The gravitational many-body PPN Lagrangian, Lorentz invariance, and the strong equivalence principle to second order

by Matthew John Benacquista

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Physics
Montana State University

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The gravitational many-body parameterized post-Newtonian (PPN) Lagrangian for compact celestial bodies is extended to second post- Newtonian order and is constrained to exhibit the invariances observed in nature — generalized Lorentz invariance, the strong equivalence principle, and the isotropy of the gravitational potential. These invariances are imposed on the Lagrangian using an empirical approach which is based on calculated observables rather than through formal procedures involving post-Newtonian approximations of transformations. When restricted in this way, the Lagrangian possesses two free parameters which can be related to light-deflection experiments and the effect of an environment of proximate matter on such experiments.

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ABSTRACT

The gravitational many-body parameterized post-Newtonian (PPN) Lagrangian for compact celestial bodies is extended to second post-Newtonian order and is constrained to exhibit the invariances observed in nature -- generalized Lorentz invariance, the strong equivalence principle, and the isotropy of the gravitational potential. These invariances are imposed on the Lagrangian using an empirical approach which is based on calculated observables rather than through formal procedures involving post-Newtonian approximations of transformations. When restricted in this way, the Lagrangian possesses two free parameters which can be related to light-deflection experiments and the effect of an environment of proximate matter on such experiments.

PARAMETERIZED POST-NEWTONIAN FORMALISM

Introduction

Since the advent of special relativity in the early part of this century attempts have been made to develop gravitational theories which incorporate the principles of relativity. At the same time, these theories have been subjected to ever-more precise experimental testing. Since the gravitational interaction is much weaker than the other fundamental forces, it is difficult to find systems which provide information about the full nature of the interaction. Thus, the majority of testing is commonly performed in weak-field systems in which the relativistic corrections to classical gravity are treated as perturbations on Newtonian gravity. The solar system is an example of such a system since the gravitational potentials are small, and the velocities of the planets are much less than the speed of light. The first post-Newtonian (PN) approximation to a theory of gravity, in which only the leading terms in an expansion of the relativisitic corrections are considered, has been sufficient for analyzing solar system experiments. A wide class of theories can be handled simultaneously in the PN approximation since the various theories differ only in their expansion coefficients. The parameterized post-Newtonian (PPN) formalism treats the expansion coefficients as phenomenological parameters to be determined by observation. In recent form, the PPN formalism uses an expansion in terms of the weak-field variables of a many-body Lagrangian.

Observations made up to the present are accurate enough to determine the first order PPN parameters to about one part in 10^3 (although some observations achieve greater accuracy and affect the second order parameters to about ten percent). The evidence indicates that the gravitational interaction obeys certain invariances such as Lorentz invariance, the strong equivalence principle, and isotropy. At order $(1/c^2)$ corrections to Newtonian gravity these invariances are sufficient to determine the structure of the interaction up to the freedom of one PPN parameter, and that one parameter is fit by observations of the defelection of light by the gravitational field of the sun. To aid in the development and analysis of the next generation of experiments which move beyond the 1PN order, this work examines the consequences of applying the observed invariances to second post-Newtonian order $(1/c^4)$, by requiring observables calculated from the 2PPN Lagrangian to exhibit the various invariances. The result of this is that the second PN order gravitational interaction is determined up to the freedom of one new second order PPN parameter if the interaction conforms to all invariances observed at the first post-Newtonian order. If the strong equivalence principle is abandoned at second order, then there remain five partially undetermined parameters which are related to rescaling effects of an environmental distribution of matter on deflection of light experiments, perihelion precession experiments, and Newton's constant G. Thus, these invariances are shown to strongly restrict the general structure of the gravitational interaction.

Historical Development

The most successful theory of gravity is the theory of general relativity developed by Einstein. When first published, it was shown to correctly predict the precession of the perihelion of Mercury. Shortly after its publication, general relativity met two other tests: the deflection of starlight by the gravitational field of the sun, and the gravitational redshift of spectral lines from the companion of Sirius. In these three classical tests the post-Newtonian predictions of general relativity were confirmed at limited levels of accuracy. The perihelion precession of Mercury is the highest accuracy test, but it relies on the assumption that the sun has negligible quadrupole moment. The precision of these measurements was high enough to permit testing of relativistic gravity at 1PN order, but insufficient at higher orders.

As the precision of experimental measurements increased and new techniques and experiments developed, it has become possible to experimentally question the assumptions and postulates that lead to general relativity: What constraints do observations impose on the gravitational interaction? This question was posed by Dicke in a very general form with few assumptions about the form of the gravitational theory (Dicke, 1964). Through analysis of particle physics and redshift experiments within the Dicke framework, the evidence indicated that a relativistic theory of gravity should relate gravity to a symmetric, second rank tensor (the metric), and the divergence of the stress-energy tensor with respect to this metric must be zero (Thorne and Will 1972). These two conditions can be interpreted as requiring spacetime to possess a metric field, that local Lorentz frames follow geodesic paths, and the special relativistic laws of physics are valid in these local Lorentz frames. Theories which fulfill these conditions are known as metric theories of gravity.

These theories can be easily generalized since the metric is relied on to describe the motions of test bodies, (and, until recently, most experimental data were analyzed under the test body limit) -- differences in the metric field for given sources serve to distinguish between alternative metric theories of gravity. The general form of the metric field is described by the PPN formalism. In this case, the metric is expanded beyond its Minkowskian form in terms of small quantities characterizing the system being analyzed. The coefficients which arise in the expansion are then determined by observation. This technique was first employed by Eddington (1922) for the restricted spherically symmetric static case and was applied to the two standard solar system tests of gravity -- light deflection and the precession of the perihelion of Mercury. Eddington took the metric field exterior to a spherically symmetric static body and expanded it in powers of 1/r to the necessary order; the coefficients arising from this expansion being constrained by observational data. This approach was later used by Robertson (1962) and Schiff (1962, 1967). Schiff introduced metric terms which incorporated rotational motion of the central body. Nordtvedt extended the formalism to more generally include multiple mass sources which were each permitted general motion (Nordtvedt 1968), and it was extended to include perfect fluid sources by Will (1971). The PPN formalism as developed by Nordtvedt and Will in the late sixties and early seventies has a systematic foundation and is treated as a "theory of theories" in its own right apart from specific relativistic field theories of gravity put forward by various authors (Nordtvedt and Will 1972, Will and Nordtvedt 1972, Will 1981).

PPN Formalism

The PPN formalism was developed for use in the analysis of gravitational systems such as the solar system in which the gravitational potentials are weak and the coordinate three-velocities of matter are small compared to the speed of light. Coordinates are chosen so that the metric becomes Minkowskian at large distances from the system. Expansion variables for the metric are dimensionless and take the following form:

$$v_i = \frac{dr_i}{cdt} \quad \text{and} \quad U_{ij} = \frac{GM_i}{|r_i - r_j|\, c^2}. \tag{1}$$

These have an "order of smallness" based upon the power of c that appears in the denominator, so v_i is O(1), U_{ij} is O(2), and the time coordinate (ct) is O(-1). To expand the line element:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$

to order O(2), g_{oo} is expanded to O(4), g_{oa} to O(3), and g_{ab} to O(2). In units where G=c=1, the general expansion to O(2) is:

$$\begin{aligned} g_{oo} = 1 - 2\sum U_i[1 &+ (1/2+\gamma+\alpha_3/2+\zeta_1)v_i^2 + \zeta_1(v_i\cdot(r\hat{} r_i))^2 \\ &+ (\alpha_1-\alpha_2-\alpha_3)w^2+\alpha_2(w\cdot(r\hat{} r_i))^2 - (2\alpha_3-\alpha_1)w\cdot v_i] \\ + 2\beta\sum U_i U_j &- 2(1-2\beta+\zeta_2)\sum U_i U_{ij} \end{aligned} \tag{2a}$$

$$g_{oa} = (1/2)\Sigma U_i[(4\gamma+3+\alpha_1-\alpha_2+\zeta_1)v_i^a + (1-\alpha_2-\zeta_1)(v_i\cdot(r-\hat{r}_i))(r-\hat{r}_i)^a + (\alpha_1-2\alpha_2)w^a + 2\alpha_2(w\cdot(r-\hat{r}_i))(r-\hat{r}_i)^a] \quad (2b)$$

$$g_{ab} = -(1 + 2\gamma\Sigma U_i)\delta_{ab}, \quad (2c)$$

where "Σ" implies a summation over all bodies, and w is the velocity of the coordinate system relative to some "preferred inertial frame". A gauge choice has been made to eliminate terms of the form:

$$\delta g_{oo} = \Sigma U_i[a(\frac{dv_i}{dt}\cdot(r-r_i)) + b(w\cdot(r-\hat{r}_i))(v_i\cdot(r-\hat{r}_i))]$$

$$\delta g_{ab} = \Sigma U_i c(r-\hat{r}_i)^a(r-\hat{r}_i)^b$$

The perfect fluid metric uses the expansion variables:

$v = \frac{dx}{dt}$ $\quad\rho$ = total mass-energy density

ρ_o = rest mass-energy density $\quad t_{ij}$ = components of the stress tensor

which appear in the following potentials:

$$U(x,t) = \int \frac{\rho(x',t)dx'}{|x-x'|}; \qquad A = \int \frac{\rho'(v'\cdot(x-x'))^2dx'}{|x-x'|^3};$$

$$\Phi_1 = \int \frac{\rho'v'^2dx'}{|x-x'|}; \quad \Phi_2 = \int \frac{\rho'U'dx'}{|x-x'|}; \quad \Phi_3 = \int \frac{\rho'\Pi'dx'}{|x-x'|}; \quad \Phi_4 = \int \frac{p'dx'}{|x-x'|};$$

$$W_a = \int \frac{\rho'(v'\cdot(x-x'))(x-x')_a}{|x-x'|^3}; \qquad V_a = \int \frac{\rho'v'_a dx'}{|x-x'|};$$

$$U_{ab} = -\frac{\partial}{\partial x^a}\frac{\partial}{\partial x^b}\int \rho'|x-x'|dx' + \delta_{ab}U$$

where $\Pi = (\rho-\rho_o)/\rho_o$ and $p = (1/3)t^i_i$. To O(2) the perfect fluid metric is:

$$g_{oo} = 1 - 2U + 2\beta U^2 - (2\gamma+2+\alpha_3+\zeta_1)\Phi_1 + \zeta_1 A \tag{3a}$$
$$- 2[(3\gamma+1-2\beta+\zeta_2)\Phi_2 + (1+\zeta_3)\Phi_3 + 3(\gamma+\zeta_4)\Phi_4]$$
$$+ (\alpha_1-\alpha_2-\alpha_3)w^2U + \alpha_2 w^a w^b U_{ab} - (2\alpha_3-\alpha_1)w^a V_a$$
$$g_{oa} = (1/2)[(4\gamma+3+\alpha_1-\alpha_2+\zeta_1)V_a + (1+\alpha_2-\zeta_1)W_a + (\alpha_1-2\alpha_2)w_a U] \tag{3b}$$
$$+ \alpha_2 w^b U_{ab}$$
$$g_{ab} = -(1 + 2\gamma U)\delta_{ab}. \tag{3c}$$

The terms:

$$\delta g_{oo} = aw^aW_a + b \iint \frac{\rho'\rho''(x-x')\cdot(x-x'')dx'dx''}{|x-x'||x-x''|^3}$$

$$\delta g_{ab} = cU_{ab}$$

have been eliminated by gauge choice.

The post-Newtonian approximation to most any metric theory of gravity can be represented by this PPN metric for particular choices of the parameters. In the "standard" relativistic theory of gravity--general relativity -- the parameters are: $\gamma=1$, $\beta=1$, and all others are zero. The physical interpretations of the parameters can be found in Will and Nordtvedt (1972) or Misner, Thorne, and Wheeler (1973).

The PPN metric is used to calculate the value of observables in existing, proposed, and hypothetical experiments, and these calculated values are compared with observations. The calculated value of an observable will generally depend upon the PPN parameters, so comparison with experiment serves to constrain acceptable values for the PPN parameters. A specific metric theory is compared with experiment by calculating the post-Newtonian approximation to the metric for the particular theory using a standard Einstein-Infeld-Hoffmann type perturbative approach and reading off the values of the PPN parameters. Those theories whose PPN parameters are outside of the ranges set by experiment can be rejected.

The PPN metric formalism has become a valuable tool for comparing metric theories with experiment and observation, and designing new experiments. But it is useful only for analyzing the dynamics of a system in which bodies have weak internal gravity which can be treated perturbatively. In general, pulsars and other exotic objects do not satisfy this condition since they possess strong internal gravity which is best treated non-perturbatively. To describe the dynamics of these compact celestial bodies, a PPN Lagrangian formalism was developed by Nordtvedt in 1985.

PPN Lagrangian For Compact Bodies

For any theory of gravity which is derivable from a Lagrangian, it is assumed that there exists an effective many-body Lagrangian which describes the positional motion of celestial bodies (compact bodies and regular bodies, as well as test bodies). If the matter is considered to be concentrated into a finite number of bodies whose internal dynamics are frozen (or neglected) and whose deviations from sphericity are negligible, then the matter variables (degrees of freedom) reduce to the coordinate trajectories for each body, $r_i(t)$, which are related to the motion of each body, but a-priori have no precise physical meaning. The effects of the gravitational field on the matter is expressed in terms of a generalized effective potential which is a function of interbody coordinates and time derivatives of the body trajectories. As an example of how this proceeds, consider a metric theory of gravity derivable from a Lagrangian of the form:

$$£ = £_{NG} + £_{G} \tag{4}$$

where $£_G$ is the part that contains no matter variables and $£_{NG}$ contains only matter variables and the metric field (Thorne 1973). The matter variables are reduced to the trajectories of each body, $r_i(t)$. The metric can, in the spirit of the PPN formalism, be expressed in terms of the matter varibles, and therefore $£_{NG}$ can be expressed entirely in terms of the trajectories, higher time derivatives of the trajectories, and various interbody coupling parameters. The field equations for any other fields associated with gravity can be solved in terms of the matter variables, so this procedure applies to $£_G$ as well with an effective many-body Lagrangian thereby resulting:

$$\int £ d^4x = \int L(r_i, v_i, a_i, \ldots) dt \tag{5}$$

Since no particular theory is assumed, then an expansion about the Newtonian Lagrangian can be constructed (if the potentials and velocities are weak) in which the expansion coefficients are considered PPN parameters. This expansion is the PPN Lagrangian which describes a broad and general class of theories of gravity.

Unlike test bodies, compact bodies do not generally follow geodesics so that a metric-based expansion procedure is inadequate for describing the motion of such bodies. On the other hand, a Lagrangian-based expansion provides the equations of motion for compact bodies as well as yielding the PPN metric from which the equations of motion for test bodies are calculated. It also provides a means of incorporating conservation laws and of imposing the invariances which observations require; and finally it gives precise meaning to the coordinate trajectories $\mathbf{r}_i(t)$ by defining distances through analysis of actual and "gedanken" experiments.

