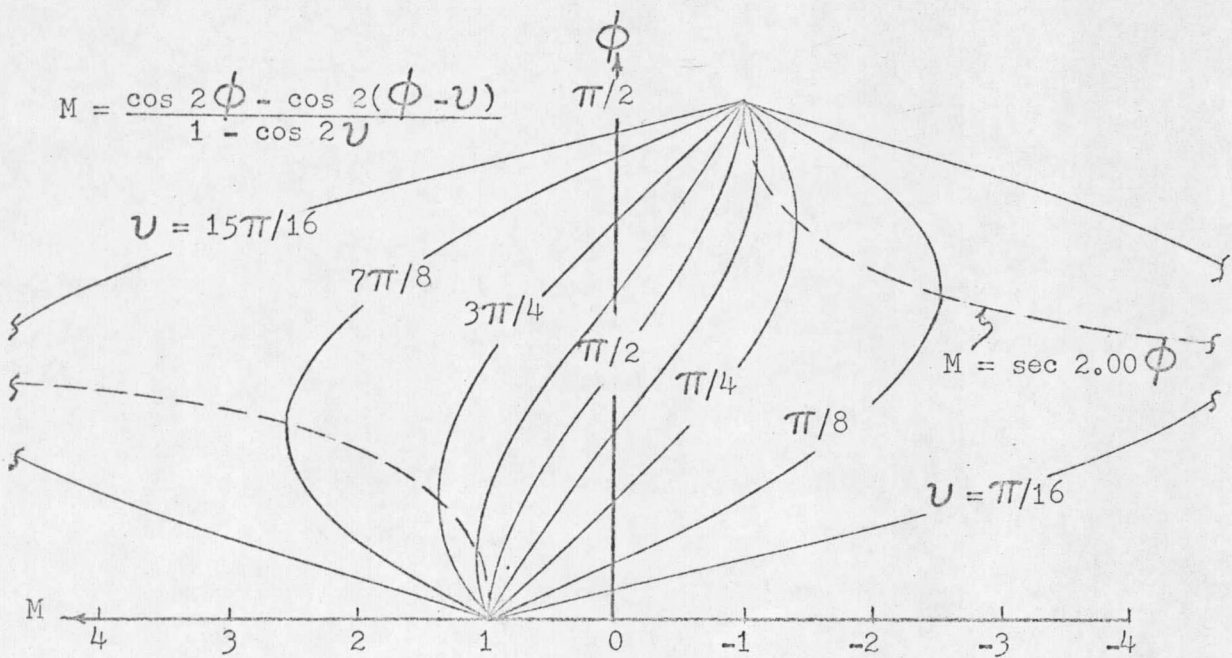


Figure 4. Variation of Term Two as a function of ν and ϕ .

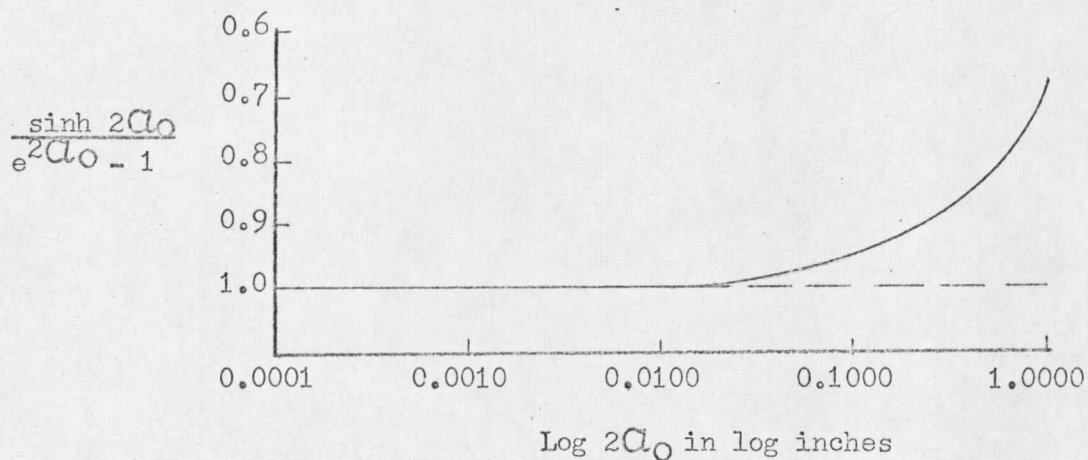


Figure 5. Comparison of $(e^{2a_0} - 1)$ with $\sinh 2a_0$.

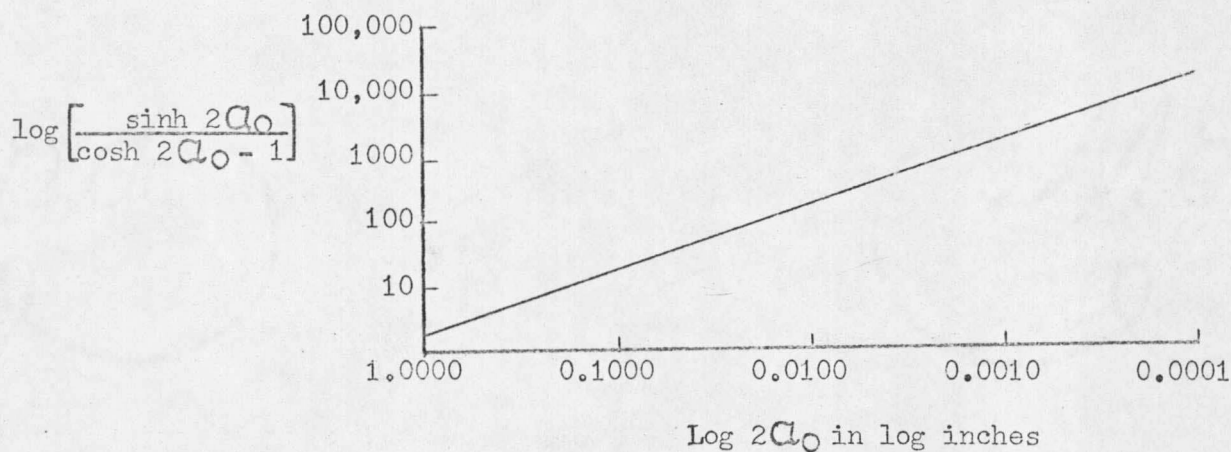


Figure 6. Variation of stress as a function of crack shape.

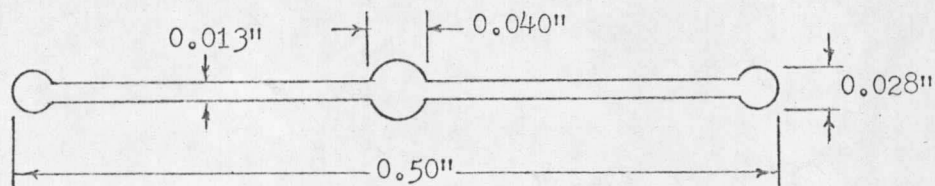


Figure 7. Standard crack.

