



A study in quantum hydrodynamics
by Herbert A Zook

A THESIS Submitted to the Graduate Faculty in Partial fulfillment of the requirements for the degree
of Master of Science in Physics
Montana State University
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Abstract:

The experimental investigations of the properties of liquid helium II are reviewed. A chronological review of the attempts to explain theoretically the properties of liquid helium II is then given. Particular attention is paid to Landau's first paper on a quantum theory of liquid helium II. A short review of the boundary layer problem in hydrodynamics is then given with attention focused on the question of transition from laminar to turbulent flow in the boundary layer. In part IV, the roton concept postulated by Landau is extended and applied to the boundary layer problem. The velocity dependence of Emmons' g factor is derived and compared with the experimentally determined velocity dependence of g . Finally, a short discussion on the nature of rotons and phonons is given.

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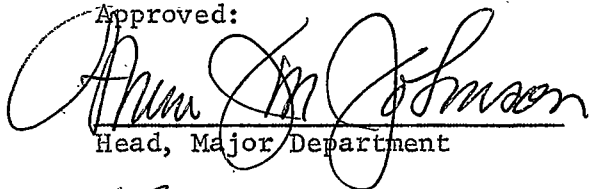
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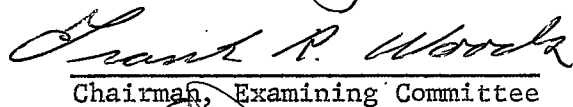
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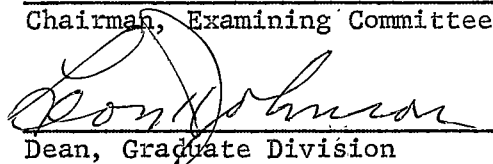
at

Montana State College

Approved:


Head, Major Department


Chairman, Examining Committee


Dean, Graduate Division

Bozeman, Montana
May, 1960

N378
Z 765
Cap. 2

- 2 -

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ABSTRACT

The experimental investigations of the properties of liquid helium II are reviewed. A chronological review of the attempts to explain theoretically the properties of liquid helium II is then given. Particular attention is paid to Landau's first paper on a quantum theory of liquid helium II. A short review of the boundary layer problem in hydrodynamics is then given with attention focused on the question of transition from laminar to turbulent flow in the boundary layer. In part IV, the roton concept postulated by Landau is extended and applied to the boundary layer problem. The velocity dependence of Emmons' g factor is derived and compared with the experimentally determined velocity dependence of g . Finally, a short discussion on the nature of rotons and phonons is given.

I. INTRODUCTION

The classical dynamical theory of gases has proved to be an extremely powerful tool in that it has shown itself capable of predicting quantitatively the macroscopic behavior of large numbers of particles, namely, a gas.

When the density of points in phase space becomes large, however, a number of discrepancies arise between theory and observation. The introduction of quantum statistics has resolved most of these discrepancies. Notable successes are the Bose-Einstein statistics of a photon gas and the Fermi-Dirac theory of electrons. As would be expected, quantum statistics gives the same results as classical statistics when the density of points in phase space is small. Mathematical and conceptual difficulties appear in either the quantum or classical statistics when the gas being investigated is not in equilibrium.

For the case of a gas in motion, one generally applies the well developed theory of hydrodynamics. In particular, for the case of a gas flowing by an object, hydrodynamics specializes in an area known as boundary layer theory. A largely unresolved problem in boundary layer theory is the treatment of transition from laminar to turbulent flow in the boundary layer.

The purpose of this paper will be to demonstrate that quantum mechanical phenomena are probably responsible for the nature of this transition.

To do this, we first review the experimental and theoretical developments in liquid helium. In view of its success, we devote considerable attention to the quantum hydrodynamical theory of liquid helium II. Landau's

first paper on this subject is, in particular, studied in some detail.

A short review of the developments in boundary layer theory is then given. Special attention is given to the treatments on transition in the boundary layer.

We then extend the concept of rotons and phonons as developed by Landau so as to apply to gases at ordinary temperatures. On this basis we are able to derive the velocity dependence of Emmons' g factor.

A complete quantum mechanical description of transition in the boundary layer has not yet been found. This paper, it is hoped, should provide a starting point for such a description.

II. LIQUID HELIUM

A. Experimental Phenomena in Liquid Helium. A research group at the University of Leiden (Holland), under the direction of Kammerlingh-Onnes, first liquified helium gas in 1908. Measurements were difficult at these low temperatures and it took a number of years to obtain accurate measurements of the properties of liquid helium--such as its density, surface tension, dielectric constant, etc.

