



The effect on final achievement in a beginning calculus course resulting from the use of programmed materials written to supplement regular classroom instruction
by William Albert Stannard

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of
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Abstract:

This investigation resulted from the belief that programmed materials written to supplement regular classroom instruction of beginning calculus would significantly improve the final achievement of students studying these materials in addition to their regular classroom instruction. The author consulted pertinent literature as a means of investigating the extent to which the general area of the problem had been explored. The literature revealed no evidence which supported or denied that programmed materials of the type specified would significantly effect the learning of beginning calculus. Hence an experiment was designed to measure the effect on final achievement in a beginning calculus course resulting from the study of programmed materials written to supplement regular classroom instruction. The first step of the experiment was the preparation of a set of preliminary programmed materials written to cover the critical areas of beginning calculus which cause students the greatest learning problems. These preliminary materials were used in a pilot study to detect flaws in construction as well as to provide the author with experience in conducting the experiment. Following the pilot study the materials were revised to correct discovered inadequacies.

An experimental group was then selected to use the revised materials as a supplement to regular classroom instruction. The students of the experimental group were volunteers from the beginning calculus course and were chosen on the basis of their mathematical achievements in previous college mathematics courses. For one college quarter the programmed materials were administered to the experimental group in addition to their regular classroom instruction. At the end of this quarter the final examination scores earned by members of the experimental group were recorded as a measure of individual student achievement.

The control group was next selected. Standards for admission into this group were similar to those for the experimental group with the exception that the control group did not have access to the programmed materials. The measure of the final achievements of the members of the control group was by means of the same final achievement examination taken by members of the experimental group. The final achievement scores of the two groups were compared by appropriate statistical tests.

As a result of reviewing literature the author concluded: (1) Modern scientific-technological pressures have brought about many changes in the emphasis and content of current mathematics. Since these changes have influenced the teaching of calculus, a need has developed for more effective calculus teaching techniques. (2) The results of research done on the effectiveness of supplementing class-room teaching of mathematics by using programmed materials has been inconclusive. From his investigation the author concluded that the effect on the final achievement of students in a beginning calculus course resulting from the study of programmed materials prepared by the author was not significantly above the corresponding achievement of students not studying such materials.

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THE EFFECT ON FINAL ACHIEVEMENT IN A BEGINNING CALCULUS
COURSE RESULTING FROM THE USE OF PROGRAMMED MATERIALS
WRITTEN TO SUPPLEMENT REGULAR CLASSROOM INSTRUCTION

by
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A thesis submitted to the Graduate Faculty in partial
fulfillment of the requirements for the degree

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ABSTRACT

This investigation resulted from the belief that programmed materials written to supplement regular classroom instruction of beginning calculus would significantly improve the final achievement of students studying these materials in addition to their regular classroom instruction. The author consulted pertinent literature as a means of investigating the extent to which the general area of the problem had been explored. The literature revealed no evidence which supported or denied that programmed materials of the type specified would significantly effect the learning of beginning calculus. Hence an experiment was designed to measure the effect on final achievement in a beginning calculus course resulting from the study of programmed materials written to supplement regular classroom instruction. The first step of the experiment was the preparation of a set of preliminary programmed materials written to cover the critical areas of beginning calculus which cause students the greatest learning problems. These preliminary materials were used in a pilot study to detect flaws in construction as well as to provide the author with experience in conducting the experiment. Following the pilot study the materials were revised to correct discovered inadequacies.

An experimental group was then selected to use the revised materials as a supplement to regular classroom instruction. The students of the experimental group were volunteers from the beginning calculus course and were chosen on the basis of their mathematical achievements in previous college mathematics courses. For one college quarter the programmed materials were administered to the experimental group in addition to their regular classroom instruction. At the end of this quarter the final examination scores earned by members of the experimental group were recorded as a measure of individual student achievement.

The control group was next selected. Standards for admission into this group were similar to those for the experimental group with the exception that the control group did not have access to the programmed materials. The measure of the final achievements of the members of the control group was by means of the same final achievement examination taken by members of the experimental group. The final achievement scores of the two groups were compared by appropriate statistical tests.

As a result of reviewing literature the author concluded: (1) Modern scientific-technological pressures have brought about many changes in the emphasis and content of current mathematics. Since these changes have influenced the teaching of calculus, a need has developed for more effective calculus teaching techniques. (2) The results of research done on the effectiveness of supplementing classroom teaching of mathematics by using programmed materials has been inconclusive.

From his investigation the author concluded that the effect on the final achievement of students in a beginning calculus course resulting from the study of programmed materials prepared by the author was not significantly above the corresponding achievement of students not studying such materials.

CHAPTER I

INTRODUCTION

There has been good reason to refer to the present age as a scientific-technological era. Never before in the history of man has the destiny of the future been so closely controlled by scientific technology. In the past twenty years single nations have come to possess, through scientific means, the machines of war capable of destroying entire civilizations. In fact the grasp of modern science has reached into the lives of people everywhere in the form of television, automobiles, satellites, electric dishwashers, medical drugs, and dozens of other areas of everyday life. Thus for the citizen of tomorrow to understand, appreciate, and properly use these tools he must be well qualified to judge them and must possess the technical ability to devise other efficient machines which will help him design a more compatible tomorrow.

However, the ability to properly judge, use, and design the complex scientific devices of this modern era has been found to be strongly dependent upon the degree of comprehension of the supporting science, mathematics. With the current shift to an increasingly automated world where tolerance for error has been found to be small, the society of today has turned to the truths of mathematics to guide the way to scientific understanding and utilization.

Naturally this heavy concentration on mathematics has had far-reaching effects. In fact, so much pressure has been brought to bear on the subject that the resulting vast changes in mathematics have been aptly described as a true revolution. This point was clearly stated in

1961 by mathematician G. Baley Price, Chairman of the Department of Mathematics at the University of Kansas, when he wrote:

The changes in mathematics in progress at the present time are so extensive, so far-reaching in their implications, and so profound that they can be described only as a revolution.¹

With such intense focus on mathematics by mathematicians and non-mathematicians throughout this country, the impact on education has been inescapable. In the past few years mathematicians and educators have found it necessary to work in close harmony in attempting to devise effective teaching programs capable of meeting modern needs. With the acceleration in scientific advancements, mathematics educators have been pressed to present the subject to the learner in the most effective way possible so that the subject matter presented will be in harmony with future expansion of mathematical knowledge. Numerous instructional improvement projects have been put into operation through federally sponsored institutes for teachers, in-service education courses, lecture series, educational television programs, and whatever other means have seemed appropriate.

Actually this revolution has repeatedly complicated itself because not only has each major advancement in mathematics implied

¹Price, G. Baley, "Progress in Mathematics and Its Implications for the Schools," in The Revolution in School Mathematics, A Report of Regional Orientation Conferences in Mathematics by the National Council of Teachers of Mathematics, p. 1.

a need for a curriculum revision at graduate level study, but each curriculum revision at the graduate level has in turn implied a need for curriculum revision at all prerequisite levels. This snowballing effect has severely challenged educational efforts to keep teaching up-to-date and in line with future needs.

Since the primary problem has become one of preparing more and more students with a broader mathematical background, it has become increasingly clear that ways need to be found to improve the transmission of mathematical knowledge from the instructor to the student at all levels. One critical level which has been found to demand such improved learning-teaching effectiveness is at the early college level where the beginning calculus is encountered.

Statement of the Problem

The problem of this study was to determine the effect on final achievement of students in a beginning calculus course who had worked through a series of lessons of programmed instruction which had been written to teach certain basic ideas in the calculus.

The existence of the problem investigated had its origin in the fact that the learning of beginning calculus by the student has been recognized to be largely governed by the teaching he receives in class, by the reading he does out of class, and by his homework. Since so many students have failed to perform satisfactorily in calculus, the learning in general has either been incomplete or the information learned has been incorrect. In view of this learning problem the

question was posed: to what extent would programmed materials written to cover the areas of calculus which cause the greatest learning problems be a significant learning aid to the student? This question in turn led to the statement of the hypothesis (H_0) with which this investigation was concerned: there is no significant difference in the final achievement of students in a beginning calculus course who have studied programmed materials as a supplement to conventional classroom instruction of topics which ordinarily cause learning problems over students who have not studied such materials.

Procedures

In order to determine if programmed materials specifically written to cover the topics which cause learning problems for the student in a beginning calculus course would be a favorable supplementary learning aid for the student the following procedures were used:

1. Pertinent literature was examined to provide background information which would orient the study in terms of past and present practices and problems involved in teaching beginning calculus and to summarize other investigations which have dealt with the effects of programmed materials which have resulted from their uses.
2. An identification was made of the topics of beginning calculus which cause students learning problems.
3. A set of programmed materials was written which possibly would assist the student in learning these difficult topics.

4. An evaluation of the effectiveness of the preliminary programmed materials was made following the administration of these materials to a pilot group. This evaluation involved a rating system whereby items of the programmed materials were ranked according to their effectiveness.

5. Sections of the preliminary programmed materials which indicated lack of teaching effectiveness were rewritten.

6. The rewritten programmed materials were administered to a carefully chosen and regulated experimental group.

7. A study of the effect of the programmed materials on final achievement was made by statistically comparing the final achieved scores of an experimental group which used the prepared materials with the final achieved scores of a control group which did not use the prepared materials.

The control and experimental groups were initially of comparable background and ability.

In order to determine the areas of beginning calculus which cause students learning problems, a list of topics covered in a typical first quarter calculus sequence was prepared. This list was given to twelve people well qualified to judge the learning problems of students in a beginning calculus sequence. This same jury then ranked the topics which in their judgment caused the greatest learning problems.

The topics of the list were then tabulated according to the frequency of items marked. By considering possible available student time it was decided that no more than seventeen evening meetings could be expected

of students desiring to work through the programmed materials which would be written. Thus the seventeen topics most frequently indicated were denoted and programmed materials were written to cover each of these areas.

The seventeen programmed lessons were administered to students requesting assistance in a beginning calculus course. These students kept a record of the items of the programmed materials with which they had difficulty. By using Holland's 10 percent permissible error ratio² those items of the materials which displayed over 10 percent error were rewritten to insure greater clarification of subject matter.

In order to test the hypothesis (H_0) an experimental group was next selected from the following quarter's beginning calculus sequence enrollment at Montana State College. On a weekly schedule during the quarter these students were provided with fifteen lessons of programmed materials written to correct the faults of the original seventeen lessons. At the end of the quarter the entire beginning calculus class was given the same comprehensive final examination for the course. Scores on this final examination were recorded and statistically studied to test the hypothesis (H_0) that there was no significant difference in the final achievement of students in a beginning calculus course who have studied programmed materials as a supplement to conventional classroom instruction of topics which ordinarily cause learning problems over students who have not studied such programmed materials. Teaching

²Fry, E. G., Teaching Machines and Programmed Instruction, p. 4.

of all of the beginning calculus students was done by five experienced and well qualified instructors, and general supervision of all teaching was carefully carried out by one individual whose extensive qualifications included forty years of college mathematics teaching and fifteen years as head of a mathematics department.

Limitations

The following limitations were imposed on the investigation:

1. The number of students in the experimental and control groups was limited to twenty-one and twenty-two respectively. These sample sizes were judged large enough to give the statistical tests sufficient discrimination power.
2. The experimental portion of the investigation was limited in time to the duration of the winter quarter of the 1964-65 college academic year. This period of investigation was judged sufficient because during that quarter the experimental group had ample time to carefully study enough programmed materials to master those topics selected as being essential to success in beginning calculus.
3. The topics of beginning calculus which were programmed for use by the experimental group were fifteen in number. The choice of these particular fifteen topics was judged appropriate because a rating system was used to identify them as being critical to the learning of beginning calculus.
4. The regular classroom instruction of the experimental and

control group was limited to instruction by five experienced and competent members of the mathematics staff at Montana State College. Supervision of the instruction was kept uniform and of high quality by a competent and experienced supervisor.

5. The measurement of the final achievements by the members of the experimental and control groups was limited to one comprehensive examination. This limitation was necessary in order that the investigation could be carried out within the structure of existing mathematics departmental policies at Montana State College.

6. The level of significance for the statistical tests was set at 5 percent. This level was judged appropriate in view of the nature of the investigation.

Under these limitations the investigation was made to discover the effects on final achievement of students in a beginning calculus course who have studied programmed materials as a supplement to conventional classroom instruction of topics which ordinarily cause learning problems over students who have not studied such programmed materials. The first step of this investigation, a review of related literature, is reported next.

CHAPTER II

REVIEW OF LITERATURE

The purpose of the review of literature was to provide background information which would orient the study in terms of past and present practices and problems involved in teaching beginning calculus and to summarize other investigations which have dealt with the effects of programmed materials resulting from their application under similar circumstances and for purposes similar to those of this study. In organizing the results of the literature reviewed four major topics were selected and correspondingly developed through reference to pertinent literature: (1) recent changes in the content and spirit of beginning calculus, (2) learning problems associated with current beginning calculus, (3) the role of programmed instruction in learning current beginning calculus, and (4) a brief summary of investigations similar in manner and purpose to that of this study. The first topic is developed in the following section.

Recent Changes in the Content and Spirit of Beginning Calculus

In the 1950's serious changes began to take place in the mathematics curriculum which had been established in the United States since the 1890's.³ One of the first major breaks with the traditional mathematics curriculum came in 1951 when the Mathematics Department

³Dyer, H. S.; Kalin, Robert; and Lord, F. M.; Problems in Mathematical Education, pp. 18-19.

at the University of Illinois reacted to the poor mathematical preparation of students in the engineering curriculum by forming a committee to specifically look into college preparatory mathematics.⁴ Typifying the reports of these investigations was the following observation:

Few freshman students had any insight into the structure of mathematics; they regarded the subject as a collection of unrelated and often inconsistent rules. . . . Textbooks were often written as though mathematics were a completed thing, there to be learned, rather than a growing subject in whose development one might take part.⁵

In an attempt to correct this shortcoming the University of Illinois Committee on School Mathematics, usually referred to as the U. I. C. S. M. , was formed with Max Beberman as its director. Soon this active group was busy writing texts and devising a curriculum to meet the modern mathematical needs of society. It was significant that the U. I. C. S. M. singled out the importance of student understanding through discovery of mathematical facts as a major factor in learning mathematics. The emphasis on this aspect of learning mathematics can be traced throughout the writings of this group in such places as the first sentence of the first page of their High School Mathematics where they state to the teacher: "We believe that students should be given an

⁴Henderson, K. B., et al., Mathematical Needs of Prospective Students, p. 5.

⁵American Association for the Advancement of Science, Science Education News; Miscellaneous Publication No. 65-7, citing one of the members of the University of Illinois Committee on School Mathematics, p. 1.

opportunity to discover a great deal of mathematics which they are expected to learn."⁶

Shortly after the U. I. C. S. M. began its attempt to design a better mathematics curriculum other groups and agencies began to show interest in these same areas. In 1952 the Board of Governors of the Mathematical Association of America appointed a Committee on the Undergraduate Program to look at what the undergraduate program in mathematics should be like.⁷ Among their recommendations which later became published in text form for the first year college calculus students was the unique feature of an intuitive approach to calculus to aid the student gain good insight into what was being presented. Secondly, they agreed on a carefully written formal development of the calculus through definitions, axioms, theorems, and mathematically complete proofs.⁸

These early efforts by the Committee on the Undergraduate Program in Mathematics and the U. I. C. S. M. set the stage for much of the mathematical spirit and curriculum content which was soon to blossom forth into what many writers referred to as a revolution in school mathematics. From this beginning the emphasis on understanding

⁶University of Illinois Committee on School Mathematics, Unit I: Arithmetic of Real Numbers, High School Mathematics, Teacher's Edition, p. I.

⁷Begle, E. G.; MacLane, Sanders; Price, G. B.; Lectures on Experimental Programs in Collegiate Mathematics, p. 4.

⁸Ibid., p. 17.

through intuition and discovery and the careful attention paid to mathematical structure has continued to dominate the mathematical scene in curriculum revision up to the time this study was made.

A later group which was very influential in continuing and emphasizing this same spirit of modern mathematics was the School Mathematics Study Group. It was formed in 1958 by the president of the American Mathematical Society with the approval of the National Council of the Teachers of Mathematics and the Mathematical Association of America.⁹ The School Mathematics Study Group singled out the importance of mathematical structure as one of their primary concerns. In fact structure was identified as being their major objective in the first sentence in the "Preface of the Teacher's Commentary" of the First Course in Algebra:

The principal objective of this First Course in Algebra is to help the student develop an understanding and appreciation of some of the algebraic structure exhibited by the real number system, and the use of this structure as a basis for the technique of algebra.¹⁰

With this strong emphasis on structure and complete understanding at all academic levels as the spirit of modern mathematics the content of mathematics courses was changed from the problem oriented approach to a logical development with student awareness

⁹Wagner, John, "The Objectives and Activities of the School Mathematics Study Group," The Mathematics Teacher 53:454, October, 1960.

¹⁰School Mathematics Study Group, First Course in Algebra, p. i.

approach. This was particularly true at the college calculus level. Agnew, Professor of Mathematics at Cornell University and well known calculus author, aptly portrayed the spirit and content of calculus prior to 1900 by stating: "Problems were the important things, and meaningful formulations of axioms, postulates, definitions, hypotheses, conclusions, and theorems either were not written or played minor roles."¹¹ Agnew went on to describe the transition which took place during the next fifty years as a direct contrast to the earlier attitudes:

Through most of the first half of the twentieth century, elementary textbooks in our subject taught unexplained but "well motivated" intuitive ideas along with their problems. Enthusiasm for this approach to calculus waned when it was realized that students were not nourished by stews in which problems, motivations, fuzzy definitions, and fuzzy theorems all boiled together while something approached something else without ever quite getting there. About the middle of the twentieth century, precise formulations of basic concepts began to occupy minor but increasingly important roles.¹²

Finally Agnew indirectly characterized the current spirit and content of modern calculus by stating in his 1962 calculus text:

Each student is expected to read the text and problems of each section as carefully as an alert physicist reads an account of a newly developed nuclear reaction, and to learn as much as he can.... The text, problems, and remarks frequently give students quite unusual opportunities and incentives to think and to become genuine authorities on developments of ideas, terminologies, notations, and theories.¹³

¹¹Agnew, R. P., Calculus, p. v.

¹²Ibid., p. v.

¹³Ibid., pp. v-vi.

As to the future of mathematical course content and spirit some extrapolations have been made. In 1963 under the financial support of the National Science Foundation twenty-five prominent and well-qualified mathematicians assembled at Cambridge, Massachusetts to assess the potential of mathematical learning and to set tentative goals towards which mathematical education should attempt to direct its efforts in the next thirty years. The goals of particular courses were specifically spelled out, and the role of structure permeated the entire system at all levels as was indicated by the statement:

We hope to make each student in the early grades truly familiar with the structure of the real number system and the basic ideas of geometry. . . . Moreover, we want to make students familiar with part of the global structure of mathematics.¹⁴

Furthermore, the Cambridge group endorsed the method of teaching mathematics which results in discovery by the students:

Even modestly endowed students can recreate larger parts of mathematics if they can remember just a few basic ideas. This fact repeatedly has been demonstrated in the classroom by the proponents of the so-called discovery method. The building of confidence in one's own analytical powers is another goal of mathematics education.¹⁵

Hence, a study of the literature pointed out that the spirit of contemporary mathematics at the level of beginning calculus as well as

¹⁴Educational Services Incorporated, The Report of the Cambridge Conference on School Mathematics, Goals for School Mathematics, p. 8.

¹⁵Ibid., p. 9.

at all levels has been largely associated with complete mathematical understanding of the subject from a structural viewpoint. That is, the modern trend in mathematics has placed a premium on clear insight into the structure of mathematics rather than on mathematical ability without understanding. Student discovery and development of mathematical truths were frequently referred to as the best means of developing this structural awareness.

In the next section the current problems associated with learning mathematics with this emphasis on structure by means of student discovery are discussed.

Learning Problems Associated with Current Beginning Calculus

The literature reviewed clearly reflected a recent swing to emphasis on mathematical structure and student development of mathematical ideas. A considerable number of authors of the present widely used calculus texts have agreed to this principle.¹⁶ However, not all of the students currently enrolled in beginning calculus have had a background in mathematics which emphasizes structure to the degree

¹⁶ Randolph, J. E., Calculus and Analytic Geometry, pp. v-vii; Protter, M. H. and Morrey, C. B., College Calculus with Analytic Geometry, pp. v-vii; Agnew, op. cit., pp. v-vii; Apostol, T. M., Calculus, Vol. 1, pp. vii-ix; Goodman, A. W., Analytic Geometry and the Calculus, pp. v-vii; Thomas, G. B., Calculus and Analytic Geometry, pp. v-vi.

modern calculus courses demand. Hence, the student whose background is entirely traditional often has had difficulty catching the spirit of modern mathematics. This fact was well substantiated when Zant, in discussing problems facing beginning college mathematics students in Oklahoma, stated:

Another problem. . . is that of the good student who has studied only traditional mathematics courses in high school. Somehow and as quickly as possible, these students must be taught the point of view towards mathematics and the basic concept involved in a modern program in secondary mathematics. If our college courses beginning at the analytics-calculus level have been modernized, then these students, even though they have studied traditional mathematics for three or more years in high school, are not prepared to enroll in such a course.¹⁷

Along with this emphasis on structure the major unifying ideas of mathematics have often been expressed using notation not commonly found in the traditional mathematics. For example, Randolph of the University of Rochester in his 1961 calculus text has stated:

The introduction into elementary courses of the well-established notation for the set of elements, each satisfying a given condition is long overdue and rapidly finding favor. Even though some books state once and for all that 'the line $x = -2$ ' means 'a point (x, y) is on the line if and only if $x = -2$ ', some students will be confused about how x and y

¹⁷ Zant, J. H., "Effects of New Mathematics Programs in the Schools on College Mathematics Courses," The American Mathematical Monthly 63:200, February, 1963.

can be both $x = -2$ and $y^2 = 4x$ in the same problem. After gradually becoming acquainted with the symbolism, then $\{(x, y) \mid x = -2\}$ and $\{(x, y) \mid y^2 = 4x\}$ forces a reading and quick comprehension where equivalent worded statements only annoy and muddle.¹⁸

The net result of this emphasis on structure when accompanied with corresponding modern notation has often compounded newness and difficulty thus resulting in learning problems for the student.

The wide spectrum of student needs and abilities were also reported as a learning problem. Evidence was plentiful which pointed out that not all students require the same type of calculus course. The mathematics majors need a firm understanding of the theory as well as a mastery of mechanical competency oriented always to their past and future training. On the other hand the engineer is required to master the application of mathematical fact to physical phenomenon which he will likely encounter in engineering problems. A third extreme is the student who takes calculus more for cultural improvement than as a prerequisite for future use. In this last case the calculus is not problem oriented but instead based on a developmental scheme via a rigorous sequence of definitions, axioms, theorems, and proofs.

The literature further revealed that many attempts have been made to meet this problem. In general the reported trend was toward special sections, honors courses, advanced placement, regular courses, and

¹⁸ Randolph, op. cit., p. vi.

cultural courses geared to fit the needs and abilities of the students. However, even with such varied alternatives the problem of differing student needs still has caused concern. For example, Brown described as follows the beginning calculus problem at Dartmouth where in spite of alternative offerings of regular calculus classes, advanced placement, and honors mathematics, students' needs were not being met: "It seems to me that the first course . . . should be as broad as possible. Admittedly we have to steer between breadth and superficiality."¹⁹

Thus, since the instructor in the average class has been challenged to direct his teaching according to the class needs, treatment of a particular topic has often not been as complete for individual needs as would be desirable. That is, structural development of the calculus may have been handled too extensively for some students whereas it was found by others too shallow.

In this same area of thought, student abilities within a particular class have been known to range widely despite attempts to group homogeneously. Jones and Pingrey made this point clear by stating: "Students in any classroom, even in those where ability grouping is used, will differ widely in both general background and special abilities and backgrounds."²⁰ Hence, since ". . . it is still the individual classroom

¹⁹Brown, B. H., "Offerings for Freshmen," American Mathematical Monthly 68:262, March, 1961.

²⁰Jones, R. S. and Pingry, R. E., "Instructional Arithmetic" in The Twenty-Fifth Yearbook of the National Council of the Teachers of Mathematics, 1960, p. 122.

teacher who must adapt instruction to the differences he finds among his pupils, "²¹ the instructor has been challenged to develop the calculus in a way which is both inspirational to excellent students and within the grasp of poor students. Frequently in an attempt to satisfy one particular ability level the instructor has sacrificed the other extremes. Thus learning problems for the neglected students have resulted.

The role programmed instruction can play in meeting these learning problems is discussed next.

The Role of Programmed Instruction in Learning Current Beginning Calculus

A search of the literature revealed that learning problems in beginning calculus are in general fourfold: (1) unfamiliar notation has provided reason for low achievement, (2) new spirit of modern mathematics has been unfamiliar to many students, (3) individual student differences have caused teaching problems, and (4) differing student needs have been found to influence teaching effectiveness. The purpose of this section will be to show the role of programmed materials as associated with these needs.

Unfamiliar notation: Research has established that students can learn effectively by programmed instruction.²² Hence, since the problem of unfamiliar mathematical notation has been shown to be a

²¹ Ibid., pp. 145-6.

²² Schramm, Wilbur, The Research on Programed Instruction, p. 3.

hinderance to learning, it follows that students should be able to learn this troublesome notation and what it means by materials programmed for that specific purpose.

The spirit of modern mathematics: Although the question of being able to learn unfamiliar mathematical notation by study of properly written programmed materials was found to have been possibly answered, the question of being able to catch the proper spirit of modern mathematics was not conclusively answered. That is, the problem as to the ability of students to properly connect an understanding of mathematical structure with logical reasoning to produce new mathematical facts was found to be unanswered in the literature. Differing opinions and statements were abundant.

Relative to this question the Department of Audiovisual Instruction of the National Education Association and the National Society for Programmed Instruction in reference to research by Cartier stated:

"Francis Cartier points out that a number of aspects of rational thinking (e. g. formal logic) can be taught by present programming techniques."²³

But in the next line the text read: "Formal logic accounts for only a part of the rational and creative processes we call thinking."²⁴

A considerable amount of skepticism as to the value of programmed instruction in teaching mathematical thinking of the type desired in modern mathematics was found in May's Programed Learning and Mathe-

²³Ofiesh, G. D. and Meierhenry, W. C., editors, Trends in Programmed Instruction, Department of Audiovisual Instruction of the N. E. A. and the National Society for Programmed Instruction, p. 415.

²⁴Ibid., p. 415.

mathematical Education. May stated emphatically: "Programed materials are incapable of eliciting the full range of behavior included in the objectives of college mathematics."²⁵ In particular May was concerned that programmed materials do not cause the student to "get out and grub" for mathematical truth; instead he contended that the student "is fed ideas intravenously drop by drop" which is not the spirit of modern mathematics."²⁶

Buck was found to agree with May in this observation because he stated that programmed instruction does not develop creative ability but merely exercises the student in drill. In particular he said:

In this I feel that a grave mistake has been made: to put it bluntly, creativity is the heart and soul of mathematics at all levels. The collection of special skills and techniques is only the raw material out of which the subject itself grows. To look at mathematics without the creative side of it, is to look at a black-and-white photograph of a Cezanne; the outlines may be there, but everything that matters is missing.²⁷

Hence, the question of programmed instruction being able to satisfactorily cause the student to grasp the spirit of modern mathematics was found to be open to question.

²⁵May, K. O., Programed Learning and Mathematical Education, p. 5.

²⁶Ibid., p. 7.

²⁷Buck, R. C., "Teaching Machines and Mathematics Programs," American Mathematical Monthly 69:554, June-July, 1962.

Individual differences: A logical question often asked was: can programmed materials sufficiently solve the individual differences problem which results from instruction being beamed at class average hence causing the slower student to be unable to follow the materials being presented in class?

In many sources the reasoning has followed the logical line that programmed instruction "tends to level the differences in learning capacities among students; while all students exposed to the program may demonstrate achievement, the gain seems to be more conspicuous among the lower portion of the class distribution."²⁸ One of the first experimental efforts which verified this belief was reported in 1934 when Little published in The Journal of Experimental Education the "Results of Use of Machines for Testing and for Drill Upon Learning in Educational Psychology."²⁹ Then in 1959 Porter confirmed the belief by experimenting with spelling and programmed instruction.³⁰

However, here again evidence was plentiful in the literature which indicated that individual differences are not significantly satisfied by a slow but sure pace due to programmed instruction. In summarizing the evidence May declared:

²⁸Lysaught, J. P. and Williams, C. M., A Guide to Programmed Instruction, p. 15.

²⁹Fry, E. B., op. cit., p. 84.

³⁰Porter, D., "Teaching Machines," Harvard Graduate School of Education Association Bulletin 3:1-5, March, 1958.

Numerous studies have indicated that self-pacing is not as helpful as one might imagine. . . . It appears that pacing is not a very important issue and that self-pacing has no necessary or unique connection with programmed materials.³¹

As added substantiation of this point Schramm has pointed out seven studies which show there is no significant difference attributable to internal or external pacing.³²

Different student needs: The final learning problem which was evident in the literature was differing student needs. Regarding mathematics and programmed learning May declares:

The real possibilities for using programmed materials to cope with individual differences lie in different directions. One is the development of large libraries of brief units focused on narrow problems, beamed to specific student difficulties, and utilizing programming devices most appropriate to the audience and difficulty.³³

Thus in summary it is clear that there has been much evidence to indicate the use of programmed materials can be beneficial in helping students learn. However, the degree to which programmed materials can be beneficial has been contested and the areas where it is of benefit has not been in total agreement.

In the next section research will be reviewed which has a bearing on learning beginning calculus by programmed materials.

³¹May, op. cit., p. 6.

³²Schramm, op. cit., p. 12.

³³May, op. cit., pp. 6-7.

Survey of Investigations Similar in Manner
and Purpose to This Study

A search of the literature revealed that no other experimental work had been carried out which measured the effect on achievement of beginning calculus students when their conventional classroom instruction was supplemented with programmed instruction. However, a number of studies had been completed which were related to the investigation with which this writer was concerned. One particular investigation reported by Lane which was of interest was concerned with different means of supplementing conventional classroom instruction of mathematics.

In 1962 Lane experimented with three means of supplementing instruction by closed circuit television of a basic mathematics course at the George Peabody College for Teachers. Lane used three types of supplementary instruction: film of assigned homework shown to students daily, daily classroom question-answer periods, and programmed instruction booklets prepared to cover the topic presented on television. The results of the study indicated that achievement of the group using programmed instruction was significantly better at the 5 percent level than the groups whose means of supplementing the televised instruction was by film or class question-answer periods.³⁴

³⁴Lane, B. R., "An Experiment with Programmed Instruction as a Supplement to Teaching College Mathematics by Closed Circuit Television," The Mathematics Teacher 57: 395, October, 1964.

Numerous other investigations have been made into the effectiveness of teaching college mathematics and mathematics related courses by programmed instruction. In reviewing the literature the following investigations have been found to reveal results pertinent to this study.

Sharpe reported at the University of Buffalo on the effectiveness of using Skinner programmed materials in place of texts in pre-calculus courses:

To date, the experiment as it has been conducted indicates a probability that programed materials may do an equivalent job, but presents no evidence that programed materials are superior. . . . Students in this experiment had programs³⁵ backed up by good instructors, yet no records were broken.

At the United States Air Force Academy Smith has studied the teaching of elementary statistics by the conventional classroom method versus the method of programmed instruction. The net result of the investigation has shown that learning achieved through study of statistics by a scrambled book program as compared to learning via conventional classroom teaching produced no significant difference in performance between experimental and control groups.³⁶

³⁵ May, op. cit., p. 5, citing Sharp, B., "Programed Mathematics, A Two Year Experiment 1962-1964," place and date unknown.

³⁶ Smith, H. N., "The Teaching of Elementary Statistics by the Conventional Classroom Method Versus the Method of Programed Instruction," Journal of Educational Research 55:417-20, 1962.

One of the first studies of the effectiveness of programmed learning in logic was done by Blyth at Hamilton College. Blyth reported the preliminary results of his investigation as encouraging. Contact hours with students were reduced by one third, level of subject matter mastery was raised, and the amount of material covered was increased.³⁷

An important study was done during the fall of 1962 by Dobyns who reported on an experiment in the teaching of college algebra. Dobyns compared teaching certain topics of college algebra by conventional methods to teaching by programmed instruction. Among his conclusions were the following:

There was no significant difference between the mean scores for the two teaching methods. . . . Significant gains made by the students using the programmed booklet covering inequalities, absolute value, coordinate systems, functions, and their graphical representations were lost after a time lapse.³⁸

Kellems also reported on the effectiveness of programmed teaching in college algebra. His study was conducted during three semesters at Indiana State College during which time he administered programmed materials in three ways. For his control group Kellems

³⁷Blyth, J. W., "Teaching Machines and Logic," American Mathematical Monthly 67:285-7, March, 1960.

³⁸Dobyns, R. A., "An Experiment in the Teaching of College Algebra," The Mathematics Teacher 57:86, February, 1964.

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Another related study was done by Heimer who conducted an experiment in which he measured the effects on achievement brought about by using different methods of instruction on college students in a beginning mathematics course. In particular he compared the effects of external and self-pacing when teaching was done by teaching machines, programmed textbooks, filmstrips, and conventional classroom teaching. No significant differences were revealed on final examination scores.⁴¹

In addition to this experiment by Heimer was one conducted by Lottes, Palmer, and Oakes who compared the effects on achievement due to differential rate of pacing programmed mathematics. No significant differences were observed in student performance or attitudes towards programmed learning when pacing was varied as much as 20 percent below and 10 percent above the average time required by a group of self-pacing students.⁴²

⁴¹Heimer, R. T., The Preparation of a Program in Contemporary Algebra and a Study of Its Effectiveness for Group Instruction Under Paced Conditions, Doctoral Dissertation, as reviewed in Dissertation Abstracts, Vol. XXIII, No. 10, April, 1963, pp. 3813-4.

⁴²Schramm, Wilbur, op. cit., p. 32 citing Lottes, J. J., Palmer, G., and Oakes, H., "An Experimental Comparison of Differential Rates of Pacing Programed Math," published by the University Division of Instructional Services, the Pennsylvania State University, 1963.

Although numerous investigations have been carried out at the high school and junior high school level, the problem of motivation often makes the results of the investigations difficult to interpret.⁴³

However, one such study at the high school level which had no mention of motivation problems and which was somewhat parallel to that of this study was done by Brown. Using the programmed booklets written by the U. I. C. S. M. Brown compared the test scores of an experimental group using programmed instructional materials with a control group not using programmed instructional materials. The experimental group in this study had conventional classroom teaching along with the programmed materials whereas the control group had only conventional classroom instruction. From this study the conclusion was reached that:

The experimental group proved to be significantly superior to the control in a test of general ability, and about the same level of superiority was maintained in eight out of nine achievement tests given during the school term. The author concludes that 'no student was penalized in his level of mathematics achievement because of having used programmed material.'⁴⁴

⁴³Randolph, P. H., "An Experiment in Programmed Instruction in Junior High School," The Mathematics Teacher 57:160-162, March, 1964; Jordy, J. L., "A Comparative Study of Methods of Teaching Plane Geometry," The Mathematics Teacher 57:472-8, November, 1964; Smith, W. I. and Moore, J. W., "Programed Materials in Mathematics for Superior Student in Rural Schools," published by Bucknell University, undated, p. 91.

⁴⁴Schramm, op. cit., p. 26, citing Brown, O. Robert, "A Comparison of Test Scores of Students Using Programed Instructional Materials with Those of Students Not Using Programed Instructional Materials," mimeographed by the University of Illinois in 1962.

The review of literature has shown that there is evidence to indicate value of using programmed instruction in place of or as a supplement to conventional classroom instruction. However, there was also evidence which indicated the gains resulting from using such programmed materials are often not significant. Also, when differences do occur, they are sometimes in favor of the conventional method or disappear with a time lapse.

A summary of this entire chapter is presented next.

Summary

In summarizing the literature reviewed four points stand out:

(1) beginning calculus has experienced a radical change in the past eight years from a problem-solution oriented attack to a careful structural attack achieved through student understanding and logical development, (2) many of the learning problems in beginning calculus have been brought about by the swing from a traditional type background to a modern approach thus leaving many students ill prepared for the degree of rigor required in modern mathematics, (3) the use of programmed instruction may reduce the imbalance in background experienced by traditionally prepared students through development of proper notation but may not entirely fill the gap in proper mathematical attitude such as is often required in mathematical reasoning, and (4) the investigations into the usefulness of programmed instruction do not consistently point to its unswerving value but instead

the majority of comparisons of its value over conventional classroom instruction indicate the differences in final learning achieved through programmed instruction are not significantly greater than those achieved by conventional classroom teaching.

In the next chapter the procedures and results of the investigation will be presented.

CHAPTER III

INVESTIGATIONAL PROCEDURES AND STATISTICAL METHODS APPLIED

The research described in this chapter was conducted as a means of investigating the effect on final achievement in a beginning calculus course resulting from the use of a series of programmed instruction written to supplement the regular classroom instruction on the topics which ordinarily cause learning problems. In reporting the investigational procedures and statistical analysis the following topics were considered to be of major importance: (1) selection of topics to be programmed, (2) preliminary writing of the programmed materials and associated student participation, (3) final writing of the programmed materials and associated student participation, (4) means of instruction of the experimental and control groups, and (5) statistical analysis used to evaluate the results of the investigation. Each of these topics is discussed in the remainder of this chapter.

Selection of Topics to be Programmed

Prior to the fall quarter of 1964 a study was conducted by the writer to establish the conditions under which topics from beginning calculus would be selected and programmed for use by students in an experimental group. This investigation included three areas: identification of topics taught in a beginning calculus course, identification of those topics which cause student learning problems, and an estimation of a logical number of programmed learning sessions which would

be convenient for the students to attend and yet would be extensive enough to guarantee an adequate coverage of material.

The first problem was to identify the topics taught in a beginning calculus course. This was done by comparing the development of the calculus in five widely used calculus texts.⁴⁵ In each of these texts the following topics appeared as introductory material: absolute value and inequalities; functions and functional notation; intersections of pairs of curves; limits; continuous functions; the derivative; simple differentiation formulas; differentiation of the product, quotient, and higher order derivatives; differentiation of the composite functions; implicit differentiation; application of the derivative to linear motion, motion under gravity, and related rates; sketching as related to increasing and decreasing functions; critical values, and extreme values of a function; Rolle's Theorem and the Mean Value Theorem; applications involving the theory of extremes; approximations by differentials; the indefinite integral; computation of area by integration; the definite integral; area as the limit of a sum; fundamental theorem of calculus; and mathematical induction.

⁴⁵ Protter, M. H. and Morrey, C. B., College Calculus with Analytic Geometry; Taylor, Angus E., Calculus with Analytic Geometry; Thomas, George B., Calculus and Analytic Geometry; Johnson, R. E. and Kiokemeister, F. L., Calculus with Analytic Geometry; Goodman, A. W., Analytic Geometry and the Calculus.

In order to identify which of the above topics cause student learning problems a questionnaire was prepared which listed all of the above topics and briefly described the intent of the investigation.⁴⁶ The persons to whom the questionnaire was directed were asked to select seventeen areas which in their opinion cause beginning calculus students the greatest learning problems.

Two factors necessitated limiting the number of areas to seventeen: (1) only a limited number of meetings could be expected of the volunteering students during the quarter, and (2) there was a potential difficulty of correlating the programmed materials with different class rates. The first factor which limited the areas to be covered to seventeen was judged essential since most of the students volunteering for the program were carrying full academic loads. Hence, it was unreasonable to expect them to meet more than twice weekly for the purposes of studying the prepared materials. The second factor was likewise judged necessary because the possibility existed that the topics selected by the jury of experienced teachers could be of the nature that course study of these topics would not occur at the beginning of the course but instead might be grouped timewise toward the end of the quarter. Hence, the decision was made to limit the potential topics selected to seventeen in number with the understanding that some modification of title and content might be necessary as the study progressed.

⁴⁶ See Appendix A.

This questionnaire was then given to twelve college educators well-qualified to judge the learning problems of students in a beginning calculus class. Ten of these persons were prominent members of the mathematics department at Montana State College with extensive experience in teaching beginning calculus. Their experience was judged by their teaching assignments and duties as mathematicians at Montana State College. The group included one department head, three supervisors of teachers of elementary mathematics courses (one of whom was also a retired departmental head and supervisor of the beginning calculus sequence), an acting departmental head, a full professor, two associate professors, one assistant professor, and one instructor.

The two non-mathematics departmental members who filled in the questionnaire were chosen from the physics and chemistry departments. Their responsibilities in their areas indicated concern for and knowledge of learning problems of students in their fields as would be reflected by inability in different phases of calculus.

Upon receiving the results of the questionnaire a frequency tabulation was made to identify those areas of beginning calculus which cause the students the greatest learning problems. The seventeen topics which were most frequently identified were:

1. Inequalities and Absolute Value
2. Limits
3. Continuity
4. The Derivative
5. Composite Functions and the Chain Rule
6. Implicit Differentiation
7. Increasing and Decreasing Functions
8. Velocity and Related Rates
9. Rolle's Theorem and the Mean Value Theorem

10. Maximum and Minimum Problems
11. Differentials
12. Indefinite Integral
13. Definite Integral
14. Area by Integration
15. Mathematical Induction
16. Area as the Limit of a Sum
17. Fundamental Theorem of Integral Calculus

Preliminary Writing of the Programmed Materials and Associated Student Participation

Three factors influenced the scheduling of the meetings during which the students would study the programmed materials. First, on examining the seventeen areas selected by the experienced jury it was apparent that the topics for which the programmed materials had to be written occurred during the development of the course about two weeks after the quarter began. Hence, the first meeting with the volunteering students was not possible until at least two weeks of the quarter had elapsed. Second, since the amount of class time ordinarily devoted to some of the initial topics was comparatively longer than for later topics, a more liberal time schedule was anticipated at the start of the quarter than at the end of the quarter. Third, there was no guarantee that all of the instructors could hold closely to the day-by-day schedule of topics to be taught. These three factors suggested that a certain amount of flexibility was needed in the administration of the programmed materials. With these problems in mind a calendar was drawn up scheduling the programmed meetings on each available Tuesday and

Thursday of each week throughout fall quarter. By allowing for holidays and final examinations exactly seventeen such Tuesdays and Thursdays were found to be available.

Having determined the topics to be programmed and the number of meetings available, it was then possible to start writing the preliminary programmed materials. Preliminary to writing each section, the topic in question was studied carefully from the point of view of the author of the textbook from which the students would be required to read their daily assignments. Then the supervisor of the beginning calculus sequence was contacted to verify the teaching techniques to be used in presenting the topic. Finally, a close contact was maintained with individual instructors to be certain as to presentation of the topic in different classes. By combining these factors with the knowledge gained as a result of consulting literature about programming, the topics were then developed by writing appropriate programmed materials.

Each topic was developed in harmony with the theory and vocabulary used in class presentation. In most cases the topic had its foundation in a certain degree of mathematical theory. Consequently, a search for a clear, precise, intuitive presentation was made by the writer in various calculus texts. Upon examination of numerous leading calculus texts a harmonious development of the topic in conjunction with notation familiar to the students and in conjunction with a sufficient background of factual information evolved.

In cases where theory was presented an intuitive prelude was programmed into the materials which made the principle in question seem plausible. In other cases where procedure was the paramount idea the development of the procedure was gradually introduced to the student with frequent reference to the overall systematic approach. In the event variations of procedure were needed, each major variation was carefully identified and systematically developed.

During the preliminary writing of the materials care was exercised to insure a careful balance among available student time, proper presentation of the topic, and sufficient development of the idea to guarantee learning. In fact it was deemed appropriate to write materials which could be satisfactorily covered by the average student in an hour or less.

After the first writing of the materials for each topic the author proceeded to other duties. Later he returned to the first writing and systematically worked through these materials. As seemingly incomplete or inappropriate sections of the materials were encountered, these sections were rewritten to correct possible faults. Eventually, as the materials became quite polished they were typed up and dittoed for student use.

As a means of checking the quality of these preliminary writings, a pilot group of students was invited to use them on specified dates during the fall quarter of 1964. This pilot group consisted of all students from all sections of the beginning calculus sequence who desired

additional help in calculus. In order to keep the number of volunteer participants in the pilot group of reasonable size no special efforts were made to encourage student participation. The availability of the materials was announced in class and from this brief invitation a sufficient number of student volunteers was obtained to accomplish the purpose of the pilot study. Because of the volunteer nature of the pilot investigation, the number of participants was not consistent throughout the quarter. However, the number was always above ten and below forty except for the last lesson when only five students completed the materials. In investigating the reason for the small interest in the last lesson it was found that the pressure of examinations at the end of the quarter and the fact that not all classes had yet covered this topic in class accounted for the poor attendance.

The purpose of the pilot study was to identify weaknesses in the preliminary writing of the programmed materials and to perfect the investigator's technique of writing programmed materials. The weaknesses in the preliminary writing were identified by instructing the student volunteers to denote by a check those items of the materials which they were unable to complete in accordance with the indicated response which followed each item. Also, a check was kept of time consumed by students in working the materials. Upon the completion of each lesson a check list was made of those items which showed 10 percent error ratio by student participants. The 10 percent figure was used in accordance to the tolerance specified by Holland.⁴⁷

⁴⁷Fry, E. B., op. cit., p. 4.

In addition to a tabulation of error frequency a close contact was maintained with student participants to gain student opinions of particular phases of the materials. These opinions were noted and evaluated so that the final writing could reflect valid criticism.

Final Writing of the Programmed Materials and Associated Student Participation

During the winter quarter of 1965 at Montana State College the final writing and use of the programmed materials was carried out.

The final writing and administration of the materials were based on the experiences gained from the use of the preliminary materials. Modifications of the preliminary materials in the main consisted of three types: (1) modification of items which reflected a 10 percent error ratio; (2) modification of two lessons to facilitate efficient use of the materials; and (3) modification of the administration of the materials.

The first type of modification of the materials was one of content and was focused on individual items of the materials. That is, those items which reflected over a 10 percent error ratio were rewritten to insure greater clarity and thus reduce the error frequency. Sometimes this simply involved changes in wording or the addition of a sketch. In a few cases it was found desirable to redevelop the concept involved by introducing smaller steps and by including more explanatory passages. Considerable discussion was used to be certain the topics were clear and complete. To guard against possible

incomplete reasoning and to insure a clear and favorable development in the rewritten portions, the sections when rewritten were presented to various members of the mathematics staff at Montana State College familiar with the program for comment and criticism. Upon agreement of a clear and consistent revision the rewritten portions were then included in the final version.

The second type of major modification of the materials also involved content. However, in this modification the focus was on efficient use of the materials from the standpoint of complete lessons. Three changes of this type were found necessary.

First, it was deemed advisable to use two meetings of the students for the first lesson because of the desirability of lengthening the time available for the introductory lesson. This was judged appropriate so as to better acquaint students with the purpose of the study and to better correlate the programmed materials with the contiguity of class presentation of corresponding topics.

Second, it was found necessary to eliminate the lesson "Fundamental Theorem of Integral Calculus" since class discussion of that lesson in the preceeding quarter was found to occur too late in the quarter to be satisfactorily supplemented by programmed instruction.