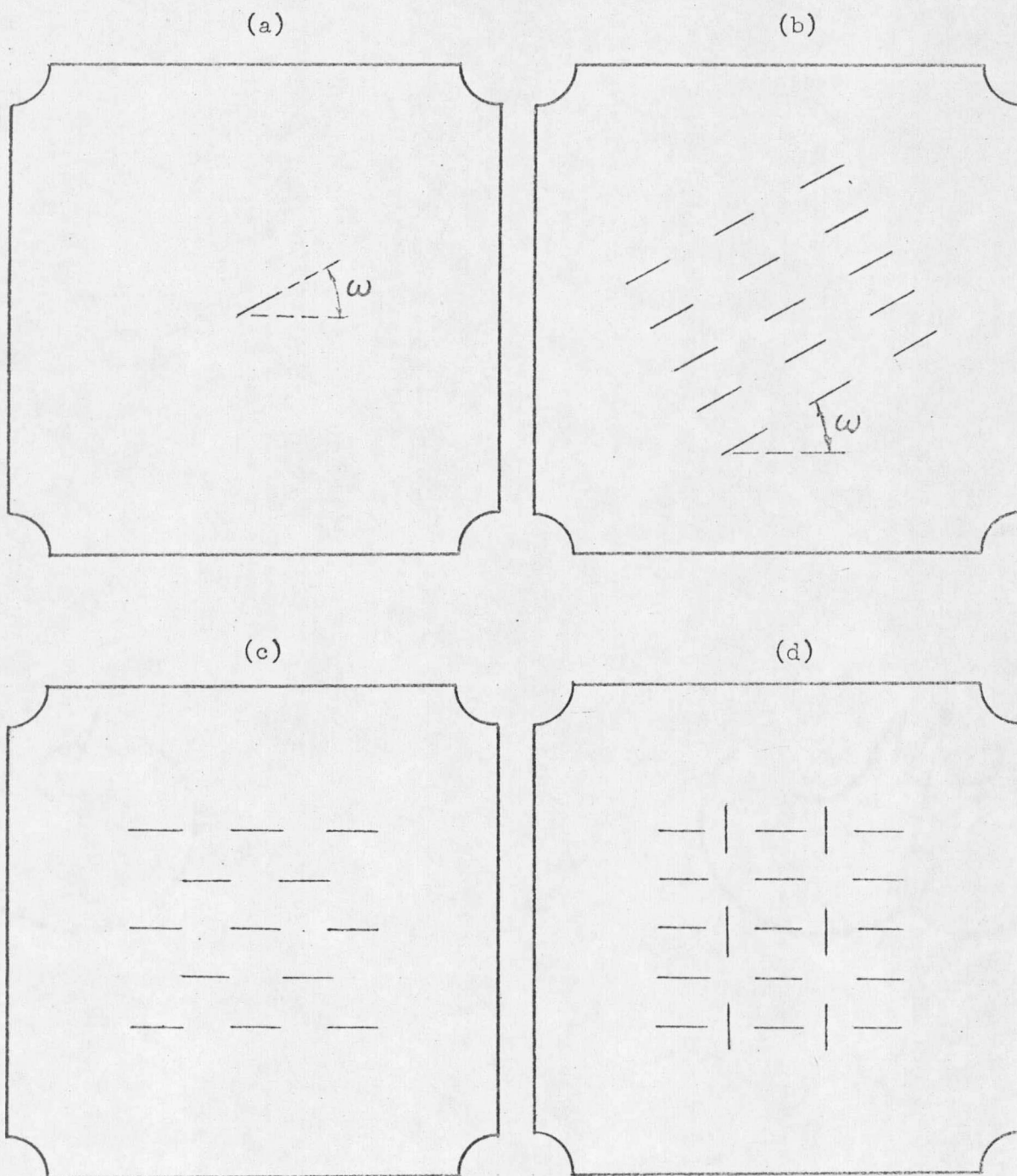


Figure 8. Crack patterns (a, single; b, c and d, multiple).

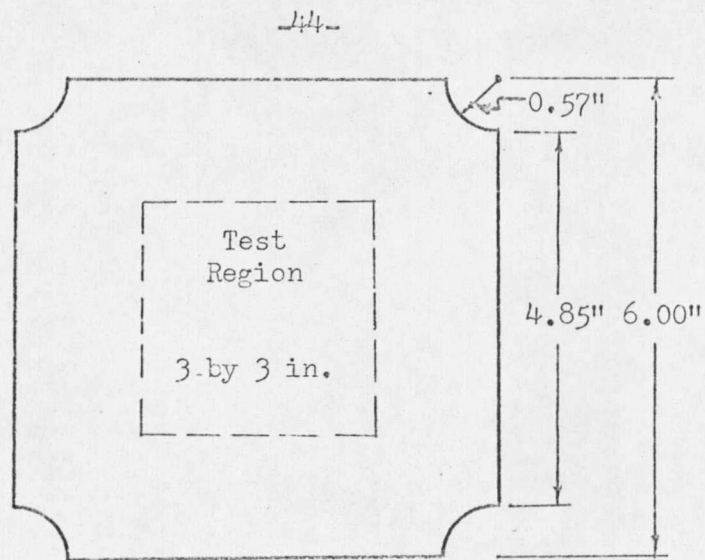


Figure 9. Outline of model showing dimensions.

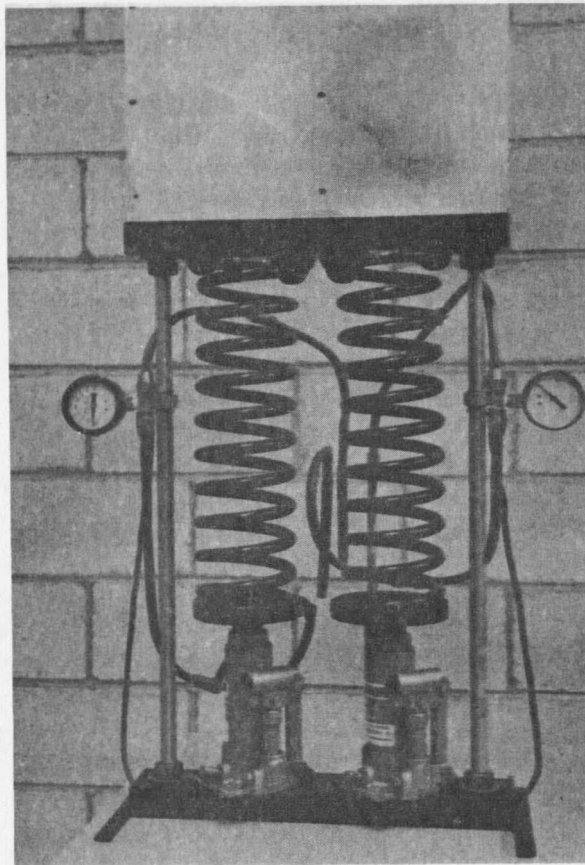


Figure 10. Model-loading machine.