In 1927 Keesom and Wolke¹¹ made a rather startling discovery, which is best told in their own words:

"When measuring the dielectric constant of liquid helium between the boiling point and 1.9°K , we observed that at a temperature almost corresponding with that at which Kammerlingh-Onnes had found a maximum in the density curve, the dielectric constant shows a sudden jump, or at least a jump made in a very small temperature region. The thought suggested itself that, at that temperature, the liquid helium transforms into another phase, liquid as well. If we call the liquid stable at the higher temperatures "liquid helium I", the liquid stable at the lower temperatures "liquid helium II", then the dielectric constant of liquid helium I should be greater than that of liquid helium II."

Thus it was first recognized that helium had two liquid phases depending upon whether the temperature was above or below a certain critical temperature T_{λ} , now called the " λ -point".

Recent measurements have shown that helium first liquified under its own vapor pressure at about 4.2°K , exists as liquid helium I down to $T_{\lambda} = 2.19^{\circ}\text{K}$, at which point it undergoes a transition to a quite different state

of matter called liquid helium II. Unless it is subjected to a pressure of about 25 atmospheres, helium II remains a liquid down to the lowest temperatures so far achieved--about 0.003 °K.

The above phenomena motivated an intense experimental study into the properties of liquid helium II and by 1940 nearly everything of importance that is now known experimentally about liquid helium II had already been observed.

Although an interesting subject in itself, we shall refrain from reviewing the voluminous experimental work on helium II and present only some of the experimental results that have aided the theoretical understanding of helium II.

At the λ -point, many of the physical properties of liquid helium undergo an abrupt and often discontinuous change. Figures 1, 2, and 3 show the experimental changes of three of these quantities near the λ -point.

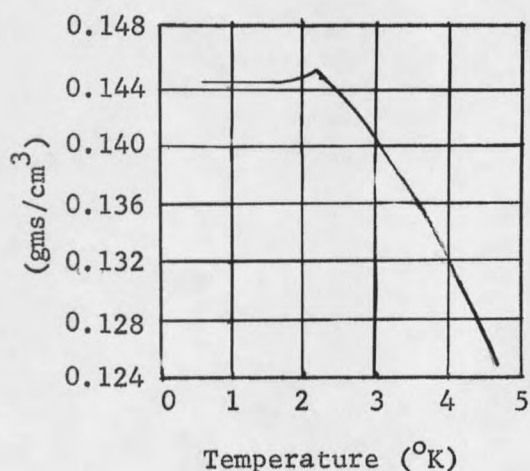


Figure 1. Density of liquid helium under its own vapor pressure

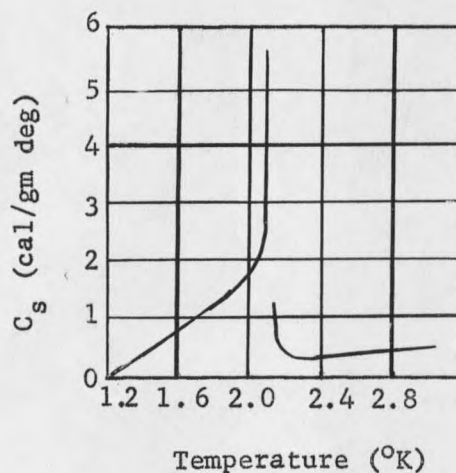


Figure 2. Specific heat of liquid helium under its own vapor pressure.

For example, the heat conductivity increases by a factor of nearly 10^6 with little more than a degree drop in temperature; the viscosity is enormously less below the λ -point; the specific heat depends upon the temperature in an entirely different manner below the λ -point than it does above it. Other quantities that show anomalies are: compressibility, sound velocity, surface tension, density, internal energy, and latent heats of fusion and vaporization.

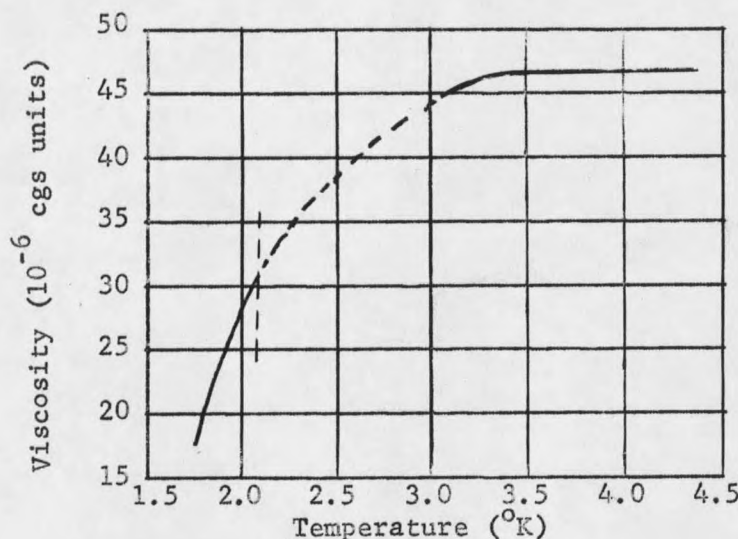


Figure 3. Viscosity of liquid helium I and II.

