



Space-based gravitational wave astrophysics
by Shane L Larson

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in
Physics

Montana State University

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Abstract:

In the near future, several Earth-based and space-based gravitational wave observatories will be completed. With this imminent change in the tools available to observe the Universe, there is much work to be done in describing the fundamental response of our instruments to gravitational radiation, understanding the sources of gravitational radiation, and determining what can be learned about the fundamental nature of the gravitational interaction from the detection of these waves.

The first original result reported here is the determination of the sensitivity limits of a spaceborne gravitational wave observatory. Determining if particular sources of gravitational radiation are detectable by a specific gravitational observatory requires knowledge of the sensitivity limits of the instrument, commonly depicted as a graph of the spectral density of the dimensionless strain vs. frequency of the gravitational wave. Previous discussions of the sensitivity have relied on approximations and heuristic arguments about the shape of the transfer function to construct a sensitivity curve. This thesis details the computation of the exact sensitivity curve given a simple set of parameters describing a space-based interferometer.

The second original result presented here is an exploration of how the mass of the graviton (the fundamental quanta of the gravitational field) might be determined from observations of known sources of gravitational radiation with a space-based interferometric observatory. Current experimental limits place an upper bound on the Compton wavelength of the graviton at $\lambda_g > 2.8 \times 10^{12}$ km, based on analysis of Yukawa violations to Newton's Universal law of gravitation in the solar system. The advent of state-of-the-art space-based gravitational wave interferometers will allow new bounds to be placed on the graviton mass by observing interacting white dwarf binary stars, such as AM CVn (AM Canum Venaticorum). If the graviton is massless, experimental uncertainty will be the limiting factor in computing bounds on the graviton mass. For space-based interferometric detectors, the predicted uncertainties in the observations of AM CVn will place an upper bound on the inverse mass of the graviton of $\lambda_g > 1.4 \times 10^{15}$ km.

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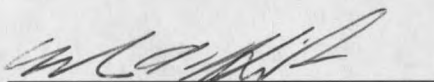
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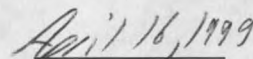
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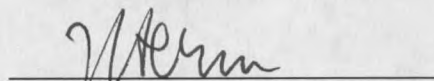
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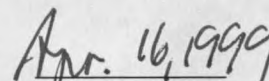

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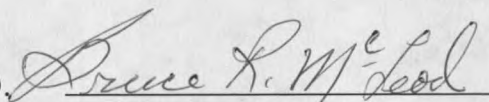
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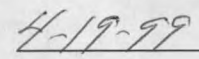

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I do not know what I appear to the world; but to myself I seem to have been only like a boy playing on a seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.

- Sir Isaac Newton

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TABLE OF CONTENTS

1	INTRODUCTION	1
1.1	The Search for Gravitational Radiation	1
1.2	Gravity and General Relativity	7
1.3	Gravitational Waves	11
2	GRAVITATIONAL RADIATION	14
2.1	Introduction	14
2.2	Linearized Einstein Field Equations	16
2.3	Gravitational Wave Solutions	20
2.4	Polarization of Gravitational Waves and the Effect on Test Particles .	26
2.5	Gravitational Wave Detectors	33
2.6	Sources of Gravitational Radiation	41
3	SENSITIVITY CURVES FOR SPACEBORNE INTERFEROMETERS	48
3.1	Introduction	48
3.2	Spacecraft and Sources of Noise	52

3.3	Doppler Tracking	55
3.4	Gravitational Wave Transfer Function	60
3.5	Sensitivity Curves	68
3.6	Long Wavelength Approximation	71
4	USING BINARY STARS TO BOUND THE MASS OF THE GRAVI- TON	74
4.1	Introduction	74
4.2	Current bounds on the graviton mass	76
4.3	Interacting Binary White Dwarfs	78
4.4	Correlation of Electromagnetic and Gravitational Observations to Mea- sure the Graviton Mass	83
4.5	Phase Delays	87
4.6	Obtaining a Mass Bound from AM <i>Canum Venaticorum</i>	95
5	SUMMARY AND CONCLUSIONS	98
5.1	Introduction	98
5.2	Instrument Design and Response	99
5.3	Astrophysics and Tests of Gravitational Theory	101
	BIBLIOGRAPHY	104

LIST OF TABLES

4.1	Properties of the six nearest interacting binary white dwarfs.	79
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LIST OF FIGURES

1.1	Schematic diagram of a LIGO ground based gravitational wave observatory.	3
1.2	Schematic diagram of the proposed OMEGA space-based gravitational wave observatory.	4
1.3	Embedding diagram showing a two dimensional surface <i>embedded</i> in a three dimensional space, making the geometry of the 2D surface readily apparent.	9
1.4	Locally, spacetime can be considered flat, as shown in (A). Gravitational waves can be pictured as oscillations of curvature superimposed on the flat background, as shown in (B).	12
2.1	An arrangement of test particles used to consider the effects of a passing gravitational wave, represented by the wave vector k^μ	29
2.2	Distinguishing polarization states using rings of test particles. (A) A circular ring of test particles lies in the $x - y$ plane. (B) Modulation of the ring shape due to passage of a $+$ polarized gravitational wave propagating down the z -axis. (C) Modulation of the ring shape due to passage of a $\times$ polarized gravitational wave propagating down the z -axis.	32
2.3	An idealized resonant gravitational wave detector. Two masses are connected by a spring of equilibrium length ℓ_o	36
2.4	Local coordinate systems for observing distant sources of gravitational radiation set up by individual observers. On each patch, it is necessary to describe the quadrupole behavior of the distant source <i>in the local coordinates</i>	44
3.1	The proposed orbit for LISA. The orbit is heliocentric, trailing the Earth by 20° , and inclined to the ecliptic by 60° . The three spacecraft remain at this inclination, orbiting a common center as the configuration orbits the Sun.	49

3.2	The tracking geometry for a Doppler gravitational wave detector. Two masses (<i>e.g.</i> , the Earth and a distant spacecraft) exchange an electromagnetic signal whose frequency is Doppler shifted by a gravitational wave propagating up the z -axis. The angle ψ is the principle polarization angle of the gravitational wave, which lies in a plane orthogonal to the propagation vector, <i>i.e.</i> , in the $x - y$ plane for waves propagating up the z -axis.	56
3.3	The transfer function of a Doppler tracking gravitational wave detector, as a function of $u = \omega\tau$. The function has been weighted by $1/u^2$. . .	60
3.4	The geometrical relationship of the interferometer to the propagation vector of a gravitational wave, used to conduct spatial averaging. The arms are designated by vectors $\mathbf{L}_1$ and $\mathbf{L}_2$ (solid black vectors), while the propagation vector of the gravitational wave is given by $\mathbf{k}$ (open white arrow). One arm of the interferometer is chosen to be aligned along the polar axis of a 2-sphere, the other arm lying an angular distance Ω away along a line of constant longitude. The angles θ_i relate the vector $\mathbf{k}$ to the arms of the interferometer, and the angle ϵ is the inclination of $\mathbf{k}$ to the plane of the interferometer.	63
3.5	The transfer function $R(u)$ is shown as a function of the dimensionless variable $u = \omega\tau$. Note that it is roughly constant at low frequencies, and has a "knee" located at $u = \omega\tau \simeq 1$	67
3.6	The sensitivity curves for the proposed OMEGA (solid line) and LISA (dashed line) observatories are shown. The low frequency rise is due to acceleration noise in each of the systems. The structure at low frequencies is a consequence of the high frequency structure in the gravitational wave transfer function, show in Figure 3.5.	71
4.1	The background of close binaries in the galaxy, plotted with the projected sensitivity of OMEGA and LISA. The six most well understood AM CVn type binaries are indicated. The assumed bandwidth is 10^{-7} Hz.	80
4.2	Finder chart to locate the cataclysmic variable, AM <i>Canum Venaticorum</i> (AM CVn), located south of Ursa Major in Canes Venatici, at coordinates RA $12^h 34^m 54.58^s$, $\delta + 37^\circ 37^m 43.4^s$	82
4.3	Schematic of a binary star system which has observable electromagnetic and gravitational wave signals. If the graviton is a massive particle, then the phase fronts of the gravitational signal will lag behind those of the electromagnetic signal.	84

- 4.4 Matter overflows from the secondary star in a coherent matter stream, shown as trajectories spiraling in towards the white dwarf (indicated by $\star$ at the center of the Roche lobe; figure adapted from [60]). Depending on the radius of the accretion disk, the matter stream will strike the disk at some angle θ which leads the line of masses; this angle is directly proportional to the phase delay parameter α_* . The dashed circles represent accretion disk radii which are 70% and 80% of the Roche radius, yielding $\sim 5^\circ$ difference in the value of θ 92
- 4.5 Schematic showing two observations of a source from opposite sides of the Earth's orbit. The radius of the Earth's orbit is r_E and the path length difference for signals propagating from far away is ξ 93
- 5.1 The sensitivity curve for the GEM gravitational wave observatory. No such mission has been proposed, but the sensitivity of the instrument may easily be estimated from a simple set of parameters. 100

CONVENTIONS

Geometrical units are used throughout this dissertation, where $c = G = 1$, except when otherwise noted. When conventional units are required, SI (*Système Internationale*) units will be employed. Sign conventions follow those of Misner, Thorne, and Wheeler [1].

For all spacetime tensor indices, Greek indices range over temporal and spatial coordinates, taking on the values 0 to 3; Latin indices range only over the spatial coordinates, 1 to 3. The Einstein summation convention is used throughout, with repeated indices summed over.

Commas are used to denote partial derivatives

$$v^\alpha{}_{,\beta} = \frac{\partial v^\alpha}{\partial x^\beta}$$

and semi-colons are used to denote the metric compatible covariant derivative, *e.g.*,

$$v^\alpha{}_{;\beta} = \frac{\partial v^\alpha}{\partial x^\beta} + \Gamma^\alpha{}_{\lambda\beta} v^\lambda.$$

ABSTRACT

In the near future, several Earth-based and space-based gravitational wave observatories will be completed. With this imminent change in the tools available to observe the Universe, there is much work to be done in describing the fundamental response of our instruments to gravitational radiation, understanding the sources of gravitational radiation, and determining what can be learned about the fundamental nature of the gravitational interaction from the detection of these waves.

The first original result reported here is the determination of the sensitivity limits of a spaceborne gravitational wave observatory. Determining if particular sources of gravitational radiation are detectable by a specific gravitational observatory requires knowledge of the sensitivity limits of the instrument, commonly depicted as a graph of the spectral density of the dimensionless strain vs. frequency of the gravitational wave. Previous discussions of the sensitivity have relied on approximations and heuristic arguments about the shape of the transfer function to construct a sensitivity curve. This thesis details the computation of the *exact* sensitivity curve given a simple set of parameters describing a space-based interferometer.

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CHAPTER 1

INTRODUCTION

1.1 The Search for Gravitational Radiation

A natural question to ask in the development of any field theory is whether or not the theory has propagating, wavelike solutions. The search for wave solutions is a familiar procedure from our experience with electromagnetism, where vacuum solutions for the electromagnetic 4-potential exist that describe electromagnetic waves (light).

Similar expectations might also be held for general relativity, the classical theory of the gravitational field. Einstein himself [2] predicted the existence of gravitational waves, based on a calculation in the linear approximation to general relativity, less than one year after his publication of the field equations in 1915.

Gravitational waves interact with matter by causing small changes in the distances between particles (in extended bodies, a change in the proper length of the object would occur). These changes in distances are typically expressed as fractional changes in length, $\Delta\ell/\ell$, and are called *strains*. Typical strains created in the vicinity of Earth by astrophysical sources are expected to be quite small (fractional changes in length

on the order of 10^{-21} and smaller), and to date, no direct detection of gravitational waves has occurred. Indirect, astrophysical evidence for the existence of gravitational waves exists from monitoring the famous Hulse-Taylor binary pulsar, PSR 1913 + 16 [3]. General relativity predicts that the period of this binary should decrease due to gravitational radiation reaction at a rate of -2.38×10^{-12} s per s. Observations of this system show that its orbital period is changing at a rate of $(-2.40 \pm 0.09) \times 10^{-12}$ s per s [4]. This value is 1.01 ± 0.04 times the rate predicted by theory, and solidifies the belief that gravitational radiation exists and has important effects on the dynamics of astrophysical systems. The Nobel Prize in Physics was awarded to Joseph Taylor and Russell Hulse in 1993 for this discovery.

Technological innovations in laser interferometry and the advent of high performance computing are making the possibility of *direct* detection of gravitational waves within the next decade a distinct reality. Currently, efforts are underway to construct a several Earth-based gravitational wave observatories using large laser interferometer systems. There have also been several proposals to design and construct interferometric gravitational wave detectors in space. In the United States, a pair of observatories known as LIGO (Laser Interferometric Gravitational-wave Observatory) are being constructed in Hanford, Washington and Livingston, Louisiana [5]. In Europe, construction has begun on a similar observatory known as VIRGO [6] near Cascina, Italy. These instruments are slated to begin operation early in the 21st century and begin a systematic search for high frequency (10 to 1000 Hz) sources of gravitational waves. Shown in Figure 1.1, a single LIGO interferometer will consist of two 4 km evacuated

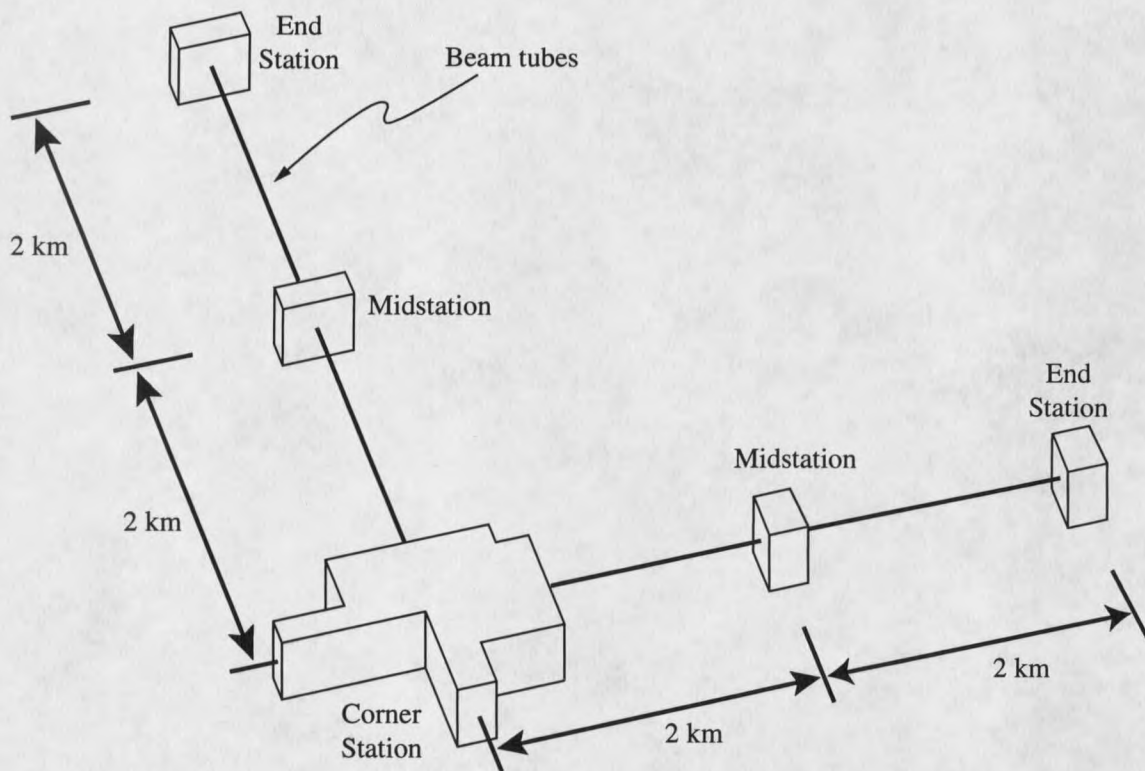


Figure 1.1: Schematic diagram of a LIGO ground based gravitational wave observatory.

arms at right angles to each other. The design of LIGO will support operation of a Michelson type interferometer, as well as several Fabry-Perot type interferometers within the 1.2 m diameter beam tubes. The midstations allow for the development, testing, and addition of equipment and secondary interferometer systems sharing the same evacuated beam lines as the primary interferometer beam. This allows the observatory to be used as a testbed for technology which will be used to increase the performance of the instrument in the coming decade. These planned improvements and modifications will greatly enhance the sensitivity and usefulness of LIGO as a gravitational observatory over the projected lifetime of the instrument.

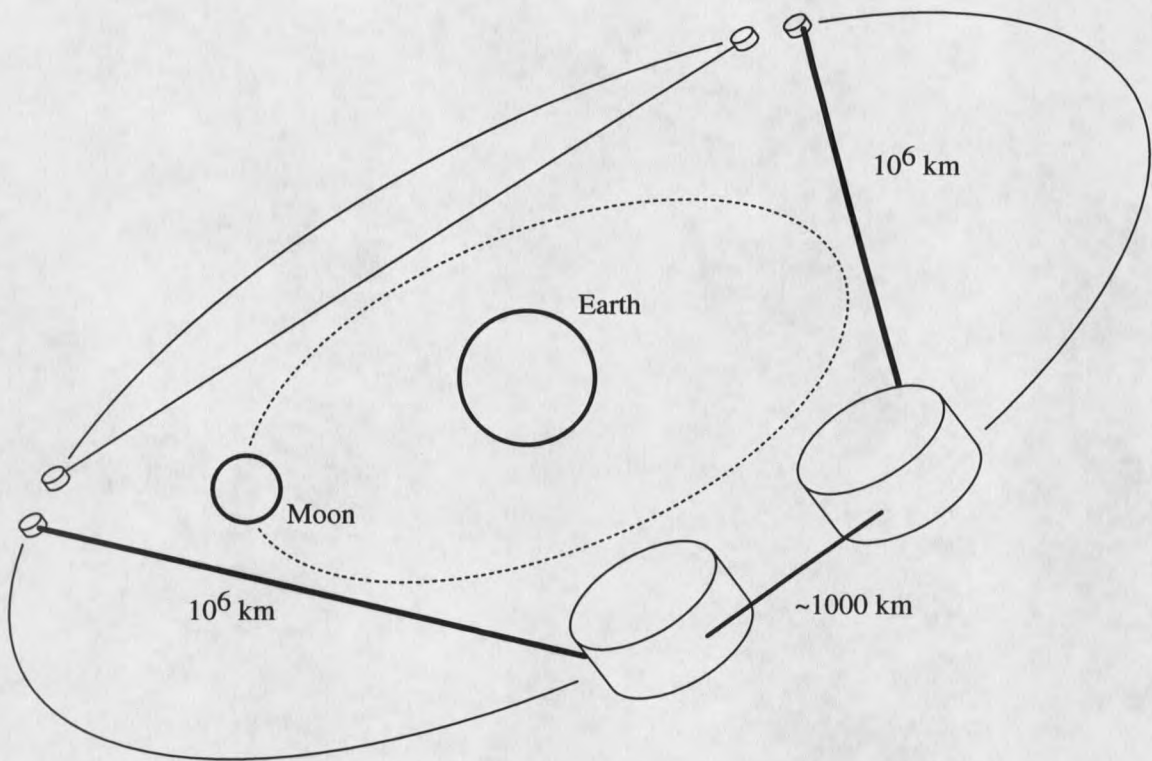


Figure 1.2: Schematic diagram of the proposed OMEGA space-based gravitational wave observatory.

Proposals have also been made to launch space-based gravitational wave observatories early in the next century. Possible designs for such an instrument include OMEGA (Orbiting Medium Explorer for Gravitational Astrophysics) [7], shown in Figure 1.2, and LISA (Laser Interferometer Space Antenna), shown in Figure 3.1 of Chapter 3 [8]. These observatories consist of a constellation of three to six spacecraft carrying lasers which allow them to be linked to form a Michelson type laser interferometer with baselines on the order of 10^9 m. These long baselines cause spaceborne interferometers to be most sensitive in the low frequency (10^{-5} to 1 Hz) gravitational waveband.

The current experimental efforts to detect gravitational waves are partly driven

by the fact that gravitational radiation presents us with a fundamentally new way of observing the Universe. It is widely believed that the advent of gravitational wave astronomy will produce a revolution in our view of the Universe, similar to the revolution electromagnetic astronomy experienced when observations in the radio waveband began in the mid-1900's [9].

Whether a revolution in astronomy ensues or not, the first detections of gravitational radiation will immediately provide us with two new and interesting opportunities to study the workings of the Cosmos.

First, because gravitational waves interact very weakly with matter (a fact which makes them hard to detect), they will propagate nearly freely out of regions of space which cannot be electromagnetically imaged. In these regions, dense matter absorbs and scatters photons, continually reprocessing them so they never actually reach distant observers (*e.g.*, this occurs in the centers of galaxies, or at the center of a collapsing star). Observation of gravitational radiation could provide information about the dynamics and structure of these regions which was previously unobtainable.

Second, studies of the gravitational radiation bathing the Earth will give us the first opportunity to study general relativity in the so-called "radiative regime." It is common practice in electromagnetic field theory to study wave solutions in regions far from the source, where the waves essentially propagate freely. This region is conventionally called the *radiation zone*. This is a convenient regime to work in since the study of the physical properties of the waves are completely separated from the problems of wave generation. The generation and propagation of gravitational

radiation is a considerably more complex problem to study than electromagnetic radiation because of the non-linear nature of general relativity. As was the case with electromagnetism, however, it is still convenient to define a series of zones which allow the study of gravitational waves independent of the sources [10].

The solar system lies well into the radiation zone of the strongest sources of gravitational radiation. For gravitational waves propagating through the vicinity of our detectors, the spacetime is reasonably flat, allowing study of the gravitational waves independently of the background curvature in the solar system. The detection and study of gravitational radiation under these conditions will provide the first direct experimental access to the radiative regime of general relativity. This is an important region to study gravitational radiation as it provides new tests of general relativity. Almost all metric theories of gravity predict the existence of gravitational waves [11]. Observations of gravitational radiation should make it possible to discern which theories are correct (*e.g.*, general relativity predicts two polarization states for gravitational radiation, whereas all other viable metric theories predict more than two; in the most general theory, six are predicted [12, 13]).

The remainder of this chapter develops a simple physical picture of gravitational waves. In Chapter 2, a formal description of gravitational waves is derived in the linearized approximation to general relativity. The polarization states of the waves are described, and the effects of the waves on the motion of test masses is described. Simple methods of detecting gravitational waves are also reviewed, and the formalism for describing the strength of gravitational waves from astrophysical sources is

described.

Chapters 3 and 4 detail the original work which is presented in this dissertation. In Chapter 3 the *exact* response of an interferometric gravitational wave detector is used to construct the sensitivity curve for a space-based gravitational wave observatory. Previous analyses of the sensitivity of spaceborne detectors have used approximations to describe the response of the observatory to incident gravitational radiation; the results derived in this chapter are the first to evaluate the response without utilizing any approximations. Chapter 4 describes a new experiment to measure the mass of the graviton by considering the correlation between the optical and gravitational wave signals of an interacting binary white dwarf system. Assuming no net phase difference between the optical and gravitational wave signals is observed, new bounds on the graviton mass will be derived from the predicted experimental uncertainties in space-based gravitational wave measurements.

In Chapter 5 conclusions and speculations on future work in this field are suggested.