A third modification of the lessons was found necessary as the quarter progressed because class presentation fell behind the biweekly programmed materials sessions. Hence, the necessity presented itself during the quarter of dropping back one lesson. As

a means of compensating for this the topic ratings by the panel of experts was reviewed and it was found that the indefinite integral was one of the topics least in need of supplementary work. Furthermore, the definite integral was a topic in great need of supplementary work. A careful examination of the preliminary writing showed that both of these goals could be best attained by altering the contents of the chapter on "Area by Integration" to include the definite integral and dropping the chapter on "The Indefinite and Definite Integral". With this final modification, which was accomplished without loss of continuity to the materials and without loss of purpose of the investigation, the writing of the materials was completed.⁴⁸

The third major change in the final use of the materials over their preliminary use involved the administration of the materials. The most significant administrative change was that the materials were used by only a select group of students, the experimental group.

The selection of this experimental group took place at the beginning of winter quarter in 1965 after the instructors of the beginning calculus sequence at Montana State College were carefully briefed on the purpose of the forthcoming study into the effect on achievement of using programmed materials as a supplement to classroom instruction of beginning calculus. Upon agreement to the procedures of administration of the materials the instructors were asked to announce the purpose of the investigation to their classes

⁴⁸ See Appendix B.

and to invite undergraduate students to volunteer for the program who had as their final mark in college algebra or college trigonometry grades of B, C, or D. Students were reminded that the nature of their participation was strictly voluntary and that no obligation to volunteer or to finish the program after starting existed. However, it was pointed out that of those starting the program the investigator desired as many to finish the program as could. Furthermore, it was noted that only students participating in the program would have access to the materials and application to start the program could not be accepted after the initial lessons had been completed.

Under these conditions students applied for acceptance into the experimental group. Screening of the candidates was carefully done on the basis of the criteria set up and an experimental group was chosen which consisted of all qualified students. The accepted students were notified of their acceptance and directed to report during the second week of classes for their orientation to the materials and their first lesson. At this first meeting a brief orientation was given as to how to use the materials and questions were answered regarding the program. An agreement was entered into whereby evening meetings would be held regularly twice weekly so that the scheduled lessons would all be worked. Also it was explained that each of these evening meetings would probably last about an hour and would be flexible enough to allow for different working rates of individual students. For a few students with conflicts on the scheduled

evenings it was necessary to schedule a third night a week so that these students would also be able to complete all of the lessons.

From that time on regularly scheduled meetings were held, progress records were kept, and individual contacts were maintained to keep students up to date with their lessons. When students dropped behind in their lessons, reminders were sent them of their delinquency and they were encouraged to make up their missed lessons.⁴⁹

In an effort to keep student advisors informed about the program and the progress of their advisees, explanations of the program and progress reports of advisees were sent to advisors at appropriate times.⁵⁰

At no time during the quarter were the programmed materials allowed to be taken from the supervision of the investigator. In the event a student needed more time to make up a lesson or do extra work, special arrangements were made with the investigator to do the materials under his supervision. In this way the materials were carefully monitored and kept available only to the experimental group.

At the end of the quarter the students who had completed the entire program were noted and their names recorded so that their final records would be easily obtained for purposes of statistical comparisons.

⁴⁹ See Appendix C.

⁵⁰ See Appendix D.

Also, at the end of the quarter the entire class roll of students completing the beginning calculus sequence was searched to identify members of the control group. That is, the backgrounds of all undergraduate students in the beginning calculus sequence were examined to determine those not in the experimental group who had taken as their last mathematics course college algebra or college trigonometry and had received a B, C, or D in that course. By restricting eligibility for selection into the control group to these standards a control group was obtained which was composed of students with backgrounds similar to those of the experimental group since that group was chosen by corresponding standards.

After carefully searching the student records all students meeting the above stated conditions were listed and were accepted as members of the control group.

Means of Instruction of the Experimental and Control Group

In order to be certain that instruction in the different beginning calculus classes was kept constant, the supervisor of instruction held weekly meetings with instructors to check on class progress and lesson presentation. In this way it was assured that all classes were quite uniform in the material presented and in the demands made of the student.

Because special mathematical notation facilitated the writing of the materials on various occasions, at each of these weekly meetings the investigator met with the instructors and supervisor to describe the progress of the study and to indicate notational usage in the materials which needed to be practiced in class to guarantee uniform familiarity.

In describing the progress of the study, the instructors and supervisor were notified of student participation by a progress chart which indicated the number of lessons each student had completed.⁵¹ In addition to the progress sheets the instructors, supervisor, and department head were supplied weekly with the lessons currently being used by the experimental group.

Through: (1) the careful monitoring of the teaching by the supervisor, (2) the competency of the instructors, (3) the close cooperation among the investigator, instructors, supervisor, and department head, and (4) the fact that the members of the experimental and control group were scattered at random throughout all sections of the beginning calculus sequence, it was assumed that uniform instruction for both experimental group and control group resulted.

⁵¹See Appendix E.

Measurement of the Final Achievements
of the Experimental and Control Groups

The measurement of the final achievement of the experimental and control groups was accomplished by the same comprehensive final examination taken by all students who had completed the beginning calculus course the quarter during which this study was made. This single test was used to measure the final achievements for two reasons. First, the testing of the control group had to be accomplished within the structure of existing practices at Montana State College. Second, it was judged possible to design a valid and reliable test which would measure the final achievements of students completing the beginning calculus course that quarter.

In order to carefully control the design of this examination an experienced and well-qualified committee of calculus instructors was appointed by the supervisor of instruction to write an appropriate test which would measure the final achievement of the students completing the course. The examination⁵² was then prepared and presented to the entire group of instructors of the beginning calculus sequence for their inspection and approval. With final approval announced by the supervisor of instruction, who was particularly well qualified to judge the validity and reliability of the test (having had over forty years of college mathematics teaching experience which

⁵² See Appendix F.

included extensive administrative responsibilities), the examination was accepted as an accurate measure of the final achievement of members of the experimental and control groups.

As a means of holding the scoring of this examination uniform, a committee was appointed by the supervisor of the beginning calculus sequence and was directed to prepare a sheet on criteria for grading each problem of the examination. This criteria for grading sheet was then given to each instructor of the beginning calculus class and he or she used it according to its purpose.

Statistical Analysis Used to Evaluate the Effect Resulting from the Use of the Programmed Materials

The problem of measuring the effect on the final achievement resulting from the use of programmed materials which were written to supplement classroom instruction involving topics which usually cause learning problems in beginning calculus was carried out by first carefully stating the hypothesis to be tested. Then appropriate statistical methods were used to test the hypothesis by comparison of final examination scores at the end of the quarter.

Statement of the hypothesis. As a prelude to actual statistical comparison of the achievement by the experimental and control groups the null hypothesis, H_0 , was stated. This hypothesis was: there is no significant difference in the final achievement of students in a beginning calculus course who have studied from programmed materials written to supplement conventional classroom instruction of topics which

ordinarily cause learning problems over students who have not studied such programmed materials. The alternative hypothesis, H_1 , was: there is a significant difference in the final achievement of students in a beginning calculus course who have studied from programmed materials written to supplement conventional classroom instruction of topics which ordinarily cause learning problems over students who have not studied such programmed materials. To test these hypotheses appropriate statistical methods were selected and applied.

Selection of statistical methods. In order to test the null hypothesis, H_0 , the question of comparability of the experimental and control groups had to be first answered. Hence, in the final analysis two major questions presented themselves: (1) were the experimental and control groups comparable as far as mathematical and intellectual backgrounds concerned? and (2) was the achievement in beginning calculus of the experimental group superior to that of the control group in view of the possible differences in mathematical and intellectual backgrounds?

In order to answer the first question the backgrounds of the students in both the experimental and control groups had to be compared. In searching for such a comparison it was found that Montana State College had devised a criterion for placement of students in the beginning calculus sequence according to their mathematical and intellectual backgrounds. This criterion was a predicted grade in beginning calculus and was a composite score based upon the following

sources: the score of the American Council of Education Psychological Examination, the score of the Minnesota Scholastic Aptitude Test (a short form of the Ohio Psychological Examination), the score on the Mechanical Comprehension Test - Form CC, a set of scores on examinations in arithmetic, algebra I, and algebra II (examinations prepared by the Mathematics Department at Montana State College), and the high school grade point average.⁵³ The fact that this scheme of predicting grades in mathematics had been used at Montana State College for over 30 years with some modification but without major revision attested to its accepted value.⁵⁴

Hence by using these predicted grades in the beginning calculus sequence as calculated by the formula adopted by Montana State College, it was possible to assign a numerical score to each member of the experimental and control group. For the purposes of this investigation this score was considered to be a comprehensive measure of the mathematical and intellectual backgrounds of the students. Thus only a statistical means was needed whereby the mathematical and intellectual backgrounds as indicated by the predicted scores could be compared.

Both of the major problems to be investigated were kept in mind in the selection of this statistical test. Namely: (1) were the

⁵³Suvak, Albert, (unpublished research at the Montana State College Testing and Counseling Service).

⁵⁴Brookhart, M. E., (unpublished statement by M. E. Brookhart, Director of the Montana State College Testing and Counseling Service).

experimental and control groups comparable as far as mathematical and intellectual backgrounds concerned? and (2) was the final achievement of the experimental group above that of the control group? Because both the question of background and the question of final achievement involved similar comparisons between the same two populations, a common statistical treatment of both questions was judged appropriate.

A decision was made to use the parametric t-test on both questions rather than non-parametric methods. This decision was based on the fact that the t-test is stronger than the alternative non-parametric methods in that the t-test is more likely to reject H_0 when H_0 is false.⁵⁵ Furthermore, the conditions required of the populations to be compared met the standards necessary for application of the t-test. These standards consisted mainly of being able to assume that the variances of the two populations being compared were equal.⁵⁶ Other standards of secondary importance such as being able to assume that the sampling was independent⁵⁷ presented no problem and were systematically accepted. Preliminary statistical investigations were necessary, however, in order to assume equality of variances of the two populations both in the case of comparing mathematical and intellectual backgrounds as well as comparing final achievement.

⁵⁵Siegel, Sidney, Nonparametric Statistics for the Behavioral Sciences, p. 19.

⁵⁶Ibid., p. 19.

⁵⁷Ibid., p. 19.

Such statistical investigation did show that one could not reject the hypothesis that the variances of the experimental and control populations in the case of comparing mathematical and intellectual backgrounds differed significantly.⁵⁸ Similarly, the same statistical procedure revealed that one could not reject the hypothesis that the variances of the experimental and control populations in the case of comparing final achievements differed significantly.⁵⁹

A condition found listed as being essential to application of the t-test was that the populations being compared must both be normally distributed.⁶⁰ However, statistician Cochran in investigating this problem stated:

The consensus from these investigations is that no serious error is introduced by non-normality in the significance levels of the F-test or the two-tailed t-test If a guess may be made about the limits of error, the true probability corresponding to the tabular 5 percent significance level may lie between 4 and 7 percent.⁶¹

⁵⁸See p. 63

⁵⁹See p. 65

⁶⁰Siegel, op. cit., p. 19.

⁶¹Cochran, W. G., "Some Consequences When the Assumptions for the Analysis of Variance Are Not Satisfied," Biometrics Bulletin, Vol. III, No. 1, December, 1946, p. 24.

Although the case is not quite so solid for the one-tailed t-test, the permissiveness of non-normality is much the same. Kendall in summarizing the results of other statisticians who had studied this problem stated that:

Their results may be broadly summarized by the statement that. . . tests on population means (i. e. "Students" t-tests for the mean of a normal population and for the difference between the means of two normal populations with the same variance) are rather insensitive to departures from normality. . . .⁶²

Both arguments that non-normality is not essential for one or two-tailed t-testing stem from the fact that the t-test is based on calculations involving sample means and one would be concerned not so much with the normality of the population, but instead with the normality of the distribution of these sample means. But by the Central Limit Theorem, these sample means are themselves approximately normally distributed.⁶³

Hence, the question of normality was not considered of major importance. Furthermore, the data tended to cause the investigator to believe the two populations to be normally distributed. Therefore, the t-test was judged appropriate for investigating the two questions: (1) were the experimental and control groups of comparable mathematical and intellectual backgrounds? and (2) was the final achievement of the experimental group significantly above from that of the control group?

⁶² Kendall, Maurice, The Advanced Theory of Statistics, Vol. II, p. 465.

⁶³ Freund, John E., Modern Elementary Statistics, p. 205.

Application of statistical methods. In order to best test the question of comparability of the experimental and control groups a null hypothesis, H_0 , was formulated: there is no significant difference in mathematical and intellectual background of the experimental group as compared to the control group. The significance level was set at 5 percent and the test, because of the nature of the null hypothesis, was a two-tailed test.

The null hypothesis was tested by application of the t-test which involved calculation of the t statistic using the predicted scores in beginning calculus as measures of the mathematical and intellectual backgrounds of members of the experimental and control groups. Since the number of students in the experimental group was 21 and the number of students of the control group was 22, the t-distribution table was consulted at the 5 percent level with $t(.025, 41) = 2.0195$.

For the calculated t less than 2.0195 or greater than -2.0195 the null hypothesis that there was no significant difference in mathematical and intellectual background of the experimental as compared to the control group could not be rejected.

Next, the second major question was investigated. Namely, a statistical study was made to investigate the relative final achievement of the experimental group as compared to the control group.

Upon receiving the scores of the students of the experimental and control groups earned on the final examination at the end of the quarter the statistical problem of comparison of the final achievement

of the experimental and control group was investigated. In recalling the null hypothesis, H_0 : there is no significant difference in the final achievement of students in a beginning calculus course who have studied from programmed materials written to supplement conventional classroom instruction of topics which ordinarily cause learning problems over students who have not studied from such programmed materials, it was clear that there too the Student t-test could be used. However, this time the t-test was one-tailed.

Again the level of significance was set at 5 percent and $t(.05, 41) = 1.6829$ was the critical value. That is, for the calculated t less than 1.6829 the null hypothesis could not be rejected that there is no significant difference in the final achievement of students in a beginning calculus course who have studied from programmed materials written to supplement conventional classroom instruction of topics which ordinarily cause learning problems over students who have not studied from such programmed materials.

Summary

In order to investigate the effect on final achievement in a beginning calculus course resulting from the use of a series of programmed instruction written to supplement the regular classroom instruction on the topics which ordinarily cause learning problems a poll was made of well-qualified mathematicians and scientists familiar with the content and learning problems of a beginning calculus course.

This survey was in turn tabulated to identify the areas of beginning calculus which cause learning problems to students.

The identification of areas causing learning problems led to the preparation of a series of lessons of programmed instruction written to supplement the regular classroom instruction of the topics identified as causing learning problems. These lessons were made available to students enrolled in the beginning calculus class at Montana State College during the fall quarter of 1964.

The students were instructed to note passages of the programmed materials which they could not satisfactorily complete. Using a permissible error ratio of 10 percent the passages causing trouble to student participants were identified.

Prior to the winter quarter of 1965 decisions were made as to appropriate modifications of the preliminary writing. These modifications included: (1) rewriting passages and supplying necessary sketches to reduce the error ratio on troublesome passages below 10 percent, (2) alterations of the lessons by lengthening the time available for the introductory lesson by dropping a lesson which came too late in the quarter to be of benefit, and by condensing the content of two chapters into one, and (3) revising the administration of the materials such that a close control could be assured over the use of the materials by the experimental group.

At the beginning of winter quarter an experimental group was chosen from students volunteering for the investigation. The criteria for selection of the experimental group involved selection of students

of demonstrated moderate achievement. Extreme levels of achievement were not allowed because of the possible skewing of final achievement results.

During the quarter as the students of the experimental group met twice weekly to complete the prepared programmed materials, a constant contact was maintained with instructors of the students and others concerned to assure uniform classroom instruction and notational usage.

At the end of the quarter a search was made of the records of all students completing the course to determine which students were qualified as members of a control group under the same criteria imposed on the admission into the experimental group. All such qualifying students were selected and the control group was thereby formed.

Also, at the end of the quarter a final examination meeting the approval of all instructors of the beginning calculus sequence was written and administered to all members of both the control and experimental group. This examination was scored in a uniform manner according to grading standards previously set down by a committee appointed by the supervisor of the beginning calculus sequence.

As a means of comparing the mathematical and intellectual backgrounds of students in the control group to those in the experimental group, the placement figures obtained by the Testing and Counseling Service at Montana State College were used. Since Montana State College had adopted a predictor of success in beginning calculus based

on previous mathematical and intellectual background, this predictor was used as a means of comparing the backgrounds of the experimental and control group. The statistical test used to compare the two groups was the Student t-test. This test was chosen due to its applicability to the data and its strength.

Similarly at the end of the quarter the null hypothesis, H_0 : there is no significant difference in the final achievement of students in a beginning calculus course who have studied from programmed materials written to supplement conventional classroom instruction of topics which ordinarily cause learning problems over students not studying from such programmed materials, was tested by the Student t-test. The raw data for this test was supplied by the scores of the final examination given to all members of the experimental and control group.

The results of the investigation are presented in the next chapter.

CHAPTER IV

EFFECT ON FINAL ACHIEVEMENT IN BEGINNING CALCULUS RESULTING FROM THE USE OF PROGRAMMED MATERIALS

In order to present the results of the investigation of the effect on final achievement by students in a beginning calculus course resulting from use of programmed materials administered as a supplement to classroom instruction of topics which ordinarily cause learning problems three areas were considered. These areas were: (1) selection of the experimental and control groups, (2) comparability of the experimental and control groups, and (3) the effect on final achievement resulting from the use of the programmed materials.

Selection of the Experimental and Control Groups

The experimental group was selected from student volunteers from the beginning calculus class at Montana State College during the winter quarter of 1965.⁶⁴ Fifty-five students made application for admittance into the experimental group. Forty-seven of these applicants were selected under the adopted criteria⁶⁵ as participants in the programmed instruction investigation. Participation in the program was strictly voluntary. Twenty-one of these forty-seven students remained with the program throughout the quarter and systematically worked all of the programmed materials. Statistical

⁶⁴See p. 43

⁶⁵See p. 42 and 43

measurements involving the effect of programmed materials were based on the final achievements of the twenty-one students who completed the entire sequence of programmed lessons.

The entire roll of students who completed the beginning calculus course was surveyed at the end of the quarter to identify those students eligible for the control group. Criteria for selection of the control group⁶⁶ reduced the potential membership to twenty-two students. Since all twenty-two of these students qualified as members of the control group, all were selected as control group members.

Thus the comparison of the final achievement of the experimental and control groups involved forty-three students. The twenty-one students in the experimental group had completed the beginning calculus course and in addition had completed all of the programmed materials. The twenty-two students in the control group had studied the same calculus course but their classroom instruction was not supplemented with programmed materials.

It was recognized by the author that the voluntary nature of the experimental group might introduce two biases which could effect the results of the investigation. First, it was foreseen that student volunteers may have applied for acceptance into the experimental group with the intent of learning calculus while not exerting a normal effort. Second, it was foreseen that the student volunteers who applied for acceptance into the experimental group might be comparatively

⁶⁶ See p. 45

more enthusiastic students with more than the average desire to learn the subject matter.

Both of these biases were beyond the control of the experimenter. However, it was hoped that the number and duration of intensive study sessions would reduce the number of students that anticipated learning calculus with little or no effort. Since twenty-six students dropped the program, there was reason to believe that those students seeking learning without effort were substantially eliminated from the experimental group.

If the second bias were present, it was not evident in the results of the investigation.

Comparability of the Experimental and Control Groups

In order to study the effect resulting from the use of programmed materials on final achievement of the experimental and control groups it was first necessary to determine the comparability of the two groups. This comparability was based on a study of composite scores which indicated the mathematical and intellectual backgrounds of the students. Nearly all of these scores had been previously calculated by the Testing and Counseling Service at Montana State College and appeared in the Montana State College Placement Test Scores and Indices, September, 1964.⁶⁷ The student scores needed for the study which

⁶⁷ Brookhart, M. E. and Suvak, Albert, Montana State College Placement Test Scores and Indices, September, 1964, 52 pp.

did not appear in the Montana State College test index booklet were calculated using the same formula as was regularly used to calculate these scores.⁶⁸

The appropriateness of the t-test was investigated as a means of comparing these scores⁶⁹ which were measures of the mathematical and intellectual backgrounds of students in the experimental and control groups. Establishment of the appropriateness of the t-test was dependent upon permissiveness of assuming equality of variances of the two sample populations.⁷⁰ Hence, the question of assuming equality of variances was investigated by testing the hypothesis: the variances of the scores measuring the mathematical and intellectual backgrounds of the experimental and control populations were the same at the 5 percent level.

⁶⁸Predicted Mathematics 121 score = .00416 (total score on A. C. E.) - .013151 (Minn. Schol. Apt. Test) +.00012 (M. S. C. Arithmetic Test) +.02755 (M. S. C. Algebra I Test) +.02061 (M. S. C. Algebra II Test) +.01250 (Mech. Comp. Test) +.06899 (High School G. P. A.) -1.16826. (This formula was developed by Albert Suvak of the Montana State College Testing and Counseling Service to predict achievement of students enrolling in Mathematics 121.)

⁶⁹See Appendix G , Table 1.

⁷⁰Siegel, op. cit., p. 19

The test applied to this hypothesis was the two-tailed F-test.⁷¹

The calculated F statistic was .65516. Since this value was between $F(.975, 20, 21) = .40816$ and $F(.025, 20, 21) = 2.42$, it was not possible to reject the hypothesis that the variances were equal at the 5 percent level. Thus it was concluded that the variances of scores measuring the mathematical and intellectual backgrounds of the experimental and control populations do not differ significantly.

The conclusion that the variances of the two populations did not differ significantly enabled the investigator to contend that the t-test was appropriate for testing the hypothesis: there was no significant difference in the mathematical and intellectual backgrounds of the experimental and control groups. This test was a two-tailed test and was accomplished by comparing the means of the sample popu-

⁷¹The steps used in computing the two-tailed F statistic at the 5 percent level are as follows:

1. Determine the number of cases in groups I and II being compared. Denote them n_1 and n_2 respectively.

2. Denote the observations in groups I and II by y_{1i} and y_{2j} respectively where $i = 1, 2, \dots, n_1$ and $j = 1, 2, \dots, n_2$.

3. Calculate: $\bar{y}_1 = \sum_{i=1}^{n_1} y_{1i}/n_1$ and $\bar{y}_2 = \sum_{j=1}^{n_2} y_{2j}/n_2$

4. Calculate: $F = \frac{\sum_{i=1}^{n_1} (y_{1i} - \bar{y}_1)^2 / (n_1 - 1)}{\sum_{j=1}^{n_2} (y_{2j} - \bar{y}_2)^2 / (n_2 - 1)}$

5. The F statistic of (4) is distributed as an F with $v_1 = n_1 - 1$ and $v_2 = n_2 - 1$ degrees of freedom.

6. Accept the null hypothesis if the calculated F in (4), F_o , is such that: $F(.975, v_1, v_2) < F_o < F(.025, v_1, v_2)$, otherwise reject the null hypothesis. (Ostle, Bernard, Statistics in Research, pp. 91-93.)

lations according to the procedure of calculating the t statistic.⁷²

It was found that the t statistic calculated from the data listed in Table A was -.69982. Since this value was smaller than $t(.025, 41) = +2.0195$ and larger than $-t(.025, 41) = -2.0195$, the hypothesis was accepted. That is, the investigator expected that 95 percent of all samples compared (the samples each consisting of an experimental and a control group) would have no significant difference in their mathematical and intellectual backgrounds.

The Effect on Final Achievement Resulting From the Use of Programmed Materials

After establishing the comparability of the experimental and

⁷²The steps used in computing the two-tailed t statistic at the 5 percent level were as follows:

1. Determine the number of cases in groups I and II being compared. Denote them n_1 and n_2 respectively.
2. Denote the observations in groups I and II by y_{1i} and y_{2j} respectively where $i = 1, 2, \dots, n_1$ and $j = 1, 2, \dots, n_2$.

3. Calculate:
$$\bar{Y}_1 = \sum_{i=1}^{n_1} y_{1i}/n_1 \text{ and } \bar{Y}_2 = \sum_{j=1}^{n_2} y_{2j}/n_2.$$

4. Calculate:
$$\sqrt{\frac{\frac{n_1}{n_1+n_2} (\bar{Y}_1 - \bar{Y}_2)^2}{\frac{\sum_{i=1}^{n_1} (y_{1i} - \bar{Y}_1)^2 + \sum_{j=1}^{n_2} (y_{2j} - \bar{Y}_2)^2}{n_1 + n_2 - 2}}}$$

5. The t statistic of (4) is distributed as a t with $n_1 + n_2 - 2$ degrees of freedom.

6. If the calculated t statistic, t_o , is such that: $-t(.025, n_1 + n_2 - 2) < t_o < t(.025, n_1 + n_2 - 2)$, accept the null hypothesis. Otherwise reject the null hypothesis. (Mood, A.M. and Graybill, F.A., Introduction to the Theory of Statistics, pp. 306-307.)

control group by statistical means it was then possible to begin investigation of the main hypothesis: there is no significant difference in the achievement of students in a beginning calculus course who have studied programmed learning as a supplement to conventional classroom instruction of topics which ordinarily cause learning problems over students who have not studied such programmed materials.

Again it was desirable to use the parametric t-test in comparing the final achievements of the control and experimental groups. Again the appropriateness of the test had to be justified by demonstrating reason for assuming equality of variance for the final achievements of the control and experimental populations. This assumption was justified as a result of studying the achieved scores of the two sample populations.⁷³ That is, in order to show reason for assuming equality of variances for the two populations a test was made of the hypothesis: there is no significant difference between the variances of the final achievement scores in beginning calculus for the experimental population and the control population. The test was a two-tailed F-test and resulted in the F statistic being calculated as 1.267. Since this value was between $F(.975, 20, 21) = .40816$ and $F(.025, 20, 21) = 2.42$, it was not possible to reject the hypothesis that the variances were equal at the 5 percent level. Thus it was assumed that the variances of the final achievement scores of the experimental and control populations did not differ significantly.

⁷³See Appendix G, Table 2.

By assuming that the variances of the two populations were not significantly different the investigator was then able to contend that the t-test was appropriate for testing the main hypothesis: there is no significant difference in the final achievement of students in a beginning calculus course who have studied programmed materials written to supplement conventional classroom instruction of topics which ordinarily cause learning problems over students who have not studied such programmed materials.

The t-test was one-tailed because of the nature of the hypothesis. On this basis the t statistic was calculated⁷⁴ from the data of Table B and was found to be $t = .60602$. Since this value was less than $t(.05, 41) = 1.6829$, the hypothesis was accepted. That is, the investigator expected that in 95 percent of all samples studied (the samples each consisted of one experimental and one control group) the experimental group would not have a final achievement mean greater than the mean of the final achievement of the control group.

Summary

Experimental and control groups were selected from the population of students in a beginning calculus course to compare the effect on final achievement in beginning calculus due to the use of programmed

⁷⁴The calculation of the one-tailed t statistic followed the procedure of calculating the two-tailed t statistic. (Ostle, op. cit., pp. 98-102.)

materials. Twenty-one of forty-seven students in the experimental group completed the programmed materials. The final achievement of these students was compared to the final achievement of the control group, consisting of twenty-two students, in order to provide reason for stating conclusions about the experimental and control populations.

In order to compare the final achievements of the experimental and control groups the comparability of the two groups had to be first established. This was done by application of the two-tailed t-test to the predicted achievement scores of the individuals in the control and experimental groups. These scores were calculated by using the formula developed and approved by the Montana State College Testing and Counseling Service. The appropriateness of the test was previously established by demonstrating permissiveness of assuming equality of variances for the two populations by F-test.

After establishing the comparability of the two groups the major hypothesis was considered: there is no significant difference in the achievement of students in a beginning calculus course who have studied programmed learning as a supplement to conventional classroom instruction of topics which ordinarily cause learning problems over students not studying such programmed materials.

This time the one-tailed t-test was needed for testing the hypothesis and hence was shown appropriate by establishing permissiveness of assuming equality of variance for the two groups by F-test. Since the one-tailed t-test was appropriate, it was applied and subsequently demonstrated the acceptability of the hypothesis.

CHAPTER V

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

The purpose of this investigation was to compare the results obtained from the use of programmed materials as a supplement to classroom instruction of topics in a beginning calculus course which ordinarily cause learning problems with results obtained in classes not using programmed materials.

The need for such a study was in part due to the changes in mathematical emphasis which have accompanied the transition into a modern scientific-technological era. These same changes in mathematical emphasis have seriously effected the need for more effective means of increasing mathematical learning-teaching effectiveness. As one means of increasing the effectiveness, this investigation concentrated on the problems associated with teaching beginning calculus at the early college level. It was believed by the author that appropriate programmed materials could be used effectively by students in a beginning calculus course to supplement their regular classroom instruction and hence improve their final achievement. The purpose of this study was to investigate this belief.

Summary

Four major points became clear to the author from the review of the literature which was carried out as a preliminary step in this investigation.

1. Beginning calculus has experienced a radical change in the past eight years from a development depending heavily on solving particular problems to a careful structural development achieved through student understanding and logical reasoning.

2. Many of the learning problems in beginning calculus have been brought about by a recent change from traditional mathematics to modern mathematics. This change has been accompanied by revolutionary teaching methods and emphasis. Thus many students whose preparation has been traditionally oriented have been unsuccessful in making a transition to the type of thinking and rigor expected in modern mathematics.

3. The use of programmed instruction possibly can facilitate correction of the imbalance in background experienced by traditionally prepared students through development of proper notation, but evidence indicated considerable disagreement as to whether programmed instruction could entirely fill the gap in developing proper mathematical attitude such as is required in mathematical reasoning.

4. The reports of the investigations into the learning value of programmed instruction have not consistently described it as being infallible. Instead, a number of comparisons of its value over conventional classroom instruction indicated that the differences in final learning achieved through programmed instruction were not significantly greater than those differences achieved by conventional classroom teaching.

In the course of reviewing the literature no reports of investigations were found which described efforts to study the effect on final

achievement of students in a beginning calculus course which had supplemented their conventional classroom instruction with programmed instruction. Hence, the author proceeded to investigate this problem. The first step in the investigation was the identification of those areas in beginning calculus which ordinarily cause learning problems. This was done by polling twelve college instructors familiar with the learning problems of students in calculus. Then a schedule of lesson involving supplementary programmed materials was drawn up to cover the topics which were identified as being troublesome.

The author consulted various standard calculus texts in order to assure a complete and effective presentation of each topic. Then he developed each topic by carefully writing a preliminary set of programmed materials. The preliminary materials were then tested for effectiveness by conducting a pilot study. This study pointed out specific needs for improved writing of certain passages, improved administrative techniques, and improvements regarding arrangements of topics to be covered. With these factors in mind the final writing of the materials was accomplished.

The author selected an experimental group during the winter quarter of 1965 at Montana State College on the basis of their previous mathematical achievement. A schedule of meetings was drawn up, and the experimental group began using the programmed materials as a supplement to regular classroom instruction. At the conclusion of the quarter a control group was chosen using criteria similar to those

used to select the experimental group. Then a final achievement examination was administered to both the control and experimental groups. Appropriate statistical procedures were applied to the test scores.

The statistical comparisons indicated that although the experimental and control populations were of comparable mathematical and intellectual backgrounds, the final achievements of students in a beginning calculus course studying programmed materials administered as a supplement to a conventional classroom instruction of topics which ordinarily cause learning problems were not significantly above the final achievements of students not studying such materials.

Conclusions

The following conclusions regarding the teaching of a beginning calculus course were evident to the investigator as a result of reviewing the literature:

1. Because scientific-technological advancements have placed a greater emphasis on the mathematical preparation of students and this emphasis has in turn influenced the manner of teaching mathematics, students in the beginning calculus courses are having trouble learning the concepts of mathematics. Hence, a need exists to improve the teaching of the basic concepts of beginning calculus.

2. The research done on the effectiveness of supplementing classroom teaching of mathematics by the use of programmed materials is inconclusive and additional research needs to be done.

The author concluded as a result of his investigation that statistically there is very little reason to believe that the effect on the final achievements of students in a beginning calculus course resulting from the study of programmed materials administered as a supplement to conventional classroom instruction of topics which ordinarily cause learning problems would be significantly above the corresponding achievements by students not studying such materials.

Recommendations

As a result of this investigation four recommendations were formulated:

1. Because a greater number of students fail to satisfactorily learn the material in a beginning calculus course, more research is needed to identify the factors contributing to this lack of learning. In particular, research needs to be done in this area with full understanding of the changing mathematical background of the students. That is, the background of the entering college students is currently changing from being oriented in traditional mathematics to an orientation involving modern mathematics.

2. Research needs to be done to find ways of improving the teaching of beginning calculus. The average beginning calculus student was found to be interested in learning the material but in many cases the learning by these same students was found to be incomplete. Thus, research into the improvement of teaching calculus and hence into the improvement of mathematical communication at the level of beginning calculus was found desirable.

3. First, evidence was found in the literature to indicate that students can learn some things by studying appropriate programmed materials. Research needs to be carried out which will identify the areas of beginning calculus which can be effectively learned by studying programmed materials.

Second, once the areas of beginning calculus which can effectively be taught through the use of programmed materials have been identified, research needs to be done to discover the most efficient way to use programmed materials so that maximum learning of beginning calculus will result.

4. The background of students in the beginning calculus course at the time of this study was in general traditionally oriented. With the current emphasis on a modern structural approach to mathematics at all levels of education, learning problems of beginning calculus will likely change as students' background reflect the modern emphasis involving notation, the structure of mathematics, and the patterns of thought essential in successful reasoning in mathematics. Hence, the investigator makes the recommendation that this investigation be

replicated at a future date when experimental and control groups can be chosen from a population of students whose background is molded by modern mathematical training.

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APPENDIX

Appendix A

Instructions to the Panel of Judges which
Selected the Topics to be Programmed

Instructions to the Panel of Judges Which
Selected the Topics to be Programmed

Certainly there are topics which students enrolled in the beginning calculus sequence are expected to master. Obviously an inappropriate amount of difficulty is experienced by many students in attaining these goals. This situation has perpetuated itself despite attempts to match student ability with course content and to improve classroom teaching. However, despite attempted solutions the problem of too many unsuccessful students still exists. Hence it is clear that further investigation into ways to improve student learning and thus success are in order. I have chosen such an investigation as a problem for my doctoral thesis and request your opinion on one important point.

You see, as a means of aiding student learning, I intend to prepare a set of programmed materials which will be focused on the critical areas of beginning calculus which display the greatest learning problems. It is intended that such supplementary materials will do much to clear up the major concepts which stand before understanding in these vital areas.

Since student time is limited, so too must the selection of topics prepared be limited. Hence, the following list of topics studied in the beginning calculus sequence is presented for your inspection. In order that the most vital areas can be concentrated on, it is requested that you select the seventeen areas which you feel deserve the greatest emphasis. In addition to you, other individuals well qualified to judge

the learning problems of students in calculus will be similarly polled. Then those topics which indicate the heaviest need for concentration will be selected as the areas to be developed.

Please check the seventeen topics which you feel are most difficult for students to master:

1. The locus problem.

What is the graph of an equation and what is the equation of a graph; slope; equations of straight lines, parallel lines, perpendicular lines; distance formula and circle; parabola; ellipse; hyperbola; related problems.

2. Functions and functional notation.

Cartesian cross product; relation; function; associated notation; related problems.

3. Intersections of pairs of curves.

Sets of points satisfying equations individually and simultaneously; introduction of extraneous solution points; related problems.

4. Inequalities and absolute value.

Definition of inequality symbols; basic theorems of inequalities; definition of absolute value; triangular inequality; related graphing; related problems.

5. Limits.

Algebraic interpretation; geometric interpretation; basic theorems of limits; related problems.

6. Continuous functions.

Definition of continuity at a point; definition of continuity over an interval; meaningful examples of continuous and discontinuous functions; related problems.

7. The derivative.

Average rates of change; instantaneous rates of change; slope of a curve at a point; average and exact velocity; definition of the derivative; mechanics of differentiation when proceeding from the definition; related problems.

8. Simple differentiation formulas.

$D_x c$; $D_x cx^n$; $D_x (u+v)$; $D_x uv$; $D_x u/v$; related problems.

9. Differentiation of the product, quotient, and higher order derivatives.
Development of product and quotient formulas; higher order derivatives; related problems.
10. Differentiation of composite functions.
Proof of chain rule; use of chain rule; related problems.
11. Implicit differentiation.
Discussion of implicit functions; related problems.
12. Application of the derivative to linear motion, motion under gravity, and related rates.
Linear motion; motion under gravity; related rate discussion; related problems.
13. Sketching as related to increasing and decreasing functions, critical values, and extreme values of a function.
Relation of the derivative to increasing and decreasing functions; use of the first derivative to isolate critical points; the use of the second derivative test; curve plotting using the derivatives; related problems.
14. Rolle's Theorem and Mean Value Theorem.
Statement and proof of Rolle's Theorem; statement and proof of Mean Value Theorem; application of Mean Value Theorem to prove other basic theorems; related problems.
15. Applications involving the theory of extremes.
Max - min problem types; problems.
16. Approximation by differentials.
Definition of differentials; geometric interpretation of differentials; approximate change and differentials; related problems.
17. The indefinite integral.
Meaning of indefinite integral; integration and differentiation formulas; constant of integration; related problems.
18. Computation of area by integration.
Concept of area; relation of the integral to area under a curve; computation of areas bounded by more complicated curves; related problems.
19. The definite integral.
Meaning of the definite integral; geometric interpretations; important properties of the definite integral; computation of the definite integral; related problems.

20. Area as the limit of a sum.
Proof that area is the limit of a sum; related problems.
21. Fundamental theorem of calculus.
Proof of the fundamental theorem; related problems.
22. Mathematical induction.
Discussion of nature of mathematical induction; examples of mathematical induction properly and improperly applied; related proofs by mathematical induction.

Appendix B

Programmed Materials Studied by
the Experimental Group

Lesson I

INEQUALITIES

1. Frequently you will find it necessary to use and understand the symbols: $>$, $<$, $\geq$, and $\leq$ in connection with the real numbers (the numbers which signify the distances to the right or left of 0 on the number line). The first symbol, $>$, is read: "is greater than". The second symbol, $<$, is read: "is less than". How would " $a > b$ " be read?

"a is greater than b".

2. How would " $a < b$ " be read?

"a is less than b".

3. The symbol $\geq$ is read: "is greater than or equal to". Correspondingly $\leq$ is read: "is less than or equal to". How would " $a \geq b$ " be read?

"a is greater than or equal to b".

4. How would " $a \leq b$ " be read?

"a is less than or equal to b".

5. Translate the following true statements into symbolic representation:
(1) 4 is greater than 1. (4) 6 is greater than or equal to 6.
(2) 4 is less than 5. (5) 5 is less than or equal to 7.
(3) 6 is greater than or equal to 5. (6) 5 is less than or equal to 5.

$$\begin{array}{l} 4 > 1 \\ 4 < 5 \\ 6 \geq 5 \end{array}$$

$$\begin{array}{l} 6 \geq 6 \\ 5 \leq 7 \\ 5 \leq 5 \end{array}$$

6. Translate the following symbolic representation into true statements:

(1) $7 > 4$	(4) $2 \leq 3$
(2) $6 < 8$	(5) $3 \geq 3$
(3) $3 \geq 2$	(6) $2 \leq 2$

 "7 is greater than 4".

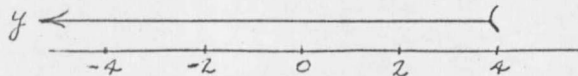
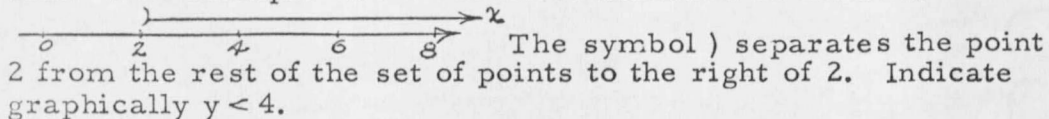
"2 is less than or equal to 3"

"6 is less than 8".

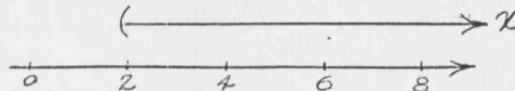
"3 is greater than or equal to 3".

"3 is greater than or equal to 2". "2 is less than or equal to 2".

7. Often such symbols are used in conjunction with variables and the value of the variable can be indicated graphically on the number line. For example " $x > 2$ " could be sketched as follows:



8. Similarly one could sketch $x \leq 6$ as: Here the symbol) indicates that 6 is included in the set of points. Indicate graphically $x \geq 2$.

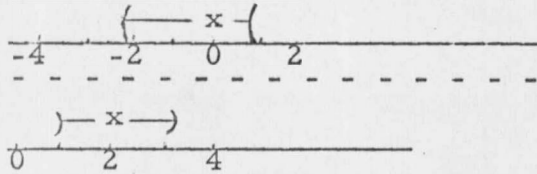


9. Sometimes in writing mathematical statements it saves time to group two statements involving inequality into one statement. For example, $6 < x$ and $x \leq 7$ could be written: $6 < x \leq 7$. Then reading from left to right: "6 is less than x is less than or equal to 7". Or to indicate that both conditions are fulfilled often this statement is read: "6 is less than x and x is less than or equal to 7", or simply: "x is greater than 6 but less than or equal to 7". What would " $3 \leq x < 5$ " mean?

x is greater than or equal to 3 but less than 5.

10. Now combining both symbols in one statement the sketch of $-2 \leq x < 1$ would appear:

Sketch: $1 < x \leq 3$.



11. What would you conclude about x if: $y \leq x \leq y$?

$x = y$

12. Could it be true that $x < y \leq x$? Why?

No. This would state that x is less than itself which is not possible for real numbers.

13. For purposes of mathematical proof it will be convenient to take as a definition that $x < y$ and $y > x$ are equivalent statements and that both mean that $y - x$ is positive or $y - x > 0$. What does $c < d$ mean?

$d - c > 0$.

14. From this simple definition and the basic properties of real numbers (the sum of two positive reals is again a positive real, the associative laws for real numbers, the commutative laws for real numbers, etc.) many facts can be proven. We shall prove that: "if $a > b$ and $c > d$, then $a + c > b + d$ ".

Why is the argument that $3 < 4$ and $5 < 6$ and surely $3 + 5 < 4 + 6$ not a valid proof of the stated theorem?

This argument is only good for one special case - not for every case as is desired.

15. In order to prove the theorem, start as follows: if $a > b$, then $a - b > 0$. Also, if $c > d$, then $c - d > 0$. Why is each statement true?

According to definition (13)

16. But from (15): $(a-b) + (c-d) > 0$. Why?

The sum of 2 positive real numbers is again a positive real number.

17. Hence from (16): $(a+c) - (b+d) > 0$. Why?

 $(a-b) + (c-d)$ is the same as $(a+c) - (b+d)$ by permissible algebraic operations on real numbers. (Actually this involves associative and commutative laws and the properties of $-$.)

18. But (17) contends that: $a+c > b+d$. Why?

The definition (13) contends that $y-x > 0$ means $y > x$.

19. Hence you have verified the theorem you set out to prove. A similar theorem is: "if $a > b$ and c is any real number, then $a+c > b+c$ ". The statements for the proof of this theorem follow. Supply the reasons.

Statements

Reasons

1. $a > b$ implies $a-b > 0$
2. $+c-c = 0$
3. $a-b+c-c > 0$
4. $(a+c)-(b+c) > 0$
5. $a+c > b+c$

- 1.
- 2.
- 3.
- 4.
- 5:

-
1. Definition of $>$
 2. Properties of real numbers
 3. Properties of real numbers
 4. Properties of real numbers
 5. Definition of $>$

20. Suppose you were asked to prove: "if $a > b$ and $c > 0$ where a , b , and c are real numbers, $ac > bc$ ", a good place to start would be with the given fact $a > b$. What can be concluded from: $a > b$?

 $a > b$ means $a-b > 0$.

21. How would you reason from the facts: $a-b > 0$ and $c > 0$, to get $c(a-b) > 0$?

 $c(a-b) > 0$ because two positive real numbers have a positive product

22. Why is it true that $c(a-b) = ac-bc$?

 Permissible algebraic operations on real numbers are used.
 (The reason really depends on the distributive and commutative laws for real numbers.)

23. Now using (21) and (22) one can say: $ac > bc$. Why?

 Definition 13 says that $y-x > 0$ means $y > x$.

24. An interesting twist to the theorem stated in (20) is achieved by setting c negative instead of positive. What would happen to the relationship between ac and bc if $a = 4$, $b = 3$, and $c = -1$?

 $ac < bc$

25. On the basis of step (24) complete the theorem: "if $a > b$ and $c < 0$,

 " "

 $ac < bc$

26. The proof of this theorem could be made to closely follow that of theorem (20) by one small step: if $c < 0$, then $-c > 0$ by the properties of real numbers. Using this innovation prove: if $a > b$ and $c < 0$, $ac < bc$.

- 1. $a > b$ implies $a-b > 0$
 2. $c < 0$ implies $-c > 0$
 3. $\therefore -c(a-b) > 0$
 4. $\therefore -ac + bc > 0$ or
 $bc - ac > 0$
 5. $\therefore bc > ac$ or $ac < bc$

1. Definition of $>$
 2. Properties of real numbers - negative of a negative is positive
 3. Product of two positives is positive
 4. Properties of real numbers - distributive and commutative laws for reals
 5. Definition of $>$

27. Now you should prove one alone to be sure you have the technique down solid. Prove: if $a < b$ and $b < c$, then $a < c$.

1. $a < b$ means $b - a > 0$
 $b < c$ means $c - b > 0$
2. $\therefore (b - a) + (c - b) > 0$
3. $\therefore c - a > 0$
4. $\therefore c > a$ or
 $a < c$

1. Definition of $<$
2. Sum of 2 positives is positive
3. Simplification of step 2 -
 actually involves the idea of an
 additive inverse
4. Definition of $>$

28. From the three theorems:

- (a) if $a > b$ and c are real numbers, $a + c > b + c$;
- (b) if $a > b$ and $c > 0$, $ac > bc$;
- (c) if $a > b$ and $c < 0$, $ac < bc$;

many inequalities of a single variable can be solved. Use theorem (a) to solve: $x - 3 > 4$.

 $x > 7$

29. Solve by using theorem (b): $3x < 5$.

 $x < 5/3$

30. Solve by using theorem (c): $-3x < 5$.

 $x > -5/3$

31. Solve: $-2x + 7 \geq 4x - 5$.

 $x \leq 2$

32. Of course there are many proofs involving inequalities which await investigation by the curious student. Let's look at another general type.

Suppose you were asked to prove: "if a and b are positive numbers, $a^2 + b^2 \geq -2ab$ ". You might start by writing $(a+b)^2 \geq 0$. Why would this be true?

 The square of any real number is positive or zero.

33. Of course this is equivalent to stating $a^2 + 2ab + b^2 \geq 0$. But from this statement one can easily conclude $a^2 + b^2 \geq -2ab$. Why?

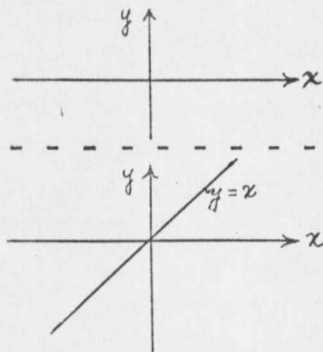
 $x - y \geq 0$ implies $x \geq y$.