B. Theoretical Developments in Liquid Helium II up to 1941. The theoretical significance of the properties of liquid helium II just discussed lies in the fact that they seemed not to obey the hydrodynamical or thermodynamical equations that had been so successful in the classical treatments of other fluids.

This led to the suspicion that quantum phenomena might be playing a major role. F. London then made the fruitful observation in 1938 that the condensation phenomenon of an ideal Bose-Einstein gas gives a specific heat

versus temperature curve similar to that of liquid helium II (see Figure 4). A number of papers then appeared as a consequence of this suggestion¹⁵. L. Tisza²³ further suggested that superfluidity, heat capacity, and the "fountain effect" are properties that a partly condensed Bose-Einstein gas might be expected to have. Therefore a partly condensed Bose-Einstein gas should behave similarly to liquid helium II near the λ -point. He later published a detailed treatment of his "two fluid" hypothesis.*

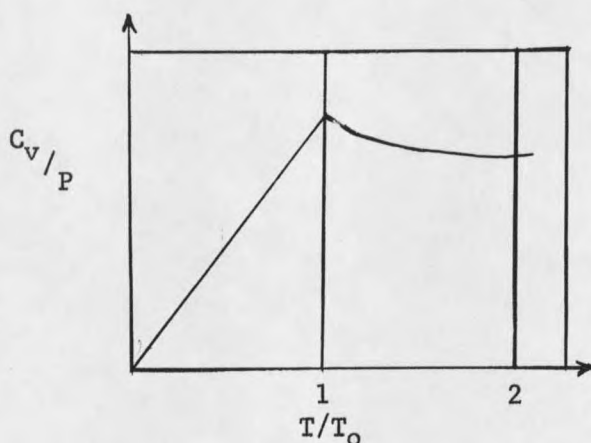


Figure 4. The condensation phenomenon of a Bose-Einstein gas.

In 1941, a brilliant paper by Landau appeared, which gave a fairly quantitative explanation of the properties of liquid helium II. His idea was that, instead of looking at the behavior of the individual atoms, one should look at the cooperative behavior of the liquid as a whole and enumerate the quantum states of this behavior¹³.

As we shall make use of his theory later on it will be of value to us to study it in some detail.

* A survey of this and other aspects of the theory up to the end of 1955 is given by J. Wilks, Repts. Prog. in Phys. 20, 38 (1957).

C. Landau's Theory of Liquid Helium. The first problem undertaken by Landau was to determine the form of the operators for the physical observables. Following Landau, we now treat this problem.

An arbitrary system of interacting particles (a liquid) can be described in classical theory by means of the density ρ and the flow of mass $\vec{J}$, which are determined in the following manner. Let $\vec{R}$ be the radius vector of an arbitrary point in space and $\vec{r}_\alpha$ -- the radius vector of a particle with a mass m_α . Then ρ is taken as

$$\rho = \sum_{\alpha} m_{\alpha} \delta(\vec{r}_{\alpha} - \vec{R}),$$

where $\delta(\vec{r}_{\alpha} - \vec{R}) = \delta(x_{\alpha} - X) \delta(y_{\alpha} - Y) \delta(z_{\alpha} - Z)$ and the δ 's are the Dirac δ -functions. The summation is extended over all the particles in the system. We note that $\int \rho dV$ gives the total mass of the system. Similarly, the density $\vec{J}$ of the flow of mass is given by

$$\vec{J} = \sum_{\alpha} m_{\alpha} \vec{v}_{\alpha} \delta(\vec{r}_{\alpha} - \vec{R}) = \sum_{\alpha} \vec{p}_{\alpha} \delta(\vec{r}_{\alpha} - \vec{R})$$

($\vec{v}_{\alpha}$, $\vec{p}_{\alpha}$ are the velocity and momentum of the particle m_{α}). It is to be emphasized that this description treats the microscopic motion and not the macroscopic motion of the liquid.

He then finds that the corresponding quantum operators for the above classical observables are

$$\hat{\rho} = \sum_{\alpha} m_{\alpha} \delta(\vec{r}_{\alpha} - \vec{R}) \quad (2.1)$$

and

$$\hat{J} = \frac{1}{2} \sum_{\alpha} \vec{p}_{\alpha} \delta(\vec{r}_{\alpha} - \vec{R}) + \delta(\vec{r}_{\alpha} - \vec{R}) \vec{p}_{\alpha} \quad (2.2)$$

where $P_\alpha = \frac{\hbar}{i} \nabla_\alpha$, the familiar momentum operator for a single particle.