CONSTRAINING TECHNIQUES

Preferred Frames

Various astronomical observations indicate that the gravitational interaction exhibits no preferred inertial frame effects and that there exists energy-momentum and angular momentum conservation. The α and ζ parameters in equations (2) and (3) signal preferred frame effects and the failure of conservation of energy-momentum. Certain types of preferred frame theories are in conflict with measurements of the time dependence of the period of the binary pulsar (1913+16) (Taylor and Weisberg, 1982). More generally, pulsar period measurements (Nordtvedt 1987b) imply that $|\alpha_1| \leq 10^{-6}$, and analysis of the alignment of the sun's spin axis with the angular momentum of the solar system (Nordtvedt, 1987a) gives $|\alpha_2| \leq 10^{-7}$. Data from observed perihelion shifts (Nordtvedt and Will, 1972) forces $|\alpha_3| \leq 10^{-5}$. Lagrangian based metric theories of gravity must have (Lee et. al., 1974):

$$-(1/2)\zeta_1 = \zeta_2 = -(3/2)\zeta_4 = \zeta_W, \text{ and } \zeta_3 = 0$$

where ζ_W is a parameter introduced by Will (1973) to include Whitehead's theory of gravity in the PPN metric. Earth-tide experiments (Warburton and Goodkind, 1976) have shown ζ_W to be less than 10^{-3}, while the sun's spin axis history implies $\zeta_W \leq 10^{-7}$ (Nordtvedt 1987a).

In addition to restricting preferred frame effects, these observations indicate a further invariance -- isotropy. Since they were made on systems which were in an environment of external matter, the environment of distant matter must not produce preferred frame or anisotropic effects. For example, the solar spin axis observation takes place in the presence of the galaxy. The rest frame of the galaxy could define a preferred frame and its concentrated matter could produce anisotropic effects by picking out a preferred direction. Since the galactic gravitational potential has a dimensionless value of 5×10^{-7} at the location of the solar system which is comparable to the restriction placed on α_2, the galaxy does not induce preferred frame effects to first order in its potential. Because the Newtonian interaction has been shown to be isotropic to 2×10^{-13} (Nordtvedt 1987a), anisotropic effects can be eliminated to second order in the galactic potential.

Since the experimental support for the absence of preferred frame or anisotropic effects is so decisive, it is efficient to restrict the freedom in the PPN Lagrangian to eliminate such effects before it is used to represent the general gravitational interaction. Procedures that eliminate preferred frame effects which are independent of the outside environment are classed as "Lorentz Invariance" constraints (LI), while those which eliminate preferred frame effects produced by the distant environment are classed as "Generalized Lorentz Invariance" constraints (GLI). Anisotropic effects can also be eliminated in the context of the GLI constraints, while the use of a Lagrangian to describe the gravitational interaction automatically insures that conservation laws are valid (including angular momentum if there is intrinsic isotropy -- no universal preferred direction.)

The Lagrangian

The PPN Lagrangian represents the dynamics of many celestial bodies in the limit that gravitational radiation is negligible (or perhaps the half retarded time, half advanced time result from the underlying field theory). The internal structure of each body is frozen so that the dynamics of each body is described solely by a trajectory degree of freedom, $\mathbf{r}_i(t)$. The Lagrangian takes the form of a 3-space scalar function of the trajectories and their time derivatives. To at least the second order of post-Newtonian approximation, the accelerations and higher derivatives of position can be removed from the Lagrangian via coordinate transformations which, in effect, push these terms into the next higher order of approximation (Damour and Schäfer 1985). Without loss of generality, the Lagrangian in some preferred inertial frame can therefore be considered as a function of inner products of interbody distances and body velocities:

$$L(r_{ij}, \mathbf{v}_i \cdot \mathbf{v}_j, \mathbf{v}_i \cdot \mathbf{r}_{ij})$$

and is required to be time-reversal symmetric.

Lorentz Invariance Constraints

Preferred frame effects are present in observational phenomena if a measurable quantity, such as period of revolution or periastron precession for a binary system, etc. allows an observer to determine the motion of a gravitational system with respect to a preferred inertial rest frame. The gravitational Lagrangian is structured so as to exclude preferred frame effects by the following procedure: two distant inertial observers -- one at rest with respect to the preferred frame, and the other at rest with respect to a system which is moving at uniform velocity w with respect to the preferred frame -- make measurements on the system using their clocks and rulers. Since their clocks and rulers are in gravity-free inertial frames, their measurements are related to each other by the Lorentz transformations. For example, if both observers measure the period of orbit for a binary system, the period measured by one observer will be time dilated when compared with the other observer's measurement. If the Lagrangian is to be consistent with observation, the calculated values of observables in one frame should be the Lorentz transform of the values calculated in another frame. Thus, the Lorentz Invariance of the Lagrangian in this work is based on the behavior of calculated observables rather than a formal "post-Galilean" transformation (Chandrasekhar and Contopoulos 1967) of the Lagrangian itself.

For any planar system with w normal to the plane of the system, the body trajectories defined by both observers will differ only by a time dilation scale factor. Any measurable quantity calculated from the Lagrangian for such a system will simply rescale by a factor of $(1\text{-}w^2)^{1/2}$ in the time dependence of the quantity. This is guaranteed if the acceleration function for any body in the system obeys

$$a_i(r_{ij},v_j(1-w^2)^{-1/2}+w) = (1-w^2)a_i(r_{ij},v_j) \qquad (7)$$

where r_{ij} and v_j are those planar values in the rest frame of the system and $w \cdot v_j = w \cdot r_{ij} = 0$. If the Lagrangian is written as:

$$L(r_{ij},v_j) = -\sum M_i(I)(1-v_i^2)^{1/2} + U(r_{ij},v_j)$$

the equations of motion for body i are:

$$\mathbf{a}_i = \frac{1}{M_i(I)}(1-v_i^2)^{-1/2}\left[\frac{\partial U}{\partial \mathbf{r}_i}-\frac{d}{dt}\frac{\partial U}{\partial \mathbf{v}_i}\right] - (1-v_i^2)^{-1}(\mathbf{v}_i\cdot\mathbf{a}_i)\mathbf{v}_i$$

Defining:

$$f(\mathbf{r}_{ij},\mathbf{v}_j(1-w^2)^{1/2}+\mathbf{w}) \equiv f'$$

for any function f gives:

$$\mathbf{a}_i' = \frac{1}{M_i(I)}(1-v_i^2)^{-1/2}(1-w^2)^{-1/2}\left[\frac{\partial U}{\partial \mathbf{r}_i}-\frac{d}{dt}\frac{\partial U}{\partial \mathbf{v}_i}\right]'$$
$$- (1-v_i^2)^{-1}[(\mathbf{v}_i\cdot\mathbf{a}_i')\mathbf{v}_i + (1-w^2)^{-1/2}((\mathbf{v}_i\cdot\mathbf{a}_i')\mathbf{w}+(\mathbf{w}\cdot\mathbf{a}_i')\mathbf{v}_i) + (1-w^2)^{-1}(\mathbf{w}\cdot\mathbf{a}_i')\mathbf{w}]$$

The constraint:

$$\mathbf{a}_i' = (1-w^2)\mathbf{a}_i$$

requires:

$$\left[\frac{\partial U}{\partial \mathbf{r}_i}-\frac{d}{dt}\frac{\partial U}{\partial \mathbf{v}_i}\right]' = \left[\frac{\partial U}{\partial \mathbf{r}_i}-\frac{d}{dt}\frac{\partial U}{\partial \mathbf{v}_i}\right] - M_i(I)(1-v_i^2)^{-3/2}(\mathbf{v}_i\cdot\mathbf{a}_i)\mathbf{v}_i$$

Noting that:

$$\left[\frac{d}{dt}\frac{\partial}{\partial \mathbf{v}_i}(\Sigma M_i(I)(1-v_i^2)^{1/2})\right]' = \left[\frac{d}{dt}\frac{\partial}{\partial \mathbf{v}_i}(\Sigma M_i(I)(1-v_i^2)^{1/2})\right](1-w^2)^{1/2}$$
$$- M_i(I)(1-v_i^2)^{-3/2}(\mathbf{v}_i\cdot\mathbf{a}_i)\mathbf{v}_i$$

forces the Lagrangian to obey:

$$\left[\frac{\partial L}{\partial r_i} - \frac{d}{dt}\frac{\partial L}{\partial v_i} \right]' = \left[\frac{\partial L}{\partial r_i} - \frac{d}{dt}\frac{\partial L}{\partial v_i} \right](1-w^2)^{1/2}.$$

This can be implemented by requiring:

$$L(r_{ij}, v_j(1-w^2)^{1/2} + w) = L(r_{ij}, v_j)(1-w^2)^{1/2} \tag{10a}$$

and:

$$\hat{w} \cdot \frac{\partial L}{\partial r_i} = \hat{w} \cdot \frac{d}{dt}\frac{\partial L}{\partial v_i} \; . \tag{10b}$$

The last constraint (eq. (10b)) can be related to a more empirical constraint on bound two-body dynamics. Motion with respect to a preferred universal frame could be detected if the plane of orbit of a two-body system were seen to precess about some direction in space (its velocity with respect to the preferred frame). Such an effect would appear, for example, as a precession of the sun's spin axis relative to the orbital angular momentum of the solar system (Nordtvedt 1987). A constraint is imposed on the Lagrangian by examining an orbit whose plane is to first approximation normal to the preferred frame velocity w, but which is perturbed in the direction of w. The angular dependence of the perturbation must be identical (up to a phase difference) with the angular dependence of the unperturbed orbit if no precession is to occur. The perturbation equations are obtained through the equations of motion derived from the Lagrangian.

If one of the bodies is much less massive than the other, the Lagrangian can be written:

$L(r,v,w)$

where r_{12}=r, v_1=v+w, and v_2=w. (If this assumption is abandoned, the clarity of the argument is hidden by notational complexity, but no additional information is gained.) Let:

$$\frac{\partial L}{\partial \mathbf{v}} \equiv F\mathbf{v} + G\mathbf{r} + K\mathbf{w} \tag{11a}$$

$$\frac{\partial L}{\partial \mathbf{r}} \equiv H\mathbf{r} + G\mathbf{v} + J\mathbf{w} \tag{11b}$$

where:

$$F = 2\,\frac{\partial L}{\partial v^2},\quad G = \frac{\partial L}{\partial(\mathbf{v}\cdot\mathbf{r})},\quad H = \frac{1}{r}\frac{\partial L}{\partial r},\quad J = \frac{\partial L}{\partial(\mathbf{w}\cdot\mathbf{r})},\quad K = \frac{\partial L}{\partial(\mathbf{w}\cdot\mathbf{v})}.$$

The conserved quantity associated with the motion around $\hat{w}$ is:

$$l \equiv \hat{\mathbf{w}}\cdot\left(\mathbf{r}\ \mathrm{x}\ \frac{\partial L}{\partial \mathbf{v}}\right) = F\hat{\mathbf{w}}\cdot(\mathbf{r}\ \mathrm{x}\ \mathbf{v}).$$

Specializing to the unperturbed orbit, with w·r=w·v=0 and using polar coördinates so that:

$$l = r^2\dot{\theta}F_o,\ \text{and}\quad \frac{d}{dt} = \frac{l}{r^2F_o}\frac{d}{d\theta}$$

with

$$F_o = F(\mathbf{r},\mathbf{v},\mathbf{w})\ \text{with}\ \mathbf{w}\cdot\mathbf{v} = \mathbf{w}\cdot\mathbf{r} = 0.$$

Define:

$$u \equiv \frac{1}{r},\quad u' \equiv \frac{d}{d\theta}u,\quad \mathbf{u}_r \equiv -\frac{d}{d\theta}\mathbf{u}_\theta,\quad \text{and}\quad \mathbf{u}_\theta \equiv \frac{d}{d\theta}\mathbf{u}_r$$

so the equations of motion:

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\frac{\partial L}{\partial v}$$

are:

$$u + u'' = \frac{-H_oF_o}{l^2u^3} + \frac{G'_o}{lu} \quad \text{and} \quad \frac{lu^2}{F_o}K'_o = J_o$$

Let $\hat{w}r = z$ be the perturbation, with the perturbation equations coming from

$$\hat{w}\cdot\left[\frac{\partial L}{\partial r}\frac{d}{dt}\frac{\partial L}{\partial v}\right] = 0$$

with $\hat{w}v = \dot{z}$ and $\hat{w}r = z$. Defining:

$$f \equiv zu$$

gives:

$$f'' + f = \frac{-F_o}{l^2u^3}\left[\frac{d}{dt}\left(z\frac{\partial K}{\partial z}\bigg|_o + \dot{z}\frac{\partial K}{\partial \dot{z}}\bigg|_o\right) - \left(z\frac{\partial J}{\partial z}\bigg|_o + \dot{z}\frac{\partial J}{\partial \dot{z}}\bigg|_o\right)\right] \quad (12)$$

to first order in z and $\dot{z}$. The requirment of no precession determines the structure of K and J in order that equation (12) has a solution $z(\theta)$ for which $f(\theta+2\pi)=f(\theta)$. However, the structure of J and K can be determined from the unperturbed equation of motion if one allows for uniform, in-plane drift of the system. Let the drift velocity be v_D, then the Lagrangian becomes:

$$L(\mathbf{r},\mathbf{v},\mathbf{w}+\mathbf{v}_D)$$

with $\mathbf{r}_{12}=\mathbf{r}$, $\mathbf{v}_1=\mathbf{v}+\mathbf{w}+\mathbf{v}_D$, and $\mathbf{v}_2=\mathbf{w}+\mathbf{v}_D$. For $\mathbf{w}\cdot\mathbf{v}=\mathbf{w}\cdot\mathbf{r}=\mathbf{w}\cdot\mathbf{v}_D=0$, the equations of motion give:

$$\frac{d}{dt}K(\mathbf{r},\mathbf{v},\mathbf{w}+\mathbf{v}_D) - J(\mathbf{r},\mathbf{v},\mathbf{w}+\mathbf{v}_D)$$

For $x \equiv \hat{\mathbf{v}}_D\cdot\mathbf{r}$ and $\dot{x} \equiv \hat{\mathbf{v}}_D\cdot\mathbf{v}$, this is:

$$\frac{d}{dt}\left(x \left.\frac{\partial K}{\partial x}\right|_o + \dot{x} \left.\frac{\partial K}{\partial \dot{x}}\right|_o \right) - \left(x \left.\frac{\partial J}{\partial x}\right|_o + \dot{x} \left.\frac{\partial J}{\partial \dot{x}}\right|_o \right) = 0 \tag{13}$$

to first order in x where:

$$\left.\frac{\partial K}{\partial x}\right|_o = \frac{\partial K(\mathbf{r},\mathbf{v},\mathbf{w}+\mathbf{v}_D)}{\partial x}$$

evaluated at $\mathbf{w}\cdot\mathbf{r} = \mathbf{w}\cdot\mathbf{v}_D = \mathbf{w}\cdot\mathbf{v} = 0$. Since $x \equiv \hat{\mathbf{v}}_D\cdot\mathbf{r}$, $z \equiv \hat{\mathbf{w}}\mathbf{r}$, and $\mathbf{w}$ and $\mathbf{v}_D$ appear in the Lagrangian as $\mathbf{w}+\mathbf{v}_D$,

$$\left.\frac{\partial K}{\partial z}\right|_o, \quad \left.\frac{\partial K}{\partial \dot{z}}\right|_o, \quad \left.\frac{\partial J}{\partial z}\right|_o, \text{ and } \quad \left.\frac{\partial J}{\partial \dot{z}}\right|_o$$

are identical to

$$\left.\frac{\partial K}{\partial x}\right|_o, \quad \left.\frac{\partial K}{\partial \dot{x}}\right|_o, \quad \left.\frac{\partial J}{\partial x}\right|_o, \text{ and } \quad \left.\frac{\partial J}{\partial \dot{x}}\right|_o$$

Therefore, the structure of K and J must be such that equation (13) is identically solved for any function of the form $f(\theta)=\mathbf{a}\cdot\mathbf{r}$ where $\mathbf{a}$ is a constant vector. Thus, equation (12) is:

$$f'' + f = 0$$

which automatically insures no precession of the orbital plane. Therefore, requiring equation (10b), which is

$$\dot{J} = \frac{d}{dt}K,$$

is equivalent to requiring no precession of the plane of a two-body orbit about the system's velocity with respect to a preferred frame.

