Dynamic constraints on Feynmans path integral
by Bung-Ning Teng

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of
MASTER OF SCIENCE in Physics
Montana State University

Abstract:
The Feynman's path integral in space-time is closely related, via classical Lagrangian theory, to Dirac's formulation of quantum mechanics. The essential point of this approach is in making use of the Lagrangian function in space-time and Hamilton's first action principle to construct the probability amplitude in conventional quantum mechanics. This formulation has, as mentioned by Feynman, mathematical and physical incompleteness. We have investigated a part of the physical incompleteness, which can be treated by using the classical technique of eliminating ignorable. coordinates.

The corresponding conjugate momenta will then provide constraint relations, which can be used to reduce the Lagrangian function and the path integral to a modified form. The paths are now restricted, via the constraint relations, to limited configuration spaces. A general modified formulation is constructed for paths in a limited space-time region. An important special case is demonstrated. In the special case, Jacobi's action principle is employed to replace Hamilton's action principle, therefore, the paths thus formulated are in space-energy rather than in space-time.

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ABSTRACT

The Feynman's path integral in space-time is closely related, via classical Lagrangian theory, to Dirac's formulation of quantum mechanics. The essential point of this approach is in making use of the Lagrangian function in space-time and Hamilton's first action principle to construct the probability amplitude in conventional quantum mechanics. This formulation has, as mentioned by Feynman, mathematical and physical incompleteness. We have investigated a part of the physical incompleteness, which can be treated by using the classical technique of eliminating ignorable coordinates. The corresponding conjugate momenta will then provide constraint relations, which can be used to reduce the Lagrangian function and the path integral to a modified form. The paths are now restricted, via the constraint relations, to limited configuration spaces. A general modified formulation is constructed for paths in a limited space-time region. An important special case is demonstrated. In the special case, Jacobi's action principle is employed to replace Hamilton's action principle, therefore, the paths thus formulated are in space-energy rather than in space-time.

I. INTRODUCTION

The path integral approach to quantum mechanics which has been developed by Dirac[1,2,3] and Feynman[4,5] represents a point of view which has proven useful for the solution of important physical problems.[6] Of equal importance, however, is the conceptual picture it provides of quantum mechanics. In conventional quantum mechanics one customarily starts with stationary states and obtains the dynamical behavior only through the application of time-dependent perturbation theory. By contrast, the primitive concept of Feynman's approach is the probability amplitude (or propagator) for the dynamical evolution of a system. Moreover, the quantum dynamics of Feynman is closely related to the dynamics of the corresponding classical system, since the classical Lagrangian is employed to construct the probability amplitude in its path integral form.

The solution of problems in classical mechanics is often facilitated by transforming the Lagrangian to eliminate the so-called ignorable coordinates which do not appear explicitly in the Lagrangian. The modified Lagrangian, thus obtained, can be used to obtain equations of motion for the reduced problem. It is the purpose of this work to discuss the manner in which this procedure of classical mechanics can be related to the corresponding quantum mechanical problem. Our method

employs the modified classical Lagrangian to construct a path integral for the probability amplitude. The discussion is restricted throughout to non-relativistic quantum mechanics and to systems with a classical analogue.

Our development proceeds in complete analogy with the logic of Dirac and Feynman. For this reason we have included in this introduction several logical outlines by means of which the analogy may be compared, step by step.

In Section II we review and summarize the relevant parts of the work of Dirac and of Feynman, in order to make clear the relation of classical mechanics to the original (or standard) space-time path integral method.

The correspondence relationship between the probability amplitude for infinitesimal times and the classical action was first derived by Dirac in 1933. For a finite time interval $t_b - t_a$, the correspondence relation can be extended to calculate the probability amplitude by means of the path integral method, which was formulated by Feynman in 1948. This probability amplitude (or kernel or propagator) in space-time is nothing but the Green's function of the time-dependent Schroedinger's equation in conventional quantum mechanics. We summarize in Outline I the logical relations which lead from classical mechanics to the probability amplitude in quantum mechanics.

Outline I

The Classical Motion can be considered as the continuous development of a contact transformation. Dirac's search for a quantum analogue of this led to a correspondence relation between the probability amplitude and the classical action, which is:

$$\langle q_b\ t_b | q_a\ t_a \rangle \sim \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} L(q,\dot{q};t)dt\right]$$

For an infinitesimal time interval, ε, this becomes

$$\langle q_b\ t_a + \varepsilon | q_a\ t_a \rangle \simeq \exp\left[\frac{i}{\hbar} L(q,\dot{q};t_a)\varepsilon\right]$$

For a finite time interval, $t_b - t_a$, the probability amplitude may also be calculated using the classical Lagrangian function by means of:

Feynman's Path Integral, which gives the probability amplitude as a sum over all paths from q_a, t_a to q_b, t_b. That is,

$$\langle q_b\ t_b | q_a\ t_a \rangle = \sum_{\text{all paths}} k \frac{1}{A^k} \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} L(q,\dot{q};t)dt\right]$$

The probability amplitude has the physical meaning of a propagator (or kernel) for the time development of the wave function:

$$\psi(q_b\ t_b) = \int \langle q_b\ t_b | q_a\ t_a \rangle\ \psi(q_a\ t_a)dq_a \ .$$

The probability amplitude is identical in:

Conventional Quantum Mechanics to the Green's function of the time-dependent Schroedinger's equation

$$(-\frac{\hbar}{i}\frac{\partial}{\partial t_b} - H_{q_b})\langle q_b\ t_b | q_a\ t_a\rangle = -\frac{\hbar}{i}\delta(q_b - q_a)\delta(t_b - t_a) \ .$$

This outline forms the basic theoretical formulation of the space-time path integral in dealing with quantum mechanics and other fields of physics. Section II is devoted to a more complete review of this standard approach.

There is an "incompleteness" of this formulation as mentioned in Feynman's original paper.[4] We quote his comments about the path integral method as follows:

> It is also incomplete from the physical standpoint. ...The present formulation, can be mathematically demonstrated to be invariant under the unitary transformation. However, it has not been formulated in such a way that it is physically obvious that it is invariant. No direct procedure has been outlined to describe measurements of quantities other than position. ... It is to be expected that the postulates can be generalized by the replacement of the idea 'paths in a region of space-time R' to 'paths of class R', or 'paths having property R.' But which properties correspond to which physical measurements has not been formulated in a general way.

This comment motivated the present work, which we hope may contribute something to completing and generalizing the path integral method.

Relating to another effort in this direction, an additional historical remark will be made. In recent years, Davies and Garrod[7,8] (1962, 1966) have followed the same ideas as Dirac and Feynman, but have used Hamilton's principle of the second kind to formulate the path integral in space-momentum rather than in space-time. This may be seen as one kind of generalization of the path integral to paths in "space-momentum" or phase space.

It is a well known fact in classical mechanics, that when a conjugate momentum is a constant of the motion, the corresponding coordinate is missing in the Lagrangian function. This missing coordinate is called an ignorable (or cyclic) coordinate. The physical meaning of this for the path integral method is that the conserved quantity, say momentum of some sort, will provide a constraint relation to limit "paths to possible regions which conserve that quantity."

In section III, we exploit this fact and show how this procedure may be carried out. The analysis is general in that we consider a general constant of the motion and do not specialize to a particular dynamical variable such as energy, angular momentum, etc.

Following the general method of eliminating ignorable coordinates and their conjugate momenta, the Lagrangian function and Hamilton's action principle are reduced to

modified forms which contain the non-ignorable coordinates and the corresponding momenta only. The conserved quantities provide constraint relations. The logical development of this procedure is summarized in Outline II, which may be compared with Outline I.

Outline II

The Modified Classical Motion can be considered as the continuous development of a contact transformation in a limited region (restricted by the constraint relations) of space-time. The quantum analogue of this leads to a correspondence relation between the modified probability amplitude and the modified action, which is

$$\langle \bar{q}_b\ t_b | \bar{q}_a\ t_a \rangle \sim \exp\left[\frac{i}{\hbar} \int_{t_a}^{t_b} \bar{L}(\bar{q},\dot{\bar{q}};t)dt\right]$$

where $\bar{q}$ are non-ignorable coordinates only.

For an infinitesimal time interval, ε, this becomes

$$\langle \bar{q}_b\ t_a+\varepsilon | \bar{q}_a\ t_a \rangle \approx \exp\left[\frac{i}{\hbar} \bar{L}(\bar{q},\dot{\bar{q}};t_a)\varepsilon\right] .$$

For a finite time interval, $t_b - t_a$, the modified probability amplitude may also be calculated using the modified Lagrangian function by means of:

The Modified Feynman's Path Integral, which gives the modified probability amplitude as a sum over limited paths from $\bar{q}_a$, t_a to $\bar{q}_b$, t_b. That is

$$\langle \bar{q}_b\ t_b | \bar{q}_a\ t_a \rangle = \sum_{\substack{k \\ \text{all limited} \\ \text{paths}}} \frac{1}{B^k} \exp\left[\frac{i}{\hbar} \int_{t_a}^{t_b} \bar{L}(\bar{q},\dot{\bar{q}};t)dt\right] .$$

The modified probability amplitude has the physical meaning of a propagator (or kernel) for the time-development of the wave function in limited regions

$$\psi(\bar{q}_b\ t_b) = \int \langle \bar{q}_b\ t_b | \bar{q}_a\ t_a \rangle\ \psi(\bar{q}_a\ t_a) d\bar{q}_a \quad .$$

The modified probability amplitude is identical in:

Conventional Quantum Mechanics to the modified Green's function of the time-dependent Schroedinger's equation

$$(-\frac{h}{i}\frac{\partial}{\partial t_b} - H\bar{q}_b)\langle \bar{q}_b\ t_b | \bar{q}_a\ t_a \rangle = -\frac{\hbar}{i}\delta(\bar{q}_b - \bar{q}_a)\delta(t_b - t_a)$$

where the ignorable coordinates are missing, and their conjugate momenta are constant.

Outline II gives the modified formulation of a limited space-time path integral (in regions restricted by conserved quantities) which is developed in detail in Section III.

In Section IV we treat an important special case of the modified path integral method using a special modified Lagrangian.

When the Hamiltonian of the system is a constant of the motion, the time "t" is an ignorable coordinate. Therefore, "t" may be eliminated as an independent variable. After the elimination of "t", the classical Lagrangian function is reduced to a function of space and energy, where the energy is a constant. Then we must use as the independent variable, an arbitrary parameter ν, which parameterizes the paths that conserve energy; i.e., "paths having the property that they conserve energy." Moreover, this parameter can be chosen to

be one of the space coordinates, which are now free from "t", but are functions of the parameter. In this case the modified Hamilton's principle, due to Jacobi, is called Jacobi's action principle of the first kind.

We again employ an outline to display the logic of the path integral analysis of this special case of energy conservation.

Outline III

The Modified Classical Path can be considered as the continuous development of a special contact transformation in a limited region of space-energy. The quantum analogue of this leads to a correspondence relation between the modified probability amplitude and the modified action, which is

$$\langle q_b\ \nu_b \mid q_a\ \nu_a\rangle \sim \exp\left[\frac{i}{\hbar}\int_{\nu}^{\nu} F(q,\dot{q},-E;\nu)d\nu\right];\ \dot{q} \equiv \frac{dq}{d\nu}$$

The time does not appear, but the energy appears as a constant. For an infinitesimal parameter interval, σ, this becomes

$$\langle q_b\ \nu_a+\sigma \mid q_a\ \nu_a\rangle \simeq \exp\left[\frac{i}{\hbar} F(q,\dot{q},-E;\nu_a)\sigma\right] .$$

For a finite parameter interval, $\nu_b-\nu_a$, the modified probability amplitude may also be calculated using the modified Lagrangian function by means of:

The Modified Feynman's Path Integral, which gives the energy conserving probability amplitude as a sum over limited paths (that conserve energy) from q_a, ν_a to q_b, ν_b. That is

$$\langle q_b\ \nu_b \mid q_a\ \nu_a\rangle = \sum_{\substack{k \\ \text{all limited} \\ \text{paths}}} \frac{1}{A^k} \exp\left[\frac{i}{\hbar}\int_{\nu_a}^{\nu_b} F(q,\dot{q},-E;\nu)d\nu\right].$$

The modified quantum amplitude has the physical meaning of the kernel for the spatial development of the wave function of constant energy,

$$\phi(q_b\ \nu_b) = \int \langle q_b\ \nu_b | q_a\ \nu_a \rangle\ \phi(q_a\ \nu_a) dq_a \quad .$$

The energy conserving probability amplitude is identical in:

Conventional Quantum Mechanics to the Green's function of the time-independent Schroedinger's equation:

$$(H - E_b)\langle q_b\ \nu_b | q_a\ \nu_a \rangle = \frac{\hbar}{i}\ \delta(q_b - q_a)\ \delta(E_b - E_a) \quad .$$

This special case of the energy-conserving spatial propagator is the subject of Section IV, and serves as a specific illustration of the use of a modified Lagrangian in the construction of a modified quantum amplitude.

In Section V, a conclusion is given.

II. THE FEYNMAN (OR ORIGINAL) PATH INTEGRAL IN SPACE-TIME

In 1933, Dirac took up the question of what corresponds in quantum mechanics to the Lagrange equations of classical mechanics. A little consideration shows, however, that one cannot expect to be able to take over the classical Lagrange equations in any very direct way. These equations involve partial derivatives of the Lagrangian with respect to the coordinates and velocities and no meaning can be given to such derivatives in quantum mechanics.

We must therefore, seek the quantum analogue of the Lagrangian theory in an indirect way. We must try to take over the ideas, but not the equations of the classical theory. Dirac's main point of view is briefly reviewed as follows:*

In classical mechanics, the action function S and the Lagrangian function L are related by the expression,

$$S = \int_{t_a}^{t_b} L(q_i, \dot{q}_i; t)\,dt \tag{2.1}$$

where q_i are the generalized coordinates and $\dot{q}_i$ are the generalized velocities, which are all functions of the time t.

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*For detail, see Dirac's original work cited in Ref. (1).

Hamilton's first action principle states that the system follows a path such that the variation of the action function (2.1) for fixed t_b and t_a is zero. In accordance with the method of the calculus of variations, with the variations $\delta q_i(t_b) = \delta q_i(t_a) = 0$ at the ends points, we then obtain from

$$\delta S = \delta \int_{t_a}^{t_b} L(q_i, \dot{q}_i; t)dt = 0 \qquad (2.2)$$

the Lagrange equations of motion in the familiar form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad . \qquad (2.3)$$

For an infinitesimal time interval, Lagrangian theory is closely related to the theory of contact transformations in classical mechanics. We shall therefore begin with a brief discussion of the analogue between classical and quantum contact transformations.

Let two sets of variables be q_i, p_i and Q_i, P_i* ($i=1,2,\ldots n$), and suppose the q's and Q's to be all independent, so that any function of the dynamical variables can be expressed in terms of them. In classical mechanics, the contact transformation equations for this case can be put in

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*In classical mechanics, q_i, Q_i represent usually two different coordinate systems, say rectangular and spherical, but here, we consider them as the same coordinate at different times. This will be seen more clearly as we follow the development .

the form

$$p_i = \frac{\partial S}{\partial q_i}, \quad P_i = -\frac{\partial S}{\partial Q_i} \tag{2.4}$$

where q_i, p_i are coordinates and momenta at time t_a say, and Q_i, P_i are coordinates and momenta at another time, say t_b. S is some function of the q's and Q's as defined in (2.1).

This connection of the action function with the contact transformation in classical mechanics can be carried over into quantum mechanics in the following way.

We take the dynamical variables in classical mechanics to correspond with the operators q, Q in quantum mechanics. In addition we consider two representations in which the q's and Q's are diagonal respectively. There will be a transformation function $\langle q' | Q' \rangle$ (or matrix element of a unitary matrix) connecting the two representations, which in the present case are actually the same representation taken at different times. The q', Q' are numerical values (eigenvalues) of the corresponding operators q, Q.

We shall now show that this unitary transformation in quantum mechanics is the quantum analogue of $\exp[\frac{iS}{\hbar}]$; where S as defined in (2.1) is the classical action function.

If α is any function of the dynamical variables in quantum theory, it will have a "mixed" matrix representation $\langle q' | \alpha | Q' \rangle$ which may be defined in terms of the usual

representation of either $\langle q'|\alpha|q''\rangle$ or $\langle Q''|\alpha|Q'\rangle$ by

$$\langle q'|\alpha|Q'\rangle = \int\langle q'|\alpha|q''\rangle dq''\langle q''|Q'\rangle = \int\langle q'|Q''\rangle dQ''\langle Q''|\alpha|Q'\rangle .$$

From the first of these definitions we obtain,

$$\langle q'|q_i|Q'\rangle = q_i'\langle q'|Q'\rangle \tag{2.5}$$

$$\langle q'|p_i|Q'\rangle = -i\hbar\frac{\partial}{\partial q_i'}\langle q'|Q'\rangle \tag{2.6}$$

and from the second:

$$\langle q'|Q_i|Q'\rangle = Q_i'\langle q'|Q'\rangle \tag{2.7}$$

$$\langle q'|P_i|Q'\rangle = +i\hbar\frac{\partial}{\partial Q'}\langle q'|Q'\rangle \tag{2.8}$$

Note the difference in sign in (2.6) and 2.8).