$\hat{J}$ has been symmetrized with respect to $\hat{P}$ and δ so as to be Hermitean.

It is easily shown for a one particle system that these operators give the expected results.

To derive some of the other operators that we shall need, we first determine the commutation rules.

Remembering that these operators are always to be thought of as operating on some wave function and in view of the δ -function, we have

$$\hat{P}_1 \hat{P}_2 - \hat{P}_2 \hat{P}_1 = 0 \quad (2.3)$$

where $\hat{P}_1 = \hat{P}(\vec{R}_1)$, $\hat{P}_2 = \hat{P}(\vec{R}_2)$.

For the sake of brevity let us consider only one term from each of the sums (2.1) and (2.2) when determining the commutation rules, as the operators corresponding to different particles commute with each other. To determine the commutation of $\hat{P}$ with $\hat{J}$ for a single particle we write

$$\hat{J}_1 \hat{P}_2 - \hat{P}_2 \hat{J}_1 = \frac{m\hbar}{2i} \left\{ \begin{aligned} & [\nabla \delta(\vec{r} - \vec{R}_1) + \delta(\vec{r} - \vec{R}_1) \nabla] \delta(\vec{r} - \vec{R}_2) - \\ & - \delta(\vec{r} - \vec{R}_2) [\nabla \delta(\vec{r} - \vec{R}_1) + \delta(\vec{r} - \vec{R}_1) \nabla] \end{aligned} \right\}.$$

We simplify the right hand expression by noticing that operators of the form

$$\delta(\vec{r} - \vec{R}_1) \nabla \delta(\vec{r} - \vec{R}_2)$$

can be transformed in the following way:

$$\delta(\vec{r}-\vec{R}_1)\nabla(\vec{r}-\vec{R}_2) = \delta(\vec{r}-\vec{R}_1)\delta(\vec{r}-\vec{R}_2)\nabla + \delta(\vec{r}-\vec{R}_1)[\nabla\delta(\vec{r}-\vec{R}_2)]$$

where the brackets indicate that ∇ operates only on $\delta(\vec{r}-\vec{R}_2)$ and not on the wave function directly. Similarly

$$\nabla\delta(\vec{r}-\vec{R}_1)\delta(\vec{r}-\vec{R}_2) = \delta(\vec{r}-\vec{R}_2)\nabla\delta(\vec{r}-\vec{R}_1) + \delta(\vec{r}-\vec{R}_1)[\nabla\delta(\vec{r}-\vec{R}_2)].$$

The result is

$$\hat{J}_1\hat{p}_2 - \hat{p}_2\hat{J}_1 = \frac{\hbar}{i} m \delta(\vec{r}-\vec{R}_1)[\nabla\delta(\vec{r}-\vec{R}_2)].$$

and in view of the factor $\delta(\vec{r}-\vec{R}_1)$ we may write this as

$$\hat{J}_1\hat{p}_2 - \hat{p}_2\hat{J}_1 = \frac{m\hbar}{i} \delta(\vec{r}-\vec{R}_1)[\nabla\delta(\vec{r}-\vec{R}_2)]$$

where ∇ is the derivative with respect to the $\vec{R}_1$ coordinates. For an arbitrary system we may now write

$$\hat{J}_1\hat{p}_2 - \hat{p}_2\hat{J}_1 = \frac{\hbar}{i} \hat{p} [\nabla\delta(\vec{R}_1-\vec{R}_2)] \quad (2.4)$$

(We note that it makes no difference whether we write $\hat{p}_1$ or $\hat{p}_2$ on the right hand side of the equation).

Let us now introduce the operator $\vec{v}$ of the velocity of the liquid according to

$$\hat{J} = \frac{1}{2}(\hat{p}\vec{v} + \vec{v}\hat{p}) \quad (2.5)$$

$$\vec{v} = \frac{1}{2}\left(\frac{1}{\hat{p}}\hat{J} + \hat{J}\frac{1}{\hat{p}}\right) \quad (2.6)$$

as it will be more convenient to use the operator $\hat{v}$ instead of the flow $\hat{J}$.