These two constraints are sufficient to determine most of the expansion coefficients in the Lagrangian, the exceptions being those coefficients which multiply terms of the form of a velocity difference between two bodies. Such terms cannot be constrained by LI constraints.

Generalized Lorentz Invariance Constraint

The environment can affect the internal dynamics of an otherwise isolated system if the value of the various mass parameters, coupling parameters, or PPN parameters in the Lagrangian are rescaled because of the proximity of matter. For example, at the Newtonian level of approximation, a free body has the Lagrangian:

$$L = M_i^{(0)}(I) + (1/2)M_i^{(2)}(I)v_i^2$$

with $M(I)^{(0)}=M(I)^{(2)}$. If the rest mass, $M(I)^{(0)}$, depends upon the environment differently than the kinetic inertial mass, $M(I)^{(2)}$, these two masses will not in general be equal and the Lagrangian will not obey the LI constraint eq. (10a). The dependence of the parameters in the Lagrangian on the environment must be examined in order to eliminate these kinds of preferred frame effects. Generalized Lorentz invariance constraints accomplish this by determining the environmental dependence of low order Lagrangian parameters in terms of higher order parameters and forcing a "rescaled" Lagrangian to obey the LI constraints.

The environment is brought into gravitational physics by considering a distribution of bodies which has a cluster of bodies -- the reduced system -- and one or more bodies located far from the cluster which are called spectator bodies (Nordtvedt 1985). The influence of the spectator bodies on the reduced system plays the role of environment modification, and the spectator bodies can be included or excluded from analysis of the reduced system. If a Lagrangian is written for the entire system (spectators plus the reduced system), the coordinates are such that the metric becomes Minkowskian far from the total system. However, if a Lagrangian is written for the reduced system alone, different coordinates are used so that the metric becomes Minkowskian at some intermediate region between the spectators and the reduced system which is yet far from the reduced system. By analogy, this is similar to analyzing the Earth-Moon system without reference to the sun by choosing coordinates which a freely falling in the sun's gravitational field. The environmental dependence is incorporated into the reduced Lagrangian by writing the Lagrangian for the entire system, changing the coordinates to those appropriate to the reduced system and neglecting that part of the Lagrangian which does not depend upon the bodies in the reduced system:

$$L_{REDUCED}dt = L'_{ENTIRE}dt' - L'_{SPECTATOR}dt'$$

The reduced Lagrangian must then satisfy the LI constraints (eqs. 10a,b). This procedure is the generalized Lorentz invariance constraint. In addition, the observed isotropy of the Newtonian interaction must be preserved in $L_{REDUCED}$.

The Strong Equivalence Principle

Lagrangians which satisfy both the LI and the GLI constraints show no preferred frame effects, but do not thereby necessarily obey the strong equivalence principle (SEP), which is an additional constraint--that the outcome of any experiment (gravitational or non-gravitational) be independent of environment. If the SEP holds, all PPN parameters and coupling parameters in the Lagrangian must be independent of the environment. In the language of the GLI constraint, the reduced Lagrangian must retain no reference to the spectator bodies. Imposing the SEP further constrains the PPN parameters, and therefore reduces the variability or freedom in the PPN Lagrangian.

SECOND ORDER INTERACTIONS

The PPN Lagrangian to Second Order

The recent analysis of the alignment of the sun's spin axis with respect to the angular momentum of the solar system constrains α_2 to a part in 10^7 (Nordtvedt 1987a). The 1PPN Lagrangian is insufficient for analysis of such phenomena since the appropriate solar and environmental variables are of order 10^{-6}. The accuracy of measurement in experimental gravity has reached a threshold at which the PPN formalism can usefully be extended to second post-Newtonian (2PPN) order. Following the procedure outlined in Chapter One, the general 2PPN Lagrangian is given by:

$$L = L^{(N-1)} + L^{(N)} + L^{(N+1)} + L^{(N+2)}$$

where $L^{(N-1)}$ is of order $(c)^2$, $L^{(N)}$ of order $(1/c)^0$, $L^{(N+1)}$ of order $(1/c)^2$, and $L^{(N+2)}$ of order $(1/c)^4$. Since empirical evidence suggests that the gravitational interaction shows no preferred frame effects and is isotropic, the Lagrangian is restricted to reflect this invariance. The additional consequences of invoking the strong equivalence principle at either first or second order are also explored.

The Lorentz Invariance and Generalized Lorentz Invariance constraints have already been imposed on the first order Lagrangian by both the Newtonian potential and 1PN corrections to the Newtonian potential. This restricted Lagrangian will be used when the invariances are imposed on $L^{(N+2)}$:

$$L = -\sum M_i(I)(1 - (1/2)v_i^2 - (1/8)v_i^4)$$
$$+ (1/2)\sum \frac{1}{r_{ij}}[\Gamma_{ij}(1 - (1/2)(v_i^2+v_j^2-v_i\cdot v_j+(v_i\cdot\hat{r}_{ij})(v_j\cdot\hat{r}_{ij})$$
$$+ (1+\gamma)M_i(I)M_j(I)(v_i-v_j)^2]$$
$$+ ((1/2)-\beta)\sum \frac{\Gamma_{ijk}}{r_{ij}r_{ik}}$$

where Γ_{ij} is a Newtonian coupling parameter between bodies i and j, Γ_{ijk} is a three-body coupling parameter, and γ and β are PPN parameters. The Newtonian coupling parameter reduces to products of appropriate masses under the test body limit of one or two of the bodies.

The general expansion of $L^{(N+2)}$ is:

$$L^{(N+2)} = \sum(1/16)M_i^{(6)}(I)v_i^6 \tag{14}$$
$$+ \sum r_{ij}[A_{ij}^{(1)}(\ddot{a}_i\cdot r_{ij}) + A_{ij}^{(2)}(\dot{a}_i\cdot\hat{r}_{ij})(v_i\cdot\hat{r}_{ij}) + A_{ij}^{(3)}(\dot{a}_i\cdot\hat{r}_{ij})(v_j\cdot\hat{r}_{ij})$$
$$+ A_{ij}^{(4)}(\dot{a}_i\cdot v_i) + A_{ij}^{(5)}(\dot{a}_i\cdot v_j) + A_{ij}^{(6)}a_i^2 + A_{ij}^{(7)}(a_i\cdot a_j)$$
$$+ A_{ij}^{(8)}(a_i\cdot\hat{r}_{ij})^2 + A_{ij}^{(9)}(a_i\cdot\hat{r}_{ij})(a_j\cdot\hat{r}_{ij})]$$

$$+ \sum\frac{1}{r_{ij}}[(a_i\cdot r_{ij})(B_{ij}^{(1)}v_i^2 + B_{ij}^{(2)}v_j^2 + B_{ij}^{(3)}(v_i\cdot v_j) + B_{ij}^{(4)}(v_i\cdot\hat{r}_{ij})^2$$
$$+ B_{ij}^{(5)}(v_j\cdot\hat{r}_{ij})^2 + B_{ij}^{(6)}(v_i\cdot\hat{r}_{ij})(v_j\cdot\hat{r}_{ij}))$$
$$+ (a_i\cdot v_i)(B_{ij}^{(7)}(v_i\cdot r_{ij}) + B_{ij}^{(8)}(v_j\cdot r_{ij}))$$
$$+ (a_i\cdot v_j)(B_{ij}^{(9)}(v_i\cdot r_{ij}) + B_{ij}^{(10)}(v_j\cdot r_{ij}))$$
$$+ C_{ij}^{(1)}v_i^4 + C_{ij}^{(2)}v_i^2v_j^2 + C_{ij}^{(3)}v_i^2(v_i\cdot v_j) + C_{ij}^{(4)}(v_i\cdot v_j)^2$$
$$+ (v_i\cdot\hat{r}_{ij})^2(C_{ij}^{(5)}v_i^2 + C_{ij}^{(6)}v_j^2 + C_{ij}^{(7)}(v_i\cdot v_j))$$

$$+ (\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})(C_{ij}^{(8)}v_i^2 + C_{ij}^{(9)}(\mathbf{v}_i\cdot\mathbf{v}_j))$$
$$+ C_{ij}^{(10)}(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^4 + C_{ij}^{(11)}(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^3(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij}) + C_{ij}^{(12)}(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^2(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})^2]$$

$$+ \sum \frac{1}{r_{ij}r_{ik}}[\mathbf{a}_i\cdot\mathbf{A}_{ijk}^{(1)} + \mathbf{a}_j\cdot\mathbf{A}_{ijk}^{(2)}$$
$$+ B_{ijk}^{(1)}v_i^2 + B_{ijk}^{(2)}v_j^2 + B_{ijk}^{(3)}(\mathbf{v}_i\cdot\mathbf{v}_j) + B_{ijk}^{(4)}(\mathbf{v}_j\cdot\mathbf{v}_k)$$
$$+ \mathbf{v}_i\cdot\mathbf{C}_{ijk}^{(1)}\cdot\mathbf{v}_i + \mathbf{v}_j\cdot\mathbf{C}_{ijk}^{(2)}\cdot\mathbf{v}_j + \mathbf{v}_i\cdot\mathbf{C}_{ijk}^{(3)}\cdot\mathbf{v}_j + \mathbf{v}_j\cdot\mathbf{C}_{ijk}^{(4)}\cdot\mathbf{v}_k]$$

$$+ \sum \frac{\Pi_{ijkl}}{r_{ij}r_{ik}r_{il}} \quad + \sum \frac{\Omega_{ijkl}}{r_{ij}r_{jk}r_{kl}}$$

The coefficients are coupling parameters which incorporate the 2PPN parameters, source strengths, and scalar functions of the interbody vectors. The accelerations and higher derivatives of $\mathbf{r}$ are removed by a coordinate transformation combined with the addition of the total time derivative of a function to the Lagrangian. The most general transformation which can be applied to the Lagrangian at this order without changing the coordinates at lower orders is given by:

$$\mathbf{r}_i' = \mathbf{r}_i + \delta\mathbf{r}_i$$

with

$$\delta\mathbf{r}_i = \sum \frac{1}{r_{ij}M_i(I)}[\mathbf{r}_{ij}(\alpha_{ij}^{(1)}v_i^2 + \alpha_{ij}^{(2)}v_j^2 + \alpha_{ij}^{(3)}(\mathbf{v}_i\cdot\mathbf{v}_j) + \alpha_{ij}^{(4)}(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^2 + \alpha_{ij}^{(5)}(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})^2$$
$$+ \alpha_{ij}^{(6)}(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij}) + \alpha_{ij}^{(7)}(\mathbf{a}_i\cdot\mathbf{r}_{ij}) + \alpha_{ij}^{(8)}(\mathbf{a}_j\cdot\mathbf{r}_{ij}))$$
$$+ \mathbf{v}_i(\alpha_{ij}^{(9)}(\mathbf{v}_i\cdot\mathbf{r}_{ij}) + \alpha_{ij}^{(10)}(\mathbf{v}_j\cdot\mathbf{r}_{ij}))$$
$$+ \mathbf{v}_j(\alpha_{ij}^{(11)}(\mathbf{v}_i\cdot\mathbf{r}_{ij}) + \alpha_{ij}^{(12)}(\mathbf{v}_j\cdot\mathbf{r}_{ij}))$$
$$+ (\alpha_{ij}^{(13)}\mathbf{a}_i + \alpha_{ij}^{(14)}\mathbf{a}_j)\mathbf{r}_{ij}]$$

$$+\sum \frac{1}{r_{ij}r_{ik}M_i(I)}[\alpha_{ijk}^{(15)}\mathbf{r}_{ij} + \alpha_{ikj}^{(15)}\mathbf{r}_{ik}]$$

$$+\sum \frac{1}{r_{ij}r_{jk}M_i(I)}[\alpha_{jik}^{(16)}\mathbf{r}_{ij} + \alpha_{jik}^{(17)}\mathbf{r}_{jk}]$$

and the addition of the total time derivative of:

$$\begin{aligned} Q = \sum \frac{1}{r_{ij}} [(\mathbf{v}_i\cdot\mathbf{r}_{ij})(q_{ij}^{(1)}v_i^2 + q_{ij}^{(2)}v_j^2 + q_{ij}^{(3)}(\mathbf{v}_i\cdot\mathbf{v}_j) + q_{ij}^{(4)}(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^2 \\ + q_{ij}^{(5)}(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})] \\ + r_{ij}^2(q_{ij}^{(6)}(\mathbf{a}_i\cdot\mathbf{v}_i) + q_{ij}^{(7)}(\mathbf{a}_i\cdot\mathbf{v}_j) + q_{ij}^{(8)}(\mathbf{a}_i\cdot\mathbf{r}_{ij})) \\ + (\mathbf{a}_i\cdot\mathbf{r}_{ij})(q_{ij}^{(9)}(\mathbf{v}_i\cdot\mathbf{r}_{ij}) + q_{ij}^{(10)}(\mathbf{v}_j\cdot\mathbf{r}_{ij}))] \end{aligned}$$

$$+\sum \frac{1}{r_{ij}r_{ik}}[q_{ijk}^{(11)}(\mathbf{v}_i\cdot\mathbf{r}_{ij}) + q_{ijk}^{(12)}(\mathbf{v}_j\cdot\mathbf{r}_{ij}) + q_{ijk}^{(13)}(\mathbf{v}_k\cdot\mathbf{r}_{ij})]$$

The coordinates used in our form of $L^{(N+1)}$ are almost universally employed in first post-Newtonian analyses of gravity, but at second order, no consensus has been reached in the choice of coordinates. This transformation proves to be very useful in comparing specific theories expressed in specific gauges to the 2PPN Lagrangian obtained in this work. The Lagrangian in the new representation differs from the original by:

$$L' = L + \sum M_i(I)(\mathbf{v}_i\cdot\delta\mathbf{v}_i) \quad - \sum \frac{\Gamma_{ij}}{r_{ij}^3}(\mathbf{r}_{ij}\cdot\delta\mathbf{r}_{ij}) \quad + \frac{dQ}{dt}$$

where $\delta\mathbf{v}_i = d(\delta\mathbf{r}_i)/dt$ and $\delta\mathbf{r}_{ij} = \delta\mathbf{r}_i - \delta\mathbf{r}_j$. By choosing particular values for the parameters in $\delta\mathbf{r}_i$ and Q, unwanted terms in the Lagrangian can be eliminated, or a particular gauge can be selected.

Before imposing the LI and GLI constraints on the Lagrangian, it is useful to separate $L^{(N+2)}$ into two pieces -- a "linear-field" (LF) piece which is linear in the potentials, and a "non-linear" (NL) piece containing the rest:

$$L^{(N+2)}=L_{(LF)}+L_{(NL)}$$

where

$$L_{(LF)} \propto \frac{M^2}{r} v^4 \quad \text{and} \quad L_{(NL)} \propto \frac{M^3}{r^2} v^2 \text{ and } \frac{M^4}{r^3}.$$

The LI and GLI constraints can be imposed on $L_{(LF)}$ without reference to $L_{(NL)}$, and the calculations involved in imposing these constraints on $L_{(NL)}$ are greatly simplified when the preferred frame effects have been removed from $L_{(LF)}$.