Equation (2.5) and (2.7) may be generalized as follows: Let $f(q)$ be any function of the q's and $g(Q)$ any function of the Q's. Then

$$\langle q'|f(q)g(Q)|Q'\rangle = \iint\langle q'|f(q)|q''\rangle dq''\langle q''|Q''\rangle dQ''\langle Q''|g(Q)|Q'\rangle$$

$$= f(q')g(Q')\langle q'|Q'\rangle .$$

Further, if $f_k(q)$ and $g_k(Q)$, $(k=1,2,\ldots m)$ denote two sets of functions of the q's and Q's respectively,

$$\langle q'|\Sigma_k f_k(q)g_k(Q)|Q'\rangle = \Sigma_k f_k(q')g_k(Q')\langle q'|Q'\rangle .$$

Thus if α is any function of the dynamical variables and we suppose it to be expressed as a function $\alpha(q,Q)$ of the q's and Q's in a "well-ordered"* way, that is, so that it consists of a sum of all terms of the form $f(q)g(Q)$, we have:

$$\langle q' | \alpha(q,Q) | Q' \rangle = \alpha(q',Q') \langle q' | Q' \rangle \ . \qquad (2.9)$$

This is a rather remarkable equation, giving us a connection between $\alpha(q,Q)$, which is a function of operators, and $\alpha(q',Q')$, which is a function of numerical variables. Moreover, it provides the basis for carrying over the contact transformation between dynamical variables in classical mechanics into a contact transformation between operators in quantum mechanics. We demonstrate this immediately below.

In conventional quantum mechanics, $\langle q' | Q' \rangle$ is the matrix element of a unitary transformation between two representations. In this analysis, we postulate that

$$\langle q' | Q' \rangle = \exp \left[\frac{iU(q',Q')}{\hbar} \right] \qquad (2.10)$$

since the right-hand side also amounts to a unitary transformation. Later we prove that the function U defined by (2.10) is the analogue of the classical action function S,

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*See Ref. (2,3) for a discussion of the importance of "well-ordered" functions in quantum mechanics.

thus linking our unitary transformation with the classical action.

If in equation (2.9) we choose $\alpha = p_i$, and apply (2.10) we get from (2.6)

$$\langle q' | p_i | Q' \rangle = \frac{\partial U(q',Q')}{\partial q_i} \langle q' | Q' \rangle .$$

By comparing this with (2.9) we obtain

$$p_i = \frac{\partial U(q,Q)}{\partial q_i}$$

as an equation between operators or dynamical variables, which holds provided $\partial U(q,Q)/\partial q_i$ is well ordered. Similarly, we get

$$P_i = - \frac{\partial U(q,Q)}{\partial Q_i}$$

which holds provided $\partial U(q,Q)/\partial Q_i$ is well-ordered. These equations are of the same form as (2.4) and show that U defined by (2.10) is the analogue of the classical action S, which is what we had to prove. We proceed to examine this analogy in more detail.

The equations of motion of classical mechanics cause the dynamical variables to vary in such a way that their values q_b, p_a at any time t_b are connected with their values q_a, p_a at another time t_a by a contact transformation, which may be put into the form

$$p_i = \frac{\partial S}{\partial q_i} \qquad P_i = -\frac{\partial S}{\partial Q_i} \tag{2.4}$$

where q, p = q_b, p_b; Q, P = q_a, p_a and S is the time integral of the Lagrangian over the range t_a to t_b along the classical path,

$$S = \int_{t_a}^{t_b} L(q_i, \dot{q}_i; t)dt \ . \tag{2.1}$$

In quantum mechanics, the q_b, p_b will still be connected with the q_a, p_a by a contact transformation (or unitary transformation) and there will be a transformation function $\langle q_b\ t_b | q_a\ t_a \rangle$ connecting the two representations in which q_b at t_b and q_a at t_a are diagonal respectively. The work of the preceding section now shows that the contact transformations in classical mechanics and in quantum mechanics have the relationship,

$$\langle q_a\ t_b | q_a\ t_a \rangle \text{ corresponds to } \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} L(q_i, \dot{q}_i; t)dt\right]^* \tag{2.11}$$

where L is the Lagrangian function. If we take t_a to differ by the infinitesimal ε from t_b, we get

$$\langle q_b\ t_a+\varepsilon | q_a\ t_a \rangle \text{ corresponds to } \exp\left[\frac{i}{\hbar} L\varepsilon\right] \tag{2.12}$$

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*Dirac's original work uses h instead of $\hbar$.

The transformation functions, i.e., $\langle q_b\ t_b | q_a\ t_a \rangle$, are very fundamental things in quantum mechanics, and it is satisfactory to find that they have their classical analogues, expressible simply in terms of the Lagrangian.

We have here the natural extension of the well-known result that the phase of the wave function corresponds to Hamilton's principle function (i.e., $S = \int_{t_a}^{t_b} L dt$) in classical mechanics.

The analogy (2.12) suggests that we ought to consider the classical Lagrangian, not as a function of the coordinates and velocities, but rather as a function of the coordinates at t_a and the coordinates at $t_a+\varepsilon$, where ε is an infinitesimal time interval.

For simplicity, we shall take the case of a single degree of freedom, although the argument applies to the general case. We shall use the notation,

$$\exp[\tfrac{i}{\hbar}\int_{t_a}^{t_b} L dt] = A(t_b, t_a)$$

so that $A(t_b, t_a)$ is the classical analogue of $\langle q_b\ t_b | q_a\ t_a \rangle$.

Suppose we divide the time interval t_a-t_b into a large number of small intervals t_a-t_1, t_1-t_2, ...t_{n-1}-t_n, t_n-t_b, by the introduction of a sequence of intermediate times t_1, t_2, ...t_n. Then

$$A(t_b, t_a) = A(t_b, t_n)A(t_n, t_{n-1}) \ldots A(t_2, t_1)A(t_1, t_a). \quad (2.13)$$

Now in the quantum theory we have

$$\langle q_b\, t_b | q_a\, t_a\rangle = \int\!\!\int \langle q_b | q_n\rangle dq_n \langle q_n | q_{n-1}\rangle dq_{n-1} \cdots$$
$$\cdots \int\!\!\int \langle q_2 | q_1\rangle dq_1 \langle q_1 | q_a\rangle \qquad (2.14)$$

where q_k denotes q at the intermediate time t_k (k=1,2,...n). (This is exactly the probability amplitude in Feynman's path integral method which will be seen later.)

Equation (2.14) at first sight does not seem to correspond properly to the equation (2.13), since on the right-hand side of (2.14) we must integrate after doing the multiplication while on the right-hand side of (2.13) there is no integration.

Let us examine this discrepancy by seeing what becomes of (2.14) when we regard t as extremely small. From the results of (2.11) and (2.12) we see that the integrand in (2.14) must be of the form $\exp[\frac{iG}{\hbar}]$ where G is a function of q_a, q_1, q_2, ...q_n, q_b which remains finite as $\hbar$ tends to zero. Let us now picture one of the intermediate q's, say q_k, as varying continuously while the others are fixed. Owing to the smallness of $\hbar$, we shall then in general have $G/\hbar$ varying extremely rapidly. This means that $\exp[\frac{iG}{\hbar}]$ will oscillate with a very high frequency about the value zero, as a result of which its integral will be practically zero. The only important part in the domain of integration of q_k is thus

that for which a comparatively large variation of q_k produces only a very small variation in G. This part is the neighborhood of a point for which G is stationary with respect to small variations in q_k.

We can apply this argument to each of the variables of integration on the right-hand side of (2.14) and obtain the result that the only important part in the domain of integration is that for which G is stationary for small variations in all the intermediate q's. But, by applying (2.11) to each of the small time sections, we see that G has for its classical analogue

$$\int_{t_n}^{t_a} L\,dt + \int_{t_{n-1}}^{t_n} L\,dt + \ldots \int_{t_1}^{t_2} L\,dt + \int_{t_a}^{t_1} L\,dt = \int_{t_a}^{t_b} L\,dt,$$

which is just the action function S, that classical mechanics requires to be stationary for small variations in all the intermediate q's. (Here Hamilton's action principle plays its role.) This shows the way in which (2.14) goes over into the classical result (2.13) when $\hbar$ becomes extremely small.

We now return to the general case when $\hbar$ is not small. We see that, for comparison with quantum theory, equation (2.13) must be interpreted in the following way. Each of the $A(t_n,t_{n-1})$ must be considered as a function of the q's at the two times to which it refers. The right-hand side of (2.13) is then a function, not only of q_b and q_a, but also of q_1, q_2, ... q_n, and in order to get from it a function of q_a

and q_b only, which we can equate to the left-hand side, we must substitute for $q_1, q_2, \ldots q_n$ their values given by Hamilton's first action principle. This process of substitution for the intermediate q's then corresponds to the process of integration over all values of q's in (2.14). (In Feynman's path integral language, this corresponds to the integration over all paths.)

Equation (2.14) contains the quantum analogue of the action principle, as may be seen more explicitly from the following argument. From (2.14) we can extract the statement (a rather trivial one) that, if we take specified values for q_a and q_b, then the importance of our considering any set of values for the intermediate q's is determined by the importance of this set of values in the integration on the right-hand side of (2.14). If we now make $\hbar$ tend to zero, this statement goes over into the classical statement that, if we take specified values for q_a and q_b, then the importance of our considering any set of values for the intermediate q's is zero unless these values make the action function stationary. This statement is one way of formulating the classical Hamilton's first action principle.

The above review gives the basic idea of the formulation of a quantum analogue of classical Lagrangian theory by Dirac.

In the following, we show the close relationship of Dirac's ideas and Feynman's path integral method.

Feynman's path integral method contains as its essential idea the concept of a probability amplitude with motion between two specified space-time points. This can be closely related to Dirac's idea in the following way.

From (2.14) we have

$$\langle q_b\ t_b|q_a\ t_a\rangle = \int..\int\langle q_b|q_{n+1}\rangle dq_{n+1}\langle q_{n+1}|q_n\rangle dq_n \\ .. \langle q_2|q_1\rangle dq_1\langle q_1|q\ \rangle, \qquad (2.14)$$

and this probability amplitude in quantum mechanics has been proved to be related to the classical action function by using the correspnndence relation.

$$\langle q_b\ t_b|q_a\ t_a\rangle \sim \exp[\frac{i}{\hbar}\int_{t_a}^{t_b} L(q,q;t)dt] \qquad (2.11)$$

What Feynman did in his work[4] was to show that for an infinitesimal time interval $t_{n+1}-t_n = \varepsilon$ this is an identity, if a proper normalization factor which depends on the system under consideration is included, that is:

$$\langle q_{n+1}\ t_{n+1}|q_n\ t_n\rangle = \frac{1}{A}\exp\ [\frac{i}{\hbar}\int_{t_n}^{t_{n+1}} L(q,\dot{q};t)dt] \qquad (2.15)$$

where A is the normalization factor.

The classical action function is expressed in the form

$$S = \sum_{n=-\infty}^{\infty} S(q_{n+1},q_n) \qquad (2.16)$$

where

$$S(q_{n+1},q_n) = \text{Min}\int_{t_n}^{t_{n+1}} L(q,q;t)dt \qquad (2.17)$$

If we put (2.15) into (2.14) and use Hamilton's first action principle as expressed in (2.17), which is the exact argument given by Dirac, we get

$$\langle q_b\ t_b | q_a\ t_a\rangle = \lim_{\epsilon\to 0}\int_R \exp\left[\frac{i}{\hbar}\sum_{n=-\infty}^{\infty} S(q_{n+1},q_n)\right]\frac{dq_{n+1}}{A}\frac{dq_n}{A}\cdots \qquad (2.18)$$

This is the probability amplitude in its path integral form, since the integration is taken over those values $q_n, q_{n+1}, \ldots$ which lie in a region R. Figure 1 shows this schematically for paths in space-time.

We proceed to show the equivalence of this path integral formulation and the conventional formulation of quantum mechanics.

How is the wave function defined from the new point of view? Does the wave function satisfy Schroedinger's equation? These two questions are answered in the following way.

We shall see that it is the possibility of expressing S as the sum as given in (2.16), and hence the amplitude as a product of contributions from successive sections of the path, which leads to the possibility of defining a quantity having the properties of a wave function.

In Figure 1, we may divide the region R into R' and R'',

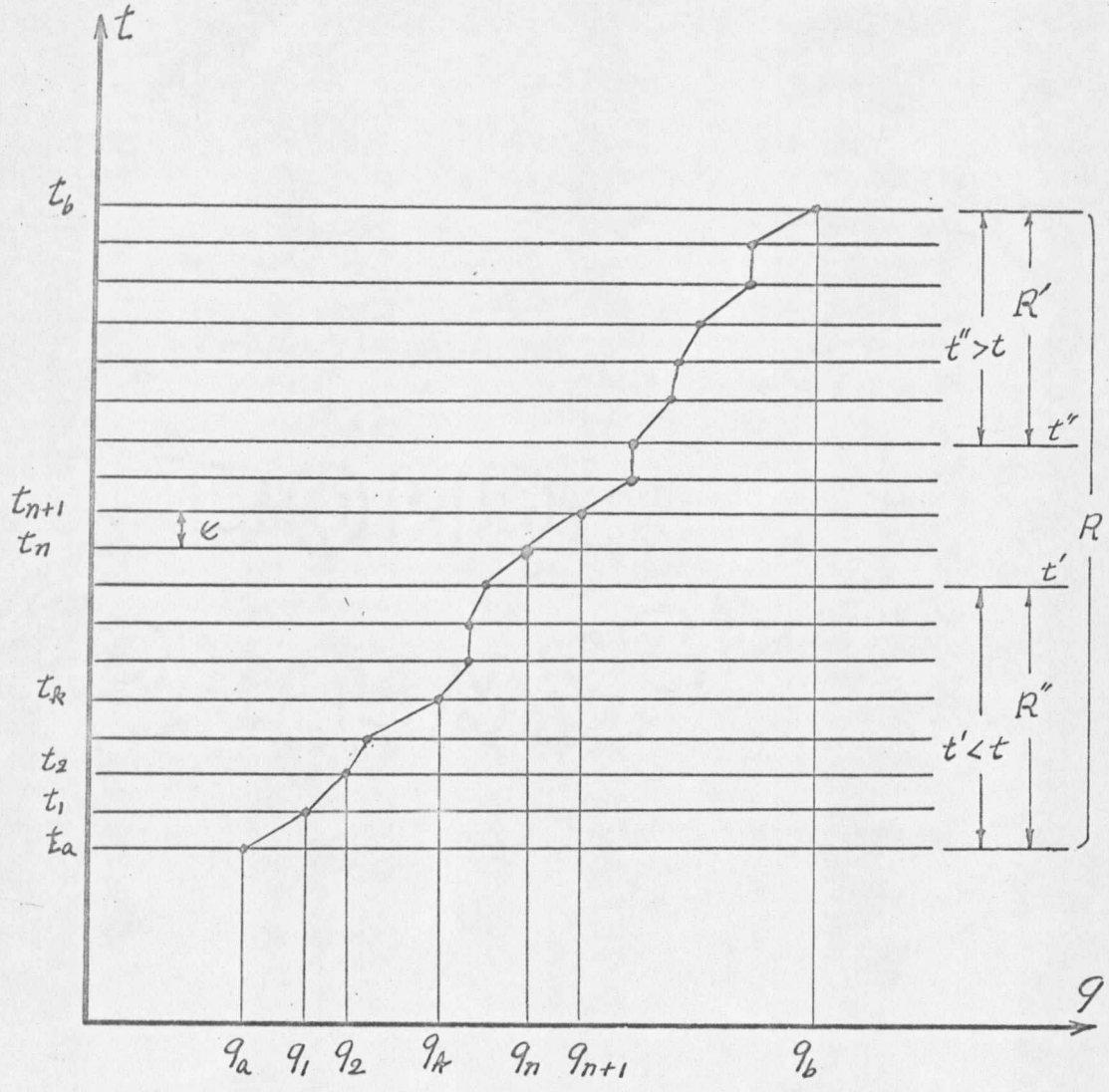

Fig. 1 Paths in Space-Time.

The sum over paths is defined as a limit, in which at first the path is specified by giving only its coordinate q at a large number of specified times separated by a very small intervals. The path sum is then an integral over all these specific coordinates. The limit is taken as $\varepsilon \to 0$.

so that (a) a region R', arbitrarily restricted in space, but lying entirely earlier in time than some t', such that $t'<t$; (b) a region R'' arbitrarily restricted in space but lying later in time than t'', such that $t''>t$; (c) the region between t' and t'' in which all the values of q coordinates are unrestricted. The region (c) is not absolutely necessary, it is convenient for letting us vary t a little without having to redefine R' and R''.

Considering t as the present, we can express $|\langle q_b\ t_b|q_a\ t_a\rangle|^2$ as the probability (or relative probability after renormalization) that the path had been in region R' and will be in region R''.