34. Now you try one. Prove: $a^2 + b^2 \geq 2ab$.

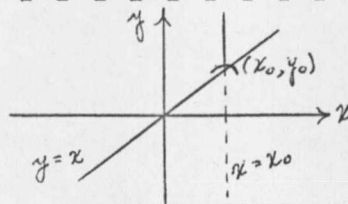
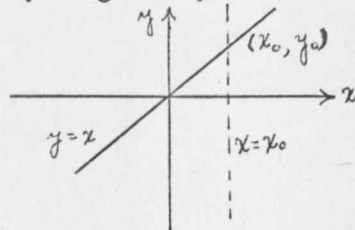
1. $(a-b)^2 \geq 0$
 2. $\therefore a^2 + b^2 \geq 2ab$

1. Square of any real is positive
 2. $x-y \geq 0$ means $x \geq y$.

35. By now you should have the general notion as to how inequality proofs proceed. So let's turn our attention now to a two dimensional geometric interpretation of inequalities. For example consider $y > x$. One might wonder if such a condition could be geometrically represented - and the answer is yes. First, sketch $y = x$.



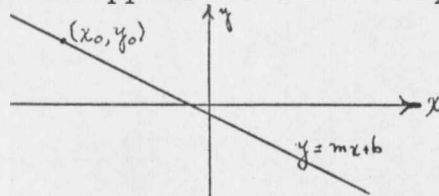
36. Now look at the sketch. Every point on that line is located such that the ordinate is exactly the same as the abscissa. On the following sketch consider in particular the point indicated on $y = x$ at $x = x_0$. On the line $x = x_0$ shade in those points where the ordinates are greater than x_0 . (i. e. Shade in those points where $y > x_0$ and y is on the line $x = x_0$.)



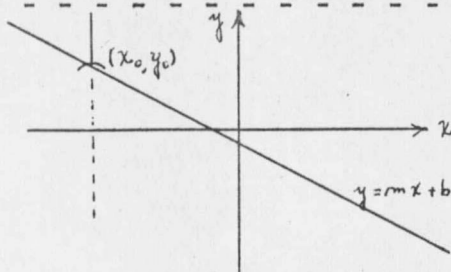
37. Of course there is nothing special about the particular position of (x_0, y_0) on the line - it could be anywhere on the line. But what is important is the relative position of the points where $y > x$! Where would all such points be?

Above the line $y = x$.

38. Let's get a little more variety in our analysis. Consider any straight line on the coordinate system - except the vertical lines. It is a well known fact from high school mathematics that every such line can be represented with an equation of the form $y = mx + b$. Our goal is to discover those points where $y > mx + b$. To do so suppose the sketch of $y = mx + b$ is as follows:



Again consider the arbitrary point (x_0, y_0) on the line and sketch in those points where $y > y_0 = mx_0 + b$. (Again consider y to be on the line $x = x_0$.)



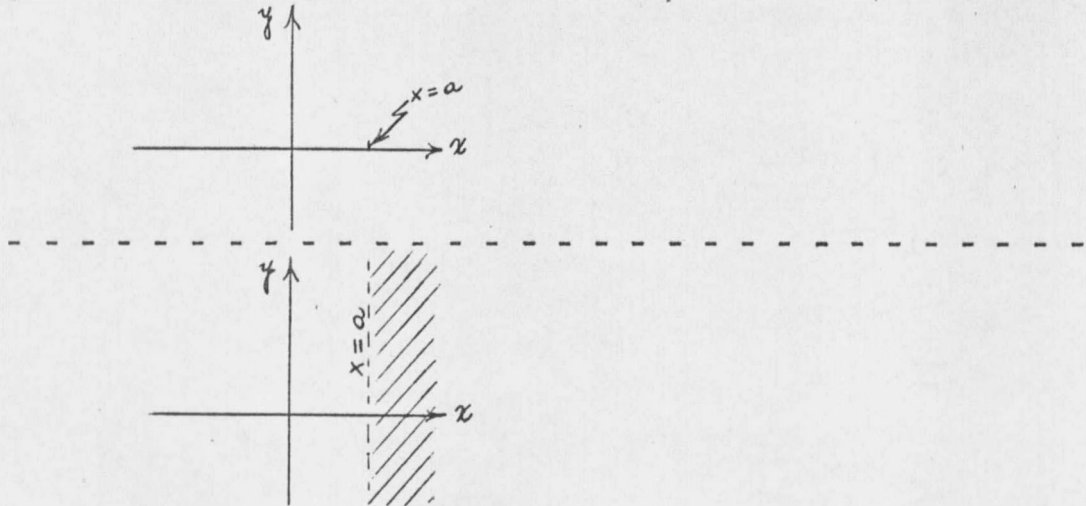
39. Now go one step further and answer the general question: where are the points which cause the inequality $y > mx + b$ to be true?

Above the line $y = mx + b$.

40. Correspondingly you should be able to answer the question: where are the points where $y < mx + b$? Do so.

Below the line $y = mx + b$.

41. Although we have neglected the case of the vertical line in our consideration, it comes as a bonus to our one-dimensional work. That is, whereas in one dimension when we wanted all points such that $x > a$, we considered the points to the right of the real number line. Now we want a two dimensional consideration of the phenomenon $x > a$. Shade the area where $x > a$.



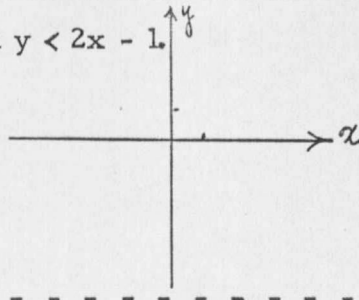
42. Also, you should be able to conclude the position of the points where $x \leq a$. Where would they be?

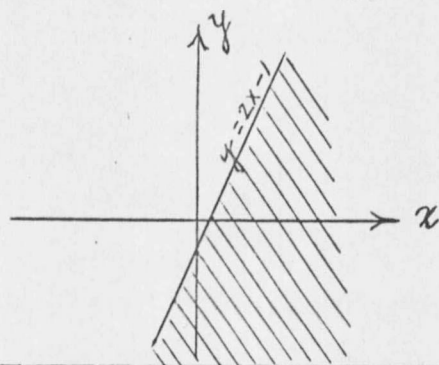
On the line $x = a$ and to the left of the line $x = a$.

43. You see! Any straight line $y = mx + b$ separates the plane into two halves where the points on one side satisfy the inequality $y > mx + b$. What inequality do the points on the other side satisfy?

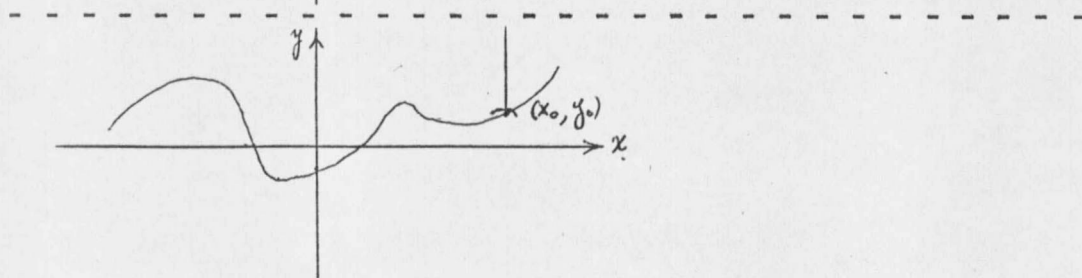
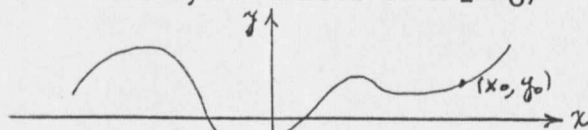
$y < mx + b$.

44. Sketch $y < 2x - 1$.





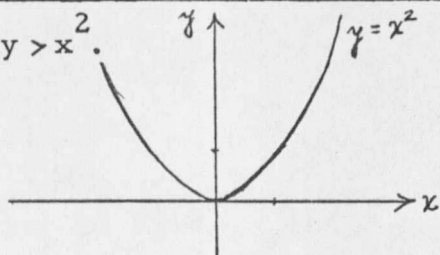
45. But what are we to do with other inequalities like $y > x^2$? To simplify our work let's just consider simple polynomial curves:
 $y = ax^n + bx^{n-1} + cx^{n-2} + \dots + k$. Suppose we had sketched
 $y = ax^n + bx^{n-1} + \dots + k$. At the point (x_0, y_0) on the curve where would $y > ax_0^n + bx_0^{n-1} + \dots + k$ be? Sketch it in.
 (Consider only ordinates on $x = x_0$)

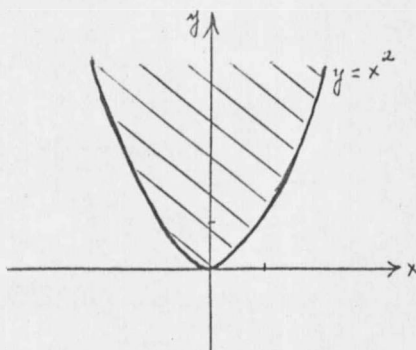


46. Now be a bit more general and specify where all points $y > ax^n + bx^{n-1} + \dots + k$ would be.

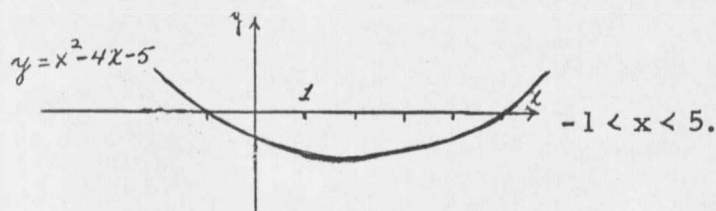
 Above the curve $y = ax^n + bx^{n-1} + \dots + k$.

47. Sketch: $y > x^2$.

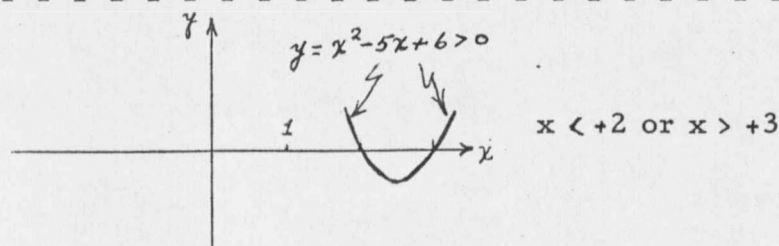




48. Now let's tie this sketching idea to specific problems involving inequalities. Suppose you are asked to solve: $x^2 - 4x - 5 < 0$. This problem could be easily transformed into an analysis of $y < x^2 - 4x - 5$. Simply set $y = x^2 - 4x - 5$, sketch it, determine what values of x force $y < x^2 - 4x - 5$, and in particular find what values of x force $y < 0$. Do so.

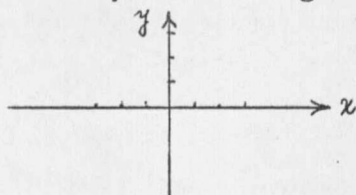


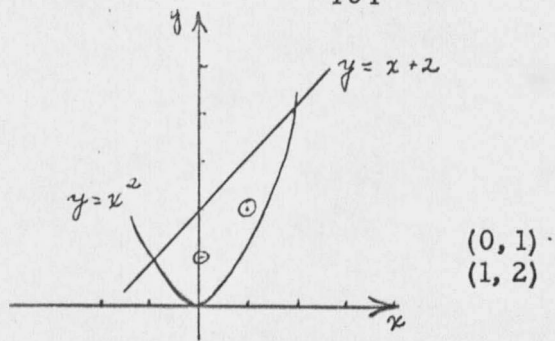
49. In the same manner solve: $x^2 - 5x + 6 > 0$.



50. Naturally all such inequality problems can be conveniently handled by non-geometric means. However, a simple geometric interpretation often leads one to a more solid understanding.

To illustrate this consider the task of finding the integral x and y where both $y < x + 2$ and $y > x^2$. This is not too easy without a little geometry, but with geometry it is not at all hard. Do it.





Lesson II

ABSOLUTE VALUE

1. A very useful concept in mathematics is that of absolute value. The symbol $| |$ is used to indicate absolute value and effects the quantity contained according to the following definition:

$$\begin{aligned} |a| &= a \text{ if } a \text{ is positive;} \\ &= -a \text{ if } a \text{ is negative;} \\ &= 0 \text{ if } a \text{ is zero.} \end{aligned}$$

That is, the symbol $| |$ serves to convert, if necessary, the value contained within them to a positive quantity - unless the contained value is zero in which case it retains that value. Evaluate each of the following:

$$\begin{aligned} \text{(a)} \quad |-5| &= ? \\ \text{(b)} \quad |+5| &= ? \\ \text{(c)} \quad |0| &= ? \end{aligned}$$

$$\begin{aligned} \text{(a)} \quad &+5 \\ \text{(b)} \quad &+5 \\ \text{(c)} \quad &0 \end{aligned}$$

-
2. Since $|x| \geq 0$, would it be legitimate to make a statement like:
 $|y-3| < -3$? Why?

No, since $|y-3| \geq 0$.

-
3. Now that you see how the absolute value symbol effects the quantity it contains, you should be able to define $|x|$ without looking back to step No. 1. Do so.

$$\begin{aligned} |x| &= x \text{ if } x \text{ is positive;} \\ &= -x \text{ if } x \text{ is negative;} \\ &= 0 \text{ if } x \text{ is zero.} \end{aligned}$$

-
4. Let's try your understanding in one more instance. What is the numerical difference between $|3x-7|$ and $|7-3x|$?

$$|3x-7| - |7-3x| = 0.$$

5. Simplify if possible:

(a) $\frac{|x^2|}{|x^{18}|}$
 (b) $\frac{x^2}{x^{18}}$

(c) $|x^3|$
 (d) $|7-x|$

(e) $|x-7|$

(a) $\frac{x^2}{x^{18}}$
 (b) $\frac{x^2}{x^{18}}$

(c) $|x^3|$
 (d) $|7-x|$ or $|x-7|$

(e) same as (d).

6. In case a problem involves a statement like $|x-a|$ you must be able to properly interpret this. Consequently, let's dwell on it for a moment.

In case $x > a$ you should be able to express the value $|x-a|$ without the absolute value symbol. Do so.

$|x-a| = x-a$ if $x > a$.

7. In case $x < a$ you should be able to express the value $|x-a|$ without the absolute value symbol. Do so.

$|x-a| = -(x-a) = a-x$ if $x < a$.

8. Now as an application of this knowledge suppose you are confronted with solving the inequality $|x-3| < 1$. Initially you should concentrate on $|x-3|$ and reason as follows:

1. If one presumes $x > 3$, the absolute value symbol can be simply dropped (since $x-3 > 0$) and the resulting inequality $x-3 < 1$ can be solved.
2. If one presumes $x < 3$, the absolute value symbol on $|x-3|$ can be removed only by prefixing $(x-3)$ with a minus sign: $-(x-3) = 3-x$. In this case the solution of $|x-3| < 1$ is boiled down to the solution of $3-x < 1$.

In the event $x > 3$ solve $|x-3| < 1$.

$x < 4$

In the event $x < 3$ solve $|x-3| < 1$.

$x > 2$

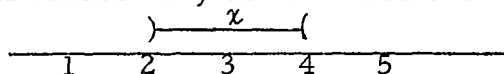
9. But what if both conditions ($x > 3$ and $x < 3$) are possible values for x !!! Then you must ask yourself what values of x would make both $x-3 < 1$ and $3-x < 1$ true. What are these values?

 $2 < x < 4$

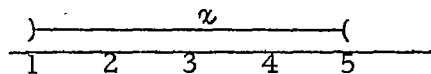
10. As a comparable problem with only a skeleton of explanation the solution of $|2x-5| < 11$ proceeds:
 if $2x > 5$, then $2x-5 < 11$ and $x < 8$;
 if $2x < 5$, then $5 - 2x < 11$ and $x > -3$.
 Thus $-3 < x < 8$. OK. Using this method solve $|5-3x| < 3$.

 If $5 > 3x$, then $5 - 3x < 3$ and $x > 2/3$;
 if $5 < 3x$, then $3x-5 < 3$ and $x < 8/3$.
 Thus $2/3 < x < 8/3$

11. Perhaps an easier consideration of the above type of problem is that of a geometric argument. Namely if $|x-3| < 1$ this means the difference between x and 3 in absolute value must be less than 1. Or stated differently it means x can't get as far as 1 unit on either side of 3. On the number line this means x must take on real values only as is indicated:



Sketch the inequality $|x-3| < 2$.



12. Pursuing this a step further consider $|x-2| < 6$. This means the magnitude of the distance from x to 2 must always be less than 6. What are the upper and lower bounds which limit the values x can assume?

 $-4 < x < 8$

13. In fact you can state that in general if $|x-a| < b$, then the values x can take on are between _____ and _____.

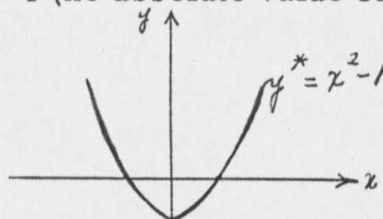
 $a-b$ and $a+b$

14. A particular inequality which is of great importance in calculus is the following: $|x-a| < \delta$. What values can x assume?

 $a - \delta < x < a + \delta$

15. Often in calculus one needs two dimensional sketches of absolute value phenomenon. Sometimes the problems are simple enough to be analyzed only in view of the definition and this gives sufficient insight to sketch the curve. Let's consider one such problem now: $y = |x^2 - 1|$.

The sketch of $y^* = x^2 - 1$ (no absolute value signs) is as follows:



Now, concentrate on the fact that y must be equivalent to $|x^2 - 1|$. What is the smallest value y can assume?

 zero

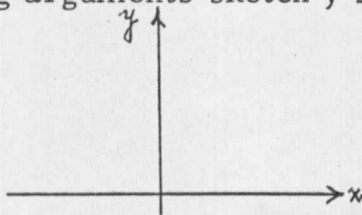
Thus y must always be no lower than the x -axis. But you can see $y^* < 0$ when $-1 < x < 1$. What then must happen to y over this same interval?

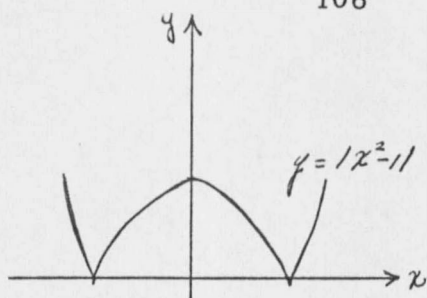
 $y = |x^2 - 1| > 0$ over this same interval.

In fact over this interval since $|y^*| = y$ you should be able to specify the relationship between y and y^* by not using the absolute value symbol. Do so.

 $y = -y^*$

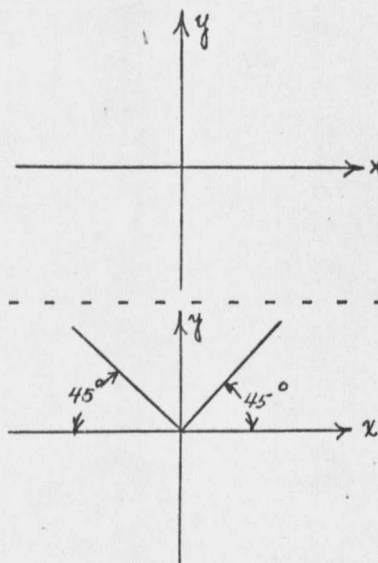
Using all the foregoing arguments sketch $y = |x^2 - 1|$.





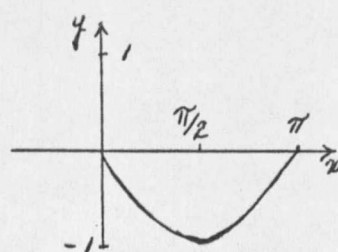
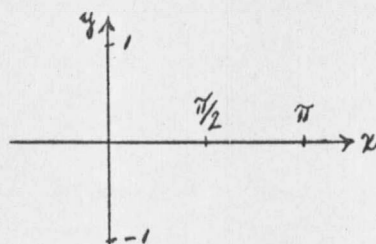
16. You see! The absolute value symbol on $x^2 - 1$ just served to reflect all the curve below the x -axis from below the axis to above it.

Sketch: $y = |x|$.



17. Thus if one has y equated to the absolute value of some expression involving x , it is safe to simply sketch the curve as though no absolute symbol were present and then reflect the positions of the curve below the x -axis across the line.

Sketch $y = -|\sin x|$ over $0 \leq x \leq \pi$.

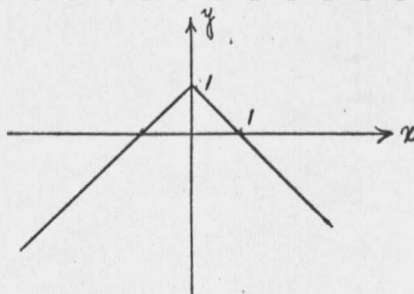
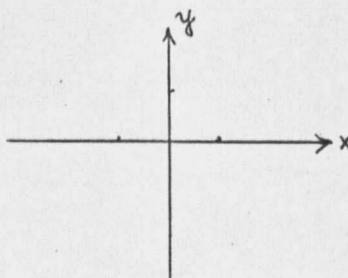


Notice the minus sign in front of the absolute value symbol! This means y is non-positive! Now you must reflect everything below the x -axis.

18. However, the task isn't always so simple. It doesn't always happen that y is a simple absolute value. It could be that y is a combination of an absolute value and a linear term. For example: $y = 1 - |x|$. In such a situation one must proceed according to cases.

- (1) If $x > 0$, $y = 1 - x$.
- (2) If $x < 0$, $y = 1 + x$.
- (3) If $x = 0$, $y = 1$.

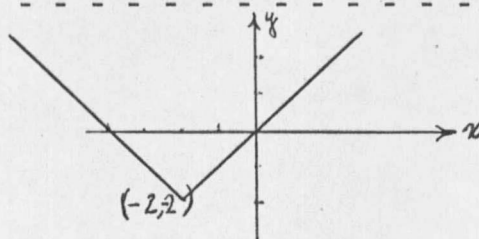
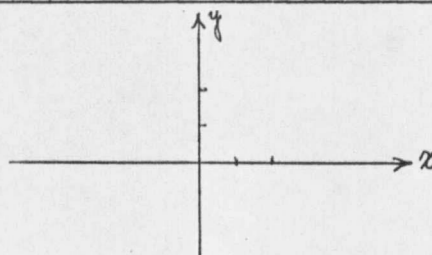
Sketch $y = 1 - |x|$.



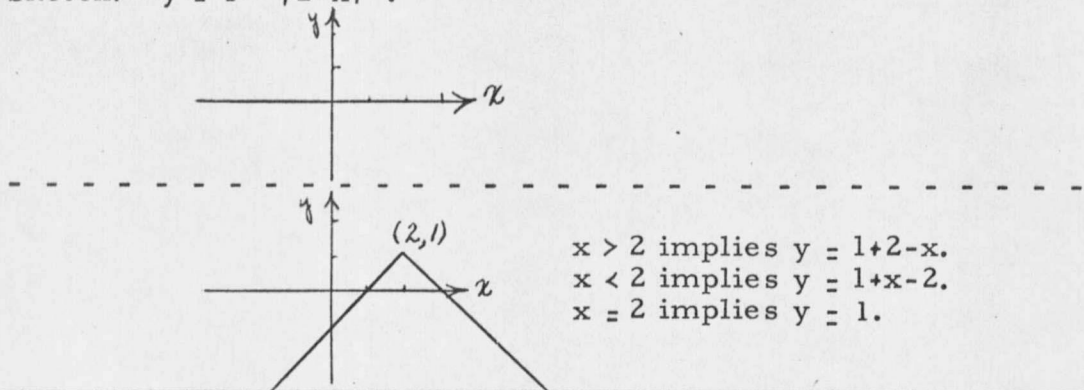
19. What three cases would you consider in sketching $y = |x+2| - 2$? And what would y be in each case?

- 1. $x > -2$; $y = x$.
- 2. $x < -2$; $y = -4 - x$.
- 3. $x = -2$; $y = -2$.

20. Sketch $y = |x + 2| - 2$.

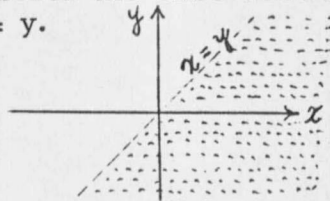


21. Sketch: $y = 1 - |2 - x|$.



22. The problem types could be even further varied. So far we have not considered the case where y is enclosed by the absolute value symbol. Let's try one.

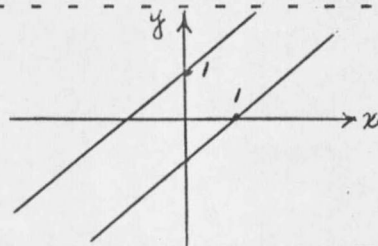
In case you are confronted with the problem: $|x - y| = 1$, the procedure is much the same as before. That is, examine cases. First, consider the case where $x > y$. This is the region to the right of $x = y$.



In this area $|x - y| = 1$ becomes $x - y = 1$ and should be plotted accordingly. In what part of the plane would one work if $x < y$?

To the left of $x = y$.

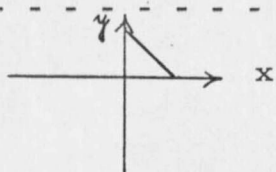
23. Sketch: $|x - y| = 1$. Use the coordinate system of number 22.



24. An even more complicated problem would involve two absolute values in the same equality: $|x| + |y| = 1$. Again the attack would be piecemeal - first consider x and y to be positive. What part of the coordinate system would be under consideration?

First quadrant

25. What would the resulting equation in this area be? Sketch it.



$$x + y = 1$$

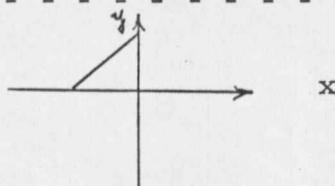
26. Now consider the case where x is negative but y positive. What part of the coordinate system would be involved?

Second quadrant

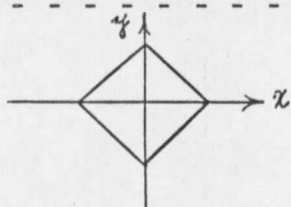
27. What equation would be considered here?

$$-x + y = 1$$

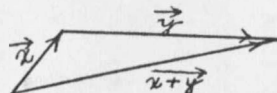
28. Sketch this portion of the figure.



29. Proceeding according to the above format complete the entire sketch of $|x| + |y| = 1$



30. One very useful inequality in calculus is the special "triangular inequality": $|x+y| \leq |x| + |y|$. This inequality can be made plausible by the geometric consideration of the triangle whose sides are taken to be vectors.



If two of the sides are denoted $\vec{x}$ and $\vec{y}$, it is clear (using the familiar addition of vector facts) that the remaining side is $\vec{x+y}$. If one designates the lengths of the three vectors by $|x|$, $|y|$, and $|x+y|$, upon what plane geometry theorem does the inequality $|x| + |y| > |x+y|$ depend?

The sum of two sides of a triangle is greater than the third.

31. In case x and y are in the same direction the triangle degenerates into a straight line. In this case $|x| + |y|$ has what relationship to $|x+y|$??

$$|x| + |y| = |x+y|$$

32. The algebraic proof of the triangular inequality is a bit more involved and less intuitive. It starts with: $(x+y)^2 = x^2 + 2xy + y^2$. But $x^2 + 2xy + y^2 \leq |x|^2 + 2|x||y| + |y|^2$. Why?

x or y may be negative hence forcing $2xy < 0$.

33. Clearly $|x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2$. Thus: $(x+y)^2 = x^2 + 2xy + y^2 \leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2$. But $|x + y|^2 = (x+y)^2$. Why?

Both are the same positive value when squared despite the fact that $(x+y)$ may have been $-|x + y|$ before squaring.

34. Considering step (33) it is clear that $|x+y|^2 \leq (|x| + |y|)^2$. Taking the positive square root of both sides yields $|x+y| \leq |x| + |y|$. Why?

If $0 < a < b$ and n is any positive integer, then $\sqrt[n]{a} < \sqrt[n]{b}$.

35. Verify the triangular inequality by choice of various values for x and y .

One such verification might be: $x = -3, y = -2; |-3-2| \leq |-3| + |-2|$.

36. Under what conditions would $|x| + |y| = |x+y|$?

- (1) $x = y = 0$
- (2) $x > 0$ and $y > 0$
- (3) $x < 0$ and $y < 0$

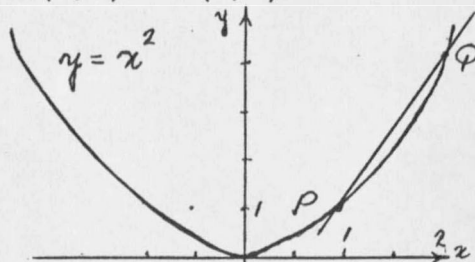
Lesson III

LIMITS

1. Undoubtedly the most important yet least understood concept in beginning calculus is the idea of limit. Consequently, the following material is planned to give you a better insight into this vital area. What concept will this material be about?

Limits

2. Suppose we consider the curve $y = x^2$ and draw a secant line PQ across this curve through the points (1, 1) and (2, 4).



Surely PQ is not a tangent to the curve at P. However, if one lets Q slide down the curve from (2, 4) towards P, the resultant secant PQ will more closely approach a position of tangency at P. As a matter of fact we could say that the closeness of the secant line to the true tangent depends on how close Q is to _____.

P

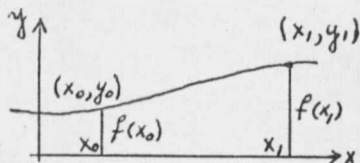
3. In brief: the difference in position between the tangent and the secant is dependent upon the closeness of Q to _____.

P

4. As a second illustration consider the old story of the frog that hops half the remaining distance to the pond's edge each day. The first day he is perhaps 16 feet away; the second day 8 feet; the third 4; etc. Although he never actually arrives at the water (since there is always half the previous remaining distance before him), he can be said to get as close as you please. In fact the frog-to-pond distance can be made as small as you please by simply insisting on a sufficient number of _____.

Jumps or days

5. The idea of a limit is much the same. The easiest consideration is a geometric one. Take $y = f(x)$ to be a simple continuous curve over an acceptable domain and take (x_0, y_0) and (x_1, y_1) to be two points on the curve over this same domain.



The magnitude of the difference of the ordinates at x_0 and x_1 would be _____.

 $|f(x_1) - f(x_0)|$ or $|y_0 - y_1|$

6. The amount of this difference can be controlled in our example by insisting on proximity of x_1 and _____.

 x_0

7. The mathematician usually specifies the amount of difference to be tolerated between y_1 and y_0 in terms the Greek letter ϵ , epsilon. That is, to be certain that the magnitude of the difference could never exceed a positive number ϵ , one could insist that:

$|y_1 - y_0| < \epsilon$. In words this inequality reads

 The absolute value of the difference between y_1 and y_0 is less than ϵ .

8. To satisfy this tolerance the reader is confronted with the task of specifying the distance x_1 is from x_0 . Let us suppose such a distance exists and let us call this distance δ , delta. Then we are saying that when $|x_1 - x_0| < \delta$ then $|y_1 - y_0| < \epsilon$.

If the distance x_1 is from x_0 is less than δ , what must be the restriction on the difference in distances of y_0 and y_1 ?

 $|y_1 - y_0| < \epsilon$.

9. Formally the mathematician frequently chooses to define a limit using this very idea. That is, y_1 approaches y_0 as a limit if and only if for any positive ϵ (no matter how small) there always exists a positive δ such that $|f(x_1) - f(x_0)| < \epsilon$ whenever

$|x_1 - x_0| < \delta$. How large can ϵ be?

 As large as is desired.

How small can ϵ be?

Arbitrarily small but positive.

10. If $y_1 \rightarrow y_0$ ($\rightarrow$ is read "approaches") as a limit, and ϵ is arbitrarily chosen to be $\frac{1}{2}$, according to the definition in (9) what must be the role of δ ?

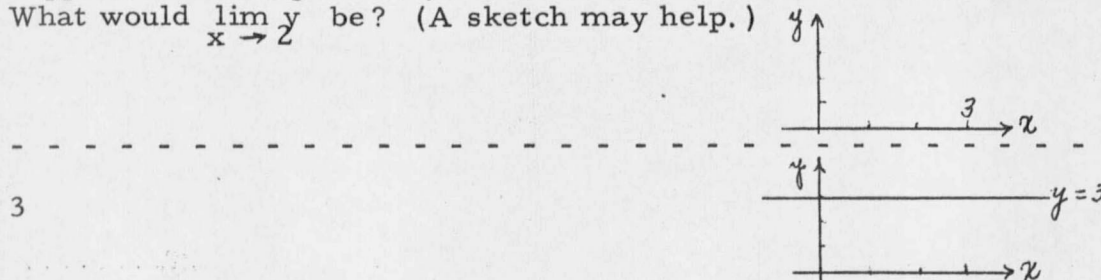
 δ must specify a safe range of variation of x_1 about x_0 such that when $|x_1 - x_0| < \delta$ then $|y_1 - y_0| < \frac{1}{2}$.

11. To indicate that $f(x)$ approaches a limit L as x approaches "a" the mathematician writes: $\lim_{x \rightarrow a} f(x) = L$. This is read: "the limit of $f(x)$ as x approaches "a" is L ".

Translate: $\lim g(y) = M$.

The limit of $g(y)$ as y approaches b is M .

12. Suppose the straight line $y = 3$ is defined over all values of x . What would $\lim_{x \rightarrow 2} y$ be? (A sketch may help.)



What δ would satisfy $\epsilon = 1$? (Hint: since $\lim_{x \rightarrow 2} y = 3$ there must be a δ such that $|y-3| < 1$ whenever $|x-2| < \delta$. Your problem is to find the δ which will force $|y-3| < 1$.)

Any δ since $|y-3| = 0$ for every δ .

What δ would satisfy any ϵ ?

Any δ because no matter where the x values are chosen the difference between the ordinates, y and 3 , must be zero which is always less than ϵ since $\epsilon > 0$.

13. As a second example consider $y = f(x) = x$. Suppose we are asked to prove: $\lim_{x \rightarrow 3} f(x) = 3$. We must show that for any $\epsilon > 0$

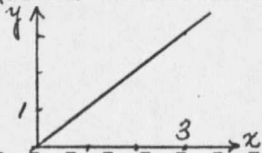
there exists a δ such that $|f(x) - 3| < \epsilon$ whenever _____.

 $|x - 3| < \delta$.

Or for any $\epsilon > 0$ we must show the existence of a δ such that
 whenever $|x - 3| < \delta$ then _____.

 $|f(x) - 3| < \epsilon$.

In particular suppose ϵ is chosen to be 1. What δ will satisfy this ϵ ? Why? (Hint: a sketch may help you decide.)

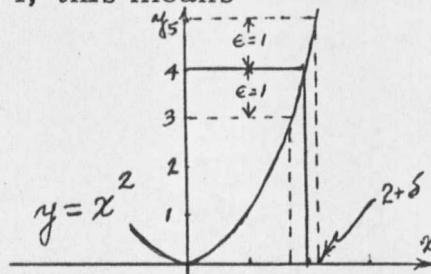


$0 < \delta \leq 1$ since $|x - 3| < 1$ implies $|f(x) - 3| < \epsilon = 1$.

If ϵ is chosen arbitrarily, what value of δ can be chosen to satisfy ϵ ?

Simply pick $\delta \leq \epsilon$.

14. Suppose $f(x) = x^2$ and we are asked to prove: $\lim_{x \rightarrow 2} f(x) = 4$. We must show that for $\epsilon > 0$ there exists a δ such that whenever $|x - 2| < \delta$ then $|f(x) - 4| < \epsilon$. If $\epsilon = 1$, this means $|f(x) - 4| < 1$ when _____.



$|x - 2| < \delta$

To find δ sketch the curve $y = f(x) = x^2$. Now take an interval one unit on either side of $y = 4$. That is, sketch $y = 3$ and $y = 5$. Now it must be true that the tops of the ordinates to the parabola be contained between these two lines.

Obviously you can force this restriction on the ordinates by simply insisting that only ordinates whose bases are close to $x = 2$ be chosen. For example, you could be certain that all the ordinates would always be shorter than 5 by insisting that x move no further to the right of $x = 2$ than to $x < \underline{\hspace{2cm}}$.

$$\sqrt{5}$$

Correspondingly you could be certain that all ordinates would be longer than 3 by insisting that x move no further to the left of $x = 2$ than $x > \underline{\hspace{2cm}}$.

$$\sqrt{3}$$

Now put the right bound and the left bound together such that whenever x is within these bounds then $|f(x)-4| < 1$. That is, what variation could you insist on which would guarantee that $|f(x)-4| < 1$?

$$\sqrt{3} < x < \sqrt{5}$$

But notice that $\sqrt{3} \doteq 1.732$ and $\sqrt{5} \doteq 2.236$ are different distances from 2 ($\sqrt{3} \doteq 1.732$ is about .268 units from 2 while $\sqrt{5} \doteq 2.236$ is only about .236 units from 2).

Consequently: $2 - .268 < x < 2 + .236$.

So actually we could force both $2 - .268 < x$ and $x < 2 + .236$ by simply insisting that $|x-2| < \underline{\hspace{2cm}}$.

$$.236$$

-
15. You see! Whenever $|x-2| < .236$ (the shorter of the interval on the left and the interval on the right) then $|x^2-4| < 1$. Notice that you had to choose δ from two values: $2 - \sqrt{3}$ and $\sqrt{5}-2$. Why would it have been incorrect to chose δ as $2 - \sqrt{3}$?

δ could not be $2 - \sqrt{3}$ since the ordinate at $2+(2 - \sqrt{3})$ is $(2+2 - \sqrt{3})^2$ or $(4 - \sqrt{3})^2 = 5.1+$. Thus at this abscissa it is not true that $|x^2 - 4| < 1$.

16. Now suppose you are asked to prove that $\lim_{x \rightarrow 3} x^2 = 9$. In particular suppose you are asked to demonstrate the existence of a δ which will confirm that $|x^2 - 9| < \epsilon$ where ϵ is chosen to be 1. Find the value δ such that whenever $|x - 3| < \delta$ then $|x^2 - 9| < \epsilon$.

$$|x - 3| < \sqrt{10} \text{ forces } |(x^2 - 9)| < 1. \text{ Thus } \delta \leq \sqrt{10}.$$

17. However, the problem of proving that a variable approach a limit does not fix the value of ϵ as we have done so far. Instead it is left as in the definition - arbitrarily small but positive. Hence, the above examples would not be complete proofs that

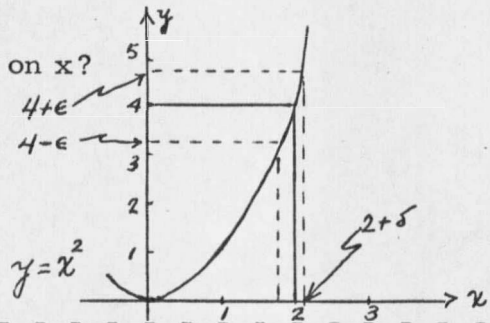
$$\lim_{x \rightarrow 2} x^2 = 4 \text{ or that } \lim_{x \rightarrow 3} x^2 = 9.$$

Instead one would have to show the δ to satisfy any arbitrarily small but positive _____.

ϵ

18. But again this can be done nicely by the same geometric argument. This time just draw lines parallel to $y = 4$ but a distance of ϵ units on both sides of $y = 4$. As before, ask yourself what values of x would keep the tops of the ordinates within these parallel boundaries.² You should decide on an abscissa on the right which will force $x^2 < 4 + \epsilon$ and an abscissa on the left which will force $4 - \epsilon < x^2$.

What are these bounds you must place on x ?



$$\sqrt{4 - \epsilon} < x < \sqrt{4 + \epsilon}$$

And as before it could be the case (and in fact is) that the variation allowed x is not of equal distance from $x = 2$. Hence, one could certainly be sure that $|x^2 - 4| < \epsilon$ by simply picking $|x - 2| < \frac{\epsilon}{4 + \epsilon}$ or $|x - 2| < \frac{\epsilon}{4 - \epsilon}$ whichever is smaller.

$$2 - \sqrt{4 - \epsilon}, \quad \sqrt{4 + \epsilon} - 2$$

To wrap it up tight you could say that $|x^2 - 4| < \epsilon$ whenever $|x - 2| < \sqrt{4 + \epsilon} - 2$ (which is in fact the smaller of the two!).

What is the value of δ in this problem?

$$\delta = \sqrt{4 + \epsilon} - 2$$

19. OK. Now you prove $\lim_{x \rightarrow 3} x^2 = 9$ by finding δ such that when $|x - 3| < \delta$ then $|x^2 - 9| < \epsilon$.

$$\delta = \sqrt{9 + \epsilon} - 3. \quad (\sqrt{9 + \epsilon} - 3 \text{ is smaller than } 3 - \sqrt{9 - \epsilon}.)$$

20. The above considerations of limit proofs have all been geometric. We shall now prove some limits by using an algebraic attack. Our procedure for limit proofs in the following section shall be entirely _____.

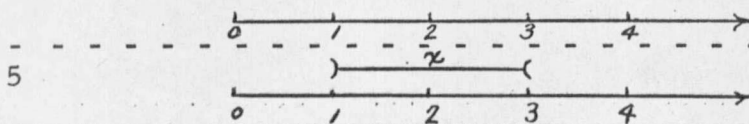
algebraic

21. To algebraically prove $\lim_{x \rightarrow 2} x^2 = 4$, consider $|x^2 - 4| < \epsilon$. Is the insistence that $|x^2 - 4| < \epsilon$ the same as the demand that $|x - 2| \cdot |x + 2| < \epsilon$? Why?

Yes, since: the product of absolute values is the same as the absolute value of the product.

22. Now: $|x - 2| \cdot |x + 2| < \epsilon$ is the same as $|x - 2| < \epsilon / |x + 2|$. Why? If $a < b$ and $c > 0$, $a/c < b/c$.

23. Thus if x is close to 2 (so close that $|x - 2| < 1$), $|x + 2|$ must certainly be less than _____. Hint: Sketch in one dimension $|x - 2| < 1$, and consider the upper bound for $|x + 2|$.



24. But under the restriction cited in (23), the inequality in (22) becomes: $|x-2| < \epsilon/5$

25. But this means δ has been found! To see why, fill in reasons for the following steps.

<u>Statements</u>	<u>Reasons</u>
1. Since $ x+2 < 5$, $1/5 < \frac{1}{ x+2 }$	1.
2. $\therefore \epsilon/5 < \frac{\epsilon}{ x+2 }$	2.
3. So, forcing $ x-2 < \epsilon/5$ forces $ x-2 < \epsilon/ x+2 $	3.
4. $\therefore x-2 \times x+2 < \epsilon$	4.
5. or $ x^2-4 < \epsilon$	5.

1. $0 < a < b$ implies $1/b < 1/a$.	
2. $a < b$, $c > 0$ implies $ac < bc$	
3. $a < b$, $b < c$ implies $a < c$	
4. see (2)	
5. $ a b = ab $	

In the above argument what value is δ if $|x-2| < 1$?

$$\delta = \epsilon/5$$

26. However, the entire previous argument depends on the fact: $|x-2| < 1$. We must be assured that our choice of δ not only implies $|x^2-4| < \epsilon$ but in fact $|x-2| < \delta$ must guarantee that $|x-2| < 1$. Indeed, if $|x-2| \not< 1$, we could never content that $|x^2-4| < \epsilon$! Hence, we must insist that $|x-2| < 1$.

In the previous argument if $\epsilon/5 > 1$, what simple value could we insist that $|x-2|$ be less than such that both $|x-2| < 1$ and $|x-2| < \epsilon/5$?

Force $|x-2| < 1$.

27. As a further example suppose $y = f(x) = x^2 - 3x + 1$. To prove:

$\lim_{x \rightarrow 2} (x^2 - 3x + 1) = -1$, it must be shown that for any _____
 there exists a δ (which is positive) such that whenever $|x - 2| < \delta$
 then _____ < _____.

 $\epsilon, |f(x) + 1|, \epsilon$.

28. Since we must show $|x^2 - 3x + 1 + 1| < \epsilon$, focus attention on this.
 Is $|x^2 - 3x + 1 + 1| < \epsilon$ the same as $|x - 2| \cdot |x - 1| < \epsilon$? And is
 this equivalent to $|x - 2| < \epsilon/|x - 1|$?

Yes. Yes.

29. If x is close to 2 (so close that $|x - 2| < 1$), then $|x - 1| < \frac{1}{2}$.

30. Thus for $|x - 2| < 1, \frac{1}{2} < \frac{1}{|x - 1|}$. Why?
 $0 < a < b$ implies $1/b < 1/a$.

31. Hence, $\frac{\epsilon}{2} < \frac{\epsilon}{|x - 1|}$. Why?

$a < b, c > 0$ implies $ac < ab$.

32. So, $|x - 2| < \epsilon/2$ implies $|x - 2| < \epsilon/|x - 1|$. Why?

$a < b, b < c$ implies $a < c$.

33. $|x - 2| \times |x - 1| < \epsilon$. Why?

$a > b, c > 0$ implies $ac > bc$.

34. So $|x^2 - 4| < \epsilon$. Why?

$$|a| |b| = |ab|$$

35. Now if it is not certain as to the magnitude of $\epsilon/2$, what value should one pick for δ such that when $|x-2| < \delta$ then

$$|(x^2 - 3x + 1) - (-1)| < \epsilon ? \quad \text{-----}$$

pick δ to be the smaller of $\epsilon/2$ and 1.

36. Now you try one all alone. Prove $\lim_{x \rightarrow 3} x^2 = 9$ by the algebraic method. (Find the δ .)

$$1. \quad |x^2 - 9| < \epsilon \text{ implies } |x-3| < \frac{\epsilon}{|x+3|}$$

$$2. \quad \text{for } |x-3| < 1, \quad |x+3| < 7$$

$$3. \quad \frac{1}{7} < \frac{1}{|x+3|} \text{ implies } \frac{\epsilon}{7} < \frac{\epsilon}{|x+3|}$$

$$4. \quad |x-3| < \epsilon/7 \text{ implies } |x-3| < \frac{\epsilon}{|x+3|}$$

$$5. \quad \therefore |x^2 - 9| < \epsilon \text{ if } |x-3| < \epsilon/7 \text{ or } 1, \text{ whichever is smaller.}$$

$$6. \quad \text{So pick } \delta = \epsilon/7 \text{ or } 1, \text{ whichever is smaller.}$$

37. Define what is meant by: $\lim_{x \rightarrow a} f(x) = L$.

$\lim_{x \rightarrow a} f(x) = L$ if and only if for every $\epsilon > 0$ there exists a positive δ such that $|f(x) - L| < \epsilon$ whenever $|x - a| < \delta$.

38. Frequently the beginning calculus student has need for seven theorems having to do with limits. The proofs of these theorems are rather involved and depend on the definition of a limit. Hence, these proofs are usually omitted in a beginning calculus class and will be omitted here. However, they are stated below for your reference since they will be used in the following work:

(a) If $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$, then $\lim_{x \rightarrow a} (f(x) + g(x)) = L + M$.

(b) If $\lim_{x \rightarrow a} f(x) = L$, then for any constant c : $\lim_{x \rightarrow a} cf(x) = cL$.

(c) If $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$, $\lim_{x \rightarrow a} f(x) g(x) = L M$.

(d) If $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$ and $M \neq 0$, $\lim_{x \rightarrow a} f(x)/g(x) = L/M$.

(e) $\lim_{x \rightarrow a} x = a$.

(f) If $P(x) = a_0 x^n - a_1 x^{n-1} - \dots - a_{n-1} x - a_n$, $\lim_{x \rightarrow a} P(x) = P(a)$.

(g) If $f(x) = N(x)/D(x)$ where $N(x)$ and $D(x)$ are polynomials and $D(a) \neq 0$, $\lim_{x \rightarrow a} f(x) = N(a)/D(a)$.

