The influence of plastic flow upon the stresses developed in short span reinforced concrete beams
by Dimitri Lambropulos

A THESIS Submited to the Graduate Committee in partial fulfillment of the requirements for the degree of Master of Science in Civil Engineering
Montana State University
© Copyright by Dimitri Lambropulos (1948)

Abstract:
In this thesis, it has been attempted to determine the actual stress variation In the concrete on the compression -side of short-span reinforced concrete beams* and to prove that the "Straight-Line Theory", adopted up to now, gives only a portion of the stresses developed,since, in the deri-vation of its formulae, the effects of plastic flow and shrinkage on the final stresses set up in the concrete are not considered.

Data and results were obtained from' tests conducted on two short beams; one with web reinforcement designed by the plastic flow method, and one assuming straight-line stress-strain variation.

The actual stress distribution on the compression side of tho reinforced concrete beams tested-, as obtained from the results shorn in Tables IV and V, are shorn in Figures 15 and 16. 4 line representing the distri-button of stress in the concrete is curved, somewhat close to the shape' of a parabola, but far from a straight line.

The capacity of the test machine was not sufficient to break tho beams; therefore, a comparison between the two web reinforcement design methods was not obtained.

THE INFLUENCE OF PLASTIC FLOW UPON THE STRESSES DEVELOPED IN SHORT SPAN REINFORCED CONCRETE BEAMS

by

DIMITRI LAMBROPULOS

A THESIS

Submitted to the Graduate Committee

in

partial fulfillment of the requirements

for the degree of

Master of Science in Civil Engineering

at

Montana State College

Approved:

R. C. DeHart
In Charge of Major Work

R. C. DeHart
Chairman, Examining Committee

J. M. Nelson
Dean, Graduate Division

Bozeman, Montana
September, 1948

N378
L17i
cop. 2

TABLE OF CONTENTS

	Page
Abstract	3
Nomenclature	4
Introduction	7
Plastic Flow and Volume Changes of Concrete	9
Stresses Developed Due to Shrinkage and Plastic Flow	16
The Plasticity Ratio	23
Effect of Plastic Flow and Volume Changes on Design	24
Plastic Theory of Reinforced Concrete Design	26
Experimental Work	32
Duscussion and Conclusion	42
Literature Cited and Consulted	50

87112

ABSTRACT

In this thesis, it has been attempted to determine the actual stress variation in the concrete on the compression side of short-span reinforced concrete beams, and to prove that the "Straight-Line Theory", adopted up to now, gives only a portion of the stresses developed, since, in the derivation of its formulae, the effects of plastic flow and shrinkage on the final stresses set up in the concrete are not considered.

Data and results were obtained from tests conducted on two short beams; one with web reinforcement designed by the plastic flow method, and one assuming straight-line stress-strain variation.

The actual stress distribution on the compression side of the reinforced concrete beams tested, as obtained from the results shown in Tables IV and V, are shown in Figures 15 and 16. A line representing the distribution of stress in the concrete is curved, somewhat close to the shape of a parabola, but far from a straight line.

The capacity of the test machine was not sufficient to break the beams; therefore, a comparison between the two web reinforcement design methods was not obtained.

NOMENCLATURE

f_s = Compressive stress in steel.

s = The shortening of the concrete block due to shrinkage if the block was not reinforced.

f_c = Tensile stress in concrete.

E_s = Modulus of elasticity of steel.

E_c = Modulus of elasticity of concrete.

A_c = Effective area of concrete.

p = Steel ratio (A_s/bd).

A = Overall area of concrete in a column.

$n = \frac{E_s}{E_c}$

Δe = Elastic shortening of column

c_p = The plastic flow of a unit length of plain concrete under unit stress from the time of loading to any given time under consideration.

dc = The plastic flow of a unit length of plain concrete for a given infinitesimal time interval dt due to a unit stress intensity.

df_c = The change of stress in concrete.

df_s = The change of stress in steel.

f_{s_1} = Actual unit steel stress combining elastic and plastic effects.

Δf_c = Total change in concrete stress due to plastic flow.

Δf_s = Total change in steel strength due to plastic flow.

f_{c_1} = The intensity of stress in the concrete at the time interval dt due both to the applied load and the effects of plastic flow up to that time, and assumed constant through the small interval dt.

f_{co} = The original concrete stress due to elastic action only.

f_{so} = The original steel stress due to elastic action only.

y = The plastic flow in millionths of an inch per inch for a unit stress of one pound per square inch.

x = Time after loading, in days.

e_1 and ϕ_1 = Constants varying with the material.

Ecx = The effective modulus of elasticity.

Ec = Instantaneous modulus of elasticity.

f_c' = Compressive strength of a concrete cylinder.

C = The total compression.

a = Depth of equivalent stress diagram.

kd = The depth from the compressive side of a beam to the neutral axis.

b = Width of beam.

c = Arm of resisting couple (plastic theory method).

M = Moment of resistance.

T = Total tension.

d' = The distance from the compression steel to the tension steel.

A_s' = Area of the compression steel.

As = Area of tension steel.

P = Applied load.

ds = A short element of distance.

V = Shearing force.

v = Intensity of shear.

ΔC = Increment in compressive force.

ΔT = Increment in tensile force.

u = Intensity of bond stress.

Σo = The sum of the bar perimeters.

b = Average Shearing intensity.

ϵ_s = Unit shear deformation.

G = Shearing modulus of elasticity for concrete.

INTRODUCTION

1. Statement and scope of problem

From several investigations on the behavior of concrete, it has been found that the elastic strains due to application of a load give only a portion of the actual strains at any time in the concrete and steel. In addition to the above stated strains, there are also strains produced even before the application of the load, due to shrinkage. A long-continuing load, such as dead load, will give increasing plastic strains as time passes. Another factor that might give an increase in the above strains is the change in temperature.

In any case, the relation between steel and concrete strains will be very different from that given by the equations derived by the standard method, due to loads only. Recently, certain designers have advocated designing for failure, the method of analysis being based upon tests of beams to failure. By this approach, the ultimate strains will include shrinkage, load and some plastic flow strains.

These additional strains due to shrinkage and flow in members in bending, result in producing a non linear stress variation in the concrete on the compression side. An assumption of a parabolic variation is frequently made which the writer feels is not justified.

In this thesis, the author has attempted to find the actual stress variation in the concrete on the compression side by tests conducted on full size beams in the laboratory at Montana State College, and to determine the shearing stress at failure.

2. Acknowledgement

Grateful acknowledgement is made to Professor R. C. DeHart for his kind assistance, interest and suggestions during and before the experimental work. Acknowledgement is also due to the Mechanical Engineering Department for the use of their equipment.

PLASTIC FLOW AND VOLUME CHANGES OF CONCRETE

It is only comparatively recently that the attention of the engineering profession has been called to the plastic action of concrete under sustained constant loads. [5] *

In the technical literature up to about twenty years ago, there was frequent reference to the strength and elasticity of concrete, but only an occasional reference to shrinkage due to drying and almost no reference to the gradual deformation under the action of sustained load, which has been variously called "creep", "time yield," and "plastic flow".

As a result of the research of recent years, there has been developed a general conception of the effect of shrinkage and plastic flow upon the behavior of concrete structures. Under constant load, the deformation of the concrete increases progressively, due to plastic flow, which may be of appreciable magnitude and cause large changes of stress from those set up initially, thus leading to marked changes in design and construction practices.

It is now generally believed that shrinkage and plastic flow are closely related phenomena, each being primarily due to changes in the amount of absorbed water in the cement gel and being but little directly influenced by the free water occupying the pore spaces within the concrete mass.

Plastic flow of concrete has been found to depend upon such factors as the magnitude of stress, the strength of the concrete, the duration

---

* Numbers refer to references given in Bibliography.

of the loading period, the humidity of the atmosphere, the age of the concrete, the characteristics of the aggregates and the quantity of cement paste. [5]

In spite of the rather extensive researches that have been made in this field, it is not yet possible quantitatively to state with any degree of certainty what is likely to be the magnitude of either plastic flow or shrinkage under the conditions which surround any given concrete structure, nor is it possible quantitatively to state with accuracy what is the effect of plastic flow and shrinkage upon the magnitude and distribution of stresses. [1]

The property of shrinkage of concrete due to drying is altogether undesirable, but while probably shrinkage can never be eliminated, there seems to be the possibility, through the proper selection of materials and methods, of reducing shrinkage to a point where it will not be a factor for serious consideration. On the whole, plastic flow does not seem to be an undesirable property. In certain reinforced concrete members, it tends to make possible more efficient use of steel, and in thin structures subjected to drying, as well as in mass structures subjected to thermal changes due to the hydration of cement, it tends to promote a more favorable distribution of stresses than would otherwise exist.