Other commutation rules are found to be

$$\hat{v}_1 \hat{p}_2 - \hat{p}_2 \hat{v}_1 = \frac{\hbar}{i} [\nabla \delta(\vec{R}_1 - \vec{R}_2)] \quad (2.7)$$

$$\hat{v}_{1i} \hat{v}_{2k} - \hat{v}_{2k} \hat{v}_{1i} = \frac{\hbar}{b} \delta(\vec{R}_1 - \vec{R}_2) \frac{1}{b} (\widehat{\text{rot } \vec{v}})_{ik} \quad (2.8)$$

where $(\widehat{\text{rot } \vec{v}})_{ik}$ denotes the difference

$$\frac{\partial v_k}{\partial x_i} - \frac{\partial v_i}{\partial x_k}$$

we also find that

$$(\widehat{\text{rot } \vec{v}})_i \hat{p}_2 - \hat{p}_2 (\widehat{\text{rot } \vec{v}})_i = 0. \quad (2.9)$$

We now show that if we apply the formulae obtained to the macroscopic motion of the liquid we get, as we would expect, the usual hydrodynamical equations of motion. The energy of a unit volume of a classical liquid is

$$\rho \frac{v^2}{2} + \rho E(\rho)$$

where $E(\rho)$ is the internal energy of a unit mass of the liquid. We suppose that the energy E depends only on the density ρ of the liquid; this corresponds to the macroscopic character of the consideration and is connected with a statistical averaging. For a microscopic investigation this supposition is, of course, invalid.

The corresponding quantum operator is

$$\frac{\hat{v} \rho \hat{v}}{2} + \rho \mathcal{E}(\rho)$$

(The operators for ρ and $\mathcal{E}$ corresponds to multiplication by these quantities.)

The Hamiltonian $\hat{H}$ of the liquid is an integral over the volume

$$\hat{H} = \int \left\{ \frac{\hat{v} \rho \hat{v}}{2} + \rho \mathcal{E}(\rho) \right\} dV \quad (2.10)$$

where dV integrates only over the coordinate $\vec{R}$ in the δ -function and not over the particle coordinates $\vec{r}$.

For the derivative of the density ρ with respect to time one has the operator

$$\hat{\rho} = \frac{i}{\hbar} (\hat{H} \rho - \rho \hat{H}).$$

Denote temporarily the coordinates of the point at which ρ is taken by the index 1, and the coordinates of the variable point in the region of integration in (2.10) by the index 2. Then

$$\hat{\rho}_1 = \int \left\{ [\hat{v}_2 \rho_2 \hat{v}_2 \rho_1 - \rho_1 \hat{v}_2 \rho_2 \hat{v}_2] + [\rho_2 \mathcal{E}(\rho_2) \rho_1 - \rho_1 \rho_2 \mathcal{E}(\rho_2)] \right\} dV_2$$

In view of (2.3) the second term under the integral sign vanishes, and the first can be written

$$\frac{1}{2} [\hat{v}_2 \rho_2 (\hat{v}_2 \rho_1 - \rho_1 \hat{v}_2) + (\hat{v}_2 \rho_1 - \rho_1 \hat{v}_2) \rho_2 \hat{v}_2]$$

or, by introducing (2.7):

$$\frac{\hbar}{2i} [\nabla \delta(\vec{R}_2 - \vec{R}_1)] (\hat{v}_2 \rho_2 + \rho_2 \hat{v}_2) = \frac{\hbar}{i} [\nabla \delta(\vec{R}_2 - \vec{R}_1)] \hat{J}_2.$$

In this way

$$\hat{P}_1 = \int [\nabla \delta(\vec{R}_1 - \vec{R}_2)] \hat{J}_2 dV_2 = - \int \delta(\vec{R}_2 - \vec{R}_1) \text{div} \hat{J}_2 dV_2 = - \text{div} \hat{J}_1$$

i.e. we come to the continuity equation in operational form:

$$\frac{\partial \hat{\rho}}{\partial t} + \text{div}(\rho \hat{v} + \hat{v} \rho) = 0. \quad (2.11)$$

In a similar way the derivative

$$\hat{v} = \frac{i}{\hbar} (\hat{H} \hat{v} - \hat{v} \hat{H})$$

can be calculated, which brings us to the equation

$$\frac{\partial \hat{v}_i}{\partial t} + \frac{1}{2} \left(\hat{v}_k \frac{\partial \hat{v}_i}{\partial x_k} + \frac{\partial \hat{v}_i}{\partial x_k} \hat{v}_k \right) = - \frac{1}{\rho} \frac{\partial}{\partial x_i} \left(\frac{d\varepsilon}{d\rho} \right) \quad (2.12)$$

i.e. Euler's equation in an operational form ($\frac{d\varepsilon}{d\rho}$ is the pressure P of the liquid).

We note that the equations (2.11), (2.12) are less general than the commutation rules (2.3)-(2.9), which are also valid for an exact microscopical investigation of the liquid..

These equations would be extremely difficult, if not impossible, to solve directly for the equations of motion of the fluid, however, an analysis of the operators that we have developed provides the clue to the spectrum of eigenvalues that we should expect. We give Landau's analysis of these operators.