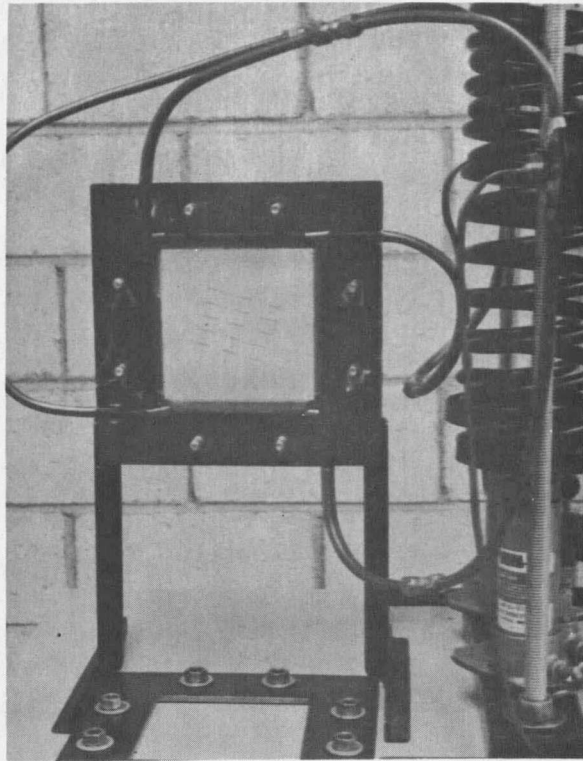


Figure 11. Model and hoses in loading frame.

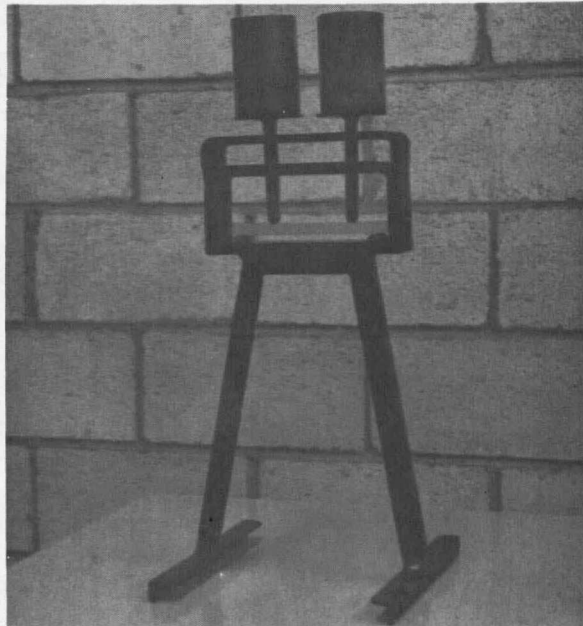


Figure 12. Beam-loading machine.

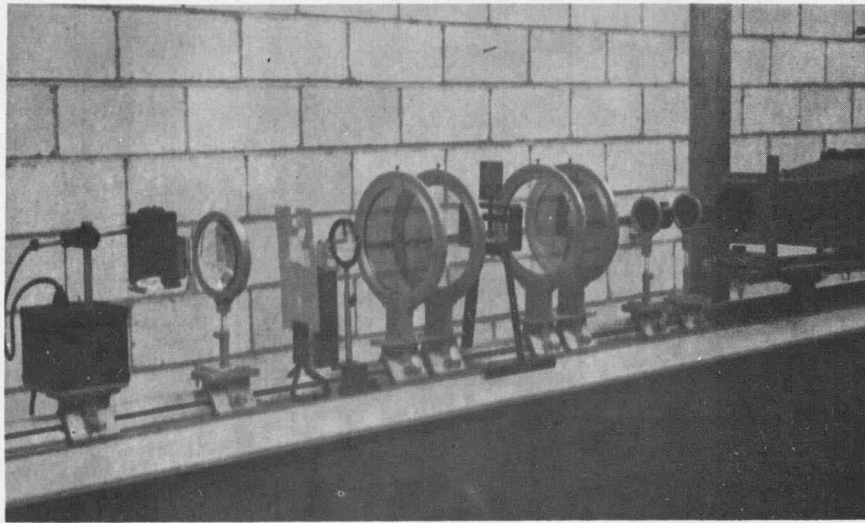


Figure 13. Polariscope and accessories.

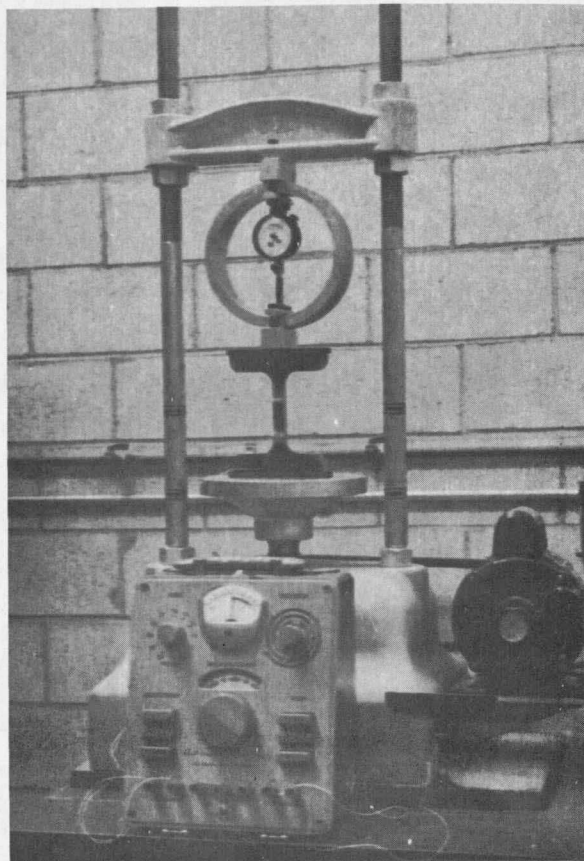


Figure 14. Steel coupon in "Soil-test" machine.

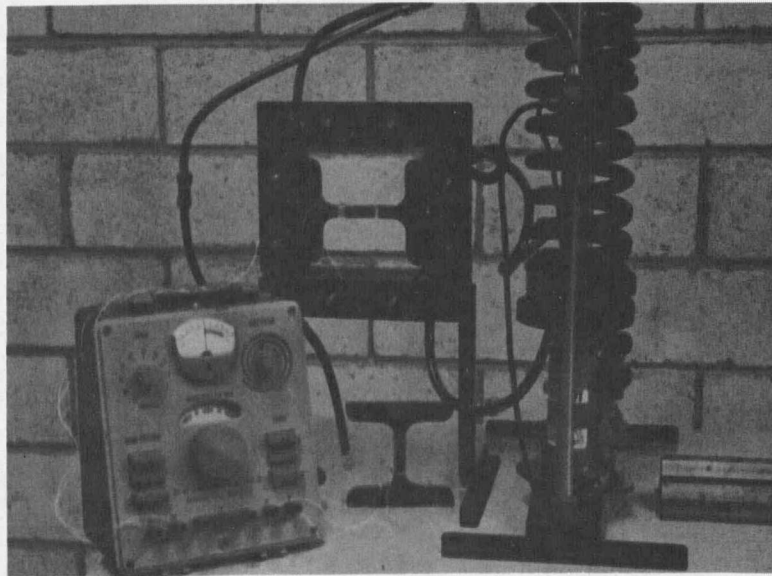


Figure 15. Steel coupon in the model-loading frame.