If $f(x) = x$ and $g(x) = x^2$, $\lim_{x \rightarrow 2} (f(x) + g(x)) = \underline{\hspace{2cm}}$. Why?

6. Theorem (a) or (f).

If $f(x) = x^2 - 3$ and $g(x) = x - 1$, $\lim_{x \rightarrow 2} \frac{f(x)}{g(x)} = \underline{\hspace{2cm}}$. Why?

1. Theorem (g) or (d).

39. $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} (x + 2) = 4$. It may seem that the division

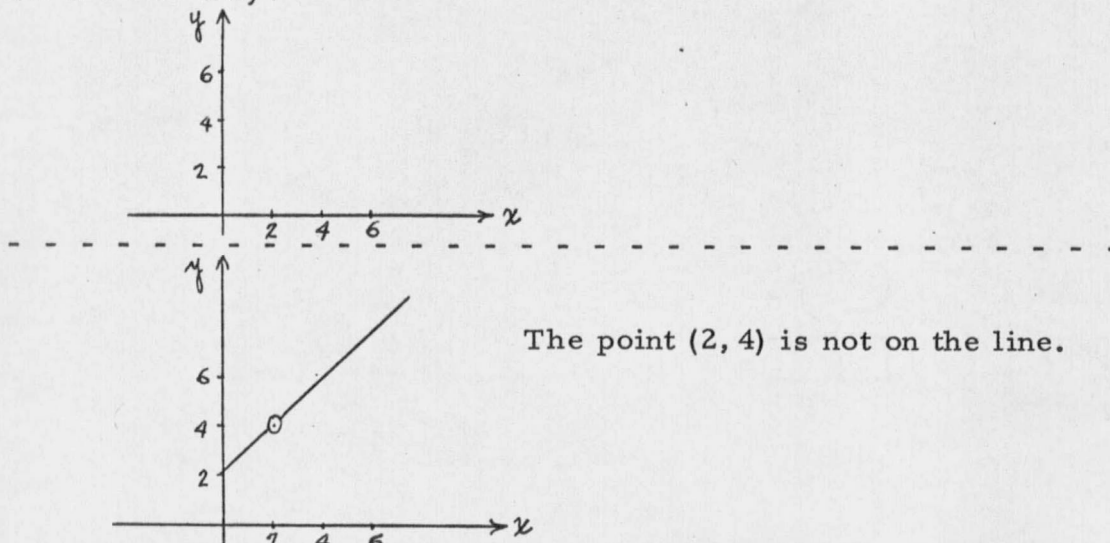
$\frac{x^2 - 4}{x - 2}$ is unallowed since when $x = 2$ both numerator and

denominator are zero. However, notice that the division is not taken when $x = 2$! The division is taken for values of x as $x \rightarrow 2$ (not as $x = 2$). Hence, the division is allowed.

$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \underline{\hspace{2cm}}$.

2.

40. Sketch: $y = \frac{x^2-4}{x-2}$ over $0 \leq x \leq 4$. What value does y take for $x = 2$? Why?



(2, 4) is not on the line since y is undefined for $x = 2$.

41. Prove: $\lim_{x \rightarrow 2} \frac{x^2-4}{x-2} = 4$.

(1) $\lim_{x \rightarrow 2} \frac{x^2-4}{x-2} = 4$ only if for every $\epsilon > 0$ there exists a δ such that $\left| \frac{x^2-4}{x-2} - 4 \right| < \epsilon$ when $|x-2| < \delta$.

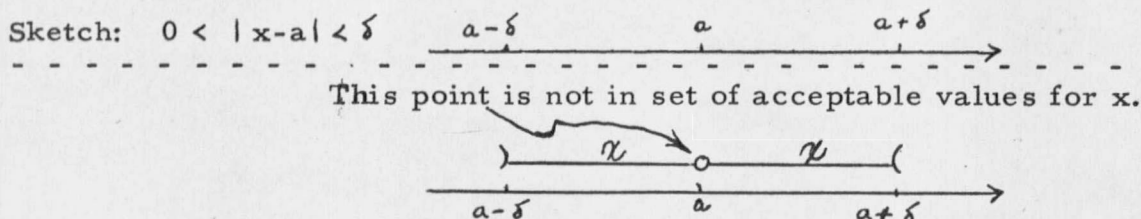
(2) $\lim_{x \rightarrow 2} \frac{x^2-4}{x-2} = 4$ only if for every $\epsilon > 0$ there exists a δ such that $|x^2-4| < \epsilon$ when $|x-2| < \delta$.

(3) Simply pick $\delta \leq \epsilon$.

42. But hold on - notice that the point (2, 4) was not even on the line! Yet $\lim_{x \rightarrow 2} \frac{x^2-4}{x-2}$ was 4. This means that x need not actually attain the position $x = 2$ and still $\lim_{x \rightarrow 2} \frac{x^2-4}{x-2} = 4$. This gives us reason to slightly modify the definition we have previously accepted. Now it can be said that $\lim_{x \rightarrow a} f(x) = L$ if and only if for every $\epsilon > 0$ there

exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$. Notice the change is in adding the restriction that $|x - a|$ be positive. That is, x now takes on values within $a - \delta$ to $a + \delta$ (one δ unit either side of a) except that x does not assume the value a .

Sketch: $0 < |x - a| < \delta$



43. Now, define: $\lim_{x \rightarrow a} f(x) = L$ in view of the above modification.

$\lim_{x \rightarrow a} f(x) = L$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

44. There are many limit problems that cause students trouble. One such type is: $\lim_{x \rightarrow \infty} \frac{5x^7 - 3}{10x^7 - 1}$. To solve this problem simply divide both numerator and denominator by x^7 and get:

$$\lim_{x \rightarrow \infty} \frac{5 - \frac{3}{x^7}}{10 - \frac{1}{x^7}}. \text{ Notice that the division is permissible for all}$$

specific values of x as $x \rightarrow \infty$. Thus $\lim_{x \rightarrow \infty} \frac{5 - 3x^{-7}}{10 - x^{-7}} = 1/2$.

$$\lim_{x \rightarrow \infty} \frac{2x^2 - 2x}{x^2 - 1} = ?$$

2.

45. Solve: $\lim_{x \rightarrow \infty} \frac{x^3 - 2x}{-x^2 - 3} = ?$ Hint: Try division by x^2 and by x^3 .

$\lim_{x \rightarrow \infty} \frac{x^3 - 2x}{-x^2 - 3} = \lim_{x \rightarrow \infty} \frac{x - 2/x}{-1 - 3/x^2} = -\infty$. Notice the division is by x^2 , the largest power of x in the denominator.

46. A very important type of limit problems is the following:

$$\lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2x \Delta x + \Delta x^2}{\Delta x} =$$

$$\lim_{\Delta x \rightarrow 0} (2x + \Delta x) = 2x.$$

Solve: $\lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} = ?$

$$3x^2$$

Lesson IV

CONTINUOUS FUNCTIONS

1. The ability of the beginning calculus student to identify situations when permissible operations can be performed is one the beginner often takes too lightly. It is because of this casual attitude that trouble is often encountered. Hence, the importance of awareness and application of the fine points of continuous functions is of paramount importance and thus basic to the development of theory which is to follow.

What is this lesson to concentrate on?

Continuous functions.

2. In order to define continuous functions, let us first concern ourselves with function. By way of review: a function is a set of ordered pairs such that no two distinct elements of the set have the same first term. Or perhaps you heard it this less distinct way: a function is a rule connecting x and y which assigns one and only one value to y for every different value of x . In either case $y = x^2$ is a good example of a function since for every value of x there is implied one and only one value of y .

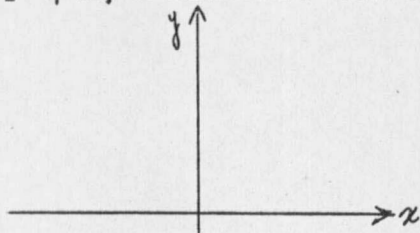
Is $y = x$ a function?

Yes. For every value of x there is one and only one value for y .

3. Is $x = y^2$ a function?

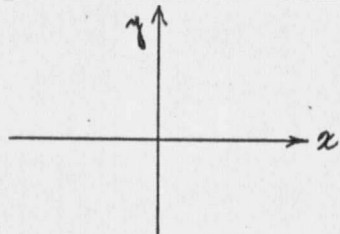
No. y is bivalued for all $x > 0$.

4. Is $y = |x|$ a function? Hint: consider the sketch of $y = |x|$.



Yes.

5. Is $|y| = x$ a function? Hint: consider the sketch of $|y| = x$.

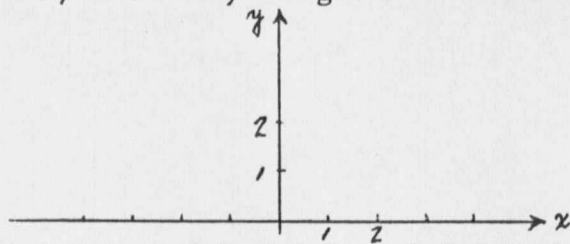


No. y is bivalued for all $x > 0$.

6. Is $y = a_0x^n + a_1x^{n-1} + \dots + a_n$ where a_i are real numbers and n is a positive integer a function?

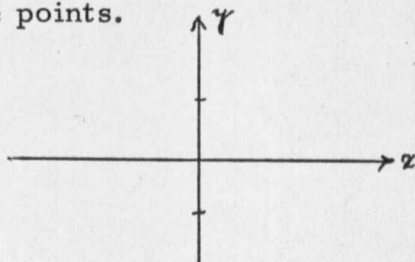
Yes.

7. Is $y = \begin{cases} 1 & \text{if } x \text{ is even} \\ 2 & \text{if } x \text{ is odd} \end{cases}$ a function? Hint: locate a few points on the coordinate system. Consider the domain (replacement values of x) to be only integral.



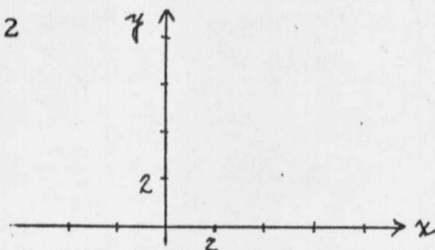
Yes. For every integral value of x one and only one value of y is implied.

8. Is $y = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational} \end{cases}$ a function? Hint: locate a few of these points.



Yes. Pick a value for x . y is then uniquely defined as either ± 1 .

9. Is $y = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 6 & \text{if } x = 2 \end{cases}$ a function? Hint: try a sketch.



Yes. y is uniquely defined for all values of x .

10. Is $y = \begin{cases} 1 & \text{when } x \text{ is rational and greater than } 1 \\ -1 & \text{when } x \text{ is irrational and less than } 1 \end{cases}$ a function?

Consider the domain (replacement values of x) to be all reals.

No. y is undefined for $x = 1$; y is undefined for $x > 1$ but irrational; y is undefined for $x < 1$ but rational.

11. A frequently used example for instructive purposes and for statistics is $y = [x]$ where $[x]$ is the greatest integer less than or equal to x . For example: $[2\frac{1}{2}] = 2$; $[1.999] = 1$; $[4.001] = 4$; $[-2\frac{1}{2}] = -3$; $[3] = 3$.

$$[7\frac{1}{2}] = ?$$

7.

$$[-.0001] = ?$$

-1.

$$[5] = ?$$

5.

12. Is $y = [x]$ a function?

Yes. y is defined uniquely for every value of x .

13. Now let's proceed to continuous functions. First consider the definition of continuity at a point: a function $y = f(x)$ is said to be continuous at the point a if and only if:

1. $y = f(x)$ is defined at $x = a$.
2. $\lim_{x \rightarrow a} f(x)$ exists.
3. $\lim_{x \rightarrow a} f(x) = f(a)$.

Actually, conditions (1) and (2) are implied in condition (3) since the presence $f(a)$ in condition (3) demands that f be meaningful at $x = a$ and since $\lim_{x \rightarrow a} f(x) = f(a)$, this implies $\lim_{x \rightarrow a} f(x)$ exists.

Thus the test of continuity depends ultimately upon the concept of limit.

The definition of continuity is essentially an application of what idea?

The idea of limit.

14. Thus if you are asked to prove that $y = f(x)$ is a continuous function at $x = a$ you must eventually prove: $\lim_{x \rightarrow a} f(x) = f(a)$.

What must one prove to demonstrate that $y = f(x)$ is continuous at $x = b$?

 $\lim_{x \rightarrow a} f(x) = f(b)$.

15. However, to prove that a function is discontinuous at a point one might be able to simply show $\lim_{x \rightarrow a} f(x) \neq f(a)$ because $f(a)$ does

not even exist! This is always the first check when one wants to investigate continuity.

What is the first prerequisite for a function to be continuous at a point?

The function must be defined at that point.

16. Suppose you are asked to investigate the continuity of $y = 1/x$ at $x = 0$. Is $y = 1/x$ continuous at $x = 0$?

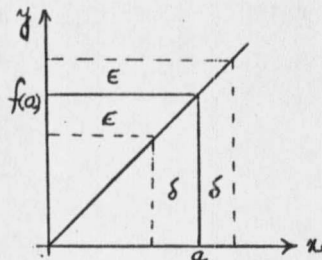
No.

Why?

 $y = 1/x$ does not exist at $x = 0$.

17. To prove that $y = x$ is a continuous function at $x = a$ it would be necessary to prove that $\lim_{x \rightarrow a} f(x) = f(a)$. And to prove this one would have to recall that: $\lim_{x \rightarrow a} f(x) = f(a)$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $0 < |x - a| < \delta$.

Find such a δ for the function $y = x$ if one exists. Refer to the figure below.



 $0 < \delta \leq \epsilon$

18. Suppose $y = x^2$. Is y continuous at $x = 2$?

 Yes.

19. To prove it using the algebraic approach one would proceed as follows:

I must show the existence of a δ such that: $|x^2 - 2^2| < \epsilon$ whenever $0 < |x - 2| < \delta$.

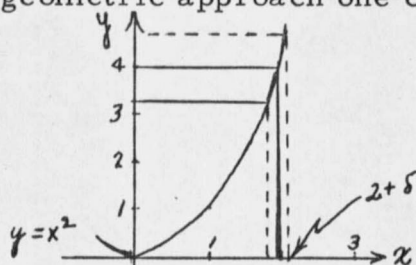
1. If $|x - 2| < 1$, $|x + 2| < 5$.
2. Now $|x^2 - 2^2| < \epsilon$ implies $|x - 2| < \epsilon / |x + 2|$.
3. From (1): if $|x + 2| < 5$, $\epsilon/5 < \epsilon / |x + 2|$.
4. Thus if I force $|x - 2| < \epsilon/5$ certainly $|x - 2| < \epsilon / |x + 2|$.
5. Therefore pick δ as $\epsilon/5$ or 1, whichever is smaller.

20. Now you prove that $y = x^2$ is continuous at $x = 3$.

-
1. I must show the existence of a δ such that: $|x^2 - 9| < \epsilon$ whenever $0 < |x - 3| < \delta$.
 2. If $|x - 3| < 1$, $|x + 3| < 7$.

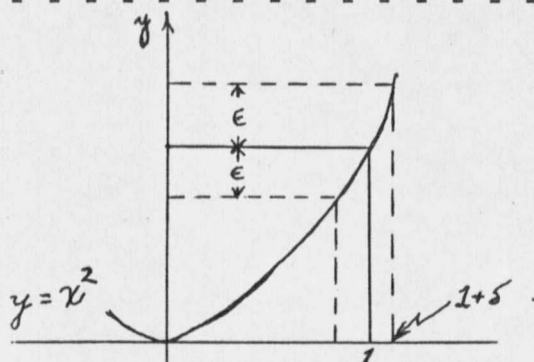
3. Now $|x^2 - 9| < \epsilon$ implies $|x-3| \cdot |x+3| < \epsilon$ or $|x-3| < \frac{\epsilon}{|x+3|}$.
4. $\frac{\epsilon}{7} < \frac{\epsilon}{|x+3|}$
5. $\therefore |x-3| < \frac{\epsilon}{7}$ implies $|x^2 - 9| < \epsilon$.
6. So pick δ the smaller of 1 and $\epsilon/7$.

21. Using a geometric approach one could prove $\lim_{x \rightarrow 2} x^2 = 4$:



$$\therefore \delta = \sqrt{4 + \epsilon} - 2.$$

Using a similar geometric approach prove $\lim_{x \rightarrow 1} x^2 = 1$.



$$\therefore \delta = \sqrt{1 + \epsilon} - 1.$$

22. Clearly $y = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$ is not a continuous function since

$$\lim_{x \rightarrow a} f(x) \neq f(a).$$

Why?

$\lim_{x \rightarrow a} f(x) \neq f(a)$ since there is no δ that would satisfy any $\epsilon < 2$.

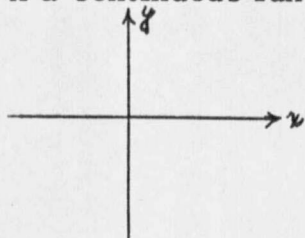
23. Is $y = \begin{cases} 1 & \text{if } x \geq 1 \\ 0 & \text{if } x < 1 \end{cases}$ a continuous function at $x = 1$? Why?

No. $\lim_{x \rightarrow 1} f(x) \neq 1$ since no δ will satisfy an $\epsilon < 1$ when x approaches 1 from the left.

24. Is $y = [x]$ a continuous function at $x = 1$? Why? ($[x]$ is largest integer $\leq x$)

No. $\lim_{x \rightarrow a} f(x) \neq f(a) = 1$ since there is no restriction on how x approach a . If x approaches 1 from the right, any $\delta \leq 1$ will satisfy any ϵ . But if x approach 1 from the left any $0 < \epsilon < 1$ cannot be satisfied no matter how small δ is chosen.

25. Is $|y| = x$ a continuous function at $x = 1$. Why? (Sketch it.)

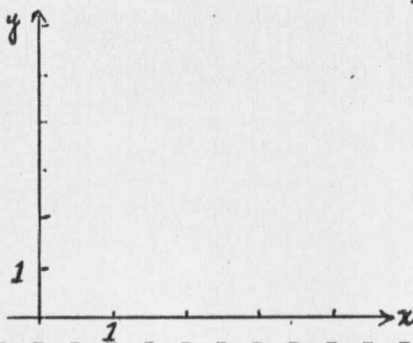


No. $|y| = x$ is not a function at $x = 1$.

26. Is $y = \frac{x^2-9}{x-3}$ a continuous function when $x = 3$? Why? (Check the three criteria for continuity.)

No. $f(x)$ does not exist at $x = 3$.

27. Is $y = \begin{cases} \frac{x^2-9}{x-3} & \text{when } x \neq 3 \\ 5 & \text{when } x = 3 \end{cases}$ a continuous function at $x = 3$? (Sketch it.)



No.

y is discontinuous at $x = 3$ since $\lim_{x \rightarrow 3} f(x) \neq 5$. Why does the limit fail?

There exists no δ which will satisfy an $0 < \epsilon < 1$.

28. The big hurdle is over. You are now ready to define continuity over an interval. It is simple: for a function to be continuous over an interval, it must be continuous at every point of that interval. Thus the problem becomes one of examining continuity at a _____.

Point.

29. Is $y = x^2$ a continuous function over $-7 < x < 9$?

Yes - since it is continuous at every point over this interval.

30. Is $y = |x|$ a continuous function over $-5 \leq x \leq 5$?

Yes - since it is continuous at every point over this interval.

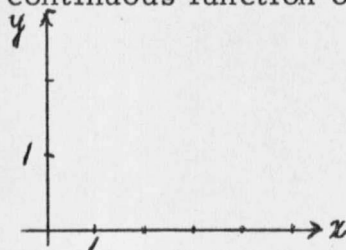
31. Is $|y| = x$ a continuous function over $0 \leq x \leq 1$?

No - since $|y| = x$ is not a function!

32. Is $y = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational} \end{cases}$ a continuous function over all x ?

No - since y is discontinuous at every x .

33. Is $y = [x]$ a continuous function over $2 \leq x < 3$? (Sketch it.)

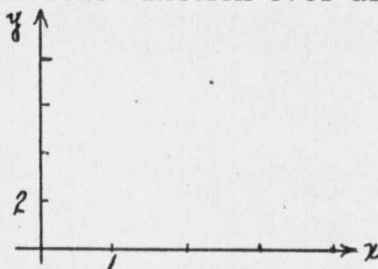


Yes. Note: y is continuous to the immediate right of $x = 2$ since $\lim_{x \rightarrow 2} [x]$ exists when x approaches 2 from the right.

34. Is $y = [x]$ a continuous function over $2 \leq x \leq 3$?

No - since $y = [x]$ is not continuous at $x = 3$.

35. Is $y = \begin{cases} \frac{x^2-9}{x-3} & \text{for } x \neq 3 \\ 7 & \text{for } x = 3 \end{cases}$ a continuous function over all x ? (Sketch it.)



No - setting $y = 7$ when $x = 3$ puts the point $(3, 7)$ onto the coordinate system but not on $y = x - 3$ as is desired. Hence there is a discontinuity at $x = 3$.

36. Is $y = \begin{cases} \frac{x^2-9}{x-3} & \text{for } x \neq 3 \\ 6 & \text{for } x = 3 \end{cases}$ a continuous function for all x ?

Yes - setting $y = 6$ when $x = 3$ puts $(3, 6)$ into the line $y = x - 3$ and the line is then continuous.

37. As a quick review: continuity over an interval depends on continuity at a _____ of the interval.

Point.

38. Continuity at a point depends on the definition of a _____.

Limit.

39. $f(x)$ is discontinuous at $x = a$ whenever $\lim_{x \rightarrow a} f(x) \neq$ _____.

$f(a)$

40. $\lim_{x \rightarrow a} f(x) \neq f(a)$ if there does not exist a _____ > 0 such that

whenever $0 < |x-a| < \underline{\hspace{2cm}}$ then $|f(x)-f(a)| < \underline{\hspace{2cm}}$ for every _____ > 0 .

$\delta, \delta, \epsilon, \epsilon$

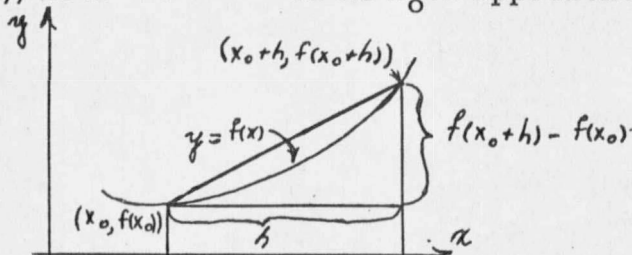
Lesson V THE DERIVATIVE

1. Basic to the calculus is the derivative. It is through this notion that critical studies of velocity, acceleration, maximum-minimum, related rate applications, and thousands of practical problems are attacked. Without the derivative there would be no calculus.

What is this lesson concerned with?

The derivative.

2. Actually the derivative is really nothing more than a limit. Perhaps one of the quickest insights into this topic is via geometry. Let us consider the continuous curve $y = f(x)$. Evaluated at any fixed point x_0 , y takes a fixed value $f(x_0)$. Then at a different abscissa, $x_0 + h$, y takes a new value $f(x_0 + h)$. If one is interested in the tangent to the curve $y = f(x)$ at x_0 , one need only consider the position approached by the secant line through $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h))$ as $h \rightarrow 0$ - - - or as $x_0 + h$ approaches the position x_0 .



To find the tangent to the curve $y = f(x)$ at $(x_0, f(x_0))$ we will be concerned with the position of the _____ through (x_0, y_0) and _____ as the point _____ slides along the curve closer and closer towards _____.

Secant, $(x_0 + h, f(x_0 + h))$, $(x_0 + h, f(x_0 + h))$, $(x_0, f(x_0))$.

3. In fact the slope of this secant would be given by:
$$\frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0}$$

at any time when x_0 and $x_0 + h$ do not coincide.

What would be the slope of the secant line through $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h))$ when $x_0 \neq x_0 + h$?

Slope of the secant =
$$\frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0}$$

4. Now it merely remains for us to consider the limiting position of the secant as $h \rightarrow 0$ if we desire to discover the condition of tangency. That is, let us consider:

$$\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{(x_0+h) - x_0}$$

This limit gives the slope of the tangent to the curve at $(x_0, f(x_0))$.

What is the slope of the tangent to the curve at $(x_0, f(x_0))$?

$$\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h}$$

5. It may seem to the casual observer that the quotient referred to in No. 4 is undefined since $h \rightarrow 0$. However this is not the case since $f(x_0+h) - f(x_0)$ also approaches zero. And as was shown in the lesson on limits, such expressions as $\lim_{x \rightarrow 2} \frac{x^2-4}{x-2}$

are quite meaningful despite the fact that the denominator approaches zero.

Can the expression: $\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h}$ exist even though $h \rightarrow 0$?

Yes.

6. Let us consider a simple example to test our belief. Suppose we want to find the slope of the straight line $y = x$ when $x_0 = 2$. We realize by previous experiences of analytic geometry that this line has a slope of 1. Now let's check it by the limiting idea.

$$\begin{aligned} \text{slope} &= \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{x_0+h - x_0}{h} = 1 \end{aligned}$$

It works as was anticipated! Now you find the slope of $y = 2x-1$ at $x_0 = 2$.

$$\begin{aligned} \text{slope} &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2(2+h)-1 - (2(2)-1)}{h} = 2. \end{aligned}$$

7. Since it works for the straight line, let's consider a tougher test; consider $y = x^2$ and find the slope at $x_0 = 3$.

$$\begin{aligned}\text{slope} &= \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(3+h)^2 - (3)^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{9+6h+h^2 - 9}{h} \\ &= 6\end{aligned}$$

Since the curve is symmetric with respect to the y-axis we would expect the slope at $x_0 = -3$ to be -6 . Prove this to the case.

$$\begin{aligned}\text{slope} &= \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(-3+h)^2 - (-3)^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{9-6h+h^2 - 9}{h} = -6\end{aligned}$$

8. As a matter of fact, the value: $\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h}$

is so important that it is labeled for easy reference. People in calculus call this limit the derivative (if the limit exists) and call the process whereby this limit is found differentiation.

What is the derivative of $y = f(x)$ at the point x_0 ?

$$\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} \quad \text{if the limit exists.}$$

What is the process of finding the derivative called?

Differentiation.

9. Let's try just one more example to be sure we really understand the process. If $y = x^2 + 2x - 1$, find the derivative of y at $x_0 = 1$:

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} &= \\ \lim_{h \rightarrow 0} \frac{(x_0+h)^2 + 2(x_0+h) - 1 - x_0^2 - 2x_0 + 1}{h} &= \\ \lim_{h \rightarrow 0} \frac{2hx_0 + h^2 + 2h}{h} &= \\ = 2x_0 + 2 = 2 + 2 = 4 \end{aligned}$$

Now you try: if $y = 3x^2 - 2x + 1$, at $x_0 = 0$ the derivative of y is ???

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{3(x_0+h)^2 - 2(x_0+h) + 1 - 3x_0^2 + 2x_0 - 1}{h} &= \\ \lim_{h \rightarrow 0} \frac{3x_0^2 + 6x_0h + 3h^2 - 2x_0 - 2h + 1 - 3x_0^2 + 2x_0 - 1}{h} &= 6x_0 - 2 = -2 \end{aligned}$$

10. Actually (as we have seen) where one specifies the derivative at a point, one is specifying the slope of the curve at that point. Thus, if one finds the derivative to appear as $2x_0 - x_0^2$ when $x_0 = 2$, the slope of the curve in question at $x_0 = 2$ is _____.
- zero

11. On the other hand if one carries through the process of differentiation without regard for fixing x at a particular point - but instead work with the arbitrary point $P(x, y)$, then one is finding an algebraic expression which represents the slope of the curve at the general point. Because the result of such differentiation results in a derived algebraic expression, the result is called the derived function.

For example, to find the derived function if $y = x^2$, one proceeds as follows:

$$\begin{aligned}
 & \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 = & \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\
 = & \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} = 2x
 \end{aligned}$$

Find the derived function of (or differentiate) $y = x^3$.

$$\begin{aligned}
 & \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 = & \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} \\
 = & \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
 = & 3x^2
 \end{aligned}$$

12. Probably at this stage you are anxious to know if there isn't a faster way to differentiate. Well, in fact there frequently are general rules which allow one to simply write down the answer. However, the beginner should never lose sight of what is behind these general rules - and each general rule depends upon the definition considered in the preceding steps.

In fact the derived function (or derivative as it is usually called) can be defined in general as: $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ when the limit

exists and every derivative (or derived function) can be shown to ultimately depend on this definition.

Define the derivative of $y = f(x)$.

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad \text{when the limit exists.}$$

13. Suppose you are asked to find the derivative of $y = |x|$ at $x = 0$. Of course your response should be: there is no derivative of $y = |x|$ at $x = 0$. Why?
-

$\lim_{h \rightarrow 0} \frac{|0+h| - |0|}{h}$ does not exist. If you don't believe this, sketch:

$$\frac{|h|}{h} = y \text{ on an } h\text{-}y \text{ axis.}$$

14. Sometimes notation connected with differentiation is incorrectly interpreted by the student. Hence, let's take a short look at some notation and clear the air.

Perhaps the most common notation denoting the derivative of $y = f(x)$ is $\frac{dy}{dx}$.

But beware, the student has no right at this time to consider $\frac{dy}{dx}$ to be anything but a symbol for the derivative. Above all the student has no right at this stage of the game to believe that $\frac{dy}{dx}$ is a quotient.

Sometimes slight variations of this notation occur. Some such are:

$$\frac{d}{dx} y = \frac{d}{dx} f(x) = \frac{df(x)}{dx}.$$

Another common notation is the use of primes to indicate the derived function: $y' = f'(x)$. And a further variation but still of considerable use is the following: $D_x y = D_x f(x)$.

Each of these notations mean the same thing: the derivative of y with respect to x and should not be interpreted as products or quotients.

Write eight notations meaning: the derivative of y with respect to x if $y = f(x)$.

$$\frac{dy}{dx} = \frac{d}{dx} f(x) = \frac{d}{dx} y = \frac{df(x)}{dx} = y' = f'(x) = D_x y = D_x f(x).$$

15. To indicate that the derivative is being evaluated at a point x_0 the notation $\left. \frac{dy}{dx} \right|_{x=x_0}$ is frequently used where the subscript type $x = x_0$ indicates the point where the derivative is evaluated. Of course an equally serviceable notation is $f'(x_0)$.

Could you invent a similarly serviceable notation to show that the derivative y' is being evaluated at $x = x_0$? The same for $f'(x)$; $D_x y$?

$$y' \Big|_{x=x_0}, \quad f'(x_0) \text{ or } f'(x) \Big|_{x=x_0}, \quad D_x y \Big|_{x=x_0}$$

16. As you may expect, the derivative can be interpreted in many ways. Whereas our approach was at first geometric, let us now take a physical situation and develop the idea of the derivative from it.

Suppose for example we were concerned with the problem of finding the velocity of a freely falling body two seconds after it dropped. We might proceed as follows: let s_t be the distance traversed at the end of t seconds and consider the formula $v = s_t/t$ where average velocity is computed by dividing the distance traveled in t seconds by the time elapsed. The problem is to find the exact velocity at the end of two seconds. Tabulating the distance traveled by a freely falling body in t seconds by $s_t = 16t^2$ one can calculate the average velocity for the period $t = 2$ to $t = 3$. This is:

$$s_3 = 16(9) = 144, \quad s_2 = 16(4) = 64$$

$$\therefore \text{avg } v = \frac{s_3 - s_2}{t_3 - t_2} = \frac{144 - 64}{3 - 2} = 80' / \text{sec.}$$

What is the average velocity over $t = 2$ to $t = 2.1$?

$$\text{avg } v = \frac{s_{2.1} - s_2}{t_{2.1} - t_2} = \frac{16(2.1)^2 - 64}{2.1 - 2} = \frac{70.57 - 64}{.1} = 65.6$$

-
17. Do you see how we are proceeding? We are calculating the average velocity by dividing the increase of s by the increment on t . Often these increments are denoted Δs and Δt respectively. Using Δs and Δt write a general formula for the average velocity over the Δt time interval.
-

$$\text{avg velocity} = \frac{\Delta s}{\Delta t}$$

18. Use the preceding notation to calculate the vacant spaces of the table below:

t	$t + \Delta t$	$s_{t=2}$	$s + \Delta s$	Δt	Δs	$\frac{\Delta s}{\Delta t}$
2	3	64	144	1	80	$80/1 = 80$
2	2.1	64	70.57	.1	6.56	65.6
2	2.01	64	64.6416	.01	.6416	64.16
2	2.001	64				

				64.064016	.001	.064016
						64.016

19. Now consider what would happen if smaller and smaller increments on t are taken. Can you guess at the velocity attained when $t = 2$?

 $v = 64' / \text{sec.}$

20. You see, in effect you have argued that the exact velocity is nothing more than the limiting value of the average velocity. Now represent symbolically the exact velocity if $\Delta s / \Delta t$ represents the average velocity.

 $\lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} = \text{exact velocity.}$

21. Now let's tie this result up with the first idea of a derivative we came across. You should see in the falling body problem that $s = f(t)$. Then $s + \Delta s = f(t + \Delta t)$.

Express Δs in terms of $f(t)$ and $f(t + \Delta t)$.

 $\Delta s = f(t + \Delta t) - f(t).$

Now express $\frac{\Delta s}{\Delta t}$ in a similar manner.

 $\frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$

Finally, if the average velocity is $\frac{\Delta s}{\Delta t}$ and the exact velocity is calculated as the limiting values of $\frac{\Delta s}{\Delta t}$, express the exact velocity in terms of the answer above.

 $\text{exact vel.} = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}$

22. But haven't you seen this before? This is abstractly the same as the derivative you developed at the start of this lesson.

Consequently could you now calculate the velocity at $t = 2$ using the derived function $s = f(t) = 16t^2$? Do so.

$$\begin{aligned}
 \text{velocity} &= \left. \frac{ds}{dt} \right|_{t=2} = \lim_{\Delta t \rightarrow 0} \frac{f(2 + \Delta t) - f(2)}{\Delta t} \\
 &= \lim_{\Delta t \rightarrow 0} \frac{16(2 + \Delta t)^2 - 16(2)^2}{\Delta t} \\
 &= \lim_{\Delta t \rightarrow 0} \frac{64 \Delta t + 16(\Delta t)^2}{\Delta t} \\
 &= 64
 \end{aligned}$$

23. Good. The derivative you see was evaluated at a point $t = 2$. Now calculate the derived function which will result in the exact velocity when calculated at any time t .

$$\begin{aligned}
 s' &= \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \\
 &= \lim_{\Delta t \rightarrow 0} \frac{16(t + \Delta t)^2 - 16t^2}{\Delta t} \\
 &= \lim_{\Delta t \rightarrow 0} \frac{32t \Delta t + 16(\Delta t)^2}{\Delta t} \\
 &= 32t
 \end{aligned}$$

24. It should now be clear what a derivative is and what differentiation is. To nail it down solid, define what is meant by:

$$\left. \frac{dy}{dx} \right|_{x=x_0} \quad \text{if } y = f(x).$$

$$\left. \frac{dy}{dx} \right|_{x=x_0} = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad \text{if the limit exists.}$$

25. What important geometric problem can be solved by direct application of $\left. \frac{dy}{dx} \right|_{x=x_0}$?

Finding the slope of the curve $y = f(x)$ at $x = x_0$.

What important physics problem is directly related to ds/dt ?

Finding the velocity of a moving body if $s = f(t)$.

26. Of course there are many other applications of the derivative but each is abstractly identical to the ideas already presented. It only remains for the reader to apply the idea of the derivative when the situation so demands. But here is where the rub comes in! (1) How do you know when the derivative should be used? (2) And if it should be used, when can it be used? In general this first question is not easy to answer since different situations demand different interpretations. However, if the change of a dependent variable is dependent upon the change of an independent variable (or variables), the exact change of the dependent variable as it is related to the independent variable(s) is calculated as a limit - this limit being the derivative.

The second question is easier. You can apply the derivative when the limit (as it is desired) exists.

Suppose the growth of a tissue is dependent upon time in the following manner: $G = f(t) = t^3 - t^2 + 1$. Could one conclude that the rate of tissue growth could be expressed as a derived function?

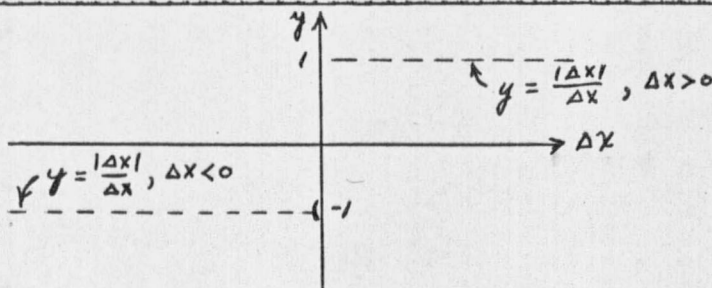
Yes.

27. Could one discuss the slope of $y = |x|$ at $x = 0$ by using the derivative?

No.

Why?

$\lim_{\Delta x \rightarrow 0} \frac{|\Delta x|}{\Delta x}$ does not exist. In case this is still bothering you, see the sketch below.



Lesson VI

COMPOSITE FUNCTIONS AND THE CHAIN RULE

1. Often the beginning student in calculus is thrown for a loss when the instructor or author uses notation which is quite convenient but unfamiliar to the student. One such situation in calculus which reputedly gives the student trouble is in connection with composite functions. Consequently, this lesson covers (in part) that problem area.

What is one thing this lesson has to do with?

Composite functions and related notation.

2. Let's start with an example: $y = \sin \sqrt{x}$. You can see we could just as easily have expressed this idea as: $y = \sin u$ and $u = \sqrt{x}$. Then clearly y is a function of u and u in turn is a function of x . In brief: $y = f(u)$ and $u = g(x)$. We could write this as: $y = f(g(x))$. This is read: "f of g of x" or "the composition of f on g of x".

Suppose $y = \phi(u)$ and $u = \theta(x)$. Express symbolically the composition function y of ϕ on θ of x .

 $y = \phi(\theta(x))$.

3. Sometimes the composition of functions can exceed two in number. For example: $y = \theta(u)$, $u = \phi(v)$, $v = \psi(x)$. Such a composition would be expressed: $y = \theta(\phi(\psi(x)))$. An example of such a composition would be:

$$\begin{aligned} y &= \sin(\tan \sqrt{x}) \\ \text{where } y &= \theta(u) = \sin u \\ u &= \tan v = \phi(v) \\ v &= \sqrt{x} = \psi(x). \end{aligned}$$

Express as a single equation the particular composition denoted by $y = \theta(\phi(\psi(x)))$ if:

$$\begin{aligned} y &= \theta(u) = u^{\frac{1}{2}} \\ u &= \phi(v) = \sin v \\ v &= \psi(x) = 1-x^2 \end{aligned}$$

 $y = \sqrt{\sin(1-x^2)}$

4. Also note that whenever there is a composition of functions, say $y = f(g(h(x)))$, for the entire system to be meaningful each h , g , and f must be meaningful in that order.

What values of x in the composition $y = f(g(h(x)))$ where $f(u) = u^{1/3}$, $u = g(v) = v^{1/2}$, $v = h(x) = 1-x^2$ would guarantee y to be real?

$$|x| \leq 1.$$

5. But now let's move on to an application of the idea of composite functions as related to differentiation.

Recall that $\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$ if the limit exists. What would $\frac{dy}{du}$ be?

$$\frac{dy}{du} = \lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u} \quad \text{if the limit exists.}$$

6. Now let's suppose $y = f(g(x))$. The question we wish to examine is this: How can one conveniently compute dy/dx when such a composition is encountered? The answer lies in an understanding of the behavior of g as it relates to x . That is, notice that if $y = f(u)$ and $u = g(x)$, as x changes then u and consequently y changes. To be more specific, suppose a particular example is chosen: Let $y = f(u) = u^2$ and $u = g(x) = x^2$.

What happens to u when x takes an increment Δx ?

$$u \text{ takes an increment } \Delta u = 2x \Delta x + (\Delta x)^2.$$

And what happens to y if u takes a corresponding increment Δu ?

$$y \text{ takes an increment } \Delta y = 2u \Delta u + (\Delta u)^2.$$

7. Let's actually calculate the change in y if $y = f(u) = u^2$ and $u = g(x) = x^2$. If x takes on an increment Δx , then y is changed to $y + \Delta y = (x + \Delta x)^2 = x^2 + 2x \Delta x + (\Delta x)^2$ or $\Delta y =$

$$2x \Delta x + (\Delta x)^2.$$

8. You see! In general if $y = f(g(x))$ where $u = g(x)$, if x is changed y is correspondingly changed through the intermediate change of u .

Calculate the change of y in terms of the change of x if $y = f(g(x))$ where $u = g(x) = x^2$ and $f(u) = u^2$.

$$y + \Delta y = f(u + \Delta u) = ((x + \Delta x)^2)^2 = x^4 + 4x^3 \Delta x + 6x^2 (\Delta x)^2 + 4x (\Delta x)^3 + (\Delta x)^4.$$

$$\text{Thus } \Delta y = 4x^3 \Delta x + 6x^2 (\Delta x)^2 + 4x (\Delta x)^3 + (\Delta x)^4.$$

9. With that as a background we are ready to look at an important consideration, namely the comparison of $\Delta y / \Delta u$ to dy/du .

Since $\lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u} = \frac{dy}{du}$ this means that for any $\epsilon > 0$ (no matter

how small!) there exists a $\delta > 0$ such that $\left| \frac{\Delta y}{\Delta u} - \frac{dy}{du} \right| < \epsilon$ whenever $0 < |\Delta u| < \delta$.

Certainly this can be interpreted as: $\frac{\Delta y}{\Delta u} = \frac{dy}{du} + e$

where e denotes the correction factor necessary to balance the equation.

Perhaps the idea can be made clearer yet. Consider $\Delta y / \Delta u$ to be the slope of the secant through the curve $u = f(u)$ at some point (u, y) whereas dy/du represents the slope of the _____ to the same curve at the same point (u, y) . Then

$\left| \frac{\Delta y}{\Delta u} - \frac{dy}{du} \right| < \epsilon$ when $0 < |\Delta u| < \delta$ specifies that the difference of the slopes of the _____ and the _____ at the point (u, y) can be made as small as is desired by simply forcing _____ to be less than _____ but positive.

tangent, secant, tangent, $|\Delta u|$, δ .

10. Thus it is clear that $\Delta y / \Delta u = dy/du + e$ is a fair appraisal of the physical representations of the graph. Indeed $\Delta y / \Delta u$ and dy/du are fixed numbers for the slopes at any given point (u, y) .

Consequently, $\Delta y = dy/du \Delta u + e \Delta u$. Why?

Multiply both sides of $\Delta y / \Delta u = dy/du + e$ by Δu , a numerical change in u .

11. But keep the air clear. We are interested in the change of y with respect to the change in x when the intermediate composition of functions is known. That is, we want as a final statement dy/dx . Now, how can you proceed from an expression of Δy to dy/dx ?
-

Divide Δy by Δx and take $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$.

12. Thus if $\Delta y = dy/du \Delta u + e \Delta u$, get $\Delta y / \Delta x$.
-

$$\Delta y / \Delta x = dy/du \Delta u / \Delta x + e \Delta u / \Delta x.$$

13. Now insist that $\Delta y / \Delta x$ proceed to a limit as $\Delta x \rightarrow 0$. But this means you are proceeding from $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$ to the resultant $\frac{dy}{dx}$.

Of course on the right hand side this means $\lim_{\Delta x \rightarrow 0} \frac{dy}{du} \frac{\Delta u}{\Delta x} + e \frac{\Delta u}{\Delta x}$

becomes $\frac{dy}{du} \cdot \frac{du}{dx} + e \frac{du}{dx}$ with a sufficient application of limit

theorems. But focus on $e du/dx$. What is the fate of this term? (Remember that e can be made as small as is desired by adjusting Δx .)

It goes to zero.

14. But let's look even deeper. Remember we are evaluating these derivatives at points. Thus you see that du/dx is merely a fixed number depending on the point at which it is evaluated. In fact then $e du/dx$ is nothing more than a number as well. But this number is regulated in magnitude by the value of e . And in turn e can be made as small as is desired by insisting on the proximity of Δx to 0. Consequently it follows that

$|e \frac{du}{dx}|$ can be made arbitrarily small by adjusting Δx toward 0.

Say, let's set $|e \frac{du}{dx}| = \epsilon$.

Hey, this means $|\frac{du}{dx} - \frac{dy}{du} \frac{\Delta u}{\Delta x}| < \epsilon$ by insisting that $0 < |\Delta x| < \delta$.

From this previous sentence what do you conclude of

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} ?$$

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{du} \cdot \frac{du}{dx}$$

15. As you have learned in class, this is an important fact. So important it is named: Chain Rule. The applications are indeed far reaching. However, before we proceed to them let's consider once again the source of the previous argument.

In step (10) we stated that $\Delta y / \Delta u = dy/du + e$. Technically speaking this equation may be meaningless since $\Delta u = 0$ is quite possible when $u = f(x)$.

Consider for example: $u = f(x) = 5$. Clearly u never changes even though x may vary at will. Hence $\Delta u = 0$ always.

Give another value for $u = f(x)$ such that Δu is always 0.

Let $u = f(x)$ be any constant.

16. But hold on! We stated that $\Delta y / \Delta u = dy/du + e$ in the initial considerations which eventually blossomed into the chain rule. Perhaps all of our arguments were built on sand because surely $\Delta y / \Delta u$ is meaningless if $\Delta u = 0$.

As a means of avoiding this trap and salvaging our work we can enjoy freedom of mind by insisting merely that $\Delta y = dy/du \Delta u + e \Delta u$.

Then using this as a point of departure we can now contend that $\Delta y / \Delta x = dy/du \cdot \Delta u / \Delta x + e \Delta u / \Delta x$.

Why can we divide by Δx with a clear conscience whereas division by Δu caused us problems?

 Δu could be zero but $|\Delta x| > 0$.

17. Now let's use this new found fact. Suppose $y = (1+x^2)^2$. Certainly one could calculate y' by simply squaring $1+x^2$ and differentiating the resultant product. However, consider $y = f(g(x))$ where $g(x) = 1+x^2$ and $f(u) = u^2$. You should notice that we can use the chain rule.

$$\begin{aligned} \text{That is: } dy/dx &= dy/du \cdot du/dx \\ &= 2u \cdot 2x = 2(1+x^2) \cdot 2x \end{aligned}$$

Suppose you are encountered with the task of differentiating $y = (2x-3)^{71}$ and you elect to consider $y = f(g(x))$. What would you take to be $g(x)$?

 $g(x) = 2x-3$

What would you assign $f(u)$ to be if $y = f(u)$?

$$y = f(u) = u^{7/8}$$

18. OK. Now use $dy/dx = dy/du \cdot du/dx$ to differentiate $y = (2x-3)^{7/8}$.

$y' = \frac{7}{8} (2x-3)^{-1/8} (2)$. This is an easier way to differentiate $(2x-3)^{7/8}$ than to first expand the binomial!

19. You see, the object of the game is to choose u such that this choice simplifies $y = f(g(x))$ to a simple relationship $y = f(u)$ which is immediately differentiable. Then when $f'(u)$ is calculated one can turn to the choice of u and apply the same thinking to it.