1.2 Gravity and General Relativity

The theory of general relativity is embodied in the Einstein field equations,

$$G_{\mu\nu} = 8\pi T_{\mu\nu} , \tag{1.1}$$

where $G_{\mu\nu}$ is the Einstein tensor, and $T_{\mu\nu}$ is the stress-energy tensor. The Einstein tensor describes the curvature of the spacetime, and the stress-energy tensor describes the matter (and energy) content of the spacetime.

The basic premise of general relativity is that there is no “gravitational force” which acts on test particles¹. From Einstein’s viewpoint, what Newton perceived to be a gravitational force is the curvature of spacetime affecting the trajectories of particles. The deflection of a particle’s path by a gravitational potential is a manifestation of mass creating curvature in the spacetime, and the particle following its straight-line² trajectory through the curved spacetime near the gravitating body. Particles are “freely falling” on the curved background of spacetime itself, and spacetime is curved by the presence of mass. This concept is readily embodied in the famous quote, “space tells matter how to move, matter tells space how to curve” [1].

A heuristic picture of these concepts can be imagined using a device called an embedding diagram, as shown in Figure 1.3. An embedding diagram represents the geometry of a lower dimensional curved space in a higher dimensional flat space. Let us consider the case where the curved space of interest is a two dimensional elastic sheet embedded in three dimensions. Placing a massive object on the sheet (*e.g.*, a heavy bearing, placed at point A as shown) *deforms* the shape of the sheet, producing some curvature. More massive objects (*e.g.*, a bowling ball, placed at point

¹Extended test bodies will feel gravitational forces due to the relative acceleration between different parts of the object (tidal forces).

²These paths are straight in the sense that the 4-acceleration of a particle on such a trajectory is zero: $a^\mu = d^2x^\mu/d\tau^2 = 0$.

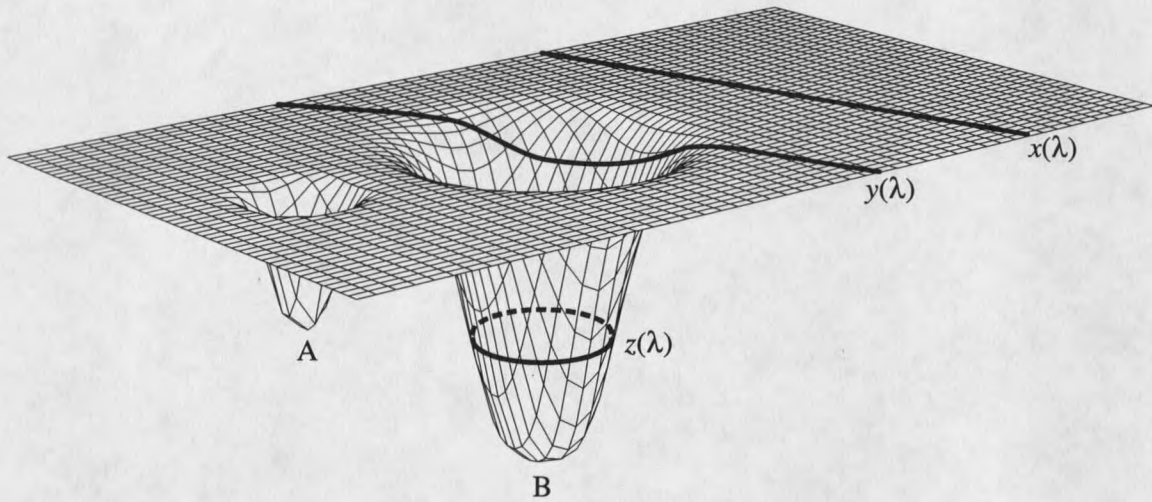


Figure 1.3: Embedding diagram showing a two dimensional surface *embedded* in a three dimensional space, making the geometry of the 2D surface readily apparent.

B as shown) produce a more serious deformation, steeply curving the sheet.

Imagine taking a ping-pong ball and rolling it across this curved landscape. The mass of the ping-pong ball will deform the sheet to some extent, but this will be much less than the deformation produced by the other objects, and one can safely ignore its contribution to the overall shape of the sheet. The ball is free to roll across the sheet, but is constrained to remain on the surface. Assume that no forces act on the ball once it is in motion (*e.g.*, there are no dissipative forces slowing the ball down)³. In the wide, flat regions of the sheet, far away from the masses, the ping-pong ball will roll across the sheet, tracing out a straight-line path such as $x(\lambda)$. This is the

³This experiment is being conducted at the surface of the Earth; gravity deforms the elastic sheet by pulling on the heavy objects, and keeps the ping-pong ball on the surface of the sheet.

natural, “force-free” path for the ball to take. Such a path is called a *geodesic*.

There are other force free paths in the space, in particular paths similar to the curve $y(\lambda)$. Imagine the ball rolls along $y(\lambda)$, which initially is parallel to $x(\lambda)$. When the ball encounters a region of non-zero curvature, the straight-line path dips into the curved region, and re-emerges along a new direction which is diverging from $x(\lambda)$. This path is also a geodesic because it is a force-free path. It is the straightest trajectory for the ball through the region of high curvature. No external forces acted on the ball to alter its trajectory; the trajectory was altered only by virtue of the fact that the ball rolled on a curved background.

Closed circular orbits are also geodesics, similar to the path shown by the curve $z(\lambda)$. Again, these paths are force-free, with the ball rolling along its trajectory unhindered by external influences, its course determined only by the curvature of the space around it.

Each of the three paths in Figure 1.3 is analogous to familiar particle trajectories described in terms of a central potential,

$$V(r) = -\frac{M}{r}, \quad (1.2)$$

where M is the mass of the central source of the potential. The path $x(\lambda)$ is that followed by a particle far from any source of gravitational attraction, the path $y(\lambda)$ is that of a particle scattering off a gravitational potential, and the path $z(\lambda)$ is that of a particle in orbit about a larger mass. One of Einstein’s great insights was to

connect the motion of freely falling objects with the geodesics of a curved spacetime, eliminating the traditional Newtonian gravitational force. The idea that the “gravitational field” is simply curvature of spacetime will be integral to our physical picture of a gravitational wave.

A complete mathematical description of geodesic motion is covered in any basic text on relativity (see for example [14]).

1.3 Gravitational Waves

Special relativity prohibits the propagation of signals at infinite speed, a restriction which applies equally to particles as well as to propagating disturbances in the gravitational field configuration. As such, in systems where the mass distribution changes, any resulting change in the gravitational field cannot be observed instantaneously by distant observers. Changes in the gravitational field will propagate away from the source at a finite speed.⁴ Far from the source, these propagating disturbances in the field are gravitational waves.

Keeping in mind that the geometry of spacetime is the fundamental concept which defines the modern description of gravity, gravitational waves are simply wavelike disturbances superimposed on the overall geometry; they are ripples in the curvature of spacetime itself, shown pictorially in Figure 1.4.

Like all waves, they have associated with them a wavelength and a frequency which

⁴In Chapter 2 it will be shown that general relativity predicts the speed of propagation to be the speed of light.

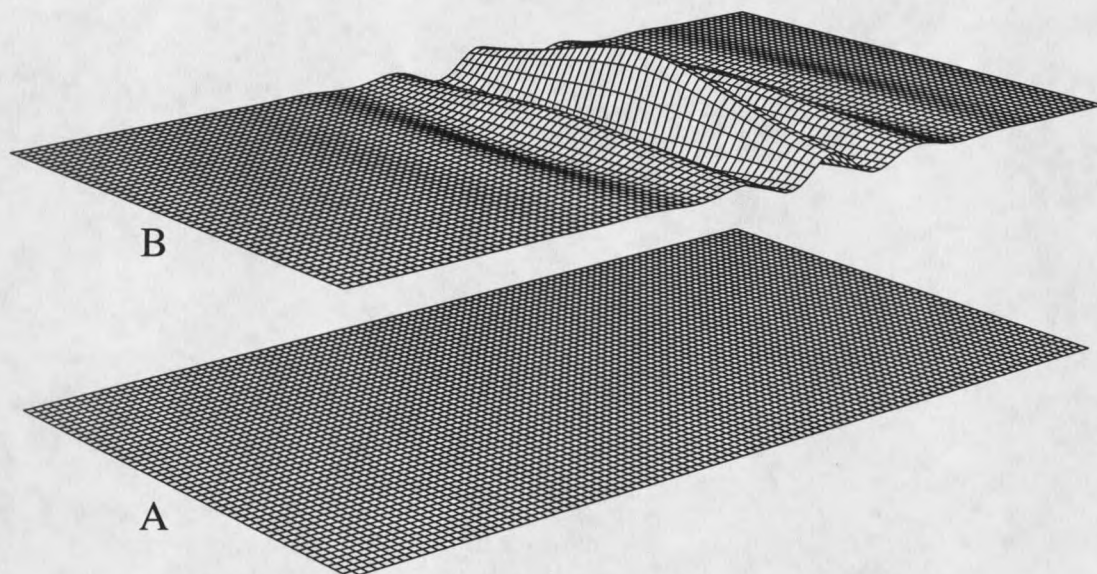


Figure 1.4: Locally, spacetime can be considered flat, as shown in (A). Gravitational waves can be pictured as oscillations of curvature superimposed on the flat background, as shown in (B).

can be directly correlated with the physical character of the source that produced the waves.

Some of the most promising sources of gravitational waves that might be detected are periodic sources, where the mass distribution is continually changing in a regular, periodic way⁵, such as astrophysical binary star systems. These candidate systems are continuous sources of radiation which should be easily detectable by currently proposed space-based observatories. The frequency of waves from periodic systems is simply related to the period of changes in the mass distribution, and the wavelength

⁵To be accurate, the mass distribution must have a time varying quadrupole moment to radiate gravitational waves. Perfectly spherical systems, for example, do *not* produce gravitational radiation (*e.g.*, a spherical star undergoing purely radial pulsations).

is related to the wave frequency through the wavespeed in the conventional manner, $c = \lambda f$. Possible sources of periodic waves are astrophysical binary systems, pulsating stars, or rapidly rotating non-spherical compact stars.

Other possible sources of gravitational radiation include “burst sources.” These sources are non-repeating, transient events which generate gravitational waves. Candidates for these types of events are the core collapse of massive stars during supernovae explosions, or the final coalescence of compact astrophysical binaries (*e.g.*, neutron star and black hole binaries).

CHAPTER 2

GRAVITATIONAL RADIATION

2.1 Introduction

Analogies are often drawn (and will be pointed out here) between gravitation and electromagnetism. Electromagnetism is known to possess radiative solutions, and it is only natural to ask if a theory of gravitation does as well. Almost immediately after his publication of the general theory of relativity, Einstein began considering wavelike solutions in the linearized formulation of the theory [2]. Over the course of the next decade, the linearized theory of gravitational waves was expanded and analyzed.

The linearized theory is a simpler framework within which to study gravitational radiation within, rather than the full non-linear theory of general relativity. Gravitational waves carry energy and momentum which contribute to the curvature of spacetime; it is generally impossible to separate the curvature contributions of the wave from the global background curvature. By studying waves in the linear theory, one can consider waves on a background spacetime, and need not consider the effect the waves would have on their own propagation.

Consideration of radiative solutions in the non-linear theory continued, however, since the linear approximation failed to adequately describe wave generation in systems which had strong self-gravity, as Eddington pointed out in 1924 [15]. Landau and Lifschitz [16] produced the first detailed study of non-linear wave generation in self-gravitating systems, but analysis of the radiative reaction problem led to doubts that gravitational waves might not carry energy and momentum.¹

Bondi [17] ended the debate by proposing a thought experiment where a gravitational wave impinges upon a frictionless rod carrying small test beads which would be allowed to move freely. The gravitational waves would modulate the position of the beads, pushing them back and forth on the rod, demonstrating that the waves did indeed carry energy (which could be deposited in a detector).²

Despite the initial difficulties in understanding the physical properties of gravitational waves, progress has been made; current theory predicts that gravitational radiation exists and will soon be within the grasp of modern detectors. Very general and rigorous treatments of linearized waves have been produced [19], and are routinely used to analyze the problems of production, propagation, and detection of gravitational waves on arbitrary curved backgrounds. Computations suggest that the gravitational radiation bathing the Earth will produce extremely weak strains ($\Delta\ell/\ell \sim 10^{-21}$) in extended systems, justifying the use of the linearized theory in analyzing the design and response of detectors within the solar system.

¹These historical points are mentioned in Section 9.1.2 of [9].

²Preskill and Thorne, in their introduction to [18], note that Feynman also proposed this experiment to demonstrate that gravitational waves carry energy.

This chapter provides a short review of the modern formalism for describing linearized gravitational waves: Section 2.2 derives the basic linearized Einstein field equations, and Section 2.3 examines plane wave solutions to the linearized equations. Section 2.4 considers the polarization states of gravitational radiation and examines the interaction of these waves with test particles, while Section 2.5 considers the application of these results to the design of gravitational wave detectors. Section 2.6 outlines the quadrupole formalism, which provides a simple method to estimate the strength of the gravitational radiation which might arrive at Earth from astrophysical sources.

2.2 Linearized Einstein Field Equations

It is conventional to initially consider gravitational radiation in the linear approximation to general relativity and look for wavelike solutions to the vacuum Einstein field equations.

Linearized gravity is derived from the assumption that the geometry of spacetime is well described by the Minkowski metric, $\eta_{\alpha\beta}$, plus small corrections, $h_{\alpha\beta}$,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} , \tag{2.1}$$

where the correction $|h_{\alpha\beta}| \ll 1$ everywhere in the spacetime.

To study the perturbations, it is useful to break them off from the background geometry, holding the geometry fixed while allowing the perturbations to propagate.

Consider a transformation of the coordinates in Eq. (2.1) to those of another Lorentz frame:

$$\begin{aligned} g_{\bar{\mu}\bar{\nu}} &= \Lambda^{\alpha}_{\bar{\mu}} \Lambda^{\beta}_{\bar{\nu}} g_{\alpha\beta} \\ &= \eta_{\bar{\mu}\bar{\nu}} + h_{\bar{\mu}\bar{\nu}} . \end{aligned} \quad (2.2)$$

Since the Minkowski metric is the same for all inertial observers, this result shows that the perturbed metric of Eq. (2.1) is described in the new frame simply as Minkowski space plus some small perturbations. These small perturbations can be related to those of Eq. (2.1) by a simple Lorentz transformation of the tensor components,

$$h_{\bar{\mu}\bar{\nu}} = \Lambda^{\alpha}_{\bar{\mu}} \Lambda^{\beta}_{\bar{\nu}} h_{\alpha\beta} , \quad (2.3)$$

justifying the interpretation of $h_{\alpha\beta}$ as a second-rank tensor field propagating on a flat background.

Indices will always be raised and lowered with the Minkowski metric. In particular,

$$h^{\alpha\beta} = \eta^{\alpha\mu} \eta^{\beta\nu} h_{\mu\nu} . \quad (2.4)$$

Using this fact, one can show the inverse spacetime metric, to linear order in h , to be

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta} \quad (2.5)$$

by requiring $\delta^{\mu}_{\nu} = g^{\alpha\mu} g_{\alpha\nu}$.

In order to write out the linearized Einstein field equations, one must first compute the curvature tensors in the case where the spacetime metric is expressed by Eq. (2.1).

With the metric of Eq. (2.1) and its inverse in Eq. (2.5), the affine connection can be computed to $\mathcal{O}(h)$:

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}\eta^{\alpha\mu} (h_{\mu\beta,\gamma} + h_{\mu\gamma,\beta} - h_{\beta\gamma,\mu}) + \mathcal{O}(h^2) . \quad (2.6)$$

The Riemann curvature tensor is defined by

$$R^\mu{}_{\alpha\nu\beta} = \Gamma^\mu{}_{\alpha\beta,\nu} - \Gamma^\mu{}_{\alpha\nu,\beta} + \Gamma^\mu{}_{\nu\lambda}\Gamma^\lambda{}_{\alpha\beta} - \Gamma^\mu{}_{\beta\lambda}\Gamma^\lambda{}_{\alpha\nu} . \quad (2.7)$$

Each of the affine products in the last two terms, will be $\mathcal{O}(h^2)$. Neglecting them and using the definition of Eq. (2.6) the Riemann tensor becomes

$$\begin{aligned} R^\mu{}_{\alpha\nu\beta} &= \Gamma^\mu{}_{\alpha\beta,\nu} - \Gamma^\mu{}_{\alpha\nu,\beta} + \mathcal{O}(h^2) \\ &= \frac{1}{2}\eta^{\mu\lambda} [h_{\lambda\beta,\alpha\nu} + h_{\alpha\nu,\lambda\beta} - h_{\lambda\nu,\alpha\beta} - h_{\alpha\beta,\lambda\nu}] + \mathcal{O}(h^2) . \end{aligned} \quad (2.8)$$

The Ricci tensor can be obtained by contracting Eq. (2.8), giving

$$\begin{aligned} R_{\alpha\beta} &= R^\mu{}_{\alpha\mu\beta} \\ &= \Gamma^\mu{}_{\alpha\beta,\mu} - \Gamma^\mu{}_{\alpha\mu,\beta} + \mathcal{O}(h^2) \\ &= \frac{1}{2} [h^\mu{}_{\beta,\alpha\mu} + h_{\alpha\mu,\mu\beta} - h_{\alpha\beta,\mu}{}^\mu - h_{,\alpha\beta}] + \mathcal{O}(h^2) , \end{aligned} \quad (2.9)$$

where $h = h_\mu{}^\mu$ is the trace of the perturbation tensor. The scalar curvature is simply

the trace of the Ricci tensor:

$$\begin{aligned} R &= R_{\alpha\beta} g^{\alpha\beta} \\ &= h^{\mu\alpha}{}_{,\alpha\mu} - h_{,\alpha}{}^{\alpha} + \mathcal{O}(h^2) . \end{aligned} \quad (2.10)$$

The linearized Einstein tensor may now be constructed from Eqs. (2.9) and (2.10):

$$\begin{aligned} G_{\alpha\beta} &= R_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} R \\ &= \frac{1}{2} [h^{\mu}{}_{\beta,\alpha\mu} + h_{\alpha\mu}{}^{\mu}{}_{\beta} - h_{\alpha\beta,\mu}{}^{\mu} - h_{,\alpha\beta} \\ &\quad - \eta_{\alpha\beta} h^{\mu\lambda}{}_{,\lambda\mu} + \eta_{\alpha\beta} h_{,\lambda}{}^{\lambda}] + \mathcal{O}(h^2) . \end{aligned} \quad (2.11)$$

This expression can be moderately simplified by choosing to work with the trace-reversed perturbations,

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} h . \quad (2.12)$$

Tracing this equation gives

$$\bar{h} = h - 2h = -h . \quad (2.13)$$

Inversion of Eq. (2.12) gives

$$h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} \bar{h} , \quad (2.14)$$

which when substituted into Eq. (2.11) yields

$$G_{\alpha\beta} = \frac{1}{2} [\bar{h}_{\alpha\mu}{}^{\mu}{}_{\beta} + \bar{h}_{\beta\mu}{}^{\mu}{}_{\alpha} - \bar{h}_{\alpha\beta,\mu}{}^{\mu} - \eta_{\alpha\beta} \bar{h}_{\mu\nu}{}^{\mu\nu}] + \mathcal{O}(h^2) . \quad (2.15)$$

In the study of electromagnetism, the field equations can be greatly simplified by the choice of a particular gauge; the choice of a gauge in no way changes the physical content of the field equations.

It can be shown that similar gauge freedom exists in gravitational theory, and that a change of gauge simply amounts to a change in coordinates. The conventional choice of gauge for linearized general relativity is called the *deDonder gauge*, and is given by

$$\bar{h}_{\mu\nu, \nu} = 0 . \quad (2.16)$$

This gauge condition is similar to the familiar Lorentz gauge condition in electromagnetism, $A_{;\mu}^{\mu} = 0$. Like Lorentz gauge, the deDonder gauge is Lorentz invariant. Imposing this gauge choice on the Einstein tensor of Eq. (2.15) leaves

$$G_{\alpha\beta} = -\frac{1}{2}\bar{h}_{\alpha\beta, \mu}{}^{\mu} + \mathcal{O}(h^2) . \quad (2.17)$$

2.3 Gravitational Wave Solutions

Taking the linearized Einstein tensor from Eq. (2.17) together with the deDonder gauge choice of Eq. (2.16), one can construct a wave equation for perturbations of empty space (vacuum solutions):

$$G_{\alpha\beta} = 0 = \frac{1}{2}\bar{h}_{\alpha\beta, \mu}{}^{\mu} . \quad (2.18)$$

The simplest solutions to Eq. (2.18) are monochromatic plane waves, described

by

$$\bar{h}_{\alpha\beta} = \Re\{A_{\alpha\beta} \exp[ik_\mu x^\mu]\} , \quad (2.19)$$

where k^μ is a constant wave vector and the symmetric tensor $A_{\alpha\beta}$ characterizes the amplitude and polarization of the wave. The linear approximation requires that $|A_{\alpha\beta}| \ll 1$. The first and second derivatives of this solution are

$$\bar{h}_{\alpha\beta,\mu} = ik_\mu \bar{h}_{\alpha\beta} \quad (2.20)$$

and

$$\bar{h}_{\alpha\beta,\mu}{}^\mu = -k_\mu k^\mu \bar{h}_{\alpha\beta} = 0 . \quad (2.21)$$

Eq. (2.21) implies

$$k_\mu k^\mu = 0 . \quad (2.22)$$

Restrictions on the form of the solutions in Eq. (2.19) are obtained by insuring that the deDonder gauge choice is obeyed. Using Eq. (2.20) to write the gauge choice of Eq. (2.16) one obtains

$$0 = \bar{h}_{\alpha\mu,\mu}{}^\mu = ik^\mu \bar{h}_{\alpha\mu} = ik^\mu A_{\alpha\mu} \exp[ik^\mu x_\mu] \quad (2.23)$$

which vanishes if and only if

$$k^\mu A_{\alpha\mu} = 0 . \quad (2.24)$$

The solutions proposed in Eq. (2.19) will only satisfy the vacuum Einstein field equa-

tions if the two conditions of Eqs. (2.24) and (2.22) are met. The null nature of k^μ implies that these waves propagate along null rays, and that the wavespeed is in fact the speed of light, c .

A general symmetric tensor $A_{\alpha\beta}$ has 10 independent components. Imposing the restriction of Eq. (2.24) on the amplitude tensor reduces the number of free components to six, which must in some way be correlated with dynamical degrees of freedom in the gravitational field. This number can be reduced after carefully examining the deDonder gauge choice. The coordinates are not unambiguously fixed by choosing the gauge proposed in Eq. (2.16). Further restrictions may be imposed by demanding that any *general* coordinate transformation also satisfy the deDonder gauge condition.

Consider an infinitesimal coordinate transformation described by a small vector field, ξ^α :

$$x'^\alpha = x^\alpha - \xi^\alpha . \quad (2.25)$$

Under this transformation of coordinates, the metric transforms as

$$g'_{\mu\nu} = \Lambda^\alpha{}_{\mu'} \Lambda^\beta{}_{\nu'} g_{\alpha\beta} , \quad (2.26)$$

where the $\Lambda^\alpha{}_{\mu'}$ are the Jacobi transformation matrices,

$$\Lambda^\alpha{}_{\mu'} = \frac{\partial x^\alpha}{\partial x'^{\mu}} = \frac{\partial}{\partial x'^{\mu}} (x'^\alpha + \xi^\alpha) = \delta^\alpha{}_{\mu'} + \xi^\alpha{}_{,\mu'} . \quad (2.27)$$

Using this definition and applying the transformation law of Eq. (2.26) to the metric

in Eq. (2.1) yields

$$g'_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} + \mathcal{O}(h^2). \quad (2.28)$$

Defining a new perturbation field by grouping the symmetric derivative in this equation with the original field $h_{\mu\nu}$,

$$h'_{\mu\nu} = h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} \quad (2.29)$$

reproduces Eq. (2.1) in the x'^α coordinates, and leaves all subsequent computations of curvature unchanged, leading to the Einstein tensor in Eq. (2.17).