Let us suppose in (2.18) that the time t is labelled by k, i.e., assume $t\equiv t_k$ in Figure 1, the index k, of course, depending upon the subdivision ε. Then the exponential may be split into a product of two factors.

$$\exp[\tfrac{i}{\hbar}\sum_{n=-\infty}^{\infty} S(q_{n+1},q_n)] = \exp[\tfrac{i}{\hbar}\sum_{n=k}^{\infty} S(q_{n+1},q_n)]$$

$$\cdot\ \exp[\tfrac{i}{\hbar}\sum_{n=-\infty}^{k-1} S(q_{n+1},q_n)]\ . \qquad (2.19)$$

The first factor contains only coordinates with index k or higher, while the second factor contains only coordinates with index k-1 or lower. This split is possible because of (2.16), which results essentially from the fact that the

Lagrangian is a function of position at different times. Now the probability amplitude can be written

$$\langle q_b\ t_b | q_a\ t_a \rangle = \int \chi^*(q_k,t_k)\psi(q_k,t_k)dq_k \qquad (2.20)$$

where

$$\psi(q_k\ t_k) = \underset{\varepsilon\to o}{\mathrm{Lim}} \int_{R'} \exp\left[\frac{i}{\hbar} \sum_{n=-\infty}^{k-1} S(q_{n+1},q_n)\right]\frac{1}{A}\,\frac{dq_{k-1}}{A}\,\frac{dq_{k-2}}{A}\cdots$$

$$\cdots\ t'<t \qquad (2.21)$$

$$\chi^*(q_k\ t_k) = \underset{\varepsilon\to o}{\mathrm{Lim}} \int_{R''} \exp\left[\frac{i}{\hbar} \sum_{n=k}^{\infty} S(q_{n+1},q_n)\right]\frac{dq_{k+1}}{A}\,\frac{dq_{k+2}}{A}\cdots$$

$$\cdots\ t''>t \qquad (2.22)$$

The symbol R' is placed on the integral for ψ to indicate that the coordinates are integrated over region R', and, for $t'<t_n<t$, over all space. In a like manner, the integral for χ^* is over R'', for $t<t_n<t''$, over all space.

The wave function $\psi(q_k,t_k)$ is sufficient to define those attributes which are left from past history which determine future behavior.

The probability will then be $\left|\int \chi^*\psi dq\right|^2$. These results agree with the principles of conventional quantum mechanics.

We approach the time-development of the system by noting that, for finite ε, equation (2.21) permits a simple recursive relation to be developed. Consider the appearance of (2.21) if we compute it at the next instant of time $t_{k+1}=t_k+\varepsilon$.

$$\psi(q_{k+1},t_k+\varepsilon) = \lim_{\varepsilon\to 0}\int_k \exp\left[\frac{i}{\hbar}\sum_{n=-\infty}^{k} S(q_{n+1},q_n)\right]\frac{dq_k}{A}\,\frac{dq_{k-1}}{A}\cdots \quad (2.23)$$

This is similar to (2.21) except for the integration over the additional variable q_k and the extra term in the sum in the exponent. From this we obtain

$$\psi(q_{k+1},t_k+\varepsilon) = \int \exp\left[\frac{i}{\hbar}S(q_{k+1},q_k)\right]\psi(q_k,t_k)\,\frac{dq_k}{A}$$

or

$$= \int \langle q_{k+1}\ t_{k+1}|q_k\ t_k\rangle \psi(q_k,t_k)dq_k \quad (2.24)$$

The quantity $\langle q_{k+1}t_{k+1}|q_k\ t_k\rangle = \frac{1}{A}\exp\left[\frac{i}{\hbar}\int_{t_k}^{t_{k+1}} L(q,q;t)dt\right]$ is the kernel (or propagator) for the infinitesimal time development of the wave function. This kernel is the Feynman identity given in (2.15), which is nothing but the probability amplitude or the unitary transformation of the same representation q at two different times, when $t_{k+1}-t_k = \varepsilon$ is infinitesimal.

We illustrate the relation of (2.24) to the Schroedinger's equation in two cases:

For the first case, we take a particle moving freely in a one-dimensional space. In this case, the action function can be chosen as

$$S(x_{n+1},x_n) = \frac{m\varepsilon}{2}\left(\frac{x_{n+1}-x_n}{\varepsilon}\right)^2 \quad (2.25)$$

Using (2.25), equation (2.24) becomes an integral equation

$$\psi(x_{k+1},t_k+\varepsilon) = \int_{-\infty}^{+\infty} \exp\left[\frac{i}{\hbar}\left(\frac{m\varepsilon}{2}\frac{(x_{k+1}-x_k)^2}{\varepsilon^2}\right)\right]\psi(x_k,t_k)\frac{d_k}{A} \quad (2.26)$$

Following the same calculation as Feynman did in his text,[5]* we are able to obtain

$$-\frac{\hbar}{i}\frac{\partial}{\partial t}\psi = \frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2\psi \quad * \quad (2.27)$$

which is the Schroedinger's equation for the free particle case, However, it is obtained directly from the path integral method.

For the second case, we take a particle moving in a potential V(x). In this case, the action function can be chosen as

$$S(x_{n+1},x_n) = \frac{m\varepsilon}{2}\left(\frac{x_{n+1}-x_n}{\varepsilon}\right)^2 - \varepsilon V(x_{n+1}) \quad ** \quad (2.25)'$$

Using (2.25)', equation (2.24) becomes an integral equation

$$\psi(x_{k+1},t_k+\varepsilon) = \int_{-\infty}^{\infty} \exp\left[\frac{i\epsilon}{\hbar}\left\{\frac{m}{2}\left(\frac{x_{k+1}-x_k}{\varepsilon}\right)^2 - V(x_{k+1})\right\}\right]\psi(x_k,t_k)\frac{dx_k}{A} \quad (2.26)'$$

From this equation, Feynman obtains

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2\psi + V(x)\psi \quad * \quad (2.27)'$$

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*For the calculation see Ref. (5), pp. 76-78.

**This form is a good approximation, for further discussion see Ref. (4).

which is the Schroedinger's equation for the problem in question.

These two cases show that most of the contribution to $\psi(x_{k+1},t+\varepsilon)$ comes from values of x_k in $\psi(x_k,t)$ which are quite close to x_{k+1}, so that the integral equation (2.26) or (2.26)' can, in the limit $\varepsilon\to 0$, be replaced by a differential equation.

In order to appreciate the concrete meaning of Feynman's amplitude, we next show that it is the Green's function of the time-dependent Schroedinger's equation, and hence the kernel which appears in the integral equation corresponding to the Schroedinger's equation. We choose the special case of a time-independent Hamiltonian, which relates to some subsequent investigations in this work.

As we have shown, the Schroedinger's equation obtained by using the path integral method, has the conventional form, (assume in one-dimension).

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2\psi + V(x)\psi \qquad (2.27)'$$

or

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = H\psi \,. \qquad (2.28)$$

We try a special solution of the form

$$\psi = f(t)\phi(x) \qquad (2.29)$$

Substitution into (2.28) gives us

$$-\frac{h}{i}\frac{f'}{f} = \frac{1}{\phi} H\phi \quad . \tag{2.30}$$

Since the variables are completely separated, so each side must be a constant, let us call this constant E. Thus the special solution is of the form

$$\psi(x,t) = f(t)\phi(x) = e^{\frac{i}{\hbar}Et}\,\phi(x) \tag{2.31}$$

where $\phi(x)$ satisfies

$$H\phi(x) = E\phi(x) \tag{2.32}$$

That is, for this special solution the wave function oscillates with a definite frequency, namely $\omega = E/\hbar$. We know that the frequency with which a wave function oscillates corresponds, in classical physics, to the energy E. For each value of E, a different particular function ϕ, a solution of (2.32), must be sought.

The probability that a particle is at x is the absolute square of the wave function ψ. That is $|\psi|^2 = |\phi|^2$ which does not depend upon the time. We say under these circumstances that the system is in a stationary state.

Now suppose that E_1 is a possible energy for which equation (2.32) has a solution $\phi_1(x)$, and E_2 corresponds to ϕ_2, etc. The general solution can be written at any time t as

$$\psi(x,t) = \sum_{n+1}^{\infty} c_n e^{\frac{-i}{\hbar} E_n t} \phi_n(x) \tag{2.33}$$

From this, we may put

$$f(x_1) \equiv \psi(x_1,t_1) = \sum_{n=1}^{\infty} c_n e^{\frac{-i}{\hbar} E_n t_1} \phi_n(x_1) = \sum_{n=1}^{\infty} a_n \phi_n(x_1) \tag{2.34}$$

since we can always make the expansion,

$$f(x_1) = \sum_{n=1}^{\infty} a_n \phi_n(x_1) \quad . \tag{2.35}$$

So we conclude

$$c_n = a_n e^{\frac{+i}{\hbar} E_n t_1} \quad . \tag{2.36}$$

Putting this into (2.33), we have, for time t_2

$$\psi(x_2,t_2) = \sum_{n=1}^{\infty} a_n e^{\frac{+i}{\hbar} E_n (t_1 - t_2)} \phi_n(x_2) \, . \tag{2.37}$$

From the orthonormality condition

$$\int_{-\infty} \phi_n^*(x) \phi_m(x) dx = \delta_{nm} \tag{2.38}$$

and equation (2.35), we obtain

$$a_n = \int_{-\infty}^{\infty} \phi_n^*(x_1) f(x_1) dx_1 = \int_{-\infty}^{\infty} \phi_n^*(x_1) \psi(x_1,t_1) dx_1 \, . \tag{2.39}$$

Substituting into (2.37) we obtain

$$\psi(x_2,t_2) = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \phi_n(x_2) \phi_n^*(x_1) e^{\frac{-i}{\hbar} E_n (t_2 - t_1)} \psi(x_1 t_1) dx_1 \tag{2.40}$$

Comparing this with (2.24), we finally obtain the desired expression for the space-time kernel, that is

$$\langle x_2\ t_2|x_1\ t_1\rangle = \sum_{n=1}^{\infty} \phi_n(x_2)\phi_n^*(x_1)e^{-\frac{i}{\hbar}E_n(t_2-t_1)} \quad \text{for } t_2>t_1$$

$$= 0 \quad \text{for } t_2<t_1 \tag{2.41}$$

This kernel is identical to the Green's function of the corresponding time-dependent Schroedinger's equation as can be shown by direct calculation as follows.

We consider the time-dependent Schroedinger's equation for the Green's function

$$(-\frac{\hbar}{i}\frac{\partial}{\partial t_2}-H_{x_2})\langle x_2\ t_2|x_1\ t_1\rangle = -\frac{\hbar}{i}\delta(x_2-x_1)\delta(t_2-t_1) . \tag{2.42}$$

It is clear that the solution depends only on x_2-x_1, t_2-t_1, therefore, we can, without loss of generality, set $x_1 = 0$, $t_1 = 0$, and

$$\langle x_2\ t_2|x_1\ t_1\rangle = \sum_{n=1}^{\infty} G_n(t_2)\phi_n(x_2)a_n(x_1) \tag{2.43}$$

and using the Fourier Transform,

$$g_n(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_n(t_2)\ e^{i\omega t_2}\ dt_2$$

$$G_n(t_2) = \int_{-\infty}^{\infty} g_n(\omega)\ e^{-i\omega t_2}\ d\omega .$$

Therefore

$$\langle x_2\ t_2|x_1\ t_1\rangle = \sum_{n=1}^{\infty}\int_{-\infty}^{\infty} g_n(\omega)e^{i\omega t_2}\ d\omega\ \phi_n(x_2)\cdot a_n(x_1) \tag{2.44}$$

Substituting (2.44) into (2.42) and using

$$\delta(x_2-x_1) = \sum_{n=1}^{\infty} \phi_n(x_2)\phi_n^*(x_1), \quad \delta(t_2-t_1) = \frac{1}{2\pi}\int_{\infty}^{\infty} e^{-i\omega t_2} d\omega$$

We have

$$[(-\frac{\hbar}{i})(-i\omega) - E_n]\sum_n \int_{-\infty}^{\infty} g_n(\omega) e^{-i\omega t_2} d\omega \; \phi_n(x_2) a_n(x_1)$$

$$= -\frac{\hbar}{i}\frac{1}{2\pi} \sum_n \int_{-\infty}^{\infty} \phi_n(x_2)\phi_n^*(x_1) e^{-i\omega t_2} d\omega$$

Therefore

$$\left[\hbar\omega - E_n\, g_n(\omega)\right] \cdot a_n(x_1) = -\frac{\hbar}{i}\frac{1}{2\pi}\phi_n^*(x_1)$$

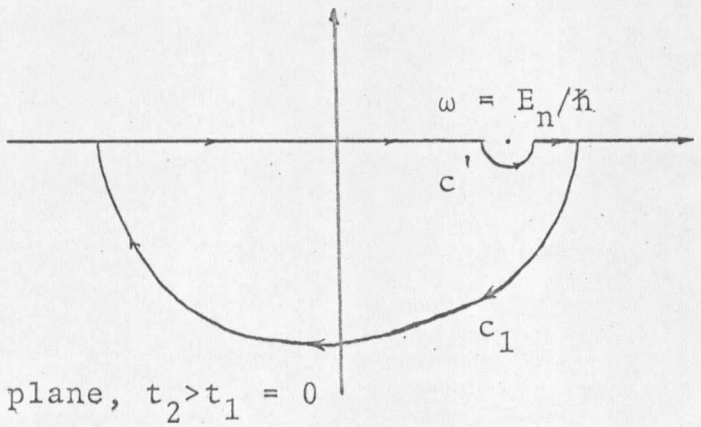
or

$$g_n(\omega) = -\frac{Q_n^*(x_1)/a_n(x_1)}{2\pi i(\omega - E_n/\hbar)}$$

Then its Fourier Transform gives

$$G_n(t_2) = [\frac{Q_n^*(x_1)}{a_n(x_1)}][-\frac{1}{2\pi_i}\int_{-\infty}^{\infty} \frac{e^{-i\omega t_2}}{(\omega - E_n/\hbar)} d\omega] \quad t_2>t_1 = 0$$

To evaluate the contour integral under the condition $t_2>t_1=0$, we choose the contour as shown:



ω- plane, $t_2>t_1 = 0$

By using Cauchy's theorem, we have

$$\int_{c_1} \frac{e^{-i\,t_2}}{(\omega-E_n/\hbar)}\,d\omega + \int_{e'} \frac{e^{-i\omega t_2}}{(\omega-E_n/\hbar)}\,d\omega = 0$$

or

$$-\int_{c_1} \frac{e^{-i\omega t_2}}{(\omega-E_n/\hbar)}\,d\omega = \int_{c'} \frac{e^{-i\omega t_2}}{(\omega-E_n/\hbar)}\,d\omega$$

and Cauchy's formula:

$$f(z_o) = \frac{1}{2\pi_i} \int_{c'} \frac{f(z)dz}{(z-z_o)} \quad .$$

Therefore, we get

$$-\int_{c_1} \frac{e^{-i\omega t_2}}{(\omega-E_n/\hbar)}\,d\omega = -\int_{-\infty}^{\infty} \frac{e^{-i\omega t_2}}{(\omega-E_n/\hbar)}\,d\omega = 2\pi i\; e^{\frac{-iE_n t_2}{\hbar}} \quad .$$

Finally we get

$$G_n(t_z) = \frac{Q_n^*(x_1)}{a_n(x_1)}\, e^{\frac{-iE_n t_2}{\hbar}} \quad .$$

Thus the Green's function is

$$\begin{aligned} \langle x_2\ t_2 | x_1\ t_1\rangle &= \sum_{n=1}^{\infty} G_n(t_2)\phi_n(x_2)\cdot a_n(x_1) \\ &= \sum_{n=1}^{\infty} \phi_n(x_2)\phi_n^*(x_1)\, e^{\frac{-iE_n}{\hbar}(t_2-t_1)} \quad \text{for } t_2>t_1 \\ &= 0 \qquad\qquad\qquad\qquad \text{for } t_2<t_1 \end{aligned} \tag{2.45}$$

which is the result that we obtained before. Thus the kernel is the Green's function of the time-dependent Schroedinger's equation in conventional quantum mechanics.

III. THE MODIFIED PATH INTEGRAL IN LIMITED SPACE-TIME

As mentioned in the introduction, we would like to make some modification of the space-time path integral method to have some characteristics of "paths of class R", or "paths having property R". This can be done in the following fashion.

We modify the Lagrangian function, and the action function as well. After this is done, the modified form of the path integral is constructed, since it depends upon the Lagrangian function. The modified form of the Lagrangian function has obviously some constraint relations associated with it. These constraint relations provide limitations on the paths, thus providing "paths of some class R" or "paths having property R". We exhibit this in detail as follows.

In classical mechanics, the Lagrangian function L is generally a function of all position coordinates q_i and the velocities $\dot{q}_i$ (i=1,2,...m). However, it may happen that a certain variable q_k does not appear in the Lagrangian function, although $\dot{q}_k$ is present. In such a case, q_k is called an

ignorable (cyclic or kinosthenic)* coordinate. When we introduce the "conjugate momentum",

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{3.1}$$

the Lagrangian equations of motion become

$$\dot{p}_i = \frac{\partial L}{\partial q_i} . \tag{3.2}$$

Now if q_k is an ignorable coordinate,

$$\frac{\partial L}{\partial q_k} = 0 . \tag{3.3}$$

So

$$p_k = \text{constant} = c_k . \tag{3.4}$$

This is the well-known fact that the momentum connected with an ignorable coordinate is a constant during the motion.