Consider the composition $y = f(g(h(x)))$

$$\text{where } v = h(x) = 2x-1,$$

$$u = g(v) = \sqrt[3]{v},$$

$$y = f(u) = u^{7/8}.$$

$$\text{Or } y = (\sqrt{2x-1})^{7/8}$$

To get dy/dx we first apply $dy/dv = dy/du \cdot du/dv$:
 $dy/dv = \frac{7}{8} u^{-1/8} \cdot \frac{1}{2v^{-1/2}}$

As a final step we proceed on as:

$$dy/dx = dy/dv \cdot dv/dx \text{ or}$$

$$dy/dx = \frac{7}{8} u^{-1/8} \cdot \frac{1}{2v^{-1/2}} \cdot 2$$

Putting it all together:

$$dy/dx = \frac{7}{8} (\sqrt{2x-1})^{-1/8} \cdot \frac{1}{2} (2x-1)^{-1/2} \cdot 2$$

Now you try one. Find y' if $y = \sqrt[3]{(3x^2-1)^{21}}$ (Don't simplify it before you start to differentiate.)

$$y' = \frac{1}{3} [(3x^2-1)^{21}]^{-2/3} \cdot 21(3x^2-1)^{20} \cdot 6x$$

20. As a further application of the chain rule consider the situation where $y = 3x^2-2x+1$ and $x = 3t-1$.

This is the usual form of parametric representation used so frequently in many situations. In this case the equalities simply state that the ordinate is dependent upon the abscissa but the abscissa in turn is dependent upon t (which could be time).

To find dy/dt , use the chain rule: $dy/dt = dy/dx \cdot dx/dt$ or $dy/dt = (6x-2) \cdot 3$.

Now you try. Suppose $z = y^{21} - 3y + 1$ and $y = 4x^2 + 3$ find dz/dx .

$$dz/dx = dz/dy \cdot dy/dx = (21y^{20} - 3) \cdot (8x).$$

21. An application of this idea might be: a point moves so that its ordinate is always twice the cube of the abscissa yet the abscissa is always the cube of the time elapsed diminished by 3. Find the change of the ordinate with respect to the time when $t = 1$.

$$y = 2x^3; x = t^3 - 3.$$

$$\begin{aligned} \therefore dy/dt &= dy/dx \cdot dx/dt = 6x^2 \cdot 3t^2 = 6(t^3 - 3)^2 \cdot 3t^2 \\ &= 6(-2)^2 \cdot 3(1) \\ &= 72. \end{aligned}$$

Now you try: A moving point receded from the x-axis directly as the square of the abscissa less 2, and the abscissa increases according to the 4th power of the time. Find the change of the ordinate with respect to the time when $t = 2$.

$$y = x^2 - 2; x = t^4$$

$$\therefore dy/dt = (2x) \cdot (4t^3) = 2(t^4) \cdot (4t^3) = 2^{10}.$$

Lesson VII

IMPLICIT DIFFERENTIATION

1. A necessary skill for every beginning calculus student is the ability to differentiate. Certainly one very important differentiation skill is the ability to differentiate "implicitly". Consequently this lesson is designed to make you more competent at differentiating "implicitly" and focuses in particular on the reasons why one can differentiate "implicitly".

This lesson concentrates on the mechanics and reasoning behind

Implicit differentiation

2. Recall that if $y = f(u)$ where in turn $u = g(x)$, then by the chain rule $dy/dx = \frac{df(u)}{du} \cdot \frac{du}{dx}$ or $dy/dx = \frac{df(u)}{du} \frac{dg(x)}{dx}$. It is this important idea upon which "implicit" differentiation is based. Consequently keep the principle of the chain rule fresh in your mind. What important idea is behind "implicit" differentiation?

chain rule

3. Suppose $y = f(x)^n$. Letting $u = f(x)$ we see that $y = u^n$. Then $y' = nu^{n-1} du/dx$ where the du/dx owes its existence to the chain rule. Similarly if $y = (g(x))^{p/q}$,

$y' = p/q \cdot g(x)^{(p/q)-1} dg(x)/dx$. Why must the term $dg(x)/dx$ be a factor of the answer?

chain rule

4. Now suppose $y = f(x)$. Clearly $y^3 = f(x)^3$ and for that matter any rational power of y is in turn reflected by this same rational power of $f(x)$. Indeed, $y^n = f(x)^n$. Of course an expression involving dy/dx is inherent in the derivative of $f(x)^n$. That is: $\frac{d}{dx} f(x)^n = n f(x)^{n-1} \cdot df(x)/dx = n f(x)^{n-1} dy/dx$.

Why is dy/dx a factor in the answer?

chain rule

But this means that $\frac{d}{dx} f(x)^3$ is $3f(x)^2 dy/dx$. Thus if $y^3 = f(x)^3$, $\frac{d}{dx} y^3$ must likewise be $3y^2 dy/dx$.

If $y^5 = f(x)^5$, find $\frac{d}{dx} f(x)^5$ and similarly find $\frac{d}{dx} y^5$.

$$\frac{d}{dx} f(x)^5 = 5f(x)^4 df(x)/dx \quad \text{and} \quad \frac{d}{dx} y^5 = 5y^4 dy/dx.$$

5. But remember $y = f(x)$ to start with in (4). Thus you see that $\frac{d}{dx} (f(x))^n = n(f(x))^{n-1} \frac{dy}{dx} = n y^{n-1} \frac{dy}{dx}$. Now you try writing down $\frac{d}{dx} y^7$ if $y = f(x)$. (First use $f(x)$ in your answer and not y .)

$$\frac{d}{dx} y^7 = 7(f(x))^6 \frac{df(x)}{dx}$$

Now use y in your answer and not $f(x)$.

$$\frac{d}{dx} y^7 = 7y^6 \frac{dy}{dx}$$

6. Now consider the last answer above: i.e.: $\frac{d}{dx} y^7 = 7y^6 \frac{dy}{dx}$.

You should see that dy/dx is a factor of the derivative and must be explicitly written down. Why is dy/dx a factor of the answer?

Chain rule

7. In the foregoing steps y was explicitly denoted to be a function of x . That is, it was declared at the outset that $y = f(x)$. However, in many cases the fact that y is a function of x is not explicitly stated but instead it is implied by the nature of the algebraic relationship involved. In such cases y is said to be "implicitly" defined as a function of x . For example: $x^2 + y^2 = r^2$. Clearly y is not denoted as an explicit function of x , but surely y can be expressed as $y = +\sqrt{r^2 - x^2}$ or $y = -\sqrt{r^2 - x^2}$, and these equations do represent y as a function of x (and for certain specified domains and ranges these equations are equivalent to $x^2 + y^2 = r^2$). Specifically which functional value y takes on is not nearly so important to us now as the fact that y can be represented as a function of x .

Would you say that y is implicitly or explicitly a function of x if:

(a) $y = 3x^2 - 9x + 1$?

explicit

$$(b) \ x^2 y^9 = y^4 + 1?$$

implicit

$$(c) \ y = \sin x - \tan x^4?$$

explicit

$$(d) \ \sin xy = 3x - 1?$$

implicit.

-
8. So, if y is explicitly represented as a function of x , one encounters no unusual problem regarding differentiation. Our concern centers on implicit representation. Let's look again at the idea. In the implicit case y is not specifically solved for in terms of x . However, the solution of y in terms of x is implied just as $x^2 + y^2 = r^2$ implies that y can be solved for in terms of x alone. That is, even though y is not solved for explicitly in terms of x , we are of the belief that given proper algebra it would be possible to transform the expressions until y is explicitly represented as a function of x : $y = f(x)$.

Solve for y explicitly as a function of x : $x + y = 1$

$$y = 1 - x$$

$$x^2 + y^2 = 1$$

$$y = \pm \sqrt{\frac{1}{x^2}} = \pm \frac{1}{x}$$

9. Do you see? Even though y is not explicitly represented as a function of x , we have a right to think that this could be the case if enough algebra were invoked. Hence, $y = f(x)$ is implied by algebraic expressions involving x , y , and constants. And this is the key to the idea of implicit differentiation: When you see y in the expression, you have a right to think of this y as being $f(x)$.

What should you consider in place of the y in $x^7 y^6 + 5y^4 + 3 = 1$?

$$y = f(x).$$

10. Now you are ready. You should see that if y is considered to be $f(x)$ in an algebraic expression, then y^7 is thought of as $(f(x))^7$.

What is $y^{\frac{1}{2}}$ thought of as?

$$(f(x))^{\frac{1}{2}}$$

11. From here the problem becomes one of easy manipulation. To differentiate y^n one simply considers $(f(x))^n$ and applies the proper rules of differentiation to $(f(x))^n$. Thus if y is implicitly a function of x and dy/dx is desired where y^5 is encountered, one simply considers $(f(x))^5$ and differentiates accordingly to get $5(f(x))^4 \frac{d f(x)}{dx}$.

Or, as it is usually written, $5 y^4 dy/dx$.

Suppose it is implied that y is a function of x and it is desired to differentiate y^{10} . Do so, writing your answer in terms of $f(x)$.

$$10(f(x))^9 \frac{d f(x)}{dx}$$

Do so writing your answer in the usual form not involving $f(x)$.

$$10y^9 dy/dx$$

12. Now let's try our luck on some specific problems. Suppose

$$x^2 + y^2 = r^2$$

is encountered and we desire y' . Differentiating

implicitly one can directly conclude: $2x + 2yy' = 0$. Thus $y' = -x/y$ or $y' = -\frac{x}{\sqrt{r^2 - x^2}}$ where the sign is chosen to fit the

particular value of y desired.

If $x^3 + y^3 = 1$, find y' .

$3x^2 + 3y^2 y' = 0$. Thus, $y' = -x^2/y^2$. If you missed this answer go back and redo #11.

13. In case of a product such as $xy = 1$, one proceeds as follows:
 $xy' + y = 0$. Where the left hand member results directly from the product formula. Consequently:

$$y' = -y/x.$$

If $x^2y^2 = 5$, find y' . (Remember that x^2y^2 is a product.)

$$x^2 2yy' + y^2 2x = 0. \text{ Thus } y' = -2xy^2 / 2x^2y = -y/x.$$

14. Now tie the ideas above together to find y' if:

$$x^5 + 4xy^3 - 3y^5 = 2.$$

$$5x^4 + 4x3y^2y' + 4y^3 - 15y^4y' = 0.$$

$$\text{Thus } y' = \frac{-(5x^4 + 4y^3)}{12xy^2 - 15y^4}.$$

15. As a particular application of implicit differentiation find the slope of the tangent to the curve $x^2 - xy + y^2 = 7$ at $(1, 2)$.

$$2x + xy' + y + 2yy' = 0. \text{ Thus } y' = \frac{-2x-y}{x+2y}$$

Hence m at $(1, 2)$ is $-4/5$.

16. Now let's prove a theorem using implicit differentiation to show its power.

Let's prove: if $y = x^{p/q}$, then $y' = p/q \cdot x^{(p/q)-1}$ where p and q are any integers ($q \neq 0$).

First, $y = x^{p/q}$ is equivalent to $y^q = x^p$. Now differentiate implicitly, simplify your result, and complete the proof.

$$\begin{aligned} qy^{q-1}y' &= px^{p-1}. \text{ Thus } y' = \frac{px^{p-1}}{qy^{q-1}} = p/q \cdot \frac{x^{p-1}}{x^{(q-1)/q}} \\ &= p/q \cdot x^{(p-1) - (q-1)/q} \\ &= p/q \cdot x^{p/q - 1} \end{aligned}$$

17. In the previous problem where did the y' come from?

chain rule

18. Good. Now let's check our procedure by investigating from a different direction. Get y' from $x^2 + y^2 = 1$ implicitly.

 $2x + 2yy' = 0$. Thus $y' = -x/y$.

19. Now prove that $x^2 + y^2 = 1$ (which leads one to the expression

$y = \sqrt{1-x^2}$ if one considers the top half of the circle)
has $dy/dx = -x/y$ also. (Do this by explicitly differentiating
 $y = \sqrt{1-x^2}$.)

 $y' = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = -x/(1-x^2)^{\frac{1}{2}} = -x/y$.

20. Notice that both methods give the same result. But the implicit differentiation is often times faster and more accurate-- particularly in cases where non-linear equations are involved.

Which method of differentiation (explicit or implicit) would you use

to get y' if: a) $y = (x^3 - 3x)^4$?

explicit

b) $x^2y^2 = x^2 - y^2$

implicit

c) $(x+y)^3 - (x-y)^3 = x^4 - y^4$

implicit

21. As a slight extension of the foregoing arguments one can easily see the application to higher order derivatives. For example, suppose y'' is desired if

$$x^2 - xy + y^2 = 3.$$

Clearly: $2x - xy' - y + 2yy' = 0$ and thus

$$2 - xy'' - y' - y' + 2yy'' + 2y'(y') = 0.$$

$$\text{So } y'' = \frac{2 - 2y' + 2(y')^2}{x - 2y}$$

Evaluate y' and y'' when $x = 1$: $x^3y + xy^3 = 2$

$$x^3y' + y^3x^2 + x^3y^2y' + y^3 = 0 \text{ or } y' = \frac{-3x^2y - y^3}{x^3 + 3xy^2}.$$

Since $x = 1$ means $y = 1$, $y' = -4/4 = -1$.

$$\begin{aligned} \text{For } y'': \quad & x^3y'' + y^3x^2 + y^6x + 3x^2y' + 3xy^2y'' + 3y'(xy^2)' + 3y^2y' = 0; \\ \text{or } & x^3y'' + 3x^2y' + 3x^2y' + 6xy + 3xy^2y'' + 6xy(y')^2 + 3y^2y' + 3y^2y' + \\ & 3y^2y' = 0; y'' = 0. \end{aligned}$$

22. As a final consideration of implicit differentiation observe that one can differentiate $y = f(x)$ explicitly and obtain $dy/dx = f'(x)$ or one can differentiate the same function $y = f(x)$ implicitly and obtain $1 \cdot dy/dx = f'(x)$.

Notice that both results are equivalent.

Differentiate $y = x^2$ both implicitly and explicitly.

Explicit: $dy/dx = 2x$.

Implicit: $1 \cdot dy/dx = 2x$

But herein lies an important insight - you see, all explicit differentiation can indeed be done implicitly! Thus all differentiation - be it implicit in nature or explicit in nature - can be done implicitly.

Can one differentiate all explicit functions in an implicit manner?

Yes.

Lesson VIII

VELOCITY AND RELATED RATES

1. The beginning calculus student is always confronted with problems involving application of the ideas of differentiation to certain physical problems. It is the purpose of this lesson to carefully look at the method of solution of these problems and apply the method to illustrative examples - particularly we will be concerned with velocity problems and related rate problems.

This lesson covers the method of solution of two major types of problems involving the ideas of differentiation. What are these problem types?

related rate and velocity

2. Suppose a particle is moved from its initial position by a force such that the distance the particle is from its initial position is entirely dependent upon time. Letting s be the distance in question and t the elapsed time, you can see that $s = f(t)$.

Indeed, suppose the distance the object is from its initial position is k times the square of the elapsed time in seconds. Under this hypothesis represent the particular distance the object is from its initial position at the end of t seconds.

 $s = k t^2$

3. In the case of a freely falling body the constant of proportionality, k , becomes 16. Now represent the distance a freely falling body falls in t seconds.

 $s = 16 t^2$

4. Now let's turn our attention to an important problem. Suppose we desire to find the velocity of a freely falling body at a particular instant - say when $t = 1$.

Perhaps we could (using the fact $s = 16t^2$) come up with the following analysis:

We could calculate the average velocity over any given time interval. Then we could cut the time interval smaller and smaller such that the average velocity approaches in fact the exact velocity at that point. How does one calculate average velocity if the distance, s , and time, t , are known? (Remember: distance = rate $\times$ time.)

$$v = s/t$$

5. OK. Let's start our approximation. Find out the total distance traveled during the time interval: 1 second to 2 seconds. (Remember: $s = 16t^2$.)

$$s = 16(2)^2 - 16(1)^2 = 48$$

6. Since distance traveled after 1 second represent increments on the distance, let's call these increments Δs . If Δt corresponding represents the time interval traversed, what expression would best represent the average velocity over this given time interval?

$$\text{avg vel.} = \Delta s / \Delta t$$

7. Calculate the average velocity over the time interval 1 second to 2 seconds.

$$\Delta s / \Delta t = 48 / 1 = 48$$

8. Now calculate Δs if the time interval is decreased from the interval 1 second to 2 seconds to the interval 1 second to 1.1 seconds.

$$\Delta s = 16(1.1)^2 - 16(1)^2 = 19.36 - 16 = 3.36$$

Now calculate the average velocity over this interval.

$$\text{avg vel.} = \Delta s / \Delta t = 3.36 / .1 = 33.6$$

9. Let's tabulate our results in a table to facilitate quick and easy observation. Fill in the blank part of the table.

s for t = 1	t + Δt	s + Δs	Δs	Δt	$\frac{\Delta s}{\Delta t}$
16 (1^2) = 16	1+1	16(2^2) = 64	64-16=48	1	48/1
16	1.1	19.36	19.36-16	.1	= 33.6
16	1.01	16.3216	16.3216-16	.01	= 32.16
16	1.001				
		16.032016	.032016	.001	32.016

10. If we keep this up in the manner we have started, do you see we will be forcing

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} ? \quad \text{What is} \quad \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} ?$$

 ds/dt (which in this case turns out to be 32)

11. In fact you see that ds/dt is the exact velocity when evaluated at $t = 1$.

What quotient represents the average velocity over a time interval Δt ?

 $\Delta s / \Delta t$

12. Hence, the secret of exact velocity problems becomes one of evaluation of ds/dt when $s = f(t)$.

Calculate the exact velocity of a freely falling stone at the end of one second by using ds/dt if $s = 16t^2$.

 $ds/dt = 32t = 32 \text{ ft/sec at } t = 1$

13. Suppose a freely falling body is inhibited in its flight by the presence of a more dense atmosphere which retards its descent 5 ft/sec. Incorporate this factor into the mathematical representation of the distance traversed in t seconds by a freely falling body ($s = 16t^2$).

 $s = 16t^2 - 5t$

14. Using this new formula find the exact velocity of the falling body after 3 seconds.

 $ds/dt = 32t - 5 = 91 \text{ ft/sec.}$

15. You see that in each of the foregoing consideration s was dependent upon t - - - $s = f(t)$ - - - and the velocity was in fact the exact rate of change of s with respect to t .

What is exact velocity if $s = f(t)$?

Velocity is the exact rate of change of s with respect to the change in t .

$$\text{exact velocity} = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = \frac{ds}{dt}$$

16. Now instead of a freely falling body let's consider the flight of a projectile. Suppose the object is pushed upward with an initial velocity of 32 ft/sec. With no gravity or air resistance, how far upward would this object rise after 1 second?
-

32 feet

2 seconds?

64 feet

t seconds?

32 t feet

17. Now we have the positive vertical component of the distance from ground zero, 32 t . But we must consider gravity for surely gravity will pull it back towards earth. But with what force? (When it falls back to earth, it is a freely falling body.)
-

the force of the freely falling body: $-16t^2$

18. Putting the components of the discussion together we see that the positive vertical component is 32 t and the negative vertical component is $16t^2$. You can now represent the height of the object at any time t after launching. Do so.
-

$$s = 32t - 16t^2$$

19. However, we can still represent the velocity at any given time t by ds/dt . Can we use this velocity for any purpose? Let's see. As the object is climbing surely the velocity is decreasing, and as the object is descending surely the velocity is increasing. But what must be true of the velocity at the exact instant the object is at the top of its flight?

The velocity must be zero.

20. What condition must be true if $ds/dt = 0$?

The object must be at the top of its flight.

21. So, if you force $ds/dt = 0$, you are in effect forcing a restriction on t . You are insisting that at $t = t_0$ seconds $ds/dt = 0$ or the velocity be zero. For what value of t is $ds/dt = 0$ if $s = 32t - 16t^2$?

 $t = 1$ since $32 - 32t = ds/dt = 0$.

22. So at what time after launch is the object in the previous argument at the top of its flight?

 $t = 1$

23. As a corollary to the problem, suppose you wish to know the maximum height attained by the object. You can see that this is easily calculated by imposing the time of maximum height on $s = 32t - 16t^2$. Do so.

 $s = 32(1) - 16(1^2) = 16$

24. Now let's be more general. Suppose the initial velocity is not $+32$ ft/sec but instead is v_0 . If again the negative vertical component is gravity alone, represent the height of the object above ground zero at time t .

 $s = v_0 t - 16t^2$

25. Using $s = v_0 t - 16t^2$ calculate the time at which the maximum height is reached.
-

$ds/dt = v_0 - 32t$. Thus at $t = v_0/32$ the maximum height is reached.

26. Now calculate the maximum height reached when $t = v_0/32$.
-

$$s = v_0(v_0/32) - 16(v_0/32)^2$$

27. Hence, if one needs to have the object rise to a height h_0 , a meaningful question is: with what initial velocity must the object be propelled upward? Find this velocity. (Remember: $s = v_0 t - 16t^2$ is the distance above ground zero.)
-

$$s = h_0 = v_0(v_0/32) - 16(v_0/32)^2 \text{ so } v_0^2 = 64 h_0 \text{ or } v_0 = 8\sqrt{h_0}$$

28. The answers to two interesting questions can be obtained with little additional effort. First, at what velocity would the object strike ground zero? Second, how much time is involved in the ascent? the descent?

To find the answer to the first of these questions consider the magnitude of s when the object strikes ground zero if $s = v_0 t - 16t^2$. Then calculate the time which would elapse before the object returns to ground zero. Do this.

$0 = v_0 t - 16t^2$ means $t = 0$ or $t = v_0/16$. Thus at $t = v_0/16$ the object returns to ground zero.

However, recall that ds/dt is always $v_0 - 32t$. Hence, what would be the velocity at the instant of impact?

Since $ds/dt = v_0 - 32t$, ds/dt at the instant of impact ($t = v_0/16$) is $v_0 - 32(v_0/16) = -v_0$.

29. Do you see! The impact velocity is always the same in magnitude as the launch velocity! Now go after the second question of (28).
-

$t = 0$ and $t = v_0/16$ are the times when the object is at ground zero. Hence, $v_0/16$ seconds elapse over the complete flight. Simply calculate the time when the object is at the apex of the flight:

$$ds/dt = 0 = v_0 - 32t.$$

But this means $t = v_0/32$ is the time elapsed in the upward flight. Consequently $v_0/16 - v_0/32 = v_0/32$ is the time consumed in the return flight.

30. How about that! Not only is the initial velocity the same in magnitude as the terminal velocity but the time for ascent is the same as that of descent! Now for a final consideration. Suppose the projectile is launched not at ground zero but at a height h feet above ground zero. Now you must insist that $s = h$ when $t = 0$. How can you do this if the initial velocity again is v_0 and the pull of gravity is still present? Do so.
-

$$s = v_0 t - 16t^2 + h$$

31. Attack the questions in order. How high will the projectile rise?
-

This question is equivalent to: when is the velocity 0?

Thus $ds/dt = 0 = v_0 - 32t$.

Consequently, $t = v_0/32$ is the time of maximum height.

So $s = v_0(v_0/32) - 16(v_0/32)^2 + h$ is the maximum height reached.

32. How should the answer in (31) compare with the answer to the corresponding question if the projectile is fired from ground zero?
-

When fired from ground zero the maximum height should be h feet less than when fired from h feet above ground zero. Check the response in (26).

33. Of course, the discussion above has been entirely centered about projectiles, etc. However, the application is not restricted there. Indeed, the idea is much more far reaching because thousands of problems identical in nature to this consideration merely await the investigation of the reader.

Let's take a few examples and see how the nature of solution is identical to the thinking we've just experienced.

First, consider the growth of a plant. Suppose the growth is dependent entirely upon time. Notice that $G = f(t)$.

In fact let's insist (for the sake of a problem) that the growth is twice the square of the time diminished by $\frac{1}{2}$ the cube of the time. (perhaps the natural growth characteristics account for the $2t^2$ and the soil food depletion accounts for $\frac{1}{2}t^3$.)

Represent the growth of the plant, G , at any given time t .

$$G = 2t^2 - \frac{1}{2}t^3.$$

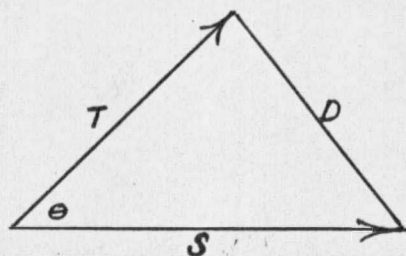
34. Now calculate the time at which the plant ceases to grow.

$$dG/dt = 4t - 3t^2/2 = 0. \text{ Thus the plant ceases to grow when } t = 8/3.$$

35. Assuming G is measured in centimeters, how tall does the plant get to be?

$$G = 2(8/3)^2 - \frac{1}{2}(8/3)^3.$$

36. You see! The problem is exactly the same as the one involving projectiles! Now try one involving a little geometry and trigonometry. Suppose a ship launches a torpedo at an angle θ from the course of the ship. Take the speed of the torpedo to be 60 ft/sec and the speed of the ship to be 40 ft/sec. The question we ask is: how fast are torpedo and ship separating after five seconds? First, draw a figure and label the relevant facts.



S = Distance ship has traveled
 T = Distance torpedo has traveled
 D = Distance from ship to torpedo

37. OK. In most related rate problems, such as this, it is usually to the advantage of the problem solver to first draw a figure representing the physical facts, label the figure with convenient notation, and then answer the question "what am I looking for?" in terms of the figure. In this case if D is the distance between the torpedo and the ship, in mathematical symbolism just what is it you are after?

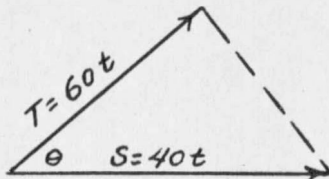
 dD/dt

38. Now that you know what it is that you are after, let's proceed to get it in an orderly manner. Obviously the question is: how could I get dD/dt ? The answer lies at first in representing the physical situation in terms of a mathematical relationship.

Let's take stock of what we know. First, the angle θ is fixed for a particular problem. But what of the distance the torpedo and the ship travel? These distances constantly change - but in a specified manner. Can you represent the distance the ship and the torpedo are from launch point at any given time? Do so.

 S = distance of ship from launch point = $40t$, t in seconds
 T = distance of torpedo from launch point = $60t$, t in seconds

39. Indeed, in this case you need to know the relationship of three sides of a triangle when two sides and the included angle are known.



First, is the triangle well defined? That is, is one and only one triangle defined for θ , S , and T ?

 Yes

40. But there is a trigonometric fact which relates D , S , T , and θ . It is the law of cosines: $a^2 = b^2 + c^2 - 2bc \cos A$. Represent our problem by stating the relationship θ , S , T , and D have to one another.

$$D^2 = S^2 + T^2 - 2ST \cos \theta$$

41. Expressing the relationship in terms of time alone one has

$$D^2 = (40t)^2 + (60t)^2 - 2(40t)(60t) \cos \theta$$

42. Remember, θ is a constant for a given launching. Now you should be able to get dD/dt . Do so. (Differentiate implicitly.)

$$(2D) dD/dt = 2(40t)40 + 2(60t)60 - 2 \cos \theta [40t(60) + 60t(40)]$$

$$\text{or } dD/dt = \frac{40t(40) + 60t(60) - \cos \theta (2400t + 2400t)}{D}$$

$$= \frac{1600t + 3600t - 4800t \cos \theta}{\sqrt{(40t)^2 + (60t)^2 - 2(40t)(60t) \cos \theta}}$$

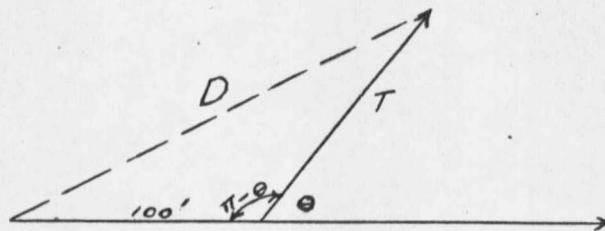
$$= \frac{5200t - 4800t \cos \theta}{\sqrt{5200t^2 - 4800t^2 \cos \theta}}$$

$$= \sqrt{5200 - 4800 \cos \theta}$$

43. But dD/dt represents the rate at which the distance between the torpedo and ship are separating. Since the value dD/dt is independent of t (i. e. is constant), how fast are the ship and the torpedo separating after 5 seconds?

$$\sqrt{5200 - 4800 \cos \theta} \text{ ft/sec}$$

44. Now let's slightly modify the problem. Suppose the ship drops a marker in the sea, travels 100 feet, and fires the torpedo as before. Now the question becomes: how fast are the torpedo and the stationary marker separating at any given time - say one second later?
Draw a figure representing the physical situation.
-



45. Again take D to be the distance from the torpedo to marker, and again use the law of cosines to establish the proper relationship among the variables.
-

$$D^2 = (60t)^2 + (100)^2 - 2(100)(60t) \cos (180^\circ - \theta)$$

46. Since $\cos (180^\circ - A) = -\cos A$, the relationship in (45) can be simplified to:
-

$$D^2 = (60t)^2 + (100)^2 + 2(100)(60t) \cos \theta$$

47. Again you want dD/dt , so go get it. (Differentiate implicitly.)
-

$$2D \, dD/dt = 2(60t)60 + 2(100)60 \cos \theta.$$

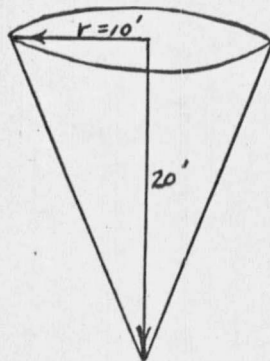
$$\text{So } dD/dt = (60t)60 + (6000) \cos \theta$$

$$\sqrt{(60t)^2 + (100)^2 + 60t(100) \cos \theta}$$

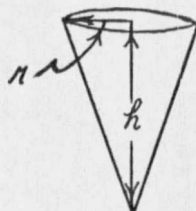
48. Notice that this time $dD/dt = f(t)$ is not a constant as was experienced before. So the question is: what is dD/dt at the instant $t = 1$? Do you see? It is merely $f(1)$. Find dD/dt at $t = 1$.
-

$$dD/dt = \frac{3600 + 6000 \cos \theta}{\sqrt{10000 + 3600 + 6000 \cos \theta}}.$$

49. Finally let's try one more type of problem. Suppose a fluid is being poured into a conical reservoir --- the dimensions of which are indicated in the diagram. If the fluid runs into the tank at the rate of 3 cu. ft/min, we want to know how fast the fluid level is rising. (See figure on next page)



Since you are interested in the volume at an arbitrary time, you probably will want to look at a figure depicting the volume of the fluid at an arbitrary time. Such a figure is:



Now using this figure, represent the volume in terms of r and h .

$$V = \pi/3 r^2 h$$

50. Now in terms of the variables in this formula, what do you want?

$$dh/dt$$

51. At this stage of the game you can proceed one of two ways. Let's take the direct route first. Since the radius of the top of the cone is 10 feet and the depth of the cone is 20 feet, what is the ratio of r to h ? (Notice that this ratio remains constant for any depth of fluid.)

$$r/h = \frac{1}{2}$$

52. Thus you should be able to solve for r in terms of h . Do so.

$$r = h/2$$

53. But this gives you a way to represent V , the volume of fluid, as a function of h alone. Do so.

$$V = (\pi/3)(h/2)^2 h = (\pi/12) \cdot h^3$$

54. Hence $V = f(h)$. Can you proceed to get dh/dt from the above formula? Do so.

$$dV/dt = (\pi/4)h^2 dh/dt \text{ or } dh/dt = 4/(\pi h^2) \cdot dV/dt$$

55. You are in the home stretch. What is the numerical value of dV/dt ?

$$dV/dt = 3$$

56. Now calculate dh/dt at the instant $h = 5$.

$$dh/dt = (4/25\pi) \cdot 3 = 12/25\pi$$

57. Going back to $V = \pi/3 r^2 h$, you should notice that V , r , and h are all functions of t . Thus you see that each of these variables is differentiable in terms of t . Now differentiate $V = (\pi/3)r^2 h$ implicitly using the product formula on the right side.

$$\frac{dV}{dt} = \frac{\pi}{3} (r^2 \frac{dh}{dt} + 2r \frac{dr}{dt} h).$$

58. Again call on geometry to verify $r = h/2$. But can you solve for dr/dt in terms of dh/dt using $r = h/2$? Do so.

$$dr/dt = 1/2 dh/dt$$

59. By combining the results of 57 and 58 you are now able to calculate dh/dt at any instant. Do so when $h = 5$.

$$3 = \pi/3 \left[\left(\frac{5}{2} \right)^2 \cdot \frac{dh}{dt} + 2 \left(\frac{5}{2} \right) \left(\frac{1}{2} \right) \cdot \frac{dh}{dt} \cdot 5 \right]$$

or $dh/dt = 12/25\pi$. The results thus prove both methods to be of equal effectiveness.

60. To summarize the method of solution of related rate problems:
1. Draw a figure representing the physical situation and label it appropriately.
 2. Decide what result you want in terms of the figure and formulate this result in terms of a derivative.

3. Select mathematical relationships which involve your given facts and will lead to the result you desire.
4. Differentiate the proper mathematical relationships so as to involve the rates of change.
5. Substitute into the formula derived from the differentiation such that proper initial conditions are satisfied.

Using no more than two or three words for each step, reproduce the main idea of the above steps.

-
1. Sketch figure.
 2. Identify desired result.
 3. Get math relationships.
 4. Differentiate.
 5. Substitute for conditions.
-

Lesson IX

INCREASING AND DECREASING CONTINUOUS FUNCTIONS

1. The knowledge of the manner in which functions behave is of crucial importance to one who must be able to think and reason in terms of functions. Consequently, this lesson is designed to cover some major aspects of functional behavior.

This lesson is written to instruct you on some of the major aspects of

functional behavior

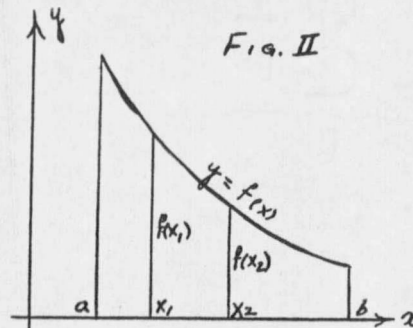
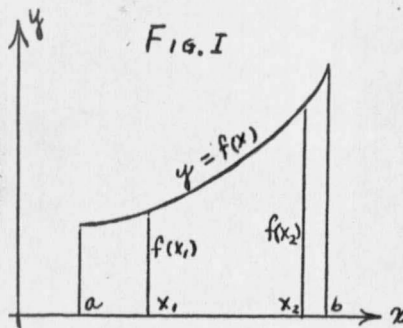
2. To start with let's straighten out some notations and definitions. First, there is the open interval and the closed interval. The open interval from a to b is denoted (a, b) whereas the closed interval from a to b is denoted $[a, b]$. Algebraically (a, b) means all real numbers, x , from a to b exclusive of each: $a < x < b$; and $[a, b]$ means all real numbers, x , from a to b inclusive of each: $a \leq x \leq b$.

Using these straight lines as number lines sketch (a, b) and $[a, b]$.

(a, b) : A horizontal line segment with endpoints labeled 'a' and 'b'.

(a, b): ; $[a, b]$

3. Now what is a decreasing function or an increasing function? Let's define an increasing function and the other should be easy. Suppose we have $y = f(x)$ defined over $[a, b]$. Then we say that $y = f(x)$ is an increasing function over $[a, b]$ if and only if for all x_1 in $[a, b]$ and all x_2 in $[a, b]$ where $a \leq x_1 < x_2 \leq b$ it follows that $f(x_2) > f(x_1)$. Technically it sounds tough but consider the geometric counterpart (figure 1).



Now you look at figure II and define a decreasing function over (a, b) .

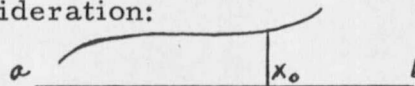
$y = f(x)$ is a decreasing function over (a, b) if and only if for all x_1 in (a, b) and all x_2 in (a, b) , $f(x_2) < f(x_1)$ whenever $x_1 < x_2$.

4. Once we have a definition of an increasing or decreasing function over an interval it is an easy matter to nail down what we mean by an increasing or decreasing function at a point. Let's do it. We say $y = f(x)$ is a decreasing function at a point x_0 if and only if there exists a positive real number δ such that $f(x)$ is decreasing over $(x_0 - \delta, x_0 + \delta)$.

Now you define an increasing function $f(x)$ at the point x_0 .

$f(x)$ is an increasing function at x_0 if and only if there exists a positive real number δ such that $f(x)$ is increasing over $(x_0 - \delta, x_0 + \delta)$.

5. What we really need now is a convenient test which we can impose on a function $y = f(x)$ to see if it is increasing or decreasing at a point. So let us design such a test. Again seek the test through the geometric consideration:



Do you see that according to our definition of increasing functions at a point that we really want the slope to be positive? Or stated slightly differently, we want the _____ to be positive at that point.

derivative

6. Indeed, it would seem that a positive derivative at a point would imply an increasing function. Let's try to prove this. Suppose $y = f(x)$ is defined over (a, b) where $a < x_0 < b$ and $f'(x_0) > 0$.

Let's prove $y = f(x)$ is increasing at x_0 .

What are the hypothesis and conclusion of this theorem? What is "given" and what is "to be proven"?)

Hypothesis: 1. $y = f(x)$ is defined over (a, b) where $a < x_0 < b$
 2. $f'(x_0) > 0$

Conclusion: $y = f(x)$ is an increasing function at x_0 .

7. But $f'(x_0) > 0$ means: (1) $\lim_{h \rightarrow 0^+} \frac{f(x_0+h) - f(x_0)}{h} > 0$ and
 (2) $\lim_{h \rightarrow 0^-} \frac{f(x_0+h) - f(x_0)}{h} > 0$.

In the first case $h > 0$ means $f(x_0+h) - f(x_0) > 0$ too.

But what does this mean as far as the relative size of $f(x_0+h)$ and $f(x_0)$ are concerned?

$$f(x_0+h) > f(x_0).$$

8. In case $h < 0$, what is the condition imposed on the relative size of $f(x_0+h)$ and $f(x)$ if $\lim_{h \rightarrow 0^-} \frac{f(x_0+h) - f(x_0)}{h} > 0$?

$$f(x_0+h) < f(x_0).$$

9. Now put the results of steps 5, 6, 7, and 8 together and state the size of interval which is insisted on in step 4 and which is guaranteed by:

$\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h}$ exists and is positive. Remember:

$\lim_{x \rightarrow a} f(x) = f(a)$ if and only if there exists a $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \quad (\text{arbitrary but positive}) \text{ whenever } 0 < |x - a| < \delta.$$

We are assured of the interval: $(x_0 - \delta, x_0 + \delta)$. Thus the interval is 2δ units wide.

10. You see! It all goes back to the definition of a limit. That is, if $\lim_{h \rightarrow 0^+} \frac{f(x_0+h) - f(x_0)}{h} > 0$, this means there exists an interval to

the right of x_0 where $\frac{f(x_0+h) - f(x_0)}{h} > 0$ (in fact we are sure this interval is δ units wide where the δ is the same δ guaranteed in the definition of the limit.) Certainly, if $\frac{f(x_0+h) - f(x_0)}{h} > 0$

and $h > 0$ this means $f(x_0+h) - f(x_0) > 0$ or $f(x_0+h) > f(x_0)$.

Over what interval are we sure that $f(x_0+h) > f(x_0)$ if

$$\lim_{h \rightarrow 0^+} \frac{f(x_0+h) - f(x_0)}{h} > 0?$$

 $(x_0, x_0 + \delta)$.

Correspondingly $\lim_{h \rightarrow 0^-} \frac{f(x_0+h) - f(x_0)}{h} > 0$ implies the existence

of an interval to the _____ of x_0 , the width being _____,

such that $\frac{f(x_0+h) - f(x_0)}{h} > 0$ also. But on this side of x_0 the

denominator is _____ and hence the numerator $f(x_0+h) - f(x_0)$ is _____. Which means _____ is smaller than

 left, δ , negative, negative, $f(x_0+h)$, $f(x_0)$.

11. All in all it is clear there exists an interval of δ units wide on each side of x_0 where $f(x)$ is increasing.

How wide is the interval on each side of x_0 where $f(x)$ is known to be increasing?

 δ

12. Clearly we have analytically proven that a positive derivative at a point implies the condition of an increasing function at that point! True, you may say that the whole argument is pointless since obviously a positive slope indicates an increasing function. But you are using intuition - not reason. Solid, logical reason must be used to prove the fact. Don't be led to accept even the seemingly obvious without proof! The person who accepts the above theorem without proof as being obvious is using _____ and is not depending on the cold, hard truth which _____ delivers.

intuition, logic or reason

13. Of course, the argument for decreasing functions is parallel to that for increasing functions. Recall that:

given: (1) $f(x)$ is defined over (a, b) where $a < x_0 < b$
(2) $f'(x_0) > 0$

results in: $y = f(x)$ is an increasing function at x_0 .

Now you state the "given" and "to prove" of the corresponding case for a decreasing function at x_0 .

Given: (1) $y = f(x)$ is defined over (a, b) where $a < x_0 < b$
(2) $f'(x_0) < 0$

to prove: $y = f(x)$ is a decreasing function at x_0 .

14. Since the proof to the suggested theorem is exactly parallel to the one previously given, we'll not retrace the steps. Let's merely apply our understanding of increasing and decreasing functions to entire intervals instead of merely at points - such as x_0 . To do this we merely demand that for $f(x)$ to be increasing over (a, b) it must be increasing at every point x_i where $a < x_i < b$. How would you insist that $f(x)$ be decreasing over (a, b) ?

 $f(x)$ must be decreasing at every point x_i where $a < x_i < b$.

15. You see! We first got increasing functions at a point and then we expanded the thinking to cover an interval, just as we did with the derivative and just as we did with continuity! Very frequently this is the pattern with analysis of functions: first we get a property of the function at a point and then we expand the property to cover an interval.

What is a common approach when one attempts to prove a property of functions?

First get the property at a point. Then expand the property to cover an interval.

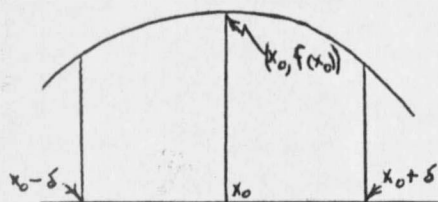
16. Now let's move on to another topic. Let's take a look at some of the critical values which a function may achieve. Indeed, consider the extreme values which may be assumed by the function. In order to be certain of precise thinking let's carefully define what we mean.

By a relative (or local maximum) we shall mean any one of the ordinates of $y = f(x)$ which obeys the following prerequisite:

if $f(x_0)$ is a relative maximum, there must exist some $\delta > 0$ such that $f(x_0) \geq f(x)$ for all x in a δ -neighborhood of x_0 .

That is, there must exist a δ -neighborhood about x_0 such that $f(x_0) \geq f(x)$ for all x in the δ -neighborhood of x_0 .

As a simple graphical illustration consider:



$f(x_0) \geq f(x)$ for all x in a δ -neighborhood of x_0 .

Now you define a relative minimum.

$f(x_0)$ is a relative minimum if there exists some $\delta > 0$ such that $f(x_0) \leq f(x)$ for all x in a δ -neighborhood of x_0 .

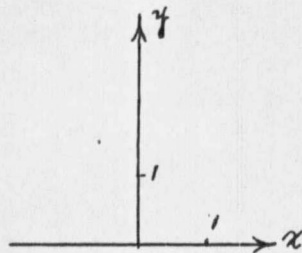
17. As a final thought consider $f(x)$ to be defined over the interval $[a, b]$. Certainly $f(x)$ may have an ordinate at $x = a$ which is greater than the ordinates to its right. We would thus like to consider $f(a)$ to be a relative maximum. Therefore let us amend our definition in (16) to include the special case of a relative maximum at $x = a$: in case $f(a) \geq f(x)$ for all x in a right-handed δ -neighborhood to $x = a$, then $f(a)$ is denoted to be a relative maximum.

Now you state an amendment which will cover the case where $f(b)$ is a relative maximum. (The situation where $f(b)$ is a relative minimum is similarly worded.)

In case $f(b) \geq f(x)$ for all x in a left-handed δ -neighborhood to $x = b$, then $f(b)$ is denoted to be a relative maximum.

18. Consider a few preliminary questions to be sure you have the idea well in mind.

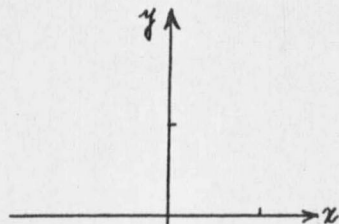
(a) for $y = x^2$ over $[-1, 1]$ what is a relative minimum?
(Sketch it.)



(b) for $y = x^2$ over $(-1, 1)$ what is a relative minimum?

0

(c) for $y = x$ over $[0, 1]$, what is a relative maximum? (Sketch it.)



1

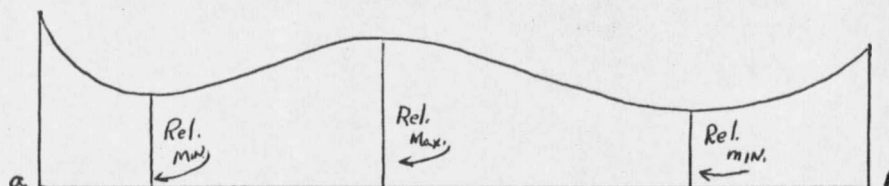
(d) for $y = x$ over $(0, 1)$, what is a relative maximum? Careful!

There is none since for $0 < x_0 < 1$ there always exists x_1 where $0 < x_0 < x_1 < 1$ and $f(x_1) > f(x_0)$.

(e) for $y = x^2$ over $[-1, 1)$, is there a relative maximum and where does it occur?

relative maximum is at $x = -1$ and is 1

19. Now we are ready to state and prove a theorem. Suppose $f(x)$ is differentiable over (a, b) . (Certainly this implies continuity over (a, b) .) Suppose our function $y = f(x)$ does in fact achieve a relative maximum or minimum somewhere in (a, b) . Consider this graphically. What would happen to the slope at the relative maximum or minimum?



The slope should be zero.

20. Now consider the converse: if the slope is zero, the curve should achieve a relative maximum or relative minimum. This would intuitively seem to be true at first glance. Furthermore, this would be a good way to analyze a function if it would work. So, let's see if we can get it. Complete the theorem you wish to have: if $y = f(x)$ has a continuous derivative over (a, b) and if $f'(x_0) = 0$ for some $a < x_0 < b$, then . . .

 $f(x_0)$ is a relative maximum or minimum.

21. But hold on. The theorem may not be true! It takes only one counter example to prove its invalidity. Consider $y = x^3$. Surely $y' = 3x^2$ is zero at $x = 0$. But what about the slope of $y = x^3$ for all non-zero x ?

slope of $y = x^3$ is always positive except at $x = 0$.

22. Thus did the theorem of (20) hold? Certainly $f'(0) = 0$.

Theorem did not hold. $f(0)$ is not a relative maximum or minimum.

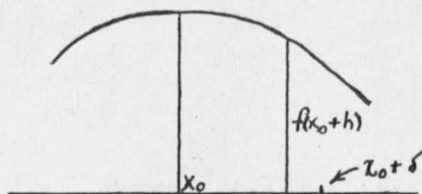
23. Well, we lost our intended theorem, but don't lose faith. Perhaps we can salvage the converse. Finish the statement of it: if $y = f(x)$ achieves a relative maximum (minimum) at x_0 when $f(x)$ has a continuous derivative over (a, b) , then $f'(x_0) = \underline{\hspace{2cm}}$ and $f(x_0)$ is the .