The Einstein tensor of Eq. (2.17) is written in terms of $\bar{h}_{\mu\nu}$. One can elucidate the effects of the infinitesimal coordinate transformation on $\bar{h}_{\mu\nu}$ by applying the results of Eq. (2.29) to yield

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} - \eta_{\mu\nu} \xi_{\lambda}{}^{\lambda}. \quad (2.30)$$

Eq. (2.17) assumes the deDonder gauge has been imposed. This gauge choice, applied to Eq. (2.30), requires that $\bar{h}'_{\mu\nu}{}^{,\nu} = 0$. Making this requirement imposes the restriction

$$\xi_{\mu,\nu}{}^{,\nu} = 0 \quad (2.31)$$

on the form of the coordinate transformation.

Eq. (2.31) is a wave equation for the proposed coordinate transformation. The solutions to this equation can be used to place further restrictions on the form of the

tensor $A_{\alpha\beta}$. Consider solutions of the form

$$\xi^\alpha = B^\alpha \exp[ik_\mu x^\mu] \quad (2.32)$$

where k^μ is the same wave vector introduced in the tensor plane wave solutions of Eq. (2.19). Using these wave solutions in Eq. (2.30) produces a description of the transformation of $A_{\alpha\beta}$,

$$A'_{\alpha\beta} = A_{\alpha\beta} + i(B_\alpha k_\beta + B_\beta k_\alpha) - ik_\lambda B^\lambda \eta_{\alpha\beta} . \quad (2.33)$$

In this expression, the B^α are completely arbitrary, since ξ^α is *any* solution to the wave equation.

The restrictions of choice are that $A_{\alpha\beta}$ be traceless,

$$A_\lambda{}^\lambda = 0 , \quad (2.34)$$

and that it be transverse to an arbitrary, fixed 4-velocity field, u^α :

$$A_{\alpha\lambda} u^\lambda = 0 . \quad (2.35)$$

By choosing a particular Lorentz frame, $u^\alpha = (1, 0, 0, 0)$ and insisting that Eq. (2.33) obey Eqs. (2.34) and (2.35) requires

$$B^\alpha = \frac{1}{k_0} (u^\alpha k_\lambda B^\lambda - B_0 k^\alpha) . \quad (2.36)$$

Since B^α is arbitrary, in all cases it can be chosen to obey Eq. (2.36), and the two restrictions in Eqs. (2.34) and (2.35) will be satisfied for all $A_{\alpha\beta}$. These two conditions constitute the “transverse-traceless” (TT) gauge. Together with the deDonder gauge of Eq. (2.16), the ten free components of $A_{\alpha\beta}$ have been reduced to two.

It is useful to construct $A_{\alpha\beta}$ in a particular Lorentz frame to examine its structure. Taking $u^\alpha = (1, 0, 0, 0)$ and a wave vector $k^\mu = \omega(1, 0, 0, 1)$, then the constraints outlined here yield

$$A_{\alpha\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (2.37)$$

The $A_{\alpha 0}$ components are zero due to Eq. (2.35). The $A_{\alpha z}$ components are zero due to Eq. (2.24). The diagonal components are equal but opposite in sign because $A_{\alpha\beta}$ is traceless, and the two remaining components are equal due to the symmetric nature of $A_{\alpha\beta}$. As will be seen in the next section, A_{xx} and A_{xy} correspond to the amplitudes of the two possible gravitational wave polarization states. For this reason, $A_{\alpha\beta}$ is often called the *polarization tensor*.

A more general derivation of these results can be obtained using the “shortwave approximation” developed by Isaacson [19]. Isaacson considered general curved spaces (as opposed to Minkowski space, used here), and detailed the theory of gravitational wave propagation in the limit where the wavelength of the radiation is much less than the radius of curvature of the background spacetime.

2.4 Polarization of Gravitational Waves and the Effect on Test Particles

As shown in Section 2.3, there are only two independent components of the tensor $A_{\alpha\beta}$, which characterizes the amplitudes of the wave solutions. The general form of $A_{\alpha\beta}$ given in Eq. (2.37) can be decomposed into linear combinations of two linearly independent basis tensors,

$$\psi_+^{\alpha\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (2.38)$$

and

$$\psi_\times^{\alpha\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (2.39)$$

These two linear polarization states are related by a 45° spatial rotation about the axis defined by the propagation vector. The rotation is given by the transformation matrix

$$\Lambda_\mu{}^\alpha = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (2.40)$$

which when applied to the basis tensor of Eq. (2.38) yields

$$\psi_+^{\mu\nu} \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \psi_\times^{\alpha\beta}. \quad (2.41)$$

The '+' and '×' designations describe the effect a purely polarized gravitational wave would have on an extended test body or system of particles. The character of each polarization can be discovered by examining the effect of the gravitational radiation on test particles.

Consider a particle which is at rest at $t = 0$ with 4-velocity $u^\alpha = (1, 0, 0, 0)$. The 4-acceleration experienced by this test particle due to the passage of a gravitational wave is

$$\begin{aligned}
 a^\alpha &= u^\beta u^\alpha{}_{;\beta} \\
 &= u^\beta u^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\beta\lambda} u^\beta u^\lambda \\
 &= \Gamma^\alpha{}_{\beta\lambda} u^\beta u^\lambda \\
 &= \frac{1}{2} g^{\alpha\sigma} (g_{\sigma\lambda,\beta} + g_{\sigma\beta,\lambda} - g_{\beta\lambda,\sigma}) u^\beta u^\lambda,
 \end{aligned} \tag{2.42}$$

where the metric is given by³

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \bar{h}_{\alpha\beta}. \tag{2.43}$$

Using this metric expression in Eq. (2.42) and expanding to first order in h yields

$$a^\alpha = \frac{1}{2} \eta^{\alpha\sigma} (\bar{h}_{\sigma\lambda,\beta} + \bar{h}_{\sigma\beta,\lambda} - \bar{h}_{\beta\lambda,\sigma}) u^\beta u^\lambda. \tag{2.44}$$

³Note that in the transverse traceless gauge the metric perturbations are equivalent to the trace reversed perturbations, $h_{\alpha\beta} = \bar{h}_{\alpha\beta}$.

Using $u^\alpha = (1, 0, 0, 0)$ for the test particle of interest,

$$a^\alpha = \frac{1}{2} \eta^{\alpha\sigma} (\bar{h}_{\sigma t,t} + \bar{h}_{\sigma t,t} - \bar{h}_{tt,\sigma}) . \quad (2.45)$$

Recalling that $\bar{h}_{\alpha\beta} = A_{\alpha\beta} \exp[ik_\mu x^\mu]$, examination of the form of the polarization tensor in Eq. (2.37) shows that all $A_{\alpha t} = 0$, and hence all $\bar{h}_{\alpha t} = 0$. So a test particle is *not accelerated* by a gravitational wave, since $a^\alpha = u^\beta u^\alpha{}_{;\beta} = 0$. Particles remain on geodesic trajectories.

Realizing that the particles are not accelerated by the passage of the gravitational wave, one's intuition might question whether there is actually physical information in $\bar{h}_{\alpha\beta}$, or if it is simply a funny gauge field. A simple way to address this issue is to compute the components of the Riemann curvature tensor. If the Riemann curvature tensor is zero, then $\bar{h}_{\alpha\beta}$ is purely gauge, and there is no interesting physics in $\bar{h}_{\alpha\beta}$. In reality, the Riemann curvature tensor is non-zero, and the $\bar{h}_{\alpha\beta}$ do represent real, wavelike solutions to the vacuum Einstein field equations.

The TT coordinates have been chosen to track the free test particles (*i.e.*, the coordinates deform with the passing wave, and individual particles remain at rest in the coordinates). To truly examine the effect of a passing gravitational wave on a collection of test particles, one must consider the observation of a physical quantity which is independent of the chosen coordinates. A particularly useful quantity is the proper spatial distance between two free test particles.

Consider the test particle arrangement shown in Figure 2.1. Along the x -axis,

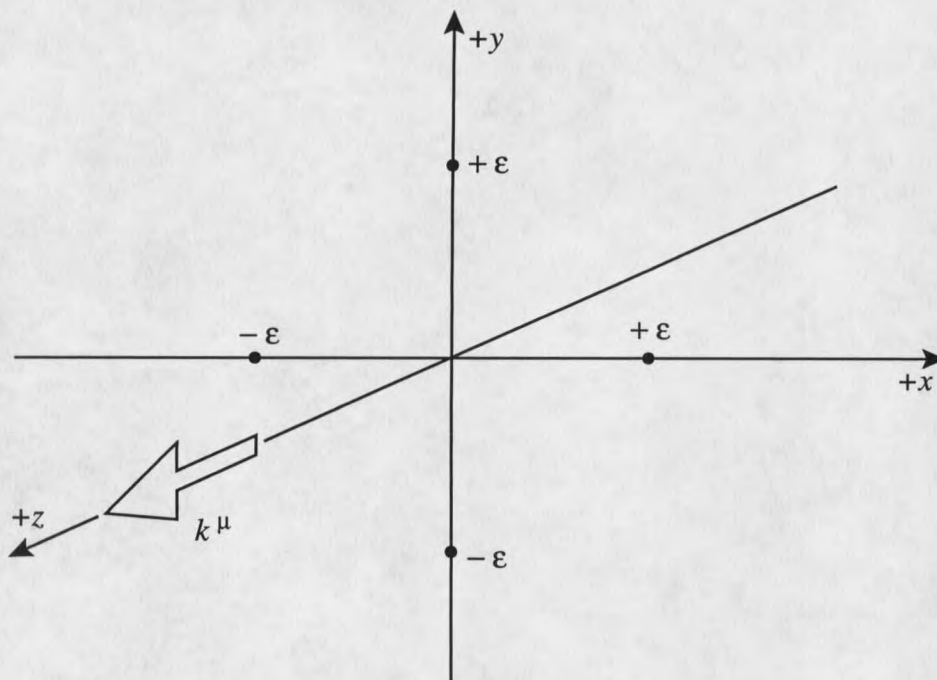


Figure 2.1: An arrangement of test particles used to consider the effects of a passing gravitational wave, represented by the wave vector k^μ .

one particle lies at spatial coordinates $x^i = (-\epsilon, 0, 0)$, and another lies at $x^i = (+\epsilon, 0, 0)$, and a gravitational wave propagates along the z -axis with wave vector $k^\mu = \omega(1, 0, 0, 1)$. As shown previously, when described in the TT gauge, the particles remain at *constant* coordinate position. The proper distance between the two test particles is given by

$$\Delta\ell_x = \int_{-\epsilon}^{+\epsilon} \sqrt{g_{xx}} dx . \quad (2.46)$$

For the situation shown in Figure 2.1, consider the metric of Eq. (2.43) and the wave solutions for $\bar{h}_{\alpha\beta}$ proposed in Eq. (2.19). Let the polarization tensor be a linear

combination of the $\psi_+^{\alpha\beta}$ basis tensor and the $\psi_\times^{\alpha\beta}$ basis tensor, giving

$$A_{\alpha\beta} = A^+ \psi_{\alpha\beta}^+ + A^\times \psi_{\alpha\beta}^\times, \quad (2.47)$$

which in matrix notation will have the form of Eq. (2.37) with $A^+ = A_{xx}$ and $A^\times = A_{xy}$. The proper distance in Eq. (2.46) becomes

$$\begin{aligned} \Delta\ell_x &= \int_{-\epsilon}^{+\epsilon} (\eta_{xx} + \bar{h}_{xx})^{1/2} dx \\ &= \int_{-\epsilon}^{+\epsilon} (1 + A^+ \cos[k_\mu x^\mu])^{1/2} dx \\ &= \int_{-\epsilon}^{+\epsilon} (1 + A^+ \cos[kz - \omega t])^{1/2} dx. \end{aligned} \quad (2.48)$$

Using the binomial expansion to expand the square root in this expression (recall $|A_{\alpha\beta}| \ll 1$ in the linear approximation) allows one to complete the integration, and the proper distance becomes

$$\begin{aligned} \Delta\ell_x &\simeq \int_{-\epsilon}^{+\epsilon} \left(1 + \frac{1}{2}A^+ \cos[kz - \omega t]\right) dx \\ &\simeq 2\epsilon \left[1 + \frac{1}{2}A^+ \cos(kz - \omega t)\right]. \end{aligned} \quad (2.49)$$

As the gravitational wave propagates past the test particles, the proper distance is 2ϵ plus a harmonic oscillation of amplitude A_{xx} .

Conducting a similar calculation for the two particles separated along the y -axis,

as shown in Figure 2.1, yields

$$\Delta\ell_y = 2\epsilon \left[1 - \frac{1}{2}A^+ \cos(kz - \omega t) \right] , \quad (2.50)$$

180° out of phase with the oscillations along the x -axis; when the proper distances along the x -axis expand, proper distances along the y -axis compress, and vice versa. The proper distances in Eqs. (2.49) and 2.50) depend *only* on the amplitude A^+ . If $A^+ = 0$, then the test masses remain separated by a constant proper distance, irrespective of the value of $A^\times$.

In Eq. (2.41) it was shown that the two polarization states are related by a 45° spatial rotation. If such a spatial rotation were applied in Figure 2.1, the test particles would lie along the axes defined by the $y = x$ and $y = -x$ lines. In this case, the proper distance between the test particles becomes

$$\Delta\ell_{xy} = 2\epsilon \left[1 + \frac{1}{2}A^\times \cos(kz - \omega t) \right] , \quad (2.51)$$

which depends *only* on the amplitude $A^\times$. If $A^\times = 0$, the test masses remain separated by a constant proper distance, irrespective of the value of A^+ .

The effect each polarization has on test particles garners the designations ‘plus’ (+) polarization for $\psi_+^{\alpha\beta}$ and ‘cross’ (×) polarization for $\psi_\times^{\alpha\beta}$. The correlation between the effect of a particular polarization and the designations becomes more readily apparent if one considers a *ring* of test particles, as shown in Figure 2.2. Consider the effect of a + polarized wave, as shown in (B) of Figure 2.2. As the wave propagates through

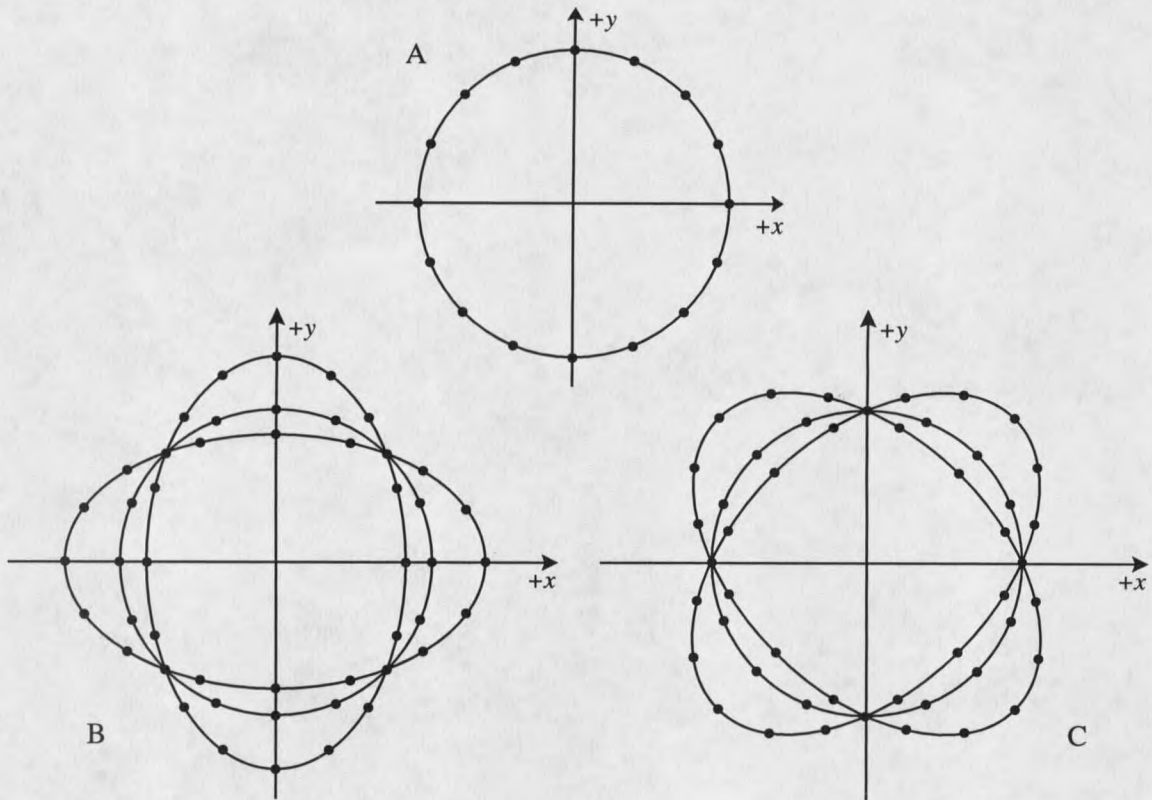


Figure 2.2: Distinguishing polarization states using rings of test particles. (A) A circular ring of test particles lies in the $x-y$ plane. (B) Modulation of the ring shape due to passage of a $+$ polarized gravitational wave propagating down the z -axis. (C) Modulation of the ring shape due to passage of a $\times$ polarized gravitational wave propagating down the z -axis.

the ring of test particles, it elongates the ring along one axis while flattening it along the other. Initially the ring might be deformed into a prolate ellipse aligned along the y -axis. Half a wave period later, the ring has an oblate shape along the x -axis, as shown. Figure 2.2 (C) shows the case for a $\times$ polarized wave, which is related to (B) by a 45° spatial rotation. In each case, particles which lie 45° away from the axes of the distorted rings remain stationary.

To this point, the description of the polarization states of gravitational waves have been expressed in terms of the *linearly* polarized states, $\psi_+^{\alpha\beta}$ and $\psi_\times^{\alpha\beta}$. One

may construct *circularly* polarized states from linear combinations of these two basis tensors. The combinations

$$\phi_L^{\alpha\beta} = \frac{1}{\sqrt{2}}(\psi_+^{\alpha\beta} - i\psi_\times^{\alpha\beta}) \quad (2.52)$$

and

$$\phi_R^{\alpha\beta} = \frac{1}{\sqrt{2}}(\psi_+^{\alpha\beta} + i\psi_\times^{\alpha\beta}) \quad (2.53)$$

correspond to normalized right- and left-circularly polarized gravitational wave states. The effect of gravitational waves in these polarization states on test particles can also be understood in terms of the ring of test particles established in Figure 2.2. The gravitational wave will permanently distort the circular ring into an elliptical state (*i.e.*, the ring will never return to its circular state, as was the case with the linearly polarized waves). For a wave in the $\phi_L^{\alpha\beta}$ state, the major axis of the ellipse will rotate around the z -axis in a counter-clockwise direction as viewed from constant z (this is often called *positive helicity*); a wave in the $\phi_R^{\alpha\beta}$ state will cause a rotation in the clockwise direction (this is called *negative helicity*) [20]. It is important to note that the test particles themselves are not rotating; they only oscillate slightly from their initial equilibrium positions, while the *shape* of the ellipse rotates.

2.5 Gravitational Wave Detectors

Understanding how gravitational waves act on particles provides the basic information needed to begin building simple instruments to detect this radiation through the

exploitation of the harmonic motion of test masses under the influence of gravitational waves, as in Eqs. (2.49 - 2.51). In this section, only the simplest of idealized detectors are reviewed to illustrate the basic principles.

The most basic detector of gravitational waves can be thought of as a pair of test masses, separated along the x -axis by a distance ℓ_o . An obvious question to ask when considering this type of detector is how it will be excited by the passage of a gravitational wave, since it was demonstrated in Section 2.4 that test particles remain on geodesic trajectories; they are unaccelerated by the passage of a gravitational wave. This is true when considering the accelerations experienced by a single test particle. Detectors, however, are composed of more than one test particle (in this case, 2 test particles). The particles will experience a *relative* acceleration due to the gravitational wave, measured by the deviation of their geodesic trajectories during the passage of the wave.

In the rest frame of the test masses, each of them has a four velocity $u^\alpha = (1, 0, 0, 0)$; they are connected by a spacelike separation vector n^α . The geodesic deviation equation is

$$\nabla_u \nabla_u n^\alpha = -R^\alpha{}_{\mu\beta\nu} u^\mu u^\nu n^\beta. \quad (2.54)$$

The affine parameter is simply time measured by the test masses, $d/d\lambda = d/dt$, and this can be written as

$$\frac{d^2 n^\alpha}{dt^2} = -R^\alpha{}_{0\beta 0} n^\beta. \quad (2.55)$$

The geodesic deviation caused by the passage of a gravitational wave may be

computed from Eq. (2.55) by using Eq. (2.8) to write the Riemann curvature tensor in terms of the gravitational wave perturbations, $h_{\mu\nu}$. The components of interest in the Riemann tensor become

$$\begin{aligned} R^\alpha{}_{0\beta 0} &= \frac{1}{2}\eta^{\alpha\lambda} [h_{\lambda 0,0\beta} + h_{0\beta,\lambda 0} - h_{\lambda\beta,00} - h_{00,\lambda\beta}] \\ &= -\frac{1}{2}h^\alpha{}_{\beta,00} , \end{aligned} \quad (2.56)$$

allowing the geodesic deviation equation to be written as

$$\frac{d^2 n^\alpha}{dt^2} = \frac{1}{2}n^\beta \frac{\partial^2}{\partial t^2} h^\alpha{}_\beta . \quad (2.57)$$

If the body vector has components $n^\alpha = (0, \ell_o, 0, 0)$ when the system is in equilibrium, then this becomes

$$\frac{\partial^2 n^x}{\partial t^2} = \frac{1}{2}\ell_o \frac{\partial^2 h_{xx}}{\partial t^2} . \quad (2.58)$$

Gravitational wave detectors then measure the geodesic deviation between two test masses, the magnitude of the deviation (*i.e.*, the time varying change in proper distance between the test masses) being a direct measure of the amplitude of the gravitational wave. In order to measure the gravitational wave amplitude, however, the deviation between the test masses must be monitored.