Let us examine what happens to Hamilton's first action principle, We have given the action integral in (2.1)

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*E. J. Routh, Dynamics of Rigid Bodies (Macmillan, 1877) calls them "absent coordinates"; H. V. Helmholtz, Journal of Math. 97 (1884) 111, calls them "cyclic coordinates"; Thomson refers to them as "kinosthenic" or "speed coordinates"; Whittaker uses the name "ignorable variables", see Ref. (9), p. 125.

$$S = \int L(q_1, q_2 \ldots q_i, \ldots q_m; \dot{q}_1, \dot{q}_2, \ldots \dot{q}_i, \ldots \dot{q}_m; t)^* \, dt. \quad (3.5)$$

Since a certain variable q_k is not present in the partial derivative $p_k = (\partial L)/(\partial \dot{q}_k)$, but $\dot{q}_k$ is present, we can obtain $\dot{q}_k$ from (3.4) as a function of non-ignorable coordinates and velocities.

For simplicity, we restrict our discussion to one single ignorable coordinate, the generalization to any number being obvious.

We arrange q_i in such a manner that the last coordinate q_m is the ignorable coordinate, so that

$$p_m = \frac{\partial L}{\partial \dot{q}_m} = c_m \quad (3.6)$$

and hence

$$\dot{q}_m = f(q_1, \ldots q_{m-1}; \dot{q}_1, \ldots \dot{q}_{m-1}, c_m; t) \quad (3.7)$$

Hamilton's first action principle requires that the action integral (3.5) vanish for arbitrary variations of the q_i between definite limits. Equation (3.6) or (3.7) actually provide the constraint relation for the system under

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*Note here q_i is different from q_n as before in (2.15), q_i stands for different coordinates, each one of which can be divided into q_n which represent the same coordinate at different times.

consideration. If we eliminate $\dot{q}_m$ by means of (3.7), our problem is reduced from m to m-1 degrees of freedom, so our path integral for the modified Lagrangian will be reduced by one degree of freedom.

Therefore, we can actually simplify the given variation problem by eliminating in advance all the ignorable coordinates. However, we have to keep in mind that the constraint relation (3.7) makes $\dot{q}_m$ a function of the non-ignorable variables in the sense that we assume the relation (3.7) to hold not only for the actual motion but for the varied motion as well.

There is no objection to this restriction of the variations of all the q_i. However, the condition that the variation of $\dot{q}_m$ must vanish at the two ends points of the range is violated because $\dot{q}_m$ is obtained from (3.7) by a quadrature. Hence, from (3.5) we have,

$$\delta S = \delta\int_{t_a}^{t_b} L\,dt = (p_m \delta q_m)_{t_a}^{t_b} \quad * \qquad (3.8)$$

But $p_m = c_m$, the ignorable momentum, is constant everywhere along the C^*-curve (a curve in configuration space). Hence we can write

$$(p_m \delta q_m)_{t_a}^{t_b} = p_m \delta\int_{t_a}^{t_b} \dot{q}_m dt = \delta\int_{t_a}^{t_b} c_m \dot{q}_m dt \ . \qquad (3.9)$$

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*See Ref. (9).

So that (3.8) becomes

$$\delta S = \delta\int_{t_a}^{t_b} (L - c_m\dot{q}_m)dt = 0 \quad . \tag{3.10}$$

Let us put $\bar{L} = L - c_m\dot{q}_m$ (3.11)

and call $\bar{L}$ the "modified Lagrangian function for the ignorable coordinate q_m." The result of our deduction can be stated as follows:

The minimizing of the original action integral S with the ignorable variable q_m can be reduced to the minimizing of the modified action integral

$$\bar{S} = \int_{t_a}^{t_b} \bar{L}(\bar{q}_r, \dot{\bar{q}}_r; t)dt \tag{3.12}$$

without the ignorable variable q_m, after eliminating $\dot{q}_m$ with the help of the constraint relation $p_m = c_m$.

The process of eliminating an ignorable coordinate may be divided into three steps:

a. Write down the equation for the conjugate momentum of the ignorable coordinate,

$$p_m = \frac{\partial L}{\partial \dot{q}_m} = c_m \quad . \tag{α}$$

b. Modify the given Lagrangian function to

$$\bar{L} = L - c_m\dot{q}_m \quad . \tag{β}$$

c. Eliminate the ignorable velocity $\dot{q}_m$ by solving equation (α) for $\dot{q}_m$ and substituting into equation (β). Then the modified $\bar{L}$ does not depend on the ignorable variable and

the original variational problem of m degrees of freedom is reduced to a new variational problem of m-1 degrees of freedom.

This reduction process remains the same if the given problem contains more than one ignorable coordinate. The modified Lagrangian function has now to be defined by

$$\bar{L} = L - \sum_{\ell=1}^{s} c_\ell \dot{q}_\ell \tag{3.13}$$

with the constraint relations

$$p_\ell \equiv \frac{\partial L}{\partial \dot{q}_\ell} = c_\ell \ , \ (\ell = 1,2,\ldots s) \tag{3.14}$$

The sum in the second term of (3.13) is extended over all the ignorable coordinates.

The modified Lagrangian $\bar{L}$ will contain all the non-ignorable coordinates after the substitution of $\dot{q}_\ell$, which are obtained by solving the constraint relations (3.14).

This process will reduce the Lagrangian to a function of the non-ignorable coordinates only. Such a Lagrangian may have advantages for the calculation of the path integral, since it will reduce the integration from m to m-s degrees of freedom.

Using the modified Lagrangian and its action function, we follow the same argument, step by step, as given by Dirac and Feynman, to formulate the modified path integral as

follows:

The contact transformation now becomes

$$\bar{p}_r = \frac{\partial \bar{L}}{\partial \bar{q}_r} \quad , \quad \bar{P} = - \frac{\partial \bar{L}}{\partial \bar{Q}_r} \quad . \tag{3.15}$$

in its classical form, where the dynamic variable $\bar{q}_r, \bar{p}_r$, etc., represent the non-ignorable variables only, and $\bar{L}$ in (3.13) is the modified Lagrangian function with the constraint relations (3.14) associated with it.

We start with these relations and follow the logic of Dirac to obtain the quantum mechanical contact transformations, which are equations between operators. That is

$$\bar{p}_r = \frac{\partial \bar{U}(\bar{q}, \bar{Q})}{\partial \bar{q}_r} \quad , \quad \bar{P}_r = - \frac{\partial \bar{U}(\bar{q}, \bar{Q})}{\partial \bar{Q}_r} \quad * \tag{3.16}$$

provided $(\partial \bar{U})/(\partial \bar{q}_r)$ and $(\partial \bar{U})/(\partial \bar{Q}_r)$ are well-ordered functions.

In classical mechanics, the equations of motion causing a set of variables $\bar{q}_r, \bar{p}_r$ to vary from one time to another can be described by a contact transformation, which may be put into the form as before

$$\bar{p}_r = \frac{\partial \bar{S}}{\partial q_r} \quad , \quad \bar{P}_r = - \frac{\partial \bar{S}}{\partial \bar{Q}_r}$$

---

*We have postulated that $\langle \bar{q}' | \bar{Q}' \rangle = \exp[i\bar{U}(\bar{q}', \bar{Q}'))/\hbar]$ as we did before in (2.10), all $\bar{q}'$, $\bar{Q}'$, represent the numerical values of the non-ignorable coordinates.

with $\bar{q}_r$, $\bar{p}_r = \bar{q}_b^*$, $\bar{p}_b$ at time t_b; $\bar{Q}_r$, $\bar{P}_r = \bar{q}_a$, $\bar{p}_a$ at time t_a; where $\bar{S}$ is the modified action function:

$$\bar{S} = \int_{t_a}^{t_b} \bar{L}(\bar{q}_r, \dot{\bar{q}}_r; t)dt. \tag{3.12}$$

In quantum mechanics, this can be represented by a unitary (or contact) transformation $\langle \bar{q}_b, t_b | \bar{q}_a, t_a \rangle$, connecting the same representation at different times in which $\bar{q}_b$ at t_b and $\bar{q}_a$ at t_a are diagonal respectively.

The previous arguments suggest a modified correspondence relationship,

$$\langle \bar{q}_b \ t_b | \bar{q}_a \ t_a \rangle \sim \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} \bar{L}(\bar{q}_r, \dot{\bar{q}}_r; t)dt\right] \tag{3.17}$$

where $\bar{L}$ is the modified Lagrangian as defined in (3.13). If we take a small time interval ε, we get

$$\langle \bar{q}_b \ t_a + \varepsilon | \bar{q}_a \ t_a \rangle \sim \exp\left[\frac{i}{\hbar} \bar{L}\varepsilon\right]. \tag{3.18}$$

We use the notation

$$\exp\left[\frac{i}{\hbar} \int_{t_a}^{t_b} \bar{L}(\bar{q}_r, \dot{\bar{q}}_r; t)dt\right] = \bar{A}(t_b, t_a)$$

so that $\bar{A}(t_b, t_a)$ is the classical analogue of $\langle \bar{q}_b \ t_b | \bar{q}_a \ t_a \rangle$.

---

* $\bar{q}_b \equiv \bar{q}_{rb}$ stand for all non-ignorable coordinates.

Suppose we divide the time interval $t_b \to t_a$ into a large number of small intervals $t_a \to t_1$, $t_1 \to t_2$, $\ldots t_{n-1} \to t_n$, $t_n \to t_b$. Then

$$\bar{A}(t_b,t_a) = \bar{A}(t_b,t_n)\ \bar{A}(t_n,t_{n-1})\ \ldots\ \bar{A}(t_2,t_1)\bar{A}(t_1,t_a). \quad (3.19)$$

Now the quantum expression is

$$\langle \bar{q}_b\ t_b | \bar{q}_a\ t_a \rangle = \int\!\!\ldots\!\int \langle \bar{q}_b | \bar{q}_n \rangle d\bar{q}_n \langle \bar{q}_n | \bar{q}_{n-1} \rangle d\bar{q}_{n-1} \ldots \langle \bar{q}_2 | \bar{q}_1 \rangle d\bar{q}_1 \langle \bar{q}_1 | \bar{q}_a \rangle^{*} \quad (3.20)$$

Here the $\bar{q}_r$ are the non-ignorable coordinates, while $\bar{q}_k$'s labelled them at different times. These $\bar{q}_r$ are related to the ignorable coordinates through the modified Lagrangian and the constraint relations given by the conservation of the momenta conjugate to the ignorable coordinates. Therefore, $\bar{q}_r$ are not so free as q_i, since the motion of $\bar{q}_r$ have to conserve "something" but the q_i do not necessarily do so.

We next use the same reasoning of Dirac to show that (3.20) goes over to (3.19) by using the modified Hamilton's first action principle in the form,

$$\delta \bar{S} = \int_{t_a}^{t_b} \bar{L} dt = 0 \quad (3.21)$$

---

*The notations are simplified, $\bar{q}_k \equiv \bar{q}_{rk}$ for all the non-ignorable coordinates $\bar{q}_r$ at time t_k is labelled by $\bar{q}_k$ instead of $\bar{q}_{rk}$.

which is now applied to the non-ignorable coordinates only.

From the relationships (3.17) and (3.18), we find that the integrand of (3.20) must be of the form $\exp(i\bar{G}/\hbar)$, where $\bar{G}$ is a function of $\bar{q}_a$, $\bar{q}_1$, $\bar{q}_2$, $..\bar{q}_n$, $\bar{q}_b$, which remains finite as $\hbar$ tends to zero. If we look at any one of the intermediate $\bar{q}$'s, say $\bar{q}_k$, the main contribution comes from a point for which $\bar{G}$ is stationary with respect to small variation in $\bar{q}_k$. This can be applied to each factor in the integrand, so that the only contribution in the domain of integration of (3.20) is that for which $\bar{G}$ is stationary for small variation in all the intermediate $\bar{q}_k$'s. But using the result of (3.17) for each of the small time intervals, we see $\bar{G}$ has for its classical analogue

$$-\int_{t_n}^{t_b} \bar{L}dt + \int_{t_{n-1}}^{t_n} \bar{L}dt + ..\int_{t_1}^{t_2} \bar{L}dt + \int_{t_a}^{t_1} \bar{L}dt = \int_{t_a}^{t_b} \bar{L}dt$$

which is just the modified action function $\bar{S}$, and the above argument is nothing but the statement of (3.21); that is the modified Hamilton's first action principle. This shows (3.20) goes over into (3.19) when $\hbar$ becomes extremely small.

Since Feynman's argument is directly related to Dirac's, his replacement of the correspondence relation with an identity becomes for the modified formulation,

$$\langle \bar{q}_{n+1} t_{n+1} | \bar{q}_n\ t_n \rangle = \frac{1}{B} \exp\left[\frac{i}{\hbar} \int_{t_n}^{t_{n+1}} \bar{L}(\bar{q}_r, \dot{\bar{q}}_r; t)dt\right] \qquad (3.22)$$

with A changed to B. The normalization must be changed to a suitable form in the system of non-ignorable coordinates.

The modified action function can be expressed as

$$\bar{S} = \sum_{n=-\infty}^{\infty} \bar{S}(\bar{q}_{n+1}, \bar{q}_n) \tag{3.23}$$

where

$$\bar{S}(\bar{q}_{n+1}, \bar{q}_n) = \text{Min} \int_{t_n}^{t_{n+1}} \bar{L}(\bar{q}_r, \dot{\bar{q}}_r; t)\, dt \tag{3.24}$$

The general modified path integral then becomes

$$\langle \bar{q}_b \; t_b | \bar{q}_a \; t_a \rangle = \lim_{\varepsilon \to 0} \int_{\bar{R}} \exp\left[\frac{i}{\hbar} \sum_{N=-\infty}^{\infty} \bar{S}(\bar{q}_{n+1}, \bar{q}_n)\right] \frac{d\bar{q}_{n+1}}{B} \frac{d\bar{q}_n}{B} \cdots \tag{3.25}$$

where the integration is taken over $\ldots \bar{q}_n, \bar{q}_{n+1}, \ldots$ which lie in a limited region $\bar{R}$. This is shown schematically in Figure 2 for paths in limited space-time.

The modified probability amplitude as expressed in (3.25) is a constrained one. That is, it is a probability amplitude for a physical event in which the possible paths lie in a region $\bar{R}$, which is limited by the constraint relations (3.14). In the one-dimensional case, the effect can hardly be seen, since if the corresponding momentum is a constant, the only coordinate is the ignorable one, which is missing in our formulation. However, for more than one dimension, the technique can obviously be applied, and we will give a concrete example later. In general, the evaluation of the reduced path

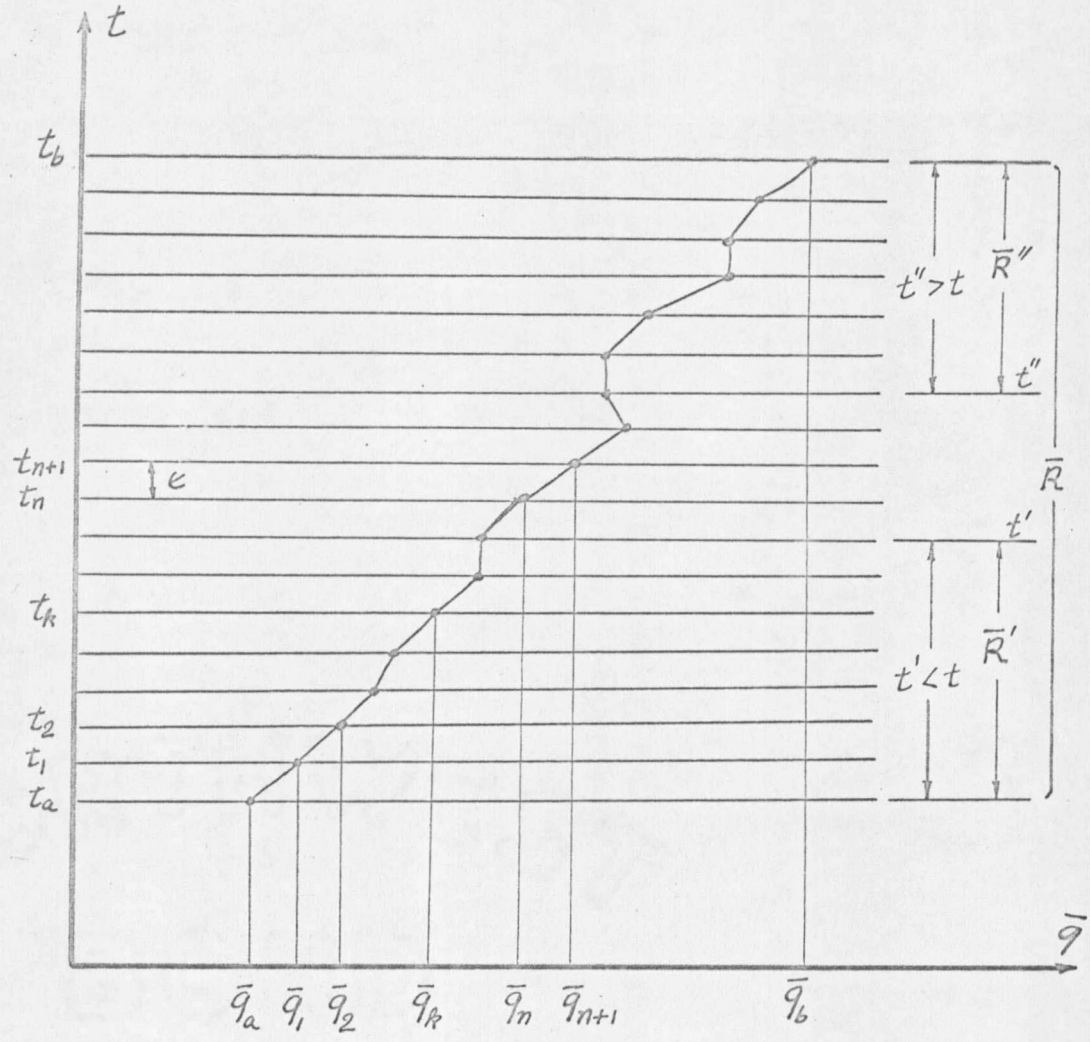

Fig. 2 Paths in Limited Space-Time:

The paths of this picture are different from Fig. 1 in that, here the paths are limited by the constraint relations, so that they conserve some quantities. The path sum extends over the non-ignorable coordinates only. In cases of two-dimensions, with one ignorable coordinate, the reduced paths will be in one-dimension only. Later, we will give a concrete example.

integral may not be easy. This is the price we have to pay under this modified formulation, since the elimination of the ignorable coordinates may reduce the kinetic energy from a quadratic form in velocities (for which the evaluation of the path integral is easy)* to a linear form in velocities. This change may make our modified path integral hard to solve, if not unsolvable. In the following we illustrate how this may come about in the modified Lagrangian.