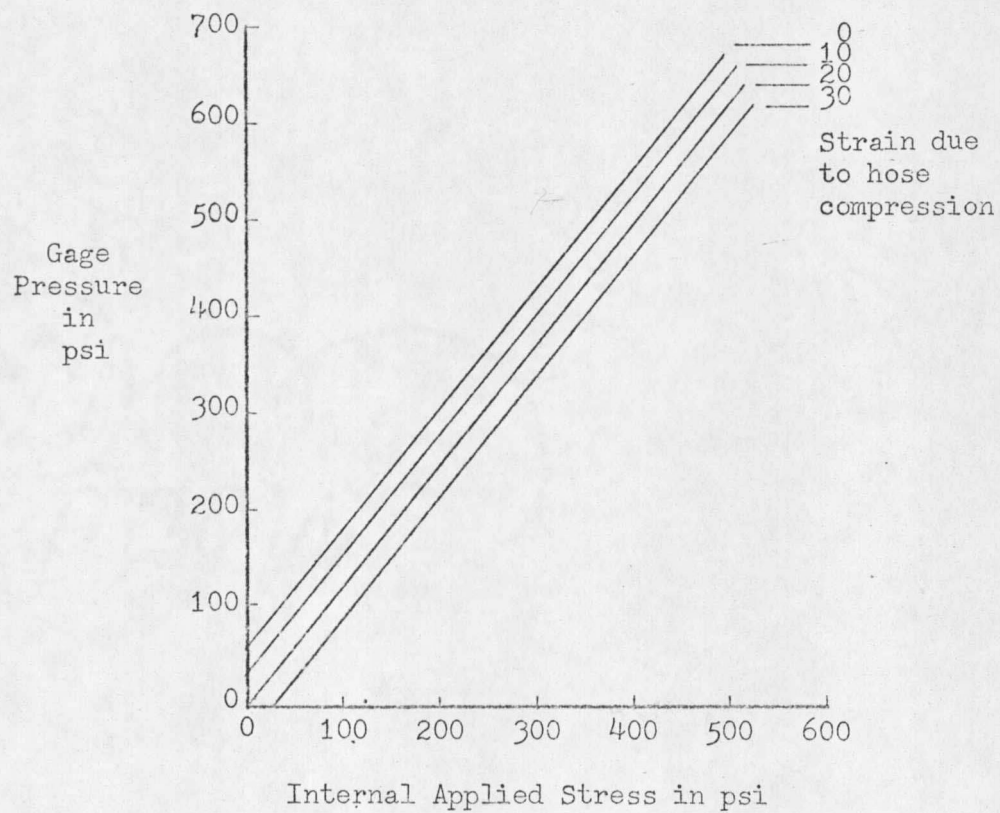


Figure 16. Calibration curves for one gage of the model-loading machine.

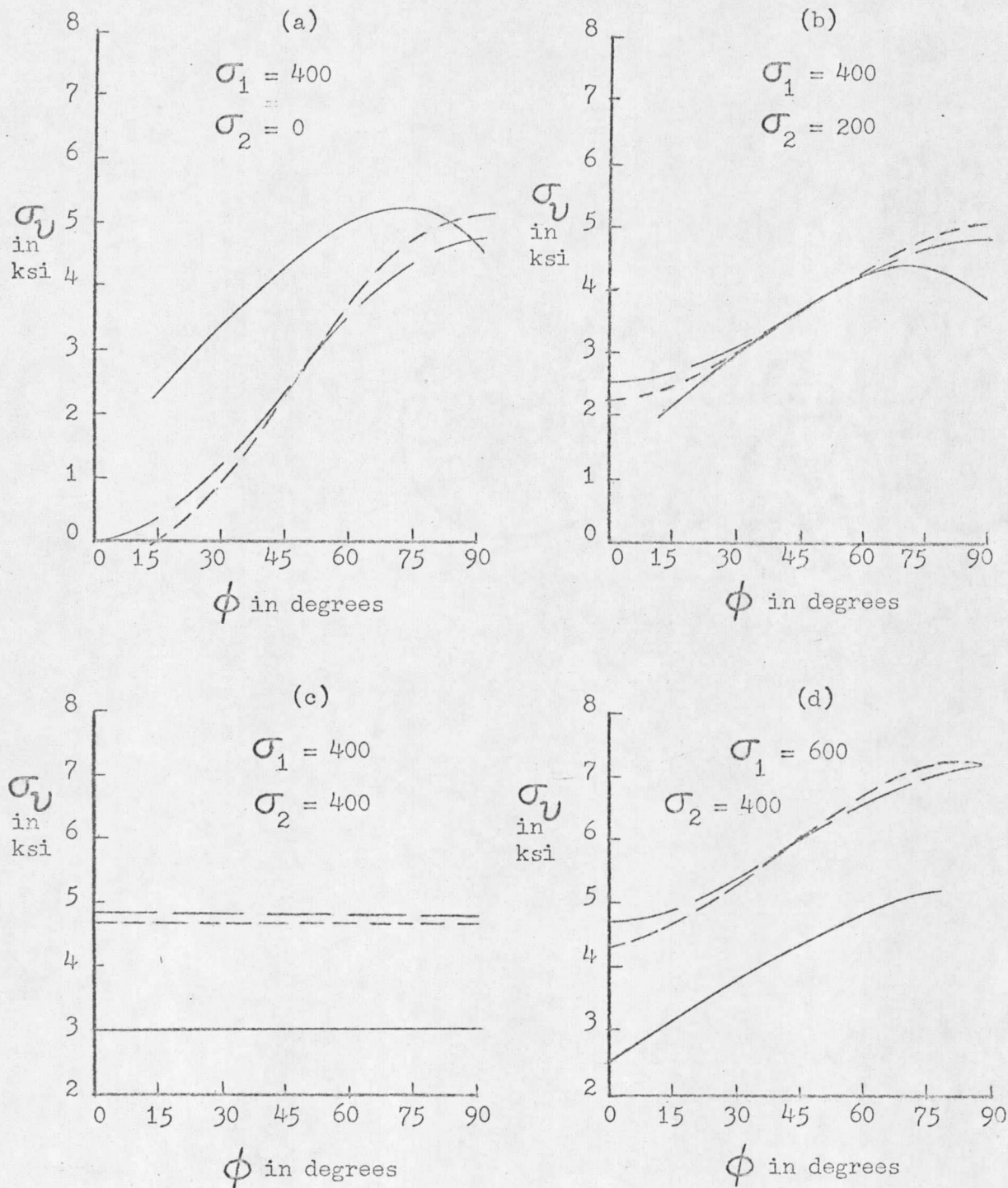


Figure 17. Variation of crack end-stress as a function of the angle between the major axis of the crack and the major applied stress. (Solid curves are experimental data, long-dashed curves represent Equation (9) and short-dashed lines represent Equation (4).)