<u>The Linear-Field Lagrangian</u>

After the acceleration terms have been removed with the appropriate coordinate choice, the linear-field Lagrangian becomes:

$$\begin{aligned} L_{(LF)} = & \sum M_i^{(6)}(I)(1/16)v_i^6 \qquad (15) \\ & + (1/2)\sum \frac{1}{r_i}[A_{ij}^{(1)}(v_i-v_j)^4 + A_{ij}^{(2)}v_i^2v_j^2 + A_{ij}^{(3)}(v_i^2+v_j^2)(v_i\cdot v_j) + A_{ij}^{(4)}(v_i\cdot v_j)^2 \\ & \quad + B_{ij}^{(1)}((v_i-v_j)\cdot\hat{r}_{ij})^2 + B_{ij}^{(2)}((v_i\cdot\hat{r}_{ij})^2v_j^2+(v_j\cdot\hat{r}_{ij})^2v_i^2) \\ & \quad + B_{ij}^{(3)}((v_i\cdot\hat{r}_{ij})^2+(v_j\cdot\hat{r}_{ij})^2)(v_i\cdot v_j) \\ & \quad + (v_i\cdot\hat{r}_{ij})(v_j\cdot\hat{r}_{ij})(B_{ij}^{(4)}(v_i^2+v_j^2) + B_{ij}^{(5)}(v_i\cdot v_j) \\ & \quad + C_{ij}^{(1)}((v_i-v_j)\cdot\hat{r}_{ij})^4 + C_{ij}^{(2)}(v_i\cdot\hat{r}_{ij})^2(v_j\cdot\hat{r}_{ij})^2 \\ & \quad + C_{ij}^{(3)}((v_i\cdot\hat{r}_{ij})^3(v_j\cdot\hat{r}_{ij})+(v_i\cdot\hat{r}_{ij})(v_j\cdot\hat{r}_{ij})^3)] \end{aligned}$$

where the coefficients $A_{ij}^{(n)}$, $B_{ij}^{(n)}$, and $C_{ij}^{(n)}$ are 2PPN coupling parameters between bodies i and j. If one of the bodies is considered to have negligible internal gravity so that its matter content is described by a single mass parameter M (its energy content in a metric theory), these parameters define mass parameters for the other body (Nordtvedt 1985). For example, in simple Newtonian gravity the analogous coupling parameter is $GM_i(G)M_j(G)$. In the limit that body i becomes a test body, its mass can be factored out leaving $GM_j(G)$. The generalization of the test body limit in post-Newtonian gravity is called the non-compact limit, thus:

$$\lim_{NCj} \frac{X_{ij}^{(n)}}{M_j} = x^{(n)} M_i(x^{(n)}) \qquad \lim_{NCj} \frac{\Gamma_{ij}^{(n)}}{M_j} = GM_i(G)$$

where $X_{ij}^{(n)}$ is any linear-field coefficient and $x^{(n)}$ is the PPN parameter associated with that coefficient. The terms involving only the velocity difference $(v_i\text{-}v_j)^n$ have been identified at the outset to facilitate imposition of the LI and GLI constraints.

The LI constraints are imposed first by requiring equation (10a) to hold, yielding:

$$\begin{aligned}
M_i^{(6)}(I) &= M_i(I) \\
A_{ij}^{(2)} + 2A_{ij}^{(3)} + A_{ij}^{(4)} &= -(1/8)\Gamma_{ij} \\
2A_{ij}^{(3)} + 2A_{ij}^{(4)} &= -(1+\gamma)M_i(I)M_j(I) + (1/4)\Gamma_{ij} \\
B_{ij}^{(2)} + B_{ij}^{(3)} &= 0 \\
2B_{ij}^{(3)} + B_{ij}^{(4)} &= -(1/4)\Gamma_{ij}
\end{aligned}$$

while requiring equation (10b) gives:

$$A_{ij}^{(3)} = (3/2)(1+\gamma)M_i(I)M_j(I) - (5/8)\Gamma_{ij}$$
$$2B_{ij}^{(3)} - B_{ij}^{(4)} = (1/2)(1+\gamma)M_i(I)M_j(I) - (1/8)\Gamma_{ij}$$
$$3B_{ij}^{(3)} + C_{ij}^{(3)} = 0$$
$$C_{ij}^{(2)} + 2C_{ij}^{(3)} = (3/8)\Gamma_{ij}$$

These two standard constraints provide eight equations, but there are nine coefficients to be constrained under imposition of LI. Thus, an "in-plane w" constraint is required to fully determine this part of the Lagrangian. The simplest procedure which accomplishes this involves the trajectory of an unbound body as it is deflected by a larger mass.

In a scattering experiment, the initial conditions measured far from the larger mass are the impact parameter, b, and the initial velocity, v_i. The final velocity, v_f, is the outcome which is also measured far from the scattering center. Since the initial conditions and the outcome of this experiment are all measured in the asymptotic gravity-free region, the final velocity as a function of v_i and b can be used as the observable quantity and the LI constraining techniques of Chapter Two employed. To linear order in the potential of the scattering center, the final velocity is calculated by integrating the acceleration of a body along its undeflected trajectory, giving the final velocity as a function of the initial conditions and the velocity of the scattering center:

$$v_f = v_f(b, v_i, v_s).$$

For simplicity, the velocity between the two observers is assumed to be normal to the initial velocity ($w \cdot v_i = 0$) and lies in the plane of the system which is defined by the trajectory of the deflected body and the location of the scattering center (which is assumed to have a much larger mass than the deflected body). For an observer at rest with respect to the scattering center, this LI constraint requires:

$$v_f(b(1-w^2)^{-1/2}, v_i(1-w^2)^{1/2} + w, w) = (1 + w \cdot v_f(0))^{-1}(\hat{w} \cdot v_f(0) + v_i(1-w^2)^{1/2} + w)$$

where $v_f(0)=v_f(b,v_i,0)$ and v_i and b are measured in the rest frame of the system. Imposing this constraint yields:

$$C_{ij}^{(3)} = 0.$$

These nine constraint equations now determine all the parameters which could produce preferred frame effects independent of the environment:

$$\begin{aligned}
A_{ij}^{(2)} &= -(1+\gamma)M_i(I)M_j(I) + (3/8)\Gamma_{ij} \\
A_{ij}^{(3)} &= (3/2)(1+\gamma)M_i(I)M_j(I) - (5/8)\Gamma_{ij} \\
A_{ij}^{(4)} &= -2(1+\gamma)M_i(I)M_j(I) + (3/4)\Gamma_{ij} \\
B_{ij}^{(4)} &= -(1/2)(1+\gamma)M_i(I)M_j(I) + (1/8)\Gamma_{ij} \\
B_{ij}^{(5)} &= (1+\gamma)M_i(I)M_j(I) - (1/2)\Gamma_{ij} \\
C_{ij}^{(2)} &= (3/8)\Gamma_{ij} \\
B_{ij}^{(2)} &= 0 \qquad B_{ij}^{(3)} = 0 \qquad C_{ij}^{(3)} = 0
\end{aligned}$$

The three remaining undetermined parameters are coefficients of terms involving velocity differences only:

$$\delta L = (1/2)\sum \frac{1}{r_{ij}} [A_{ij}^{(1)}(v_i-v_j)^4 + B_{ij}^{(1)}((v_i-v_j)\cdot\hat{r}_{ij})^2(v_i-v_j)^2 + C_{ij}^{(1)}((v_i-v_j)\cdot\hat{r}_{ij})^4]$$

and require use of the GLI constraint.

To impose the GLI constraint on the Lagrangian at this order, the spectator bodies are assumed at rest and their rescaling effects kept to first order in the potentials. At some intermediate region between the reduced system and the spectator bodies the metric is rescaled to first order in the spectator's potentials. The metric needed for rescaling comes from the 1PPN Lagrangian and is given by (Nordtvedt 1985):

$$g_{oo} = 1 - 2\sum \frac{M_s(G)}{R_s} \quad \text{and} \quad g_{ab} = -\left(1 + 2\sum \frac{1}{P_s}[(1+\gamma)M_s(I) - M_s(G)]\right)\delta_{ab} \qquad (16)$$

The coordinate rescaling which puts this metric into Minkowski form is:

$$t' = t(g_{oo})^{1/2}$$

and

$$r' = r(|g_{ab}|)^{1/2}$$

Following the procedure outlined in Chapter Two, the reduced Lagrangian gives:

$$M_i^{(4)}(I)' = M_i(I) + \sum \frac{1}{R_s}(8(A_{is}{}^{(1)} + B_{is}{}^{(1)}(\hat{v}_i \cdot \hat{R}_s)^2 + C_{is}{}^{(1)}(\hat{v}_i \cdot \hat{R}_s)^4) \\ - 4(1+\gamma)M_i(I)M_s(I) + M_i(I)M_s(G))$$

but we have from previous work (Nordtvedt 1985):

$$M_i^{(0)}(I)' = M_i(I) + \sum \frac{1}{R_s}(M_i(I)M_s(G) - \Gamma_{is}).$$

Imposing the LI constraints on this Lagrangian requires:

$$A_{ij}{}^{(1)} + B_{ij}{}^{(1)}(\hat{v}_i \cdot \hat{r}_{ij})^2 + C_{ij}{}^{(1)}(\hat{v}_i \cdot \hat{r}_{ij})^4 = (1/2)(1+\gamma)M_i(I)M_j(I) - (1/8)\Gamma_{ij}$$

which forces:

$$A_{ij}^{(1)} = (1/2)(1+\gamma)M_i(I)M_j(I) - (1/8)\Gamma_{ij}$$
$$B_{ij}^{(1)} = C_{ij}^{(1)} = 0$$

The 2PPN linear-field Lagrangian is then:

$$\begin{aligned} L = -\ &\Sigma M_i(I)(1 - (1/2)v_i^2 - (1/8)v_i^4 - (1/16)v_i^6) \\ &+ (1/2)\Sigma \frac{1}{r_{ij}}[\Gamma_{ij}(1 - (1/2)(v_i^2 + v_j^2 - v_i \cdot v_j) \\ &\qquad - (1/8)(v_i^4 + v_j^4 - v_i^2 v_j^2 + (v_i^2+v_j^2)v_i \cdot v_j - 2(v_i \cdot v_j)^2 \\ &\qquad - (1/2)(v_i \cdot \hat{r}_{ij})(v_j \cdot \hat{r}_{ij})(1 - (1/4)(v_i^2 + v_j^2 - 4v_i \cdot v_j)) \\ &\qquad - (3/8)(v_i \cdot \hat{r}_{ij})^2(v_j \cdot \hat{r}_{ij})^2) \\ &\qquad + (1+\gamma)M_i(I)M_j(I)((v_i - v_j)^2(1 - (1/2)(v_i \cdot \hat{r}_{ij})(v_j \cdot \hat{r}_{ij})) \\ &\qquad\qquad + (1/2)(v_i^4 + v_j^4 - (v_i^2+v_j^2)v_i \cdot v_j))] \end{aligned}$$

if it is to show no preferred frame effects even in the presence of spectators, i.e. if it is to have generalized Lorentz invariance.

The metric can be found by using this Lagrangian to obtain the equation of motion for a non-compact test body and comparing this with a geodesic equation, and identifying the metric components. In some coordinate systems this procedure cannot be implemented, but convenient coordinates can be chosen which put the Lagrangian for a test body into the explicit form of a line element (proper time interval):

$$\int L dt = \int \left(g_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} \right)^{1/2} dt.$$

The needed transformation is:

$$\delta \mathbf{r}_i = \sum \frac{1}{r_{ij}M_i(I)}((1/2)(1+\gamma)M_i(I)M_j(I)-(3/8)\Gamma_{ij})(v_j^2\mathbf{r}_{ij} - 2(\mathbf{v}_j\cdot\mathbf{r}_{ij})\mathbf{v}_i), \tag{17}$$

combined with the addition of the total time derivative of:

$$Q = -\sum \frac{1}{r_{ij}}((1/2)(1+\gamma)M_i(I)M_j(I)-(3/8)\Gamma_{ij})((\mathbf{v}_i\cdot\mathbf{r}_{ij})v_j^2 - (\mathbf{v}_j\cdot\mathbf{r}_{ij})v_i^2) \tag{18}$$

to the Lagrangian. The Lagrangian is altered by:

$$\begin{aligned}\delta L = (1/2)\sum \frac{1}{r_{ij}}[&-(3/8)\Gamma_{ij}(2v_i^2v_j^2 - (v_i^2+v_j^2)\mathbf{v}_i\cdot\mathbf{v}_j + (\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})(v_i^2+v_j^2) \\ &+ (\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^2v_j^2+(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})^2v_i^2) \\ &(1/2)(1+\gamma)M_i(I)M_j(I)(2v_i^2v_j^2 - (v_i^2+v_j^2)\mathbf{v}_i\cdot\mathbf{v}_j +(\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})(v_i^2+v_j^2) \\ &+ (\mathbf{v}_i\cdot\hat{\mathbf{r}}_{ij})^2v_j^2+(\mathbf{v}_j\cdot\hat{\mathbf{r}}_{ij})^2v_i^2)]\end{aligned}$$

and gives the "linear-field" line element for a metric:

$$g_{oo} = 1 - 2\sum \frac{1}{r_i}[M_i(G)(1-(1/2)v_i^2-(1/8)v_i^4) + (1+\gamma)M_i(I)v_i^2(1+(1/2)v_i^2)]$$

$$g_{oa} = \sum\frac{1}{r_i}[(1/2)M_i(G)(1+(1/2)v_i^2)(\mathbf{v}_i-(\mathbf{v}_i\cdot\hat{\mathbf{r}}_i)\hat{\mathbf{r}}_i) - 2(1+\gamma)M_i(I)(1+(1/2)v_i^2)\mathbf{v}_i]$$

$$\begin{aligned}g_{ab} = -(1 - 2\sum \frac{1}{r_i}&[M_i(G)(1+(3/8)(v_i^2-(\mathbf{v}_i\cdot\hat{\mathbf{r}}_i)^2)) \\ &- (1+\gamma)M_i(I)(1+(1/2)(v_i^2-(\mathbf{v}_i\cdot\hat{\mathbf{r}}_i)^2))])\delta_{ab} \\ -(1/2)\sum \frac{1}{r_i}&[M_i(G)(\mathbf{v}_i\mathbf{v}_i - (\mathbf{v}_i\cdot\hat{\mathbf{r}}_i)(\mathbf{v}_i\hat{\mathbf{r}}_i+\hat{\mathbf{r}}_i\mathbf{v}_i) + (3/4)(v_i^2+(\mathbf{v}_i\cdot\hat{\mathbf{r}}_i)^2)\hat{\mathbf{r}}_i\hat{\mathbf{r}}_i) \\ &- 2(1+\gamma)M_i(I)(v_i^2\hat{\mathbf{r}}_i\hat{\mathbf{r}}_i - (\mathbf{v}_i\cdot\hat{\mathbf{r}}_i)(\mathbf{v}_i\hat{\mathbf{r}}_i+\hat{\mathbf{r}}_i\mathbf{v}_i))]\end{aligned}$$

When written in these "metric" coordinates, the Lagrangian does not satisfy the constraint imposed in equation (10b) since the coordinate transformation introduces body-dependent coordinate alterations in the direction of any uniform velocity which is added to all bodies of the system.

The linear-field Lagrangian has now been constrained to show no preferred frame effects, and as a result it has been uniquely determined in terms of PPN parameters already present in the first order Lagrangian. *Generalized Lorentz invariance implies that there are no new degrees of freedom (i.e. PPN parameters) in the linearized gravitational interaction at the 2PN level.* This constrained Lagrangian can now be used in imposing the LI and GLI constraints on the non-linear Lagrangian.