Recent findings of investigations carried on at the University of California are as follows. [1]

Plastic Flow Under Long-Time Loading

In tests under long-time loading, it was found that even after 10 years some plastic flow was still discernible in plain concretes under

sustained stresses of several hundred pounds per square inch; however, over 95 percent of the probable total flow took place within 5 years. In a series of tests on reinforced concrete columns under load for more than 5 years, it was found that, in terms of stresses in the steel or concrete, the movements due to drying shrinkage or to plastic flow are of practical importance during the first year.

The plastic flow of plain concrete cylinders, which have been under load for 10 years, are shown in Figure 1. One group of specimens was loaded at the age of 28 days, and the other at the age of 3 months; prior to loading, the concrete was moist cured, and after loading, it was stored in air at 70° F. The sustained compressive stresses were 300, 600, 900 and 1200 pounds per square inch.

In table I are summarized the results of tests upon reinforced concrete columns under load for about 5½ years. There are shown the stresses in the longitudinal steel and in the concrete due to the combined effect of elastic and plastic strains produced by the sustained load and of length changes due to causes other than applied load. As flow progressed, more and more load was transferred from the concrete to the steel, and the rate of flow decreased in greater proportion than would be the case for plain concrete sustaining the same initial stress.

It will be noted that while under conditions of wet storage, the stress in the steel has not greatly increased during the period of sustained load; under dry conditions for the columns with the smaller percentage of reinforcement the stress in the steel has increased more than five times and for the columns with the larger percentage of reinforcement, the concrete is actually in tension.

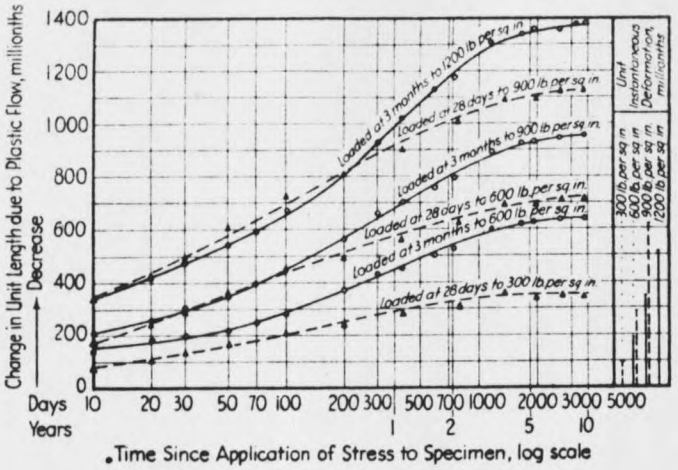


FIG. 1.—Flow Under Long-Sustained Stress

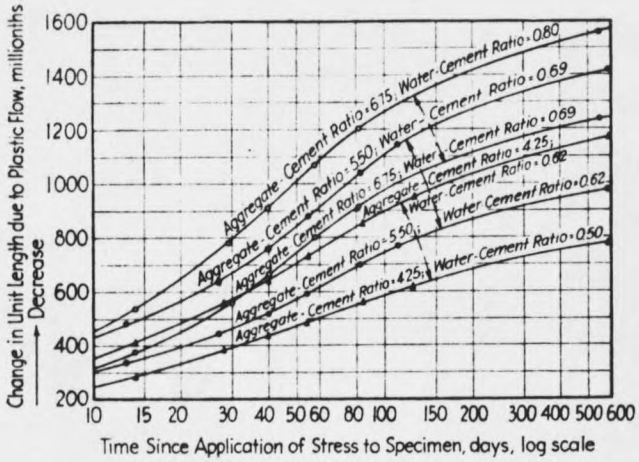


FIG. 2.—Effect of Aggregate-Cement Ratio and Water-Cement Ratio Upon Flow

Effect of Aggregate-Cement Ratio and Water-Cement Ratio upon Plastic Flow.

Tests to determine the effect of water-cement ratio and aggregate-cement ratio upon plastic flow indicated that the more the cement paste, the less the flow and that this variable was very important. Within the range of normal concretes tested, it was observed that with pastes with the same water cement ratio, the flow was practically proportional to the paste content of the concrete.

For a more thorough understanding of the effects of the water-cement and aggregate-cement ratios upon the plastic flow, refer to Figure 2.

Flow in Axial Tension and Compression

In tests to compare the flow of concrete in axial tension and compression, it was observed that, at least during the early ages, the flow under tensile stress was greater than that under compressive stress. At the later ages, the rate of flow was less under tensile than under compressive stress.

The results of a series of tests to study the flow of concrete under sustained tension and compression are shown in Figures 3 and 4. [1]

Fiber Strains Under Sustained Flexural Load

Similar results as those obtained in determining the flow in axial tension and compression, were found in tests to determine the fiber strains in plain concrete beams under constant sustained bending moment.

Effect of Fineness and Composition of Cement upon Plastic Flow.

In tests to determine the effect of fineness and composition of cement upon plastic flow, it was found that a low-heat type * of portland cement

* A low-heat cement has a low percentage of tricalcium aluminate, and a

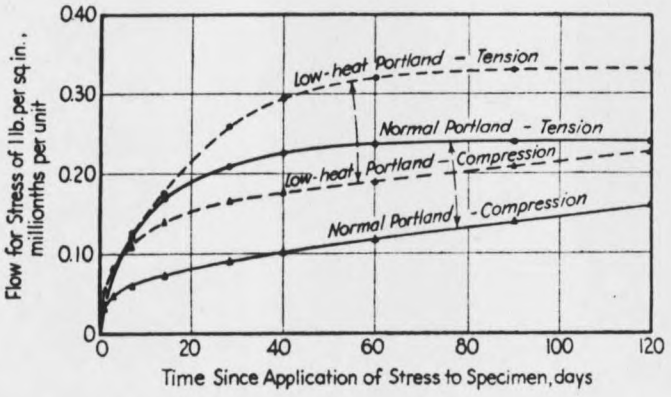


FIG. 3.—Flow in Compression and Tension—Mass Cured Concrete.

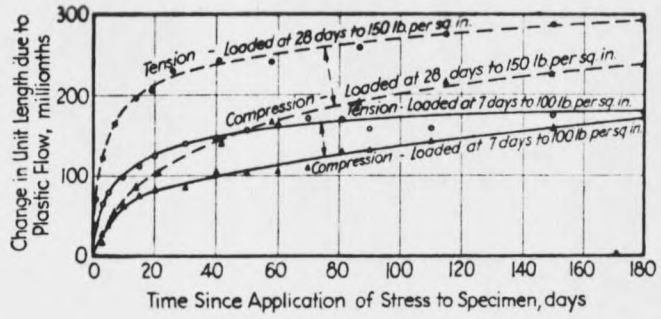


FIG. 4.—Flow in Compression and Tension—Standard-Cured Concrete.

exhibited markedly greater plastic properties, particularly at the early ages, than a normal portland cement.

Thermal Stress Studies.

In tests made under mass-curing conditions, the stresses developed due to thermal changes in large concrete cylinders under complete axial restraint, were measured. [1] It was indicated that the degree to which flow takes place has an important influence upon the stresses developed.

---

relatively low percentage of tricalcium silicate which liberates medium amount of heat during hydration, but has a relatively high percentage of dicalcium silicate which liberates a small amount of heat during hydration. It is slow in hardening and has a satisfactory ultimate strength. Its main purpose is to reduce the amount of heat generated by the hydrating cement thus decreasing the amount of expansion and subsequent contraction of the concrete.

STRESSES DEVELOPED DUE TO SHRINKAGE AND PLASTIC FLOW

Shrinkage

As was mentioned before, the drying out of concrete varies with time and with the exposure of the member. A plain concrete specimen will shrink as it dries out but there will be no stress in the member due to shrinkage. Steel, on the other hand, does not shrink as the concrete dries out, so its restraint tends to reduce the volume change of reinforced concrete. After a certain time, the concrete will have tensile stresses and the steel will be in compression.

To compute the effect of shrinkage upon the stresses in a reinforced concrete member, consider Figure 5, which shows a member of unit length, of a certain cross-sectional area A, and symmetrically reinforced with steel in amount pA. If the member were without reinforcement, the shortenings due to shrinkage would be s but the action of steel reduces the actual change as shown in the figure by a certain amount.

From Figure 5, it can be easily seen that the final length of the steel and concrete must be the same, so equating the strains, we get:

$$\frac{fs}{Es} = s - \frac{fc}{Ec} \qquad (1)$$

or

$$s = \frac{fs}{Es} + \frac{fc}{Ec} \qquad (2)$$

where:

fc = tensile stress in concrete

fs = compressive stress in steel

The stresses at a given section must be in equilibrium, so equating the compressive force on the steel to the tension in the concrete, we have:

$$f_s pA = f_c Ac = f_c (1-p) A \quad * \qquad (3)$$

Solving Equation 3 for fs

$$f_s = \frac{f_c(1-p)A}{pA} = \frac{f_c(1-p)}{p}$$

Substituting this value of fs in Equation 1

$$\frac{\frac{f_c(1-p)}{p}}{nE_c} = s - \frac{f_c}{E_c}$$

Solving this for fc

$$f_c = \frac{np}{1 + p(n-1)} sE_c \qquad (4)$$

Similarly,

$$f_s = \frac{1-p}{1 + (n-1)p} sE_s \qquad (5)$$

The use of these relations is difficult because of the impossibility of knowing exactly the value of the shrinkage coefficient s. Since the length of the member was assumed to be unity, s is given in inches per inch; a fair value is approximately 0.0004 for protected members in buildings.