0, relative maximum (minimum)

24. Now for the proof. Suppose $f(x)$ achieves a relative maximum at x_0 . This means: $f(x_0+h) - f(x_0) \leq 0$ for all $h < \delta$ where the δ is the same δ guaranteed by the derivative. Now consider:

$$(1) \lim_{h \rightarrow 0^+} \frac{f(x_0+h) - f(x_0)}{h} \quad \text{and} \quad (2) \lim_{h \rightarrow 0^-} \frac{f(x_0+h) - f(x_0)}{h}.$$

In case (1): $\frac{f(x_0+h) - f(x_0)}{h} \leq 0$. Why? Remember: $h \rightarrow 0^+$.



$$f(x_0+h) - f(x_0) \leq 0 \text{ and } h > 0 \text{ implies } \frac{f(x_0+h) - f(x_0)}{h} \leq 0$$

25. Thus $\lim_{h \rightarrow 0^+} \frac{f(x_0+h) - f(x_0)}{h} \leq 0$. Now focus on case (2):

$$\lim_{h \rightarrow 0^-} \frac{f(x_0+h) - f(x_0)}{h}. \quad \text{Certainly } f(x_0+h) - f(x_0) \leq 0 \text{ again.}$$

But this time $h < 0$. So what do you conclude about $\frac{f(x_0+h) - f(x_0)}{h}$?

$$\frac{f(x_0+h) - f(x_0)}{h} \geq 0.$$

26. Now what statement can you make regarding $\lim_{h \rightarrow 0^-} \frac{f(x_0+h) - f(x_0)}{h}$?
-

$$\lim_{h \rightarrow 0^-} \frac{f(x_0+h) - f(x_0)}{h} \geq 0.$$

27. But in the general derivative the left hand limit and right hand limit must be the same! Consequently, what can you conclude with respect to $\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h}$?
-

$$\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} = 0$$

28. Stated slightly differently what must you conclude about $f'(x_0)$?

 $f'(x_0) = 0$

29. OK. Let's review our learning. (Assume $f(x)$ to be differentiable over (a, b) and $a < x_0 < b$.)

1. If $f'(x_0) = 0$, this does not necessarily mean $f(x_0)$ is a relative maximum (or minimum).
2. If $f(x_0)$ is a relative maximum (or minimum) then $f'(x_0) = 0$.

If $f'(x_0) = 0$, certainly it could be true that $f(x_0)$ is a relative maximum (or minimum). Give an example of such a case.

 One example would be $y = x^2$ over $(-1, 1)$.

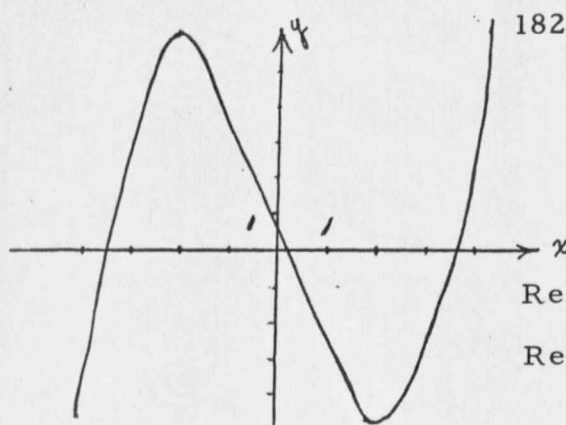
On the other hand if $f'(x_0) = 0$, it may not be true that $f(x_0)$ is a relative maximum (or minimum). Give an example of such a case.

 $y = x^3$ over $(-1, 1)$.

However, one can always claim that if $f(x_0)$ is a relative maximum or minimum then _____.

 $f'(x_0) = 0$.

30. Now let's apply what we've learned to a few problems. If given $y = f(x)$ over $[a, b]$, our first inclination is to differentiate so as to get $f'(x)$. By setting $f'(x) = 0$ we may find some critical points - but each has to be checked out. Finally, we check $f(a)$ and $f(b)$ for relative maximum and relative minimum values. Using this as a basis find: all relative maximums and all relative minimums of $y = x^3/3 - 4x + 2/3$ over $[-4, 4]$. Sketch the curve.
-



Rel. max: 6 (occurs at $x = -2, 4$)

Rel. min: $-14/3$ (occurs at $x = -4, 2$)

31. Modify your answers in (30) if the interval is changed to $(-4, 4)$.

Rel. max: 6 (occurs at $x = -2$ only)

Rel. min: $-14/3$ (occurs at $x = 2$ only)

32. Modify your answers in (30) if the interval is changed to $[-3, 4]$.

Rel. max: 6 (occurs at $x = -2, 4$)

Rel. min: $3 \frac{2}{3}, -14/3$ (occurs at $x = -3, 2$)

33. Modify your answers in (30) if the interval is changed to $(-3, 4]$.

Rel. max: 6 (occurs at $x = -2, 4$)

Rel. min: $-14/3$ (occurs at $x = 2$)

34. Modify your answers in (30) if the interval is changed to $(-3, 4)$.

Rel. max: 6 (occurs at $x = -2$)

Rel. min: $-14/3$ (occurs at $x = 2$)

35. Modify your answers in (30) if the interval is changed to $[-3, 3]$.

Rel. max: $-2 \frac{1}{3}, 6$ (occurs at $x = 3, -2$)

Rel. min: $3 \frac{2}{3}, -14/3$ (occurs at $x = -3, 2$)

36. So far in our discussion we have been careful to insist that $f(x)$ be differentiable over (a, b) and that our consideration of maxima and minima be restricted to this well-behaved region. However, suppose $f(x)$ is not differentiable over (a, b) . We should wonder how the maxima-minima situation could then be altered.

Consider $y = |x|$ over $(-1, 1)$. Certainly this curve is not differentiable at $(0, 0)$ and yet $f(0)$ is certainly a relative minima.

Can relative maxima (or minima) occur at a point x_0 even though $f'(x_0) \neq 0$?

Yes.

How?

If the function is not differentiable at a point, it could very easily happen that f evaluated at this point is a relative maximum (or minimum).

- 37. Give an example of a function which has a relative maximum at $(0, 0)$ but whose derivative is not zero at this point. (Hint: consider $y = |x|$.)

One such example is $y = -|x|$.

- 38. Thus, we should summarize our findings so far:

1. If $y = f(x)$ is differentiable over (a, b) and if $f'(x_0) = 0$ when $a < x_0 < b$, then $f(x_0) \dots$

need not but may be a relative maximum or minimum.

- 2. If $y = f(x)$ is differentiable over (a, b) and if $f(x_0)$ is a relative maximum or minimum when $a < x_0 < b$, then what is true of $f'(x_0)$?

 $f'(x_0) = 0$

- 3. If $f(x)$ is a continuous curve over (a, b) under what conditions could a relative maximum or minimum occur at x_0 where $a < x_0 < b$ and $f'(x_0) \neq 0$?

Only if $f(x)$ is not differentiable at x_0 .

39. Whereas it is true that setting the first derivative equal to zero may or may not lead one to the discovery of relative maxima or minima one should not discount the process as entirely valueless. For example if $y' = (x-3)^3 (x-4)^4 (x-5)^5$, possible candidates for relative maxima or minima are $x = 3, 4, 5$ because as x takes on these values, $f'(x) = 0$.

What are potential maxima or minima if $y' = (x-2)^3 (x-3)^4 (x+4)^5$?

 $x = 2, 3, -4$

40. Now we must see if we can devise a means whereby maxima and minima can be identified if the first derivative is known. Again consider the example: $y' = (x-3)^3 (x-4)^4 (x-5)^5$. Clearly the slope of the curve at the point x is given by y' . Now let's examine this slope around each of the points 3, 4, and 5.

Our procedure is to evaluate the slope just to the right and left of each value of x which causes the first derivative to be -----

zero.

41. Consider $x < 3$.

If $x < 3$, $x-3 < 0$. Hence $(x-3)^3 < 0$.

If $x < 3$, $x-4 < 0$. Hence $(x-4)^4 > 0$.

If $x < 3$, $x-5 < 0$. Hence $(x-5)^5 < 0$.

Thus $(x-3)^3 (x-4)^4 (x-5)^5$ is ----- than zero.

greater

42. What is the slope of the curve in question to the left of $x = 3$?

 $m > 0$

43. Now for $3 < x < 4$.

If $x > 3$, $x-3 > 0$. Hence $(x-3)^3 > 0$.

If $x < 4$, $x-4 < 0$. Hence $(x-4)^4 > 0$.

If $x < 4$, $x-5 < 0$. Hence $(x-5)^5 < 0$.

Thus $(x-3)^3 (x-4)^4 (x-5)^5$ is ----- than zero.

less

44. What is the slope of the curve in question to the immediate right of $x = 3$?

 $m < 0$

45. If the slope to the left of $x = 3$ is positive, to the right of $x = 3$ is negative, and is zero at $x = 3$, what do you conclude must be the nature of the curve at $x = 3$?

 The curve must achieve a relative maximum at $x = 3$.

46. Now let's try $x = 4$.

If $3 < x < 4$, we have already found the slope to be negative from (43).

If $4 < x < 5$: $(x-3)^3 > 0$, $(x-4)^4 > 0$, $(x-5)^5 < 0$.

Thus the slope of the curve to the right of $x = 4$ is -----.

 negative

47. But if the slope to the right of $x = 4$ is negative, to the left of $x = 4$ is positive, and the slope is zero at $x = 4$, what can we conclude about the nature of the curve at $x = 4$?

 We cannot assert that a relative maximum or minimum is achieved at $x = 4$.

48. What about the slope to the right of $x = 5$?

 The slope to the right of $x = 5$ is positive.

49. What can you conclude about the nature of the curve at $x = 5$?

 The curve has a relative minima at $x = 5$.

50. Use the foregoing method to discover relative maxima and minima if $y' = (x-2)^3 (x-3)^4 (x+4)^5$.

Since m at $x < -4$ is positive and m at $-4 < x < 2$ is negative, the curve in question must achieve a relative maximum at $x = -4$.

Since m at $2 < x < 3$ is positive, the curve in question must achieve a relative minimum at $x = 2$.

Since m at $x > 3$ is positive, we cannot assert the curve achieves either a relative maximum or minimum at $x = 3$.

51. So - despite the fact that setting the first derivative equal to zero may be fruitless, in attempting to identify relative maximum or minimum points, an analysis based on the first derivative may sometimes clearly identify such critical points. Hence we should make certain we clearly understand the role of the first derivative in determining relative maxima and minima.

1. Does finding those values which make the first derivative equal to zero guarantee that each of these values is either a relative max or a relative min?

no

2. If a relative max or min occurs at a point x_0 (where $a < x_0 < b$ and $f(x)$ is differentiable over (a, b)), does this mean the derivative must be zero at x_0 ?

yes

3. Can a relative max or min occur at a point and the derivative not be zero at that point?

yes

4. State two ways this could occur.

-
1. A rel. max or min could occur at the end of a closed interval where the derivative is not zero.
 2. A function may be continuous but not differentiable at a point and thus a rel. max or min could exist at the point despite the fact that the derivative is not zero at the point.

5. Can you sometimes use the first derivative to find relative max or min at points where the first derivative is zero.

yes

6. How?

Examine the slope to the right and left of the point to determine relative max or min conditions.

Lesson X

ROLLE'S THEOREM AND THE MEAN VALUE THEOREM

1. A foundation stone to a great deal of theory in the calculus is the Mean Value Theorem. It is a key to many important concepts which are basic to the calculus. Consequently we will point our investigation in this lesson towards a complete mastery of this vital theorem.

What theorem is this lesson concerned with?

Mean Value Theorem

2. However, in order to prove the Mean Value Theorem one needs a lemma. This lemma is called Rolle's Theorem. Rolle's Theorem is in fact a special case of the Mean Value Theorem and it is through Rolle's Theorem that the truth of the Mean Value Theorem can be deduced.

Upon what theorem does the Mean Value Theorem depend?

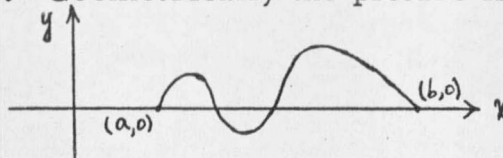
Rolle's Theorem

3. So let's start with Rolle's Theorem. Let's prove it first and then dispense with the real object - the Mean Value Theorem.

What theorem are we ultimately interested in proving in this lesson?

Mean Value Theorem

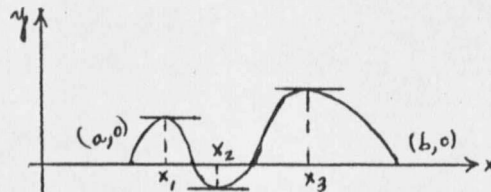
4. Very loosely stated, Rolle's theorem says: if a continuous function has the starting and ending ordinates both zero, then there must exist at least one place somewhere between the starting and finishing points where the slope of the curve is zero - providing the curve has no sharp peaks. Geometrically the picture is:



where the starting and ending ordinates are at $(a, 0)$ and $(b, 0)$. You will notice that the curve seems to be continuous over the interval.

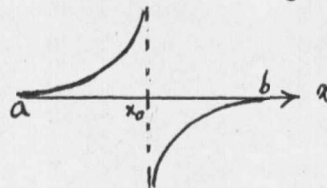
Where would you pick the abscissa(s) such that the slope(s) of the curve evaluated at this (these) point(s) would satisfy Rolle's theorem?

Do so on the previous figure.



Pick points at x_1, x_2, x_3 .

5. Although we need to fully understand all three of the conditions of Rolle's Theorem, we shall first concentrate on those which are least obvious: (1) $f(x)$ must be continuous over $[a, b]$; and (2) $f(x)$ must be differentiable over (a, b) . Let's consider continuity first. Suppose $y = f(x)$ were discontinuous at x_0 where $a < x_0 < b$ as is illustrated

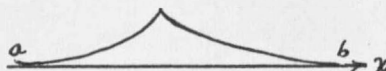


In this case is there a point $(\xi, 0)$ between a and b where $f'(\xi) = 0$?

no

Second, the curve will have to be a smooth curve possessing a derivative everywhere over (a, b) . Suppose for purposes of illustration that $f(x)$ were not everywhere differentiable over (a, b) even though continuous over $[a, b]$.

Certainly this could happen as is illustrated:



What is the value of the derivative (slope) at the point of the cusp? (Remember, we seek a point between a and b where the slope of $y = f(x)$ is zero.)

The derivative (slope) does not exist at the point of the cusp.

Thus in the illustration is there a point (ξ , 0) where $f'(\xi) = 0$ and yet $a < \xi < b$?

no

6. So, technically stated Rolle's Theorem says:

- If (1) $y = f(x)$ is continuous over $[a, b]$,
 (2) $y = f(x)$ is differentiable over (a, b) ,
 (3) $f(a) = f(b) = 0$;

then there exists a number ξ where $a < \xi < b$ such that $f'(\xi) = 0$.

What are the hypothesis of Rolle's Theorem? (Refer back to the above figures if you need to).

- 1. $y = f(x)$ is continuous over $[a, b]$.
 2. $y = f(x)$ is differentiable over (a, b) ,
 3. $f(a) = f(b) = 0$.

7. What is the conclusion of Rolle's theorem?

There exists a point (ξ , 0) where $a < \xi < b$ such that $f'(\xi) = 0$.

8. Now reconsider the hypothesis and conclusion in the spirit of honest evaluation. You will see that although figures aid the understanding really no figure is needed for the theorem. The theorem can in fact be stated with no reference to a figure at all. Indeed, the argument can be made entirely on the basis of the real numbers alone.

Is a figure really needed to state Rolle's Theorem?

no

9. Now let's get the proof. On the basis of the hypotheses clearly either one of two cases exist: (1) $f(x) \equiv 0$ for all x over $[a, b]$ or
 (2) $f(x) \not\equiv 0$ for all x over $[a, b]$.

What could be the nature of $f(x)$ under the hypotheses?

- (1) $f(x) \equiv 0$ for all x over $[a, b]$ or
 (2) $f(x) \not\equiv 0$ for all x over $[a, b]$.

10. If $f(x) \equiv 0$ for all x over $[a, b]$ then ξ can be very easily chosen. Where could you choose ξ such that $f'(\xi) = 0$?

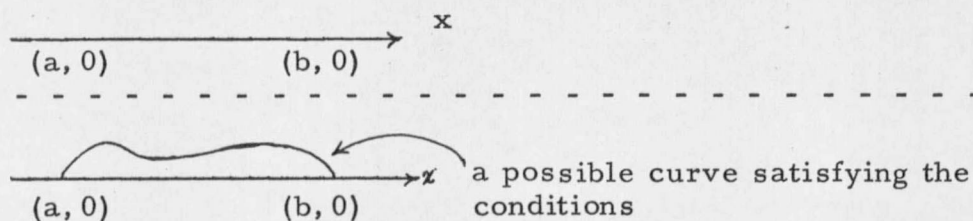
Choose ξ anywhere between a and b .

11. If $f(x) \not\equiv 0$ then our task is a wee bit more touchy. If $f(x) \not\equiv 0$ then either $f(x) > 0$ or $f(x) < 0$ somewhere over (a, b) . But in either case $f(x)$ must achieve at least one maximum or one minimum value over (a, b) . (Actually this statement itself begs to be proven - - - and can be! - - - but the proof is beyond our present means. However, intuition can be called upon to guarantee its plausibility. If you don't believe it, go to step 11a. If you do believe it, continue with this step.)

What must be true of $f(x)$ if $f(x)$ is continuous over $[a, b]$ and $f(a) = f(b) = 0$?

 $f(x)$ must achieve a minimum or maximum value somewhere over (a, b) .

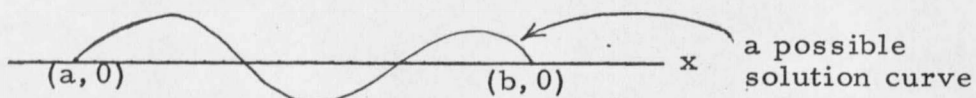
- 11a. Consider $y = f(x)$ starting at $(a, 0)$ and ending at $(b, 0)$. Remember $y = f(x)$ is entirely differentiable over (a, b) so no cusps can be encountered. Indeed, $y = f(x)$ is a well-behaved curve. Draw a curve from $(a, 0)$ to $(b, 0)$ such that it is continuous, a function, and has a derivative everywhere between $(a, 0)$ and $(b, 0)$.



- 11b. You can see that no matter how you try to draw the curve it must take on a maximum or minimum value somewhere within $(a, 0)$ to $(b, 0)$. Can the curve take on both a maximum and a minimum value over (a, b) neither of which is zero?

yes

Sketch such a curve.



11c. Return now to remainder of step (11).

12. But $f(x)$ is everywhere differentiable over (a, b) ! So what does this mean about $f'(x)$ at the abscissa where $f(x)$ did achieve the maximum or minimum value?

$f'(x)$ must be zero!

13. So simply pick ξ to be that abscissa where $f'(x) = 0$. What numerical value is $f'(\xi)$ equal to?

$f'(\xi) = 0$

14. Where is ξ located with respect to a and b ?

$a < \xi < b$.

15. Do you see? We have proven the existence of a ξ between a and b such that $f'(\xi) = 0$. Now restate the hypothesis and conclusion of Rolle's Theorem to make sure you have it firmly in mind.

If (1) $f(x)$ is continuous over $[a, b]$

(2) $f(x)$ is differentiable over (a, b)

(3) $f(a) = f(b) = 0$

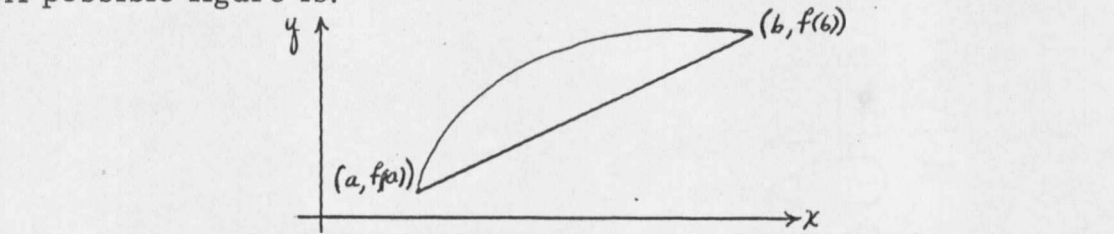
then there exists a number ξ where $a < \xi < b$ such that $f'(\xi) = 0$.

16. With this as a tool let's go after the big game - - - let's prove the Mean Value Theorem. You ask what it is about? Well, loosely worded it states that a continuous curve over some closed interval - the curve possessing a slope everywhere within the interval - has at least one point on it within the interval where the tangent to the curve at that point is parallel to the secant across the curve from its starting point to its ending point. It's easy to see in terms of a figure so let's design one.

- (1) Pick two points $(a, f(a))$ and $(b, f(b))$, in the space below. (Pick them about 2" apart and not vertical or horizontal since such a choice is most rewarding in terms of illustration. Label the points with coordinates $(a, f(a))$ and $(b, f(b))$).
- (2) Draw a smooth curve representing a function which is: continuous over $[a, b]$ (has no breaks); and differentiable over (a, b) (no cusps).
- (3) Draw a secant line from $(a, f(a))$ to $(b, f(b))$.

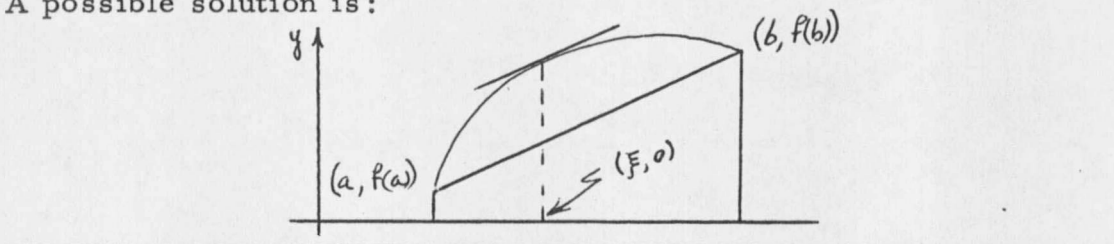


A possible figure is:



17. Now on your figure locate a point $(\bar{x}, 0)$ where $a < \bar{x} < b$ such that the slope of the curve is parallel to the slope of the secant from $(a, f(a))$ to $(b, f(b))$. Draw the tangent.

A possible solution is:



18. What is the slope of the secant from $(a, f(a))$ to $(b, f(b))$?

$$\frac{f(b) - f(a)}{b - a}$$

19. What must the slope of the tangent be at $(\xi, f(\xi))$ (in terms of the derivative)?

$$f'(\xi)$$

20. Now you should be ready to carefully state the Mean Value Theorem. What hypotheses would you insist on?

-
- (1) $f(x)$ is continuous over $[a, b]$
 (2) $f(x)$ is differentiable over (a, b) .
-

21. What would the conclusion be?

There exists a point $(\xi, 0)$ where $a < \xi < b$ such that:

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

22. Now state the entire Mean Value Theorem.

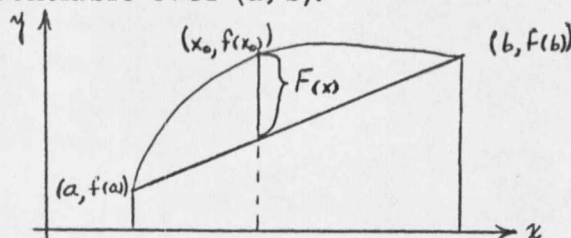
If (1) $f(x)$ is continuous over $[a, b]$
 and (2) $f(x)$ is differentiable over (a, b) , then

$$f'(\xi) = \frac{f(b) - f(a)}{b - a} \quad \text{where } a < \xi < b.$$

23. Now for a proof. We shall reduce the entire argument to Rolle's Theorem by carefully designing a function which fits the hypotheses of Rolle's Theorem but which in turn proves the Mean Value Theorem. So the task we face is to design such a function. Before we actually design this mystery function (let's call it $F(x)$ from here on), let's recall what its properties must be. What three things must be true of $F(x)$?

-
- (1) $F(x)$ must be continuous over $[a, b]$,
 (2) $F(x)$ must be differentiable over (a, b) ,
 (3) $F(a) = F(b) = 0$.
-

24. Now suppose we select $F(x_0)$ to be the vertical distance between the point $(x_0, f(x_0))$ on the curve $y = f(x)$ and the corresponding point on the secant line connecting $(a, f(a))$ and $(b, f(b))$ as the figure below describes. Remember that $y = f(x)$ is continuous over $[a, b]$ and differentiable over (a, b) .



What is $F(x)$?

The length of the segment between the curve and the secant line.

Using a two-point form write the equation of the line joining $(a, f(a))$ and $(b, f(b))$ and solve for y .

$$\frac{f(b) - f(a)}{b - a} = \frac{y - f(a)}{x - a} \quad \text{or} \quad \frac{f(b) - f(a)}{b - a} = \frac{y - f(a)}{x - a}$$

hence

$$y = f(b) + (x - b) \frac{f(b) - f(a)}{b - a} \quad \text{or} \quad y = f(a) + (x - a) \frac{f(b) - f(a)}{b - a}$$

25. Now if $y = f(b) + (x - b) \frac{f(b) - f(a)}{b - a}$ or $y = f(a) + (x - a) \frac{f(b) - f(a)}{b - a}$

represents the equation of the secant line, at x_0 what is the ordinate to the secant line?

$$y_0 = f(b) + (x_0 - b) \frac{f(b) - f(a)}{b - a} \quad \text{or} \quad y_0 = f(a) + (x_0 - a) \frac{f(b) - f(a)}{b - a}$$

26. Calculate $F(x_0)$ as (24) defines it.

$$F(x_0) = f(x_0) - y_0 = f(x_0) - \left[f(b) + (x_0 - b) \frac{f(b) - f(a)}{b - a} \right]$$

$$\text{or} \quad f(x_0) - f(a) - (x_0 - a) \frac{f(b) - f(a)}{b - a}$$

27. But let's look at $F(x_0)$ more carefully. If x_0 were at a ($x_0 = a$), what would be the value of $F(x_0)$?

$$0 \text{ since } F(x_0) = F(a) = f(a) - f(b) - (a-b) \frac{f(b)-f(a)}{b-a} = 0$$

(Using the other equation for the line gives the same result.)

28. If x_0 were at b ($x_0 = b$), what would be the value of $F(x_0)$?

$$0 \text{ since } F(x_0) = F(b) = f(b) - f(b) - (b-b) \frac{f(b)-f(a)}{b-a} = 0$$

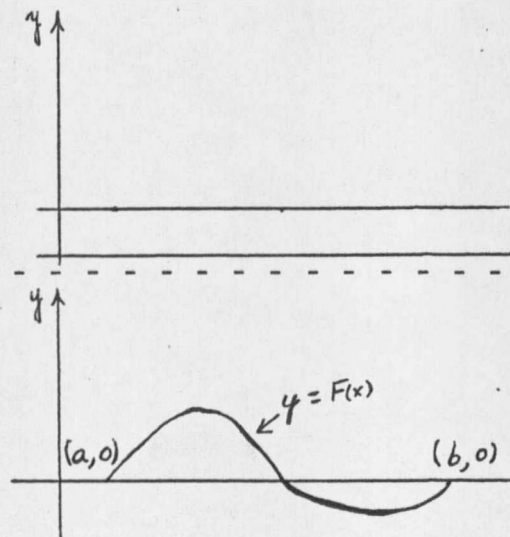
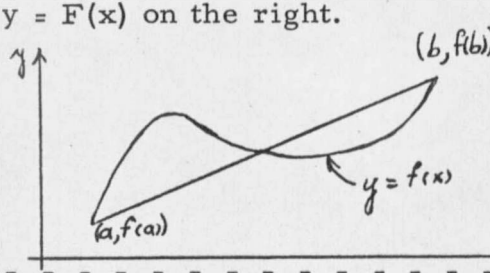
(Using the other equation for the line gives the same result.)

29. It may be that F is the function we seek! To find out for sure, define $y = f(x)$ carefully over $[a, b]$. (We want a function which satisfies Rolle's Theorem.)

$$F(x) = f(x) - f(b) - (x-b) \frac{f(b)-f(a)}{b-a} \quad \text{or}$$

$$F(x) = f(x) - f(a) - (x-a) \frac{f(b)-f(a)}{b-a} .$$

30. Perhaps it may even be worthwhile to sketch $y = F(x)$. Suppose $y = f(x)$ is as it appears in the sketch on the left. Now you sketch $y = F(x)$ on the right.



31. OK. It appears from the sketch that $y = F(x)$ will in fact fit the Rolle's Theorem conditions. Let's check them one at a time.

(1) First is $y = F(x)$ a function over $[a, b]$?

yes

(2) Is $y = F(x)$ continuous over $[a, b]$?

yes

(3) Is $y = F(x)$ differentiable over (a, b) ?

Yes, because $f(x) - f(b) - (x-b) \frac{f(b) - f(a)}{b-a}$

or $f(x) - f(a) - (x-a) \frac{f(b) - f(a)}{b-a}$
--

is differentiable over (a, b) .

(4) Is $F(a) = F(b) = 0$?

Yes, see steps 27 and 28.

32. So since $y = F(x)$ fits Rolle's Theorem we are sure there exists a ξ where $a < \xi < b$ such that $F'(\xi) = 0$. But this means:

$$d/dx \left[f(x) - f(b) - (x-b) \frac{f(b) - f(a)}{b-a} \right] = 0 \text{ when } x = \xi \text{ and } a < \xi < b$$

or $d/dx \left[f(x) - f(a) - (x-a) \frac{f(b) - f(a)}{b-a} \right] = 0 \text{ when } x = \xi \text{ and } a < \xi < b$

Carry out the differentiation and substitution and see what results.

$$f'(\xi) - \frac{f(b) - f(a)}{b-a} = 0$$

33. Clearly $f'(\xi) = \frac{f(b) - f(a)}{b-a}$ for the exact ξ that is guaranteed by

Rolle's Theorem! The key ideas in the proof are two in number:

(1) we manufactured a function $F(x)$ which obeyed the conditions of Rolle's Theorem; (2) we applied Rolle's Theorem to the manufactured function to be assured of the ξ where $a < \xi < b$. Of

particular interest is the fact that the whole argument can be done entirely independent of any geometric consideration. When one departs from the concrete visualization to the abstract analytic approach, one becomes aware of the power of this idea.

Can this theorem be proven independently of geometric considerations?

yes

34. Well, you have proven the ξ exists if given the function and the interval. Let's see if you can find it. Suppose $y = f(x) = x$ over $[1, 3]$. Where is the ξ guaranteed by the Mean Value Theorem?

 $a < \xi < b$ thus $1 < \xi < 3$ [anywhere in $(1, 3)$]

35. If $y = f(x) = x^2$ over $(1, 3)$, where is the ξ guaranteed by the Mean Value Theorem? Sketch the curve on scratch paper and reconsider the problem.

slope of secant is 4 since $(9-1)/2 = 4$

$y' = f'(x) = 2x$
Thus $2\xi = 4$ and $\xi = 2$.

36. Now prove that for the parabola $y = x^2$ over $[a, b]$ the ξ is always midway between a and b . Sketch the curve on scratch paper and reconsider the problem.

 $f'(x) = 2x \quad \therefore \xi = \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}$

37. So you can find the ξ given a proper function and interval. However, the beginning calculus student may wonder as to the real value of this theorem. Well, its use is largely theoretic. The Mean Value Theorem is mainly used as a springboard to other important theory.

What is the main use of the Mean Value Theorem?

A background of fact for other important theory.

38. Let's prove one very useful theorem by using the Mean Value Theorem as a platform. Consider a function $y = f(x)$ which has a continuous derivative over $[a, b]$ and which has a second derivative existing over (a, b) . The claim is that these conditions when united with the fact that $f''(x) > 0$ over (a, b) are sufficient to guarantee that the sketch of $y = f(x)$ is concave upward over (a, b) .

What are the hypotheses of this theorem?

-
- (1) $y = f(x)$ has a continuous derivative over $[a, b]$
 - (2) $f''(x)$ exists over (a, b)
 - (3) $f''(x) > 0$ for $a < x < b$.

What is the conclusion?

$y = f(x)$ is concave upward over (a, b) .

39. On the basis of the Mean Value Theorem it can be said:

$$f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \text{where } a \leq x_1 < x_2 \leq b. \quad \text{Where is}$$

ξ located?

$$x_1 < \xi < x_2$$

40. Now consider the derived function. It can also be said:

$$f''(\eta) = \frac{f'(\xi_2) - f'(\xi_1)}{\xi_2 - \xi_1} \quad \text{Where } \xi_1 < \eta < \xi_2. \quad \text{Why?}$$

This is the mean value theorem applied to the first derived function over $[\xi_1, \xi_2]$.

41. Order by using inequality signs $a, b, x_1, x_2, \xi, \eta$ as used in the preceding steps.

$$a \leq x_1 < \xi < \eta < x_2 \leq b$$

42. Now in step 40 solve for $f'(\xi)$.

$$f'(\xi) = f'(x_2) - f''(\eta)(x_2 - \xi)$$

43. Taking this value of $f'(\xi)$ substitute it for $f'(\xi)$ in the equality of step 39.

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_2) - f''(\eta)(x_2 - \xi).$$

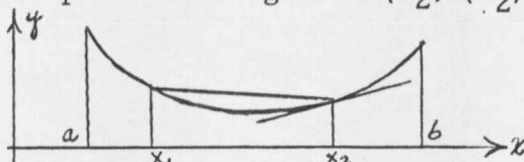
44. But by the third hypothesis of (38) $f''(\eta) > 0$. Moreover $x_2 - \xi > 0$. So what of $f''(\eta)(x_2 - \xi)$?

$$f''(\eta)(x_2 - \xi) > 0.$$

45. Consequently if we remove the product $f''(\eta)(x_2 - \xi)$ from the right hand side of the result in step 43, it appears that an inequality would result. State the inequality.

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} < f'(x_2)$$

46. But this says the slope of the secant from $(x_1, f(x_1))$ to $(x_2, f(x_2))$ is always less than the slope of the tangent at $(x_2, f(x_2))$ where $a \leq x_1 < x_2 \leq b$. Look at it graphically:



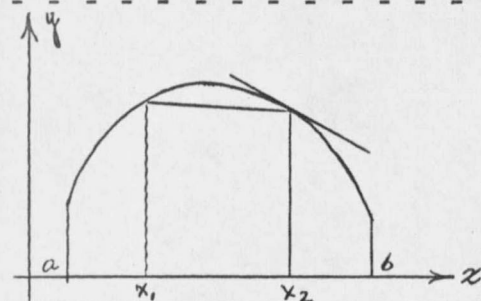
So, you see, as x_1 and x_2 take on different values over $[a, b]$ so that $x_1 < x_2$ it is clear the curve is always bending upward. Thus the theorem is proved. Suppose one desires to show: if $f'' < 0$ over the interval (with the same restrictions on f), then the curve is concave downward. The proof would go the same with one change:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_2) - f''(\eta)(x_2 - \xi) \text{ would be transformed into}$$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \stackrel{?}{\geq} f'(x_2). \text{ (Which sign: } > \text{ or } < \text{ ?)}$$

47. In effect this would mean the slope of the secant from $(x_1, f(x_1))$ to $(x_2, f(x_2))$ would be always greater than the slope of the tangent to the curve at $(x_2, f(x_2))$. That is, the curve is bending downward as one proceeds from left to right.

Sketch a representative figure to assure yourself of this fact.



48. But since you have the theorem, let's try to use it. Suppose when given $y = f(x)$ it is apparent that $f'(x_0) = 0$ and $f''(x_0) < 0$. What would you conclude of $f(x_0)$?

$f(x_0)$ is a relative maximum

If $f(x)$ is given and $f'(x_0) = 0$ and $f''(x_0) > 0$, what would you conclude of $f(x_0)$?

$f(x_0)$ is a relative minimum

49. Use this test to find the relative maxima and minima for $y = x^3/3 - x$.

$$f'(x) = x^2 - 1$$

$$\text{so } x = \pm 1 \text{ implies } f'(x) = m = 0.$$

$$\text{But } f''(x) = 2x$$

$$\text{and } f''(1) = 2 > 0$$

$$f''(-1) = -2 < 0.$$

Thus $f(1)$ is a relative minimum and $f(-1)$ is a relative maximum.

50. Nail it down solid by once more stating this 2nd derivative test.

If $y = f(x)$ has $f(x_0) = 0$ and $f''(x_0) > 0$, then $f(x_0)$ is a relative minimum.

If $y = f(x)$ has $f(x_0) = 0$ and $f''(x_0) < 0$, then $f(x_0)$ is a relative maximum.

Lesson XI

APPLICATIONS OF DIFFERENTIATION TO MAX-MIN PROBLEMS

1. A type of problem which is nicely solved by the calculus is the max-min problem. Since problems of this type frequently are very important in application, let's look at them here.

What kind of problems which are nicely handled by the calculus will we look into in this lesson?

Max-min problems.

2. Actually each of the problems is ultimately reducible to an analysis of a sketched curve. So let's focus on curve sketching briefly.

Max-min problems ultimately become problems in analysis of

curve sketching.

3. For example, suppose $y = x^3/3 - 4x$. Let us assume we are pressed to find the sketch of the curve. To start with find the interval(s) in which the slope is increasing.
-

$$dy/dx = m = x^2 - 4$$

$$\therefore m > 0 \text{ over } |x| > 2.$$

Where is the slope decreasing?

$$|x| < 2$$

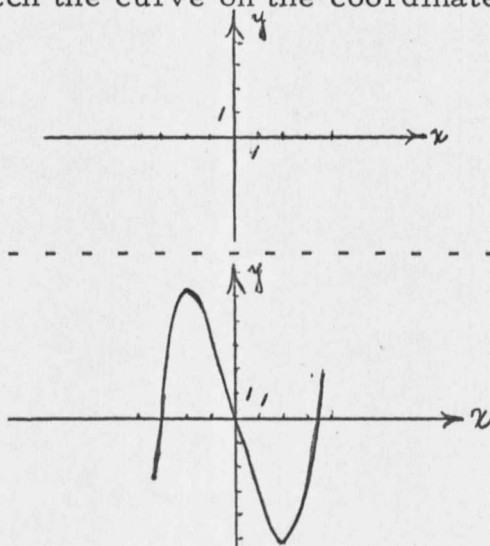
Where is the slope 0?

$$|x| = 2$$

4. Already we have established some important facts. Now let's get the intercepts. Do so.

$y = x(x^2/3 - 4)$ which implies $x = 0$ or $x^2 = 12$ in which case
 $x = \pm 2\sqrt{3}$.

5. Now sketch the curve on the coordinate system.



6. Now look at the sketch and answer these questions.

If $y = x^3/3 - 4x$, where is a relative maximum?

at $x = -2$

Where is a relative minimum?

at $x = 2$

7. Sure, these are easy. But now let's complicate matters a bit. Let's find a relative max over $[-3, 0]$.

Clearly the relative max is at $x = -2$.

What about a relative max over $[0, 3]$?

A relative max occurs at $x = 3$ and at $x = 0$.

8. You see! The max can occur at the end of the interval. Indeed, suppose the same question is asked for $[2, 3]$.
-

No relative max exists since $x = 3$ is not in the interval.

9. How about a relative max over $(-3, 3)$?
-

Relative max exists at $x = -2$ only.

10. As a last consideration suppose a relative max is to be determined over $[-3, 3]$. Where is it (or are they)?
-

Relative max occurs at $x = -2$ and at $x = 3$.

11. In the previous problem what is the largest value y takes on over $[-3, 3]$?
-

Now one must make a choice of the relative max values. In this case $f(-2) > f(3)$. Thus $f(-2)$ is the answer.

12. Now, you see frequently the answer to the question "where does the maximum (minimum) value occur?" is answered not within the interval but at the end of the interval. Indeed, whereas many solutions of the first equation may tell you where potential max-min points exist, one needs a strong test to really examine which points actually result in max values and which min values. You have such a test. It is the second derivative test.

State the second derivative test.

If $f''(x_0) > 0$, the curve is concave up at x_0 , if $f''(x_0) < 0$ the curve is concave down at x_0 .

13. This is where the beginning student usually falls in error in working max-min problem - - - he usually forgets to apply the second derivative test.

In what respect does the beginning student usually fail to become competent as far as the solution of max-min problems are concerned?

In testing solutions with the second derivative test.

14. Now that you know of the big pitfall, let's go after a few typical problems to get a view of the procedure.

First, consider a simple one. Suppose we are asked to find two real numbers whose sum is 4 and whose product is a maximum.

Let's make sure we know what it is we want to maximize. What is it?

the product of the two numbers.

15. All right. Now use common sense for a moment. Suppose one of the numbers were 4. What would the other factor have to be?

zero

16. In this case the product would be _____.

zero

17. Now if the other number were 0, what would the first factor be? The product?

4, zero.

18. The influence of the size of either factor influences not only the other factor, but, more important, influences the magnitude of the product. Let us say it again: the change in a factor induces a corresponding change in the product. Furthermore, given a value for a factor, one and only one value of the product is implied.

Technically it is clear that the product is a function of either

factor

19. Let's see if we can actually design the function in question. First represent the product P in terms of the two factors x and y .

$$P = xy$$

20. The problem also specifies that the sum of the factors must be 4. Translate this condition into mathematical language.

$$x + y = 4$$

21. Now put steps (19) and (20) together to get $P = f(x)$.

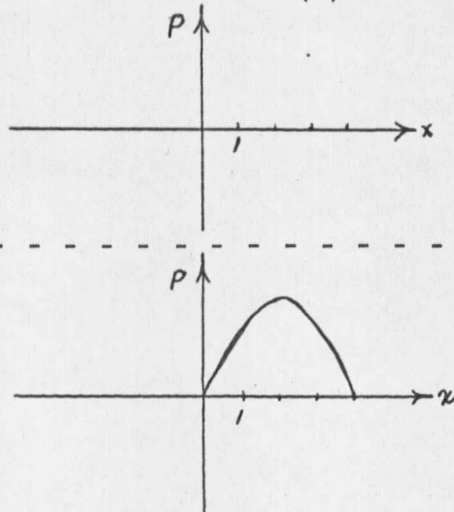
$$P = f(x) = x(4-x) = 4x - x^2$$

22. You see! Now it is an easy problem to maximize P in terms of x . Do so.

$$dP/dx = 4 - 2x$$

$$\therefore dP/dx = 0 \text{ means } 2x = 4 \text{ or } x = 2.$$

23. As a further consideration sketch $P = f(x) = 4x - x^2$



24. See! The problem is ultimately reducible to an analysis of a geometric sketch! Furthermore, we know (even without sketching) that $P = f(x)$ has a max at $x = 2$ since $P''(2) < 0$. Thus the curve is concave down. Calculate $P''(2)$.

$$P''(x) = -2$$

25. Suppose we further restrict x according to: $1 \leq x \leq 1\frac{1}{2}$.

Now what value of x will maximize the product P ?

 $x = 1\frac{1}{2}$

Since $P'' < 0$ over $1 \leq x \leq 1\frac{1}{2}$ and $P' = 0$ over $1 \leq x \leq 1\frac{1}{2}$,
 $P(1\frac{1}{2}) > P(x)$ for all other x in $[1, 1\frac{1}{2}]$.

26. As a slight modification of the above problem, suppose x is defined only over $[1, 1\frac{1}{2})$. What of the maximum value of P now?

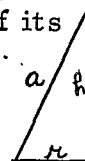
 No maximum value of P exists.

27. In retrospect we see that at first P was expressed in terms of two variables and eventually P was expressed in terms of one variable due to a side condition which allowed one variable to depend on the other. This is usually the case in max-min problems. So keep this in mind: (1) decide on what it is you want to maximize or minimize; (2) express this variable as a function of only one variable; (3) roughly sketch the relationship in (2); (4) apply the calculus to get results you desire; (5) check your answers in (4) by using d^2x/dy^2 to be sure you have the proper response.

What should be your scheme of solution?

- (1) Decide on what you want to maximize or minimize.
 (2) Express the variable as a function of only one variable.
 (3) Roughly sketch the relationship in (2).
 (4) Apply the calculus to get the results you desire.
 (5) Check your answers in (4) by using the second derivative.

28. With this method of procedure go after the following problem: a right triangle of fixed hypotenuse is to be rotated about one of its legs to generate a cone of maximum volume. What does the height of the cone have to be to produce the desired results?



- (1) What do you want to maximize?

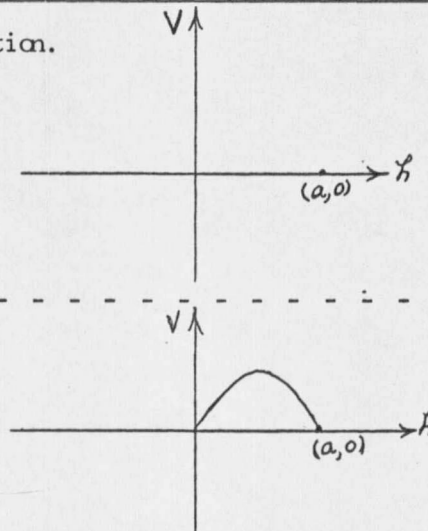
 Volume of cone.

(2) Formulate the function

$$V = 1/3 \cdot \pi r^2 h, \quad a^2 = r^2 + h^2.$$

$$\therefore V = 1/3 \cdot \pi (a^2 - h^2)h.$$

(3) Roughly sketch the function.



Now use the calculus to achieve the desired result.

(4) Apply the derivative $dV/dh = \pi/3 \cdot (a^2 - h^2) + \pi/3 \cdot [h(-2h)]$

$$\therefore a^2 - h^2 = 2h^2 \quad \text{or} \quad 3h^2 = a^2$$

$$\therefore h = \frac{a\sqrt{3}}{3}$$

(5) Check the answer with d^2y/dx^2

$$V'' = -\pi/3 (2h) - \pi/3 (4h)$$

$$V'' \left(\frac{a\sqrt{3}}{3} \right) < 0. \quad \text{Thus max.}$$

29. As a final test consider the saga of the wire bender. The problem is to form a square and a circle from a piece of wire of length L such that the enclosed area is maximized. Find the dimensions of the figures which will do the job. This time proceed under your own power. But beware!

$$L = 4s + 2\pi r \text{ hence } r = \frac{L - 4s}{2\pi}$$

$$\begin{aligned} A &= s^2 + \pi r^2 \\ &= s^2 + \frac{L^2 - 8Ls + 16s^2}{4\pi} \end{aligned}$$

$$dA/ds = 2s - 2L/\pi + 8s/\pi \quad \text{which means } s = L/(\pi + 4)$$

30. Did you finish with $s = L/(\pi + 4)$? If so, you are right only in that you know how to differentiate. But you are wrong if you conclude that $s = L/(\pi + 4)$ results in a maximum area. Indeed, $A'' = 2 + 8/\pi > 0$. So what do you conclude about $s = L/(\pi + 4)$ now?
-

$s = L/(\pi + 4)$ results in a minimum area - not a maximum.