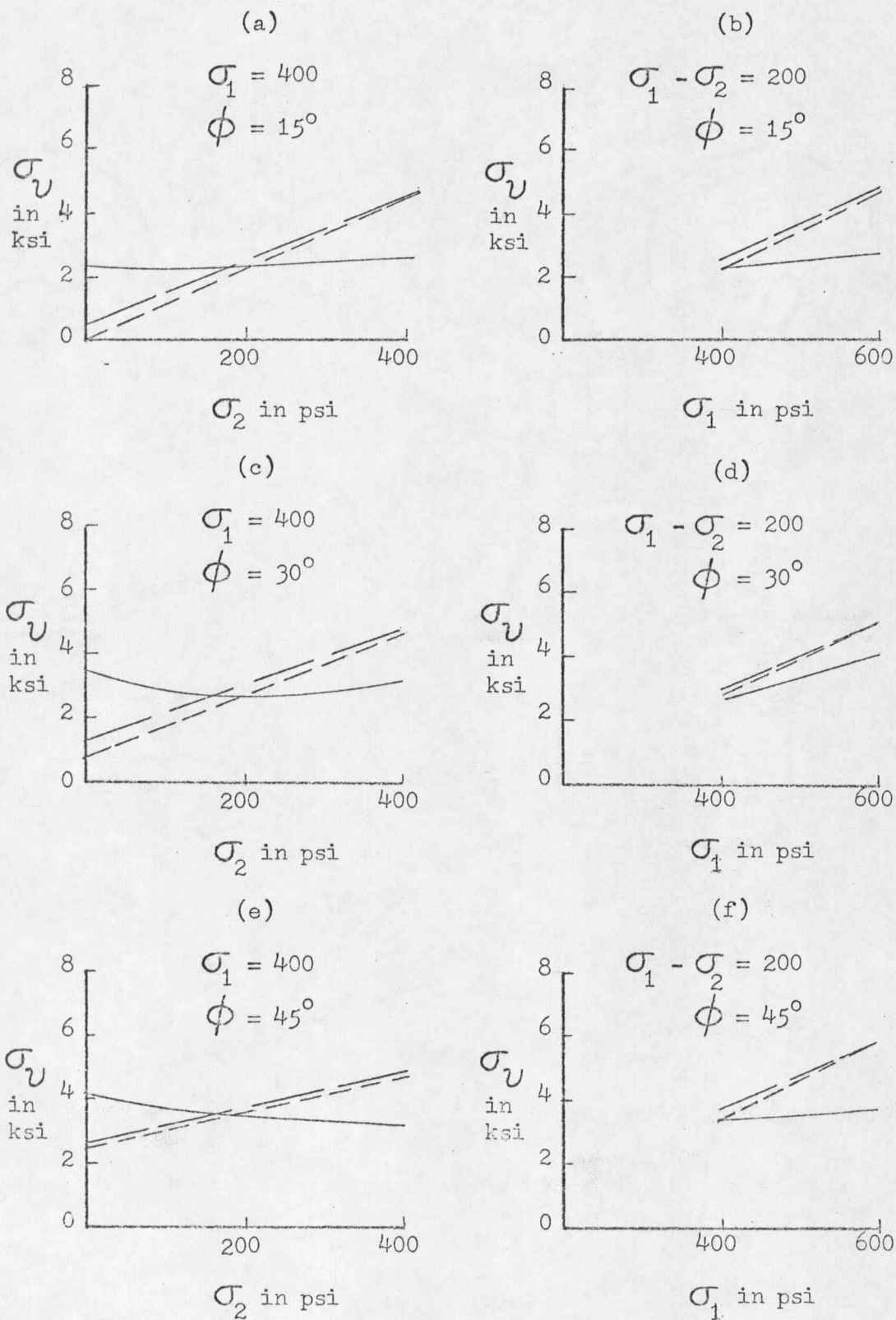


Figure. 18 Caption on next page.

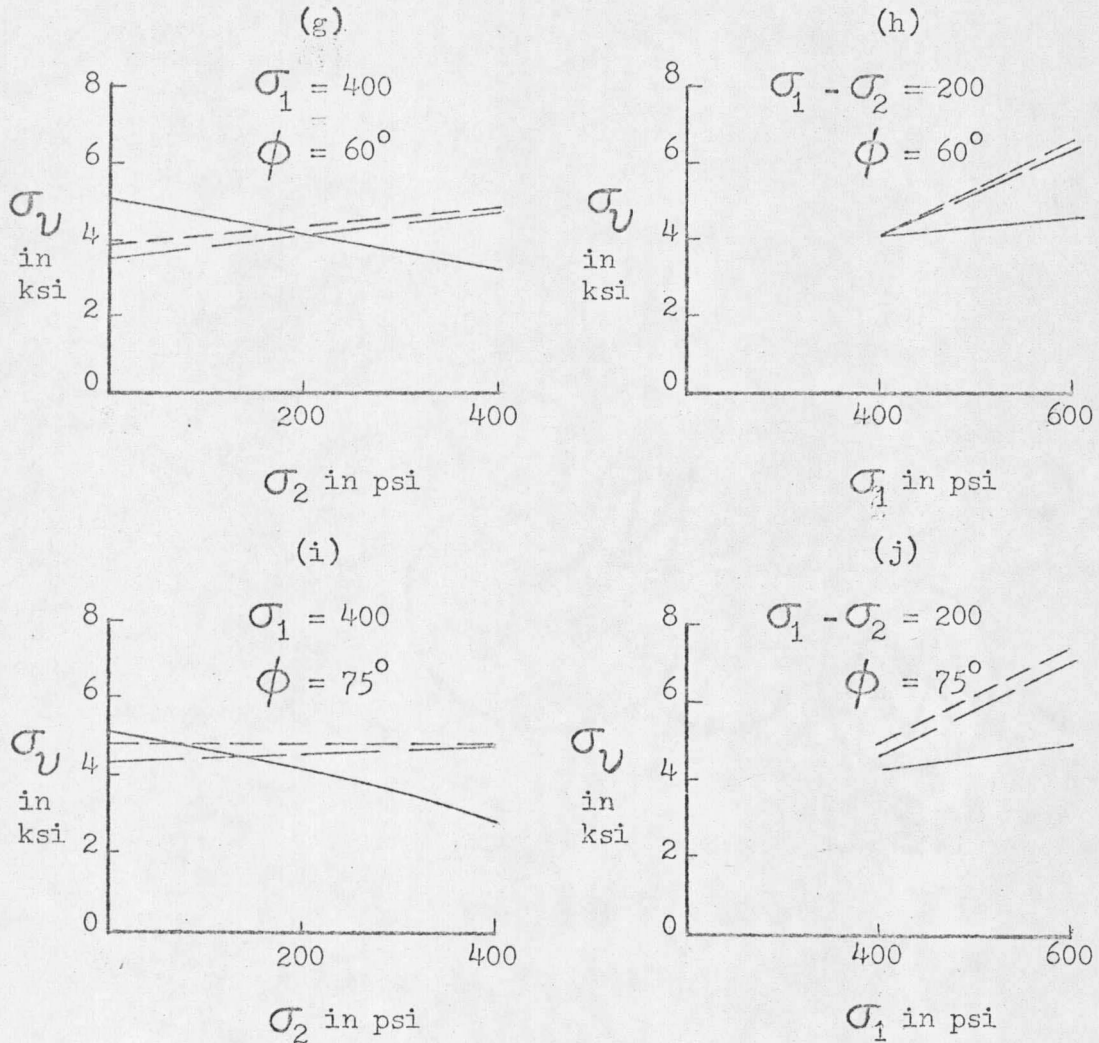


Figure 18. Variation of stress at the end of cracks with change in both the difference between and sum of the applied stresses. (The solid curves represent experimental data, the long-dashed curves represent values of σ_v calculated from Equation (9) and the short-dashed curves represent σ_v values calculated from Equation (4).)