Plastic Flow

If a constant load is maintained on a concrete member, there is a slow increase in the initial deformation due to flow which results in a progressive transfer of stress from concrete to steel, a process which lasts for about 6 years or more before equilibrium is reached.

The effect of plastic flow on the concrete stress may be approximately

---

* in columns, $p = \frac{A_s}{A}$ and $A_s = pA$, but $A_c = A - A_s$; so $A_c = A - pA = (1-p)A$.

analyzed as follows. The modulus of elasticity of concrete increases with the age of the concrete. In the derivation below, it is assumed to be constant.

Considering Figure 6, which shows a block of concrete with a central reinforcing rod, a load P is applied axially which makes the concrete block shorten by an amount Δe. Owing to the plastic action, which increases with time, the column still shortens as time goes on. Using the following notation:

c_p = The plastic flow of a unit length of plain concrete under unit stress from the time of loading to any given time under consideration.

dc = The plastic flow of a unit length of plain concrete for a given infinitesimal time interval dt due to a unit stress intensity.

f_c = The intensity of stress in the concrete at the time interval dt due both to the applied load and the effects of plastic flow up to that time, and assumed constant through the small interval dt.

If the block were of plain concrete, it would shorten plastically an amount $f_c dc$ in addition to the original elastic deformation. Assuming perfect bond between the steel and concrete, the shortening of the concrete becomes less due to the resistance offered by the steel.

Let dfc be the change of stress in concrete and dfs be the change of stress in steel. The final lengths of both the steel and concrete will be the same. Then,

$$f_c dc + \frac{dfc}{Ec} = \frac{dfs}{Es} \quad * \qquad (6)$$

---

* Plus sign denotes compression and minus sign tension.

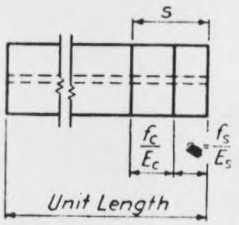

s
$\frac{f_c}{E_c}$
$\frac{f_s}{E_s}$
Unit Length

FIG. 5

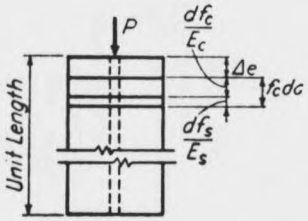

P
$\frac{df_c}{E_c}$
Δe
$f_c dc$
$\frac{df_s}{E_s}$
Unit Length

FIG 6

Equating the total change in load carried by the steel to the change carried by the concrete,

$$Asdfs = -Acdfc \qquad (7)$$

Solving equations 6 and 7 by eliminating dfs, we obtain

$$\frac{dfc}{fc} = \frac{-dc}{\frac{Ac}{AsEs} + \frac{1}{Ec}}$$

Integrating

$$\ln fc_1 = \frac{-c}{\frac{Ac}{AsEs} + \frac{1}{Ec}} + k$$

when $c = o$, fc becomes fc_o, that is, the original concrete stress due to elastic action only, and $k = \ln fc_o$. Hence,

$$\ln fc_1 = \frac{-c}{\frac{Ac}{AsEs} + \frac{1}{Ec}} + \ln fc_o$$

or

$$\frac{fc_1}{fc_o} = e^{-\frac{-c}{\frac{Ac}{AsEs} + \frac{1}{Ec}}}$$

and

$$fc_1 = \frac{fco}{e^{-\frac{fc}{\frac{Ac}{AsEs} + \frac{1}{Ec}}}}$$

Substituting $\frac{Es}{Ec} = n$; $As = pA$; and $Ac = A(1-p)$, the above equation reduces to

$$fc_1 = \frac{fco}{e^{-\frac{Escp}{p(n-1)+1}}} \qquad (8)$$

Let fs, equal actual unit steel stress combining elastic and plastic effects; fso equal original steel stress from elastic action only; $\triangle fc$ equal total change in concrete stress due to plastic flow, and $\triangle fs$ equal

total change in steel stress due to plastic flow.

$$fs, = fso + \Delta fs$$

$$-\Delta fc = fco - fc; \text{ but } fc, = \frac{fco}{e - \frac{c}{\frac{Ac}{AsEs} + \frac{1}{Ec}}}$$

$$-\Delta fc = fco - \frac{fco}{e - \frac{c}{\frac{Ac}{AsEs} + \frac{1}{Ec}}}$$

$$-\Delta fc = fco \left(1 - \frac{1}{e - \frac{c}{\frac{Ac}{AsEs} + \frac{1}{Ec}}}\right) \qquad (9)$$

$$\Delta fs = -\Delta fc \frac{Ac}{As}$$

and since $fco = \frac{fso}{n}$, we have

$$\Delta fs = fso \frac{Ac}{nAs} \left(1 - \frac{1}{e - \frac{c}{\frac{Ac}{AsEs} + \frac{1}{Ec}}}\right) \qquad (10)$$

The final stresses may be approximately found by giving n in the ordinary formulae an increased value, a fact indicated by examining equation 8.

If this is rewritten and the denominator expanded into a series, $(e^x = 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \text{-------})$, and the first two terms taken (because the other terms are rather negligible), the result is:

$$fc, = \frac{fco}{1 + \frac{c}{\frac{Ac}{AsEs} + \frac{1}{Ec}}} = \frac{fco\,(Ac + \frac{Es}{Ec}\,As)}{Ac + (\frac{Es}{Ec} + cEs)As} = \frac{P}{Ac + nrAs} \qquad (11)$$

Here, the numerator represents the original elastic concrete stress by the transformed area with $n = \frac{Es}{Ec}$; this equals the load on the column. The denominator represents the transformed area, using a modified "n," called "nr." This combines elastic and plastic effects, with $nr = \frac{Es}{Ec} + cEs$.

According to Glanville, to take care of the effect of plastic flow, the same method as for purely elastic deformation may be used with an increased value of "n" of about 40 without an appreciable error.

THE PLASTICITY RATIO

This subject is mainly based on the hypothesis that the stress-strain diagram for concrete under short time loading may be idealized to consist of two linear parts, one representing elastic behavior and the other representing plastic behavior.

The first part is measured by what we know as the "modulus ratio", which is the ratio of the modulus of elasticity of steel to that of the concrete. *

The second part of the diagram, on the other hand, is measured by the "plasticity ratio", which is defined to be the ratio of the plastic strain to the total at rupture of the concrete.

An empirical equation has been derived by Professor V. P. Jensen of the University of Illinois to express the relationship between the plasticity ratio and the compressive strength of concrete.

The idealized stress-strain curves for concrete of Professor V. P. Jensen are shown in Figure 7 for reference.

---

* By modulus of elasticity of concrete, its original modulus of elasticity is meant, as it varies with time.

EFFECT OF PLASTIC FLOW AND VOLUME CHANGES ON DESIGN

In a routine reinforced concrete design, the designer bases his work on the accepted methods and standard specifications. With the march of time, more and more knowledge has been acquired on the properties and behavior of concrete and consequently the accepted methods and specifications must be altered to comply with the new findings in this field.

With reference to T. R. Shank's paper, C. T. Morris, in his paper on the "Effects of plastic flow and volume changes on design", gives the following useful formulae and information.

In general, the plastic flow of concrete may be expressed by an equation of this form,

$$y = c, \sqrt[a,]{x} \qquad (12)$$

in which:

y = The plastic flow in millionths of an inch per inch for a unit stress of one pound per square inch.

x = Time after loading in days.

a, and c, = Constants varying with the material.

The shape of the curve as expressed by the above equation is fairly accurate under controlled conditions for time up to one year after loading; for longer periods, y should be progressively reduced until, after a period of five years, no further plastic flow need be considered. The rate of plastic flow in air is quite sensitive to changes in humidity.

The plastic flow of concrete loaded at 28 days for periods of five years or more will not exceed the flow at one year by more than 30 percent, and for concretes loaded at 3 months, the excess over the flow at one year

may reach 40 percent.

Plastic flow data so far available may be expected to be as much as 50 percent in error when considering external conditions and any specific concrete mixture.