A simple modification to monitor the deviation between the test masses is to connect them by a spring which has an equilibrium length ℓ_o along the x -axis, as shown in Figure 2.3. The spring has a spring constant k and a damping coefficient of

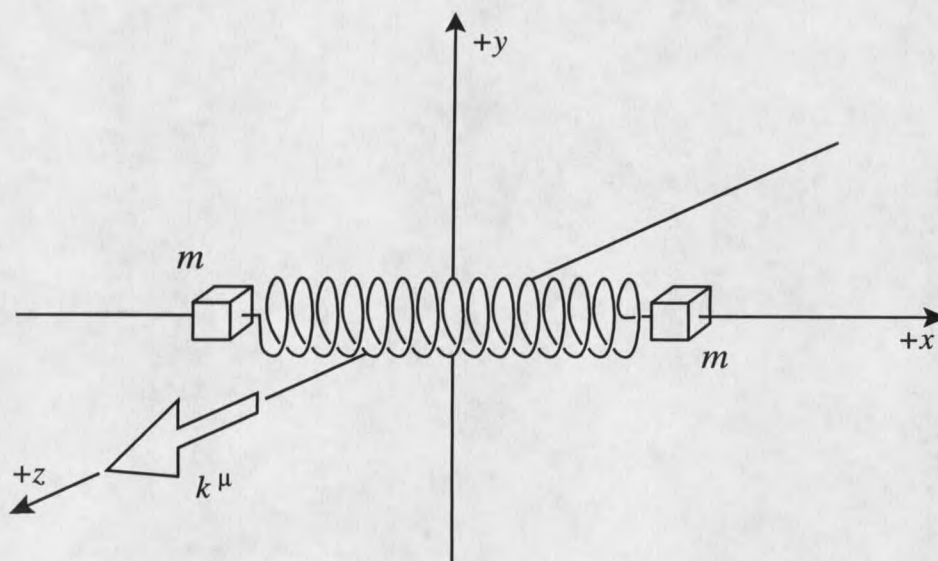


Figure 2.3: An idealized resonant gravitational wave detector. Two masses are connected by a spring of equilibrium length ℓ_o .

κ . The restoring force of the spring is the only non-gravitational force acting on the masses. When a gravitational wave is incident on the detector, the proper distance between the test masses will change, compressing or stretching the spring. The spring will begin to oscillate and slowly return to its equilibrium state; these oscillations can be monitored to determine the characteristics of the gravitational wave that caused the excitation.

To analyze the response of the detector shown in Figure 2.3, consider the test masses to lie at positions x_1 and x_2 along the x -axis. The separation of the test masses can be written as the sum of the equilibrium proper length, ℓ_o , and a small deviation vector, ξ so that $\ell = x_2 - x_1 = \ell_o + \xi$.

If the spring experiences a time dependent change in its proper length, $\ell(t)$, then

the equations of motion for each of the two masses are

$$\begin{aligned} m \frac{\partial^2 x_1}{\partial t^2} &= -k(\ell_o - \ell) - \kappa \frac{\partial(\ell_o - \ell)}{\partial t} \\ &= k\xi + \kappa \dot{\xi}, \end{aligned} \quad (2.59)$$

$$\begin{aligned} m \frac{\partial^2 x_2}{\partial t^2} &= -k(\ell - \ell_o) - \kappa \frac{\partial(\ell - \ell_o)}{\partial t} \\ &= -k\xi - \kappa \dot{\xi}, \end{aligned} \quad (2.60)$$

where the linear restoring force of the spring is represented by the first term on the right hand side, and the damping of the oscillator is proportional to the first derivative of the deviation shown in the second term. These two equations can be differenced to produce an equation of motion for the deviation ξ , yielding

$$m(\ddot{x}_2 - \ddot{x}_1) = -2k\xi - 2\kappa\dot{\xi}. \quad (2.61)$$

The term $(\ddot{x}_2 - \ddot{x}_1)$ can be re-expressed in terms of the deviation ξ by realizing this oscillator will be driven by the time dependent change in the proper distance between the two masses. As previously shown, if a gravitational wave is propagating along the z -axis, then the proper distance between test masses separated along the x -axis will vary in proportion to the amplitude of the wave. Using Eq. (2.49), the separation between the test masses can be written as

$$\ell(t) = (x_2 - x_1) \left[1 + \frac{1}{2} \bar{h}_{xx} \right] + \mathcal{O}(h^2). \quad (2.62)$$

Using the definition of the deviation, $\xi = \ell - \ell_o$, with Eq. (2.62), one may obtain an expression for the difference in the coordinate positions,

$$\begin{aligned} x_2 - x_1 &= \frac{\xi + \ell_o}{1 + (1/2)\bar{h}_{xx}} \\ &= \xi + \ell_o - \frac{1}{2}\bar{h}_{xx} + \mathcal{O}(h^2), \end{aligned} \quad (2.63)$$

where a binomial expansion has been used, keeping terms only to $\mathcal{O}(h^2)$. Taking two time derivatives of this expression yields

$$\ddot{x}_2 - \ddot{x}_1 = \ddot{\xi} - \frac{1}{2}\ell_o \frac{\partial^2 \bar{h}_{xx}}{\partial t^2}, \quad (2.64)$$

which when used in Eq. (2.61) gives,

$$\ddot{\xi} + \frac{2\kappa}{m}\dot{\xi} + \frac{2k}{m}\xi = \frac{1}{2}\ell_o \frac{\partial^2 \bar{h}_{xx}}{\partial t^2}. \quad (2.65)$$

This is the equation of a damped, driven harmonic oscillator with resonant frequency $\omega_o^2 = 2k/m$, and describes the response of the detector shown in Figure 2.3 to gravitational radiation.

Simple gravitational wave detectors, like the one described above, were originally considered by Weber [21], who also suggested that the test masses and spring could be replaced by a distributed system, such as a solid bar. The traditional design for a gravitational wave bar antenna is a massive suspended cylinder. The first bar antenna built by Weber at the University of Maryland was an aluminum cylinder,

~ 0.61 m in diameter and 1.5 m long, with a mass of 1400 kg. Distributed systems like this will be excited by incident gravitational waves just as the simple spring oscillator described above; they are most sensitive to gravitational radiation which has frequencies near the resonant frequencies of the bar. For the bar described above, the resonant frequency is $\omega_o \sim v_s/\ell \sim 1.6$ kHz, where v_s is the speed of sound in the bar, and ℓ is its length. By the mid-1960s Weber had constructed the first resonant bar antenna, and was observing strains at the thermal limit of the antenna [22].

Bar detectors are intrinsically *narrow band* detectors, with bandwidths of only a few hertz around the central frequency in the kilohertz band; their sensitivity to gravitational waves peaks at frequencies which correspond to their fundamental vibrational modes and the successive harmonics of the fundamental mode. Bar technology is pursued by several groups around the world⁴, and new bar systems (such as large, resonant spherical shells, rather than a cylindrical bar, [24] and [25]) are still being proposed.

Using electromagnetic signals to probe the spacetime curvature between two free masses is the basic scheme of many gravitational wave detectors, including pulsar timing [26, 27], Doppler tracking of spacecraft⁵ [33], and the emerging field of interferometric gravitational wave astronomy.

⁴Currently, the most sensitive bar antenna, called ALLEGRO, is maintained by the experimental gravity group at Louisiana State University. The ALLEGRO antenna is a 3 m, 2300 kg aluminum bar. Operating at a temperature of $T = 6$ mK, this bar has a strain sensitivity of $h^{rms} \simeq 6 \times 10^{-19}$ [23].

⁵Doppler tracking of U.S. spacecraft as a method of gravitational wave detection has been an ongoing endeavor for many years. Projects have been carried out with the Viking [28], Voyager [29], Pioneer [30, 31], and Ulysses [32] spacecraft.

The resonant mass detector can be modified by replacing the spring in the system with an electromagnetic signal (such as a laser) which is reflected off the masses. In mechanical oscillators (the spring/mass system shown in Figure 2.3, or a distributed body like a bar antenna, where the properties of the solid provide the restoring force between the ends of the antenna), gravitational waves deposit energy that excites the vibrational modes in the detector, which can then be detected. If the mechanical connection between the ends of the bar are removed (*e.g.*, remove the spring from Figure 2.3, leaving two free masses), the change in proper distance between the two masses due to the passage of a gravitational wave may still be detected using a transmitted electromagnetic signal since the number of wavelengths that fit between the masses will change. For two mass systems, this effect produces a Doppler shift in the transmitted signal [33, 35]. Simple, Michelson interferometer detectors can be devised by monitoring three test masses oriented on orthogonal axes in an 'L' shape. A gravitational wave incident on this three mass detector will induce different changes in proper length in each arm, which can be detected by interfering the electromagnetic signals exchanged between each pair of masses. The first interferometer to be used for gravitational wave detection was built at the Hughes Research Laboratory by Forward [36]. Detection strategies with interferometers have rapidly led to the proposal and development of more complex and sensitive interferometric systems, such as the multi-pass Michelson interferometer [37] and the Fabry-Perot Michelson [38].

Interferometers (to contrast with the bar detectors described above) are *broad band* detectors; they are sensitive to a wide range of frequencies which will cause a fringe

shift in the output beam. The proposed space-based interferometers are sensitive from $\sim 10^{-5}$ Hz to ~ 1 Hz. Ground based instruments, such as LIGO, are sensitive from ~ 1 Hz to ~ 1000 Hz.

Despite the dominance of bars and interferometers in the arena of modern detection strategies, a great deal of attention has been given over the years to the development of other ideas for detectors of gravitational radiation. The same basic ideas of resonant detection are applied to a variety of laboratory apparatuses or astrophysical and terrestrial systems which can be monitored for deviations from their expected behavior due to interaction with gravitational radiation. A variety of these mechanical detectors are described in Chapter 37 of [1].

2.6 Sources of Gravitational Radiation

The analysis of candidate sources of gravitational waves has been part of the focus of an enormous enterprise of theoretical efforts aimed at supporting the new generation of interferometric gravitational wave detectors. Particular attention has been paid to astrophysical binary systems, which over the course of time evolve via emission of gravitational radiation, slowing spiraling together. A variety of astrophysical binaries are expected to be observable by interferometric detectors. For example, spaceborne detectors are expected to be sensitive to the inspiral and coalescence of massive ($\sim 10^7 M_\odot$ to $\sim 10^5 M_\odot$) black hole binaries, as well as the gravitational wave emission from compact binary star systems which are far from coalescence (*e.g.*, white dwarf binaries, neutron star binaries). By contrast, ground-based detectors will be

sensitive to the merger of smaller black hole binaries ($\sim 10M_{\odot}$), as well as the coalescence of compact binary stars (*e.g.*, neutron star/neutron star binaries). Precisely understanding the structure and evolution of the gravitational radiation emitted from these astrophysical systems is essential to the data reduction techniques ('matched filtering') that will be used to extract information from the interferometer data streams [39]⁶.

Simple estimates of the strength of gravitational radiation emitted by a particular source may be obtained in linearized general relativity from the quadrupole approximation, which was originally derived by Einstein [40]. Within the quadrupole approximation, the amplitude of a gravitational wave, h_{ij} , is proportional to the second time derivative of the quadrupole moment of the source generating the waves,

$$h_{ij} = \frac{2}{r} \frac{\partial^2}{\partial t^2} \Xi_{ij}(t - r) , \quad (2.66)$$

where r is the distance to the source, and Ξ is the trace free quadrupole tensor (or 'reduced quadrupole tensor'), defined by

$$\Xi_{ij}(t) = \int \rho(t) \left[x_i x_j - \frac{1}{3} x^2 \delta_{ij} \right] d^3 x . \quad (2.67)$$

where $\rho(t)$ is the mass density at the source. The evaluation of the wave amplitude in Eq. (2.66) occurs at a distance r from the source, and the reduced quadrupole tensor is evaluated at the retarded time $(t - r)$.

⁶This reference gives a general introduction to the concept of matched filtering, then applies it to the data analysis problem for the LISA space-based interferometer

In order to assess the strain a distant observer might observe from a particular source of gravitational radiation, it is useful to use the wave equation form of the Einstein field equations. Using the Einstein tensor of Eq. (2.17), the field equations are

$$G_{\alpha\beta} = -\frac{1}{2}\square\bar{h}_{\alpha\beta} = 8\pi T_{\alpha\beta} , \quad (2.68)$$

where $T_{\alpha\beta}$ is the stress energy tensor of the source producing the waves, and $\square$ is the flat space wave operator.

For sources with low internal velocities (“Newtonian” sources, with $v \ll 1$), the stress energy tensor is connected to the quadrupole moment of the source through Eq. (2.67), where $\rho(t) \simeq T^{00}$. For observers who are far from a source of gravitational waves (of frequency ω), the general solutions to this wave equation are found to be⁷

$$\bar{h}_{ij}(t) = -\frac{2\omega}{r}\Xi_{ij}(t)e^{i\omega r} . \quad (2.69)$$

When observing gravitational radiation from a distant source, it is convenient to write the metric perturbations in the transverse-traceless gauge. If an observer defines local coordinates such that the gravitational radiation is propagating along the z -axis, as shown in Figure 2.4, then Eq. (2.69) may be written in the form

$$\bar{h}_{zi} = 0 , \quad (2.70)$$

$$\bar{h}_{xx} = -\bar{h}_{yy} = -\frac{\omega^2}{r}(\Xi_{xx} - \Xi_{yy})e^{i\omega r} , \quad (2.71)$$

⁷A concise derivation of this result can be found in Chapter 9 of [14].

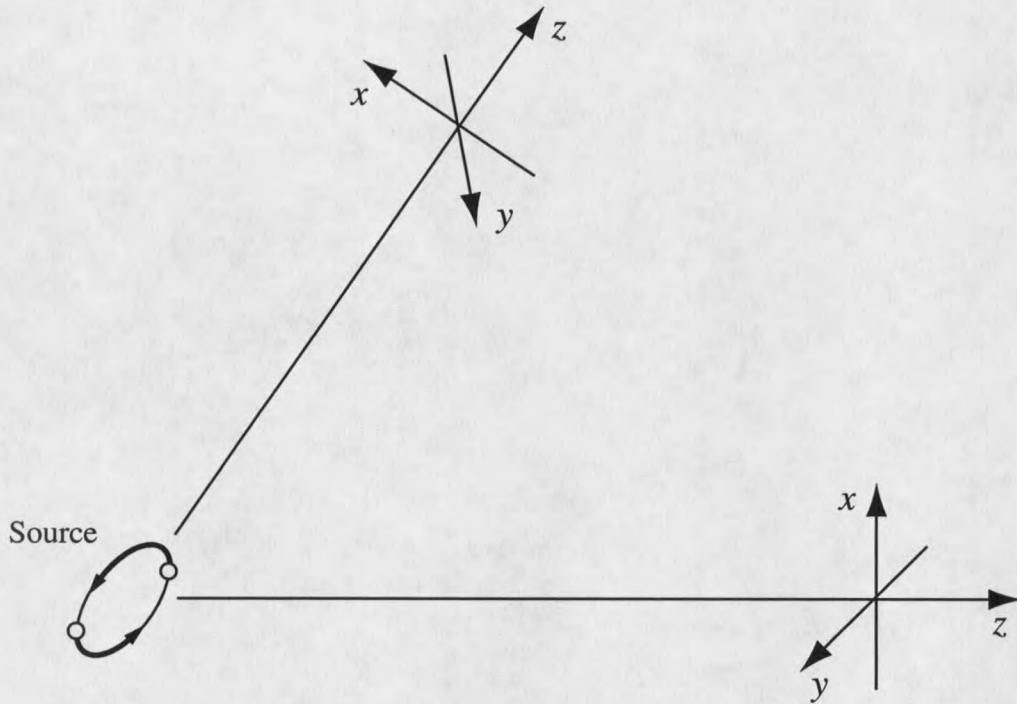


Figure 2.4: Local coordinate systems for observing distant sources of gravitational radiation set up by individual observers. On each patch, it is necessary to describe the quadrupole behavior of the distant source *in the local coordinates*.

and

$$\bar{h}_{xy} = -\frac{2\omega^2}{r}\Xi_{xy}e^{i\omega r} . \quad (2.72)$$

Care must be taken when using Eqs. (2.70) - (2.72), as the components of the reduced quadrupole tensor must be written in terms of the local coordinate patch, even though the source is a great distance away ⁸.

One of the most promising sources of gravitational waves for modern detectors is the astrophysical binary system. Simple estimates of the strength of the gravitational radiation being emitted by a binary system can be made using Eqs. (2.66) and (2.70)-

⁸This assumes the waves have propagated freely from the source to the observer; the observer's coordinate system extends all the way to the source. An example of this procedure is given in Chapter 4, Section 4.4.

(2.72).

Consider, for simplicity, an equal mass binary in a circular orbit of diameter ℓ_o lying in the $x - y$ plane. From Kepler's laws, the orbital frequency can be written as $\omega_o^2 = 2m/\ell_o^3$, and can be used to write the positions of the stars as a function of time:

$$x_1(t) = \frac{\ell_o}{2} \cos \omega_o t \quad y_1(t) = \frac{\ell_o}{2} \sin \omega_o t , \quad (2.73)$$

$$x_2(t) = -\frac{\ell_o}{2} \cos \omega_o t \quad y_2(t) = -\frac{\ell_o}{2} \sin \omega_o t . \quad (2.74)$$

Using these expressions, the components of the reduced quadrupole tensor are found to be

$$\Xi_{xx} = m(x_1^2 + x_2^2) = \frac{m}{4} \ell_o^2 (1 + \cos 2\omega_o t) , \quad (2.75)$$

$$\Xi_{yy} = m(y_1^2 + y_2^2) = \frac{m}{4} \ell_o^2 (1 - \cos 2\omega_o t) , \quad (2.76)$$

$$\Xi_{zz} = -(\Xi_{xx} + \Xi_{yy}) = -\frac{m}{2} \ell_o^2 , \quad (2.77)$$

$$\Xi_{xy} = m(x_1 y_1 + x_2 y_2) = \frac{m}{4} \ell_o^2 \sin 2\omega_o t . \quad (2.78)$$

and all other components equal to zero. With the elements of the quadrupole tensor in hand, Eqs. (2.70)-(2.72) can be used to write out the gravitational wave amplitudes for observers viewing the system face on, yielding

$$\bar{h}_{xx} = -\bar{h}_{yy} = -\frac{2m\omega_o^2 \ell_o^2}{r} e^{i2\omega_o(r-t)} , \quad (2.79)$$

and

$$\bar{h}_{xy} = -\frac{i2m\omega_o^2\ell_o^2}{r}e^{i2\omega_o(r-t)} . \quad (2.80)$$

Note that the frequency of the gravitational radiation is twice the orbital frequency of the binary, $\omega_{gw} = 2\omega_o$. Similar estimates of the gravitational wave amplitudes can be made for observers on other coordinate patches at a variety of inclinations to the binary, so long as the reduced quadrupole moment is computed in the local coordinate system.

The gravitational wave luminosity is another quantity that is often useful to know, and can be derived from the quadrupole approximation (see, for example [1]),

$$L_{gw} = \frac{1}{5}\langle\ddot{\Xi}_{ij}\ddot{\Xi}^{ij}\rangle , \quad (2.81)$$

where the $\langle \rangle$ denotes spatial averaging over several wavelengths of the gravitational waves. This has been studied for general binary systems, and can be written in the form [41]

$$L_{gw} = \frac{32}{5}\frac{(m_1m_2)^2(m_1+m_2)}{a^5}f(e) , \quad (2.82)$$

where a is the semi-major axis, e is the eccentricity, and $f(e)$ is the eccentricity function, given by

$$f(e) = \frac{1 + be^2 + ce^4}{(1 - e^2)^{7/2}} , \quad (2.83)$$

with $b = 73/24$ and $c = 37/96$. For the equal mass binary star system in a circular

orbit ($e = 0$), Eq. (2.82) reduces to

$$L_{gw} = \frac{64}{5} \frac{(m)^3}{a^5}, \quad (2.84)$$

By considering numerical estimates of Eqs. (2.79), (2.80) and (2.82), one can consider if astrophysical binary systems will even be visible to current detectors. Taking the magnitude of the metric perturbations yields

$$|h| = \frac{(2m)^{5/3} \omega^{2/3}}{r}. \quad (2.85)$$

A particularly interesting known binary system is the binary pulsar PSR 1913 + 16. This binary has $m \simeq 1.4M_{\odot}$, and the orbital period $\tau \simeq 7.75$ hr yields a frequency $\omega \simeq 2 \times 10^{-4} \text{ rad s}^{-1}$ [4]. At a distance $r \simeq 5$ kpc, Eq. (2.85) gives a dimensionless strain of

$$|h|_{\text{binary pulsar}} \sim 10^{-23}, \quad (2.86)$$

which is an order of magnitude smaller than the target strain sensitivity $|h| \sim 10^{-21}$ to $|h| \sim 10^{-22}$ [42] of the current generation of interferometric detectors.

The methods outlined here are excellent for obtaining quick estimates of the gravitational radiation that might be received from general astrophysical binaries. Data analysis for modern observatories, however, will depend on much better descriptions of gravitational radiation than those presented here.

CHAPTER 3

SENSITIVITY CURVES FOR SPACEBORNE INTERFEROMETERS

3.1 Introduction

Just as with electromagnetic astronomical observatories, building a gravitational wave observatory in space will provide astronomers with information which is distinctly different from the data that can be obtained with ground based instrumentation. While the operational principles of a spaceborne observatory are similar to those of Earth based systems, building instruments in space will pose a unique set of challenges and problems which will have to be overcome.

Consider the proposed LISA observatory, whose orbit is shown in Figure 3.1. Any orbiting observatory will have to be able to track the distant spacecraft with lasers over distances of millions of kilometers ¹. The various signals from the different arms of the constellation of spacecraft will have to be combined to form conventional interferometer signal data, and that data will have to be analyzed to account for the

¹The proposed baseline for LISA is $\ell = 5 \times 10^6$ km. The proposed baseline for OMEGA is $\ell = 1 \times 10^6$ km.

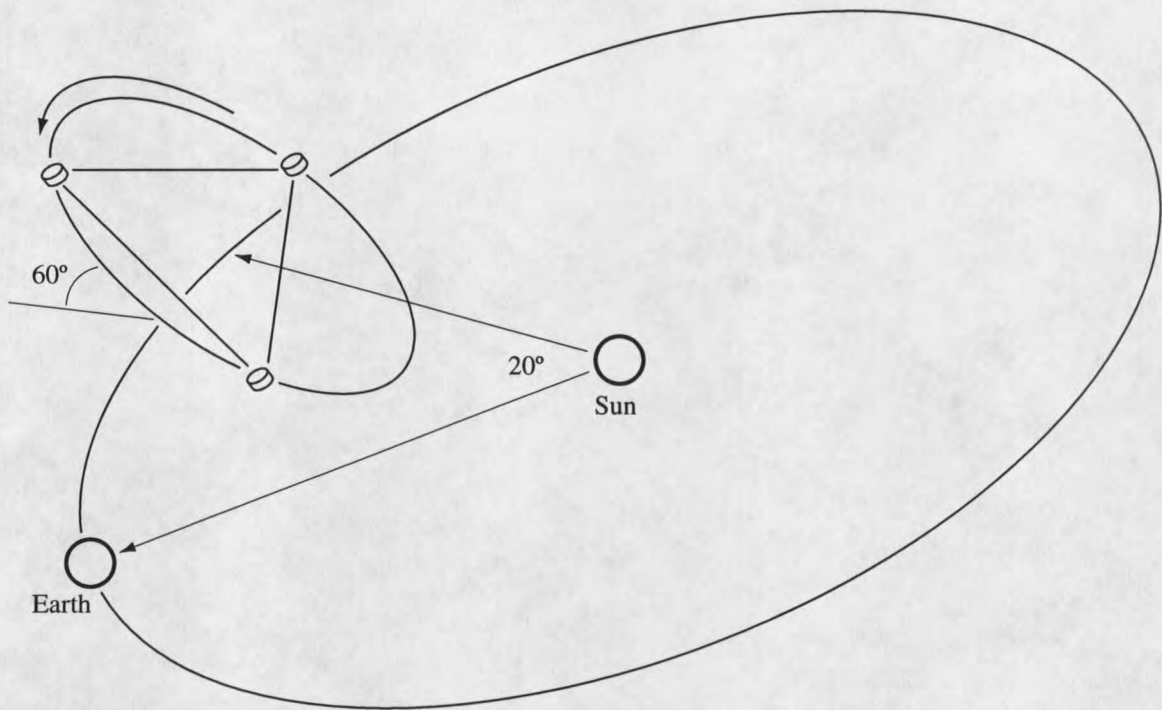


Figure 3.1: The proposed orbit for LISA. The orbit is heliocentric, trailing the Earth by 20° , and inclined to the ecliptic by 60° . The three spacecraft remain at this inclination, orbiting a common center as the configuration orbits the Sun.

motion of the spacecraft in their orbits.