Suppose the kinetic energy T of the original system can be written as follows.

$$T = \frac{1}{2} \sum_{t,k=1}^{m} a_{i,k}\, \dot{q}_i\, \dot{q}_k = \frac{1}{2} \sum_{i,k=1}^{m-1} a_{i,k}\, \dot{q}_i\, \dot{q}_k + \sum_{i=1}^{m-1} a_{i,m}\, \dot{q}_i\, \dot{q}_m + \frac{1}{2} a_{m,m}\, \dot{q}_m^2 \qquad (3.26)$$

We have separated here the corresponding ignorable velocity from the other velocities. Now the ignorable momentum becomes

$$p_m = \frac{\partial T}{\partial \dot{q}_m} = \sum_{i=1}^{m-1} a_{i,m}\, \dot{q}_i + a_{m,m}\, \dot{q}_m = c_m \qquad (3.27)$$

if the potential $V(q_i,t)$ is independent of the velocities.

We obtain for the modified Lagrangian function,

$$\bar{L} = L - c_m \dot{q}_m = \frac{1}{2} \sum_{i,k=1}^{m-1} a_{i,k}\, \dot{q}_i\, \dot{q}_k - \frac{1}{2} a_{m,m}\, \dot{q}_m^2 - V. \qquad (3.28)$$

---

*A general discussion is given in Ref. (5) pp. 58-61.

The reduction process is not yet accomplished since we have to eliminate q_m with the help of the constraint relation (3.27). After the elimination of q_m, we obtain

$$\bar{L} = \frac{1}{2} \sum_{i,k=1}^{m-1} a_{i,k} \dot{q}_i \dot{q}_k - \frac{1}{2a_{m,m}} \sum_{i=1}^{m-1} a_{i,m}^2 \dot{q}_i^2 + \frac{c_m}{a_{m,m}} \sum_{i=1}^{m-1} a_{i,m} \dot{q}_i - \bar{V} , \tag{3.29}$$

where $\bar{V}$ may be called an effective potential, which contains two terms,

$$\bar{V} = V + \frac{1}{2} \frac{c_m^2}{a_{m,m}} . \tag{3.30}$$

The remaining terms, in $\bar{L}$, however, are of an unusual nature. They can be considered as part of the kinetic energy T, however, T instead of being quadratic is now linear in velocities.

In our modified path integral method, we are dealing with the modified Lagrangian function, which is now reduced to the form of (3.29). If the velocity-dependent part can be reduced to a quadratic form again, then the evaluation of the modified path integral of the remaining non-ignorable coordinates can be done easily, and we will get a definite answer to the problem under study. Otherwise, the modified path integral is hard to evaluate.

We next examine Feynman's ideas to formulate the modified wave function. Using the same argument as given in Section II,

we may divide $\bar{R}$ into three regions as shown in Figure 2. The integrand in (3.25) can be rewritten as

$$\exp\left[\frac{i}{\hbar}\sum_{n=-\infty}^{\infty}\bar{S}(\bar{q}_{n+1},\bar{q}_n)\right] = \exp\left[\frac{i}{\hbar}\sum_{n=k}^{\infty}\bar{S}(\bar{q}_{n+1},\bar{q}_n)\right] \cdot \exp\left[\frac{i}{\hbar}\sum_{n=-\infty}^{k-1}\bar{S}(\bar{q}_{n+1},\bar{q}_n)\right] \tag{3.31}$$

This split is possible because of (3.23), which results from the fact that the modified Lagrangian is also a function of position at different times. Therefore, equation (3.25) can be written as

$$\langle\bar{q}_b\ t_b|\bar{q}_a\ t_a\rangle = \int\chi^*(\bar{q}_k,t_k)\psi(\bar{q}_k,t_k)d\bar{q}_k \tag{3.32}$$

where

$$\psi(\bar{q}_k,t_k) = \lim_{\varepsilon\to o}\int_{\bar{R}'}\exp\left[\frac{i}{\hbar}\sum_{n=-\infty}^{k-1}\bar{S}(\bar{q}_{n+1},\bar{q}_n)\frac{1}{B}\right]\frac{d\bar{q}_{k-1}}{B}\,\frac{d\bar{q}_{k-2}}{B}\,..\ ..\ t'<t \tag{3.33}$$

$$\chi^*(\bar{q}_k,t_k) = \lim_{\varepsilon\to 0}\int_{\bar{R}''}\exp\left[\frac{i}{\hbar}\sum_{n=k}^{\infty}\bar{S}(\bar{q}_{n+1},\bar{q}_n)\right]\frac{d\bar{q}_{k+1}}{B}\,\frac{d\bar{q}_{k+2}}{B}\,..\ ..\ t''>t \tag{3.34}$$

The wave function $\psi(\bar{q}_k,t)$ is sufficient to define those attributes which are left from past history which determine future behavior for all these non-ignorable coordinates.

The probability of finding the system in region $\bar{R}$ is defined by $\left|\int\chi^*\psi\ dq\right|^2$. These results agree with the principles of conventional quantum mechanics.

For an infinitesimal time interval, the change of the wave function $\psi(q_k,t)$ of (3.33) in the configuration space of non-ignorable coordinates can be written as

$$\psi(\bar{q}_{k+1},t_k+\varepsilon) = \int d\bar{q}_k <\bar{q}_{k+1}\ t_{k+1}|\bar{q}_k\ t_k>\psi(\bar{q}_k,t_k) \qquad (3.35)$$

where the kernel in the limited space is:

$$<\bar{q}_{k+1}\ t_{k+1}|\bar{q}_k\ t_k> = \frac{1}{B}\exp[\frac{i}{\hbar}\int_{t_k}^{t_{k+1}}\bar{L}(\bar{q}_r,\dot{\bar{q}}_r;t)dt] \qquad (3.36)$$

this is the same equation as given in Feynman's identity (3.22).

Consequently, we will also be able to get the result that this modified kernel is identical to the modified Green's function of the time-dependent Schroedinger's equation in a modified form, since the Hamiltonian of the system will be modified as well as the Lagrangian. We obtain

$$(-\frac{\hbar}{i}\frac{\partial}{\partial t_b} - H_{\bar{q}_b})<\bar{q}_b\ t_b|\bar{q}_a\ t_a> = -\frac{\hbar}{i}\delta(\bar{q}_b-\bar{q}_a)\delta(t_b-t_a) \qquad (3.37)$$

where $H_{\bar{q}_b}$ is the modified Hamiltonian, in which the ignorable coordinates are completely eliminated. Thus the modified path integral formulation and conventional quantum mechanics are equivalent.

The above formal construction of the modified path integral in a limited region of configuration space can be seen explicitly in the following concrete sample.

Let us consider the motion of a particle in a uniform gravitational field in two-dimensions.

The ordinary Lagrangian function expressed in Cartesian coordinates is in the form

$$L = \frac{m}{2}(\dot{x}^2+\dot{y}^2) - mgy \tag{3.38}$$

The wave function in the ordinary path integral method should be written in the form

$$\psi(x_{k+1},y_{k+1};t_k+\varepsilon) = \int_{-\infty}^{\infty}\frac{dx_k}{A}\int_0^{\infty}\frac{dy_k}{A}\exp\left[\frac{i}{\hbar}\int_{t_k}^{t_k+\varepsilon} L(x,y,\dot{x},\dot{y};t)dt\right]\times$$

$$\psi(x_k\ y_k\ t). \tag{3.39}$$

From (3.38) we see that x is an ignorable coordinate. Therefore, we will be able to employ our modified path integral method. This enables us to reduce the ordinary path integral from two-dimensions to one-dimension.

Following the procedure of elimination as given previously, we obtain

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} = c_x\ . \tag{3.40}$$

Solving for $\dot{x}$ we get

$$\dot{x} = \frac{c_x}{m}\ . \tag{3.41}$$

The modified Lagrangian function has the form

$$L = L - c_x \dot{x} = \frac{m}{2}\left[\left(\frac{c_x}{m}\right)^2 + \dot{y}^2\right] - mgy - c_x\left(\frac{c_x}{m}\right)$$

$$= \frac{m}{2}\dot{y}^2 - mgy - \frac{c_x^2}{2m} = \frac{1}{2}m\dot{y}^2 - V(y) . \qquad (3.42)$$

Here we see our effective potential $\bar{V}(y)$

$$\bar{V}(y) = mgy + \frac{c_x^2}{2m} \qquad (3.43)$$

Therefore, the wave function in the modified path integral formulation has the form

$$\psi(y_{k+1}, t_k+\varepsilon) = \int_0^\infty \frac{dy_k}{A'} \exp\left[\frac{i}{\hbar}\int_{t_k}^{t_k+\epsilon} L(y,\dot{y};t_k)dt_k\right]\psi(y_k\ t_k) \qquad (3.44)$$

Note the ignorable coordinate x is completely eliminated.

The quantum mechanics, just like the corresponding classical mechanics describes a motion in which the momentum conjugate to the ignorable coordinate has a fixed, constant value.

If we use (3.42) and put it into (3.44), it is a completely one-dimensional problem. Following the same calculation as done by Feynman, we get the Schroedinger's equation in a modified form,

$$-\frac{\hbar}{i}\frac{\partial}{\partial t}\psi = \frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right)^2\psi + \bar{V}(y)\psi \qquad (3.45)$$

where

$$\bar{V}(y) = mgy + \frac{c_x^2}{2m} = V(y) + \frac{c_x^2}{2m} \qquad (3.46)$$

Our constraint relation (3.40) is built into the effective potential $\bar{V}(y)$, which in this case contain an additional constant term in comparison with the original potential $V(y)$.

The modified Hamiltonian is of the form

$$H_y = \bar{T} + \bar{V} = \frac{1}{2} m\dot{y}^2 + mgy + \frac{c_x^2}{2m} \qquad (3.47)$$

We proceed now to illustrate by a specific example the calculation of the path integral, both in the standard (or original) form of Feynman and in our modified form containing constraints.

For this purpose, we choose again the case of a particle moving in a uniform gravitational field in two-dimensions.

The ordinary Lagrangian function for this case as given in (3.38), that is

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy \qquad (3.48)$$

Using the general expression

$$L = L\left(\frac{q_i - q_{i-1}}{\varepsilon}, \frac{q_i + q_{i-1}}{2}, \frac{t_i + t_{i-1}}{2}\right)^* \qquad (3.49)$$

---

*See Ref.(5), p. 38.

we can express the sum of (2.17)

$$S = \sum_{i=-\infty}^{\infty} S(q_i, q_{i-1}) . \tag{3.50}$$

For our present case, we get from (2.18) the probability amplitude

$$\langle x_b\ y_b\ t_b | x_a\ y_a\ t_a \rangle = \lim_{\varepsilon \to 0} \int\int \ldots \int \exp\{\frac{im}{2\hbar\varepsilon} \sum_{i=1}^{N} [(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2]\} \exp\{\frac{-img\varepsilon}{2\hbar} \sum_{i=1}^{N} (y_i + y_{i-1})\} dx_1 \ldots dx_{N-1} \times (\frac{2\pi i\hbar\varepsilon}{m})^{-N/2} dy_1 \ldots dy_{N-1} (\frac{2\pi i\hbar\varepsilon}{m})^{-N/2} \cdot 2^N \tag{3.51}$$

We first evaluate one of the integrals and develop a recursive relation.

$$\langle 1 \rangle = \int_{-\infty}^{\infty} (\frac{2\pi i\hbar\varepsilon}{m})^{-2/2} dx_1 \exp\{\frac{im}{2\hbar\varepsilon}[(x_2 - x_1)^2 \cdot (x_1 - x_a)^2]\} \times \int_{0}^{\infty} (\frac{2\pi i\hbar\varepsilon}{m})^{-2/2} dy_1 \exp\{\frac{im}{2\hbar}[(y_2 - y_1)^2 + (y_1 - y_a)^2]\} \times \exp\{\frac{-img\varepsilon}{2\hbar}[(y_2 + y_1) + (y_1 + y_a)]\}$$

The first factor can be integrated over x_1, by using the formula

$$\int_{-\infty}^{\infty} \exp[a(x_1 - x)^2 + b(x_2 - x)^2] dx = \sqrt{\frac{-\pi}{a+b}} \exp[\frac{ab}{a+b}(x_1 - x_2)]^2 .$$

We get for the first factor in <1>

$$\int_{-\infty}^{\infty} \left(\frac{2\pi i\hbar\varepsilon}{m}\right) dx_1 \exp\left\{\frac{im}{2\hbar\varepsilon}\left[(x_2-x_1)^2 + (x_a-x_1)^2\right]\right\}$$
$$= \left[\frac{2\pi i\hbar\cdot 2\varepsilon}{m}\right]^{-1/2} \exp\left\{\frac{im}{2\hbar\cdot 2\varepsilon}(x_2-x_a)^2\right\}$$

Now we proceed to evaluate the second factor in <1>. After a simple calculation by integrating over y_1 we get

$$\int_0^{\infty}\left[\frac{2\pi i\hbar\varepsilon}{m}\right]^{-2/2}\left[dy_1 \exp\left\{\frac{im}{2\hbar\varepsilon}\left[(y_2-y_1)^2\right] + (y_1-y_a)^2\right]\right.$$
$$\left. - \frac{img\varepsilon}{2\hbar}\left[(y_2+y_1) + (y_1+y_a)\right]\right\} = \frac{1}{2}\left[\frac{2\pi\hbar i\cdot 2\varepsilon}{m}\right]^{-1/2} \times$$
$$\exp\left\{\frac{im}{2\hbar\cdot 2\varepsilon}(y_2-y_a)^2 - \frac{img\cdot 2\varepsilon}{2\hbar}(y_2+y_a)\right\}\exp\left\{\frac{-img\varepsilon^3}{4\hbar}\right\}$$

Therefore, we obtain

$$<1> = \left[\frac{2\pi i\hbar\cdot 2\varepsilon}{m}\right]^{-1/2} \exp\left\{\frac{im}{2\hbar\cdot 2\varepsilon}(x_2-x_a)^2\right\}\times$$
$$\frac{1}{2}\left[\frac{2\pi i\hbar\cdot 2\varepsilon}{m}\right]^{-1/2} \exp\left\{\frac{im}{2\hbar\cdot 2\varepsilon}(y_2-y_a)^2 - \frac{img\cdot 2\varepsilon}{2\hbar}(y_2+y_a)\right\}\times$$
$$\exp\left\{\frac{-img\varepsilon^3}{4\hbar}\right\}$$

We multiply this by the factor

$$\left[\frac{2\pi i\hbar\varepsilon}{m}\right]^{-1/2} \exp\left\{\frac{im}{2\hbar\varepsilon}(x_3-x_2)^2\right\}\left[\frac{2\pi i\hbar\varepsilon}{m}\right]^{-1/2}\times$$
$$\exp\left\{\frac{im}{2\hbar\varepsilon}(y_3-y_2)^2\right\}\cdot\exp\left\{\frac{-img\varepsilon}{2\hbar}(y_3+y_2)\right\}$$

And integrating again, this time over x_2, y_2, we get the result

$$<2> = \left[\frac{2\pi i\hbar\cdot 3\varepsilon}{m}\right]^{-1/2} \exp\{\frac{im}{2\hbar\cdot 3\varepsilon}(x_3-x_a)^2\}\times$$

$$\frac{1}{2}2\left[\frac{2\pi i\hbar\cdot 3\varepsilon}{m}\right]^{-1/2} \exp\{\frac{im}{2\hbar\cdot 3\varepsilon}(y_3-y_a)^2 - \frac{img\cdot 3\varepsilon}{2\hbar}(y_3+y_a)$$

$$- \frac{mg\varepsilon^3}{4\hbar}(3+1)\}$$

We continue this kind of calculation for N-steps. Finally we obtain

$$<x_b\ y_b\ t_b|x_a\ y_a\ t_a> = \underset{\varepsilon\to 0}{\text{Lim}}\ \left[\frac{2\pi i\hbar\cdot N\varepsilon}{m}\right]^{-1/2}$$

$$\exp\{\frac{im}{2\hbar\cdot N\varepsilon}(x_b-x_a)^2\} \times \frac{2N}{2^N}\left[\frac{2\pi i\hbar\cdot N\varepsilon}{m}\right]^{-1/2}$$

$$\exp\{\frac{im}{2\hbar\cdot N\varepsilon}(y_b-y_a)^2 - \frac{img\cdot N\varepsilon}{2\hbar}(y_b+y_a)\} \times$$

$$\exp\{-\frac{mg\varepsilon^3}{4\hbar}(\ldots 15+10+6+3+1)\}\ .$$

After taking the limit $\varepsilon\to 0$, we obtain for the probability amplitude in space-time,

$$<x_b\ y_b\ t_b|x_a\ y_a\ t_a> = \left[\frac{m}{2\pi i\hbar\cdot T}\right]^{-1} \exp\{\frac{im}{2\hbar\cdot T}[(x_b-x_a)^2$$

$$+ (y_b-y_a)^2]\} \exp\{-\frac{img\cdot T}{2\hbar}(y_b+y_a)\} \qquad (3.52)$$

where $T \equiv t_b-t_a = N\varepsilon$.