The plastic flow for ordinary portland cement concretes made from ordinary aggregate materials, natural sands, natural gravels and limestones, loaded in air at 28 days may be dxpressed with sufficient accuracy by:

$$y = 0.13 \sqrt[3]{x} \qquad (13)$$

For the same concretes in water, equation (12) may be written

$$y = 0.089 \sqrt[4]{x} \qquad (14)$$

To take into account, the effect of plastic flow, a reduced modulus of elasticity should be used, obtained by the following equation.

$$Ecx = \frac{Ec}{1 + Ecy} \qquad (15)$$

in which:

Ecx = The effective modulus of elasticity.

Ec = Instantaneous modulus of elasticity.

y = Plastic flow deformation for the time under consideration obtained from equations (12), (13), or (14).

PLASTIC THEORY OF REINFORCED CONCRETE DESIGN

Since concrete is not a purely elastic material, the theory of elasticity which assumes a straight-line distribution of stresses, is too inflexible and inaccurate to be entirely satisfactory.

As it was mentioned before, it is rather impossible to determine the plastic flow and shrinkage effects with any accuracy for any given structure, so it becomes a hopeless task to attempt to determine the stresses set up under a given load with any degree of accuracy.

It is, however, more practicable to adopt the "plastic theory" instead of the "straight-line theory" for design purposes, because it takes into account the plasticity of the material.[5] The equations derived from this theory are much simpler than the standard formulae and agree better with test results. They are based on the cylinder strength of the concrete and the yield strength of the steel without the use of modulus ratio n,[a, 7, a, 9] which has been shown to be meaningless.

The classic conception of the stress-strain curve is that of a parabola with its lower part approximately straight and ending at the top tangent to the horizontal. The lower third of the curve being nearly straight, it formed the basis of the straight-line formula in which working stresses are used. The entire curve was used in the parabolic formula based on ultimate strength. Both of the formulae involve Young's modulus of elasticity of the concrete, which is inaccurate because the stress-strain relation is an indeterminate variable depending on the concrete, the manner and intensity of loading and on time. Change in strain with time is primarily due to plastic flow and shrinkage. On account of the variability

of this stress-strain relation, it is not possible to determine the unit stresses at any particular point in the concrete member by measuring strains, except to a very limited degree.

An almost universal misconception of the behavior of concrete in flexure is the assumption that at maximum load, the highest unit stress is at the outer surface. The descending branch (see Figure 8) of the compression stress-strain curve has been generally disregarded because it is difficult to measure and has been considered of no particular interest.

Rectangular Beams

The stress distribution in a reinforced concrete at failure is assumed to have the shape of the cylinder stress-strain curve as shown in Figure 9. The total compression C, is the area bounded by the curve. The line of action of C lies through the center of gravity of this area. For the sake of simplicity in deriving a workable expression, the actual stress curve for the concrete may be replaced by an equivalent rectangular area of width equal to $0.85f'_c$ (f'_c being the strength of a test cylinder) and a depth equal to a (as shown in Figure 9). The location of the center of gravity of this rectangle corresponds closely with that of the actual area.

The actual irregular stress block is replaced only for simplicity with a rectangular one of equal size having an average stress intensity of $0.85f'_c$, without meaning that this method is based on an assumption of rectangular stress distribution. The use of a rectangle is merely a mathematical device to approximate the effect of true distribution and to simplify the calculations.

It should be noted that the depth "a" does not correspond with the depth

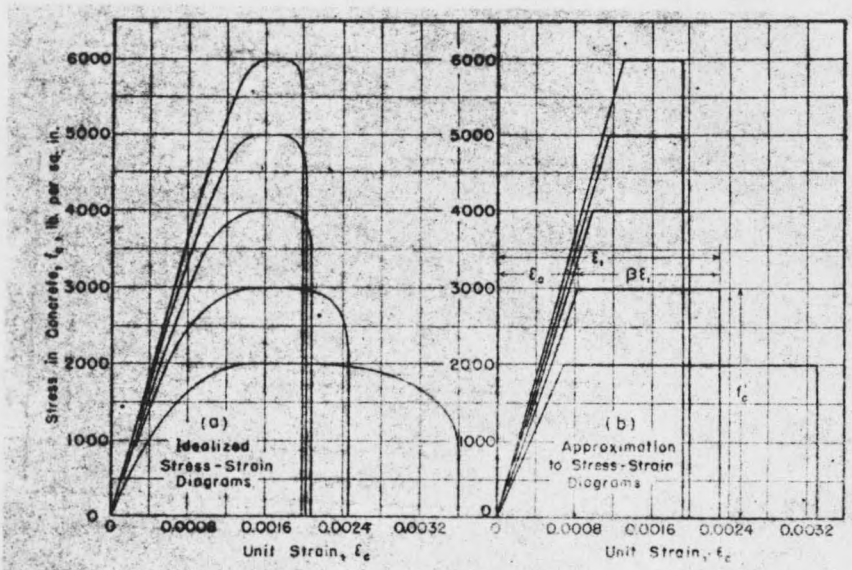


Fig. 7—Idealized stress-strain curves for concrete of different strengths

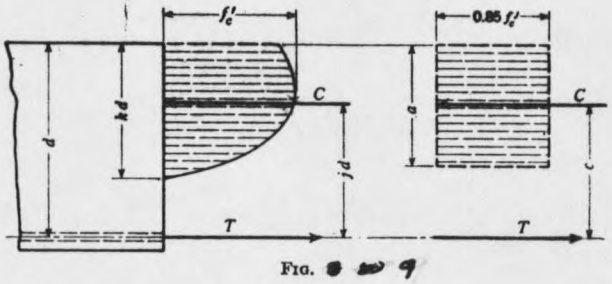


FIG. 8 and 9

to the neutral axis, kd, and it bears no definite relation to it. The value, kd, is determined by the strains at the top and bottom of the beam, whereas "a" is determined by the strength of the material and is less than the distance kd.

Considering Figure 10, since T equals C

$$a = \frac{Asfs}{0.85bfc'} \qquad (16)$$

Since $C = 0.85bfc^1$ (area bounded by curve)

$$M = Cc = 0.85fc'bac \qquad (17)$$

in which

As = Area of tensile steel.

fs = Yield point stress of steel.

b = Width of beam.

fc' = Compressive strength of concrete cylinder.

A study of test results [10] shows that we may take a/d = 0.537, and c/d = 0.732, which results in

$$M = 0.33fc'bd^2 \qquad (18)$$

If the beam is reinforced in compression as well as in tension, Figure 11, the compression distance being placed at a distance d^1 from that in tension, the moment of resistance will be increased, becoming

$$M = 0.33fc'bd^2 + fsAs'd' \qquad (19)$$

where fs is the elastic limit stress as before and As^1 is the area of the compression steel.

$$M = Tc = fsAs0.732d$$

$$As = \frac{M}{0.732fsd} = \frac{0.33fc'bd^2}{0.732fsd} = 0.456\frac{fc'}{fs}bd \qquad (20)$$

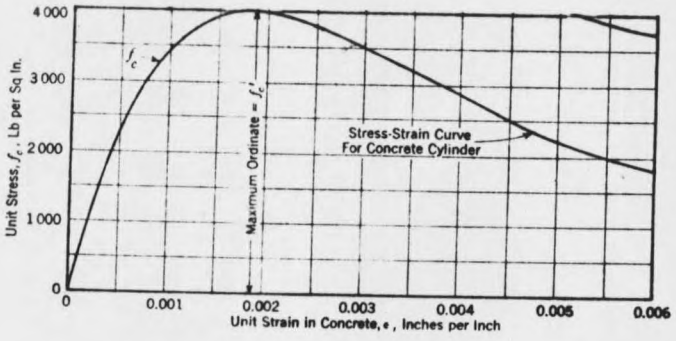

4 000
3 000
2 000
1 000
0
f_c
Maximum Ordinate = f_c'
Stress-Strain Curve
For Concrete Cylinder
Unit Stress, f_c, Lb per Sq In.
0
0.001
0.002
0.003
0.004
0.005
0.006
Unit Strain in Concrete, ϵ, Inches per Inch

FIG. 8

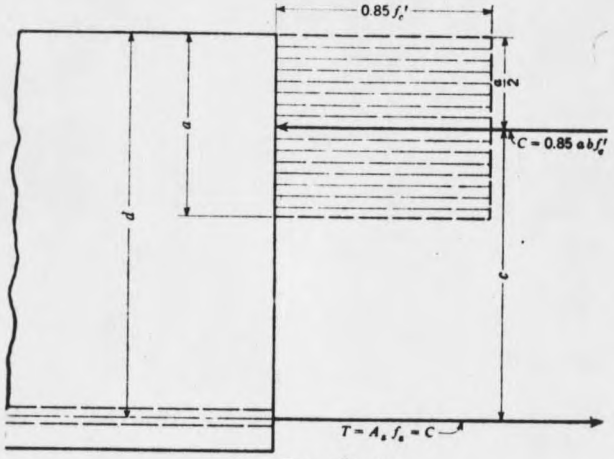

0.85 f_c'
a
d
c
$C = 0.85\, a\, b f_c'$
$T = A_s f_s = C$

FIG. 9 10

Since in the plastic theory the designs are done on the basis of ultimate strength, it is necessary to apply a definite factor of safety suitable to the conditions. It is based on the yield point of steel and the cylinder strength of concrete.