Not only will space based missions expand our reach into the low frequency bands (10^{-5} to 1 Hz; ground based systems will be observing in the high frequency band, from 10 to 1000 Hz), but there are *known sources* of gravitational radiation which should be visible to these observatories. Interacting binary white dwarf systems should be detected by a mission like LISA or OMEGA several minutes after observations begin. Since these sources have already been observed optically (as opposed to the more speculative neutron star/black hole binaries the ground based systems will be sensitive to), there is little doubt that they will be observable by a space-based mission, giving immediate confirmation of the existence of gravitational radiation.

With the possibility of launching a space based observatory early in the next century, an essential task to be completed is understanding the sources of noise in any given instrument, and the response of an instrument to various sources of gravitational radiation. This information is important for predicting what types of sources will be visible, where improvements in design can be made to reduce noise, and what preparations will need to be made in anticipation of analyzing the signal data returned by the observatory. One of the most common tools for understanding the response of a space based observatory is a sensitivity curve (or 'noise curve'), which represents the overall sensitivity of the interferometer, at a given frequency, across the entire band of observation.

In Section 3.2, the sources of noise in spacecraft interferometer systems are described. Section 3.3 covers the Doppler tracking formalism (which the interferometer analysis is largely based upon).

In Sections 3.4 and 3.5² the problem of describing the sensitivity of a space-based gravitational wave observatory is examined. Conventional sensitivity curves are shown as a plot of the root spectral density of the strain in the interferometer vs. the frequency of the gravitational waves. The overall curve has a well shape which has traditionally been represented as a three part curve: a steep rise at low frequencies, a flat "floor" at intermediate frequencies, and another rise at high frequencies. Here the sensitivity limits of an observatory are computed exactly, producing a smooth,

²The work presented in this section is being prepared for publication in collaboration with W. A. Hiscock at Montana State University-Bozeman, and R. W. Hellings at the Jet Propulsion Laboratory.

continuous curve across the entire frequency band of interest. To compute the precise shape of the sensitivity curve, the exact form of the spectral density must be derived, where the spectral density is the signal power in the interferometer as a function of gravitational wave frequency. For an interferometric detector, the spectral density is computed from the spectral density of the noise in the instrument and the gravitational wave transfer function, which describes the connection between the power spectrum of the interferometer signal and the power spectra of gravitational waves which are incident on the detector.

The *exact* gravitational wave transfer function is derived, and applied to the generation of sensitivity curves. This analysis shows that the shape of the noise curve for the interferometer is given at low frequencies by the acceleration noise in the system, and that the mid-frequency floor and high frequency rise in the curve are a direct consequence of the form of the gravitational wave transfer function and the shot noise in the instrument. The “knee” where the mid-frequency floor turns upward is found to occur at $\omega\tau \sim 1$ for any interferometer of interest. In addition, the sensitivity curve shows rapidly varying high frequency structure, which arises from the form of the transfer function. Specific curves for both OMEGA and LISA are shown.

In Section 3.6 the “long wavelength” approximation that previous work in this field has relied upon is derived and shown to coincide with the long-wavelength limit of the exact results in Sections 3.4 and 3.5.

3.2 Spacecraft and Sources of Noise

The first task in describing the response of a gravitational wave observatory to a spectrum of gravitational waves is characterizing the sources of noise present in the system. Ground based observatories are plagued by a variety of noise sources, but the dominant noise sources are seismic noise at low frequencies, thermal suspension noise (in the isolation systems which suspend the mirrors in the interferometers) at intermediate frequencies, and photon shot noise at high frequencies [42]. By contrast, space-based observatories are dominated by acceleration noise at low frequencies, and photon shot noise at higher frequencies [7, 8].

Noise in spaceborne interferometers systems can be broadly classified into two categories: noise which is *common* to both arms of the interferometer, and noise which is *independent* in the two arms.

The largest source of common noise in a space-based interferometer system is laser phase noise, caused by fluctuations in the laser system. In space based interferometers, this noise cannot be eliminated by simply subtracting the strain signal between the two arms because the arms are not of equal length, as they are in traditional Michelson interferometers. In a spaceborne observatory such as OMEGA, with arm lengths on the order of 10^6 km, the arms can differ in length by 1 – 2%. Select methods of data reduction have been developed to overcome this problem, and allow the common phase noise in the interferometer arms to be greatly reduced.

One suggested data reduction method, proposed by Giampieri and collaborators

[43, 44], has been studied for space interferometry for some time. This method estimates the size of the laser phase noise by assuming it dominates the interferometer signal. An estimate of the phase noise is made from the Fourier transform of the signal, and subtracted from the detector output. The unfortunate effect of this processing is the introduction of poles into the Fourier reconstruction of the data stream. These poles are purely artifacts of the data reduction method.

A preferred reduction procedure was recently proposed by Tinto and Armstrong [45], which works in the time domain. This method combines the strain in the interferometer arms at different times to precisely cancel the laser phase noise, and avoids the singular behavior of previous analyses. One begins by writing the strain signals, including noise, as

$$s_i(t) = z_i(t) + p(t) - p(t - 2\tau_i) + n_i(t) , \quad (3.1)$$

where $s_1(t)$ and $s_2(t)$ are the two noisy strain signals in the individual arms of the interferometer, $z_i(t)$ are the strains produced by gravitational waves, $p(t)$ is the laser phase noise which is common to the two arms, $n_i(t)$ is the strain noise in the i^{th} arm produced by all other noise sources, and τ_i is the one-way light travel time in the i^{th} arm. Simple algebra then confirms that the combination

$$\Sigma(t) = s_1(t) - s_1(t - 2\tau_2) - s_2(t) + s_2(t - 2\tau_1) , \quad (3.2)$$

is devoid of laser phase noise:

$$\begin{aligned}\Sigma(t) = & z_1(t) - z_2(t) - z_1(t - 2\tau_2) + z_2(t - 2\tau_1) \\ & + n_1(t) + n_1(t - 2\tau_2) + n_2(t) + n_2(t - 2\tau_1) .\end{aligned}\quad (3.3)$$

The independent noise sources are uncorrelated with one another, and so add together when the signal subtraction is completed, as shown in Eq. (3.3).

It is expected that the noise common to both arms of the interferometer may be significantly reduced by one of these proposed data analysis procedures, and that the sensitivity of spaceborne observatories will be limited by independent noise sources. The primary sources of independent noise in space based systems are shot noise in the laser system, and acceleration noise (produced by parasitic forces on the proof mass).

The strain spectral density of the laser shot noise is ³ [46]

$$S_z = \frac{4}{\pi^4} \frac{h\nu_o}{\epsilon^2 D^4 P_t} \frac{1}{\nu_o^4} , \quad (3.4)$$

where r is the distance between the transmitter and receiver, D is the optics diameter⁴, ϵ is the telescope efficiency, P_t is the transmitted power, ν_o is the frequency of the tracking signal, and h is Planck's constant.

³To restore conventional units to this expression, multiply by a factor of c^4 .

⁴For space based interferometers, the laser is transmitted through and received by an onboard telescope.

The strain spectral density of spacecraft acceleration noise is given by⁵ [46]

$$S_z = \frac{S_a}{\omega^4 r^2}, \quad (3.5)$$

where S_a is the acceleration noise spectral density, and $\omega = 2\pi f_{gw}$ is the gravitational wave frequency.

Each of these spectral densities is used in the computation of the noise limiting the sensitivity of a given gravitational wave detector, outlined in Section 3.5.

3.3 Doppler Tracking

The formalism for deriving the sensitivity of a gravitational wave interferometer derived in Section 3.4 will be based on combining the individual gravitational wave signals produced in each arm. The signal in a single arm can be derived through a simple application of the Doppler formalism used to analyze spacecraft tracking signals for evidence of gravitational waves.

Consider the tracking geometry depicted in Figure 3.2, which shows an observer (a free mass, m_1) at the origin of coordinates, tracking a distant spacecraft (or other free mass, m_2 in the $x-z$ plane) via a transmitted electromagnetic signal of frequency ν_o . The signal is transmitted from m_1 , transponded of m_2 , and received back at m_1 .

A gravitational wave propagating between two masses will produce a Doppler shift

⁵To restore conventional units to this expression, divide by a factor of c^2 .

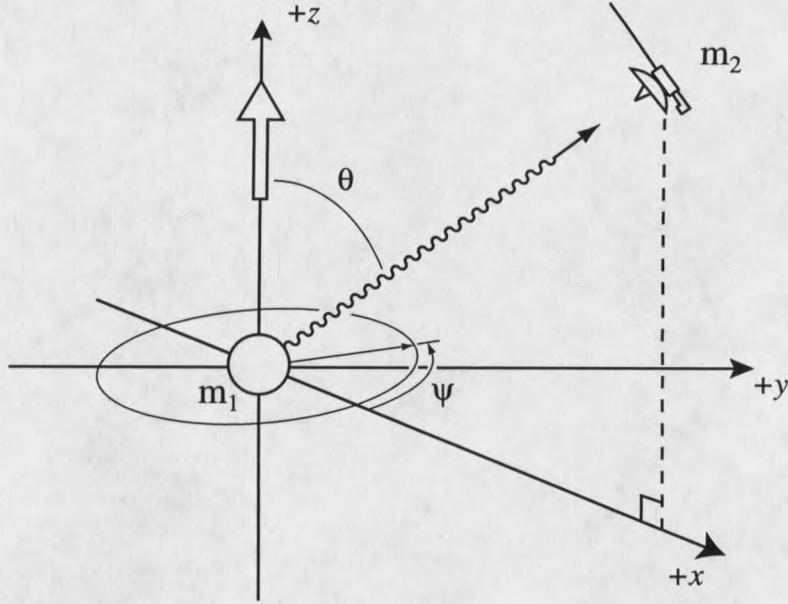


Figure 3.2: The tracking geometry for a Doppler gravitational wave detector. Two masses (*e.g.*, the Earth and a distant spacecraft) exchange an electromagnetic signal whose frequency is Doppler shifted by a gravitational wave propagating up the z -axis. The angle ψ is the principle polarization angle of the gravitational wave, which lies in a plane orthogonal to the propagation vector, *i.e.*, in the $x - y$ plane for waves propagating up the z -axis.

in the electromagnetic signal they exchange of (see, for example, [33], [34], [35])

$$y = \frac{\Delta\nu}{\nu_o} = \frac{1}{2} \cos 2\psi \times [(1 - \cos \theta) h(t) + 2 \cos \theta h(t - \tau - \tau \cos \theta) - (1 + \cos \theta) h(t - 2\tau)], \quad (3.6)$$

where $h(t)$ is the dimensionless amplitude of the gravitational wave, τ is the light travel time between the masses, θ is the polar angle between the propagation vector of the gravitational wave and the line-of-sight between the masses, and ψ is the principal polarization angle of the gravitational wave. The principal polarization angle is measured from the x -axis of the coordinate system to the axis of the gravitational

wave polarization state (the axis along which a + or $\times$ polarized wave would elongate or compress a ring of test particles; see Section 2.4).

It is illustrative to derive the Doppler gravitational wave transfer function from this quantity, since the derivation for the interferometer is constructed in an analogous way. The method presented here closely follows [35].

All the time dependence in Eq. (3.6) is embedded in the time varying amplitude of the gravitational wave, $h(t)$. It is useful to write $h(t)$ in terms of its Fourier transform $\tilde{h}(\omega)$. If the Doppler record is sampled for a time T then $h(t)$ is related to its Fourier transform by

$$h(t) = \frac{\sqrt{T}}{2\pi} \int_{-\infty}^{+\infty} \tilde{h}(\omega) e^{i\omega t} d\omega, \quad (3.7)$$

where the $\sqrt{T}$ normalization factor is used to keep the power spectrum roughly independent of time. Using this definition of the Fourier transform, the Doppler shift can be written as

$$\begin{aligned} y(t) = & \frac{\sqrt{T}}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{2} \cos(2\psi) \tilde{h}(\omega) \\ & \times \left[(1 - \mu) + 2\mu e^{-i\omega\tau(1+\mu)} - (1 + \mu) e^{-i2\omega\tau} \right] e^{i\omega t} d\omega, \end{aligned} \quad (3.8)$$

where $\mu = \cos \theta$. The average power in the Doppler signal is obtained by squaring and time averaging this quantity, yielding

$$\begin{aligned} \langle y^2 \rangle &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |y|^2 dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt y(t) y^*(t), \end{aligned} \quad (3.9)$$

where $y^*(t)$ is the complex conjugate of Eq. (3.8). Taking the real part of this expression and utilizing the δ -function identity

$$\delta(\omega - \omega') = \frac{1}{\pi} \int_0^\infty dt \cos[(\omega - \omega')t] , \quad (3.10)$$

to evaluate the ω' integral in the definition of $y^*(t)$, Eq. (3.9) becomes

$$\begin{aligned} \langle y^2 \rangle = & \frac{1}{4\pi} \int_{-\infty}^{+\infty} d\omega \cos^2(2\psi) \tilde{h}^2(\omega) \left[\mu^2 + \sin^2(\omega\tau) + \mu^2 \cos^2(\omega\tau) \right. \\ & \left. - 2\mu \sin(\omega\tau) \sin(\mu\omega\tau) - 2\mu^2 \cos(\omega\tau) \cos(\mu\omega\tau) \right] . \end{aligned} \quad (3.11)$$

This expression depends on the angular relation between the detector and the source of the gravitational waves, giving the power from a particular direction θ relative to the line of sight between the masses. Of more practical use when analyzing the response characteristics of a detector is the *isotropic power*, obtained by averaging over all directions and polarizations,

$$\overline{\langle y^2 \rangle} = \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^{+1} d\psi \, d\mu \langle y^2 \rangle . \quad (3.12)$$

The angular averages are straightforward to evaluate, yielding

$$\overline{\langle y^2 \rangle} = \frac{1}{8\pi} \int_{-\infty}^{+\infty} d\omega \tilde{h}^2(\omega) \left(1 - \frac{1}{3} \cos 2\omega\tau - \frac{3}{(\omega\tau)^2} - \frac{\cos 2\omega\tau}{(\omega\tau)^2} + \frac{2 \sin 2\omega\tau}{(\omega\tau)^3} \right) . \quad (3.13)$$

To compute the gravitational wave transfer function from the isotropic power, one defines the spectral density of gravitational wave amplitude as $S_h(\omega) = |\tilde{h}(\omega)|^2$ such

that

$$\langle h^2 \rangle = \frac{1}{T} \int_0^\infty h(t)^2 dt = \frac{1}{2\pi} \int_0^\infty S_h(\omega) d\omega . \quad (3.14)$$

The spectral density of the isotropic power is defined as

$$\overline{\langle y^2 \rangle} = \frac{1}{2\pi} \int_0^\infty S_{\bar{y}}(\omega) d\omega . \quad (3.15)$$

With these definitions, Eq. (3.13) can be written as

$$S_{\bar{y}} = S_h R(\omega\tau) , \quad (3.16)$$

where $R(\omega\tau)$ is the gravitational wave transfer function (also called the “response function”) and is given by

$$R(u) = \frac{1}{2} \left(1 - \frac{1}{3} \cos 2u - \frac{3}{u^2} - \frac{\cos 2u}{u^2} + \frac{2 \sin 2u}{u^3} \right) , \quad (3.17)$$

where $u = \omega\tau$. Taking the low frequency limit of this function, $u \ll 1$, gives

$$R(u) \sim \frac{4}{15} u^2 . \quad (3.18)$$

The Doppler tracking transfer function is plotted in Figure 3.3. The function has been normalized by a factor of $1/u^2$ to facilitate comparison with the interferometer transfer function derived in Section 3.4.

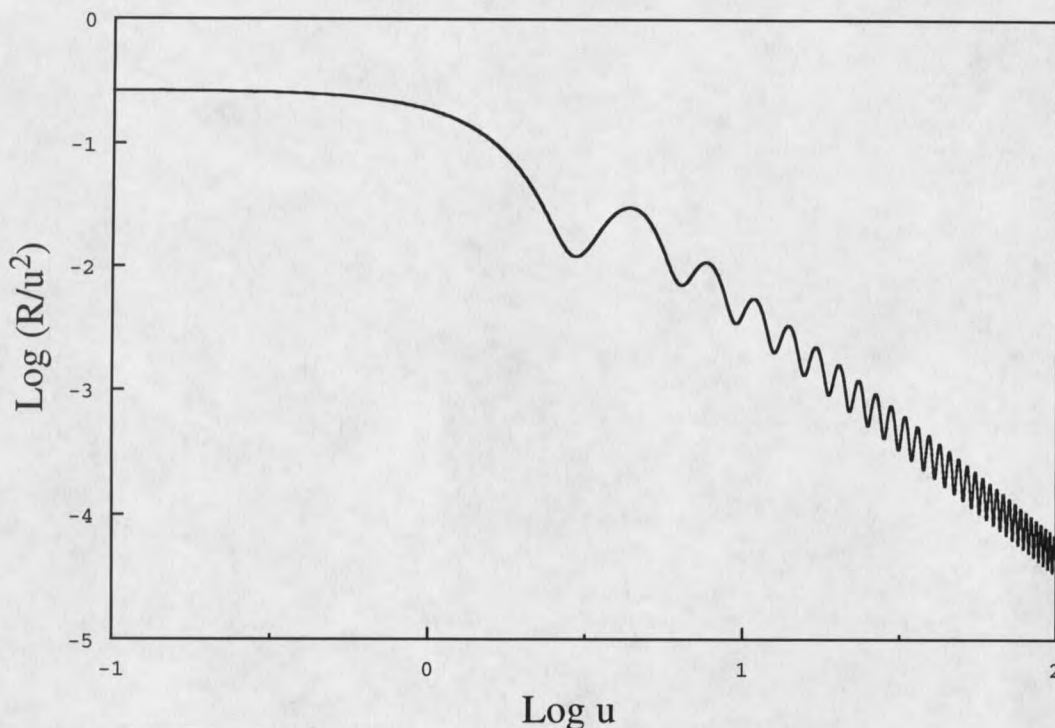


Figure 3.3: The transfer function of a Doppler tracking gravitational wave detector, as a function of $u = \omega\tau$. The function has been weighted by $1/u^2$.

3.4 Gravitational Wave Transfer Function

Previous estimates of the sensitivity of interferometric detectors have resorted to working in the long wavelength approximation (reviewed in Section 3.6), which assumes the wavelength of the gravitational radiation incident on a detector is much longer than the baseline of the detector.

For ground-based detectors such as LIGO or VIRGO, the long wavelength limit is a good approximation: the interferometer arms are several kilometers in length, while the instruments are sensitive to high frequency gravitational radiation with wavelengths of $\lambda \simeq 3 \times 10^5$ km ($f = 1$ Hz) to $\lambda \simeq 30$ km ($f = 10^4$ Hz). In contrast, this is a poor approximation for space-based observatories like OMEGA or LISA.

These detectors have arm lengths on the order of 10^6 km, and are sensitive to low frequency gravitational waves with wavelengths $\lambda \simeq 3 \times 10^{10}$ km ($f = 10^{-5}$ Hz) to $\lambda \simeq 3 \times 10^5$ km ($f = 1$ Hz).

With serious consideration being given to construction of space-based gravitational wave observatories, a precise understanding of the sensitivity of the proposed instruments is needed, requiring the long wavelength approximation be abandoned, and *exact* expressions for the response be derived.

Deriving the exact response of a laser interferometer also begins with the Doppler signal in a single arm. The Doppler signal in an interferometer arm may be exactly integrated with respect to time to find the phase shift read out by the laser tracking system,

$$\Delta\phi(t, \theta_i, \psi_i) = \int \Delta\nu(t, \theta_i, \psi_i) dt. \quad (3.19)$$

It is convenient to define a quantity which is the phase of Eq. (3.19) divided by the laser frequency, ν_o , and light travel time of each arm, τ . Beginning with the Doppler signal of Eq. (3.8), which has been written in terms of the Fourier transform of $h(t)$, this yields

$$\begin{aligned} z_i(t, \theta_i, \psi_i) &= \frac{\phi(t, \theta, \psi)}{\nu_o \tau} = \frac{\sqrt{T}}{4\pi} \int_{-\infty}^{+\infty} d\omega \cos 2\psi \tilde{h}(\omega) \\ &\times \left[(1 - \mu) + 2\mu e^{-i\omega\tau(1+\mu)} - (1 + \mu) e^{-i2\omega\tau} \right] \frac{1}{\omega} e^{i\omega t}, \end{aligned} \quad (3.20)$$

where the $1/\omega$ in the Fourier integral arises from the time integration carried out in Eq. (3.19). One may compute the *exact* gravitational wave transfer function from Eq.

(3.20). At high frequencies, this signal is very complicated due to the contributions of the three h terms which enter at different times. At low frequencies, this quantity reduces to a pure spatial strain, and will thus loosely be referred to as the gravitational wave strain in a single interferometer arm.

The interferometer signal, Δ , is created by subtracting the gravitational wave strain produced in each arm:

$$\Delta(t) = z_1(t) - z_2(t) . \quad (3.21)$$

Consider the geometry shown in Figure 3.4. The vectors $\mathbf{L}_1$ and $\mathbf{L}_2$ point along the arms of the interferometer, and the vector $\mathbf{k}$ points along the propagation vector of the gravitational wave. The angles θ_i measure the separation between the arm vectors and the propagation vector. The value Ω is the opening angle of the interferometer, and ϵ is the inclination of the propagation vector to the plane of the interferometer.

Not shown is the principal polarization vector of the gravitational wave, which lies in a plane perpendicular to $\mathbf{k}$. The polarization angles, ψ_i , are measured from the point where the plane of the i^{th} arm and the propagation vector intersects the plane containing the principal polarization vector. The angle α in Figure 3.4 is simply the difference of the two polarization angles, $\alpha = \psi_2 - \psi_1$.