The Non-Linear Lagrangian

The non-linear part of the 2PPN Lagrangian includes three and four body interactions. It is therefore possible that the expansion coefficients in a general $L_{(NL)}$ can be more complex than dimensionless numbers; they can be scalar functions of the multiple interbody vectors ($\mathbf{r}_{ij}$). The general acceleration independent non-linear Lagrangian can be written:

$$
\begin{aligned}
L_{(NL)} = \sum \frac{1}{r_{ij}r_{ik}} [& \Psi_{ijk}{}^{(1)} v_i{}^2 + \Psi_{ijk}{}^{(2)} v_j{}^2 + \Psi_{ikj}{}^{(2)} v_k{}^2 \\
& + \Psi_{ijk}{}^{(3)} \mathbf{v}_{ij}\cdot\mathbf{v}_{jk} - \Psi_{ikj}{}^{(3)} \mathbf{v}_{ik}\cdot\mathbf{v}_{jk} + \Psi_{ijk}{}^{(4)} \mathbf{v}_{ij}\cdot\mathbf{v}_{ik} \\
& + \mathbf{v}_i\cdot\mathbf{A}_{ijk}\cdot\mathbf{v}_i + \mathbf{v}_{ij}\cdot\mathbf{B}_{ijk}\cdot\mathbf{v}_{ij} + \mathbf{v}_{ik}\cdot\mathbf{B}_{ikj}\cdot\mathbf{v}_{ik} \\
& + \mathbf{v}_i\cdot\mathbf{C}_{ijk}\cdot\mathbf{v}_j + \mathbf{v}_i\cdot\mathbf{C}_{ikj}\cdot\mathbf{v}_k + \mathbf{v}_{ij}\cdot\mathbf{D}_{ijk}\cdot\mathbf{v}_{ik}]
\end{aligned}
$$

$$
+ \sum \frac{\Pi_{ijkl}}{r_{ij}r_{ik}r_{il}} \quad + \sum \frac{\Omega_{ijkl}}{r_{ij}r_{jk}r_{kl}}
$$

where A_{ijk}, B_{ijk}, C_{ijk}, and D_{ijk} are matrices built from outer products of the interbody unit vectors whose matrix elements are dimensionless scalar functions of the interbody vectors. Similarly $\Psi_{ijk}{}^{(n)}$, Π_{ijkl}, and Ω_{ijkl} are such scalar functions. The coordinate freedom:

$$\mathbf{r}_i' = \mathbf{r}_i + \sum \frac{1}{r_{ij} r_{ik} M_i(I)} (\alpha_{ijk}{}^{(1)} \mathbf{r}_{ij} + \alpha_{ikj}{}^{(1)} \mathbf{r}_{ik}) + \sum \frac{1}{r_{ij} r_{jk} M_i(I)} (\alpha_{jik}{}^{(2)} \mathbf{r}_{ij} + \alpha_{jik}{}^{(3)} \mathbf{r}_{jk})$$

eliminates some of the Lagrangian terms, leaving:

$$\begin{aligned} L_{(NL)} = \sum \frac{1}{r_{ij} r_{ik}} [&\Psi_{ijk}{}^{(1)} v_i{}^2 + \Psi_{ijk}{}^{(2)} \mathbf{v}_{ij} \cdot \mathbf{v}_{jk} - \Psi_{ikj}{}^{(2)} \mathbf{v}_{ik} \cdot \mathbf{v}_{jk} \\ &+ \mathbf{v}_i \cdot \mathbf{A}_{ijk} \cdot \mathbf{v}_i + \mathbf{v}_{ij} \cdot \mathbf{B}_{ijk} \cdot \mathbf{v}_{ij} + \mathbf{v}_{ik} \cdot \mathbf{B}_{ikj} \cdot \mathbf{v}_{ik} \\ &+ \mathbf{v}_i \cdot \mathbf{C}_{ijk} \cdot \mathbf{v}_j + \mathbf{v}_i \cdot \mathbf{C}_{ikj} \cdot \mathbf{v}_k + \mathbf{v}_{ij} \cdot \mathbf{D}_{ijk} \cdot \mathbf{v}_{ik}] \\ + \sum \frac{\Pi_{ijkl}}{r_{ij} r_{ik} r_{il}} &+ \sum \frac{\Omega_{ijkl}}{r_{ij} r_{jk} r_{kl}} . \end{aligned} \tag{19}$$

The coefficients have the following symmetries in their indices:

$$\Psi_{ijk}{}^{(1)} = \Psi_{ikj}{}^{(1)}; \quad \mathbf{A}_{ijk} = \mathbf{A}_{ikj}; \quad \mathbf{D}_{ijk} = \mathbf{D}^T{}_{ikj};$$
$$\Pi_{ijkl} = \Pi_{i(jkl)}; \quad \Omega_{ijkl} = \Omega_{lkji}$$

and A_{ijk} and B_{ijk} are symmetric matrices. Each of the matrix parameters can be expressed as a sum of four "basis" outer products as:

$$\begin{aligned} \mathbf{A}_{ijk} &= A_{ijk}{}^{(1)} \hat{\mathbf{r}}_{ij} \hat{\mathbf{r}}_{ij} + A_{ikj}{}^{(1)} \hat{\mathbf{r}}_{ik} \hat{\mathbf{r}}_{ik} + A_{ijk}{}^{(2)} (\hat{\mathbf{r}}_{ij} \hat{\mathbf{r}}_{ik} + \hat{\mathbf{r}}_{ik} \hat{\mathbf{r}}_{ij}) \\ \mathbf{B}_{ijk} &= B_{ijk}{}^{(1)} \hat{\mathbf{r}}_{ij} \hat{\mathbf{r}}_{ij} + B_{ijk}{}^{(2)} \hat{\mathbf{r}}_{ik} \hat{\mathbf{r}}_{ik} + B_{ijk}{}^{(3)} (\hat{\mathbf{r}}_{ij} \hat{\mathbf{r}}_{ik} + \hat{\mathbf{r}}_{ik} \hat{\mathbf{r}}_{ij}) \\ \mathbf{C}_{ijk} &= C_{ijk}{}^{(1)} \hat{\mathbf{r}}_{ij} \hat{\mathbf{r}}_{ij} + C_{ijk}{}^{(2)} \hat{\mathbf{r}}_{ik} \hat{\mathbf{r}}_{ik} + C_{ijk}{}^{(3)} \hat{\mathbf{r}}_{ij} \hat{\mathbf{r}}_{ik} + C_{ijk}{}^{(4)} \hat{\mathbf{r}}_{ik} \hat{\mathbf{r}}_{ij} \\ \mathbf{D}_{ijk} &= D_{ijk}{}^{(1)} \hat{\mathbf{r}}_{ij} \hat{\mathbf{r}}_{ij} + D_{ikj}{}^{(1)} \hat{\mathbf{r}}_{ik} \hat{\mathbf{r}}_{ik} + D_{ijk}{}^{(2)} \hat{\mathbf{r}}_{ij} \hat{\mathbf{r}}_{ik} + D_{ijk}{}^{(3)} \hat{\mathbf{r}}_{ik} \hat{\mathbf{r}}_{ij} \end{aligned}$$

with $A_{ijk}^{(2)}$, $D_{ijk}^{(2)}$ and $D_{ijk}^{(3)}$ symmetric under interchange of j and k. The coefficients are non-linear coupling parameters and also define various reduced coupling parameters in the non-compact limit of one or more of the bodies. In general, the non-linear coupling parameters are functions of the locations of all three bodies as well as the matter content of each body and therefore the reduced parameters retain information about the location of a body even after the information of its matter content has been removed by taking the non-compact limit. The notation of an underlined subscript is introduced to indicate which body is taken to the non-compact limit -- for example:

$$A_{\underline{i}jk}^{(1)} \equiv \frac{1}{M_i} \lim_{NCi} A_{ijk}^{(1)}.$$

Since a reduced coupling parameter still depends on the location of any non-compact bodies, their indices are kept, although *its dimension has been reduced by a power of* [M] *for each non-compact body* -- e.g.: $A_{\underline{i}jk}^{(1)}$ has dimension $[M]^2$ since all dependence on the matter content of body i has been removed. In the limit that all bodies become non-compact, there remain the "bare" coupling parameters:

$$X_{\underline{i}\,\underline{j}\,\underline{k}} = f_{ijk}(x)$$

where $X_{ijk}^{(n)}$ represents any of the three-body parameters. The bare coupling parameters for the four-body coefficients are:

$$\Pi_{\underline{i}\,\underline{j}\,\underline{k}\,\underline{l}} = f_{ijkl}(\pi) \text{ and } \Omega_{\underline{i}\,\underline{j}\,\underline{k}\,\underline{l}} = f_{ijkl}(\omega).$$

When the non-linear metric field $g_{\alpha\beta}(r)$ is calculated, the reduced coupling parameters will appear in the source potentials, and thus includes their dependence on the metric field variable (r). A test body placed at r, necessarily taken to the non-compact limit, will provide this information. The index "_" indicates the use of a generic test body, so that $X_{\_ij}(r) = X_{tij}$ with $r_t = r$.

Imposing the first LI constraint (eq. (10a)) on this Lagrangian gives:

$$\Psi_{ijk}^{(1)} = -(1/2)((1/2)-\beta)\Gamma_{ijk} \tag{20}$$

Since $\Psi_{ijk}^{(1)}$ is merely a constant, it simplifies the imposition of the second LI constraint (eq. (10b)) which is the primary reason for the gauge choice adopted for $L_{(NL)}$. The second LI constraint requires:

$$C_{ijk}^{(1)} = -[((1/2)-\beta)\Gamma_{ijk} - (1/8)\frac{r_{ij}}{r_{ik}}\frac{\Gamma_{ij}\Gamma_{ik}}{M_i(I)}(\hat{r}_{ij}\cdot\hat{r}_{ik})] \tag{21a}$$

$$C_{ijk}^{(3)} = +\frac{r_{ij}}{r_{ik}}[(1/2)(1+\gamma)\Gamma_{ik}M_j(I) - (1/8)\frac{\Gamma_{ij}\Gamma_{ik}}{M_i(I)}] \tag{21b}$$

$$C_{ijk}^{(2)} = C_{ijk}^{(4)} = 0 \tag{21c}$$

$$A_{ijk}^{(1)} = -(1/2)C_{ijk}^{(1)} \tag{21d}$$

$$A_{ijk}^{(2)} = -(C_{ijk}^{(3)} + C_{ikj}^{(3)}) \tag{21e}$$

The remaining coefficients, $\Psi_{ijk}^{(2)}$, B_{ijk}, and D_{ijk}, multiply terms which depend only on interbody velocities and therefore cannot be fixed by the LI constraints. The Lagrangian fulfilling equations (20) and (21) will show no preferred frame effects in the absence of spectator matter.

The GLI constraint is imposed on $L_{(NL)}$ in a two step process. The Lagrangian's body kinetic terms (which persist in the absence of gravity) must be examined to second order in the spectator potentials if new constraints are to be placed on $L_{(NL)}$. This requires knowledge of the metric (in particular g_{ab}) to this order. Since $L_{(NL)}$ is needed to evaluate g_{ab} at second order in the potentials, the non-linear Lagrangian should be as fully constrained as possible before this constraint is applied. Thus the GLI constraint is most effectively applied to $L_{(NL)}$ by first analyzing the two-body interaction terms and then applying the constraining techniques to the kinetic terms.

The GLI constraint requires that the reduced Lagrangian satisfy the LI constraints when written in the rescaled coordinates, although not necessarily in the standard gauge of $L^{(N+1)}$. The spectators themselves allow a further gauge freedom using the positions of the spectators relative to the reduced system. The GLI constraint is therefore applied to $L_{(NL)}$ by finding the reduced Lagrangian in the rescaled coordinates and requiring this to have the same form as the 1PPN Lagrangian (modulo a gauge transformation). The gauge transformation needed is:

$$\delta r_i = \Sigma \frac{1}{r_{ij} R_s M_i(I)} (\alpha_{ijs} r_{ij} + \beta_{ijs}(r_{ij} \cdot \hat{R}_s)\hat{R}_s)$$

where α_{ijk} and β_{ijk} are dimensionless scalar functions of r_{ij} and R_s. This modifies the two-body interaction terms in the Lagrangian by:

$$\delta L = \Sigma M_i(I) v_i \cdot \delta v_i.$$

The two-body interaction terms in the Lagrangian must have the form of:

$$L = (1/2)\Sigma \frac{1}{r_{ij}} [\Gamma'_{ij}(1 - (1/2)(v_i^2 + v_j^2 - v_i \cdot v_j + (v_i \cdot \hat{r}_{ij})(v_j \cdot \hat{r}_{ij}))) + (1+\gamma)G'M'_i(I)M'_j(I)(v_i - v_j)^2]$$

where Γ'_{ij}, $M'_i(I)$, γ', and G' are the values of those parameters which have been rescaled by the new environment which now includes the spectator bodies. Except for γ, all of these parameter rescalings have already been determined through imposing the GLI constraints on the 1PPN Lagrangian and are (Nordtvedt 1985):

$$\Gamma'_{ij} = \Gamma_{ij} + \sum \frac{1}{R_s}[(1+\gamma)\Gamma_{ij}M_s(I) + (1-2\beta)(\Gamma_{ijs}+\Gamma_{jis})]$$

$$M'_i(I) = M_i(I) + \sum \frac{1}{R_s}(M_i(I)M_s(G) - \Gamma_{is}) .$$

$$G' = G + \sum \frac{1}{R_s}[(1+\gamma)M_s(I) + 2(1-2\beta)M_s(\beta)]$$

where G=1. Requiring the equality:

$$L_{REDUCED}=L+\delta L$$

gives:

$$\Psi_{isj}^{(2)} + \Psi_{jsi}^{(2)} = (1/2)(1+\gamma)\Gamma_{ij}M_s(I) \tag{22}$$

and

$$\begin{aligned} B_{isj} + B_{jsi} = & -(1/2)(1+\gamma)\Gamma_{ij}M_s(I)\hat{r}_{ij}\hat{r}_{ij} - (1/2)[(1/2)-\beta)(\Gamma_{ijs}+\Gamma_{jis})])\hat{R}_s\hat{R}_s \\ & + (1/16)\ \frac{R_s}{r_{ij}}\Gamma_{ij}\left(\frac{\Gamma_{is}}{M_i(I)} - \frac{\Gamma_{js}}{M_j(I)}\right)((\hat{r}_{ij}\cdot\hat{R}_s)\hat{R}_s\hat{R}_s+(\hat{r}_{ij}\hat{R}_s + \hat{R}_s\hat{r}_{ij})). \end{aligned} \tag{23}$$

It also defines how γ is rescaled in the presence of proximate matter:

$$\begin{aligned} (\gamma'-\gamma) = & -\sum \frac{1}{R_s}[2(1+\gamma)((1+\gamma)M_s(I) + M_s(G) + (1-2\beta)M_s(\beta)-(\Gamma_{js}/M_j(I)+\Gamma_{is}/M(I)) \\ & - (1/2)\Gamma_{ij}M_s(I)/(M_i(I)M_j(I))) - (1/2)(1-2\beta)(\Gamma_{ijs}+\Gamma_{jis})/M_i(I)M_j(I) \\ & + \{\Psi_{ijs}^{(2)}+\Psi_{jis}^{(2)}-\hat{r}_{ij}\cdot[2(B_{ijs}+B_{jis})+(D^T_{ijs}+D_{jis})]\cdot\hat{r}_{ij}\}/M_i(I)M_j(I)] \end{aligned} \tag{24}$$

Since $(\gamma'-\gamma)$ must be independent of the reduced system body coordinates $\hat{r}_i$, an additional constraint on the structure of B_{ijk}, D_{ijk} and $\Psi_{ijk}^{(2)}$ is obtained:

$$\Psi_{ijs}^{(2)}+\Psi_{jis}^{(2)}-\hat{r}_{ij}\cdot[2(B_{ijs}+B_{jis})+(D^T_{ijs}+D_{jis})]\cdot\hat{r}_{ij} = \lambda\Lambda_s M_i(I)M_j(I) \qquad (25)$$

where λ is a numerical parameter and Λ_s is a mass parameter. Equations (22), (23) and (25) do not actually provide specific values for the parameters $\Psi_{ijk}^{(2)}$ and B_{ijk} because these parameters are given in the limit that $|r_i-R_s|\approx|r_j-R_s|\approx R_s$.