From the standpoint of economy, the design of plastic theory method seems to be more economical than the straight-line method because the sections of members thus found require less area of concrete and more area of steel, a factor indicating that the designs based on the plastic theory method are cheaper since steel is less expensive than concrete in the United States. Other important factors that affect the economy of a structure designed by this method considerably are: the headroom saving, the overall building height, which is reduced due to the smaller sections and slabs used, the dead load reduction, etc.

EXPERIMENTAL WORK

As stated in the foreword, the scope of this work was to find the actual stress variation in the concrete on the compression side of beams.

For this purpose, two short beams, each four feet long, were designed to carry a load of 15,000 pounds.*

The strength of the concrete was taken as 2000 pounds per square inch with a water-cement ratio of 8 gallons per sack of cement and a cement-sand-stone proportion of 1-2½-3½ by volume.

Both of the beams (as shown in the design below) were designed with the plastic theory method except for the stirrups, which for one beam were designed with the standard straight-line method and for the other one with the plastic theory method.

Due to improper forms, one of the beams was poured to be slightly wider at the middle part, a factor that the writer thinks will not cause appreciable error in the final test of the beam.

To find the actual strength of the concrete, 5 standard size cylinders were made of the same water-cement ratio and same cement-find aggregate, coarse-aggregate proportion.

Derivation of Shearing and Bond Formulae

The variation of shear intensity in a rectangular reinforced concrete beam may be ascertained by considering a small portion of such a beam between any two sections a small distance, ds, apart, as shown in Figure 13 the breadth of beam being taken as b inches. The forces acting on this

---

* Dead weight is neglected.

TABLE I.—UNIT STRESSES IN REINFORCED-CONCRETE COLUMNS.

Nominal Strength of Concrete at 28 Days, lb. per sq. in.	Steel Ratio, per cent	Total Load Applied to Column, lb.	Unit Stresses, lb. per sq. in.							
			At Time of Application of Load		1 yr. Under Load		3 yr. Under Load		5½ yr. Under Load	
			Steel	Concrete	Steel	Concrete	Steel	Concrete	Steel	Concrete
COLUMNS STORED IN AIR OF 50 PER CENT RELATIVE HUMIDITY AT 70 F.										
2000	5.0	22 300	9660	875	26 900	−20[a]	27 400	−50[a]	28 000	−75[a]
2000	1.9	14 200	6540	610	34 800	60	35 700	45	37 100	15
4000	1.9	21 800	7860	975	37 500	395	40 400	340	41 700	315
COLUMNS STORED UNDER WATER AT 70 F.										
2000	5.0	19 200	7200	810	10 050	665	10 890	620	11 400	590
2000	1.9	13 650	5460	605	7 980	555	9 060	535	9 480	525
4000	1.9	20 600	7320	925	10 590	865	11 670	840	12 120	835

[a] Minus sign indicates tension.

NOTE.—Flow tests series No. 8. Specimens, reinforced-concrete columns 5 in. in diameter by 48 in. long. Aggregate 0 to ¾-in. local gravel. Fineness modulus, 4.47. Cement, normal portland. Aggregate-cement ratio, by weight (*a*) for 2000-lb. concrete, 8.40; (*b*) for 4000-lb. concrete, 4.80. Water-cement ratio, by weight (*a*) for 2000-lb. concrete, 0.83; (*b*) for 4000-lb. concrete, 0.53. Consistency, 6-in. slump. Curing, 21 days moist at 70 F. (21 C.), then stored as indicated. Age at loading, 28 days. Column loads determined in accordance with provisions of 1924 Report of Joint Committee on Specifications for Concrete and Reinforced Concrete.

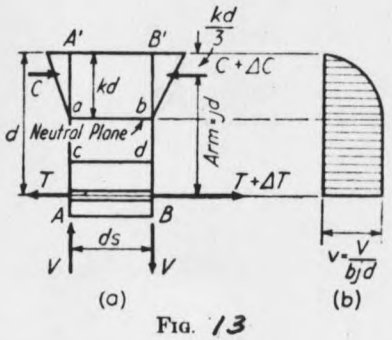


FIG. 13

bit of beam consist of the normal C and T and the shear V; it is assumed that the moment at BB' is larger than that at AA', and that the sections are so close together that the two shears may be considered equal. These forces are in equilibrium, and if the condition $\Sigma M = o$ is applied, there results:

$$\Delta T \times jd = Vds$$

The tendency of the small portion of the beam, cdBA, to be pulled to the right is resisted by the horizontal shear on the cd plane, which may be expressed as the intensity of shear, v, assumed to be uniform, multiplied by the area bds. Then

$$v \cdot b \cdot ds = \Delta T$$

Combining these two equations given:

$$v \cdot b \cdot ds = \frac{Vds}{jd}$$

and

$$v = \frac{V}{bjd}$$

Consideration of Figure 13 shows that the total bond, the grip of the concrete on the tension reinforcement, which prevents slippage, equals the difference in total steel stress at the two sections, A and B: $\Delta T = v \cdot b \cdot ds$. Assuming uniform intensity, u, of bond stress, we have also $\Delta T = u \cdot \Sigma o \cdot ds$, where o equals the sum of the bar perimeters, this results in:

$$u = \frac{vb}{\Sigma o} \qquad \text{or} \qquad u = \frac{V}{jd\Sigma o}$$

The transformation of the above formulae so that they can be applied in the design by the plastic theory method is shown in the part under the sub-heading of "Design of Beams" when checking for shear and bond.

Design of Beams

$$M = 0.85 fc' bac$$

$$M = \tfrac{1}{2} \times 15000 \times 2 \times 12 = 130000 \text{ inch-pounds}$$

Assuming d = 7 inches and substituting a = 0.537d, c = 0.732d and fc' = 0.45 x 2000 (factor of safety) the above equation becomes,

$$180000 = 0.85 \times 0.45 \times 2000 \times 7 \times 0.537d \times 0.732d$$

$$180000 = 2100d^2$$

$$d = \sqrt{\frac{180000}{2100}} = 9.25 \text{ inches} \quad \underline{\text{Use 11 inches}}$$

$$M = Tc = fsAsC$$

$$180000 = 20000As \times 0.732 \times 11$$

$$As = \frac{180000}{161000} = 1.12 \text{ square inches.}$$

Use 6-$\frac{1}{2}$ inch ϕ rods with an area of 1.20 square inches. Check for shear (Standard method)

$$v = \frac{V}{bjd} = \frac{7500}{7 \times 0.866 \times 11} = 112 \text{ pounds per square inch.}$$

Using deformed bars allowable is

$$v = 0.02fc = 0.02 \times 2000 = 40 \text{ pounds per square inch.}$$

$40 < 112$ web reinforcement needed. $112 - 40 = 72$ pounds per inch^2 to be carried by stirrups or $72 \times 2 \times 7 \times 12 = 12100$ pounds (half span). Using $\frac{1}{2}$ inch ϕ U-stirrups, each stirrup can carry

$$2 \times 0.05 \times 20000 = 2000 \text{ pounds.}$$

$$\text{Number of stirrups needed} = \frac{12100}{2000} = 6.05$$

Use 6-$\frac{1}{2}$ inch ϕ U-stirrups for half of span.

$$20000 \times 0.05 \times 2 = 72 \times 7S$$

$$S = \frac{2000}{504} = 3.97 \text{ inches for spacing of stirrups.}$$

Check for bond:

$$u = \frac{V}{jd\Sigma o} = \frac{7500}{0.866 \times 11 \times 6\pi 0.5} = 83.5 \text{ pounds per inch}^2$$

$83.5 < 112$ o.k.

Check for shear (Plastic theory method)

$$v = \frac{V}{bjd} \quad \text{or} \quad \frac{V}{bc} \qquad \text{(See figure 10)}$$

but $c = 0.732d$, therefore:

$$v = \frac{V}{0.732bd} = \frac{7500}{0.732 \times 7 \times 11} = 133 \text{ pounds per inch}^2$$

$133 - 40 = 93$ pounds per inch^2 to be carried by stirrups

or $93 \times 2 \times 7 \times 12 = 15640$ pounds.

Using again the same U-stirrups, each stirrup carries,

$20000 \times 2 \times 0.05 = 2000$ pounds.

Number of stirrups needed,

$$\frac{15640}{2000} = 7.82 \text{ stirrups}$$

Use 8-$\frac{1}{2}$ inch$\emptyset$ stirrups for half span,

$$2000 \times 0.05 \times 2 = 93 \times S$$

$$S = \frac{2000}{650} = 3.08 \text{ inches for spacing.}$$

Check for bond,

$$u = \frac{V}{0\Sigma o} = \frac{V}{0.752d\Sigma o}$$

$$= \frac{7500}{0.732 \times 11 \times 9.42} = 98.7 \text{ pounds per inch}^2$$

$98.7 < 112$ pounds per inch^2 o.k.