The average power in the interferometer is given by

$$\langle \Delta^2 \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |\Delta|^2 dt , \quad (3.22)$$

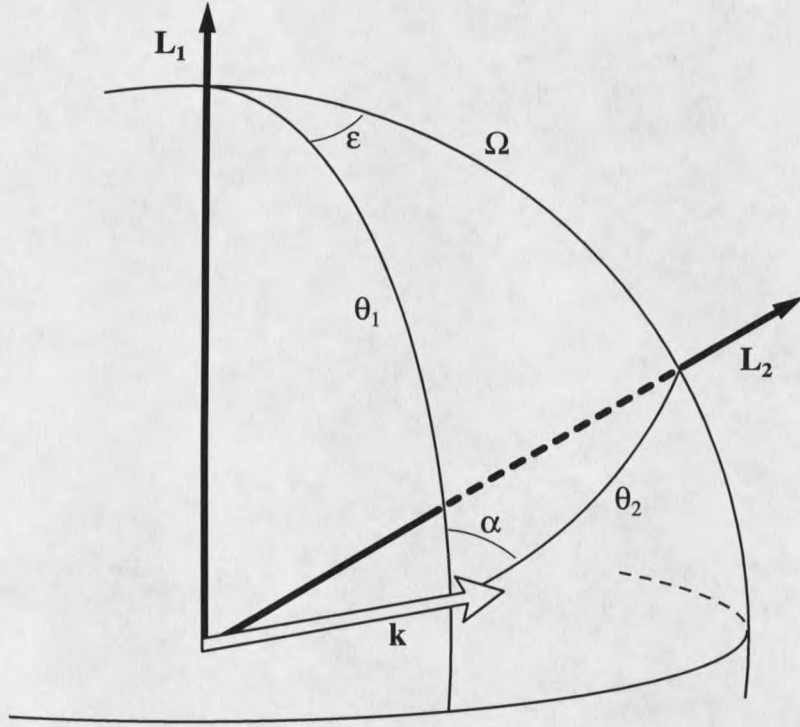


Figure 3.4: The geometrical relationship of the interferometer to the propagation vector of a gravitational wave, used to conduct spatial averaging. The arms are designated by vectors $\mathbf{L}_1$ and $\mathbf{L}_2$ (solid black vectors), while the propagation vector of the gravitational wave is given by $\mathbf{k}$ (open white arrow). One arm of the interferometer is chosen to be aligned along the polar axis of a 2-sphere, the other arm lying an angular distance Ω away along a line of constant longitude. The angles θ_i relate the vector $\mathbf{k}$ to the arms of the interferometer, and the angle ϵ is the inclination of $\mathbf{k}$ to the plane of the interferometer.

where Δ is expressed by Eq. (3.21). Using the definition of z from Eq. (3.20) this can

be expanded to yield

$$\langle \Delta^2 \rangle = \frac{1}{2\pi} \int_0^\infty d\omega \tilde{h}^2(\omega) \frac{1}{(\omega\tau)^2} [T_1(\omega) + T_2(\omega) - 2T_3(\omega)] , \quad (3.23)$$

where

$$\begin{aligned} T_1(\omega) &= \cos^2(2\psi_1) \left[\mu_1^2 (1 + \cos^2(\omega\tau)) + \sin^2(\omega\tau) \right. \\ &\quad \left. - 2\mu_1^2 \cos(\omega\tau) \cos(\omega\tau\mu_1) - 2\mu_1 \sin(\omega\tau) \sin(\omega\tau\mu_1) \right] , \end{aligned} \quad (3.24)$$

$$\begin{aligned}
T_2(\omega) = & \cos^2(2\psi_2) \left[\mu_2^2 (1 + \cos^2(\omega\tau)) + \sin^2(\omega\tau) \right. \\
& \left. - 2\mu_2^2 \cos(\omega\tau) \cos(\omega\tau\mu_2) - 2\mu_2 \sin(\omega\tau) \sin(\omega\tau\mu_2) \right] , \quad (3.25)
\end{aligned}$$

$$T_3(\omega) = \cos(2\psi_1) \cos(2\psi_2) \eta(\omega) , \quad (3.26)$$

with $\mu_i = \cos \theta_i$, and the quantity

$$\begin{aligned}
\eta(\omega, \theta_1, \theta_2) = & [\cos(\omega\tau) - \cos(\omega\tau\mu_1)] [\cos(\omega\tau) - \cos(\omega\tau\mu_2)] \mu_1\mu_2 \\
& + [\sin(\omega\tau) - \mu_1 \sin(\omega\tau\mu_1)] [\sin(\omega\tau) - \mu_2 \sin(\omega\tau\mu_2)] , \quad (3.27)
\end{aligned}$$

has been defined for convenience. The expression for the power in the detector, as given by Eq. (3.23), is a complicated function of frequency and of the orientation between the propagation vector of the gravitational wave and the interferometer. It represents the antenna pattern for a laser interferometer gravitational wave detector.

To characterize the average sensitivity of the instrument it is customary to consider the isotropic power, obtained by averaging the antenna pattern over all propagation vectors and all polarizations:

$$\overline{\langle \Delta^2 \rangle} = \frac{1}{8\pi^2} \int_0^{2\pi} d\psi \int_0^{2\pi} d\epsilon \int_0^\pi \sin \theta d\theta \langle \Delta^2 \rangle . \quad (3.28)$$

Since the variables (θ_1, ψ_1) and (θ_2, ψ_2) that figure in Eq. (3.23) each locate the same propagation vector of the gravitational wave, they are not independent of one another.

Thus, in order to complete the averaging described by Eq. (3.28), one must write Δ^2 in terms of a single set of angular coordinates.

However, examination of Eqs. (3.24-3.26) shows that T_1 depends only on the (θ_1, ψ_1) variables, while T_2 depends only on (θ_2, ψ_2) . This allows the integration of Eq. (3.28) to be performed very easily for the T_1 and T_2 terms without converting to a common set of angular variables. In each case, the averaging of the T_i terms over the (θ_i, ψ_i) gives precisely the same value:

$$\begin{aligned} \overline{T_1} &= \overline{T_2} = \frac{1}{8\pi^2} \int d\psi_1 d\epsilon d\theta_1 \sin \theta_1 T_1 \\ &= \frac{1}{2} \left[\left(1 + \cos^2 \omega\tau\right) \left(\frac{1}{3} - \frac{2}{(\omega\tau)^2}\right) + \sin^2 \omega\tau \right. \\ &\quad \left. + \frac{4}{(\omega\tau)^3} \sin \omega\tau \cos \omega\tau \right] . \end{aligned} \quad (3.29)$$

The average isotropic power can then be expressed as

$$\overline{\langle \Delta^2 \rangle} = \frac{1}{\pi} \int_0^\infty d\omega \tilde{h}^2(\omega) \frac{1}{(\omega\tau)^2} [\overline{T_1} - \overline{T_3}] . \quad (3.30)$$

To complete the integration, T_3 , which depends on both (θ_1, ψ_1) and (θ_2, ψ_2) , must be expressed in terms of a single set of angular variables. One may choose to eliminate (θ_2, ψ_2) in favor of (θ_1, ψ_1) by using conventional spherical trigonometry in Figure 3.4. For the polarization angle, the relationship is particularly simple, $\psi_2 = \psi_1 + \alpha$, and the integration over ψ_1 may be carried out analytically, giving,

$$\overline{T_3} = \frac{1}{8\pi^2} \int d\psi_1 d\epsilon d\theta_1 \sin \theta_1 T_3 = \frac{1}{8\pi} \int d\epsilon d\theta_1 \sin \theta_1 \cos 2\alpha \eta(\omega, \theta_1, \theta_2) . \quad (3.31)$$

The function $\eta(\omega, \theta_1, \theta_2)$ in Eq. (3.27) has terms containing $\mu_2 = \cos \theta_2$, which must be re-expressed in terms of the integration variable θ_1 . The relationship between θ_1 and θ_2 is given by

$$\cos \theta_2 = \cos \Omega \cos \theta_1 + \sin \Omega \sin \theta_1 \cos \epsilon, \quad (3.32)$$

where Ω is the opening angle of the interferometer, and ϵ is the inclination of the gravitational wave propagation vector to the interferometer. Due to the complexity of Eq. (3.27) when Eq. (3.32) is substituted into it, I have not been able to calculate an expression for $\overline{T_3}$ analytically. Therefore, when the gravitational wave transfer function is calculated below, this integral will be evaluated numerically.

Calculation of the gravitational wave transfer function begins by defining the spectral density of the gravitational wave amplitude as $S_h(\omega) = |\tilde{h}(\omega)|^2$ so that

$$\langle h^2 \rangle = \frac{1}{T} \int_0^\infty h(t)^2 dt = \frac{1}{2\pi} \int_0^\infty S_h(\omega) d\omega. \quad (3.33)$$

Similarly, the spectral density of the isotropic power is defined such that

$$\overline{\langle \Delta^2 \rangle} = \frac{1}{2\pi} \int_0^\infty S_{\Delta}(\omega) d\omega. \quad (3.34)$$

This allows Eq. (3.30) to be written as $S_{\Delta} = S_h R(\omega)$, where $R(\omega)$ is the gravitational wave transfer function and is given by

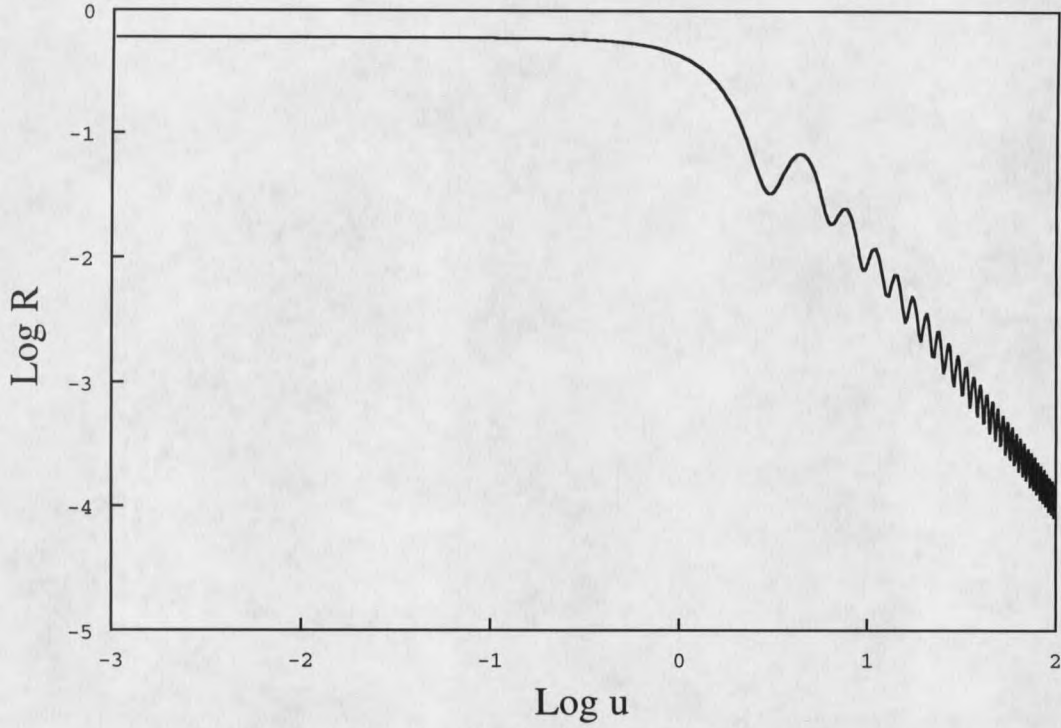


Figure 3.5: The transfer function $R(u)$ is shown as a function of the dimensionless variable $u = \omega\tau$. Note that it is roughly constant at low frequencies, and has a “knee” located at $u = \omega\tau \simeq 1$.

$$\begin{aligned}
 R(\omega) &= 2 \frac{1}{(\omega\tau)^2} [\overline{T_1} - \overline{T_3}] \\
 &= \frac{1}{(\omega\tau)^2} \left[(1 + \cos^2 \omega\tau) \left(\frac{1}{3} - \frac{2}{(\omega\tau)^2} \right) \right. \\
 &\quad + \sin^2 \omega\tau + \frac{4}{(\omega\tau)^3} \sin \omega\tau \cos \omega\tau \\
 &\quad \left. - \frac{1}{4\pi} \int d\epsilon \, d\theta_1 \, \sin \theta_1 \, (1 - 2 \sin^2 \alpha) \, \eta(\omega) \right] . \quad (3.35)
 \end{aligned}$$

The exact transfer function, with the last integral evaluated numerically, is shown in Figure 3.5 as a function of the dimensionless quantity $u = \omega\tau$.

Taking the limit of small $\omega\tau$ in Eq. (3.35) will yield a low frequency limit for $S_{\bar{\Delta}}$

of

$$S_{\bar{\Delta}} = \frac{4}{5} (1 - \cos^2 \Omega) S_h, \quad (3.36)$$

which is in agreement with previous results in the literature [47]⁶. The low frequency limit is reviewed in detail in Section 3.6.

3.5 Sensitivity Curves

The sensitivity curve for a gravitational wave observatory is obtained from $S_{\bar{\Delta}} = S_h R(\omega)$ in the case where the signal to noise ratio in the instrument is unity (that is, the spectral density of the isotropic power, $S_{\bar{\Delta}}$, is given purely by noise). As previously discussed in Sections 3.2, the sensitivity of the instrument will be limited by laser shot noise and by acceleration noise. These sources of noise are uncorrelated and will not cancel out through subtractions in the data reduction process. As was shown in Eq. (3.3), the data reduction process produces 4 terms which correspond to the independent noise (sum of laser shot noise and acceleration noise) in the interferometer. Each of these terms, however, represents the noise in a round trip signal between two probes on a single arm of the interferometer, and therefore contains noise in both the transponding far spacecraft, and the local spacecraft; each $n_i(t)$ is, in fact, *two* independent noise sources. Due to this fact, the process of data reduction yields a

⁶In the notation of this paper, the interferometer signal has been multiplied by the laser period and divided by the arm length of the interferometer, as described in Eq. (3.20). To agree with the expressions for the spectral density of the interferometer signal S_{δ} in the literature, this factor must be accounted for above, so that $S_{\delta} = (\tau\nu)^2 S_{\bar{\Delta}}$. See Eq. (3.46) in Section 3.6.

numerical factor of 8 when combining the spectral noise densities, giving

$$S_h = \frac{S_{\bar{\Delta}}}{R} = \frac{8S_{noise}}{R} \quad (3.37)$$

where the quantity S_{noise} is the sum of the strain noise spectral density due to acceleration described by Eq. (3.5), and the spectral density of the shot noise in the detector described by Eq. (3.4).

The spectral amplitude of the strain is simply the square root of this, giving

$$h_f = \sqrt{S_h} = 2\sqrt{2}\sqrt{\frac{S_{noise}}{R}}. \quad (3.38)$$

This is the effective strain in the interferometer for a signal to noise ratio of 1 (*i.e.*, the only strain seen in the interferometer is a result of the instrumental noise).

The proposed OMEGA and LISA observatories are currently slated to use identical laser systems, and should have similar acceleration noise characteristics. The only significant difference between the two (for the purpose of calculating the instrument sensitivity) will be the arm length of the interferometers, $\tau = 1 \times 10^9$ m for OMEGA, and $\tau = 5 \times 10^9$ m for LISA.

Both systems have proposed to use $D = 30$ cm optical telescopes in the laser tracking systems, with Nd:YAG lasers ($\lambda = 1064$ nm, or $\nu_o = 2.8 \times 10^{14}$ Hz) transmitting $P_t = 1$ W of power. For this calculation, it is assumed the telescopes are perfectly efficient, $\epsilon = 1$. Using these parameters, the value of the spectral density of

the shot noise (with conventional units restored) is

$$S_z = 1.23 \times 10^{-42} \frac{1}{\text{Hz}} . \quad (3.39)$$

The expected value of the spectral density of the acceleration noise (with conventional units restored) is [43]

$$\sqrt{S_a} = 1 \times 10^{-15} \frac{\text{m}}{\text{s}^2 \text{Hz}^{1/2}} , \quad (3.40)$$

which when used in Eq. (3.5) for the proposed OMEGA and LISA arm lengths yields

$$S_z = 6.42 \times 10^{-52} \frac{\text{Hz}^3}{f_{gw}^4} \quad \text{OMEGA} , \quad (3.41)$$

and

$$S_z = 2.57 \times 10^{-53} \frac{\text{Hz}^3}{f_{gw}^4} \quad \text{LISA} . \quad (3.42)$$

Using these parameters, the sensitivity curve for each observatory is computed using Eq. (3.38) and is shown in Figure 3.6.

The width of the mid-frequency floor is a function of the arm length of the interferometer, which governs where the “knee” at high frequencies occurs, at $\omega\tau \simeq 1$. This is the point at which the transfer function shown in Figure 3.5 turns over, and directly enters into the computation of the shape of the sensitivity curve through Eq. (3.38).

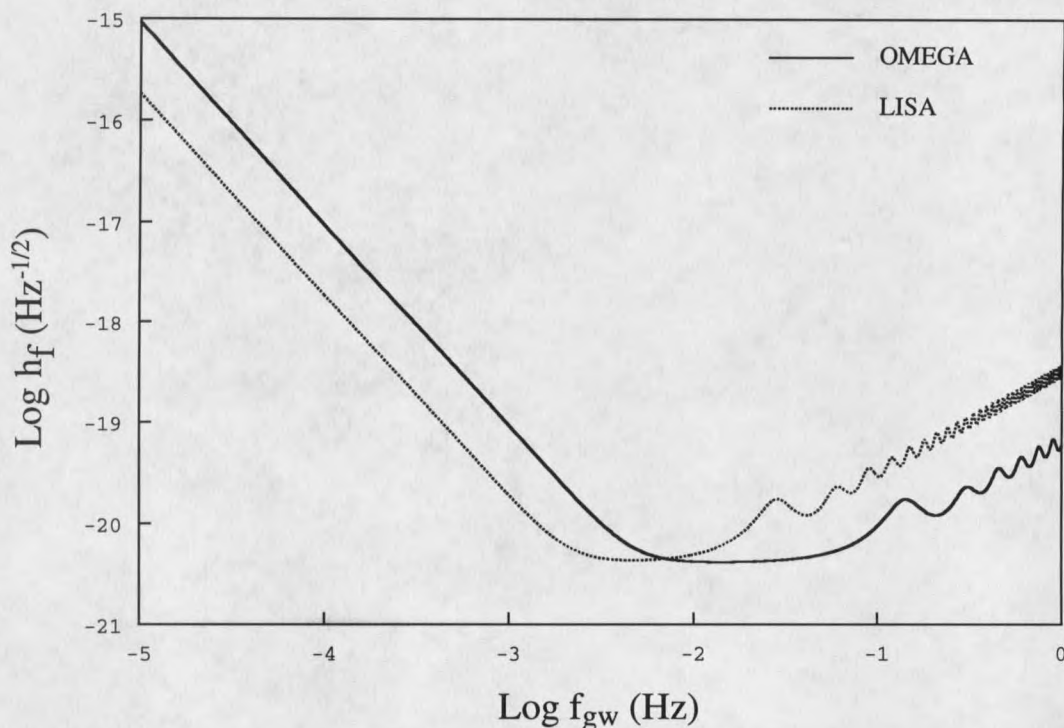


Figure 3.6: The sensitivity curves for the proposed OMEGA (solid line) and LISA (dashed line) observatories are shown. The low frequency rise is due to acceleration noise in each of the systems. The structure at low frequencies is a consequence of the high frequency structure in the gravitational wave transfer function, shown in Figure 3.5.

3.6 Long Wavelength Approximation

Interferometric systems detect gravitational waves through shifts in the relative phase of the electromagnetic signal in each of the interferometer arms. These types of systems have become the leading design for gravitational wave detectors, but analyses to characterize the limits of their performance have, to date, depended on using the ‘long wavelength approximation,’ where the wavelength of the gravitational wave is assumed to be much longer than the baseline of the interferometer arms (see, for example, [47]).

In the next section, the *exact* interferometer gravitational wave transfer function is computed. For comparison, an interferometer is described here in terms of the long wavelength approximation.

The phase of the signal in a given interferometer arm may be obtained from the Doppler formalism described in Section 3.3. If the gravitational wave period is *long* compared to the light-travel time of the electromagnetic signal (*i.e.*, the gravitational wavelength is long compared to the baseline of the interferometer), τ , then the right hand side of Eq. (3.6) may be approximated by expanding h in a Taylor series about t , yielding

$$y \simeq \tau \cos 2\psi \sin^2 \theta \dot{h}(t) . \quad (3.43)$$

To obtain the change in phase associated with the Doppler shift, one integrates this with respect to time and multiplies by the reference frequency of the laser, ν_o , giving

$$\Delta\phi = \nu_o \tau \cos 2\psi \sin^2 \theta h(t) . \quad (3.44)$$

The output of an interferometric system is obtained by letting the signal in each of the arms interfere, producing the difference in phase between the two arms:

$$\delta = \Delta\phi_1 - \Delta\phi_2 = \nu_o \tau h(t) \left[\cos 2\psi_1 \sin^2 \theta_1 - \cos 2\psi_2 \sin^2 \theta_2 \right] . \quad (3.45)$$

The indices labeling the angles correspond to the angular measures between each arm of the interferometer and the propagation vector of the gravitational wave (see Figure 3.4 in Section 3.4 for reference).

In precise analogy to the method shown for computing the power spectrum of the Doppler signal shown in Eqs. (3.7) – (3.16), the spectral density for the interferometer is found to be [47]⁷

$$S_{\bar{f}} = \frac{4}{5} \nu_o^2 \tau^2 (1 - \cos^2 \Omega) S_h, \quad (3.46)$$

where Ω is the opening angle of the interferometer.

Note that if one assumes a form $S_{\bar{f}} = R S_h$, then the transfer function in this approximation may be written

$$R = \frac{4}{5} \nu_o^2 \tau^2 (1 - \cos^2 \Omega), \quad (3.47)$$

which is completely independent of the gravitational wave frequency. As was shown in the last section, only at low frequencies (long wavelengths) does the transfer function exhibit constant behavior; at higher frequencies it has important and rapidly varying structure.

⁷The procedure to complete the spatial averaging, leading to the isotropic power, is precisely the same as the one that was outlined in Section 3.4, with the exception that the integration can be done analytically in the long wavelength approximation; cf. Eq. (3.35) and Figure 3.5.

CHAPTER 4

USING BINARY STARS TO BOUND THE MASS OF THE GRAVITON

4.1 Introduction

From very general considerations [18], it can be shown that general relativity is the natural, self-consistent formulation of the theory of a massless, self-interacting, spin-2 field (which is called the graviton). As was shown in Chapter 2, within the context of the linearized approximation to general relativity, gravitational radiation is expected to propagate at the speed of light; the graviton is massless. This is an essential feature of general relativity which is not shared in all metric theories of gravity [11].

With a theoretical prediction that the graviton is a massless particle, the stage is set for experimental testing of the validity of the theory. General relativity has stood up to a number of detailed tests since its introduction in 1915, including the three classic solar system tests: deflection of light, perihelion precession, and the time-delay of starlight.¹ In the solar system, each of these tests may be analyzed within the post-

¹The nomenclature employed here is the same as [11], where the delay of starlight is used as a third classical test, rather than the gravitational redshift originally proposed by Einstein.

Newtonian expansion of other metric theories of gravity (besides general relativity), but cannot yield conclusive evidence in favor of one theory of gravity over another. The reason is that by changing adjustable parameters in each of these theories, they can be made to agree with the results of general relativity (which successfully explains all solar system tests) to a high degree of accuracy (from 1% to parts per 10^{-7} [11]).

Studies of gravitational radiation provide new methods of distinguishing between theories by determining the number of polarization states (discussed in Section 2.4), and by measuring the speed with which gravitational waves propagate. In this chapter, a new method of measuring the graviton mass is described using space-based gravitational wave observations in correlation with optical observations of known binary star systems.

Current experimental bounds on the mass of the graviton are reviewed in Section 4.2, and the proposed sources to be monitored (interacting binary white dwarfs) are discussed in Section 4.3. In Sections 4.4 and 4.5 a new experiment is proposed to measure the mass of the graviton by correlating gravitational wave and optical observations of interacting binary white dwarfs, and an expression for the graviton mass is derived. In Section 4.6, the experimental uncertainty predicted for space-based interferometers is used to estimate how precise a bound could be placed on the graviton mass from observations of the prototype interacting binary white dwarf (AM CVn).