In order to impose the GLI constraint at second order in the spectators' potentials, the metric must be obtained to that order. The "metric" coordinates (eqs. 17 and 18) which put $L_{(LF)}$ into the form of a line element also change the non-linear Lagrangian by:

$$\delta L_{(NL)} = -(1/2)\sum \frac{\Gamma_{ij}}{r_{ij}^3}(r_{ij}\cdot\delta r_{ij})$$

which is:

$$\delta L = -(1/2)\sum \frac{1}{r_{ik}^2}\left((1+\gamma)\Gamma_{ij}M_k(I) - 3\frac{\Gamma_{ij}\Gamma_{ik}}{M_i(I)}\right)[(\hat{r}_{ij}\cdot\hat{r}_{ik})v_k^2 - 2(v_i\cdot\hat{r}_{ij})(v_k\cdot\hat{r}_{ik})]$$

In these coordinates, the Lagrangian is of the form of a line element, but it does not provide a metric which is diagonal for non-moving bodies--consequently the coordinate change associated with the reduced Lagrangian will not be a simple coordinate rescaling. However, an additional coordinate change puts the metric into a diagonal form:

$$\delta r_i = \sum \frac{1}{r_{ij} r_{ik} M_i(I)} (2\alpha_{ijk} + \beta_{ijk} - \gamma_{ikj}) r_{ij} + \sum \frac{1}{r_{ij} r_{jk} M_i(I)} (\beta_{jik} r_{ij} + \gamma_{jik} r_{jk}) \tag{26}$$

where α_{ijk}, β_{ijk}, and γ_{ijk} are defined by:

$$\left(r_{ij} \frac{d}{dt} - (v_{ij} \cdot \hat{r}_{ij}) - \frac{r_{ij}}{r_{ik}} (v_{ik} \cdot \hat{r}_{ik}) \right) \alpha_{ijk} = ((1/2) C_{ijk}^{(1)} - C_{ikj}^{(3)})(v_{ij} \cdot r_{ij}) - C_{ijk}^{(3)} (v_{ij} \cdot \hat{r}_{ik}) - C_{ikj}^{(3)} (v_{ik} \cdot \hat{r}_{ij}) \tag{27}$$

$$2\left(r_{ij} \frac{d}{dt} - (v_{ij} \cdot \hat{r}_{ij}) - \frac{r_{ij}}{r_{ik}} (v_{ik} \cdot \hat{r}_{ik}) \right) \beta_{ijk} = -2B_{ijk}^{(1)} (v_{ij} \cdot \hat{r}_{ij}) - B_{ijk}^{(3)} (v_{ij} \cdot \hat{r}_{ik}) - D_{ijk}^{(1)} (v_{ik} \cdot \hat{r}_{ij}) - D_{ijk}^{(2)} (v_{ik} \cdot \hat{r}_{ik}) \tag{28}$$

$$2\left(r_{ij} \frac{d}{dt} - (v_{ij} \cdot \hat{r}_{ij}) - \frac{r_{ij}}{r_{ik}} (v_{ik} \cdot \hat{r}_{ik}) \right) \gamma_{ijk} = D_{ijk}^{(1)} (v_{ij} \cdot \hat{r}_{ij}) - D_{ijk}^{(3)} (v_{ij} \cdot \hat{r}_{ik}) + 2B_{ikj}^{(2)} (v_{ik} \cdot \hat{r}_{ij}) + B_{ikj}^{(3)} (v_{ik} \cdot \hat{r}_{ik}) \tag{29}$$

In "metric" coordinates, the non-linear part of the Lagrangian is:

$$\begin{aligned} L_{(NL)} = \sum \frac{1}{r_{ij} r_{ik}} \Big[& ((1/2) - \beta) \Gamma_{ijk} (1 - (1/2)(v_i^2 + (v_i \cdot \hat{r}_{ij})(v_j \cdot \hat{r}_{ij}) + (v_i \cdot \hat{r}_{ik})(v_k \cdot \hat{r}_{ik}))) \\ & + (1/8) \frac{r_{ij}}{r_{ik}} \frac{\Gamma_{ij} \Gamma_{ik}}{M_i(I)} ((\hat{r}_{ij} \cdot \hat{r}_{ik})(3v_j^2 + (v_i \cdot \hat{r}_{ij})(v_j \cdot \hat{r}_{ij})) - 6(v_i \cdot \hat{r}_{ik})(v_j \cdot \hat{r}_{ij}) \\ & \qquad + 2(v_i \cdot \hat{r}_{ik})(v_{jk} \cdot \hat{r}_{ik})) \\ & - (1/2) \frac{r_{ij}}{r_{ik}} (1+\gamma) \Gamma_{ik} M_j(I) ((\hat{r}_{ij} \cdot \hat{r}_{ik}) v_j^2 - 2(v_i \cdot \hat{r}_{ik})(v_j \cdot \hat{r}_{ij}) \\ & \qquad + 2(v_i \cdot \hat{r}_{ik})(v_{jk} \cdot \hat{r}_{ik})) \\ & + 2\alpha_{ijk}(v_i^2 - v_i \cdot v_j) + \beta_{ijk} v_{ij}^2 + 2\Psi_{ijk}^{(2)} v_{ij} \cdot v_{jk} - \gamma_{ijk} v_{ij} \cdot v_{ik} \\ & + \sum \frac{\Pi'_{ijkl}}{r_{ij} r_{ik} r_{il}} + \sum \frac{\Omega_{ijkl}}{r_{ij} r_{jk} r_{kl}} \end{aligned}$$

where

$$\Pi'_{ijkl} = \Pi_{ijkl} + \frac{\Gamma_{ij}}{M_i(I)} \left(\frac{r_{ik}}{r_{ij}} (2\alpha_{ikl} + \beta_{ikl} - \gamma_{ilk} + \frac{r_{il}}{r_{kl}} \beta_{kil}) \hat{r}_{ij} \cdot \hat{r}_{ik} + \frac{r_{kl}}{r_{ij}} \gamma_{kil} \hat{r}_{ij} \cdot \hat{r}_{kl} \right) \tag{30}$$

symmetrized under interchange of j,k, and l. To reduce the size of the metric, the term-wise symmetry under interchange of body labels in the sums over bodies in the metric has not been retained, (although the sums themselves are still independent of the labelling choice). In this form, the metric is:

$$
\begin{aligned}
g_{oo} = 1 &- 2\sum \frac{1}{r_i}\left[M_i(G)(1-(1/2)v_i^2-(1/8)v_i^4) + (1+\gamma)M_i(I)v_i^2(1+(1/2)v_i^2)\right] \qquad (31)\\
&- 2\sum \frac{1}{r_i r_j}\Big[((1/2)-\beta)\lambda_{ij} - (1/2)M_i(G)M_j(G)(1-(1-(3/4)\ \frac{r_i}{r_j}(\hat{r}_i\cdot\hat{r}_j))v_i^2 \\
&\qquad - (1+\gamma)M_i(I)M_j(G)(1+(1/2)\ \frac{r_i}{r_j}(\hat{r}_i\cdot\hat{r}_j))v_i^2 \\
&\qquad - 2\Psi_{\_ij}^{(2)}(v_i^2-v_i\cdot v_j) + \beta_{\_ij}v_i^2 - \gamma_{\_ij}v_i\cdot v_j\Big] \\
&- 4\sum \frac{1}{r_i r_{ij}}\Big[((1/2)-\beta)\lambda'_{ij}(1-(1/2)(v_i^2+(v_i\cdot\hat{r}_{ij})(v_j\cdot\hat{r}_{ij}))) \\
&\qquad + (1/4)(1+\gamma)M_i(G)M_j(I)\ \frac{r_{ij}}{r_i}(v_j^2-2(v_i\cdot\hat{r}_i)(v_j\cdot\hat{r}_{ij})+2(v_i\cdot\hat{r}_{ij})(v_{ij}\cdot\hat{r}_{ij})) \\
&\qquad - (1/16)\Gamma_{ij}\ \frac{M_i(G)}{M_i(I)}\frac{r_{ij}}{r_i}((\hat{r}_i\cdot\hat{r}_{ij})(v_j^2-(v_i\cdot\hat{r}_{ij})(v_j\cdot\hat{r}_{ij})) - 6(v_i\cdot\hat{r}_i)(v_j\cdot\hat{r}_{ij}) \\
&\qquad\qquad 2(v_i\cdot\hat{r}_{ij})(v_{ij}\cdot\hat{r}_{ij})) \\
&\qquad + \alpha_{i\_j}v_i^2 + \alpha_{ij\_}(v_i^2-v_i\cdot v_j) + (1/2)\beta_{i\_j}v_i^2 \\
&\qquad + (1/2)\beta_{ij\_}(v_i^2-v_j^2-v_i\cdot v_j) - \Psi_{i\_j}^{(2)}v_i\cdot v_j - \Psi_{ij\_}^{(2)}(v_j^2-v_i\cdot v_j) \\
&\qquad - (1/2)\gamma_{i\_j}(v_i^2-v_i\cdot v_j) - (1/2)\gamma_{ij\_}(v_i^2-v_i\cdot v_j)\Big] \\
&- 2\sum\Bigg[\frac{1}{r_i r_j r_k}(\Pi'_{\_ijk} - ((1/2)-\beta)\lambda_{ij}M_k(G)) + \frac{3}{r_i r_{ij} r_{ik}}\Pi'_{i\_jk} \\
&\qquad + \frac{2}{r_i r_j r_{jk}}(\Omega_{i\_jk} - ((1/2)-\beta)M_i(G)\lambda'_{jk}) + \frac{2}{r_i r_{ij} r_{jk}}\Omega_{\_ijk}\Bigg]
\end{aligned}
$$

$$g_{oa} = - \sum \frac{1}{r_i} [(1/2)M_i(G)(1+(1/2)v_i^2)(v_i-(v_i\cdot\hat{r}_i)\hat{r}_i) - 2(1+\gamma)M_i(I)(1+(1/2)v_i^2)v_i] \quad (32)$$

$$- \sum \frac{1}{r_i r_j} [- ((1/2)-\beta)\lambda_{ij}(v_i\cdot\hat{r}_i)\hat{r}_i$$

$$- (1/2)M_i(G)M_j(G)(v_i-(1+(1/4) \quad \frac{r_i}{r_j}(\hat{r}_i\cdot\hat{r}_j))(v_i\cdot\hat{r}_i)\hat{r}_i$$

$$-(3/2) \frac{r_i}{r_j}(v_i\cdot\hat{r}_i)\hat{r}_j+(1/4) \quad \frac{r_i}{r_j}(v_{ij}\cdot\hat{r}_i)\hat{r}_i)$$

$$+ (1+\gamma)M_i(I)M_j(G)(2v_i+ \quad \frac{r_i}{r_j}((3/4)(v_i\cdot\hat{r}_i)\hat{r}_j-(1/2)(v_{ij}\cdot\hat{r}_i)\hat{r}_i) - 2\alpha_{\_ij}v_i$$

$$+ 2(\Psi_{\_ij}^{(2)}+\Psi_{\_ji}^{(2)})v_i - 2\beta_{\_ij}v_i + (\gamma_{\_ij}+\gamma_{\_ji})v_i]$$

$$- \sum \frac{1}{r_i r_{ij}} [-(1/2)((1/2)-\beta)\lambda'_{ij}(v_i\cdot\hat{r}_i)\hat{r}_i$$

$$+ (1/16) \quad \frac{r_i}{r_{ij}}\Gamma_{ij} \quad \frac{M_i(G)}{M_i(I)}((\hat{r}_i\cdot\hat{r}_{ij})(v_i\cdot\hat{r}_i)+6(v_i\cdot\hat{r}_{ij})+2(v_i\cdot\hat{r}_i))\hat{r}_i$$

$$- (1/2) \quad \frac{r_i}{r_{ij}}(1+\gamma)\Gamma_{ij}((v_i\cdot\hat{r}_{ij})+(v_i\cdot\hat{r}_i)\hat{r}_i) - \alpha_{i\_j}v_i + \Psi_{i\_j}^{(2)}(v_i+v_j)$$

$$-\Psi_{ij\_}^{(2)}(v_i-v_j) - \beta_{i\_j}v_i + (1/2)(\gamma_{i\_j}+\gamma_{ij\_})(v_i-v_j)]$$

$$g_{ab} = - \delta_{ab}(1 - 2\sum \quad \frac{1}{r_i}[M_i(G)(1+(3/8)(v_i^2-(v_i\cdot\hat{r}_i)^2)) \quad (33)$$

$$- (1+\gamma)M_i(I)(1+(1/2)(v_i^2-(v_i\cdot\hat{r}_i)^2))]$$

$$- 2\sum \frac{1}{r_i r_j}[((1/2)-\beta)\lambda_{ij} - (1/2)M_i(G)M_j(G) + (1+\gamma)M_i(I)M_j(G)$$

$$- 2\alpha_{\_ij} - \beta_{\_ij} + \gamma_{\_ij}]$$

$$+ 4\sum \frac{1}{r_i r_{ij}}[- (1/2)((1/2)-\beta)\lambda'_{ij} - \quad (3/8) \frac{r_i}{r_{ij}}\Gamma_{ij} \quad \frac{M_i(G)}{M_i(I)}(\hat{r}_i\cdot\hat{r}_{ij})$$

$$+ \frac{r_i}{r_{ij}}(1/2)(1+\gamma)\Gamma_{ij}(\hat{r}_i\cdot\hat{r}_{ij}) - 2\Psi_{i\_j}^{(2)} + \beta_{i\_j}])$$

$$- 2\sum \frac{1}{r_i}[M_i(G)((1/4)v_iv_i + (3/8)(v_i^2+(v_i\cdot\hat{r}_i)^2)\hat{r}_i\hat{r}_i$$
$$- (1/4)(v_i\cdot\hat{r}_i)(v_i\hat{r}_i+\hat{r}_iv_i))$$
$$- (1/2)(1+\gamma)M_i(I)(v_i^2\hat{r}_i\hat{r}_i-(v_i\cdot\hat{r}_i)(v_i\hat{r}_i+\hat{r}_iv_i))].$$

If the spectators' motion is negligible, the appropriate coordinate change for second order rescaling is:

$$t'=t(g_{oo})^{1/2} \text{ and } r'=r(-g_{kl})^{1/2}$$

where

$$g_{oo} = 1 - 2\sum \frac{1}{r_i}M_i(G) - 2\sum\frac{1}{r_ir_j}[((1/2)-\beta)\lambda_{ij}-(1/2)M_i(G)M_j(G)] -4\sum\frac{1}{r_ir_{ij}}((1/2)-\beta)\lambda'_{ij}$$

$$g_{ab} = - \delta_{ab}(1 - 2\sum \frac{1}{r_i}[M_i(G) - (1+\gamma)M_i(I)]$$
$$- 2\sum\frac{1}{r_ir_j}[((1/2)-\beta)\lambda_{ij} - (1/2)M_i(G)M_j(G) + (1+\gamma)M_i(I)M_j(G)$$
$$- 2\alpha_{\_ij} - ß_{\_ij} + \gamma_{\_ij}]$$
$$+ 4\sum\frac{1}{r_ir_{ij}}[-(1/2)((1/2)-\beta)\lambda'_{ij} - (3/8)\frac{r_i}{r_{ij}}\Gamma_{ij}\frac{M_i(G)}{M_i(I)}(\hat{r}_i\cdot\hat{r}_{ij})$$
$$+ \frac{r_i}{r_{ij}}(1/2)(1+\gamma)\Gamma_{ij}(\hat{r}_i\cdot\hat{r}_{ij}) - 2\Psi_{i\_j}^{(2)} + ß_{i\_j}])$$