In the classical hydrodynamics of ideal liquids it is shown that if, at a certain moment of time, the motion is irrotational ($\text{rot} \vec{v} = 0$) in the whole volume of the liquid, it will be irrotational for all other moments of time (Lagrange's theorem)*. It appears that this classical theorem finds its analogy in quantum mechanics. According to the commutation rules (2.9), $\text{rot} \hat{v}$ always commutes with the density ρ . The components of $\text{rot} \hat{v}$, however do not commute, generally speaking, either with each other or with the velocity $\hat{v}$ (when the operation rot is applied to the equation (2.8) the right hand side does not vanish). Therefore $\text{rot} \hat{v}$ does not, generally speaking, commute with the Hamiltonian, i.e. is not conserved.

An exception is the case when over the whole volume $\text{rot} \hat{v} = 0$. In this case we have zero in the right hand side of (2.8) and $\text{rot} \hat{v}$ commutes with ρ and $\hat{v}$ and, therefore, also with the Hamiltonian.

In this way $\text{rot} \hat{v}$ is conserved if it is zero. In other words a quantum liquid always possesses stationary states in which $\text{rot} \hat{v}$ equals zero over the whole volume of the liquid. Such a state might be called by analogy to classical hydrodynamics, a state of irrotational motion of the liquid.

Concerning these results an analogy can be made with the angular momentum $\vec{M}$ in quantum mechanics**. The commutation of two components of $\vec{M}$ with each other leads to the third component of $\vec{M}$, with the result that all of the components of $\vec{M}$ commute with each other if they are all equal to zero. It is also known that there exist no states with an infinitely

* e.g. see Leigh Page, Introduction to Theoretical Physics, Third edition, (D. Van Nostrand Co., 1952) p. 248.

** A discussion of this is given by Landau and Lifschitz, Quantum Mechanics (Pergamon Press LTD., 1958) pp. 78-80.

small angular momentum, its first non-zero eigenvalues are of the order of $\hbar$. This is a consequence of the fact that the commutation rules are inhomogeneous--their left-hand sides are quadratic in $\vec{M}$ and the right hand sides are linear.

A similar statement can be advanced concerning $\text{rot} \vec{v}$ in quantum hydrodynamics. Namely, no states can exist in which $\text{rot} \vec{v}$ would be non-zero, but arbitrarily small over the whole volume of the liquid. In other words, between the states of potential ($\text{rot} \vec{v} = 0$) and vortex ($\text{rot} \vec{v} \neq 0$) motions of a quantum liquid there is no continuous transition.

From this the principle features of the energy spectrum of a liquid directly follow. The presence of a gap between the states of the potential and vortex motions means that between the lowest energy levels of vortex and potential motions a certain finite energy level must exist. As to the question which of these two lies lower apparently both cases are logically possible. It will be shown below that we get the phenomenon of superfluidity if we suppose that the normal level of the potential motions lies lower than the normal level of the vortex motions. Hence we must suppose that this very case exists in liquid helium. It must be remarked, however, that, as only one quantum liquid exists, liquid helium, the question as to whether such a distribution of the levels and hence the property of superfluidity is a general property of a quantum liquid cannot be solved experimentally.

This brings us to the following picture of the distribution of the levels in the energy spectrum of liquid helium (it must be emphasized that we do not here refer to the levels for single helium atoms but to the levels

corresponding to the states of the whole liquid). This spectrum is made up of two superimposed continuous spectra. One of them corresponds to the potential motions and the other to vortex motions. The lowest level of the vortex spectrum is situated above the lowest level of the potential spectrum, this latter level being the normal, unexcited state of a liquid; the energy interval between these two levels we denote by Δ .

The value of the energy gap Δ cannot be calculated exactly. Its order of magnitude is

$$\Delta \sim \hbar^2 \left(\frac{\rho^2}{m^3} \right)^{1/3} \quad (2.13)$$

(m being the mass of a helium atom and ρ the density of the liquid). This is the only quantity with the dimensions of energy which can be built up from m , ρ and $\hbar$.

Landau then supposes that every weakly excited state of the fluid can be regarded as an aggregate of "elementary excitations"--the elementary excitations of the potential motion he calls "phonons" and the elementary excitations of the vortex motion he calls "rotons". He further supposes that if the roton and phonon density is not too great that they may be treated as an ideal gas which obeys Bose statistics.