4.2 Current bounds on the graviton mass

Currently the best bounds² on the graviton mass come from looking for violations of Newtonian gravity in surveys of planetary motions in the solar system. If gravity were described by a massive field, such a survey might detect this as a deviation from the Newtonian description of gravity. Deviations from the Newtonian potential will have Yukawa modifications of the form

$$V(r) = -\frac{M}{r} \exp(-r/\lambda_g) \quad (4.1)$$

where M is the mass of the source of the potential, $\lambda_g = h/m_g$ (the Compton wavelength of the graviton) is the cutoff radius of the Yukawa term, and h is Planck's constant. The best bound on the graviton mass from planetary motion surveys is obtained by using Kepler's third law to describe the observed orbit of Mars, yielding $\lambda_g > 2.8 \times 10^{12}$ km ($m_g < 4.4 \times 10^{-22}$ eV) [48].

Another bound on the graviton mass can be established by considering the motions of galaxies in bound clusters [49], yielding $\lambda_g > 6 \times 10^{19}$ km ($m_g < 2 \times 10^{-29}$ eV). This bound, while stronger than solar system estimates, is not as reliable due to uncertainty about the matter content of the Universe on large scales (*e.g.*, the amount and nature of dark matter is widely debated, and uncertain at best).

Recent work [50] has suggested that the mass of the graviton could be bounded using gravitational wave observations. If the graviton is a massive particle, then the

²By "best" here I have indicated my assessment of the reliability of the methods used to estimate the bounds.

speed of propagation of a gravitational wave will depend on its frequency. This can easily be seen by considering wavelike solutions to the Klein-Gordon equation

$$(\square - m^2)h = 0 , \quad (4.2)$$

which describes the dynamics of massive fields. Taking solutions of the form $h = h_o \exp[ik_\mu x^\mu]$ and evaluating Eq. (4.2) yields the dispersion relation³

$$\omega^2 = k^2 + m^2 . \quad (4.3)$$

The group velocity is

$$\begin{aligned} v_g &= \frac{\partial \omega}{\partial k} \\ &= \sqrt{1 - \frac{m^2}{\omega^2}} , \end{aligned} \quad (4.4)$$

showing that lower frequency waves will propagate slower than high frequency waves when the graviton mass m is non-zero. This dispersion of gravitational waves could in principle be detected in the signal from compact binary systems using the matched filtering techniques⁴ required for data analysis in the current generation of laser interferometer detectors. As binary systems evolve, they will slowly spiral together due to the emission of gravitational radiation. Over the course of time, the frequency of the binary orbit rises, ramping up rapidly in the late stages of the evolution, just

³Eqs. (4.3) and (4.4) are written in units where $\hbar = c = G = 1$.

⁴A detailed explanation of matched filtering may be found in [39].

prior to coalescence. Interferometric observatories should be sensitive to this binary evolution. Space-based detectors will be sensitive to coalescence of massive ($\sim 10^7$ to $\sim 10^5 M_\odot$) binary black holes, as well as the gravitational wave emission from compact binary star systems which are far from coalescence (*e.g.*, white dwarf binaries). Ground-based detectors will be sensitive to the merger of smaller black hole binaries ($\sim 10 M_\odot$), as well as the coalescence of compact binary stars (*e.g.*, neutron star/neutron star binaries). If the graviton is a massive particle, then the observed signal will not perfectly match with theoretical templates computed using general relativity theory, in which the graviton is massless, due to dispersion of the gravitational waves. By using matched filtering of inspiral waveforms, this dispersion could be bounded, thereby bounding the mass of the graviton. LIGO could bound the graviton mass at $\lambda_g > 6.0 \times 10^{12}$ km ($m_g < 2.1 \times 10^{-22}$ eV) from observation of the inspiral of two $10 M_\odot$ black holes. A space-based interferometer observing the inspiral of two $10^7 M_\odot$ black holes could bound the graviton mass at $\lambda_g > 6.9 \times 10^{16}$ km ($m_g < 1.8 \times 10^{-26}$ eV).

4.3 Interacting Binary White Dwarfs

Estimates suggest the galaxy is populated by $\sim 10^7$ close binary star systems [51]⁵. The sheer numbers of these systems could have profound consequences for space-borne gravitational wave observatories. Close binary systems with periods $P_{orb} \sim 10^5$

⁵This reference estimates the gravitational radiation from several different binary components in the galaxy.

Table 4.1: Properties of the six nearest interacting binary white dwarfs.

Name	Secondary Mass ($M_{\odot}$)	Orbital Period (s)	Distance (pc)	Strain h ($\times 10^{-22}$)
AM CVn	0.044	1028.73	101	5.27
EC15330-1403	~ 0.04	1119	165	5.36
CR Boo	~ 0.03	1491	157	2.82
V803 Cen	~ 0.03	1611	~ 100	4.20
CP Eri	~ 0.03	1724	~ 100	4.02
GP Com	~ 0.02	2791.2	165	1.77

s to $P_{orb} \sim 1$ s are expected to be easily observable by space-based observatories. Estimates of the combined signal from these binaries suggest they will produce a stochastic background [52] which rises well above the low frequency detection sensitivity of proposed space-based interferometers, as shown in Figure 4.1. Particularly strong sources could rise up above this background and be observable by a spaceborne observatory. One class of such sources are helium cataclysmic variable (HeCV) stars, often called AM CVn binaries after the prototype star, *AM Canum Venaticorum*. The properties of the six nearest HeCVs are shown in Table 4.1 [46], and the predicted signals are plotted in Figure 4.1 as solid black diamonds. Estimates have suggested that if the models of the space-density of close binaries in the galaxy are correct, roughly 5000 of these sources should be individually detectable by a space-based detector like OMEGA or LISA [8].

As will be discussed in Section 4.4 below, being able to simultaneously monitor the gravitational wave signal and the electromagnetic light curve from these star systems will provide a new method of bounding (or measuring) the mass of the graviton.

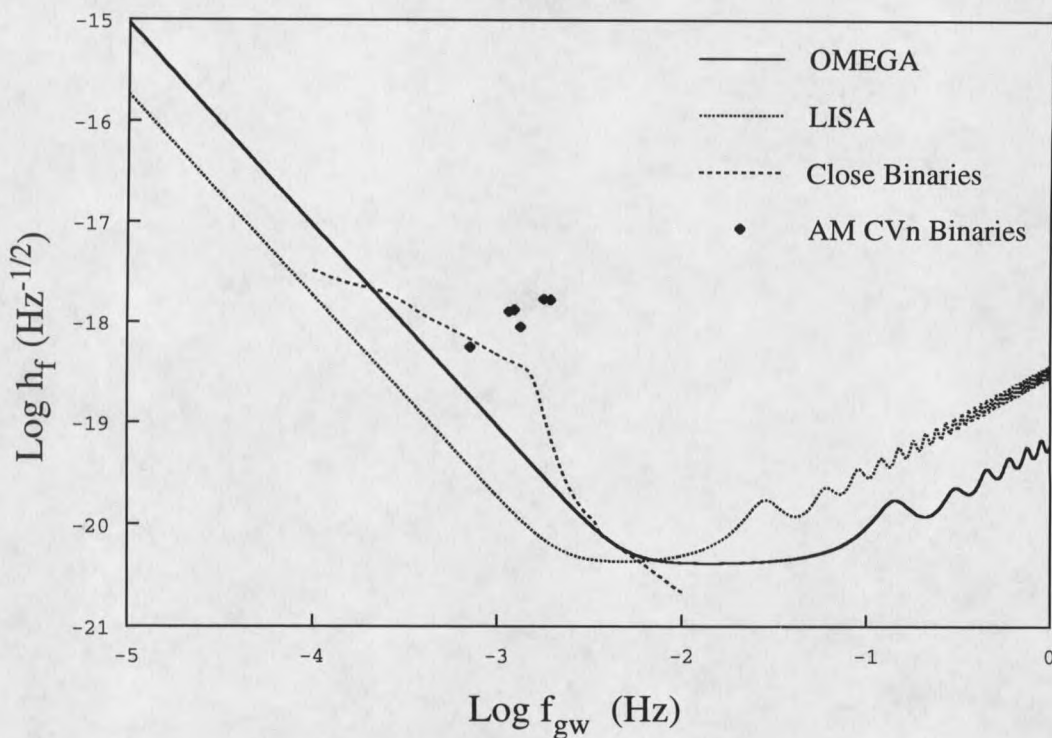


Figure 4.1: The background of close binaries in the galaxy, plotted with the projected sensitivity of OMEGA and LISA. The six most well understood AM CVn type binaries are indicated. The assumed bandwidth is 10^{-7} Hz.

In order to carry out such investigations, it is necessary to understand the basic orbital motions of the stellar components and mechanisms of electromagnetic signal production in these binaries.

Currently the best models for HeCVs describe a star system where the secondary star (the lower mass companion in the binary system, usually a degenerate helium dwarf star) has expanded to fill its Roche lobe, and the primary star (the larger mass, compact white dwarf) lies at the core of an accretion disk. Matter overflows from the secondary Roche lobe and streams onto the accretion disk, creating a hot spot which outputs a strong electromagnetic signal.

There are several mechanisms whereby the electromagnetic light curve of an HeCV

could be modulated from the perspective of observers on Earth. The simplest modulation model is for stars that lie at high inclination angles and periodically eclipse, partially or wholly. The eclipse phase could dim the stellar components, part of the primary emission regions of the accretion disk, or the Roche lobe. Another possible mechanism for variation in the light curve can be associated with the secondary star, which has expanded to fill its Roche lobe. The star will appear brighter when the largest surface area is presented to the observer along the projected line of sight. This will occur twice in each orbit, when the line of sight is perpendicular to the line of centers. Signatures in the light curve due to projected area effects are called “ellipsoidal variations,” and are most easily observed in infrared wavelengths [53]. The proposed model for the source of variation in the AM CVn light curve [55] is a hot spot on the accretion disk which radiates approximately radially outward from the disk. As the binary orbits, this hot spot alternately turns towards and away from distant observers, leading to a modulation of the light curve⁶.

The prototype HeCV, AM CVn, is easily visible to ground based optical telescopes⁷. AM CVn’s position is shown in Figure 4.2. This binary is expected to be very luminous in gravitational waves, and should be visible to proposed space-based interferometers several minutes after they come online. With the ability to monitor this star in both the gravitational and electromagnetic wavebands, it is an optimal source with which to test the propagation speed of the graviton.

⁶This is a so-called “flashlight” mechanism.

⁷At magnitude 14, it should be visible to telescopes roughly 8” in diameter and larger.

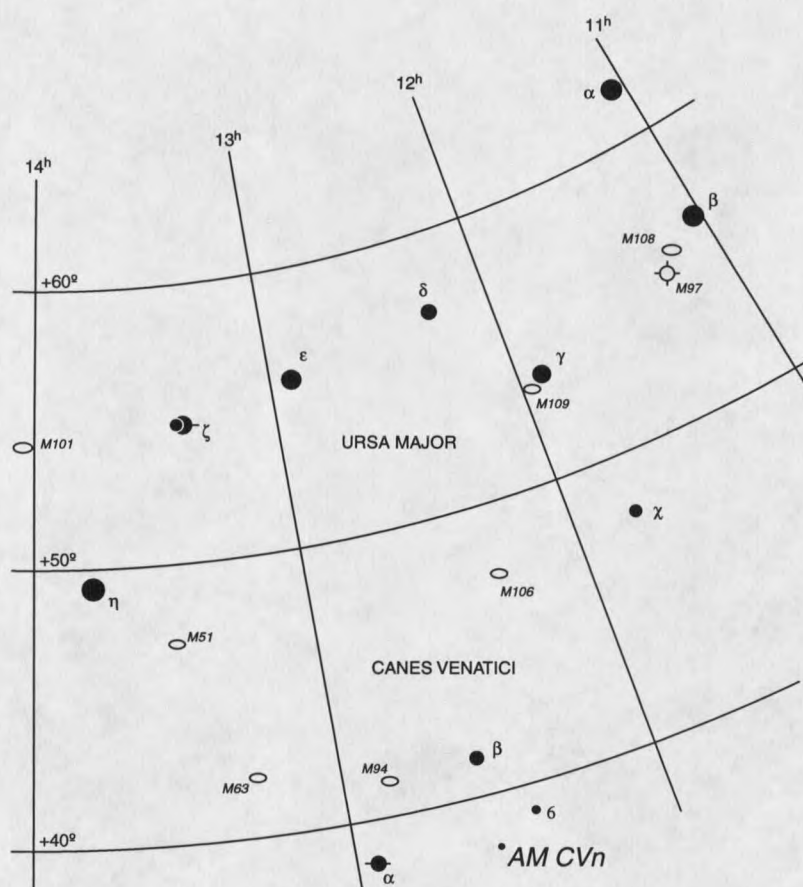


Figure 4.2: Finder chart to locate the cataclysmic variable, AM *Canum Venaticorum* (AM CVn), located south of Ursa Major in Canes Venatici, at coordinates RA $12^h 34^m 54.58^s$, $\delta + 37^\circ 37' 43.4''$.

A detailed theoretical model of AM CVn has been worked out, describing a variety of signals which are present in the photometric data [54, 56, 57]. This model suggests that AM CVn is a member of a class of variable stars which have periodic features in the light curve known as “superhumps” [58, 59]. The superhump mechanism proposes the existence of an eccentric precessing accretion disk, with a precession period which is slightly longer than the orbital period of the binary. Knowing the superhump period, P_{sh} , and the precessional period of the accretion disk (apsidal advance), P_{aa} ,

one may predict the *orbital* period as

$$P_{orb}^{-1} = P_{sh}^{-1} + P_{aa}^{-1} . \quad (4.5)$$

Photometry of AM CVn shows the existence of a superhump signature at $P_{sh} = 1051.2$ s, and the period of the accretion disk precession at $P_{aa} = 13.38$ hr. Using Eq. (4.5) this model has been used to predict a binary orbital period of $P_{orb} = 1028.77 \pm 0.18$ s for AM CVn [57]. Photometric observations by the CBA⁸ have recently confirmed an orbital period of $P_{orb} = 1028.7325 \pm 0.0004$ s [55].

With the ability to conduct ground-based photometric monitoring of the orbital signal, the phase comparison developed in Sections 4.4 and 4.5 can be carried out as soon as a spaceborne gravitational wave observatory comes online.

4.4 Correlation of Electromagnetic and Gravitational Observations to Measure the Graviton Mass

As noted in Section 4.3, it is expected that the galaxy harbors a large population of interacting binary white dwarf stars. These systems are expected to be easily detectable by space-based gravitational wave detectors, and the optical light curves for several of these stars are known, obtained from ground-based optical photometry. Because these systems are expected to be observable in the gravitational spectrum (in

⁸The Center for Backyard Astrophysics (CBA) is a network of amateur astronomers (equipped with CCD photometry equipment) who monitor variable stars. The network is managed by professional astronomers at Columbia University.

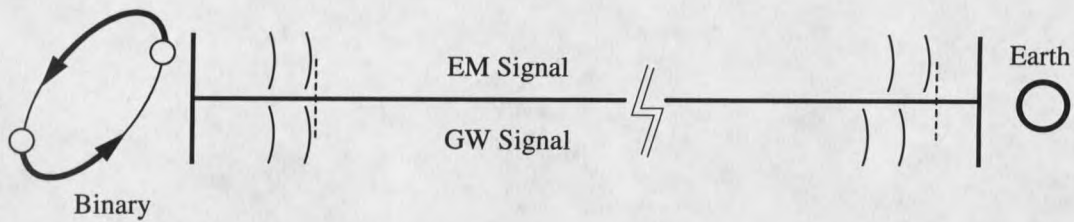


Figure 4.3: Schematic of a binary star system which has observable electromagnetic and gravitational wave signals. If the graviton is a massive particle, then the phase fronts of the gravitational signal will lag behind those of the electromagnetic signal.

addition to the optical spectrum), astronomers will have an excellent opportunity to directly compare the propagation speed of electromagnetic and gravitational waves.

Consider the schematic diagram shown in Figure 4.3. The electromagnetic signal which propagates to observers on the Earth is simply the light curve, tracing out the orbital motion of the stars in the binary system. The phase fronts of the light curve are represented in the top half of the diagram. The binary star system will also emit gravitational radiation which could be monitored by Earth bound observers as well. The phase fronts of the gravitational wave signal are represented in the lower half of the diagram⁹.

Suppose the two signals are emitted in phase at the source as shown. If the graviton is a massive particle, then the gravitational waves propagate at a speed $v_g < 1$, and the phase fronts will *lag* behind the electromagnetic phase fronts when the signals arrive at Earth, as shown. By measuring the lag between the phase fronts, the mass of the graviton can be measured or bounded.

⁹As shown in Section 2.6 of Chapter 2, the frequency of the gravitational radiation is actually twice the orbital frequency of the binary. For clarity, only half the gravitational phase fronts have been drawn in the diagram.

To determine the lag between the two signals, the phase of each signal must be measured. Consider a binary with orbital frequency ω_o at a distance D from Earth. Assuming the photon to be massless ($v = c = 1$), the phase of the signal being monitored by astronomers on Earth will be

$$\phi_{em} = \omega_o D + A , \quad (4.6)$$

where the term A encodes a variety of effects (discussed in Section 4.5) which could create phase delays between the electromagnetic and gravitational signals that are being monitored. By contrast, a massive graviton ($v_g < 1$) will arrive at Earth with a phase of

$$\phi_{gw} = 2\omega_o \frac{D}{v_g} . \quad (4.7)$$

The phase lag between the optical and gravitational signals is constructed from these two phases:

$$\begin{aligned} \Delta &= \frac{1}{2}\phi_{gw} - \phi_{em} \\ &= \omega_o D \left(\frac{1}{v_g} - 1 + \alpha \right) , \end{aligned} \quad (4.8)$$

where $\alpha = A/(\omega_o D)$. The factor of $1/2$ insures that the phase subtraction is done between two signals with the *same* frequency. It is convenient to define the fractional change in the phase as

$$\epsilon = \frac{\Delta}{\omega_o D} = \left(\frac{1}{v_g} - 1 + \alpha \right) . \quad (4.9)$$

The speed of the graviton, v_g , can be eliminated in favor of the graviton mass, m_g , by using the relativistic energy, $E^2 = p^2 + m^2$. Utilizing the identities for relativistic momentum and energy, $p = \gamma m v_g$ and $E = \gamma m$, this yields

$$v_g = 1 - \left(\frac{m_g}{E} \right)^2. \quad (4.10)$$

Writing the energy in Eq. (4.10) as $E = \hbar\omega$, and identifying the Compton wavelength of the graviton as $\lambda_g = h/m_g$ this becomes

$$v_g = 1 - \left(\frac{2\pi}{\lambda_g \omega} \right)^2. \quad (4.11)$$

Taking this definition of v_g and substituting into Eq. (4.9), the Compton wavelength of the graviton as a function of the fractional phase lag is found to be

$$\lambda_g = \frac{\pi}{\omega_o} \sqrt{\frac{1}{2} \left(1 + \frac{1}{\epsilon - \alpha} \right)}. \quad (4.12)$$

An obvious question to ask about this analysis is how to determine whether or not the phase difference between the two signals is undetectably large (*e.g.*, $\Delta = 2\pi + \xi$, where ξ is a small quantity). One can eliminate these doubts because strong bounds on the graviton mass already exist. The largest observable phase shift which is consistent with current bounds can be computed by simply evaluating Eq. (4.12) with λ_g equal to the bound of interest. For the bound on the graviton mass given by solar system constraints, this procedure yields a maximum fractional phase change of $\epsilon = 1.5 \times 10^{-9}$. This is for the source AM CVn (discussed in Section 4.3), which lies

at a distance of $D = 101$ pc; this value of ϵ indicates a maximum phase difference of $\Delta = 9.6 \times 10^{-2}$ for AM CVn. I have let $\alpha \rightarrow 0$ here for convenience; if the measured phase difference were larger than this value, it would indicate that $\alpha \neq 0$.

4.5 Phase Delays

In order to evaluate Eq. (4.12), one must not only measure the phase lag between the two signals, but an estimate must be made for the value of α .

The parameter α can be written as the sum of two primary sources of delay between the two phases:

$$\alpha = \alpha_{atm} + \alpha_*, \quad (4.13)$$

where α_{atm} is a phase lag due to the index of refraction of the Earth's atmosphere, and α_* is a phase lag which depends on the specific astrophysical model of the binary system.

Phase lags such as α_{atm} will also enter the electromagnetic signal during propagation through the interstellar medium, which in principle has non-unit index of refraction. In practice, however, over the column depth between the observer and the source, the effect of the Earth's atmosphere will be the dominant source of lag in the electromagnetic phase fronts.

Simple estimates of the value of α_{atm} can be made by computing the electromagnetic phase delay due to propagation through a modeled exponential atmosphere,

with a density profile

$$\rho(r) = \rho_o \exp(-r/h_s) , \quad (4.14)$$

where h_s is the scale height of the atmosphere. If the index of refraction, n , varies linearly with the density, then

$$n(r) = 1 + \eta \exp(-r/h_s) , \quad (4.15)$$

where $\eta = (n_{atm} - n_{vac}) = (n_{atm} - 1)$ is the difference in index of refraction between the atmosphere and vacuum.

The phase lag in the electromagnetic signal is due to the change in propagation speed when the photons enter a medium with non-unit index of refraction. The index of refraction is defined in terms of the propagation speed v as

$$v = \frac{dr}{dt} = \frac{1}{n(r)} . \quad (4.16)$$

Eq. (4.16) can be integrated using Eq. (4.15) to obtain

$$\int_0^{t_{transit}} dt = \int_0^{r_o} dr [1 + \eta \exp(-r/h_s)] , \quad (4.17)$$

where $t_{transit}$ is the time it takes a photon to transit the atmosphere, and r_o is the height at which the effects of the atmosphere become negligible. Completing the integration yields

$$t_{transit} = r_o + h_s \eta (1 - e^{-r_o/h_s}) . \quad (4.18)$$

In order to compute a phase delay, one is interested in the time by which the photons are *delayed* by the atmosphere, which is

$$\begin{aligned} t_{delay} &= t_{transit} - t_{vacuum} \\ &= t_{transit} - r_o, \end{aligned} \quad (4.19)$$

where t_{vacuum} is the time it would take a photon to travel the same distance (*i.e.*, the atmospheric depth) in vacuum.

The phase delay introduced by this effect is obtained by multiplying the delay time by the frequency of the signal being observed, yielding

$$\phi_{delay} = \omega_o t_{delay}. \quad (4.20)$$

The parameter α_{atm} is obtained from the phase delay by dividing by the reference phase, $\omega_o D$ (as in Eq. (4.8)), giving

$$\alpha_{atm} = \frac{\phi_{delay}}{\omega_o D} = \frac{t_{delay}}{D}. \quad (4.21)$$

The parameter α_* is a measure of the initial phase lag between the gravitational wave and electromagnetic signals. It is an astrophysical quantity which indicates the relative phase difference between the peaks in the light curve of the binary, and the peaks in the quadrupolar gravitational radiation pattern.