The rescaling of the kinetic terms in the reduced Lagrangian are then:

$$M_i'^{(0)}(I) = M_i(I) + \sum\frac{1}{R_s}(M_i(I)M_s(G)-\Gamma_{is})$$
$$+ \sum\frac{1}{R_sR_{s'}}[((1/2)-ß)(M_i(I)\lambda_{ss'}-\Gamma_{iss'})$$
$$+ M_{s'}(G)M_i(I)M_s(G)-(1/2)(M_{s'}(G)\Gamma_{is}+M_s(G)\Gamma_{is'})]$$
$$+ \sum\frac{1}{R_sR_{ss'}}[(1-2\beta)(M_i(I)\lambda'_{ss'}-\Gamma_{sis'})]$$

and

$$M_i'^{(2)}(I) = M_i(I) + \sum \frac{1}{R_s}(M_i(I)M_s(G)-\Gamma_{is})$$

$$+ \sum \frac{1}{R_sR_{s'}}[((1/2)-ß)(M_i(I)\lambda_{ss'}-\Gamma_{iss'}) - 4(M_i(I)\alpha_{\underline{i}ss'}-\alpha_{iss'})$$
$$+ (M_{s'}(G)-2(1+\gamma)M_{s'}(I))(M_i(I)M_s(G)-\Gamma_{is})$$
$$- 2(M_i(I)ß_{\underline{i}ss'}-ß_{iss'}) + 2(M_i(I)\gamma_{\underline{i}ss'}-\gamma_{iss'})]$$

$$- \sum \frac{1}{R_sR_{ss'}}[(3/4)\frac{R_s}{R_{ss'}}(\hat{R}_s\cdot\hat{R}_{ss'})\frac{\Gamma_{ss'}}{M_s(I)}(M_i(I)M_s(G)-\Gamma_{is})$$
$$+ 4(M_i(I)\Psi_{s\underline{i}s'}{}^{(2)}-\Psi_{sis'}{}^{(2)})$$
$$- 2(M_i(I)ß_{s\underline{i}s'}-ß_{sis'})]$$

Generalized Lorentz invariance requires:

$$M_i'^{(0)}(I) = M_i'^{(2)}(I),$$

so,

$$(M_i(I)ß_{\underline{i}ss'}-ß_{iss'}) - (M_i(I)\gamma_{\underline{i}ss'}-\gamma_{iss'}) = -2(M_i(I)\alpha_{\underline{i}ss'}-\alpha_{iss'}) - (1+\gamma)M_{s'}(I))(M_i(I)M_s(G)-\Gamma_{is}) \tag{34}$$

and

$$4(M_i(I)\Psi_{s\underline{i}s'}{}^{(2)}-\Psi_{sis'}{}^{(2)}) - 2(M_i(I)\beta_{s\underline{i}s'}-\beta_{sis'}) = (1-2\beta)(M_i(I)\lambda'_{ss'}-\Gamma_{sis'}) + [(3/4)\frac{R_s}{R_{ss'}}(\hat{R}_s\cdot\hat{R}_{ss'})\frac{\Gamma_{ss'}}{M_s(I)}](M_i(I)M_s(G)-\Gamma_{is}) \tag{35}$$

A further constraint is the requirement that the Newtonian interaction remain isotropic in the presence of spectators. This restricts the form of the bare coupling parameters. If the rescaled Newtonian parameter (Γ_{ij}) is to be isotropic to second order in the spectators' potentials, then $3(\Pi'_{ijss'}+\Pi'_{jiss'})+2\Omega_{sijs'}$ αvδ $2(\Omega_{ijss'}+\Omega_{jiss'})$ must be independent of the body coordinate r_{ij}. Imposing isotropy on the 1PPN coupling parameter (Γ_{ijk}) to first order in the spectators' potentials forces $3\Pi'_{ijks}+(1/2)(\Omega_{jiks}+\Omega_{kijs}+\Omega_{jkis}+\Omega_{kjis})$ to be independent of the location of all four bodies. The non-compact limit of these expressions will be used later, so we here define:

$$3(\Pi'_{\underline{i}\underline{j}ss'}+\Pi'_{\underline{j}\underline{i}ss'})+2\Omega_{s\underline{i}\underline{j}s'} = F_{ss'}[(\hat{R}_s\cdot\hat{R}_{s'}),(\hat{R}_s\cdot\hat{R}_{ss'}),(\hat{R}_{s'}\cdot\hat{R}_{ss'})] \tag{36}$$

$$2(\Omega_{\underline{i}\underline{j}ss'}+\Omega_{\underline{j}\underline{i}ss'}) = G_{ss'}[(\hat{R}_s\cdot\hat{R}_{s'}),(\hat{R}_s\cdot\hat{R}_{ss'}),(\hat{R}_{s'}\cdot\hat{R}_{ss'})] \tag{37}$$

where $F_{ss'}$ and $G_{ss'}$ are coupling functions involving only the spectators, and:

$$3\Pi'_{\underline{i}\underline{j}\underline{k}s} + (1/2)(\Omega_{\underline{j}\underline{i}\underline{k}s} + \Omega_{\underline{k}\underline{i}\underline{j}s} + \Omega_{\underline{j}\underline{k}\underline{i}s} + \Omega_{\underline{k}\underline{j}\underline{i}s}) = HM_s(H) \tag{38}$$

where H is a numerical parameter, independent of the body locations and $M_s(H)$ is the mass associated with that parameter.

Although specific values of the PPN parameters $\Psi_{ijk}^{(2)}$, B_{ijk} ,and D_{ijk} have not been given, the two constraints (eqs. 34 and 35) provide constraints on two of these parameters in their general form (unlike equations (22), (23) and (25) which only give limiting values). The gravitational interaction which shows no preferred frame effects is described by the 2PPN parameters:

$$\Psi_{ijk}^{(2)},\ \beta_{ijk},\ \gamma_{ijk},\ \Pi'_{ijkl},\ \text{and}\ \Omega_{ijkl}$$

which are subject to the constraints given by equations (34-38). Since the Newtonian gravitational interaction has been shown to be isotropic to a part in 10^{13} and the sun's potential is of order 10^{-6}, the 2PPN coupling parameters (Π_{ijkl} and Ω_{ijkl}) must produce isotropic rescaling to about ten percent. The 2PPN parameters are related to the environmental rescaling of the 1PPN parameters γ and ß and the gravitational constant G. The rescaling of γ is given in equation (24) and the rescalings of ß and G are:

$$(\beta'-\beta) = -\sum \frac{1}{R_s}\left[\left((1/2)-\beta\right)\left(M_s(G)-2(1+\gamma)M_s(I)\right) + HM_s(H)\right] \tag{39}$$

$$\begin{aligned}(G'-G) = {} & \sum \frac{1}{R_s}\left[(1+\gamma)M_s(I) + 2(1-2\beta)M_s(\beta)\right] \\ & +\sum \frac{1}{R_s R_{s'}}\Big[(1+\gamma)M_{s'}(I)\big(M_s(G)-(1/2)(1+\gamma)M_s(I)+2(1-2\beta)M_{s'}(\beta)\big) + F_{ss'} \\ & \qquad + 2\alpha_{ss'} + \beta_{ss'} - \gamma_{ss'}\Big] \\ & +\sum \frac{1}{R_s R_{ss'}}\Bigg[\frac{R_s}{R_{ss'}}\Gamma_{ss'}\left((1+\gamma) - \frac{3}{16}\frac{M_s(G)}{M_s(I)}\right)(\hat{R}_s\cdot\hat{R}_{ss'}) \\ & \qquad + \Psi_{ss'}^{(2)} + (1/2)\beta_{ss'} + G_{ss'}\Bigg].\end{aligned} \tag{40}$$

The Strong Equivalence Principle

The strong equivalence principle constrains the PPN Lagrangian to be independent of an environment of spectators. When applied to the Lorentz invariant 1PPN Lagrangian, the SEP requires (Nordtvedt 1985):

$$4\beta - 3 - \gamma = 0 \tag{41}$$

which has been observationally verified to a part in 10^3. The SEP has the weakest experimental support of any of the invariances discussed previously. Because there is no decisive evidence that it must hold at second PN order, the second order consequences of imposing the SEP to 1PN order will be discussed before it is applied to the 2PPN Lagrangian.

If the SEP is valid only to 1PN order, then equation (41) is satisfied and therefore the Newtonian and 1PPN coupling parameters reduce to products of inertial masses. This requires that the linear-field 2PPN coupling parameters be similarly reduced, so that:

$$X_{ij}^{(n)} = x^{(n)} M_i(I) M_j(I)$$

with

$$\begin{array}{llllll}
a^{(1)} & = (3/8)+(\gamma/2) & b^{(4)} & = -((3/8)+(\gamma/2)) & c^{(2)} & = 3/8 \\
a^{(2)} & = -((5/8)+\gamma) & b^{(5)} & = (1/2)+\gamma & & \\
a^{(3)} & = (7/8)+(3\gamma/2) & & & & \\
a^{(4)} & = -((5/4)+2\gamma). & & & &
\end{array}$$

The non-linear parameters, $\Psi_{ijk}^{(1)}$, A_{ijk}, and C_{ijk}, must be:

$$\begin{aligned}
\Psi_{ijk}^{(1)} &= (1/8)(1+\gamma)M_i(I)M_j(I)M_k(I) \\
C_{ijk}^{(1)} &= (1/8)(2(1+\gamma) - (r_{ij}/r_{ik})(\hat{r}_{ij}\cdot\hat{r}_{ik}))M_i(I)M_j(I)M_k(I) \\
C_{ijk}^{(3)} &= ((3/8)+(\gamma/2))(r_{ij}/r_{ik})M_i(I)M_j(I)M_k(I) \\
A_{ijk}^{(1)} &= -(1/2)C_{ijk}^{(1)} \\
A_{ijk}^{(2)} &= -(C_{ijk}^{(3)}+C_{ikj}^{(3)}).
\end{aligned}$$

The constraint equations required by GLI become:

$$\Psi_{isj}^{(2)}+\Psi_{jsi}^{(2)} = (1/2)(1+\gamma)M_i(I)M_j(I)M_s(I) \tag{42a}$$

$$B_{isj}+B_{jsi} = -(1/8)(1+\gamma)(4\hat{r}_{ij}\hat{r}_{ij} - \hat{R}_s\hat{R}_s)M_i(I)M_j(I)M_s(I) \tag{42b}$$

$$(M_i(I)\beta_{\underline{i}ss'}-\beta_{iss'}) - (M_i(I)\gamma_{\underline{i}ss'}-\gamma_{iss'}) = -2(M_i(I)\alpha_{\underline{i}ss'}-\alpha_{iss'} \tag{42c}$$

$$4(M_i(I)\Psi_{s\underline{i}s'}^{(2)}-\Psi_{sis'}^{(2)}) - 2(M_i(I)\beta_{s\underline{i}s'}-\beta_{sis'}) = 0, \tag{42d}$$

and the rescalings of G, γ, and ß are:

$$(G'-G) = -\sum \frac{1}{R_s R_{s'}} [(1/2)(1+\gamma)(1+3\gamma)M_s(I)M_{s'}(I) - F_{ss}' - 2\alpha_{\_ss'} - ß_{\_ss'} + \gamma_{\_ss'}] \quad (43a)$$

$$+\sum \frac{1}{R_s R_{ss'}} \left[\frac{R_s}{R_{ss'}} ((1/8)+(\gamma/2))M_s(I)M_{s'}(I)(\hat{R}_s \cdot \hat{R}_{ss'}) + \Psi_{s\_s'}^{(2)} + (1/2)\beta_{s\_s'} + G_{ss'} \right]$$

$$(\gamma' - \gamma) == -\sum \frac{1}{R_s} [(1+\gamma)^2 M_s(I) + \lambda\Lambda_s] \quad (43b)$$

$$(ß'-ß) = -\sum \frac{1}{R_s} [(1/4)(1+\gamma)(1+2\gamma)M_s(I)) + HM_s(H)]. \quad (43c)$$

If the SEP holds at 2PN order, then the rescalings of G, γ and ß must vanish, so:

$$\lambda\Lambda_s = -(1+\gamma)^2 M_s(I) \quad (44a)$$

$$F_{ss'} = (1/2)(1+\gamma)(1+3\gamma)M_s(I)M_{s'}(I) - 2\alpha_{\_ss'} - \beta_{\_ss'} + \gamma_{\_ss'} \quad (44b)$$

$$G_{ss'} = (1/8)(1+4\gamma)(R_s/R_{ss'})(\hat{R}_s \cdot \hat{R}_{ss'}) + 2\Psi_{s\_s'}^{(2)} - \beta_{s\_s'} \quad (44c)$$

$$H = -(1/4)(1+\gamma)(1+2\gamma) \quad (44d)$$

and $M_s(H) = M_s(I)$. These give limiting constraints on the free parameters remaining after Lorentz invariance. Since the inertial mass is rescaled to third order in the spectators' potentials by:

$$\delta M'^{(0)}_i(I) = \sum \frac{1}{R_s R_{s'} R_{s''}} [M_i(I)\lambda_{ss's''}(\pi') - \Pi'_{iss's''} + (M_i(I)M_s(G) - \Gamma_{is})M_{s'}(G)M_{s''}(G) + (1-2\beta)(M_i(I)\lambda_{ss'}M_{s''}(G) - (1/2)\Gamma_{is}\lambda_{s's''} - (1/2)\Gamma_{iss'}M_{s''}(G))]$$

$$+\sum \frac{2}{R_s R_{s'} R_{ss''}} [M_i(I)\lambda'_{s'ss''}(\omega) - \Omega_{s'iss''} + (1-2\beta)(M_i(I)\lambda'_{ss''}M_{s'}(G)-(1/2)\Gamma_{is'}\lambda'_{ss''}-(1/2)\Gamma_{sis''}M_{s'}(G))]$$

$$+\sum \frac{3}{R_s R_{ss'} R_{ss''}} [M_i(I)\Pi'_{\underline{si}\,s's''} - \Pi'_{sis's''}]$$

$$+\sum \frac{2}{R_s R_{ss'} R_{s's''}} [M_i(I)\Omega_{\underline{i}\,ss's''} - \Omega_{iss's''}].$$

and the SEP requires $M'(I) = M(I)$, then:

$$\Pi'_{jikl} = M_i(I)\Pi'_{\underline{ji}\,kl}$$
$$\Omega_{ijkl} = M_i(I)\Omega_{\underline{i}\,jkl}.$$

Taking the non-compact limit of various spectators gives:

$$\Pi'_{ijkl} = f_{ijkl}(\pi')M_i(I)M_j(I)M_k(I)M_l(I)$$
$$\Omega_{ijkl} = f_{ijkl}(\omega)M_i(I)M_j(I)M_k(I)M_l(I).$$

If it is further assumed that the four-body gravitational interaction is simple, i.e.:

$$f_{ijkl}(\pi') = \pi'$$
$$f_{ijkl}(\omega) = \omega$$

then equations (44b-d) can be solved, giving:

$$\pi' = (1/3)[(1+\gamma)((3/4)+2\gamma) - (2f_{\_ss'}(\alpha)+f_{\_ss'}(\beta)-f_{\_ss'}(\gamma))] \qquad (45a)$$

$$\omega = -(1/4)(1+\gamma)(2+5\gamma) + (1/2)(2f_{\_ss'}(\alpha)+f_{\_ss'}(\beta)-f_{\_ss'}(\gamma)) \qquad (45b)$$

$$(2f_{\_ss'}(\alpha)+f_{\_ss'}(\beta)-f_{\_ss'}(\gamma)) - (2f_{s\_s'}(\Psi^{(2)})-f_{s\_s'}(\beta)) = \qquad (45c)$$
$$(1/2)(1+\gamma)(2+5\gamma) + (1/8)(1+4\gamma)(R_s/R_{ss'})(\hat{R}_s\cdot\hat{R}_{ss'})$$

which requires the combination $(2f_{\_ss'}(\alpha)+f_{\_ss'}(\beta)-f_{\_ss'}(\gamma))$ to be a numerical parameter:

$$(2f_{\_ss'}(\alpha)+f_{\_ss'}(\beta)-f_{\_ss'}(\gamma)) = \sigma.$$

Furthermore, the functional dependence of $(f_{s\_s'}(\beta)-2f_{s\_s'}(\Psi^{(2)}))$ must be:

$$(1/8)(1+4g)(R_s/R_{ss'})(\hat{R}_s\cdot\hat{R}_{ss'}).$$

This leaves only γ_{ijk} with no restrictions. Thus, under the imposition of the strong equivalence principle and simplicity of Π_{ijkl} and Ω_{ijkl}, the number of free parameters in the 2PPN Lagrangian reduces to two -- γ and γ_{ijk} where γ_{ijk} is defined in equation (29). All other parameters and bare coupling functions are at least partially fixed by equations (42) and (45).