31. So the problem still remains. We must still discover the dimensions of the figures which result in maximum area. Remember, this is a problem in functional analysis. So the task is really no different from finding the extreme points of a function - but over a closed interval! What values of s are permissible for our problem? (Remember the length of the wire is L .)
-

$$0 \leq s \leq L/4.$$

32. Consequently we need to examine the end points of the curve $A = f(s)$ to be sure of their nature. Do so.
-

$$\begin{aligned} A &= f(0) = L^2/4\pi \\ A &= f(L/4) = L^2/16 + \frac{L^2 - 8L \cdot \frac{L}{4} + 16 \left(\frac{L}{4}\right)^2}{4\pi} \\ &= L^2/16 \end{aligned}$$

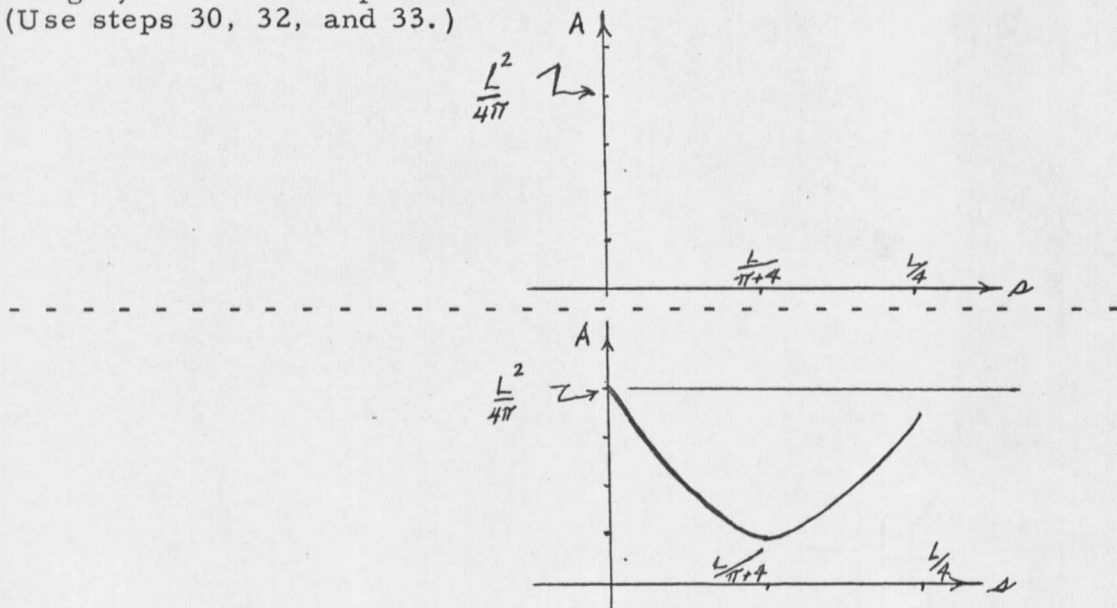
33. But since $L^2/4\pi > L^2/16$, it appears $s = 0$ is a better candidate for the desired dimension than is $L/4$.

As a final thought consider A'' once again. Notice that $A'' = 2 + 8/\pi > 0$ indicates the nature of the curve as being concave

 over $[0, L/4]$.

 upward

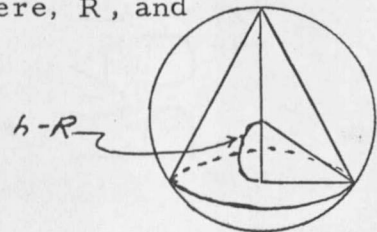
34. Roughly sketch the dependence of A on s .
 (Use steps 30, 32, and 33.)



35. Let's consider another type of max-min problem - geometric. Suppose we wish to inscribe a right circular cone within a sphere of fixed radius. Our problem may be to find the largest cone which can be so positioned. If so, our first task is to identify what we want to maximize. Do so.

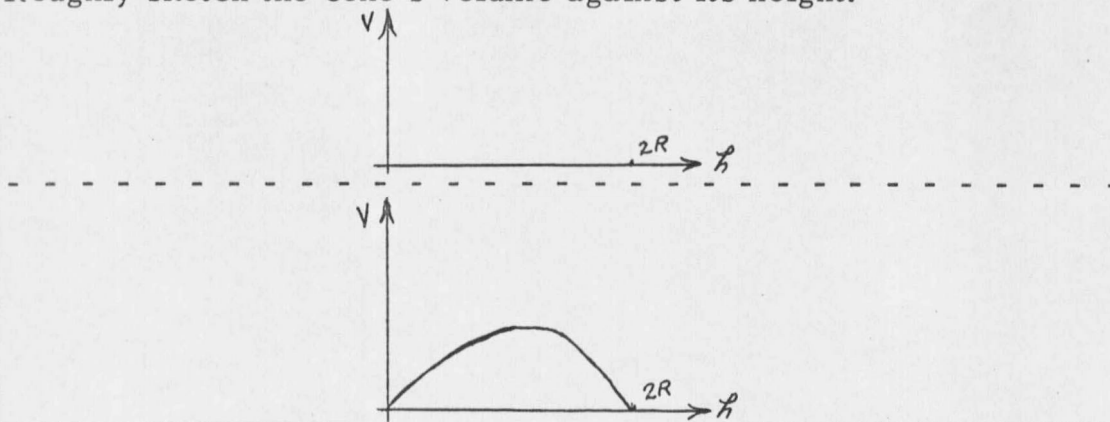
 We want to maximize the volume of the cone.

36. Considering the figure on the right, express the volume of the cone in terms of the radius of the sphere, R , and the height of the cone, h .



$$\begin{aligned}\text{Volume of cone} &= \pi \frac{(\sqrt{R^2 - (h-R)^2})^2 h}{3} \\ &= \pi/3 (2h^2 R - h^3)\end{aligned}$$

37. Roughly sketch the cone's volume against its height.



38. Find the change in the cone's volume with respect to its height.

$$dV/dh = \pi/3 (4hR - 3h^2).$$

39. Maximize the volume.

$$dV/dh = 0 = \pi/3 (4hR - 3h^2) \text{ implies } h = 0 \text{ or } h = 4R/3$$

40. Now check your result in No. 39 to be sure it gives a maximum volume.

$$V'' = \pi/3 (4R - 6h) \text{ implies } V'' (4R/3) = \pi/3 (4R - 8R) < 0 \text{ which guarantees } h = 4R/3 \text{ to be a relative maximum}$$

41. You see! The procedure remains constant despite the variation of problem type.

Using three or four words to describe each step which one uses in solving max-min problems outline the procedure.

1. Identify quantity to be maximized.
2. Express quantity in one variable.
3. Roughly sketch relationship.
4. Differentiate and maximize.
5. Check answer by y'' .

42. As a final problem suppose we are presented with the task of finding the points on $x^2 - y^2 = 1$ which are nearest $(4, 0)$.

Picking (x, y) to be an arbitrary point on the curve, what are you to minimize?

You must minimize the distance, d , from (x, y) to $(4, 0)$.

43. Now express d as a function of x alone.

$$d = \sqrt{(x-4)^2 + y^2 - 1}$$

44. Hold on. You may now see a faster route to the solution. Notice that minimizing d is equivalent to minimizing d^2 . Hence, since the differentiation of d^2 is easier than the differentiation of d , what will you now minimize?

$$d^2 = (x-4)^2 + x^2 - 1$$

45. Complete the problem.

$$dd^2/dx = 2(x-4) + 2x = 4x-8.$$

Thus $x = 2$ seems to be the desired abscissa.

$$d^2 d^2/dx^2 = 4 > 0. \text{ Thus } x = 2 \text{ is the desired abscissa}$$

Thus the distance from $(4, 0)$ to $(2, \sqrt{3})$ is the desired distance or

the distance is
$$\sqrt{2^2 + (-\sqrt{3})^2} = \sqrt{7}.$$

Lesson XII

Differentials

1. The differential is an important topic in calculus which the student often fails to recognize as vital. Consequently, we shall concern ourselves in this lesson with the theory and use of differentials.

What is this lesson about?

Differentials.

2. To start with we should recall that prior to this topic the symbol $\lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ dy/dx has been interpreted by us as $\lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$. In other words we have specifically associated with dy/dx a function of some type. In fact the notation y' , $f'(x)$, and $D_x y$ has probably been used by your instructor--and in each case you have interpreted this symbol as a function of x (the derived function of x). The point to be made clear is that dy/dx has been referred to as a derived function and has never been interpreted as a fraction with numerator dx and denominator dy .

In the past dy/dx has been referred to as a _____.
Under no circumstance have you considered it to be a _____.

derived function, fraction.

3. However, the time has come for us to give new meaning to dy/dx . Now we want to show that it can in fact be divided into parts which do give it meaning as a fraction.

In this lesson we intend to broaden our interpretation of _____
so that we can consider it to be a _____.

 dy/dx , fraction.

4. We shall do the above job by convenient definitions. First, we shall define dx . By dx we shall mean Δx where the Δx is the same Δx as appears in $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$. What is dx ?

 $dx = \Delta x$

5. dx shall be referred to as "the differential of x ". Translate " dx ".

 " dx " is "the differential of x ".

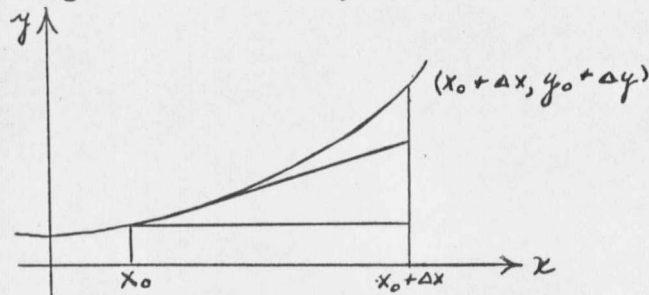
6. But shall we define dy as Δy ? No. The story is not that simple. You see, we want our definitions to be consistent with all we have done so far--and, in fact, what we will do. Thus we must be cautious in proceeding.

Should we define dy as Δy ?

 no

7. To get a clear meaning of the problem let's go to a geometric interpretation. In the figure below notice that as an increment Δx is taken past x_0 , a corresponding increment is noted in y .

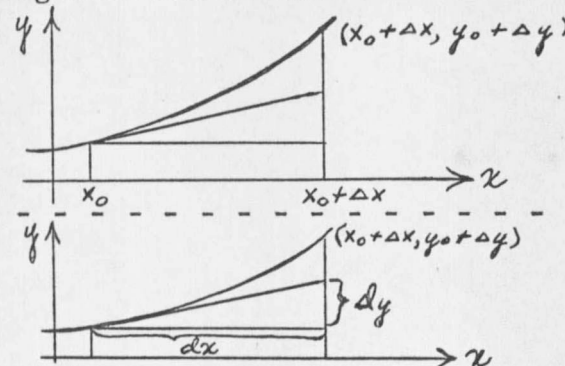
According to the figure what would you call this increment?



 Δy

8. But now let's see where the dx and dy fit into the picture. To accomplish this recall that dy/dx is the tangent to the curve. Also recall that $dx = \Delta x$.

Now label on the figure below the segment you would choose to be dy such that consistency with previous mathematics is achieved. Also label the segment denoted dx .



9. Now you will notice that dy is not in general the same quantity as Δy . In fact, in the geometric interpretation above Δy is _____ than dy .

greater

10. But we are still faced with the problem of a proper definition of dy such that this definition is consistent with the rest of the calculus. Of course, in a definition we are free to describe as we see fit just so the consequences do not in any way conflict with the remainder of our mathematical structure. So here we go. We'll just arbitrarily (but carefully) assign a meaning to dy --and it shall be up to us to make sure no conflicts arise with work we have done or with work we do come upon.

By definition: $dy = f'(x)dx$. What is the meaning of dy ?

 $dy = f'(x)dx$

11. But hold on. We may be pulling a fast one. If $dy = f'(x)dx$; the observant student may say: " $dy = \frac{dy}{dx} dx$; cancel out dx ; and

thus $dy = dy$. Hence, you've not defined dy at all". But such is not the case! I wrote $f'(x)$ intentionally rather than dy/dx to guard against just such an interpretation. You see, when we define something new, we must always define it in terms of something we understand and are familiar with already.

What is the meaning of $f'(x)$ in the definition of dy ?

 $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ or is the derived function of $f(x)$.

12. So you see $dy = f'(x)dx$ is a meaningful definition in the sense that we already know the meaning of $f'(x)$ and dx .

Now you define dy again but this time explain the meaning of each part of the definition.

 $dy = f'(x)dx$ where $f'(x)$ is the derived function and $dx = \Delta x$.

13. Let's look at it from a different angle now. Let's compare dy and Δy . Certainly we know $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$. But this means that for small values of Δx : $\frac{\Delta y}{\Delta x}$ approximates $\frac{dy}{dx}$ or $\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$.

In fact the approximation can be eliminated by adding an error factor, e , to the right side of the "inequation". That is,

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} + e$$

Now since Δx is a non-zero quantity we are within accepted rights to multiply both sides of the resultant equation by Δx . Do so.

$$\Delta y = \frac{dy}{dx} \Delta x + e \Delta x$$

14. Since $\Delta x = dx$ and since $dy/dx = f'(x)$ we could rewrite Δy using this new notation. Do so.

$$\Delta y = f'(x)dx + e dx$$

15. Now recall that $dy = f'(x)dx$. Hence, what must the difference of dy and Δy be?

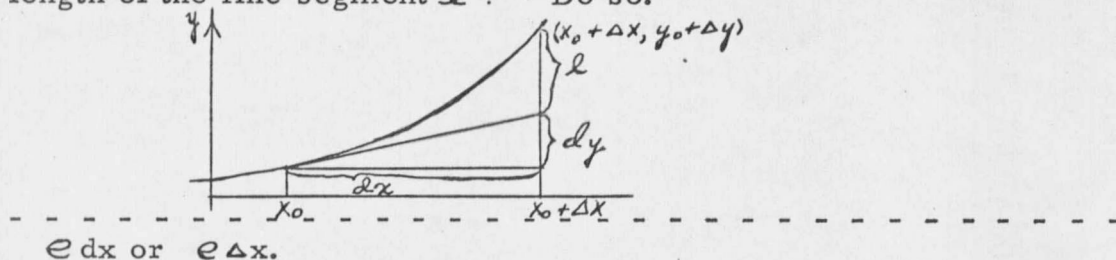
$$\Delta y - dy = e dx \text{ or } e \Delta x$$

16. Clearly as Δx goes to 0, $e \Delta x$ will approach zero faster since e too is dependent upon Δx . Hence $e \Delta x$ is of the second order of smallness and is likely to become small much faster than is either e or Δx .

Suppose for illustration that $e = \frac{1}{10}$ and $\Delta x = \frac{1}{100}$. What would be the value of $e \Delta x$?

$$e \Delta x = \frac{1}{1000} \text{ which is indeed less than either } \frac{1}{10} \text{ or } \frac{1}{100}.$$

17. Again go back to the geometry of the problem. In the light of the foregoing argument can you assign a correct value label for the length of the line segment ℓ ? Do so.



18. Now we have dx and dy within our sphere of meaningful mathematics. Now we can think "fraction" when we see dy/dx . Thus such things as the familiar chain rule become even more meaningful since $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ can now be thought of in a fractional manner and we can cancel the du 's to get true equality. Also, $D_x y = \frac{1}{D_y x}$ can now be handled from the fractional consideration: $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

to lend new confidence to the equality.

Express $D_x y$ using the differentials dy , dx , du , and dv in chain rule fashion.

$$D_x y = \frac{dy}{du} \frac{du}{dv} \frac{dv}{dx} \text{ or } \frac{dy}{dv} \frac{dv}{du} \frac{du}{dx}.$$

19. Furthermore the derivative formulas once learned by the student now can be considered in briefer differential notation. For example:

$$\frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx}, \text{ where } u \text{ and } v \text{ are differentiable}$$

functions of x now becomes $d(u+v) = du + dv$.

Write the differential equivalent for the product formula and the quotient formula.

$$\begin{aligned} \text{(a)} \quad d(uv) &= u dv + v du \\ \text{(b)} \quad d(u/v) &= \frac{v du - u dv}{v^2} \end{aligned}$$

20. That completed the theory. You ask for applications? OK. Let's turn to applications. Recall that $dy \doteq \Delta y$; $y + \Delta y = f(x + \Delta x)$ when $y = f(x)$.

How would $y + \Delta y = f(x + \Delta x)$ compare to $y + dy$?

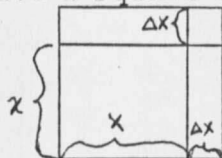
$$y + \Delta y = f(x + \Delta x) \doteq y + dy$$

21. If $y = f(x)$, $dy = f'(x)dx = f'(x) \Delta x$. Thus $y + f'(x) \Delta x \doteq$ _____.
 $f(x + \Delta x)$ or $y + \Delta y$

22. Taking a specific case, suppose we desire to approximate $y + \Delta y$ if $y = x^2$. From the definition $y + f'(x) \Delta x \doteq y + \Delta y$. Hence $y + \Delta y \doteq$ _____.
 $x^2 + 2x \Delta x$

23. But $y + \Delta y = (x + \Delta x)^2 = x^2 + 2x \Delta x + (\Delta x)^2$. What must be the error in the approximation?
 $edx = (\Delta x)^2$

24. In terms of a geometric figure let's see what the whole argument means. We have a square of side x and area x^2 .



We consider a growth of the sides of the square to size $x + \Delta x$. The resultant area is certainly $(x + \Delta x)^2 = x^2 + 2x \Delta x + (\Delta x)^2$. But our approximation of $(x + \Delta x)^2$ was $x^2 + 2x \Delta x$. In terms of the figure what is the magnitude of edx , the error?

$$edx = (\Delta x)^2$$

25. To get the full force of the argument, try it with numbers on the corresponding problem. Suppose we desire to approximate $(20.01)^2$. You see - - - we could set the side of the square to be 20 and take $dx = \Delta x = .01$. Then the approximation is:

$$\begin{aligned}
 y + \Delta y \doteq y + dy &= x^2 + f'(x)dx \\
 &= x^2 + f'(x) \Delta x \\
 &= x^2 + 2x \Delta x \\
 &= (20)^2 + 2(20)(.01) \\
 &= 400 + .40 = 400.4
 \end{aligned}$$

By how much does the approximation differ from the true value?

 .0001.

26. You see! Differentials can be used to get approximations for certain numerical quantities. However, in each case the student should not lose sight of the principle argument:

$$\Delta y \doteq dy = f'(x)dx \text{ or } f'(x)\Delta x.$$

thus $y + \Delta y = y + dy = y + f'(x)\Delta x$.

An approximation for $y + \Delta y$ is _____.

 $y + dy = y + f'(x)\Delta x.$

27. In each problem where an approximation is desired, one is first confronted with the problem of selecting $f(x)$ which will fit the situation at hand. Recall the case where we approximated $(20.01)^2$. We used as our function here $y = x^2$. You see - - - x^2 fits the idea of squaring 20.01. Furthermore, $\Delta x = .01$ fits conveniently the increment of .01 on 20.

Suppose we needed to calculate $\sqrt{26}$. Here we need a function which will reflect the argument of taking the square root. Express such a function.

 $y = x^{\frac{1}{2}}$

28. Suppose the problem were to approximate $\sqrt[3]{28}$. What functional relationship would you choose?

 $y = x^{\frac{1}{3}}$

29. Lastly, suppose you are to calculate $\sin 31^\circ$. What functional relationship would you use?

 $y = \sin x$

30. You see! The functional relationship is chosen to fit the problem. But what of Δx ? It must be chosen too. But this is the job of the problem solver - to choose Δx convenient for easy solution of the problem and yet small such that a sufficient degree of accuracy can be enjoyed.

Clearly if one needs to approximate $\sqrt{26}$, one would insist on Δx being 1 such that convenient calculation of $f'(25)$ could be accomplished.

Go ahead and approximate $\sqrt{26}$ by using differentials.

$$\begin{aligned} y &= x^{\frac{1}{2}} \\ \therefore y + \Delta y &\approx y + dy = \sqrt{x} + f'(x) \Delta x \\ &= \sqrt{25} + \frac{1}{2\sqrt{25}} (1) \\ &= 5 + \frac{1}{10} = 5.1 \end{aligned}$$

31. Now let's check this approximation to see how close it was. Square 5.1.

 26.01

32. Not too bad considering that dx was chosen so large, 1. You may wonder what happens when one makes a mistake and accidentally chooses dx and x inconveniently. Let's look and see. Suppose in the last problem x is chosen not as 25 but as 23. Now try to calculate an approximation of $\sqrt{26}$.

 $y + \Delta y \approx y + dy = \sqrt{23} + \frac{3}{2\sqrt{23}}$. You see! You are in worse trouble than when you started.

33. How about the situation where one selects x conveniently but Δx is correspondingly too large? Let's again look at such a problem. Suppose you are asked to approximate $\sqrt{24}$. Certainly one could proceed by setting $y = x^{\frac{1}{2}}$; consider $y + \Delta y \approx y + dy = x^{\frac{1}{2}} + \frac{1}{2\sqrt{x}} dx$;
 then set $x = 16$ and $dx = 8$. What is the resultant approximation?
-

$$y + dy = \sqrt{16} + 8/2\sqrt{16} = 4 + 1 = 5$$

34. True, 5 isn't too far off. But let's use a little common sense and select dx to be of smaller magnitude. Remember the error in these approximations is $\mathcal{O}dx$. Hence if dx is chosen small in magnitude then $\mathcal{O}$ is small in magnitude, and so $\mathcal{O}dx$ will be even smaller in magnitude yet. But how could you choose dx in approximating $\sqrt{24}$ such that $f'(x)$ would still be easy to calculate and yet dx would be smaller than before?
-

Take $dx = -1$

35. OK. With $dx = -1$ approximate $\sqrt{24}$.
-

$$\begin{aligned}\sqrt{24} &\approx y + \Delta y \approx y + dy = y + f'(x)dx \\ &= \sqrt{25} + -1/2\sqrt{25} \\ &= 5 - \frac{1}{10} \\ &= 4.9\end{aligned}$$

36. Now square 4.9 to check its accuracy:
-

$$24.01$$

37. See! By a choice of $dx = -1$ instead of 8 we are much closer to the desired result- - - in fact 100 times closer!

Notice also that dx can be chosen to be a negative number. It need not be positive.

With these pitfalls in mind specify the function you would set up and the choice you would make for dx so as to approximate the following:

$$\sqrt[3]{28}$$

$$y = \sqrt[3]{x}, \quad dx = 1$$

$$(1.001)^7$$

$$y = x^7, \quad dx = .001.$$

$$\sqrt[3]{.126}$$

$$y = \sqrt[3]{x}, \quad dx = .001$$

$$\sin 31^\circ$$

$$y = \sin x, \quad dx = 1^\circ \text{ or } \frac{\pi}{180}$$

38. With this you should have a fairly good feeling for approximations by using differentials. To nail it down solid let's work a couple more problems.

Calculate an approximation of $(1.001)^7$.

$$\begin{aligned} y &= x^7 \\ y + \Delta y &\approx y + dy = y + f'(x)dx \\ &= 1^7 + 7(1)^6(.001) \\ &= 1 + .007 \\ &= 1.007 \end{aligned}$$

39. OK. Now try an approximation of $(.999)^7$.
-

$$\begin{aligned} y + \Delta y &\approx y + dy = y + f'(x)dx \\ &= 1^7 + 7(1)^6(-.001) \\ &= 1 - .007 \\ &= .993 \end{aligned}$$

40. True, differentials can be used for a good purpose in their application to approximations. However, the most important use they serve is theoretical in that meaning is clearly attached to dx and dy even when they are considered separately and not as dy/dx .

Is there any contradiction to our previous understanding of dy/dx when we consider the dy and dx as terms of a fraction?

no

Define dy and dx .

 $dx = \Delta x$

$$dy = f'(x) \Delta x.$$

Lesson XIII

AREA BY INTEGRATION

1. In class we learned that $\int_a^b f(x)dx = F(b) - F(a)$ where $dF(x) = f(x)dx$.
What is $\int_a^b f(x)dx$?

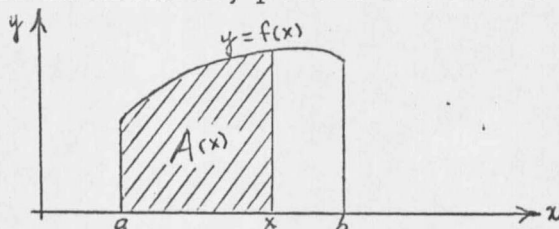
$$\int_a^b f(x)dx = F(b) - F(a) \text{ where } dF(x) = f(x)dx.$$

2. In this lesson we want to tie the idea of the definite integral to a corresponding geometric area. We will accomplish the job by solving a simple differential equation.
In this lesson we intend to associate the _____ integral to a corresponding geometric _____.

definite, area.

3. So let's get on with the task. Suppose $y = f(x)$ is a continuous, positive, function over the closed interval from a to b .

Now consider an arbitrary point x where x is between a and b .



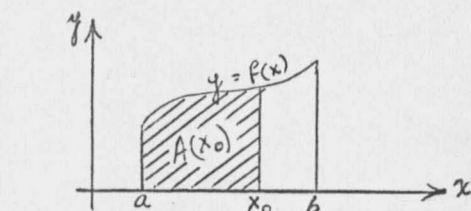
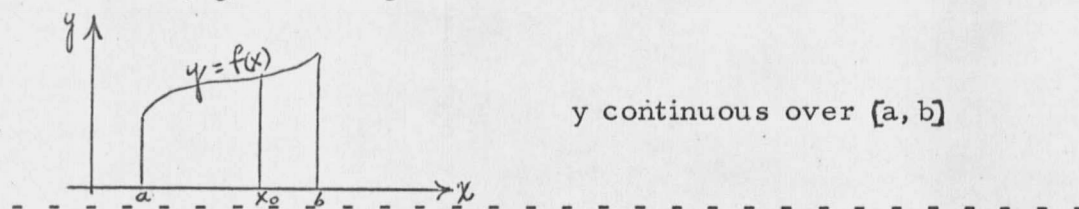
Now consider the area bounded by $x = a$ on the left, by the ordinate at x on the right, above by the curve $y = f(x)$, and below by $y = 0$. Clearly the position of x dictates the amount of area so described. in fact for each value of x that is taken between a and b one specific area is denoted.

Suppose one now considers the set of ordered pairs, $(x, A(x))$, formed by taking as the first term x , an abscissa between a and b , and by taking as the second term $A(x)$, the amount of area cut off by the ordinates at a and x , above by $y = f(x)$, and below by $y = 0$.

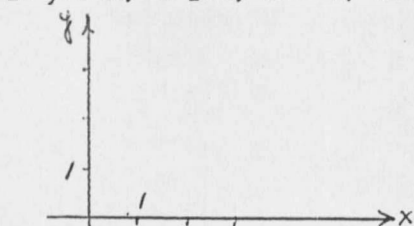
Is this set of ordered pairs a function?

yes

4. Shade in $A(x_0)$ in the figure below.



5. If $y = x$, $a = 2$, $b = 4$; what is $(x, A(x))$ at $x = 3$? Sketch it.



$(x, A(x))$ is $(3, 2\frac{1}{2})$

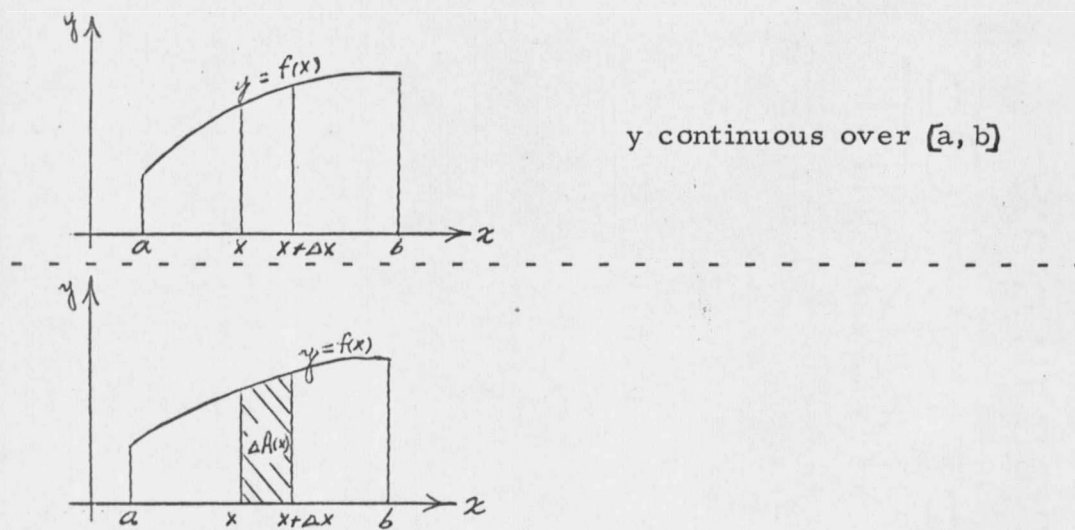
6. You should now see clearly that A is truly a function of x . Our next task is to design a situation which will result in the solution of a differential equation, the solution of which will relate $\int f(x)dx$ and $A(x)$.

We will relate $\int f(x)dx$ and $A(x)$ by the solution of a _____

_____ differential equation

7. To accomplish the job consider x to be at some arbitrary position between a and b . Take Δx to be some increment (positive to start with) on x . Obviously such an increment on x will produce a corresponding increment on $A(x)$, $\Delta A(x)$.

If $x + \Delta x$ is positioned as is indicated on the figure below, shade in the $\Delta A(x)$ area.



8. Now recall the important fact about a continuous function over a closed interval: a continuous function over a closed interval attains a maximum and a minimum value, M and m respectively, over the same interval.

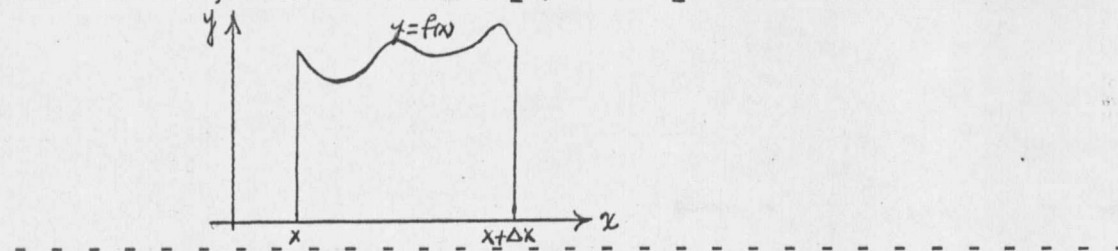
Since $y = f(x)$ is continuous over $[a, b]$, is $y = f(x)$ continuous over $[x, x + \Delta x]$?

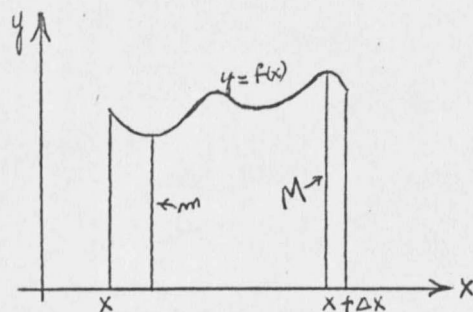
yes

Since y is continuous over $[x, x + \Delta x]$, is y bounded above and below over $[x, x + \Delta x]$?

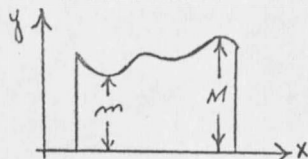
yes

In the following diagram sketch and label an ordinate m such that $y \geq m$ for all x in $[x, x + \Delta x]$. Also do the same for an ordinate M such that $y \leq M$ for all x in $[x, x + \Delta x]$.





9. That is all we need! Now notice the area $\Delta A(x)$.



Circumscribe a rectangle about $\Delta A(x)$. What is its area?

 $M \Delta x$, where M is the maximum ordinate over $[x, x + \Delta x]$.

Inscribe a rectangle in $\Delta A(x)$. What is its area?

 $m \Delta x$, where m is the minimum ordinate over $[x, x + \Delta x]$.

10. Clearly $\Delta A(x)$ is caught between $m \Delta x$ and $M \Delta x$! Represent this fact in terms of inequalities.

 $m \Delta x \leq \Delta A(x) \leq M \Delta x$

11. Since Δx is positive we can divide the inequality in (10) by Δx and obtain:

 $m \leq \frac{\Delta A(x)}{\Delta x} \leq M$

12. Now take the limit of $m \leq \frac{\Delta A(x)}{\Delta x} \leq M$ as $\Delta x \rightarrow 0$. Geometrically what will happen to m and M as $\Delta x \rightarrow 0$? Remember the ordinate at x is $f(x)$!

 They will become closer and closer together --- and in fact will both approach $f(x)$.

13. Again look at $m \leq \frac{\Delta A(x)}{\Delta x} \leq M$. What will become of $\frac{\Delta A(x)}{\Delta x}$ in the limit as $\Delta x \rightarrow 0$?

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta A(x)}{\Delta x} = \frac{dA(x)}{dx}$$

14. But if $m \rightarrow f(x)$, $M \rightarrow f(x)$, and $\frac{\Delta A(x)}{\Delta x} \rightarrow \frac{dA(x)}{dx}$, what can you conclude will be the net result of $\lim_{\Delta x \rightarrow 0} m \leq \frac{\Delta A(x)}{\Delta x} \leq M$?

$$\frac{dA(x)}{dx} = f(x).$$

15. That's it! We now have the differential equation we were after. Clearly: $dA(x) = f(x)dx$ so $\int dA(x) = \int f(x)dx$. Integrate: $\int dA(x)$.

$$\int dA(x) = A(x) + c$$

16. Thus you can connect $\int f(x)dx$ and $A(x)$. Do so.

$$\int f(x)dx = A(x) + c$$

17. You see! $\int f(x)dx$ is the area function. In order to calculate the area for any specific value of x we will now need only to calculate the constant of integration. To do so we will use an initial condition. Indeed, what reasonable condition should we impose on the area function, $\int f(x)dx$, when $x = a$? (Refer to the sketch in No. 3.)

We would like to have $\int f(x)dx$ be zero.

18. But this means when $x = a$, $0 = A(a) + c$. What must the constant of integration for the area function turn out to be?

$$c = -A(a)$$

19. That's the story. The area between abscissas a and x must be $A(x) - A(a)$. Now if one let's x take the position at b , clearly the area from the abscissa at a to the abscissa at b is _____.

$$A(b) - A(a)$$

20. Only one hole remains to be patched up. We insisted at first that Δx be positive. However, we must now make certain Δx could be either positive or negative. (After all, there is no restriction on how $\Delta x \rightarrow 0$ in $\lim_{\Delta x \rightarrow 0} \frac{\Delta A(x)}{\Delta x}$!)

When one divides the result of step 10 by a negative Δx , the inequality is changed. How?

$$m \geq \frac{\Delta A(x)}{\Delta x} \geq M$$

21. However, when we again take the limit as $\Delta x \rightarrow 0$, we are forced to the conclusion: $\frac{dA(x)}{dx} = f(x)$. In either case the differential equation $dA(x)/dx = f(x)$ ultimately boils down in solution to (solve the differential equation.)

$$dA(x)/dx = f(x) \text{ implies } \int dA(x) = \int f(x)dx$$

or $A(x) + c = \int f(x)dx$

22. To put the icing on the cake recall that $\int f(x)dx = F(b) - F(a)$. Clearly in our case $\int dA(x) = \int f(x)dx$ or $dA(x)/dx = f(x)$ implies that the $A(x)$ is exactly the same $F(x)$ which we first learned about.

So in terms of the area function can you represent $\int_a^b f(x)dx$? Do so.

$$\int_a^b f(x)dx = A(b) - A(a)$$

23. In review we see four important facts were used to prove the relation between the integral and the area:
1. A continuous function over a closed interval achieves a maximum and a minimum value within the interval.
 2. An inequality can be set up involving m , M , Δx , and $\Delta A(x)$.
 3. The limit of the inequality results in a solveable differential equation.

4. The initial condition on the area function at $x = a$ can be used to evaluate the constant of integration and hence give a formula for the area under the curve over the closed interval.

What is the result of the first two facts?

$$m \Delta x \leq \Delta A(x) \leq M \Delta x$$

What is the result of the third fact?

$$dA(x)/dx = f(x) \text{ or } A(x) + c = \int f(x)dx$$

What is the result of the fourth fact?

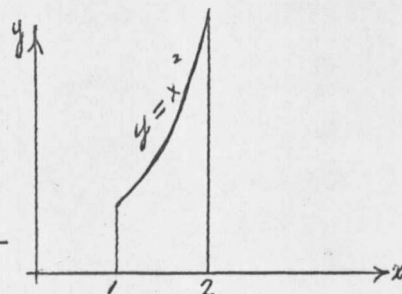
$$A(a) + c = 0 \text{ or } c = -A(a)$$

$$\text{Hence } \int_a^x f(x)dx = A(x) - A(a)$$

$$\text{or } \int_a^b f(x)dx = A(b) - A(a)$$

24. Now let's put the theorem to use. Let's find the area bounded by $x = 1$, $x = 2$, $y = 0$, and $y = x^2$.

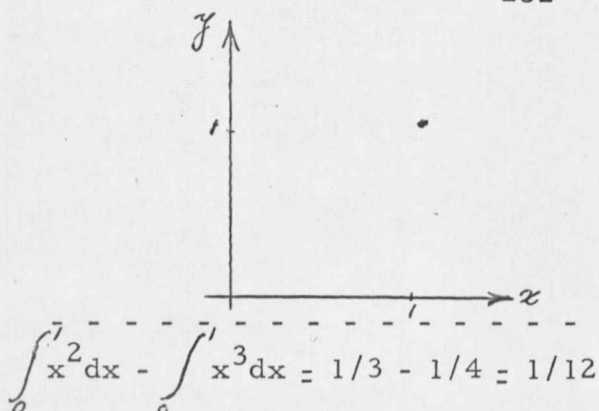
$$\text{Area} = \int_1^2 x^2 dx = \left. \frac{x^3}{3} \right|_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$



Now you find the area bounded by $y = x^3$, $x = 1$, $x = 2$, and $y = 0$.

$$\text{Area} = \int_1^2 x^3 dx = \left. \frac{x^4}{4} \right|_1^2 = 16/4 - 1/4 = 15/4$$

25. You see, even slight modifications of the principles allow a much wider application. For example, consider the area bounded between $y = x^2$ and $y = x^3$. Clearly one could calculate the area under x^2 and subtract from it the area under x^3 . Do so. (Watch the limits of integration.) Sketch the problem.



26. Of course you may wonder in the above case if $\int_0^1 x^2 dx - \int_0^1 x^3 dx = \int_0^1 (x^2 - x^3) dx$. Let's examine this in the general case since it seems obvious in the particular case. We wish to prove (or disprove): $\int_a^b f(x) dx + \int_a^b g(x) dx = \int_a^b (f(x) + g(x)) dx$.

Suppose $\int_a^b f(x) dx = F(b) - F(a)$ and $\int_a^b g(x) dx = G(b) - G(a)$.

In order to establish the truth of the suspected fact we must show that:

$$\int_a^b (f(x) + g(x)) dx = \dots \quad (\text{Get your answer in terms of } F \text{ and } G.)$$

We need $\int_a^b (f(x) + g(x)) dx = (F(b) + G(b)) - (F(a) + G(a))$.

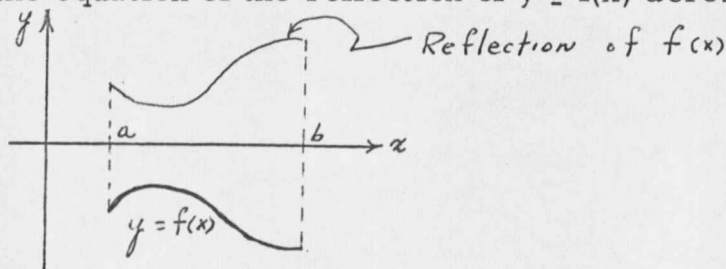
27. With the given information and the knowledge of what you need to complete the proof, write the essential steps of the proof. Start with the definition of $\int_a^b f(x) dx$ and $\int_a^b g(x) dx$.

$$\begin{aligned} \int_a^b f(x) dx + \int_a^b g(x) dx &= F(b) - F(a) + G(b) - G(a) \\ &= (F(b) + G(b)) - (F(a) + G(a)) \\ &= \int_a^b (f(x) + g(x)) dx. \end{aligned}$$

28. Use the theorem you have just proven to write $\int_1^3 (x^2 - 2x) dx - \int_1^3 (x^2 + x) dx$ as one integral.

$$\int_1^3 (2x^2 - x) dx$$

29. Now we need to extend our horizons again. Remember that the restriction on $f(x)$ in all previous arguments was that $f(x)$ be positive. We now want to see if we could calculate the area bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$ if $f(x)$ is negative over the closed interval from a to b . To get the measure of the area by our previous method we could merely consider the mirror image of the curve across the x -axis. If $y = f(x)$ is negative over $[a, b]$, what is the equation of the reflection of $y = f(x)$ across the x -axis?



$$y = -f(x)$$

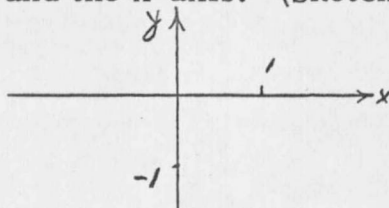
30. Using this idea we can evaluate $\int_a^b -f(x)dx$ to get the measure of the area in question. But again this suggests a general theorem:

$$\int_a^b -f(x)dx = - \int_a^b f(x)dx. \text{ Or even stronger yet: } \int_a^b cf(x)dx = c \int_a^b f(x)dx. \text{ Take } \int_a^b f(x)dx = F(b) - F(a) \text{ and prove:}$$

$$\int_a^b cf(x)dx = c \int_a^b f(x)dx. \text{ (Start with } c \int_a^b f(x)dx = \dots \text{.)}$$

$$c \int_a^b f(x)dx = c [F(b) - F(a)] = cF(b) - cF(a) = \int_a^b cf(x)dx.$$

31. Use the theorem you have just proven to find the area bounded by $y = x^2 - 1$ and the x -axis. (Sketch the figure involved.)

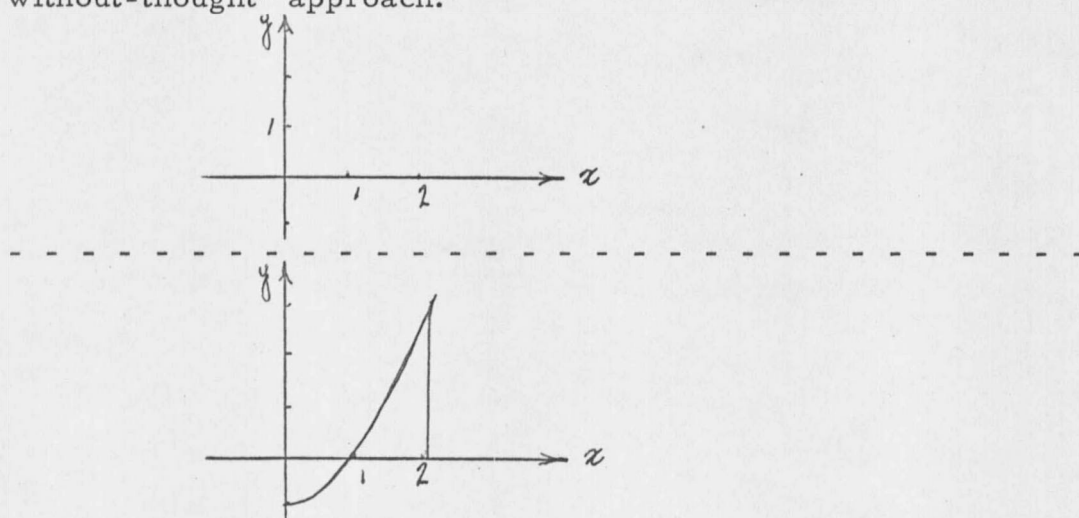


$$\text{Area} = \int_{-1}^1 (1 - x^2)dx = \left[x - \frac{x^3}{3} \right]_{-1}^1 = 2/3 - (-2/3) = 4/3.$$

32. Hence it appears that one can conveniently evaluate areas bounded by $y = f(x)$, $x = a$, $x = b$, and the x -axis by mere integration. But some problems involve a wee bit of insight - - - don't become too mechanical. For example, if you are asked to find the area bounded by $y = x^2 - 1$, $x = 0$, $x = 2$ and the x -axis, the solution is not

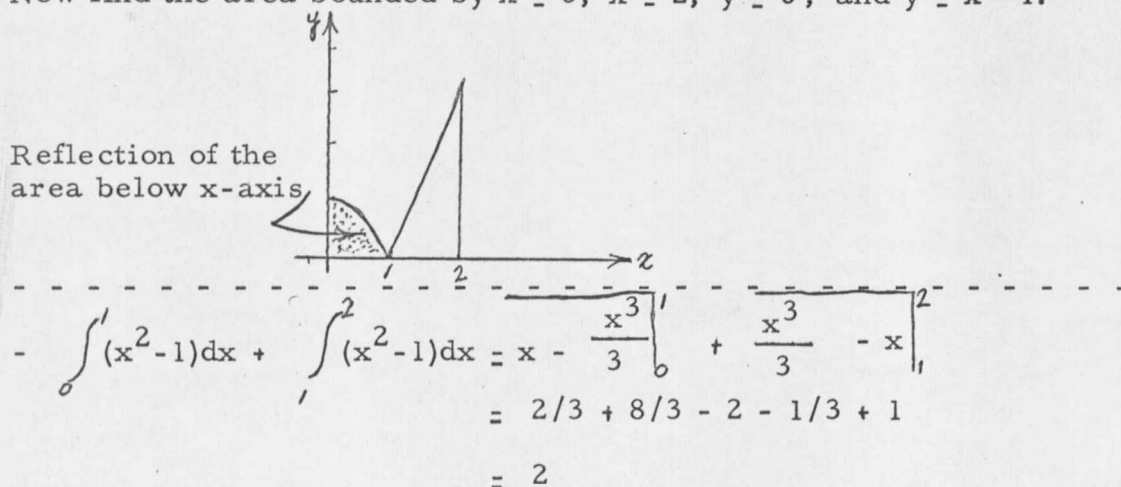
$$\int_0^2 (x^2 - 1) dx = \left[\frac{x^3}{3} - x \right]_0^2 = 2/3: \text{ Sketch } y = x^2 - 1 \text{ over}$$

$[0, 2]$ and this should show you the error in a "mechanical-without-thought" approach.



33. You see! The mechanical method automatically subtracted the area below the x -axis from the positive area! You need to calculate the magnitude of the area below the x -axis in terms of a separate integral by considering $-(x^2 - 1)$ rather than $x^2 - 1$.

Now find the area bounded by $x = 0$, $x = 2$, $y = 0$, and $y = x^2 - 1$.



34. As a matter of vocabulary a distinction is made between the values $\int_0^2 (x^2 - 1) dx$ and $-\int_0^1 (x^2 - 1) dx + \int_1^2 (x^2 - 1) dx$. The first accounts for the negative area by automatically subtracting this area from the positive area. Hence this result which reflects concern for signed areas is called the algebraic area. On the other hand $-\int_0^1 (x^2 - 1) dx + \int_1^2 (x^2 - 1) dx$ results in the magnitude of the area defined. Hence it is strictly a geometric approach and the so found area is called the geometric area.