To determine the value of α_* , it is necessary to understand the alignment of the

quadrupolar radiation pattern with regards to the positions of the stellar masses in the binary system. Consider the equal mass binary star system detailed in Section 2.6. Eqs. (2.70)-(2.72) express the gravitational wave signal seen by an observer far from the source, with the radiation propagating up the z -axis of a local TT coordinate patch. These results may be used to express what an observer in the plane of the binary orbit (inclination $\iota = 90^\circ$) would observe. Consider a distant observer lying along the binary system's x -axis. Since the components of the quadrupole tensor in Eq. (2.78) have been expressed in the coordinate system of the binary (with the stars lying in the $x - y$ plane), it is convenient to rewrite Eqs. (2.70)-(2.72) in the coordinates of the star system. This can easily be accomplished in these expressions by renaming the coordinates: $z \rightarrow x$, $x \rightarrow y$, and $y \rightarrow z$. For a source a distance r away, this procedure yields

$$\bar{h}_{xi} = 0 , \quad (4.22)$$

$$\bar{h}_{yy} = -\bar{h}_{zz} = -\frac{4\omega^2}{r}(\Xi_{yy} - \Xi_{zz})e^{i2\omega r} = \frac{m\ell_o^2\omega^2}{r}e^{i2\omega r}(\cos 2\omega t - 3) , \quad (4.23)$$

and

$$\bar{h}_{yz} = -\frac{2\omega^2}{r}\Xi_{yz}e^{i2\omega r} = 0 . \quad (4.24)$$

Comparing the gravitational signal of Eq. (4.23) with the equation of the binary orbit, Eq. (2.74), one sees that the signal peaks along the axis connecting the masses of the binary system.

This demonstrates that the quadrupolar radiation pattern will peak along the line of masses in the binary system, and also 90° away from the line of masses (since the

frequency of the gravitational radiation is $\omega_{gw} = 2\omega_{bin}$). If the primary variation in the binary light curve is associated with the transit of the line of masses across the observer's line of sight (*e.g.*, the system is an eclipsing binary), then it is straightforward to assign $\alpha_* = 0$, assigning no initial delay between the emission of the gravitational wave and electromagnetic signal.

For AM CVn, the orbital period imprints on the light curve through the bright spot of the matter stream from the secondary Roche lobe overflow striking the accretion disk. Studies of the Roche lobe overflow [60] show that the transferred material remains in a coherent matter stream which spirals in towards the primary star. The point at which the matter stream strikes the accretion disk will determine the value of α_* , which in these systems measures the amount by which the hot spot leads the line of masses, as shown in Figure 4.4. Estimates of the size of the primary accretion disk in HeCV type systems suggest that disk radii will be around 75% of the primary Roche radius [61], but this estimate is only certain to within about 10%. This uncertainty makes it virtually impossible to estimate α_* for use in Eq. (4.12). Future missions, such as the Space Interferometry Mission [62] and the Terrestrial Planet Finder [63] may make direct optical imaging of close binary systems (like AM CVn) possible, which would allow for a direct measurement of α_* .

For cases when α_* cannot be accurately determined, the dependence of the Compton wavelength on this parameter (or on α_{atm}) can be eliminated by subtraction of two observations of the source. Consider the situation shown in Figure 4.5, where the gravitational and electromagnetic signals are monitored when the Earth lies on one

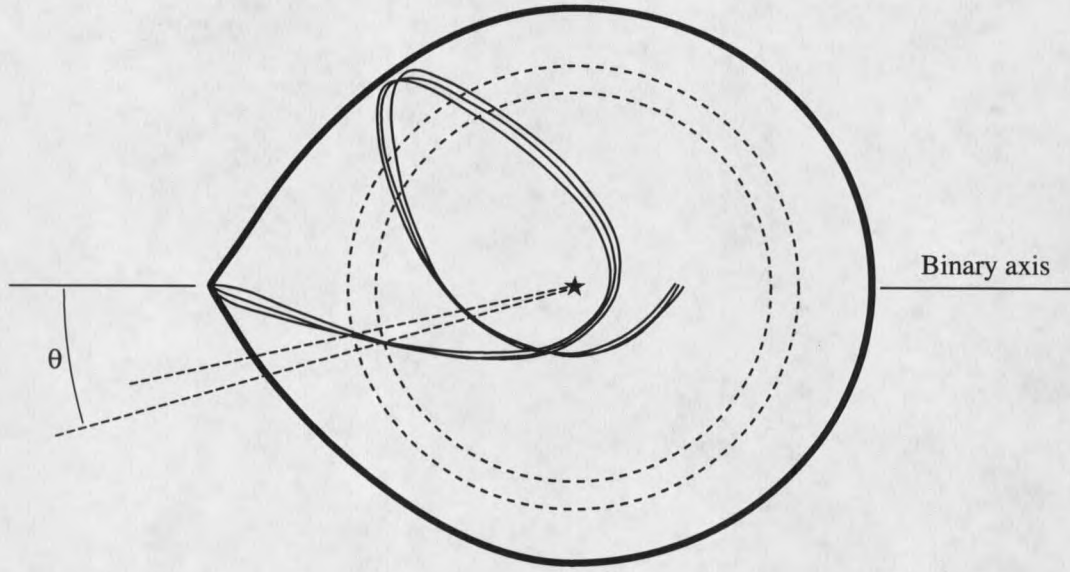


Figure 4.4: Matter overflows from the secondary star in a coherent matter stream, shown as trajectories spiraling in towards the white dwarf (indicated by $\star$ at the center of the Roche lobe; figure adapted from [60]). Depending on the radius of the accretion disk, the matter stream will strike the disk at some angle θ which leads the line of masses; this angle is directly proportional to the phase delay parameter α_* . The dashed circles represent accretion disk radii which are 70% and 80% of the Roche radius, yielding $\sim 5^\circ$ difference in the value of θ .

side of its orbit, and again six months later, when it lies on the other side of its orbit.

When the Earth is in position 1, the phase difference between the electromagnetic and gravitational wave signal can be written

$$\Delta_1 = \frac{\omega_o D}{v_g} - \omega_o D + A . \quad (4.25)$$

Similarly, when the Earth is in position 2, the phase difference may be written

$$\Delta_2 = \frac{\omega_o (D + \xi)}{v_g} - \omega_o (D + \xi) + A , \quad (4.26)$$

where ξ is the path length difference for the time of flight between the two measure-

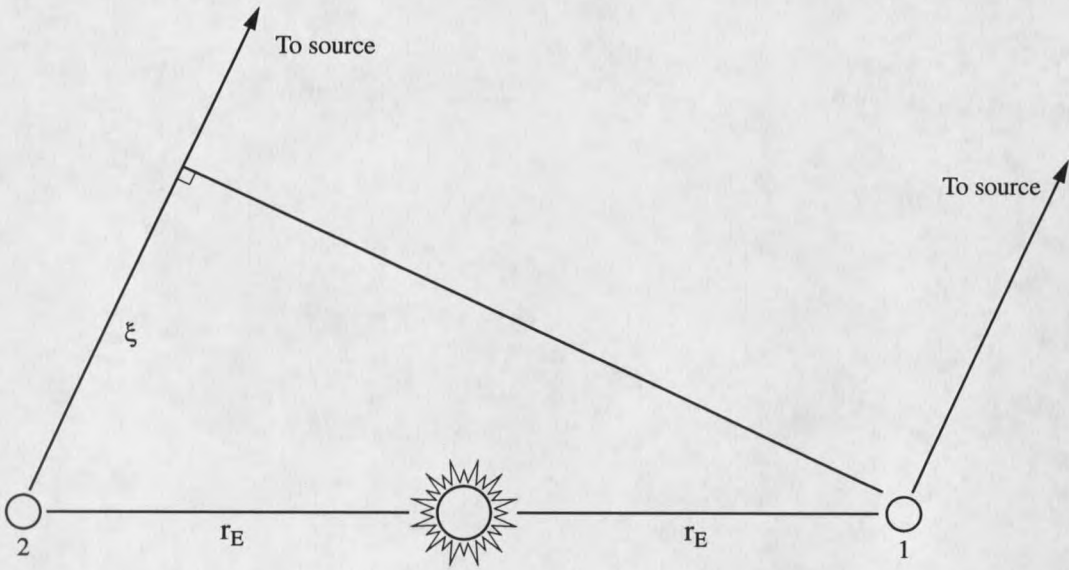


Figure 4.5: Schematic showing two observations of a source from opposite sides of the Earth's orbit. The radius of the Earth's orbit is r_E and the path length difference for signals propagating from far away is ξ .

ment. Subtraction eliminates the unknown quantity A , yielding

$$\Delta_2 - \Delta_1 = \omega_o \xi \left(\frac{1}{v_g} - 1 \right) . \quad (4.27)$$

Defining the fractional change in phase from this quantity gives

$$\epsilon = \frac{\Delta_2 - \Delta_1}{\omega_o \xi} = \frac{1}{v_g} - 1 , \quad (4.28)$$

which in terms of the Compton wavelength becomes

$$\lambda_g = \frac{\pi}{\omega_o} \sqrt{\frac{1}{2} \left(1 + \frac{1}{\epsilon} \right)} . \quad (4.29)$$

The unknown parameter, α , has been eliminated from the expression, at the cost of defining the Compton wavelength of the graviton over a characteristic distance ξ ,

which is much less than the original characteristic distance D . In terms of Figure 4.5, this amounts to measuring the phase lag between the two signals over the time it takes to cross the Earth's orbit (at most), as opposed to the time it takes to propagate over the Earth-source distance.

Throughout this analysis, it has been assumed that the photon is a massless particle. The current bound on the photon mass is $m_\gamma < 2 \times 10^{-16}$ eV [64], which is much larger than the current bounds on the graviton mass (*c.f.*, the solar system bound on the graviton mass is $m_g < 4.4 \times 10^{-22}$ eV). Is it then justifiable to treat the photon as a massless particle, while at the same time treating the graviton as a massive particle?

The resolution to this question can be understood by examining the partition of energy between the rest mass and kinetic energy of a particle being received from a distant binary. From the relativistic energy, $E^2 = p^2 + m^2$, one may write the velocity of any particle as

$$v^2 = 1 - \frac{m^2}{E^2} . \quad (4.30)$$

If $m \ll E$, then Eq. (4.30) implies

$$\epsilon = 1 - v \simeq \frac{1}{2} \frac{m^2}{E^2} , \quad (4.31)$$

where ϵ parameterizes the difference between the velocity of the particle and c .

For optical photons ($\lambda \sim 500$ nm) received from a binary star system, the characteristic energy is $E_\gamma \simeq 2.48$ eV. For this energy, $\epsilon_\gamma \leq 3 \times 10^{-33}$. Similar considerations

may be applied to the gravitons received from the same binary systems. In this case, the frequency of the gravitational waves is $f \sim 10^{-3}$ Hz, giving a characteristic energy for a single graviton of $E_g \sim 4 \times 10^{-18}$ eV. Using the solar system bound on the graviton mass, $m_g < 4.4 \times 10^{-22}$ eV, yields $\epsilon_g \leq 1 \times 10^{-8}$.

For the current bounds on the photon and graviton masses, $\epsilon_\gamma \ll \epsilon_g$. If the bound on the mass of the graviton is not drastically improved (*e.g.*, decreasing the bound on m_g by 12 orders of magnitude, such that $\epsilon_g \sim \epsilon_\gamma$), then the effect of a non-zero mass on the propagation of a particle will be much more significant for gravitons than photons in the analysis presented in this section. This justifies the treatment of the photons as massless particles and the gravitons as massive particles.

4.6 Obtaining a Mass Bound from AM *Canum Venaticorum*

In order to estimate a bound on the graviton mass, we will assume a null result for the measurement of the phase difference, Δ , between the two signals. The size of Δ (and therefore ϵ) will then be limited only by the uncertainty in the measurements of the phase. Combining the uncertainty of the gravitational phase measurements with the electromagnetic phase measurements in quadrature yields

$$\Delta \rightarrow \delta\Delta = \sqrt{\delta\phi_{em}^2 + \frac{1}{4}\delta\phi_{gw}^2}, \quad (4.32)$$

where $\delta\phi_{em}$ and $\delta\phi_{gw}$ are the uncertainties in each of the phases.

For observations with a space-based interferometer, the error in phase measure-

ments can be estimated as the ratio between the sampling time and the total integration time. For observatories such as OMEGA and LISA, the sampling time is expected to be 1 s, with total integration times of 10^7 s, yielding $\delta\phi_{gw} = 10^{-7}$. The CBA reports a 0.0004s uncertainty over the 1028.7325s period of AM CVn, yielding a phase uncertainty of $\delta\phi_{em} = 4 \times 10^{-7}$.

A numerical estimate of the delay introduced by the atmosphere must also be obtained. Assuming the Earth's atmosphere to be in hydrostatic equilibrium gives a scale height $h_s = 8540$ m. Taking the atmospheric depth to be $r_o = 10h_s$, Eq. (4.19) gives $t_{delay} = 2.8 \times 10^{-5}$ s. For the source AM CVn (discussed in Section 4.3), which lies at a distance $D = 101$ pc, Eq. (4.21) yields $\alpha_{atm} = 9.1 \times 10^{-24}$. This value is much smaller than the expected accuracy, $\delta\Delta$ in the phase measurements and can be neglected.

For the case of AM CVn, the value of α_* is still not known, so bounds on the graviton mass must be derived from Eq. (4.29). As shown in Figure 4.5, the characteristic distance ξ is simply the path length difference between the two measurements. If the Earth's axial inclination is $\iota = 23.4^\circ$, and the binary system lies at a declination δ , and a right ascension Υ , then the inclination β of the system to the plane of the Earth's orbit is defined by [65]

$$\sin \beta = \cos \iota \cdot \sin \delta - \sin \iota \cdot \cos \delta \cdot \sin \Upsilon . \quad (4.33)$$

With this, the characteristic distance is $\xi = 2r_E \cos(\beta)$. For AM CVn, which lies

at declination $\delta + 37^\circ 37^m 43.4^s$ and right ascension $\Upsilon 12^h 34^m 54.58^s$, this yields $\xi = 2.38 \times 10^{11}$ m. Using this value, Eq. (4.29) gives a bound

$$\lambda_g > 5 \times 10^{14} \text{ m} = 5 \times 10^{11} \text{ km} , \quad (4.34)$$

or $m_g < 2 \times 10^{-21}$ eV.

If the value of α_* could be determined precisely (*e.g.*, by monitoring ellipsoidal variations the light curve in the infrared, as suggested in Section 4.3, or with future observations by SIM [62] or TPF [63]), such that the uncertainties $\delta\alpha_* \lesssim 10^{-7}$, then Eq. (4.12) could be used to bound the graviton mass. The distance to AM CVn from Earth is $D = 101$ pc, yielding a bound

$$\lambda_g > 1 \times 10^{18} \text{ m} = 1 \times 10^{15} \text{ km} , \quad (4.35)$$

or $m_g < 1 \times 10^{-24}$ eV.

CHAPTER 5

SUMMARY AND CONCLUSIONS

5.1 Introduction

When the search for gravitational radiation began in the early 1960s with the resonant bar antennae of Joseph Weber, the outlook for gravitational wave astronomy was bright and optimistic. As the end of this millennium approaches, there has yet to be a confirmed detection of gravitational waves originating from an astrophysical system (or any other source, for that matter). Nevertheless, the field of gravitational wave astronomy is poised on the brink of momentous events. The technology being designed and put to use today is, to the best of our knowledge, perfectly capable of detecting gravitational radiation from a variety of astrophysical sources.

As the time nears when large-scale interferometric observatories become viable instruments for observing the Universe, the prominence of general relativity in the arena of astrophysics will undoubtedly grow. In order to firmly establish the value of gravitational wave astronomy as an observational science, it is necessary to solidify the theoretical foundations of what gravitational observatories can do, and what they

might be able to see.

5.2 Instrument Design and Response

Constructing the sensitivity curves for a spaceborne observatory, such as those shown in Figure 3.6, is the first step in understanding the response of the instrument to gravitational radiation. The formalism developed here can be used to determine the sensitivity of *any* space-based interferometer simply in terms of the essential parameters which describe the overall design of the instrument. Not only does this allow one to quickly and accurately assess the performance of one proposed observatory compared to another, but it also provides a quick and easy method for considering new observatory designs.

Carefully detailing the response of any new instrument not only allows us to propose new and exciting experiments (such as bounding the mass of the graviton), but prepares us for the inevitable detection of unexplainable signals from distant astrophysical sources, and provides a clear idea of how to improve our instrumentation for the construction of the next generation of observatories.

As an example, consider the GEM (Gravitational Explorer Mission) observatory [66]. Designed to cut costs by reducing the drag free requirements (δa), reducing the size of the optics (D), and only requiring a geosynchronous orbit, GEM would have the following parameter description:

$$\ell = 10^6 \text{ m} ,$$

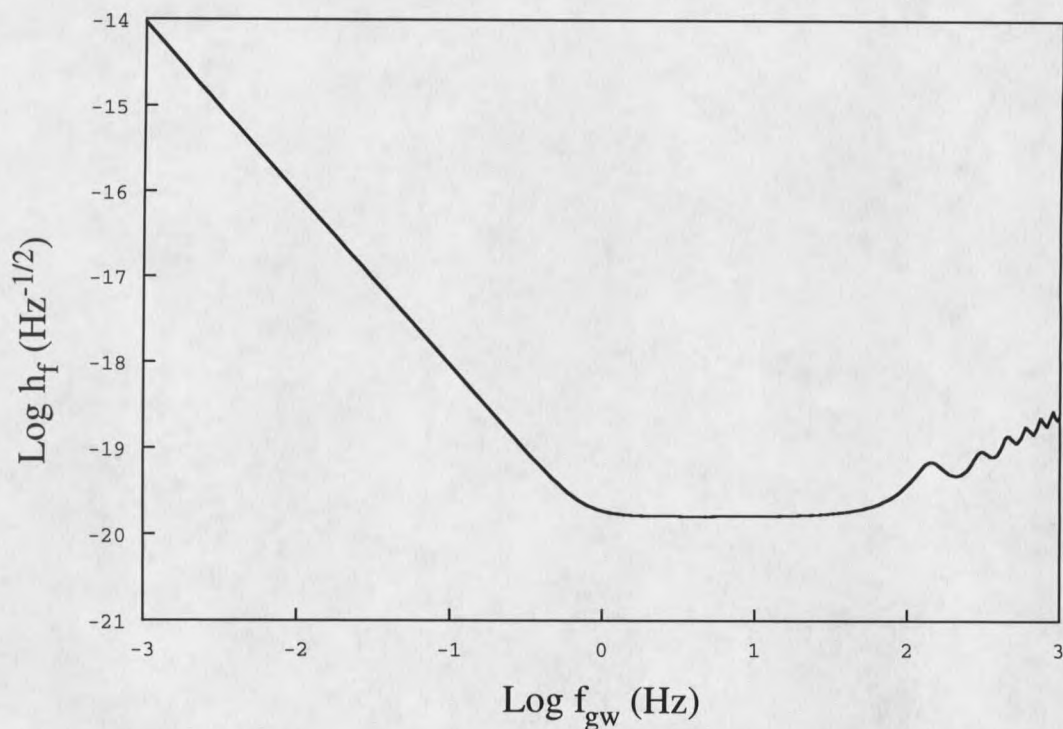


Figure 5.1: The sensitivity curve for the GEM gravitational wave observatory. No such mission has been proposed, but the sensitivity of the instrument may easily be estimated from a simple set of parameters.

$$D = 15 \text{ cm} ,$$

$$P = 1 \text{ W} ,$$

$$\delta a = 10^{-13} \text{ m s}^2 \text{Hz}^{-1/2} .$$

The overall sensitivity limits for GEM are shown in Figure 5.1, computed using the results of Chapter 3. Unlike other proposed space-based observatories, GEM would operate in a higher frequency waveband. The formalism presented here allows a rapid assessment of what our sensitivity in this region of the gravitational wave spectrum would be, and considerations can now be turned to the issue of whether or not any sources might exist in this regime which would be observable to GEM.

5.3 Astrophysics and Tests of Gravitational Theory

Currently the largest focus in gravitational wave research is aimed at design and data analysis for detectors. Once these instruments are online, the primary effort will be to confirm the existence of gravitational waves. After the initial detection of gravitational waves, however, the challenge will be for the field to evolve into a productive observational science which makes firm contact with astrophysics, complementing the broad base of electromagnetic observations already supporting that field.

Work to this end has already started on a variety of fronts. In particular, one of the most important capabilities of a gravitational wave observatory will be to localize a source of gravitational waves in the sky, and several analyses have been performed for spaceborne observatories [67]. The ability to locate the source of gravitational waves will allow the coordination of observations with electromagnetic telescopes, to maximize the amount of information gathered about specific sources. These types of simultaneous observations in the electromagnetic spectrum (*e.g.*, simultaneous optical and UV observations) have proved to be fruitful, and there is no reason to expect that the addition of gravitational wave observations will not add important useful information. For example, recent work [68] has suggested that observation of gravitational radiation from white dwarf binaries in globular clusters could provide an important calibration for the distance ladder used to estimate cosmological distances.¹

Inspiring astrophysical binaries are being modeled in great detail because they

¹For an excellent summation of the distance ladder and how it is built, see [69].

are expected to be a primary source for ground-based interferometric detectors, such as LIGO. Gravitational wave observations will provide a way to determine the binary orbital parameters for systems which are unresolvable by current electromagnetic telescopes [70]. Direct measurements of the structure of these star systems can be made through an application of the work outlined in Chapter 4 if one assumes that general relativity is the correct description of gravity.

General relativity predicts that the graviton is a massless particle. If the graviton *is* massless (*i.e.*, if it propagates with the speed of light, $v_g = c = 1$), then any phase difference measured between the gravitational wave and electromagnetic signal must be due to other effects. Setting $v_g = 1$ in Eq. (4.9) yields

$$\epsilon = \frac{\Delta}{\omega_o D} = \alpha , \quad (5.1)$$

showing that the difference in phase is simply an indicator of the value of $\alpha = \alpha_{atm} + \alpha_*$. As was shown in Chapter 4, the value of α_{atm} is expected to be negligible ($\alpha_{atm} = 9.1 \times 10^{-24}$ for AM CVn). In the cases where α_{atm} can be ignored, the measured phase difference will be a direct measure of the value of A , which is the amount of phase by which the electromagnetic signal leads the line of masses in the binary systems. With good models of the matter stream from the secondary Roche lobe overflow, a measurement such as this (unobtainable by any other means²) could allow an accurate determination of the accretion disk radius and the refinement of physical models for

²Optical interferometry missions such as the Space Interferometry Mission [62] and the Terrestrial Planet Finder [63] could resolve these systems, and measure A directly.

HeCV type stars.

Possible experiments to measure the mass of the graviton also demonstrates the possibility of using gravitational wave observations to contribute to basic tests of our understanding of the gravitational interaction. The opportunity to use gravitational wave astronomy to expand the experimental tests of gravitation should prove fruitful.

It is expected that study of other important parameters will also be possible with gravitational wave observations. In particular, the addition of gravitational radiation to the repertoire of spectra used to study the Universe should provide new methods to test cosmological models [71], and new ways to measure cosmological parameters such as the Hubble parameter H_0 [72] and the cosmological constant Λ [73].

With a growing number of ground-based detectors on the verge of operation, and plans for building large spaceborne detectors, the era of gravitational wave astronomy is on the horizon. Most of the observations described above will be part of the initial gravitational wave searches, but surprises will undoubtedly arise as gravitational wave astronomy develops, challenging the best theories we have to explain the phenomena we observe. Our imaginations are certainly not up to the task of cataloging every source of gravitational radiation in the Cosmos before we begin our observations. There will be ample work for many years to come.

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