This is the explicit expression for the probability amplitude, which is obtained directly from the standard (or original) space-time path integral calculation.

We now introduce the constraint of constant momentum conjugate to the x coordinate and evaluate the path integral

in our modified formulation.

The modified Lagrangian function for this case as given in (3.42), that is

$$\bar{L} = \frac{1}{2} m\dot{y}^2 - mgy - \frac{c_x^2}{2m} \tag{3.53}$$

We get from (3.25) the modified probability amplitude for our present case

$$\begin{aligned} \langle y_b\ t_b | y_a\ t_a \rangle = \lim_{\varepsilon \to 0} \int \cdots \int \exp\left\{ \frac{im}{2\hbar\varepsilon} \sum_{i=1}^{N} (y_i - y_{i-1})^2 \right\} \times \\ \cdot \exp\left\{ \frac{-img\varepsilon}{2\hbar} \sum_{i=1}^{N} (y_i + y_{i-1}) \right\} \times \exp\left[\frac{-ic_x^2 \cdot N\varepsilon}{2m\hbar} \right] dy_1 \cdots \\ \cdots dy_{N-1} \left[\frac{2\pi i\hbar\varepsilon}{m} \right]^{-N/2} \cdot 2^N \end{aligned} \tag{3.54}$$

After making a calculation as we did before, we are able to obtain the modified probability amplitude in the form,

$$\begin{aligned} \langle y_b\ t_b | y_a\ t_a \rangle = \left[\frac{m}{2\pi i\hbar \cdot T} \right]^{-1/2} \exp\left\{ \frac{-ic_x^2 \cdot T}{2\hbar m} \right\} \times \\ \exp\left\{ \frac{im}{2\hbar \cdot T} (y_b - y_a)^2 \right\} \times \exp\left\{ \frac{-img \cdot T}{2\hbar} (y_b + y_a) \right\} \end{aligned} \tag{3.55}$$

where $T \equiv t_b - t_a = N\varepsilon$.

This is the explicit expression for the modified probability amplitude, which is obtained directly from the modified space-time path integral calculation.

In comparing (3.52) with (3.55), we see that the normalization factors are different, in addition, equation (3.55) does not contain the ignorable coordinate x, by

contrast, it contains the factor provided by the constraint relation.

Notice that equation (3.52) is the probability amplitude (or propagator) in two-dimensional space-time, which amounts to the unitary transformation between y_b x_b at t_b and y_a x_a at t_a. However, equation (3.55) is the modified probability amplitude (or propagator) in limited one-dimensional space-time, which amounts to the unitary transformation between y_b at t_b and y_a at t_a of the non-ignorable coordinate. The ignorable coordinate x provides a constraint, which has already been built into the modified probability amplitude through the additional factor containing c_x^2.

The normalized probability amplitudes are

$$\left[\frac{2\pi i\hbar\cdot T}{m}\right]\langle y_b x_b\ t_b | y_a\ x_a\ t_a\rangle = \exp\left\{\frac{im}{2\hbar\cdot T}\ (x_b-x_a)^2+(y_b-y_a)^2\right\}\times$$

$$\exp\left\{-\frac{img\cdot T}{2\hbar}(y_b+y_a)\right\} \qquad (3.52)'$$

$$\left[\frac{2\pi i\hbar\cdot T}{m}\right]^{+1/2}\langle y_b t_b | y_a\ t_a\rangle = \exp\left\{-\frac{ic_x^2\cdot T}{2\hbar m}\right\}\exp\left\{\frac{im}{2\hbar\cdot T}(y_b-y_a)^2\right\}\times$$

$$\exp\left\{-\frac{img\cdot T}{2\hbar}(y_b+y_a)\right\} \qquad (3.55)'$$

respectively. These results are what we desired to obtain.

We will return to this special example problem at the end of Section IV, where it is used to illustrate the

probability amplitude with the constraint of constant total energy.

IV. THE IMPORTANT SPECIAL CASE OF THE MODIFIED PATH INTEGRAL IN SPACE-ENERGY

There is an important special case to which we will apply the modified path integral method. This is the case when the Hamiltonian of the system under consideration is a constant of the motion. Then, the time "t" does not appear explicitly in the Hamiltonian, so "t" is an ignorable coordinate. Therefore, "t" may be eliminated as an independent variable. After the elimination of "t", the classical Lagrangian function and consequently the modified path integral are reduced to a function of space-energy instead of space-time.

Standard mechanics uses the time "t" as an independent variable or parameter, and ignorable coordinates must be dependent variables. Therefore, we first show how the time "t" may be treated as a dependent variable if an additional parameter is introduced.

Let us consider all m+1 variables, $q_1,\ldots q_i,\ldots q_m$, and t, as given functions of some arbitrary parameter ν. The system has now m+1 degrees of freedom. Denoting derivatives with respect to ν by a prime, the action integral appears as follows:

$$S = \int_{\nu_1}^{\nu_2} L\left(q_1,\ldots q_m;\ \frac{q_1'}{t'},\ldots \frac{q_m'}{t'}\right)\ t'\ d\nu;\quad t' \equiv \frac{dt}{d\nu},\ q_i' \equiv \frac{dq_i}{d\nu} \qquad (4.1)$$

The time "t" is an ignorable coordinate in L, since only t' appears but not "t" itself.

We make use of the theorem that the conjugate momentum associated with an ignorable coordinate is a constant of the motion to construct the modified action integral.

For this purpose, let us form p_t, the momentum, which is conjugate to the time "t",

$$P_t = \frac{\partial(Lt')}{\partial t'} = L - \left(\sum_{i=1}^{m} \frac{\partial L}{\partial \dot{q}_i} \frac{q_i'}{t'^2}\right) t' = L - \sum_{i=1}^{m} p_i \dot{q}_i$$

$$= -\left(\sum_{i=1}^{m} p_i \dot{q}_i - L\right) \quad * \qquad (4.2)$$

The expression in the last parenthesis of (4.2) is exactly the total energy of the system under consideration. We denote it by Λ. For the usual mechanical system Λ is the sum of kinetic energy and potential energy, T + V. We then have the important theorem -- which holds whether the system is conservative or not -- that the momentum associated with the time "t" is the negative of the total energy.

If "t" is ignorable, i.e., if our system is conservative, then Λ is a constant, and we obtain from (4.2),

$$p_t = -\Lambda = -E = \text{constant}. \qquad (4.3)$$

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*Our Lagrangian in a new form $L_{m+1}=Lt'$, our new conjugate momentum is $p_i=(\partial L_{m+1})/(\partial q_i')$, the $i=m+1$ th coordinate is t, so that the $i=m+1$ th momentum is $p_t=(\partial L_{m+1})/(\partial t')=(\partial Lt')/(\partial t')$; $\dot{q}_i \equiv (dq_i)/(dt)$; $q_i' \equiv (dq_i)/(d\nu)$; $p_i=(\partial L)/(\partial \dot{q}_i)$.

Therefore, we can eliminate "t" from the original variational problem and obtain a new one which results in an equation that determines the path in space but says nothing about the time-development of the motion, since the time "t" appears nowhere.

According to the general scheme of eliminating ignorable coordinates we first modify the Lagrangian function. For the present case, equation (β) will be in the form

$$\bar{L} = Lt' - p_t t' = (L-p_t)t' = \sum_{i=1}^{m} p_i \dot{q}_i t' \qquad (4.4)$$

where we get the last equality from (4.2).

The modified action integral becomes

$$S = \int_{\nu_1}^{\nu_2} \sum_{i=1}^{m} p_i \dot{q}_i t' \, d\nu \, . \qquad (4.5)$$

From the relation that T is quadratic in $\dot{q}_i$, i.e.,

$$T = \frac{1}{2} \sum_{i=1}^{m} a_{i,k} \dot{q}_i \dot{q}_k ; \; p_i = \frac{\partial T}{\partial \dot{q}_i} = \sum_{k=1}^{m} a_{i,k} \dot{q}_k ,$$

we obtain

$$\sum_{i=1}^{m} p_i \dot{q}_i = \sum_{i,k=1}^{m} a_{i,k} \dot{q}_i \dot{q}_k = 2T \, . \qquad (4.6)$$

Therefore, we have

$$S = \int_{\nu_1}^{\nu_2} 2T \, t' \, d\nu \, . \qquad (4.7)$$

In the literature of the eighteenth century, this integral frequently appears in the form

$$\bar{S} = \int_{t_1}^{t_2} 2T \, dt. \tag{4.8}$$

But Jacobi pointed out that this is not satisfactory, because the time "t" may not be used as an independent variable in the present variational problem, when the total energy is a constant of the motion, and "t" is the ignorable coordinate.

Indeed, our process of reduction is not finished yet. We have to eliminate the conjugate momentum p_t of "t" by using the constraint relation,

$$p_t = - E = -(T + V) \tag{4.9}$$

Without the last step of elimination by using (4.9), the modified action function (4.7) cannot be employed directly in the variational problem. The elimination is done as follows:

We may choose a symbolic C-point to represent the mechanical system in configuration space. The kinetic energy of the system point can be written as the kinetic energy of a single particle of mass = 1.*

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*This imaginary particle is the C-point in the configuration space which symbolizes the position of the mechanical system.

$$T = \frac{1}{2} \sum_{i,k=1}^{m} a_{i,k}(q_1,\ldots q_m)\dot{q}_i\ \dot{q}_k = \frac{1}{2}\left(\frac{\tilde{ds}}{d\tau}\right)^2 \qquad (4.10)$$

where $\tilde{ds}$ is a line-element of the configuration space. We thus define the line-element $\tilde{ds}$ as

$$\tilde{ds}^2 \equiv \sum_{i,k=1}^{m} a_{i,k}\, dq_i\, dq_i^{*} . \qquad (4.11)$$

The quantities $a_{i,k}$, which are given functions of the q_i's, are the elements of a metric tensor.

For our present case, the independnnt variable of the system is no longer t but ν, so we write

$$T = \frac{1}{2}\left(\frac{\tilde{ds}}{d\nu}\right)^2 / t'^2 . \qquad (4.12)$$

Making use of the constraint relation (4.9), we get

$$t' = \frac{\left(\frac{\tilde{ds}}{d\nu}\right)}{(2T)^{1/2}} = \frac{(\tilde{ds}/d\nu)}{[2(E-V)]^{1/2}} . \qquad (4.13)$$

Finally the modified action becomes,

$$S = \int_{\nu_1}^{\nu_2} 2T\ t'\, d\nu = \int_{\nu_1}^{\nu_2} \sqrt{2(E-V)}\ \frac{\tilde{ds}}{d\nu}\, d\nu \qquad (4.14)$$

$$= \text{or} \int_{\nu_1}^{\nu_2} \sqrt{2(E-V)}\, \tilde{ds} \qquad (4.15)$$

We may define

$$F(q,q',-E;\nu) \equiv \sqrt{2(E-V)}\ \frac{\tilde{ds}}{d\nu} \qquad (4.16)$$

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*See Ref. (9), pp. 17-24.

as a modified Lagrangian function. Then

$$\bar{S} = \int_{\nu_1}^{\nu_2} F(q, q', -E; \nu) d\nu \quad * \qquad (4.17)$$

We have now completed the elimination process, and obtained the action integral in a modified form. We call this last form, equation (4.17), Jacobi's action integral.

The time "t" does not appear in $\bar{S}$. Moreover, the line-element $\tilde{ds}$** is not a complete differential and it would be quite wrong to believe that the integrand of $\bar{S}$ is $\sqrt{2(E-V)}$; $\tilde{ds}$ corresponds to the differential of the independent variable. In order to prevent this misunderstanding we put a tilde above ds. Some parameter must be chosen as an independent variable. In particular, we may take ν to be one of the q_i, which are now functions of ν instead of t.

The principle of minimizing the integral (4.14) or (4.17) in order to find the paths of the mechanical system is called Jacobi's first action principle. This principle determines the paths of the C-point in configuration space at constant energy, but says nothing about the time-development of the

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* See Ref. (10), p. 234.

** $\tilde{ds}$ represents a non-integrable differential, which can not be considered as the infinitesimal change of something, like dq is "d of q". $\tilde{ds}$ has to be conceived as an infinitesimal expression and not as "d of s". See Ref. (9), p. 18, footnote for more details.

motion since "t" appears nowhere.

Putting $a_{i,k}=m_{i,k}$, we get from (4.11) the line-element

$$\widehat{ds} = \sqrt{\sum_{i,k=1}^{m} m_{i,k} \frac{dq_i dq_k}{d\nu \; d\nu}} \; d\nu . \tag{4.18}$$

Therefore, the modified action integral is

$$\bar{S} = \int_{\nu_1}^{\nu_2} \sqrt{2(E-V)} \sqrt{\sum_{i,k=1} m_{i,k} \frac{dq_i}{d\nu} \frac{dq_k}{d\nu}} \, d\nu \tag{4.19}$$

or

$$\bar{S} = \int_{\nu_1}^{\nu_2} F(q_i, q_i', -E\nu) d\nu; \; q_i' \equiv \frac{dq_i}{d\nu} \tag{4.20}$$

where

$$F = \sqrt{2(E-V)} \sqrt{\sum_{i,k=1}^{m} m_{i,m} \frac{dq_i}{d\nu} \frac{dq_k}{d\nu}} \; . \tag{4.21}$$

Applying Jacobi's first action principle, the paths of the system are determined by the vanishing of the variation,

$$\delta\bar{S} = \delta \int_{\nu_1}^{\nu_2} F d\nu = 0 . \tag{4.22}$$

Finally, we get the modified Euler-Lagrange equations in parametric form:

$$\frac{d}{d\nu} \left(\frac{\partial F}{\partial q_i'}\right) - \frac{\partial F}{\partial q_i} = 0 \tag{4.23}$$

There is a question, what is the reason that allowed us to use the Euler-Lagrange equations when only paths of constant energy are allowed in variations?

Let us examine the line-element as given in (4.19), which does not contain time t. E is a constant and V is a function of q_i only. Therefore, the time "t" does not appear in the modified action integral (4.20). Hence Jacobi's first action principle can be used to furnish differential equations for the path. Moreover, in the present case, the parameter "ν" is a geometrical property of the paths which conserve energy. Therefore, it can be so chosen as to be constant during the displacements which characterize the δ-variation* with respect to ν. In contrast, in Hamilton's first action principle, "t" is the time-development of the motion, therefore, it can be so chosen as to be constant during the displacements which characterize the δ-variation with respect to time t.

We now make use of the modified action integral and Jacobi's first action principle to formulate the path integral in space-energy.

Dirac's idea can be applied to this special case. The contact transformation, in classical mechanics and quantum mechanics, connect two sets of dynamical variables at different parameters instead of at different times.

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*Goldstein uses Δ-variation with respect to ν, to distinguish it from δ-variation with respect to time t. See Ref. (10) pp. 228-234.

The classical transformation equations for the present case can be put into the same forms as before

$$p_i = \frac{\partial \bar{S}}{\partial q_i} \quad , \quad P_i = - \frac{\partial \bar{S}}{\partial Q_i} \quad . \tag{4.24}$$

Now $\bar{S}$ is defined in (4.19), and $p_k(\nu)$, $q_k(\nu)$; $P_k(\nu)$, $Q_k(\nu)$, are all functions of the parameter ν. Later we will choose $q_k(\nu)$ and $Q_k(\nu)$ to be the same representation at different parameters.

There is also a quantum mechanical transformation function $<q'|Q'>$ connecting two representations, in which q's and Q's are diagonal respectively.

We shall demonstrate that this transformation function is the quantum analogue of $\exp(i\bar{S}/\hbar)$, where $\bar{S}$ is defined in (4.19).

Suppose α is any function (or functional) of dynamical variables. It will have a "mixed" representation $<q'|\alpha|Q'>$, which may be defined in terms of either of the usual representations $<q'|\alpha|q''>$, $<Q'|\alpha|Q''>$, (from now on all the q's and Q's are functions of the parameter ν instead of time t).