A section of the beam is shown in Figure 12.

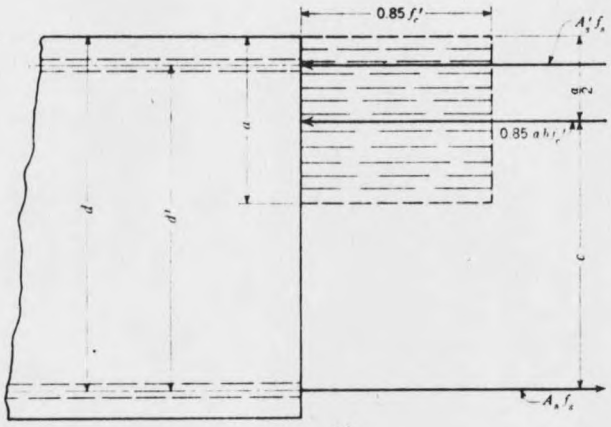


FIG. 11

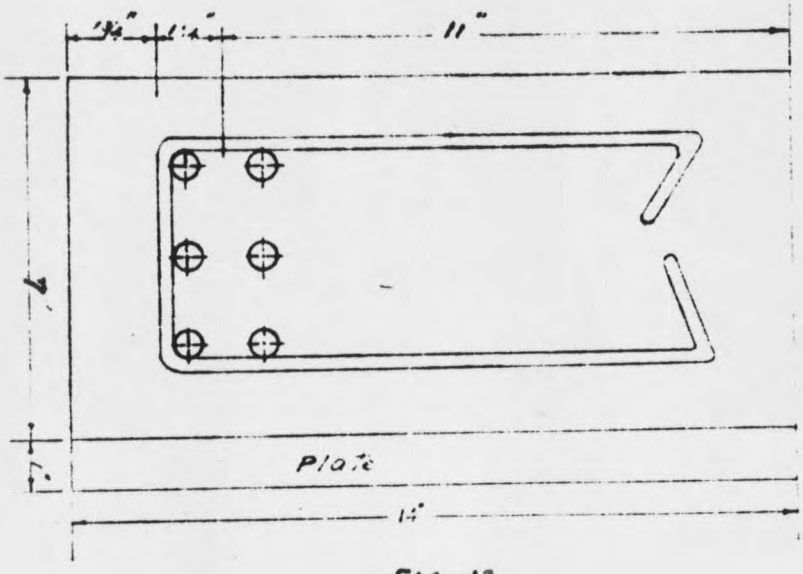


FIG. 12

Table II - 8 Stirrup Beam

Load in Pounds	Strain in Steel in/in	Strain in Concrete in in./in.		
		1" from top	3" from top	5" from top
0	0	0	0	0
4000	0.00004	0.0001	0.0001	0.0001
6000	0.00007	0.0002	0.0002	0.0001
8000	0.00012	0.0003	0.0004	0.0003
10000	0.00018	0.0004	0.0005	0.0003
12000	0.00025	0.0005	0.0006	0.0004
14000	0.00031	0.0007	0.0006	0.0005
16000	0.00038	0.0009	0.0007	0.0006
18000	0.00045	0.0010	0.0009	0.0008
20000	0.00053	0.0012	0.0010	0.0009

Table III - 6 Stirrup Beam

Load in Pounds	Strain in Steel in./in.	Strain in Concrete in in./in.		
		1" from top	3" from top	5" from top
0	0	0	0	0
4000	0.00003	0.0001	0.0001	0.0001
6000	0.00008	0.0002	0.0003	0.0002
8000	0.00016	0.0003	0.0004	0.0003
12750	0.00033	0.0005	0.0005	0.0005
16000	0.00044	0.0008	0.0008	0.0007
18000	0.00052	0.0010	0.0009	0.0008
20000	0.00059	0.0011	0.0011	0.0009

Table IV - 8 Stirrup Beam

Load in Pounds	Computed Theoretical Stress of Steel in lbs/in^2	Computed Theoretical Stress of Concrete in lbs/in^2	Computed Actual Stress of Steel in lbs/in^2	Computed Actual Stress of Concrete in lbs/in^2		
				1" from top	3" from top	5" from top
0	0	0	0	0	0	0
4000	4150	142	1200	140	140	140
6000	6220	212	2100	280	280	140
8000	8300	283	3600	420	560	420
10000	10360	354	5400	560	700	420
12000	12420	424	7500	700	840	560
14000	14500	495	9300	980	840	700
16000	16600	565	11400	1260	980	840
18000	18670	636	13500	1400	1260	1120
20000	20700	705	15900	1680	1400	980

The actual stresses in the steel and concrete were computed by multiplying their moduli of elasticity, which were taken as 30×10^6 psi and 1.4×10^6 psi respectively, by the measured strains.

In computing the actual concrete stresses, the shearing deformation was not subtracted from the measured strains.

Table V - 6 Stirrup Beam

$E_s = 30 \times 10^6$ $E_c = 1.4 \times 10^6$

Load in Pounds	Computed Theoretical Stress of Steel in lbs/in^2	Computed Theoretical Stress of Concrete in lbs/in^2	Computed Actual Stress of Steel in lbs/in^2	Computed Actual Stress of Concrete in lbs/in^2		
				1" from top	3" from top	5" from top
0	0	0	0	0	0	0
4000	4150	142	900	140	140	140
6000	6220	212	2400	280	280	280
8000	8300	283	4800	420	560	420
12750	13200	450	9900	700	700	700
16000	16600	565	13200	1120	1120	980
18000	18670	636	15600	1400	1260	1120
20000	20700	705	17700	1540	1540	1260

The actual stresses in the steel and concrete were computed by multiplying their moduli of elasticity, which were taken as 30×10^6 psi and 1.4×10^6 psi respectively, by the measured strains.

In computing the actual concrete stresses, the shearing deformation was not subtracted from the measured strains.

DISCUSSION AND CONCLUSION

From the data and results obtained from the tests of the two beams, the following things were observed.

With reference to tables IV and V, it is seen that at low stresses there was a great difference between the computed stresses and the actual stresses in the steel. This is primarily due to the fact that at low stresses a considerable amount of the tension was taken by the concrete. As the load increased, this difference started decreasing because almost all of the tension was carried by steel and very little or none by the concrete, which started cracking and consequently could not carry any more tension.

For the concrete, at low stresses both the computed and the actual stresses were close to each other, whereas at higher loads, the actual stresses were much greater than the computed ones. This was probably due to the fact that at high stresses, the concrete strains started increasing much more rapidly than the stresses. A glance at Figure 14 would show that at high stresses, the stress-strain curve of concrete starts getting flatter resulting in a greater strain-stress ratio.

From Figures 15 and 16, it can be seen that the neutral axis of the cross-section of the beams shifted upwards as more load was applied. This occured because at higher stresses, all of the tension was taken by the steel and consequently, the strains were much greater than those obtained at low stresses where a considerable amount of tension was taken by the concrete.