To find the temperature dependence of the heat capacity of the fluid it is supposed that the energy added is used only to excite potential and vortex motions in the fluid. Thus, additional energy must either create new phonons and rotons or it must raise the old ones to new levels of excitation. From this it is found, that the heat capacity of liquid helium consists of two parts--a phonon part (C_{ph}) and a roton part (C_r). The

phonon part turns out to be the Debye heat capacity

$$C_{ph} = 4.4 \times 10^{-3} T^3 \text{ cal/deg.}$$

The roton part of the heat capacity (C_r) has, owing to the presence of the energy gap Δ between the normal and vortex levels, an exponential temperature dependence. If we consider the roton density as small then Bose-Einstein statistics reduce to Maxwell-Boltzmann statistics (for sufficiently low temperatures this should be true). The free energy of the gas with an undetermined number of particles (the number of rotons is itself a temperature function determined from the condition of the minimum of the free energy) is

$$F_r = -KT V \int e^{-\frac{\epsilon}{KT}} \frac{d\tau_p}{(2\pi\hbar)^3}$$

$$(d\tau_p = dP_x dP_y dP_z)$$

(V is the volume, K = Boltzmann's constant, T is the absolute temperature and $\epsilon = \Delta + \frac{\vec{P}^2}{2\mu}$ where $\vec{P}$ and μ are the roton momentum and effective mass respectively). Below we shall refer the free energy to 1 gm of helium corresponding to which we shall put $V = \frac{1}{\rho}$ (ρ being the helium density), so that

$$F_r = -\frac{KT}{\rho(2\pi\hbar)^3} \int e^{-\frac{\epsilon}{KT}} d\tau_p \quad (2.14)$$

the number of rotons is

$$N_r = \frac{1}{\rho(2\pi\hbar)^3} \int e^{-\frac{\epsilon}{KT}} d\tau_p \quad (2.15)$$

and using $\epsilon = \Delta + \frac{\vec{p}^2}{2\mu}$ we get

$$F_r = -\frac{KT}{p} \left(\frac{\mu KT}{2\pi\hbar^2} \right)^{3/2} e^{-\frac{\Delta}{KT}} \quad (2.16)$$

$$N_r = \frac{1}{p} \left(\frac{\mu KT}{2\pi\hbar^2} \right)^{3/2} e^{-\frac{\Delta}{KT}} \quad (2.17)$$

The entropy is

$$S_r = -\frac{\partial F_r}{\partial T} = \frac{1}{p} \left(\frac{\mu K}{2\pi\hbar^2} \right)^{3/2} T^{1/2} \Delta e^{-\frac{\Delta}{KT}} \left(1 + \frac{5KT}{2\Delta} \right) \quad (2.18)$$

and the roton heat capacity is

$$C_r = T \frac{\partial S_r}{\partial T} = \frac{1}{p} \left(\frac{\mu}{2\pi\hbar^2} \right)^{3/2} \frac{K^{1/2} \Delta^2}{T^{1/2}} e^{-\frac{\Delta}{KT}} \left[1 + 3 \frac{KT}{\Delta} + \frac{15}{4} \left(\frac{KT}{\Delta} \right)^2 \right] \quad (2.19)$$

Using experimental data¹⁰ for specific heat one finds for μ and Δ the values:

$$\frac{\Delta}{K} = 8-9 \text{ } ^\circ\text{K}, \quad \mu \approx 8 m_{\text{He}}$$

(m_{He} is the mass of a helium atom).

From this foundation Landau then considers some conceptual experiments from which he is able to derive the minimum velocities with which the fluid must flow past an object in order to excite the phonon (elementary sound excitation) and roton (elementary vortex excitation) spectra.

Although the derivations are elementary, they are lengthy, so we shall only state the results. To excite the first level of the phonon spectrum he finds that the relative object-to-fluid velocity must exceed that of the velocity of sound in the liquid $|\vec{v}| > C$, where C is the velocity of

sound). To excite the first level of the roton spectrum we must have

$$|\vec{v}| > \sqrt{\frac{2\Delta}{\mu}}, \text{ where } \mu \text{ is the "effective roton mass".}$$

We now have the quantum hydrodynamical foundation for the treatment we shall take later on in this paper. It is interesting to note, in passing, that Landau was able to explain, to a good approximation, most of the anomalous properties of liquid helium such as superfluidity, heat capacity, etc., with the theory we have just outlined. His superfluid liquid was the part of the liquid that was not excited, while the liquid with the phonon and roton excitations exhibited the viscosity of a normal liquid.

D. Later Developments in Quantum Hydrodynamics. After the energy spectrum of liquid helium postulated by Landau had proven to be so successful in explaining and predicting many properties of liquid helium II, numerous attempts were made to give this energy spectrum a more secure foundation.

We do not propose to give a detailed analysis of these attempts nor even to survey the field. However, it may prove enlightening to touch upon some of the later work.