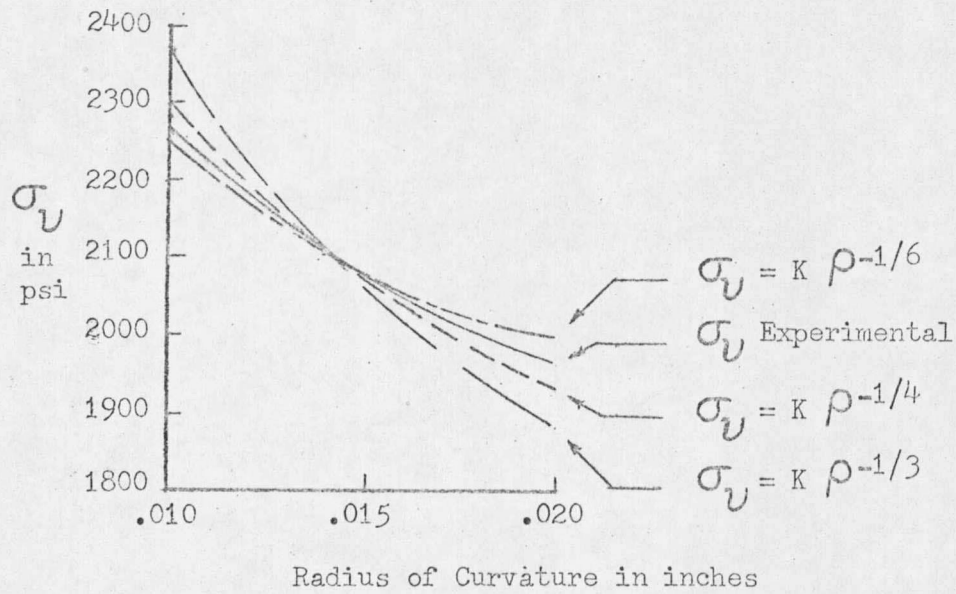


Figure 19. A comparison of experimental data with various curves for the relationship between crack-end radius of curvature and crack length.

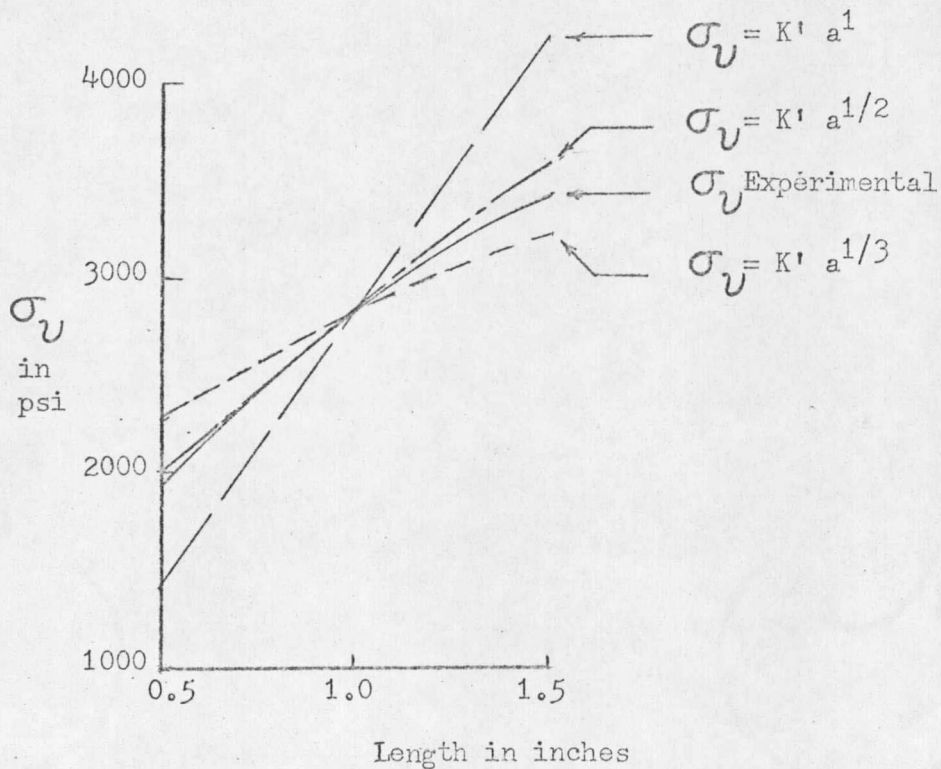


Figure 20. A comparison of experimental data with various curves for the relationship between crack length and stress at the crack end.