The actual distance from the top of the beam to the neutral axis was

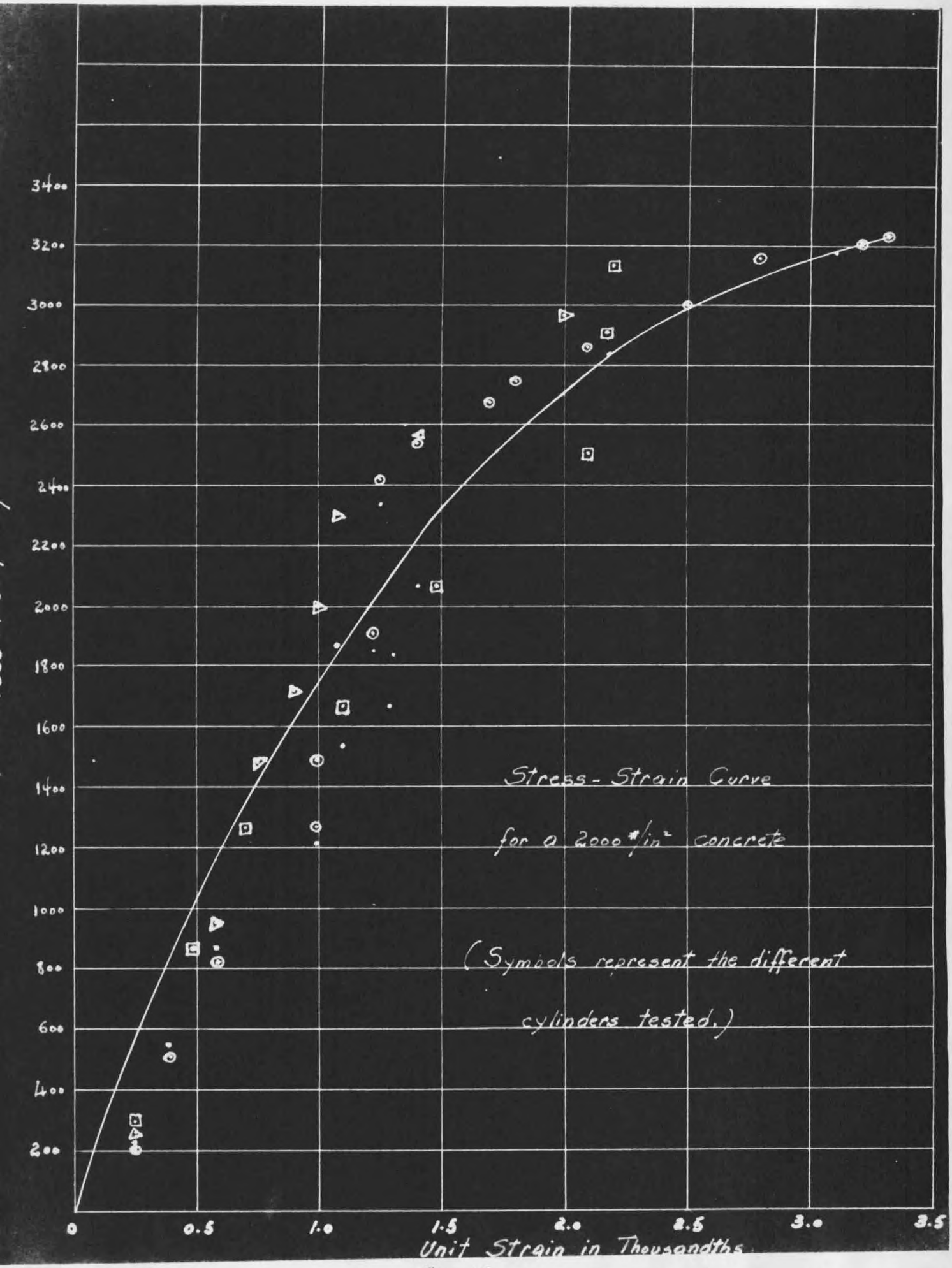
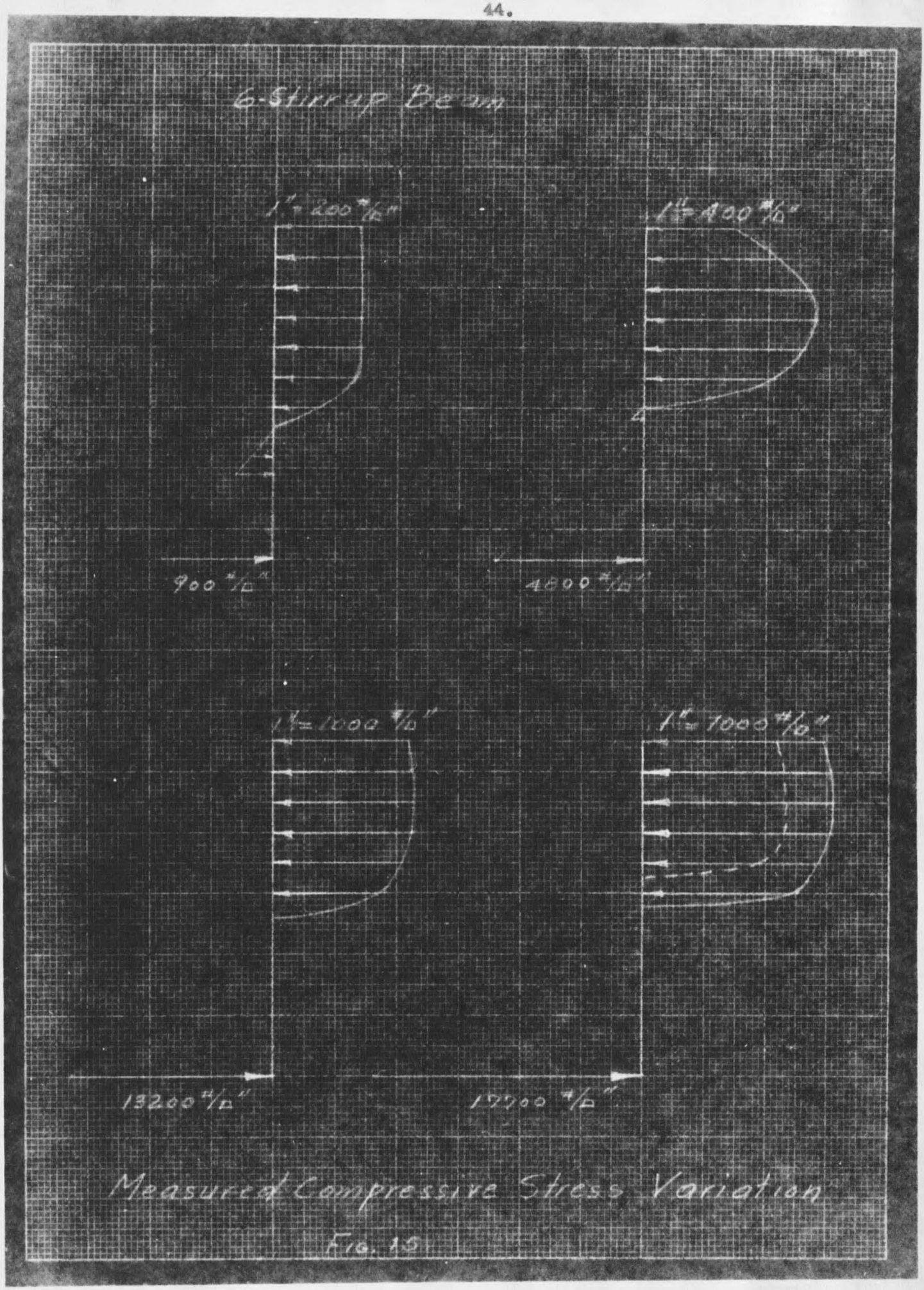
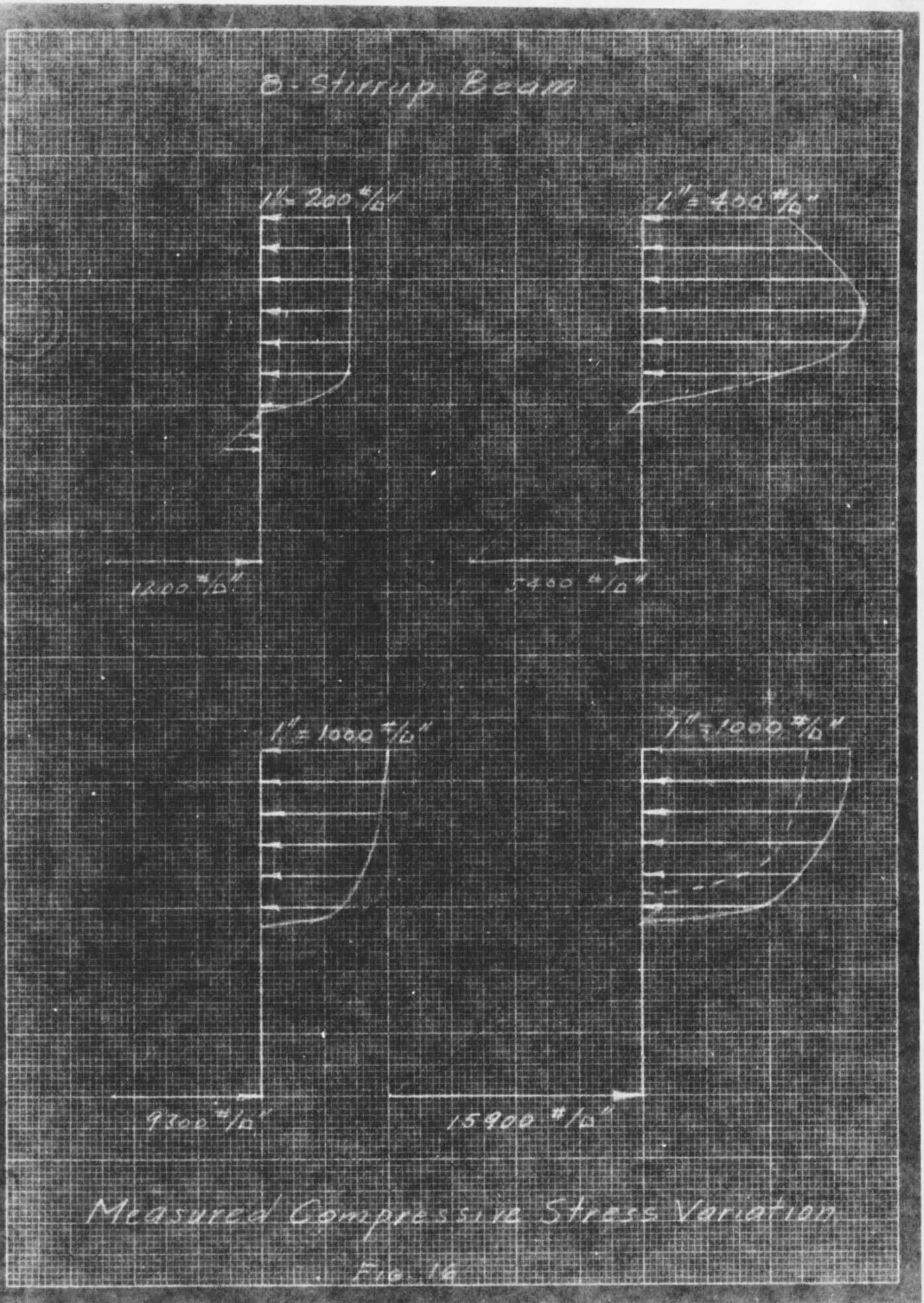

3400
3200
3000
2800
2600
2400
2200
2000
1800
1600
1400
1200
1000
800
600
400
200
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
Unit Strain in Thousandths
Stress - Strain Curve
for a 2000 #/in² concrete
(Symbols represent the different
cylinders tested.)