The Two-Body Lagrangian

For many applications of this Lagrangian, the system under analysis will be a two-body system, in which case the coupling parameters of the non-linear part of the Lagrangian all reduce to numerical parameters. The two-body limit of the non-linear Lagrangian (eq. 19) is:

$$\begin{aligned} L_{(NL)} = \frac{1}{r^2}\,[&((1/2)-\beta)\Gamma_{122} + \Psi_{122}^{(1)}v_1^{\,2} \\ &+ v_1\cdot A_{122}\cdot v_1 + 2v_1\cdot C_{122}\cdot v_2 + v_{12}\cdot(2B_{122}+D_{122})\cdot v_{12}] \\ + \frac{1}{r^3}\,[&\Pi_{1222} + \Omega_{1212}] \qquad + 1\leftrightarrow 2, \end{aligned}$$

and the coordinate transformation which diagonalizes the metric (eq. 26) becomes:

$$\delta r_1 = \frac{1}{M_1(I)r^2}\,[(2\alpha_{122}+\beta_{122}-\gamma_{122}) + (\beta_{211}-\gamma_{211})]r$$

with:

$\alpha_{122} = -(1/4)(C_{122}^{(1)}+2C_{122}^{(3)})$ and $\beta_{122}-\gamma_{122} = (1/2)\hat{r}\cdot(2B_{122}+D_{122})\cdot\hat{r}$.

If these new coordinates are adopted and Υ_{122} is defined to be $\beta_{122}-\gamma_{122}$, then the non-linear Lagrangian is:

$$L_{(NL)} = \frac{1}{r^2}[((1/2)-\beta)\Gamma_{122} + \Psi_{122}^{(1)}v_1^2 - ((1/2)C_{122}^{(1)}+C_{122}^{(3)})(v_1^2-v_1\cdot v_2) + \Upsilon_{122}v_{12}^2 + (C_{122}^{(1)}+2C_{122}^{(3)})(v_1\cdot\hat{r})(v_2\cdot\hat{r})$$
$$+ \frac{1}{r^3}[(\Pi_{1222} + (\Gamma_{12}/M_1(I))((1/2)C_{122}^{(1)}+C_{122}^{(3)}-\Upsilon_{122})) + (\Omega_{1212} - (\Gamma_{12}/M_1(I))\Upsilon_{211})] \qquad + 1\leftrightarrow 2.$$

The LI constraints give:

$$C_{122}^{(1)}+2C_{122}^{(3)} = -((1/2)-\beta)\Gamma_{122} - (1/8)((\Gamma_{12})^2/M_1(I)) + ((1+\gamma)/2)\Gamma_{12}M_2(I).$$

When both spectators are taken to the non-compact limit, the applicable GLI constraint (eq. 34) requires:

$$2\upsilon(M(\upsilon)-M(I)) = (\beta-(1/2))(M(\beta)-M(I)) - \gamma(M(G)-M(I)) +(9/8)(M(I)-(M(G)^2/M(I)),$$

where $\Upsilon_{1\underline{22}} = \upsilon M_1(\upsilon)$. If the strong equivalence principle is imposed at 1PN order only, then:

$$C_{122}^{(1)}+2C_{122}^{(3)} = (1/16)(9+10\gamma)M_1(I)M_2(I)^2$$
$$\Upsilon_{122} = \upsilon M_1(I)M_2(I)^2,$$

and the non-linear Lagrangian is:

$$L_{(NL)} = \frac{M_1M_2^2}{r^2}[((1/2)-\beta)(1 - (1/2)v_1\cdot v_2 - (v_1\cdot\hat{r})(v_2\cdot\hat{r})) + \upsilon(v_1-v_2)^2 - ((7/16)+(\gamma/2))(v_1^2 - v_1\cdot v_2 - 2(v_1\cdot\hat{r})(v_2\cdot\hat{r})$$

$$+\frac{1}{r^3}[\Pi'_{1222}+\Omega'_{1212}] \qquad\qquad +1\leftrightarrow 2,$$

where

$$\Pi'_{1222} = \Pi_{1222} + (\Gamma_{12}/M_1(I))((1/2)C_{122}^{(1)}+C_{122}^{(3)}-\Upsilon_{122})$$
$$\Omega'_{1212} = \Omega_{1212} - (\Gamma_{12}/M_1(I))\Upsilon_{211}.$$

When the SEP is imposed to 2PN order, the parameters Π'_{1222} and Ω'_{1212} are given by:

$$\Pi'_{1222} = \pi M_1(I)M_2(I)^3$$
$$\Omega'_{1212} = \omega M_1(I)^2 M_2(I)^2$$

with:

$$\pi = (1/3)[(1+\gamma)((3/4)+2\gamma) + ((9/16)+(5\gamma/8)) - \upsilon]$$
$$\omega = -(1/4)[(1+\gamma)(2+5\gamma) - ((9/8)+(5\gamma/4)) + 2\upsilon],$$

leaving only two parameters, γ and υ.

Conclusion

Experiments are now being designed to exploit new technology to perform solar system tests of gravitation to second order. Along with continued observation of the binary pulsar, future experimental gravity will provide tighter limits on the observed values of the PPN parameters. It is important that these experiments most effectively probe the gravitational interaction. This work has explored the 2PN gravitational interaction via a 2PPN many-body Lagrangian expansion. The freedom in the nature of the interaction (represented by 2PPN parameters in the Lagrangian) is restricted by the imposition of several empirically based invariances on the observables calculated from the Lagrangian. The initial freedom of twenty-one gauge independent parameters in the 2PN expansion is reduced to two partially determined parameters and three free parameters under a generalized Lorentz invariance which requires that no preferred frame effects be detected in a system which may be in an environment of spectator matter. This invariance is supported by observation to an accuracy ranging from a part in 10^5 to one in 10^7. When the isotropy of the first order, static gravitational interaction is also required (which is supported to a part in 10^{13}), two of the remaining three free parameters are partially restricted. The four partially determined parameters are futher constrained by the imposition of the strong equivalence principle to 1PN order, which has been confirmed to a tenth of a percent. Although at 2PN order the evidence for the SEP is minimal and of order ten percent for isotropy, these two invariances are also imposed on the Lagrangian and further restrict three of the partially determined parameters while fixing two of them with the assumption of a simple scalar form for the four-body parameters. The Lagrangian which fulfills all of these invariances has the form:

$$L = - \sum M_i(I)(1 - (1/2)v_i^2 - (1/8)v_i^4 - (1/16)v_i^6)$$

$$\begin{aligned}+ (1/2)\sum \frac{M_iM_j}{r_{ij}} [1 - &(1/2)(v_i^2 + v_j^2 - v_i \cdot v_j) \\ &- (1/8)(v_i^4 + v_j^4 - v_i^2 v_j^2 + (v_i^2+v_j^2)v_i \cdot v_j - 2(v_i \cdot v_j)^2 \\ &- (1/2)(v_i \cdot \hat{r}_{ij})(v_j \cdot \hat{r}_{ij})(1 - (1/4)(v_i^2 + v_j^2 - 4v_i \cdot v_j)) \\ &- (3/8)(v_i \cdot \hat{r}_{ij})^2(v_j \cdot \hat{r}_{ij})^2) \\ &+ (1+\gamma)((v_i - v_j)^2(1 - (1/2)(v_i \cdot \hat{r}_{ij})(v_j \cdot \hat{r}_{ij})) \\ &\quad +(1/2)(v_i^4 + v_j^4 - (v_i^2+v_j^2)v_i \cdot v_j))]\end{aligned}$$

$$+ \sum \frac{M_i M_j M_k}{r_{ij} r_{ik}} [((1/2)-\beta)(1 - (1/2)(v_i^2 + (v_i \cdot \hat{r}_{ij})(v_j \cdot \hat{r}_{ij}) + (v_i \cdot \hat{r}_{ik})(v_k \cdot \hat{r}_{ik})))$$

$$+ (1/8) \quad \frac{r_{ij}}{r_{ik}} (v_i \cdot \hat{r}_{ij})(v_j \cdot \hat{r}_{ij}) + 2f_{ijk}(\alpha)(v_i^2 - v_i \cdot v_j)$$

$$- (2f_{ijk}(\Psi^{(2)}) - f_{ijk}(\beta))v_{ij}^2 + (2f_{ijk}(\Psi^{(2)}) - f_{ijk}(\gamma))v_{ij} \cdot v_{ik}]$$

$$- \sum \frac{M_i M_j M_k M_l}{r_{ij} r_{ik} r_{il}} (1/3)[(1+\gamma)((3/4)+2\gamma) - \sigma]$$

$$- \sum \frac{M_i M_j M_k M_l}{r_{ij} r_{jk} r_{kl}} (1/4)(1+\gamma)(2+5\gamma) + (1/2)\sigma].$$

The parameters $f_{ijk}(\alpha)$, $f_{ijk}(\beta)$, and $f_{ijk}(\gamma)$ are defined by:

$$\hat{O}f_{ijk}(\alpha) = - [(1/2)((1/2)-\beta) - (1/16) \quad \frac{r_{ij}}{r_{ik}} (\hat{r}_{ij} \cdot \hat{r}_{ik}) + \frac{r_{ik}}{r_{ij}} ((3/8)+(\gamma/2))](v_{ij} \cdot \hat{r}_{ij})$$

$$+ \frac{r_{ij}}{r_{ik}} [(3/8)+(\gamma/2)](v_{ij} \cdot \hat{r}_{ik}) + \quad \frac{r_{ik}}{r_{ij}} [(3/8)+(\gamma/2)](v_{ij} \cdot \hat{r}_{ik})$$

$$\hat{O}f_{ijk}(\beta) = (1/2)[-2B_{ijk}^{(1)}(v_{ij} \cdot \hat{r}_{ij}) - B_{ijk}^{(3)}(v_{ij} \cdot \hat{r}_{ik}) - D_{ijk}^{(1)}(v_{ik} \cdot \hat{r}_{ij}) - D_{ijk}^{(2)}(v_{ik} \cdot \hat{r}_{ik})]$$

$$\hat{O}f_{ikj}(\gamma) = (1/2)[D_{ijk}^{(1)}(v_{ij} \cdot \hat{r}_{ij}) - D_{ijk}^{(3)}(v_{ij} \cdot \hat{r}_{ik}) + 2B_{ikj}^{(2)}(v_{ik} \cdot \hat{r}_{ij}) + B_{ikj}^{(3)}(v_{ik} \cdot \hat{r}_{ik})]$$

where

$$\hat{O} \equiv \left(r_{ij} \quad \frac{d}{dt} - (v_{ij} \cdot \hat{r}_{ij}) - \quad \frac{r_{ij}}{r_{ik}} (v_{ik} \cdot \hat{r}_{ik}) \right).$$

The parameters $\Psi_{ijk}^{(2)}$ and B_{ijk} are constrained to satisfy:

$$\Psi_{isj}^{(2)} + \Psi_{jsi}^{(2)} = (1/2)(1+\gamma)M_i M_j M_s$$

$$B_{isj} + B_{jsi} = (1/4)(1+\gamma)M_i M_j M_s \hat{R}_s \hat{R}_s$$

$$\lambda\Lambda_s = -(1+\gamma)^2 M_s(I) = Y_{ijs}^{(2)} + \Psi_{jis}^{(2)}$$

$$- \hat{r}_{ij} \cdot [2(B_{ijs} + B_{jis}) + (D^T_{ijs} + D_{jis})] \cdot \hat{r}_{ij}.$$

For comparison, the 2PN Lagrangian for general relativity can be found in Damour and Schäfer (1985), although it is expressed in a different gauge.

The partially restricted parameters in the many-body Lagrangian are not known exactly because the additional complexity introduced by three bodies allows the parameters in the non-linear Lagrangian to be functions of the interbody vectors and thus the GLI constraints can only give limiting values of the parameters' functional form. In the two-body Lagrangian, this complexity disappears, allowing the two-body counterparts to the partially restricted parameters to be fully determined. The two-body Lagrangian has only two free parameters, γ (introduced at 1PN order) and υ (which first appears at 2PN order), if it refelects all of the invariances discussed in this work. Comparison of the two-body Lagrangian with the equivalent general relativity Lagrangian given by Grishchuk and Kopejkin (1986) reveals that the general relativity value for υ is (47/16). The spherically symmetric, static metric expanded to 2PN order depends on the new PPN parameter υ:

$$g_{oo} = 1 - 2\frac{M}{r} + (3+\gamma)/2\frac{M^2}{r^2} - [(3/8)+(7\gamma/4)+(4\gamma^2/3)-(2\upsilon/3)]\frac{M^3}{r^3}$$

$$g_{ab} = -\delta_{ab}[1 + 2\gamma\frac{M}{r} - [(13/8)+(11\gamma)/4 - 2\upsilon]\frac{M^2}{r^2}],$$

and therefore could be determined in second order light deflection experiments.

If the SEP does not hold at the 2PN order, then the remaining 2PPN parameters measure the rescaling of the Newtonian and 1PPN parameters:

$$(G'-G) = -\sum \frac{1}{R_s R_{s'}} [(1/2)(1+\gamma)(1+3\gamma)M_s(I)M_{s'}(I) - F_{ss}' - 2\alpha_{\_ss'} - \beta_{\_ss'} + \gamma_{\_ss'}]$$

$$+\sum \frac{1}{R_s R_{ss'}} \left[\frac{R_s}{R_{ss'}} ((1/8)+(\gamma/2))M_s(I)M_{s'}(I)(\hat{R}_s \cdot \hat{R}_{ss'}) + \Psi_{s\_s'}^{(2)} + (1/2)\beta_{s\_s'} + G_{ss'} \right]$$

$$(\gamma'-\gamma) = -\sum \frac{1}{R_s} [(1+\gamma)^2 M_s(I) + \lambda\Lambda_s]$$

$$(\beta'-\beta) = -\sum \frac{1}{R_s} [(1/4)(1+\gamma)(1+2\gamma)M_s(I)) + HM_s(H)].$$

Observational evidence for the SEP is fairly weak at second order. Since the linear-field Lagrangian is completely determined without invoking the SEP, this work shows that the most interesting physics in future studies of the gravitational interaction at second post-Newtonian order will be found in experiments which probe systems with strong gravitational potentials in which the consequences of 2PN order dynamics of bodies or light rays are measurable.

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