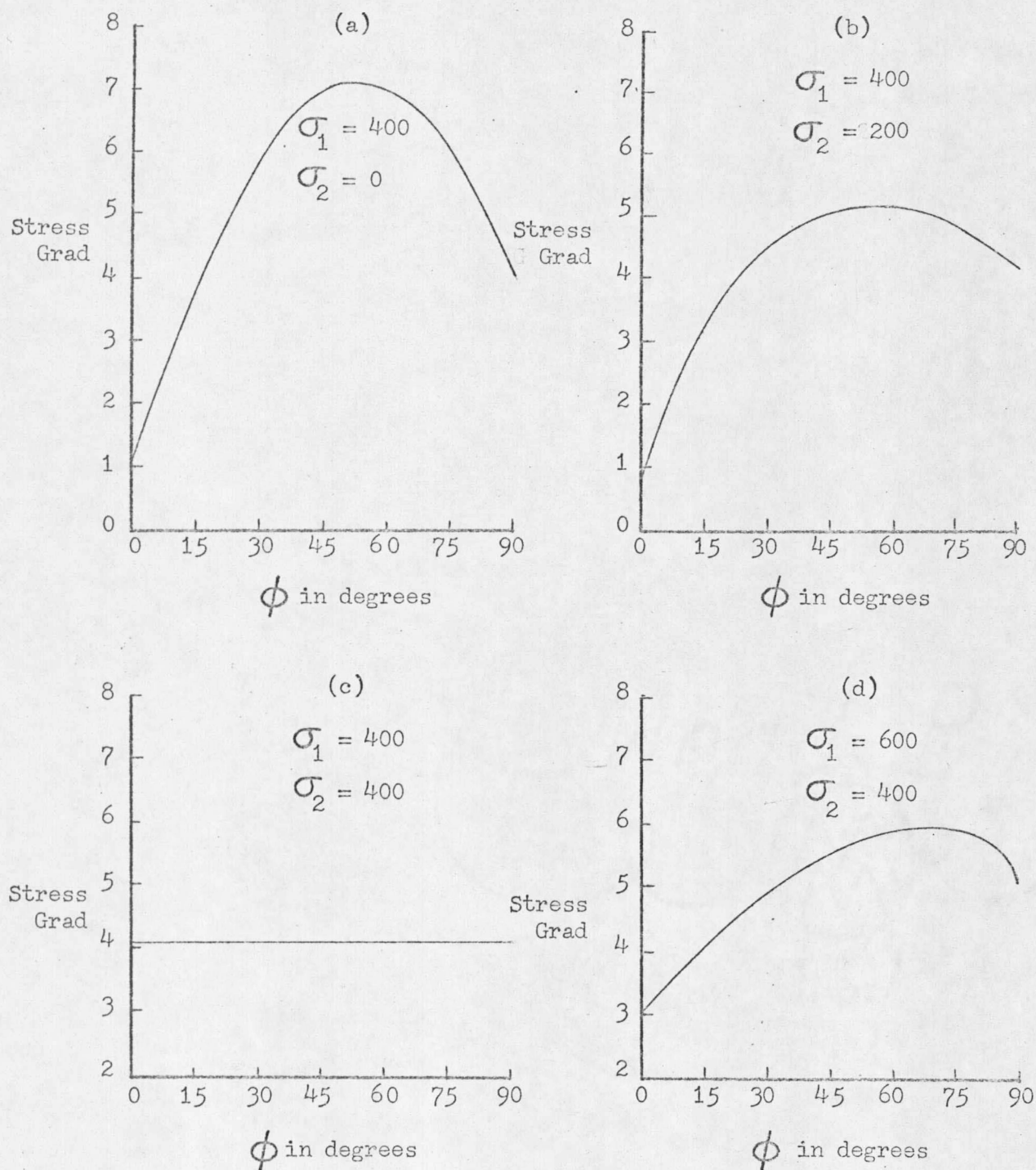


Figure 21. Variation of crack stress due to change in angle between the elliptical major axis and the major applied stress for a multiple-crack system.

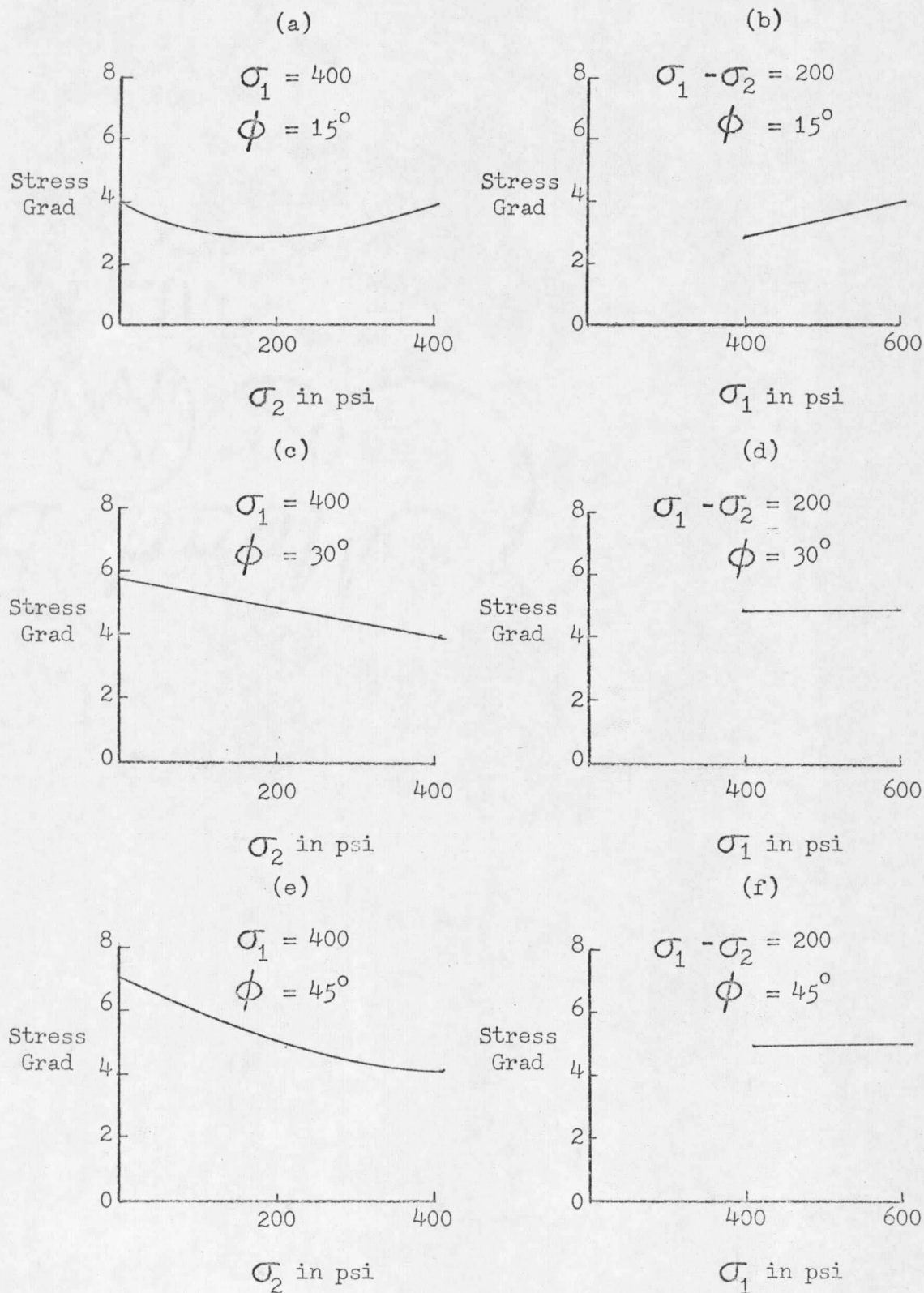


Figure 22. Caption on next page.