Find the algebraic area bounded by $y = x^3$, $x = -1$, $x = 1$, $y = 0$.

$$-\int_{-1}^1 x^3 dx = 0$$

35. Now find the geometric area bounded by $y = x^3$, $y = 0$, $x = -1$, $x = +1$.

$$2 \int_0^1 x^3 dx = \frac{1}{2}.$$

36. As a last thought we should prove a final but important theorem. Suppose $y = f(x)$ is continuous over the larger of the closed intervals $[a, b]$, and $[a, c]$: If $\int_a^b f(x) dx = F(b) - F(a)$, prove

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx. \text{ Start with the right hand side.}$$

$$\begin{aligned} \int_a^c f(x) dx + \int_c^b f(x) dx &= (F(c) - F(a)) + (F(b) - F(c)) \\ &= F(b) + F(c) - F(c) - F(a) \\ &= F(b) - F(a) \\ &= \int_a^b f(x) dx. \end{aligned}$$

37. You see! The theorems are all similar in proof and all depend on

$$\int_a^b f(x) dx = F(b) - F(a).$$

Use the theorem you have just proven to write $\int_1^3 f(x) dx + \int_3^4 f(x) dx$ as a single integral.

$$\int_1^4 f(x) dx$$

Lesson XIV

SUMMATION NOTATION AND MATHEMATICAL INDUCTION

- One important phase of beginning calculus has to do with summation notation. This notation is frequently used by staticians and computer programmers as well as other people in mathematically oriented fields. Hence, we want to look into this new notation in this lesson.

What important notation will be examined in this lesson?

 summation notation

- Actually the notation is simply an abbreviated way of expressing a sum, the abbreviation being the Greek letter Σ which means: "the sum of". Of course, you may ask, "the sum of what?" and the answer lies in the expression following Σ . For example: Σk means: "the sum of k". But again you may ask, "What values does k take on?" and the answer is given by limits on the summation symbol. For example: $\sum_{k=1}^n k$ means: form the sum of k as k takes on integral values $k = 1, 2, 3, \dots, n$.

Find $\sum_{k=1}^3 k$.

 $1 + 2 + 3 = 6$

- Actually the sum need not always start at 1 or terminate at n. It could start at any integral value and end at any integral value. For example $\sum_{k=3}^5 k$ would mean: form the sum of 3, 4, and 5.

Find $\sum_{k=10}^{12} k$.

 $10 + 11 + 12 = 33$

- Of course the serious person expects more versatility from notation than has so far been exhibited. And indeed we insist on this. After all, we could desire to sum values other than consecutive integers!

Hence, let's define $\sum_{k=a}^b f(k)$ as $f(a) + f(a+1) + f(a+2) + f(a+3) + \dots + f(b)$,

where the a and b are integral with $b \geq a$.

What would $\sum_{k=1}^3 f(k)$ be?

 $f(1) + f(2) + f(3)$

5. Actually the real versatility comes in the design of f . You see, $f(x)$ could be x^2 , $\ln x$, $\tan x$, ... Calculate $\sum_{k=2}^4 f(k)$ if $f(x) = x^2$.

 $2^2 + 3^2 + 4^2 = 29$

6. As a final thought with respect to the notation notice that $\sum_{i=1}^n f(i)$ and $\sum_{k=1}^n f(k)$ are exactly the same. Write out the first two terms of each sum to verify this.

 $\sum_{i=1}^n f(i) = f(1) + f(2) + \dots; \sum_{k=1}^n f(k) = f(1) + f(2) + \dots$

7. You see! The role of the i or k is simply to carry the proper numerical label for f . These "index" numbers simply keep track of the manner in which f progresses.

Does the letter j occur in the expanded sum $\sum_{j=1}^n f(j)$?

 no

8. Now let's look at some of the properties of summation notation. First, it is true that $\sum_{i=1}^n c f(i) = c \sum_{i=1}^n f(i)$. Write out a proof of this fact using expansions of each notation.

 $\sum_{i=1}^n c f(i) = c f(1) + c f(2) + \dots + c f(n) = c(f(1) + f(2) + \dots + f(n)) = c \sum_{i=1}^n f(i).$

9. Consider $\sum_{i=1}^n (f(i) + g(i))$. It seems reasonable to believe the sum could be considered: $\sum_{i=1}^n f(i) + \sum_{i=1}^n g(i)$. Use expansions of the notation to prove the theorem.

$$\begin{aligned} \sum_{i=1}^n (f(i) + g(i)) &= f(1) + g(1) + f(2) + g(2) + \dots + f(n) + g(n) \\ &= (f(1) + f(2) + \dots + f(n)) + (g(1) + g(2) + \dots + g(n)) \\ &= \sum_{i=1}^n f(i) + \sum_{i=1}^n g(i). \end{aligned}$$

10. Now what about $\sum_{i=1}^n f(i) g(i)$? Could this be $\sum_{i=1}^n f(i) \sum_{i=1}^n g(i)$?

Expand to the first two terms of each sum before you answer.

$$\sum_{i=1}^n f(i) g(i) = f(1) g(1) + f(2) g(2) + \dots$$

$$\sum_{i=1}^n f(i) = f(1) + f(2) + \dots$$

$$\sum_{i=1}^n g(i) = g(1) + g(2) + \dots$$

Now you should be able to judge the truth of the conjecture. Do so.

$$\sum_{i=1}^n f(i) g(i) \neq \sum_{i=1}^n g(i) \sum_{i=1}^n g(i)$$

11. You see! It doesn't always pay to act without thinking. You must be aware of the consequences of your assertion! With this hand-slapping judge the truth of:

$$\frac{d}{dx} \sum_{k=1}^n x^k = \sum_{k=1}^n k x^{k-1} \quad \text{and prove your assertion.}$$

$$\frac{d}{dx} \sum_{k=1}^n x^k = \frac{d}{dx} (x^1 + x^2 + \dots + x^n) = 1 + 2x + 3x^2 + \dots + nx^{n-1} = \sum_{k=1}^n k x^{k-1}$$

12. Here's an easy one. If $1 < j < n$, prove: $\sum_{k=1}^j f(k) + \sum_{k=j+1}^n f(k) = \sum_{k=1}^n f(k)$.

$$\begin{aligned} \sum_{k=1}^j f(k) + \sum_{k=j+1}^n f(k) &= f(1) + f(2) + f(3) + \dots + f(j) + f(j+1) + f(j+2) + \dots \\ &+ f(n) = \sum_{k=1}^n f(k). \end{aligned}$$

13. Also, one often finds that the advantage of changing limits of summation is important. For example,

$$\sum_{k=1}^n f(x) = \sum_{k=2}^{n+1} f(k-1) \quad \text{since} \quad \sum_{k=2}^{n+1} f(k-1) = f(2-1) + f(3-1) + \dots + f(n+1-1) = \sum_{k=1}^n f(k).$$

Suppose one wished to change the limits of $\sum_{i=1}^n f(i)$ to run not from 1 to n but from -1 to n-2.

Express such a sum using proper notation.

$$\sum_{i=1}^n f(i) = \sum_{i=-1}^{n-2} f(i+2)$$

14. Try one more of this type. If you desire to change $\sum_{i=6}^{n+1} f(i)$ to the same sum but with summation limits starting at 3, what would you have? (Hint: write out the first term and the last term of $\sum_{i=6}^{n+1} f(i)$ and of your answer.)
-

$$\sum_{i=6}^{n+1} f(i) = \sum_{i=3}^{n-2} f(i+3)$$

15. Now for an application of summation notation. Frequently the mathematician finds he must simplify an expression involving sums in which summation notation is used. One such expression might be: $\sum_{i=1}^n (x_i)^2 + \sum_{j=1}^{n+1} (y_j)^2$. Of course it would be easier to work with the entire expression if it were written as one term - not as two. Do so using as the sum: $\sum_{k=3} \dots$
-

$$\sum_{i=1}^n (x_i)^2 + \sum_{j=2}^{n+1} (y_j)^2 = \sum_{k=3}^{n+2} ((x_{k-2})^2 + (y_{k-1})^2).$$

16. As a last exercise in summation notation consider the sum:

$$\sum_{i=1}^n f(i+1) - f(i).$$

Write out the first two and last two elements of this sum.

$$\sum_{i=1}^n (f(i-1) - f(i)) = f(2) - f(1) + f(3) - f(2) + \dots + f(n) - f(n-1) + f(n+1) - f(n)$$

Now looking at this sum carefully you should see that it collapses. To what value does it collapse?

$$f(n+1) - f(1)$$

17. We are now ready to turn our attention to proofs by mathematical induction. Such proofs are entirely dependent upon the positive integers. That is, if you agree (1) there is a first positive integer and (2) that every positive integer has a successor, you must admit the validity of proofs by mathematical induction.

Proofs by mathematical induction are dependent upon the properties of the _____.

positive integers

18. You see, we will assert the truth of a statement. Then we will:
- (1) prove our assertion true in the first case (P_1).
 - (2) prove that the truth of the assertion in the general k th case (P_k) implies the truth of the assertion in the k plus first case (P_{k+1}).
- It's very simple. You see, if we have two arguments on our side:
- (1) the assertion is true in case 1,
 - (2) the truth of the assertion in case k implies the truth of the assertion in case $k + 1$;

we can claim the truth of the assertion in every case.

If an assertion is true in the first case and it has been established that the truth of the k th proposition implies the truth of the $(k+1)$ st proposition, why would the assertion be true in the second case?

The truth of case I implies the truth of case II.

But then if case II is true, why would case III be true?

The truth of case II implies the truth of case III.

But then if case III is true, why would case IV be true?

The truth of case III implies the truth of case IV.

19. You see! Because the positive integers each have a successor, the assertion will be true in every case.

To lend even more credibility to the argument let the proposition which you desire to prove be denoted by P and the instance in which it is considered by a subscript on P . That is, P_1 is the proposition in the first case; P_2 in the second case; P_k in the k th case.

Suppose: (1) P_1 is true; (2) the truth of P_k implies the truth of P_{k+1} for all positive integral k .

We now claim P is true for all k .

- By indirect proof:
1. Either P is true for all k or P is not true for all k .
 2. Suppose P is not true for all k .
 3. Suppose the first case in which P fails is n . (That is, P is true for all $m < n$.)
 4. But then P is true for $n-1$.
 5. We claim P is now true for n . Why?
-

By the second accepted fact: the truth of P_k implies the truth of P_{k+1} means the truth of P_{n-1} implies the truth of P_n .

6. Hence since this argument produced a contradiction to the hypothesis that P was not true in the n th case it follows that P is true for all k .

List the two conditions upon which proofs by mathematical induction depend.

1. P_1 must be true.
 2. The truth of P_k must imply the truth of P_{k+1} .
-

20. In order to make a proof by mathematical induction you must prove two things. What are they?
-

1. P_1 must be true.
 2. The truth of P_k must imply the truth of P_{k+1} .
-

21. Let's look at a couple of examples of incorrect application of mathematical induction. First, consider P to be: every two consecutive positive integers are equal.

Proof: the truth of P_k implies the truth of P_{k+1} since:

if $k = k+1$
 and certainly $1 = 1$,
 then $k + 1 = (k+1) + 1$.

Thus every positive integer is equal to its successor.

Yes! If P_k is true, P_{k+1} is true also. But the proof fails in one important respect.

What is it?

Every proof by mathematical induction must show P_1 to be true, but $1 \neq 2$. Hence P_1 fails to be true.

22. Consider P to be the proposition: $\sum_{i=1}^n i = n^2$.
 Certainly P_1 is true. That is, $\sum_{i=1}^1 i = 1^2 = 1$.

But to prove P true one must also show that the truth of P_k implies the truth of P_{k+1} . Let's try it.

Assume P_k true. i.e. assume $\sum_{i=1}^k i = k^2$.

Now to show the truth of P_{k+1} follows, what must we prove?

$$\sum_{i=1}^{k+1} i = (k+1)^2$$

But this is not a true mathematical consequence of P_k since if

$\sum_{i=1}^k i = k^2$ then $\sum_{i=1}^{k+1} i = \underline{\hspace{2cm}}$ which is not $(k+1)^2$.

$$k^2 + k + 1$$

23. Hence you see that every proof by mathematical induction must include two steps:

- (1) P_1 must be proven true.
- (2) if P_k is true, the truth of P_{k+1} must be a consequence.

Why could one not prove P : $\sum_{k=1}^n (a_k b_k)^2 = \sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2$? (When in doubt, write out a few terms of the sum.)

Although P_1 holds, the truth of P_k does not imply the truth of P_{k+1} .

24. Usually the tough part in a proof by mathematical induction is the proof of the transitive property, that is, to prove: P_k implies P_{k+1} true. So let's look at this a little closer.

We always assume P_k to be true and try to show the truth P_{k+1} follows. Hence if P_k is the proposition: $\sum_{i=1}^k 2i = k(k+1)$, what must we show to be true?

$$\sum_{i=1}^{k+1} 2i = (k+1)((k+1) + 1)$$

25. Do so. (Remember: you must proceed from $\sum_{i=1}^k 2i$ to $\sum_{i=1}^{k+1} 2i$. To do this you must add one term to $\sum_{i=1}^k 2i$.)
-

By assumption: $\sum_{i=1}^k 2i = k(k+1) = k^2 + k$

But $2(k+1) = 2(k+1)$

Thus $\sum_{i=1}^{k+1} 2i = k^2 + 3k + 2 = (k+1)((k+1)+1).$

26. Once more, if P is the proposition: $\sum_{i=1}^n \frac{1}{(3i-2)(3i+1)} = \frac{n}{3n-1}$, what is P_k ? And what must P_{k+1} be?

$$P_k: \sum_{i=1}^k \frac{1}{(3i-2)(3i+1)} = \frac{k}{3k+1}$$

$$P_{k+1}: \sum_{i=1}^{k+1} \frac{1}{(3i-2)(3i+1)} = \frac{k+1}{3k+4}$$

27. Prove the proposition P in (26) is true.

$$P_1 \text{ is true since: } \sum_{i=1}^1 \frac{1}{(3i-2)(3i+1)} = \frac{1}{1(4)}$$

$$\text{and } \frac{1}{(3)1+1} = \frac{1}{4}$$

$$\text{Now assume } P_k \text{ true: } \sum_{i=1}^k \frac{1}{(3i-2)(3i+1)} = \frac{k}{3k+1}$$

$$\begin{aligned} \text{but } \sum_{i=1}^{k+1} \frac{1}{(3i-2)(3i+1)} &= \sum_{i=1}^k \frac{1}{(3i-2)(3i+1)} + \frac{1}{(3k+1)(3k+4)} = \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)} \\ &= \frac{k(3k+4)+1}{(3k+1)(3k+4)} = \frac{(3k^2+4k+1)}{(3k+1)(3k+4)} = \frac{(3k+1)(k+1)}{(3k+1)(3k+4)} = \frac{k+1}{3k+4}. \end{aligned}$$

∴ P is true.

28. As a last algebraic problem consider as the proposition P: $2^n > 2n + 1$. This time the assertion is that P_1 and P_2 are false, but that P is true for all other values of $n > 2$.

To prove P_1 and P_2 false is easy: $2 \nless 2 + 1$ and $2^2 \nless 4 + 1$. And it is as easy to show P_3 true: $2^3 > 6 + 1$.

But to prove the transitive property is not quite so easy. That is, if $2^k > 2k + 1$ you must prove _____.

$$2^{k+1} > 2k + 3 \text{ or } 2(k+1) + 1$$

Do so using the hint: $2^k > 2$. (Surely this is true for $k \geq 3$.)

Since $2^k > 2k + 1$, $2^{k+1} > 2k + 3$

Since unequals ($2^k > 2k+1$) added to unequals ($2^k > 2$) in the same order results in unequals ($2^{k+1} > 2k + 3$) in that same order.

29. Now let's try one problem which is geometric in nature. Suppose we desire to prove P: a straight line is divided into $n+1$ segments by n distinct points of division. First, is P_1 true?
-

Yes.

Now assume P_k true. Does P_{k+1} follow?

Yes. If the line is divided by k distinct points into $k+1$ pieces, the addition of one more distinct point introduces one more segment. So $k+1$ distinct points yield $k+2$ parts of the line.

Has P been proven true on the basis of the foregoing argument?

Yes.

30. As a quick review, state the two conditions which must be verified when one makes a proof by mathematical induction.
-

1. P_1 must be true.

2. The truth of P_k must imply the truth of P_{k+1} .

Lesson XV

AREA AS THE LIMIT OF A SUM

1. In this lesson we want to demonstrate that area under a curve can be expressed as the limit of a sum. Since this interpretation is essential for the true understanding of the definite integral, we want to examine it with care.

In this lesson we want to show that area under a curve can be expressed as the limit of a _____.

sum

-
2. To start, let us consider a continuous function, $y = f(x)$, over the closed interval a to b . Furthermore, assume y to be positive over $[a, b]$ for simplicity.

What restrictions have we imposed so far on y ?

 y is continuous over $[a, b]$, y is positive over $[a, b]$.

-
3. Now insert abscissas $a = x_0 < x_1 < x_2 < x_3 < \dots < x_n = b$ such that $x_i - x_{i-1} = \Delta x$ for $i = 1, 2, \dots, n$. That is, insert division points along $[a, b]$ which divide $[a, b]$ into n equal pieces.

How long is Δx ?

Length of Δx is $(b-a)/n$.

-
4. Now since y is continuous over $[a, b]$, clearly y is continuous over $[x_{i-1}, x_i]$ where $i = 1, 2, \dots, n$. From this we can use the fact that a continuous function over a closed interval must assume a maximum and a minimum value somewhere over the same interval.

In particular (with this fact in mind) what can you say about f over $[a = x_0, x_1]$?

 f must achieve a maximum and a minimum value somewhere over $[x_0, x_1]$

What of f over $[x_{i-1}, x_i]$?

f must achieve a maximum and a minimum value somewhere over $[x_{i-1}, x_i]$.

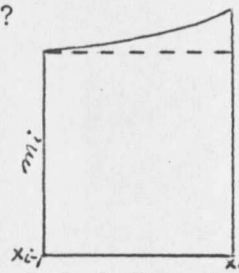
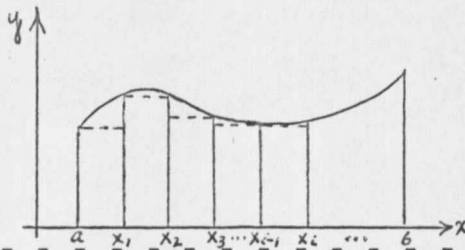
5. In fact let's name the maximum and minimum ordinates which exist in each subinterval. Let's call the minimum ordinate in $[x_{i-1}, x_i]$ by m_i and let's call the maximum ordinate in $[x_{i-1}, x_i]$ by M_i .

What is the minimum ordinate in $[x_4, x_5]$? The maximum ordinate in $[x_{12}, x_{13}]$?

m_5, M_{13}

6. That did it! Now we can proceed with full steam. As a means of illustration consider the figure below. In each sector bounded by $y = 0$, $y = f(x)$, $x = x_i$, $x = x_{i-1}$ inscribe a rectangle ($i = 1, 2, \dots, n$).

What must the area of the i th rectangle be?



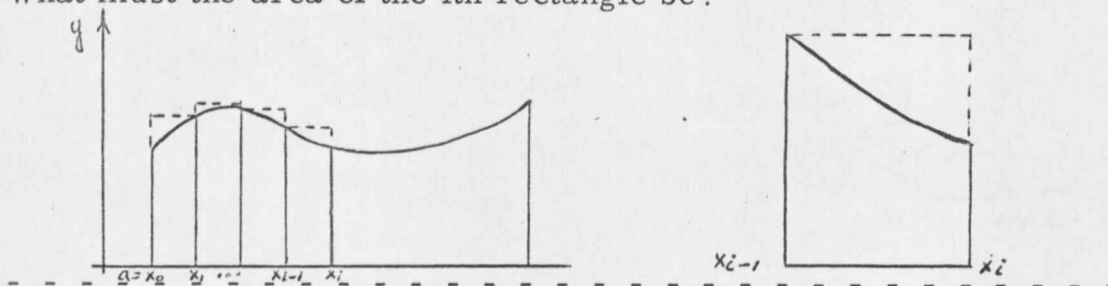
Area of rectangle $i = m_i \Delta x$.

7. Call the sum of all such inscribed rectangles, $\underline{S}$. Calculate $\underline{S}$.
-

$$\underline{S} = \sum_{i=1}^n m_i \Delta x$$

8. Now reconsider $y = f(x)$ with its partitions as described previously. This time circumscribe a rectangle about each sector bounded by $y = 0$, $y = f(x)$, $x = x_{i-1}$, $x = x_i$ ($i = 1, 2, \dots, n$).

What must the area of the i th rectangle be?



Area of the i th rectangle is: $M_i \Delta x$.

9. Calculate the sum of all such circumscribed rectangles, $\bar{S}$.

$$\bar{S} = \sum_{i=1}^n M_i \Delta x.$$

10. Clearly $\underline{S} \leq$ true area. Clearly $\bar{S} \geq$ true area.

But we know: true area is $\int_a^b f(x) dx$.

Express in one statement of inequality the relationship of $\underline{S}$ to

$\int_a^b f(x) dx$ to $\bar{S}$.

$$\underline{S} \leq \int_a^b f(x) dx \leq \bar{S}.$$

11. There, we have boxed in $\int_a^b f(x) dx$ by $\underline{S}$ and $\bar{S}$. File this for reference since we will need it shortly.

Now let's turn our attention again to $f(x)$. Clearly if $f(x)$ is continuous over $[a, b]$, then $f(x)$ is continuous over $[x_{i-1}, x_i]$ and there exists a maximum ordinate, M_i , and a minimum ordinate, m_i , over $[x_{i-1}, x_i]$ ($i = 1, 2, \dots, n$).

However, suppose ξ_i is any point in $[x_{i-1}, x_i]$, ($i = 1, 2, \dots, n$).

What must be the relationship of m_i to $f(\xi_i)$ to M_i ?

$$m_i \leq f(\xi_i) \leq M_i.$$

12. But if $m_i \leq f(\xi_i) \leq M_i$, $i = 1, 2, \dots, n$, then $m_i \Delta x \leq f(\xi_i) \Delta x \leq M_i \Delta x$. Why?

Both sides of an inequality may be multiplied by the same positive value and the state of inequality remains constant.

13. But again if $m_i \Delta x \leq f(\xi_i) \Delta x \leq M_i \Delta x$ for each $i = 1, 2, \dots, n$, then one could form the sum of these inequalities as $i = 1, 2, \dots, n$. Do so.

$$\sum_{i=1}^n m_i \Delta x \leq \sum_{i=1}^n f(\xi_i) \Delta x \leq \sum_{i=1}^n M_i \Delta x.$$

14. But this result involves our familiar $\underline{S}$ and $\bar{S}$. Express the result of (13) using $\underline{S}$ and $\bar{S}$.

$$\underline{S} \leq \sum_{i=1}^n f(\xi_i) \Delta x \leq \bar{S}$$

15. With this result we have boxed $\sum_{i=1}^n f(\xi_i) \Delta x$ in by $\underline{S}$ and $\bar{S}$ also. We will now show $\lim_{\substack{n \rightarrow \infty \\ \Delta x \rightarrow 0}} |\underline{S} - \bar{S}| < \epsilon$, where ϵ is a positive, arbitrarily small value.

If we can do this, what must be true of: $\left| \sum_{i=1}^n f(\xi_i) \Delta x - \int_a^b f(x) dx \right|$?

$\left| \sum_{i=1}^n f(\xi_i) \Delta x - \int_a^b f(x) dx \right| < \epsilon$ where ϵ is a positive, arbitrary small value.

16. This would be a worthy result indeed for then we could state:

$$\lim_{\substack{n \rightarrow \infty \\ \Delta x \rightarrow 0}} \sum_{i=1}^n f(\xi_i) \Delta x = \int_a^b f(x) dx. \quad \text{Why?}$$

If for any positive ϵ no matter how small there exists a positive δ (in this case $n > N$ or $\Delta x < \delta$) such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$ (when $n > N$ or $\Delta x < \delta$), then $\lim_{x \rightarrow a} f(x) = L$.

17. So here we go. We must show: $\lim_{\substack{n \rightarrow \infty \\ \Delta x \rightarrow 0}} |\underline{S} - \overline{S}| = 0$ or
 equivalently $\lim_{\substack{n \rightarrow \infty \\ \Delta x \rightarrow 0}} \underline{S} = \lim_{\substack{n \rightarrow \infty \\ \Delta x \rightarrow 0}} \overline{S}$

Consider $\underline{S} - \overline{S}$. $\underline{S} = m_1 \Delta x + m_2 \Delta x + \dots + m_n \Delta x$
 $\overline{S} = M_1 \Delta x + M_2 \Delta x + \dots + M_n \Delta x.$

Calculate $|\overline{S} - \underline{S}|$.

$$\overline{S} - \underline{S} = (M_1 - m_1) \Delta x + (M_2 - m_2) \Delta x + (M_3 - m_3) \Delta x + \dots + (M_n - m_n) \Delta x.$$

18. However, since $f(x)$ is continuous over each subinterval of $[a, b]$ we can force the difference between each M_i and m_i to be as small as we please by simply forcing the number of subdivisions to be large enough! It seems worth our effort to force this difference between M_i and m_i to be the constant ϵ' for each $i = 1, 2, \dots, n$.

Using this fact: $M_i - m_i < \epsilon'$ for each $i = 1, 2, \dots, n$, how small can the difference $|\overline{S} - \underline{S}|$ be made?

$$|\overline{S} - \underline{S}| < \epsilon' \Delta x + \epsilon' \Delta x + \dots + \epsilon' \Delta x = n \epsilon' \Delta x.$$

19. But look again at $n \epsilon' \Delta x$. This can be written $\epsilon' (n \Delta x)$. Consult the figure in (6) and state a substitute value for $n \Delta x$.

$$n \Delta x = b - a$$

20. Using this substitute value for $n \Delta x$ how small can $|\overline{S} - \underline{S}|$ be made?

$$|\overline{S} - \underline{S}| < \epsilon' (b - a).$$

21. Remember: $\lim_{\substack{n \rightarrow \infty \\ \Delta x \rightarrow 0}} \bar{S} = \lim_{\substack{n \rightarrow \infty \\ \Delta x \rightarrow 0}} \underline{S}$ if for every $\epsilon > 0$ there exists an N such that if $n > N$ then $|\bar{S} - \underline{S}| < \epsilon$.
 It is highly desirable at this stage to rename the value $\epsilon'(b-a)$.
 What label would you affix to it?

 $\epsilon'(b-a) = \epsilon$, the epsilon in the definition of a limit.

22. That did it. Now we know $\lim_{\substack{n \rightarrow \infty \\ \Delta x \rightarrow 0}} \bar{S} = \lim_{\substack{n \rightarrow \infty \\ \Delta x \rightarrow 0}} \underline{S}$.
 But hold on. We have shown $\lim_{\substack{n \rightarrow \infty \\ \Delta x \rightarrow 0}} \bar{S} = \lim_{\substack{n \rightarrow \infty \\ \Delta x \rightarrow 0}} \underline{S}$ and yet

$$\underline{S} \leq \sum_{i=1}^n f(\xi_i) \Delta x \leq \bar{S} \text{ and}$$

$$\underline{S} \leq \int_a^b f(x) dx \leq \bar{S}.$$

What conclusion can you draw with respect to

$$\lim_{\substack{n \rightarrow \infty \\ \Delta x \rightarrow 0}} \underline{S} \text{ and } \lim_{\substack{n \rightarrow \infty \\ \Delta x \rightarrow 0}} \sum_{i=1}^n f(\xi_i) \Delta x \text{ and}$$

$$\lim_{\substack{n \rightarrow \infty \\ \Delta x \rightarrow 0}} \bar{S} \text{ and } \int_a^b f(x) dx?$$

 These limits are all equal!

23. How do you know that $\lim_{\substack{n \rightarrow \infty \\ \Delta x \rightarrow 0}} \underline{S} = \int_a^b f(x) dx$?

 There exists a number N such that when $n > N$ then $|\bar{S} - \underline{S}| < \epsilon$, ϵ arbitrary but positive. Thus when challenged with a particular ϵ in proving $|\underline{S} - \int_a^b f(x) dx| < \epsilon$ or $\lim_{\substack{n \rightarrow \infty \\ \Delta x \rightarrow 0}} \underline{S} = \int_a^b f(x) dx$, simply pick the same N which satisfied this same ϵ in showing:

$|\bar{S} - \underline{S}| < \epsilon$. (Or simply pick the number of divisions large enough such that $\Delta x < \delta$ where δ is the value referred to in No. 21.)

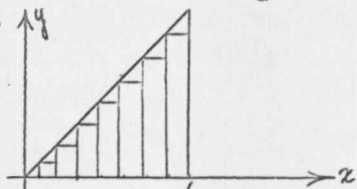
How do you know that $\lim_{\substack{n \rightarrow \infty \\ \Delta x \rightarrow 0}} \sum_{i=1}^n f(\xi_i) \Delta x = \int_a^b f(x) dx$?

There exists a number N such that when $n > N$ then $|\bar{S} - \underline{S}| < \epsilon$, ϵ arbitrary but positive. Thus when challenged with a particular ϵ in proving $|\sum_{i=1}^n f(\xi_i) \Delta x - \int_a^b f(x) dx| < \epsilon$, simply pick the same N which satisfied this ϵ in showing $|\bar{S} - \underline{S}| < \epsilon$.

24. Now let's take a special case and show $S = \int_a^b f(x) dx$. Consider $y = x$ over $[0, 1]$ and divide $[0, 1]$ into n equal subdivisions. How wide is each subdivision?

$1/n$

25. Next we must calculate the height of the rectangles over each $[x_{i-1}, x_i]$.



What is the altitude of the rectangle over $[0, 1/n]$?

zero

Over $[1/n, 2/n]$?

$1/n$

Over $[\frac{i}{n}, \frac{2}{n}]$?

i/n

26. Find $\underline{S}$ as a sum. Express in summation notation.

$$\underline{S} = 0/n \cdot 1/n + 1/n \cdot 1/n + 2/n \cdot 1/n + \dots + (n-1/n) \cdot 1/n.$$

$$= \frac{1}{n^2} \sum_{i=0}^{n-1} i.$$

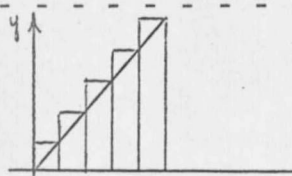
27. Since $\sum_{i=1}^n i = \frac{n(n+1)}{2}$, calculate $\sum_{i=0}^{n-1} i$.

$$\sum_{i=0}^{n-1} i = \frac{(n-1)(n)}{2}$$

28. So $\underline{S} = \frac{1}{n^2} \frac{(n-1)(n)}{2}$. Calculate $\lim_{n \rightarrow \infty} \underline{S}$.

$$\lim_{n \rightarrow \infty} \frac{n^2 - n}{2n^2} = \frac{1}{2}$$

29. Of course $\overline{S}$ can be calculated in a similar manner. Do it for practice.



$$\overline{S} = 1/n \cdot 1/n + 1/n \cdot 2/n + 1/n \cdot 3/n + \dots + 1/n \cdot n/n = \sum_{i=1}^n i \cdot \frac{1}{n^2}$$

$$= \frac{1}{n^2} \frac{n(n+1)}{2} = \frac{n^2 + n}{2n^2} \therefore \lim_{n \rightarrow \infty} \overline{S} = \frac{1}{2}.$$

30. If you really understand it, you should be able to spot the difficulties in calculating the area under $y = x^2$ from $x = 0$ to $x = 2$. Let's see if you are that good. Using n equal subdivision of $[0, 2]$, how wide is each subdivision? (A sketch may help.)

$$2/n$$

31. Suppose we want $\bar{S}$. What are the successive heights of the rectangles?

$$(2/n)^2, (4/n)^2, (6/n)^2, \dots, (2n/n)^2.$$

32. Represent $\bar{S}$ in summation notation.

$$\bar{S} = \sum_{i=1}^n 2/n \cdot (2i/n)^2 = 8/n^3 \cdot \sum_{i=1}^n i^2$$

33. Using the fact that $\sum_{i=1}^n i^2 = \frac{n(2n+1)(n+1)}{6}$ we can now

calculate $\lim_{n \rightarrow \infty} \bar{S}$. Do so.

$$\lim_{n \rightarrow \infty} \frac{8}{n^3} \cdot \frac{2n^3 + 3n^2 + n}{6} = 8/3.$$

34. But what of $\sum_{i=1}^n f(\xi_i) \Delta x$? We haven't used it yet? On the contrary we have calculated this sum three times! But each time we selected the ξ_i to be in a very special position. For instance in inscribing the rectangles inside the sectors of $y = x$, we picked the ξ_i on the _____ of the interval $[x_{i-1}, x_i]$.
left

Similarly in circumscribing rectangles about the sectors of $y = x$, we picked the ξ_i on the _____ of the interval $[x_{i-1}, x_i]$.

right

35. Now let's turn our attention to a slight - but important - modification of the above arguments. Suppose that $[a, b]$ is again partitioned by $a = x_0 < x_1 < x_2 < x_3 < \dots < x_n = b$, but this time the points of division are not taken to be equally spaced but are arbitrarily spaced over $[a, b]$.

If $[a, b]$ is $[0, 20]$, would it be possible to have $x_i - x_{i-1} = 10$ (for some i)?

Yes. Just pick x_i , $i = 0, 1, 2, \dots, i-1$, to occur in $[0, 1]$.

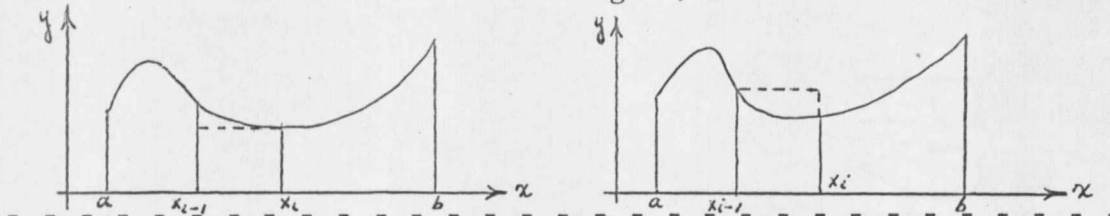
Then pick x_i , $i = i, i+1, \dots, n$ to occur in $[19, 20]$.

This will certainly do the job!

36. As a means of notation call the interval from x_{i-1} to x_i by Δx_i , $i = 1, 2, \dots, n$. How would the interval from x_7 to x_8 be denoted?
-

Δx_8

37. Again suppose $f(x)$ is continuous and positive over $[a, b]$. Clearly we are assured as before of minimum and maximum ordinates in $[x_{i-1}, x_i]$. Call them m_i and M_i respectively. Using this information calculate the area of the inscribed rectangles, $\underline{S}$, and the area of the circumscribed rectangles, $\bar{S}$.



$$\underline{S} = m_1 \Delta x_1 + m_2 \Delta x_2 + \dots + m_n \Delta x_n = \sum_{i=1}^n m_i \Delta x_i$$

$$\bar{S} = M_1 \Delta x_1 + M_2 \Delta x_2 + \dots + M_n \Delta x_n = \sum_{i=1}^n M_i \Delta x_i$$

38. Again $\underline{S} \leq \int_a^b f(x) dx \leq \bar{S}$. So let's see if $\underline{S}$ and $\bar{S}$ again converge to $\int_a^b f(x) dx$ even with the subdivisions of unequal size. To test this we must examine a slightly different limit than before. This time we must insist on more than before because $\lim_{n \rightarrow \infty}$ only assures us of the fact that more divisions are being imposed on $[a, b]$. They could in fact all be imposed onto (x_7, x_8) ! Then surely $\underline{S}$ and $\bar{S}$ would not converge!

Why could all the new points of division be imposed on (x_7, x_8) ?

There exists an infinity of points on (x_7, x_8) .

39. To remedy this malady let's insist that the divisions are always taken so that the largest remaining subdivision is the recipient of the next division point. Let's indicate this sly choice of division as follows: $\lim_{\|\Delta\| \rightarrow 0}$ where $\|\Delta\|$ is read "the norm of Δ " and indicates that we are taking the limit as the largest subdivision, Δx_i where $i = 1, 2, \dots, n$, is to approach 0.

Translate into English and tell the meaning of $\lim_{\|\Delta\| \rightarrow 0} \underline{S}$.

$\lim_{\|\Delta\| \rightarrow 0} \underline{S}$ is read: "limit of $\underline{S}$ as the norm of Δ goes to 0" and means we are to consider the limit of $\underline{S}$ as the largest subdivision of $[a, b]$ goes to zero.

40. Thus we now are ready for $\underline{S}$ and $\overline{S}$. We must show $\lim_{\|\Delta\| \rightarrow 0} \underline{S} = \lim_{\|\Delta\| \rightarrow 0} \overline{S}$ or $\lim_{\|\Delta\| \rightarrow 0} (\underline{S} - \overline{S}) = 0$. To do this consider $\overline{S} - \underline{S}$. Calculate: $\overline{S} - \underline{S}$ as a sum.
-

$$\overline{S} - \underline{S} = (M_1 - m_1) \Delta x_1 + (M_2 - m_2) \Delta x_2 + \dots + (M_n - m_n) \Delta x_n.$$

41. But again $M_i - m_i$, $i = 1, 2, \dots, n$, can be made as small as is desired by insisting that $\|\Delta\| \rightarrow 0$. If $M_i - m_i$, $i = 1, 2, \dots, n$, is in fact made less than ϵ' , what is the limitation on $\overline{S} - \underline{S}$? (Express the difference in summation notation.)
-

$$\overline{S} - \underline{S} = \epsilon' \Delta x_1 + \epsilon' \Delta x_2 + \dots + \epsilon' \Delta x_n =$$

42. But clearly $\sum_{i=1}^n \epsilon' \Delta x_i = \epsilon' \sum_{i=1}^n \Delta x_i$. Now refer to the figure in No. 37 and evaluate $\sum_{i=1}^n \Delta x_i$.
-

$$\sum_{i=1}^n \Delta x_i = b - a$$

43. So, you see! We can now specify that $|\overline{S} - \underline{S}| < \epsilon' (b-a)$ or better yet (since ϵ' can be made arbitrarily small by controlling the size of $\|\Delta\|$) we can set $\epsilon' (b-a) = \epsilon$. But this means that $|\overline{S} - \underline{S}| < \epsilon$ --- where ϵ itself is any arbitrarily small but positive number. Of course we can effect this restriction on

$|\bar{S} - \underline{S}|$ by forcing $\|\Delta\|$ to be less than some positive number δ .

Under these circumstances what conclusion must one reach about

$$\lim_{\|\Delta\| \rightarrow 0} \bar{S} \text{ and } \lim_{\|\Delta\| \rightarrow 0} \underline{S} ?$$

$$\lim_{\|\Delta\| \rightarrow 0} \bar{S} = \lim_{\|\Delta\| \rightarrow 0} \underline{S}$$

44. But since $\underline{S} \leq \int_a^b f(x)dx \leq \bar{S}$ by the precise ϵ argument used when the subdivisions were of equal length, it is a sure bet that
- $$\lim_{\|\Delta\| \rightarrow 0} \underline{S} = \lim_{\|\Delta\| \rightarrow 0} \bar{S} = \int_a^b f(x)dx.$$

Indeed to show that $\lim_{\|\Delta\| \rightarrow 0} \underline{S} = \int_a^b f(x)dx$ it suffices to say that

$|\underline{S} - \int_a^b f(x)dx|$ can be made less than an arbitrary ϵ by simply forcing $\|\Delta\|$ to be less than δ where the δ can be chosen to be the same delta which forced $|\bar{S} - \underline{S}| < \epsilon$, where ϵ is the same ϵ as appeared before.

To prove: $\lim_{\|\Delta\| \rightarrow 0} \bar{S} = \int_a^b f(x)dx$ we only need to demonstrate $|\bar{S} - \int_a^b f(x)dx| < \epsilon$ by choosing a suitable restriction. What is this restriction?

$|\bar{S} - \int_a^b f(x)dx| < \epsilon$ if $\|\Delta\| < \delta$ where δ is the same δ appearing in the $|\bar{S} - \underline{S}| < \epsilon$ argument for that particular ϵ .

45. Finally we must show that despite arbitrary partitioning of $[a, b]$ we have a finite sum which is analogous to the $\sum_{i=1}^n f(\xi_i) \Delta x_i$ in the case of equal divisions. Obviously our new sum will grow out of the same arguments as the old one. That is, for $x_{i-1} \leq \xi_i \leq x_i$, $i = 1, 2, \dots, n$, the following is true: $m_i \leq f(\xi_i) \leq M_i$. But this means $m_i \Delta x_i \leq f(\xi_i) \Delta x_i \leq M_i \Delta x_i$ for $i = 1, 2, \dots, n$. (Notice: this inequality is different from the case of equal divisions in that Δx previously was a constant multiplier for every $m_i \leq f(\xi_i) \leq M_i$. In the present case the $m_i \leq f(\xi_i) \leq M_i$ is multiplied through by the corresponding Δx_i subinterval!)

Now represent the sum of the inequalities $m_i \Delta x_i \leq f(\xi_i) \Delta x_i \leq M_i \Delta x_i$ using summation notation.

$$\sum_{i=1}^n m_i \Delta x_i \leq \sum_{i=1}^n f(\xi_i) \Delta x_i \leq \sum_{i=1}^n M_i \Delta x_i$$

46. See it! The outside sums are our old friends S and $\bar{S}$ respectively. Thus we've found the sum analogous to the corresponding sum in the case of equal divisions. Now we only need to show that

$$\lim_{\|\Delta\| \rightarrow 0} \underline{S} = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i = \lim_{\|\Delta\| \rightarrow 0} \bar{S}$$

But we already know $\lim_{\|\Delta\| \rightarrow 0} \underline{S} = \lim_{\|\Delta\| \rightarrow 0} \bar{S}$.

Hence, since $\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i$ is boxed in by these values, what can we conclude?

$$\lim_{\|\Delta\| \rightarrow 0} \underline{S} = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i = \lim_{\|\Delta\| \rightarrow 0} \bar{S}$$

47. As a last observation we can use the result of (46) and the statement in (44) to pull all the threads together. Do so.

$$\lim_{\|\Delta\| \rightarrow 0} \underline{S} = \lim_{\|\Delta\| \rightarrow 0} \bar{S} = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i = \int_a^b f(x) dx$$

48. So the argument is complete. To nail it up solid let's review the main points.

1. Define $\int_a^b f(x) dx$.

$$\int_a^b f(x) dx = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i \text{ where:}$$

$$1. a = x_0 < x_1 < \dots < x_n = b$$

$$2. x_{i-1} < \xi_i < x_i$$

$$3. \Delta x_i = x_i - x_{i-1}$$

CONT.

4. $\|\Delta\| = \max. \text{ of } \Delta x_i$

2. If there exists a function $F(x)$ such that $F'(x) = f(x)$, then

$\int_a^b f(x) dx = \underline{\hspace{2cm}}$

$F(b) - F(a)$

Appendix C

Notices Sent to Students Missing Classes During
Which Programmed Materials Were Studied

Notices Sent to Students Missing Classes During
Which Programmed Materials were Studied

I notice that you were unable to attend the programmed materials class _____ Monday _____ Tuesday _____ Thursday evening. I hope you can drop by my office to arrange to go over this lesson you missed. I want to caution you that I will be unable to allow students to arbitrarily come to these classes.

So _____ if you miss a class, be sure to see me about it. I realize that on special occasions certain unusual circumstances may not permit you to be in class. If and when such problems do arise, I will do all I can to arrange a make-up time which will fit both your schedule and my free time.

W. A. Stannard
Office 212 New Chem Bldg
Office hours: 2-5 daily

Appendix D

Notices Sent to Student Advisors Indicating
Advisee Progress with Programmed Materials

Notices Sent to Student Advisors Indicating Advisee Progress with Programmed Materials

Math 121 Programmed Materials Report

Student

Instructor _____

Advisor

Class Hour

Lessons Completed

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
---	---	---	---	---	---	---	---	---	----	----	----	----	----	----	----

Appendix E

Progress Chart of Students Studying
Programmed Materials

Progress Chart of Students Studying Programmed Materials

To: Advisor No. 1 Instructor No. 1 Instructor No. 4
 Advisor No. 2 Instructor No. 2 Instructor No. 5
 Department Head Instructor No. 3 Supervisor

The following students in the Mathematics 121 classes are involved in the programmed materials investigation. I will keep this progress chart up to date and in your hands weekly.

Meeting Number

[illegible]

Appendix F

Final Examination Taken by the Experimental
and Control Groups

Final Examination Taken by the
Experimental and Control Groups

Winter 1965

Math 121
Final

Name _____

Instructor _____

Show all work neatly. Indicate answers clearly. Do no work on backs of pages.

- I. Find an equation of the straight line through (2, 4) and (-3, 0). Simplify your answer to the form $Ax + By + C = 0$.
- II. Write an equation satisfied by the coordinates of a point that moves so that it is equidistant from the line $x = 4$ and the point (0, -4). Sketch this curve.
- III. Given $y = x^2 - 2x$ find dy/dx from the definition. (Δ - process). Show all steps clearly.
- IV. Given $\lim_{x \rightarrow -3} (-2x + 1) = 7$ find a δ that will satisfy the definition of a limit when $\epsilon = 2/3$.
- V. For each of the following find dy/dx by use of appropriate formulas.
 $y = (x+1)^5 (2x + 1)^4$; $y = \frac{2x + 1}{3x^2 + 2}$
- VI. A rancher wishes to fence off a rectangular area from a corner of a larger rectangular fenced field. If he has 1000 linear feet of fencing material what should be the dimensions of this corner piece to enclose a maximum area? (Show test verifying that this is a maximum).
- VII. Given $y = 3x^2$, $0 \leq x \leq 2$. Subdivide the interval into 4 equal subintervals and compute the upper sum (S_n) and the lower sum (s_n).
- VIII. Evaluate each of the following:
 $\int_2^3 3(t^2 + 4) dt =$ $\int_0^1 (4 + 3u^2)^2 u du =$
- IX. Prove $d/dx x^n = nx^{n-1}$, where n is a positive integer, by mathematical induction on n . (Assume it is known that for $y = x$, $dy/dx = 1$.)

(Also you may assume any other derivative formulas from this course.)

- X. Find the area bounded by the curves $y = x^2$ and $y = x$.

Appendix G

Tables

TABLE 1. PREDICTED SCORES OF STUDENTS IN THE EXPERIMENTAL AND CONTROL GROUPS ENROLLING IN MATHEMATICS 121*

Experimental Group		Control Group	
Student number	Predicted Math 121 Score	Student number	Predicted Math 121 Score
1	27**	1	39
2	26	2	29
3	15	3	24
4	26	4	26
5	26	5	19
6	33	6	29
7	17	7	21
8	21	8	20
9	20	9	26
10	17	10	18
11	21	11	14
12	24	12	27
13	13	13	26
14	22	14	20
15	19	15	12
16	23	16	29
17	28	17	24
18	23	18	23
19	25	19	18
20	30	20	27
21	19	21	33
		22	20

*See footnote on p.62 for procedure of calculating these predicted scores by the Montana State College Testing and Counseling Service.

**These scores were calculated to predict the grade of the student in Mathematics 121 according to: A as 40, B as 30, C as 20, D as 10, and F as 00. The usual interpretation is: A as 4.0, B as 3.0, C as 2.0, D as 1.0, and F as 0.0; however, decimal points were not reported in the Montana State College Testing and Counseling Service figures so do not appear in the table.

TABLE 2. FINAL ACHIEVEMENT SCORES IN MATHEMATICS 121 OF STUDENTS IN THE EXPERIMENTAL AND CONTROL GROUPS

Experimental Group		Control Group	
Student number	Achieved Score	Student number	Achieved score
1	169*	1	133
2	122	2	135
3	110	3	119
4	181	4	170
5	120	5	143
6	140	6	125
7	136	7	80
8	157	8	134
9	126	9	134
10	141	10	144
11	103	11	131
12	174	12	152
13	104	13	151
14	85	14	138
15	155	15	147
16	172	16	134
17	147	17	140
18	144	18	147
19	139	19	109
20	140	20	142
21	134	21	74
		22	158

*These achieved scores were reported on the basis of a total score of 200.

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Stannard, W. A.

The effect on final achievement ...

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