FIG. 14



Measured Compressive Stress Variation

FIG. 15



Measured Compressive Stress Variation

Fig. 16

smaller than the one computed by the "Straight - line method". This difference in the position of the neutral axis is due to the effect of plastic flow, which results in a redistribution of the concrete stresses producing a shorter moment arm and consequently, raising the steel stresses for a given applied moment.

The compressive forces in the beams were greater than the tensile forces. This difference in tension and compression stresses may be due to:

a) reading greater strains of concrete than the actual ones due to the difficulties encountered in measuring those strains by the provided means.

b) reading smaller strains of steel because of some possible slippage between the extensometer and the steel bar.

c) Using a greater value for the modulus of elasticity of concrete, a factor that is very hard to determine with any accuracy due to each inconsistency, which would result in greater stresses in the concrete.

In computing the stresses of concrete, the shearing deformation was taken into consideration. At high loads, due to the great shearing stresses developed, every element of the beam is distorted, as shown in Figure 17, which results in additional strains in the concrete.

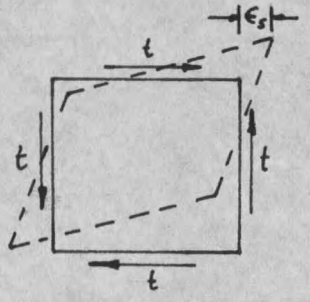


Fig. 17

The strains were computed by the formula

$$\epsilon_s = \frac{t}{G} \quad \text{where}$$

ϵ_s = unit shear deformation

t = average shearing intensity computed from $t = \frac{V}{bjd}$, and

G = shearing modulus of elasticity for concrete.

In figures 15 and 16, the solid lines represent the stress distribution on the compression side of the reinforced concrete beams when the shearing deformations were neglected, and the dotted lines show the stress variation when the effect of the shearing deformation was taken into account.

The applied moments were greater than the resisting ones. In this case, the resisting moment was computed as "Tension times lever arm" because the writer felt that the steel strains were more reliable. For the most favorable condition, the error was about 15 percent, because of reasons mentioned above. In computing the theoretical moments, the span of the beams was taken as 3 feet and 4 inches, which was the clear distance between the supports.

Another factor that might affect the measured strains slightly is shrinkage, because as the beams dried out there were some tensile stresses set up in the concrete and compressive stresses in the steel before the load was applied.

Conclusions

The results obtained from the two beams, if not very satisfactory, were quite helpful in determining the actual stress variation on the compression side of reinforced concrete beams and in proving that the "Straight-Line Theory" is not much justified.

Designing with the plastic theory method seems to be more accurate because it is based on the ultimate strength of reinforced concrete beams, a value little affected by shrinkage and flow since there is a large redistribution of stress with the large strains previous to failure. On the other hand, the straight line theory is rather approximate because it is almost impossible to predict with any accuracy the effects of shrinkage and plastic flow and thus determine with any exactitude the stresses set up under any given load.

A better way of getting more accurate results concerning the distribution of compressive stresses in reinforced concrete beams, would be to attach two or three extensometers at different depths of a beam in such a way that moving of the measuring device for each reading will be avoided. In the author's opinion, if such a thing can be done, the possibility of error in reading the dial every time that it is set on the beam between the two plates, as shown in Figure 18, will be eliminated.

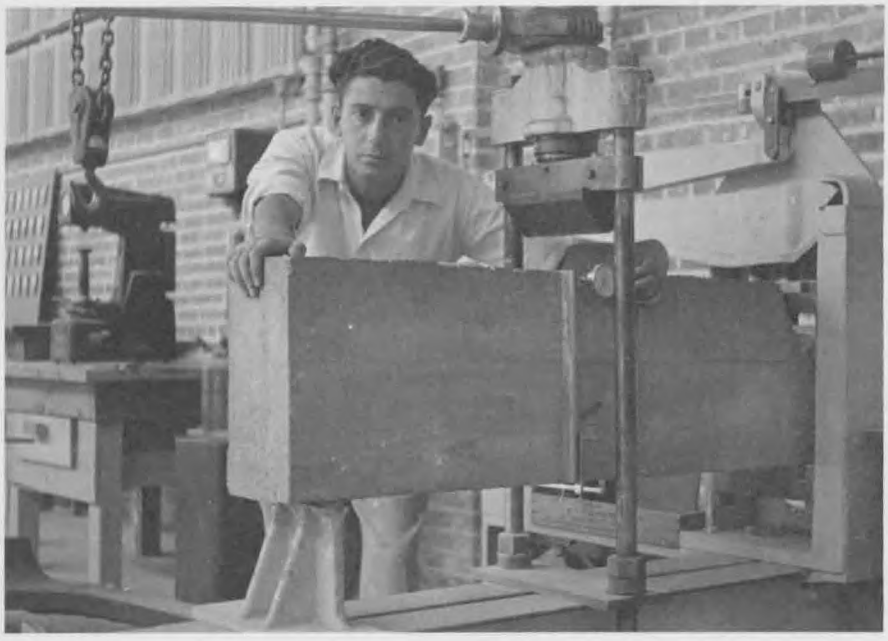


FIG. 18

LITERATURE CITED AND CONSULTED

1. Davis, R. E., Davis, H. E., Brown, Elwood, "Plastic Flow and Volume Changes of Concrete", Proceedings, A.S.T.M., Part II, Vol. 37, 1937, p. 318.

2. Jensen, V. P., "The Plasticity Ratio of Concrete and its Effect on the Ultimate Strength of Beams", Journal American Concrete Institute, June, 1943, p. 565.

3. Moris, Clyde T., "Effect of Plastic Flow and Volume Changes on Design", Journal, American Concrete Institute, November-December, 1936, p. 123.

4. Peabody, Dean, "Reinforced Concrete Structures", John Wiley and Sons, New York, 1946.

5. Sutherland, Hale, Reese, Raymond C, "Reinforced Concrete Design", John Wiley and Sons, New York, 1943.

6. Whitney, Charles S., "Plastic Theory of Reinforced Concrete Design", Proceedings, A.S.T.M., Vol. 66, 1940, p. 1749.

7. Saliger, Rudolf, "Bruchzustand und Sicherheit im Eisenbeton-balken", Beton und Eisen, October, 1936, p. 317.

8. Hajnal-Kanyi, K. and discussion by others, "The Modular Ratio - A New Method of Design Omitting m", Concrete and Constructional Engineering, January.

9. Coppee, R., "Considerations sur le Calcul et la Securite des Pieces Flechies Moments de Rupture", Publications, International Association for Bridge and Structural Engineering, Vol. 3, 1935, p. 19.

10. Slater, W. A., Lyse, Inge, "Compressive Strength of Concrete in Flexure", Journal, American Concrete Institute, June, 1930, p. 831.

11. Emperger, Fritz V., "Der Beiwert'n", Beton und Eisen, Vol. 35, No. 19, October 5, 1936, pp. 324-432.

12. Cox, Kenneth C., "Tests of Reinforced Concrete Beams with Recommendations for Attaining Balanced Design", ACI Journal, September, 1941, pp. 65-80. Duscussion, ACI Journal, June, 1942, pp. (80-1) to (80-11), Proceeding V. 37.

13. Shank, J. R., "The Mechanics of Plastic Flow of Concrete", Journal, American Concrete Institute, November-December, 1935, Proceedings, Vol. 32, p. 149.

MONTANA STATE UNIVERSITY LIBRARIES
3 1762 10014690 9

[redacted]
N378
L178
cop.2

87112

Lambropulos, Dimitri
The influence of
plastic flow ...

[redacted] 87112

N378
L178
cop. 2