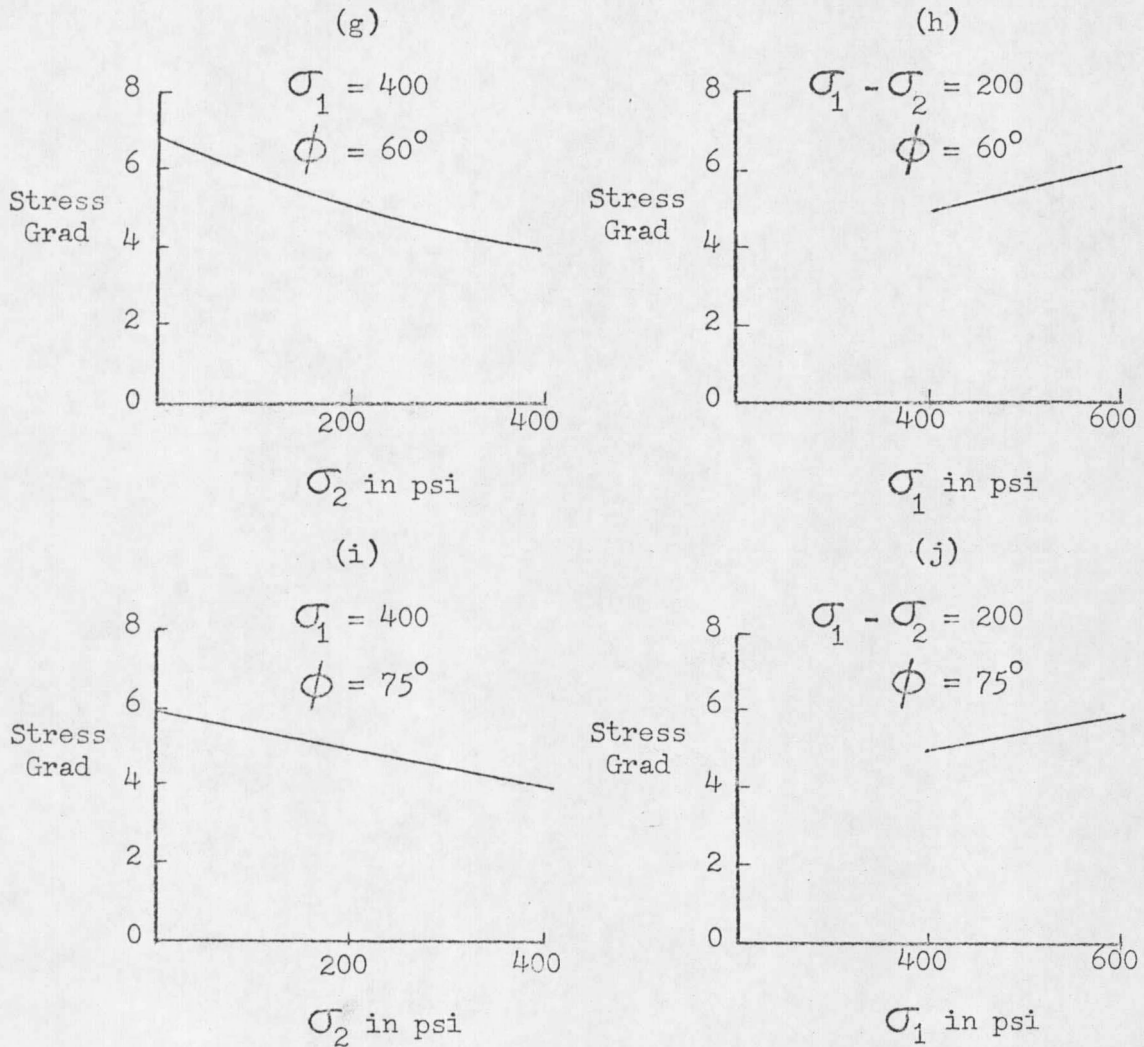


Figure 22. Variation of crack end-stress as a function of the sum of and the difference between the applied stresses for a multiple-crack system.

APPENDIX

APPENDIX A

The development of the interrelationship of the elliptical parameters defined in Figure 3 is shown here. From Durelli and Murray (1943),

$$x = c \cosh \alpha \cos \nu$$

and

$$y = c \sinh \alpha \sin \nu$$

where the boundary conditions are:

- 1) at $y = 0$, $x = a$ and $\alpha = \alpha_0$, $\nu = 0$;
- 2) at $y = b$, $x = 0$ and $\alpha = \alpha_0$, $\nu = \pi/2$.

Substituting the boundary conditions for x and y gives

$$a = c \cosh \alpha_0 \tag{1}$$

and

$$b = c \sinh \alpha_0. \tag{2}$$

Squaring both Equations (1) and (2) and subtracting b^2 from a^2 gives

$$a^2 - b^2 = c^2 [\cosh^2 \alpha_0 - \sinh^2 \alpha_0]$$

and by the identity, $\cosh^2 \alpha_0 - \sinh^2 \alpha_0 = 1$,

$$c = \sqrt{a^2 - b^2}. \tag{3}$$

Substituting c from Equation (3) into Equation (2) gives an expression

for α_0 in terms of a and b ,

$$\alpha_0 = \sinh^{-1} \frac{b}{\sqrt{a^2 - b^2}}. \tag{4}$$

APPENDIX B

The relationship between the radius of curvature, ρ , of an ellipse and the parameters a and b is derived here. The equation for radius of curvature is

$$\rho = \frac{[1 + (y')^2]^{3/2}}{|y''|},$$

and the equation of the ellipse being studied is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or

$$y = \frac{b}{a} \sqrt{a^2 - x^2},$$

where

$$y' = -\frac{bx}{a\sqrt{a^2 - x^2}}$$

and

$$y'' = -\frac{b}{a} \left[\frac{1}{\sqrt{a^2 - x^2}} + \frac{x^2}{(a^2 - x^2)^{3/2}} \right].$$

Substituting y' and y'' into ρ gives

$$\rho = \frac{\left[1 + \left(-\frac{bx}{a\sqrt{a^2 - x^2}} \right)^2 \right]^{3/2}}{\frac{b}{a} \left[\frac{(a^2 - x^2)}{(a^2 - x^2)^{3/2}} + \frac{x^2}{(a^2 - x^2)^{3/2}} \right]},$$

and simplification leaves

$$\rho = \left[\frac{a^2 - x^2 + \frac{b^2 x^2}{a^2}}{(ba)^{2/3}} \right]^{3/2}.$$

When the condition that x equals a is substituted into ρ , the relationship between radius of curvature at the end and the elliptical parameters is

obtained,

$$\rho = \frac{b^2}{a} \cdot$$

By combining the results of Appendix A with those obtained here, $2\alpha_o$ for the standard crack may be calculated.

$$a = .50$$

$$\frac{b^2}{a} = .014$$

$$b = \sqrt{.0070} = .0837$$

$$\alpha_o = \sinh^{-1} \frac{b}{\sqrt{a^2 - b^2}} = 0.170$$

$$2\alpha_o = 0.340$$

APPENDIX C

The model-material calibration was done by counting the fringes between points of known stress conditions and correlating the fringe order with maximum shear stress. The relationship between these two quantities is expressed by the stress-optic law (See Chapter III). Known conditions of stress in a beam in pure bending occur at the neutral axis where the maximum shear stress is zero and at the extreme fiber where the maximum shear stress equals one-half the normal stress. Table I summarizes the calculation of λ/c from the equation

$$\lambda/c = t(\sigma_1 - \sigma_2)/n.$$

TABLE I. CALCULATION OF PLASTICS SENSITIVITY

Date	Beam	Trial	Section Modulus in ⁴	Moment #in	Normal Stress psi	Fringe Order	λ/c psi-in/fringe
Oct. 1	1	1	0.007874	6.25	794	6.0	36.9
		2	0.007874	12.50	1588	11.8	37.5
Jan. 14	1	1	0.007874	12.50	1588	9.7	45.7
	2	1	0.010560	15.00	1420	8.5	46.4

A linear graph of λ/c versus time was plotted and the λ/c value that applied to each model was picked from the curve. Then the thickness effect of each model was taken into account and the stress-fringe values were calculated in Table II.

TABLE II. CALCULATION OF SHEAR-FRINGE VALUES

Model	λ/c psi-in/fringe	Thickness in.	Fringe Values psi/fringe
1	38	0.265	287
2	39	0.264	295
3	41	0.273	301
4	41	0.265	309
5	42	0.261	322
6	42	0.261	322
7	42	0.280	300
8	43	0.261	330
9	43	0.259	332
10	43	0.281	306
11	44	0.263	335

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