That is

$$<q'|\alpha|Q'> \equiv \int <q'|\alpha|q''>dq''<q''|Q'> = \int <q'|Q''>dQ''<Q''|\alpha|Q'>. \tag{4.25}$$

From the first of these definitions we obtain

$$\langle q' | q_i | Q' \rangle = \int \langle q' | q_i | q'' \rangle \, dq'' \langle q'' | Q' \rangle \, .$$

Since

$$\langle q' | q_i | q'' \rangle = q_i' \, \delta(q' - q'')$$

therefore

$$\langle q' | q_i | Q' \rangle = q_i' \int \delta(q' - q'') dq'' \langle q'' | Q' \rangle = q_i' \langle q' | Q' \rangle \qquad (4.26)$$

Similarly, one obtains the following relations for the other mixed matrix representations:

$$\langle q' | p_i | Q' \rangle = \frac{\hbar}{i} \int \delta'(q' - q'') dq'' \langle q'' | Q' \rangle = -i\hbar \frac{\partial}{\partial q_i'} \langle q' | Q' \rangle \qquad (4.27)$$

$$\langle q' | Q_i | Q' \rangle = Q_i' \int \langle q' | Q'' \rangle dQ'' \, \delta(Q' - Q'') = Q_i' \langle q' | Q' \rangle \qquad (4.28)$$

$$\langle q' | P_i | Q' \rangle = -\frac{\hbar}{i} \int \langle q' | Q'' \rangle dQ'' \, \delta'(Q' - Q'') = +i\hbar \frac{\partial}{\partial Q_i'} \langle q' | Q' \rangle \qquad (4.29)$$

Note the different sign in (4.27) and (4.29).

Consequently, for two functions $f(q)$ and $g(Q)$ of dynamical variables, we obtain

$$\langle q' | f(q) g(Q) | Q' \rangle = \iint \langle q' | f(q) | q'' \rangle dq'' \langle q'' | Q'' \rangle dQ'' \langle Q'' | g(Q) | Q' \rangle$$

$$= f(q') g(Q') \langle q' | Q' \rangle$$

Furthermore, for two sets of functions $f_k(q)$ and $g_k(Q)$, $(k=1,2,\ldots r)$, the result is

$$\langle q' | \Sigma_k f_k(q) g_k(Q) | Q' \rangle = \Sigma_k f_k(q') g_k(Q') \langle q' | Q' \rangle \ .$$

Thus, for a function $\alpha(q,Q)$ of the dynamical variables, which expressed in a "well-ordered" way, so that it consists a sum of terms like $f(q)g(Q)$, we obtain the result

$$\langle q' | \alpha(q,Q) | Q' \rangle = \alpha(q',Q') \langle q' | Q' \rangle \qquad (4.30)$$

This relation connects $\alpha(q,Q)$, which is a function of operators, and $\alpha(q',Q')$, which is a function of numerical variables. The underlying parameter of transformation is ν instead of t.

We follow Dirac to make the postulate that

$$\langle q' | Q' \rangle = \exp\left[\frac{i\bar{U}(q',Q')}{\hbar}\right] \qquad (4.31)$$

since each side amounts to a unitary transformation, where $\bar{U}$ is a new function of the $q^{\prime}$s and $Q^{\prime}$s.

Using this postulation, we get from (4.27) for $\alpha = p_i$

$$\langle q' | p_i | Q' \rangle = \frac{\partial \bar{U}(q', Q')}{\partial q_i'} \langle q' | Q' \rangle$$

By comparing with (4.30), we obtain a relation between operators,

$$p_i = \frac{\partial \bar{U}(q,Q)}{\partial q_i}$$

which holds provided $\partial \bar{U}(q,Q)/\partial q_i$ is well-ordered, where ν is

the independent variable (parameter).

Similarly, applying the result of (4.29) for $\alpha = P_i$ and comparing with (4.30), we obtain another relation between operators.

$$P_i = - \frac{\partial \bar{U}(q,Q)}{\partial Q_i}$$

which holds provided $(\partial U(q,Q))/(\partial Q_i)$ is well-ordered in the modified space.

These equations are of the same form as (4.24), thus showing that $\bar{U}$ defined by (4.31) is the classical analogue of the action function $\bar{S}$, that is

$$\bar{U} \equiv \bar{S} = \int_{\nu_1}^{\nu_2} \sqrt{2(E-V)} \sqrt{\sum_{i,k}^{m} m_{i,k} \frac{dq_i}{d\nu} \frac{dq_k}{d\nu}} \; d\nu$$

We find therefore, the result that the classical equation of the path causes the dynamical variables to vary from one parameter ν_a to another ν_b in accordance with the contact transformation relations $p_i = \frac{\partial \bar{S}}{\partial q_i}$, $P_i = \frac{\partial \bar{S}}{\partial Q_i}$, with $q,p = q(\nu_a), p(\nu_a)$; $Q,P = q(\nu_b), p(\nu_b)$. Here $q(\nu_a)$, $q(\nu_b)$ are the same representation at different parameters, and $\bar{S}$ is the modified action function, i.e., the integral over the parameter of the modified Lagrangian F over a range ν_a and ν_b.

The quantum mechanical contact transformation functions $\langle q_b \; \nu_b | q_a \nu_a \rangle$ connect two representations in which $q(\nu_a)$ and $q(\nu_b)$ are diagonal respectively.

The work of the preceding section shows a correspondence relationship of the classical and quantum mechanical contact transformation,

$$\langle q_b\ \nu_b | q_a\ \nu_a \rangle \sim \exp[\tfrac{i}{\hbar} \int_{\nu_a}^{\nu_b} F d\nu]. \tag{4.32}$$

For ν_a differing infinitesimally from ν_b, we obtain

$$\langle q_b\ \nu_a+\sigma | q_a\ \nu_a \rangle \simeq \exp[\tfrac{i}{\hbar} F\sigma] \tag{4.33}$$

where σ is the infinitesimal change in parameter ν.

It is satisfactory to find that the quantum transformation functions have their classical analogues, which can be expressed simply as a modified action integral. This is a manifestation of the well-known result, familiar from the JWKB method, that the phase of the wave function corresponds to Jacobi's action integral.

For simplicity, we continue using one-dimension. We adopt the notation,

$$\exp(\tfrac{i}{\hbar} \int_{\nu_a}^{\nu_b} F d\nu) = B(\nu_b, \nu_a)$$

so that $B(\nu_b,\nu_a)$ is the classical analogue of $\langle q_b\ \nu_b | q_a\ \nu_a \rangle$.

Using the same treatment as given before, we divide $\nu_b \to \nu_a$ into small sections $\nu_a \to \nu_1$, $\nu_1 \to \nu_2, \ldots \nu_{m-1} \to \nu_m$, $\nu_m \to \nu_b$, then

$$B(\nu_b,\nu_a) = B(\nu_b,\nu_m)B(\nu_m,\nu_{m-1}) \ldots B(\nu_2,\nu_1)B(\nu_1,\nu_a) \tag{4.34}$$

In quantum mechanics, we have

$$\langle q_b\ \nu_b | q_a\ \nu_a\rangle = \int\cdots\int \langle q_b|q_m\rangle dq_m \langle q_m|q_{m-1}\rangle dq_{m-1}$$

$$\cdots \langle q_2|q_1\rangle dq_1 \langle q_1|q_a\rangle \qquad (4.35)$$

where q_k denote q at the intermediate ν_k (k=1,2,...m).

Now we show how in the correspondence limit (4.35) goes over into (4.34) by means of Jacobi's action principle. From its appearance (4.35) does not give us the confidence that it corresponds properly to (4.34), for on the right-hand side of (4.35) we have to integrate after performing the multiplication while on the right-hand side of (4.34) there is no integration.

Let us examine this discrepancy by asking what becomes of (4.35) when we take ν as extremely small. From the relationship of (4.32) and (4.33) we find the integrand in (4.35) has to be the form $\exp(\frac{i\bar{G}}{h})$ where $\bar{G}$ is a function of $q(\nu_a)$, $q(\nu_1)\ldots q(\nu_m), q(\nu_b)$, which remains finite as $\hbar$ tends to zero.

Consider one of the intermediate q's, say $q(\nu_k)$, more carefully. On account of the smallness of $\hbar$, we shall have $\bar{G}/\hbar$ varying extremely rapidly. This indicates that $\exp(\frac{i\bar{G}}{\hbar})$ will oscillate rapidly about zero. As a result, the integral of this kind of integrand is obviously zero. Therefore, the only important contribution comes from the domain of the

$q(\nu_k)$ integration for which a comparatively large variation in $q(\nu_k)$ produces only a very small variation in $\bar{G}$. This is the contribution from the neighborhood of the point for which $\bar{G}$ is stationary with respect to small variation in $q(\nu_k)$. In other words, for small $\hbar$, the only contribution to the integral comes from the range $q(\nu_k)$ on or near the classical path.

This argument applies to any one of the intermediate q's in the right-hand side of (4.35) and we obtain the result that the only important contribution to the integral comes from the path for which $\bar{G}$ is stationary for small variations in all the intermediate q's. But by applying (4.33) to each of the small parameter sections, we find that $\bar{G}$ has for its classical analogue

$$\int_{\nu_m}^{\nu_b} F d\nu + \int_{\nu_{m-1}}^{\nu_m} F d\nu + \ldots \int_{\nu_1}^{\nu_2} F d\nu + \int_{\nu_a}^{\nu_1} F d\nu = \int_{\nu_a}^{\nu_b} F d\nu$$

which is just the modified action function $\bar{S}$, which Jacobi's first action principle requires to be stationary for small variation in all the q's. We thus prove (4.35) goes over into the classical result of (3.34) when $\hbar$ becomes extremely small. For the general case, when $\hbar$ is not small, we have to explain (4.34) in the following way.

Each of the quantities $B(q_k, q_{k-1})$ has to be considered as a function of the q's at the two ends parameters to which

it refers. The right-hand side is therefore a function of all the $q(\nu_a), q(\nu_1), \ldots q(\nu_m), q(\nu_b)$. In order to obtain from it a function of $q(\nu_a), q(\nu_b)$ only, we have to substitute for $q(\nu_1), q(\nu_2), \ldots q(\nu_m)$ their values given by Jacobi's action principle. This process of substitution for the intermediate q's then corresponds to the integration over all values of the q's in (4.35).

Equation (4.35) contains the quantum analogue of Jacobi's action principle, as may be seen more explicitly from the following argument.

From (4.35), we extract a statement that, if values of $q(\nu_a), q(\nu_b)$ are specified, then the importance of our considering any set of values for the intermediate q's is determined by the importance of that set of values in the integration on the right-hand side of (4.35).

If we now let $\hbar$ tend to zero, this statement goes over into the classical statement that, if the values of $q(\nu_a)$, $q(\nu_b)$ are specified, then the importance of our considering any set of the values for the intermediate q's is zero unless these values make the modified action function $\bar{S}$ stationary. This statement is one way of formulating the classical Jacobi's action principle.

So far, we have repeated the contents of Dirac's work. The only variety was to replace the time "t" by a parameter

ν, now $t(\nu)$ can be seen as an additional dependent coordinate. But "t" is an ignorable coordinate in the conservative system under consideration. After the reduction, "t" is not contained in the modified Lagrangian function F at all. Therefore, the action principle can be written

$$\delta\bar{S} = \delta\int_{\nu_1}^{\nu_2} F d\nu = 0; \text{ with } F = F(q,\dot{q},-E;\nu)$$

instead of the more familiar form,

$$\delta S = \delta\int_{t_1}^{t} L dt = 0; \text{ with } L = L(q,\dot{q};t).$$

Now F satisfies the modified Euler-Lagrange equations

$$\frac{d}{d}(\frac{\partial F}{\partial q_i'}) = \frac{\partial F}{\partial q_i} = 0$$

which are the consequence of Jacobi's action principle. And L satisfies the ordinary Euler-Lagrange equations

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0$$

which are the consequence of Hamilton's action principle. Moreover, in Jacobi's action principle, we may let the parameter ν be any one of the coordinates, which are now free from the time "t".

We have thus established a correspondence relation between a time-free quantum mechanics and Jacobi's action principle.

We now proceed to use Feynman's idea in formulating a modified path integral in space-energy.

Feynman's identity can be taken over by using a new normalization factor $\mathcal{A}$ instead of A. That is

$$\langle q_{m+1}, \nu_{m+1} | q_m, \nu_m \rangle \equiv \frac{1}{\mathcal{A}} \exp\left[\frac{i}{\hbar}\int_{\nu_m}^{\nu_{m+1}} F(q,q',-E;\nu)d\nu\right] \quad (4.36)$$

The modified action function is expressed in the form

$$\bar{S} = \sum_{m=-\infty}^{\infty} \bar{S}(q_{m=1}, q_m) \quad (4.37)$$

where

$$\bar{S}(q_{m+1}, q_m) = \text{Min}\int_{\nu_m}^{\nu_{m+1}} F(q,q',-E;\nu)d\nu \quad (4.38)$$

We put (4.36) into (4.35) and use the Jacobi's first action principle as expressed in (4.38), which is the argument we gave above. We obtain the modified path integral in space-energy

$$\langle q_b\ \nu_b | q_a\ \nu_a \rangle = \lim_{\sigma \to 0} \int_{\mathcal{R}} \exp\left[\frac{i}{\hbar} \sum_{m=-\infty}^{\infty} \bar{S}(q_{m+1}, q_m)\right] \frac{dq_{m+1}}{\mathcal{A}} \frac{dq_m}{\mathcal{A}} \cdots \quad (4.39)$$

This is a probability amplitude, which is now expressed in a path integral form, since the integration is taken over these values $q_m, q_{m+1}, \ldots$ which lie in a special region $\mathcal{R}$.

Figure 3 shows this schematically for paths in space-energy configuration.

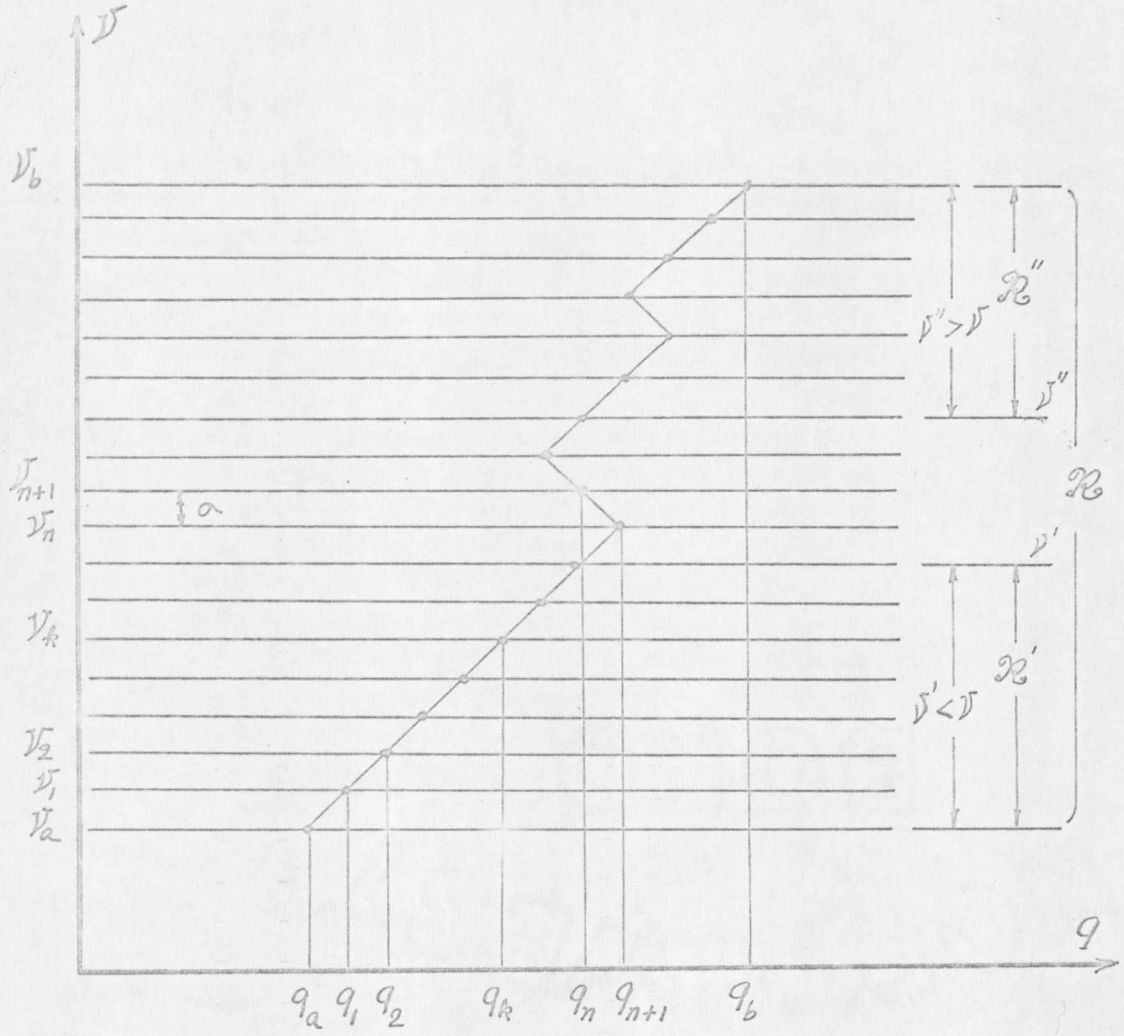

Figure 3 Paths in Space-Energy

The paths of this picture are different from those of Fig. 1 and Fig. 2 in that, here the paths are limited to those which conserve energy, the conjugate "momentum" of the ignorable coordinate time "t". Therefore, the time "t" does not appear. In contrast, the parameter ν takes its place. The path sum is an integral over those specific coordinate values which conserve energy. The limit is taken as $\sigma \to 0$.

We continue to illustrate the equivalence of this special formulation and the conventional formulation of quantum mechanics.

We first give the definition of the wave function in the new formulation and then show that it satisfies the time-independent Schroedinger's equation.

We shall find that is is possible to express $\bar{S}$ as the sum given in (4.37), and hence the probability amplitude as a product of contributions from successive sections of the path, which allows us to define a quantity having the properties of a wave function.

The argument of the classification $\mathcal{R}$ into three regions is the same as given before, the only change made is to replace t by ν. This is shown on the Figure 3. Then the exponential relation in (4.39) may be put into the form

$$\exp[\tfrac{i}{\hbar}\sum_{m=-\infty}^{\infty}\bar{S}(q_{m+1},q_m)] = \exp[\tfrac{i}{\hbar}\sum_{m=k}^{\infty}\bar{S}(q_{m-1},q_m)]$$

$$\exp[\tfrac{i}{\hbar}\sum_{m=-\infty}^{k-1}\bar{S}(q_{m+1},q_m)] \qquad (4.40)$$

This split is possible because of (4.37), which results from the fact that the modified function F is a function of positions at different parameter ν_k's. Therefore we obtain the probability amplitude in the following form

$$\langle q_b\ \nu_b | q_a\ \nu_a \rangle = \int \chi^*(q_k\ \nu_k)\psi(q_k\ \nu_k)dq_k \tag{4.41}$$

where

$$\psi(q_k\ \nu_k) = \underset{\sigma\to 0}{\text{Lim}} \int_{\mathcal{R}'} \exp\left[\frac{i}{\hbar}\sum_{m=-\infty}^{k-1} S(q_{m+1}, q_m)\right] \frac{1}{\mathcal{A}}\frac{dq_{k-1}}{\mathcal{A}}\frac{dq_{k-1}}{\mathcal{A}} \cdots \nu' < \nu \tag{4.42}$$

$$\chi^*(q_k\ \nu_k) = \underset{\sigma\to 0}{\text{Lim}} \int_{\mathcal{R}''} \exp\left[\frac{i}{\hbar}\sum_{m=k}^{\infty} S(q_{m+1}, q_m)\right] \frac{dq_{k+1}}{\mathcal{A}}\frac{dq_{k+2}}{\mathcal{A}} \cdots \nu'' > \nu \tag{4.43}$$

The $\mathcal{R}'$ indicates that the coordinates are integrated over the region $\mathcal{R}'$ for $\nu' < \nu_m < \nu$, over limited space; and $\mathcal{R}''$ is over $\mathcal{R}''$, for $\nu'' > \nu_m > \nu$, over limited space.

The probability of finding the system at a certain position will be $\left|\int \chi^* \psi dq\right|^2$. All these results agree with the principles of conventional quantum mechanics.

We approach the evolution of the path of the system point in configuration space (now in space-energy) by seeing that, for a small finite σ, equation (4.42) permits us to construct a recursive relation. That is

$$\psi(q_{k+1}, \nu_k + \sigma) = \underset{\sigma\to 0}{\text{Lim}} \int_{\mathcal{R}'} \exp\left[\frac{i}{\hbar}\sum_{m=-\infty}^{k} \bar{S}(q_{k+1}, q_k)\right] \frac{dq_k}{\mathcal{A}}\frac{dq_{k-1}}{\mathcal{A}}\frac{dq_{k-2}}{\mathcal{A}} \cdots \qquad \nu' < \nu \tag{4.22}'$$

In comparing (4.42) with (4.42)' we immediately find

$$\psi(q_{k+1},\nu_k+\sigma) = \int \exp\left[\frac{i}{\hbar}\bar{S}(q_{k+1},q_k)\right]\cdot\psi(q_k,\nu_k)\,\frac{dq_k}{\mathcal{A}}$$

$$\overset{\text{or}}{=} \int \langle q_{k+1},\nu_{k+1}|q_k,\nu_k\rangle\,\psi(q_k,\nu_k)\,dq_k \qquad (4.44)$$

The quantity

$$\langle q_{k+1},\nu_{k+1}|q_k,\nu_k\rangle = \frac{1}{\mathcal{A}}\exp\left[\frac{i}{\hbar}\int_{\nu_k}^{\nu_k+\sigma} F(q,q',-E;\nu)\,d\nu\right]$$

is the kernel (or propagator) for the change of the wave function when the parameter changes infinitesimally. This kernel is Feynman's identity as defined in (4.36), which is exactly the unitary transformation of the same representation q at two different parameters, if the difference of them $\nu_{k+1}-\nu_k = \sigma$ is small but finite.

We give here two concrete examples as we did in Section II, in order to show that relationship of (4.44) and the corresponding Schroedinger's equation.

For the free particle case (using one dimension): the parameter ν is chosen to be the coordinate x, therefore (4.44) becomes an integral equation of the form

$$\phi(x) = \int_{-\infty}^{\infty}\frac{1}{\mathcal{A}}\exp\left[\frac{i}{\hbar}\int_y^x \sqrt{2mE}\,d\nu\right]\cdot\phi(y)\,dy \qquad (4.45)$$

since $F = \sqrt{2mE}$.

Using the fact that $\frac{d}{dx}\int_y^x G d\theta = G$, we obtain

$$\frac{d\phi}{dx} = \frac{i}{\hbar}\sqrt{2mE}\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}\exp\left[\frac{i}{\hbar}\int_y^x\sqrt{2mE}\,d\nu\right]\cdot\phi(y)dy \qquad (4.46)$$

$$= \frac{i}{\hbar}\sqrt{2mE}\,\phi(x) = \frac{i}{\hbar}p_x\phi(x);\ \text{with } p_x = \sqrt{2mE}$$

Therefore

$$\frac{\hbar}{i}\frac{d\phi(x)}{dx} = p_x\cdot\phi(x) \qquad (4.45)$$

Taking the derivative of (4.46) with respect to x again, we obtain

$$\frac{d^2\phi}{dx^2} = \left[\frac{i}{\hbar}\sqrt{2mE}\right]^2\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}\exp\left[\frac{i}{\hbar}\int_y^x\sqrt{2mE}\,d\nu\right]\cdot\phi(y)\,dy$$

$$= -\frac{2m}{\hbar^2}E\,\phi(x)$$

Therefore

$$-\frac{h^2}{2m}\frac{d^2\phi(x)}{dx^2} = E\,\phi(x) \qquad (4.48)$$

a result we expect to have, since it is the time-independent Schroedinger's equation of a free particle moving in one-dinemsion, when the energy is a constant E. Consequently (4.48) gives us a desired result without time "t" involved.

For the second example, consider a particle moving in a one-dimensional potential V(x):

Then (4.44) will become an integral equation

$$\phi(x) = \int_{-\infty}^{\infty} \frac{1}{\mathcal{A}} \exp\left[\frac{i}{\hbar} \int_{y}^{x} \sqrt{2m[E-V(\nu)]}\, d\nu\right] \phi(y)dy \qquad (4.49)$$

since $F = \sqrt{2m[E-V(\nu)]}$.

For this case, if the interval $x-y = \sigma$ is extremely small, as it is in our formulation, we may take $p_x = \sqrt{2m\ E-V(x)}$ to be a constant.

If this is achieved, then

$$\phi(x) = \int_{-\infty}^{\infty} \frac{1}{\mathcal{A}} \exp\left\{\frac{i}{\hbar} \sqrt{2m[E-V(x)]}\ (x-y)\right\}\phi(y)dy \qquad (4.50)$$

After differentiating (4.50) once with respect to x, we obtain

$$\frac{\hbar}{i} \frac{d}{dx}\ \phi(x) = p_x\ \phi(x);\ \text{with}\ p_x = \sqrt{2m[E-V(x)]} \qquad (4.51)$$

And differentiating (4.50) twice with respect to x, we obtain

$$-\frac{\hbar^2}{2m} \frac{d^2\phi(x)}{dx^2} + V(x)\phi(x) = E\ \phi(x) \qquad (4.52)$$

This is the desired time-independent Schroedinger's equation. When the energy is conserved in the system, the time "t" does not appear.

The apparent inconsistency of considering $x-y = \sigma$ small, and integrating over y with x fixed may be resolved by recalling that the rapid oscillations of the exponential factor eliminate contributions except in the immediate

neighborhood of x.

From these two examples, we see, the integral equations (4.45) and (4.49) are equivalent to the corresponding Schroedinger's equations (4.48) and (4.52), respectively, a direct consequence of the space-energy path integral formulation.

For the sake of completeness, we exhibit the explicit form of the space-energy kernel, which can also be shown to be the Green's function of the time-independent Schroedinger's equation.

We have shown in Section II that, for a constant Hamiltonian, the kernel in space-time is in the form of (2.41)

$$\begin{aligned} \langle x_2\ t_2 | x_1\ t_1\rangle &= \sum_{n=1}^{\infty} \phi_n(x_2)\phi_n^*(x_1)\ e^{-i/\hbar E_n(t_2-t_1)} \qquad \text{for } t_2>t_1 \\ &= 0 \qquad\qquad\qquad\qquad\qquad \text{for } t_2<t_1 \end{aligned} \tag{4.41}$$

We want to eliminate the time coordinate, therefore, the space-time kernel will be reduced to the space-energy kernel.

If the time variable only is transformed but not the spatial ones, we obtain by definition

$$\langle x_2\ E(\nu_2) | x_1\ E(\nu_1)\rangle^* \equiv \iint e^{i/\hbar E_n t_2}\ \langle x_2\ t_2 | x_1\ t_1\rangle\ e^{-i/\hbar E_n t_1}\ dt_2\ dt_1 \tag{4.53}$$

*$\langle x_2\ E_2|x_1\ E_1\rangle \equiv \langle x_2\ E(\nu_2)|x_1\ E(\nu_1)\rangle$ is the probability amplitude or the kernel, which is a unitary transformation of the same representation at different parameter ν_1 and ν_2, which conserve energy E.

Substituting (2.41) into (4.53), we obtain

$$\langle x_2\ E(\nu_2)|x_1\ E(\nu_1)\rangle = \int_{-\infty}^{\infty} dt_1 \int_{t_1}^{\infty} dt_2\ e^{+i/\hbar E_n t_2}$$

$$\sum_{n=1}^{\infty} \phi_n(x_2)\ \phi_n(x_1) e^{-i/\hbar E_n (t_2-t_1)} \cdot e^{-i/\hbar E_n t_1}$$

Making the substitution $t_2 = t_1 + \tau$, $dt_2 = d\tau$, keeping t_1 fixed

$$\langle x_2\ E(\nu_2)|x_1\ E(\nu_1)\rangle = \int_{-\infty}^{\infty} dt_1\ e^{i/\hbar (E_2-E_1)t_1}$$

$$\int_0^{\infty} d\tau\ e^{i/\hbar E_2 \tau} \sum_{n=1}^{\infty} \phi_n(x_2)\ \phi_n^*(x_1)$$

$$= 2\pi\hbar\ \delta(E(\nu_2)-E(\nu_1)) \sum_{n=1}^{\infty} \phi_n(x_2)\ \phi_n^*(x_1) \int_0^{\infty} e^{i/\hbar (E_2-E_1)\tau} d\tau$$

After the evaluation of the last factor, we obtain the kernel in space-energy.

$$\langle x_2\ E(\nu_2)|x_1\ E(\nu_1)\rangle = 2\pi\hbar i \delta\left[E(\nu_2)-E(\nu_1)\right]$$

$$\sum_{n=1}^{\infty} \frac{\phi_n(x_2)\phi_n^*(x_1)}{\left[E(\nu_2)-E(\nu_n)+i\varepsilon\right]} \quad \text{for } \nu_2 > \nu_1 \qquad (4.45)$$

$$= 0 \qquad \text{for } \nu_2 < \nu_1$$

This can be shown to be the Green's function of the time-independent Schroedinger's equation in the form

$$\left(-\frac{\hbar^2 d^2}{2m dx_2^2} + V - E_2\right)\langle x_2\ E_2|x_1\ E_1\rangle = +\frac{\hbar}{i}\delta(x_2-x_1)\delta(E_2-E_1). \tag{4.55}$$

Letting

$$\langle x_2\ E_2|x_1\ E_1\rangle \equiv \langle x_2\ E(\nu_2)|x_1\ E(\nu_1)\rangle = \sum_n a_n(x_1)\phi_n(x_2). \tag{4.56}$$

Substituting (4.56) into (4.55), we get

$$\Sigma a_n(x_1)(E_n-E_2)\phi_n(x_2) = +\frac{\hbar}{i}\,\delta(x_2-x_1)\delta(E_2-E_1) \tag{4.57}$$

since $$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx_2^2} + V\right]\phi_n(x_2) = E_n\ \phi_n(x_2).$$

Recall

$$\delta(x_2-x_1) = \sum_n \phi_n(x_2)\phi_n^*(x_1)\ . \tag{4.58}$$

Substituting (4.58) into (4.57), we get

$$\sum_n a_n(x_1)(E_n-E_2)\ \phi_n(x_2) = +\frac{\hbar}{i}\sum_n \phi_n(x_1)\delta(E_2-E_1)\ . \tag{4.59}$$

Comparing both sides, we have

$$a_n(x_1) = i\,\hbar\,\delta(E_2-E_1)\ \frac{\phi_n^*(x_1)}{[E_2-E_n+i\varepsilon]}\ .$$

Finally, we obtain from (4.56)

$$\langle x_2\ E(\nu_2)|x_1\ E(\nu_1)\rangle = i\hbar[\delta E(\nu_2)-E(\nu_1)]$$
$$\sum_{n=1}^{\infty} \frac{\phi_n(x_2)\phi_n^*(x_1)}{[E(\nu_2)-E(\nu_n)+i\varepsilon]} \qquad \text{for } \nu_2'>\nu_1$$
$$= 0 \qquad \text{for } \nu_2<\nu_1 \qquad (4.60)$$

This is exactly equation (4.54).

We have, therefore, shown the close relationship of the space-energy path integral method and conventional quantum mechanics.

It is desireable to calculate the path integral directly in the new formulation, and compare it with the old ones. For this purpose, we choose the example of Section III. That is the motion of a particle in a uniform gravitational field in two-dimensions.

The ordinary Hamiltonian function expressed in Cartesian coordinates is in the form:

$$H = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + mgy \quad .$$

For a conservative system, the Hamiltonian H = E is a constant of the motion, the modified Lagrangian function F is

$$F = \sqrt{2m(E-V)} \quad = \sqrt{2m(E-mgy)} \quad .$$

The special probability amplitude should be given by the expression:

$$\langle y_b\ \nu_b | y_a\ \nu_a \rangle = \lim_{\sigma\to 0}\int\int..\int \exp\left\{\frac{i\sigma}{\hbar}\sum_{i=1}^{N}\sqrt{2m\left[E-mg\left(\frac{y_i+y_{i+1}}{2}\right)\right]}\right\}\times$$

$$dy_1\ ..\ dy_{N-1}(\mathcal{A})^{-N/2}. \qquad (4.61)$$

The calculation should be carried out as follows:

$$\langle 1\rangle = \int_0^\infty dy_1(\mathcal{A})^{-2/2}\ \exp\left[\frac{i\sigma}{\hbar}\left\{\sqrt{2m\left[E-mg\left(\frac{y_2+y_1}{2}\right)\right]}\right.\right.$$

$$\left.\left.+\sqrt{2m\left[E-mg\left(\frac{y_1+y_a}{2}\right)\right]}\right\}\right].$$

We have tried to calculate this, but have not been able to get a definite answer. Therefore, we have to leave it open. That is the price we have to pay for the modified path integral formulation. However, the physics of this formulation is quite clear, and provides in principle a direct extension of the standard path integral method to obtain the time-independent quantum mechanics of energy conserving systems.

V. CONCLUSION

The development of the path integral method in space-time was reviewed briefly following the ideas of Dirac and Feynman. We found the physical "incompleteness" of that formulation, mentioned by Feynman, can be treated by using the classical technique of eliminating ignorable coordinates in the system under consideration. That is, whenever there is an ignorable coordinate in the Lagrangian function, the corresponding conjugate momentum will provide a constraint relation. With the help of this relation, the Lagrangian function can be reduced to a modified form. This new Lagrangian function can then be taken to form a modified action function, which together with the corresponding action principle can be employed to form a modified path integral constructed of -- "the paths in a limited region" or "the paths having some property." "That region" or "some property" is clearly related to the constraint relation provided by the conservation of the conjugate momentum associated with the ignorable coordinate.

It is to be noted that the general modified form of path integral uses the time "t" as an independent variable in the variational problem, even though the modified Lagrangian reduces the freedom of the paths to some degree. Consequently

the characteristics of the space-time path integral are not changed too much. A concrete example of a particle moving in two-dimensions under a uniform gravitational field was given to illustrate this method.

Since for systems with a constant Hamiltonian the time "t" may be considered as an ignorable coordinate, we applied our technique to this case to derive the time-independent Schroedinger's equation via the path integral method. This special case illustrates the close connection established with classical mechanics because the point of departure is Jacobi's action principle instead of Hamilton's action principle. The time "t" is missing in this formulation, but instead we must introduce a parameter "γ" which parameterizes the energy conserving paths of the system.

Our technique clarifies the relation of classical mechanics to quantum mechanics, since we found that not only the customary space-time Lagrangian may be used to quantize a system via the path-integral method; but also any transformed or modified Lagrangian, obtained by the elimination of ignorable coordinates, can equally well serve in the path-integral method to quantize the system.

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