In 1952, Hiroshi Itô⁹ introduced (almost simultaneously with other authors) a transformation due to Clebsch^{*} into quantum hydrodynamics. The transformation is

$$\vec{v} = -\nabla\phi - \lambda\nabla\mu \tag{2.20}$$

(If the fluid velocity is irrotational then $\vec{v} = -\nabla\phi$ and the motion is said to be describable by a "potential function" ϕ .) $\lambda\nabla\mu$ represents

^{*} A development of this transformation has been done by S. H. Lamb, Hydrodynamics, Sixth edition, (Cambridge University Press, 1932) p. 248.

the contribution of the rotational part of the velocity. He uses the variational principle on the Lagrangian density of the fluid. He regards the field quantities ϕ, λ, μ as being the quantities which should be quantized--this being analogous to the procedure in quantum electrodynamics.

As this idea is treated more completely by Thellung²² in a later paper (1956), we shall pass to a short review of his work.

He writes, in agreement with other authors, the Hamiltonian as

$$H = \int d^3x \left(\frac{1}{2} \vec{v} \rho \vec{v} + E_P(\rho) \right)$$

and assumes for $\vec{v}$

$$\vec{v} = -\nabla \phi - \frac{i\hbar}{2\rho} \{ \psi^* \nabla \psi - (\nabla \psi^*) \psi \}$$

which is another form of the Clebsch transformation originally obtained by Ziman. (The difference now is that ρ has been elevated to the rank of a field variable for the velocity). He assumes for commutation relations

$$[\rho(x), \phi(x')] = \frac{\hbar}{i} \delta(x-x'), \quad [\psi(x), \psi^*(x)] = \delta(x-x')$$

all other commutators vanishing.

He expands the reciprocal of the density and the energy density as

$$\frac{1}{\rho} = \frac{1}{\rho_0} - \frac{\rho - \rho_0}{\rho_0^2} + \frac{(\rho - \rho_0)^2}{\rho_0^3} - \dots$$

and

$$E_P(\rho) = \frac{1}{2!} \frac{c_0^2}{\rho_0} (\rho - \rho_0)^2 + \frac{1}{3!} \left(\frac{d}{d\rho} \frac{c^2}{\rho} \right)_0 (\rho - \rho_0)^3 + \frac{1}{4!} \left(\frac{d^2}{d\rho^2} \frac{c^2}{\rho} \right)_0 (\rho - \rho_0)^4 + \dots$$

(Where c_0 and ρ_0 are equilibrium values for the sound velocity and density respectively).

He is then able to write his Hamiltonian as a sum of linear operators which he breaks up and labels as follows:

$$H_{ph} = \int d^3x \left\{ \frac{1}{2} \rho_0 (\nabla \phi)^2 + \frac{1}{2} \frac{c_0^2}{\rho_0} (\rho - \rho_0)^2 \right\}$$

$$H_{rot} = \int d^3x \left(-\frac{\hbar^2}{2\rho_0} \right) \{ \psi^* \nabla \psi - (\nabla \psi^*) \psi \}$$

$$H_{int} = \int d^3x \frac{i\hbar}{2} \nabla \phi \{ \psi^* \nabla \psi - (\nabla \psi^*) \psi \}$$

$$H'_{ph} = \int d^3x \left\{ \frac{1}{2} \nabla \phi (\rho - \rho_0) \nabla \phi + \frac{1}{3!} \left(\frac{d}{d\rho} \frac{c_0^2}{\rho} \right) (\rho - \rho_0)^3 \right\}$$

plus other perturbation terms. (a single prime denotes the first perturbation to the Hamiltonian).

The interesting part about this development is that the energy to be attributed to the potential, vortex, and interaction spectrum respectively is explicitly stated. The perturbation terms then give us the corrections that must be made to obtain the correct spectrum. The field quantities, as in Itô's work, are regarded as of fundamental importance. These quantities are then treated by the usual methods of second quantization.

Thellung's treatment is not without its imperfections however. This is because the series diverge and one has to argue for the reasonableness of cutting the series off after a certain maximum wave-number has been reached. This leads to a renormalization problem. An anomaly develops

here because the effective roton mass depends upon how the renormalization is done. If one attaches physical significance to an "effective roton mass" then it is certainly arbitrary to have it depend upon mathematical manipulations.

It is interesting to note that these theoretical treatments have, in general, concerned themselves only with incompressible, nonviscous fluids.

Thellung's treatment is representative of much of the work done in the period 1952 to 1956.

We shall mention one other author in connection with recent work in quantum hydrodynamics.

R. P. Feynman⁷ has attempted to deduce the atomistic behavior of a quantum fluid from first principles of quantum mechanics. In particular, he has attempted to justify the existence of Landau's rotons by a consideration of the uncertainty principle in quantum mechanics. In his own words: "Consider, for example, a tiny region of the liquid, say a cube 3 or 4 atomic spacings on a side. If the atoms in this region are confined in this region (so we have a submicroscopic sample of liquid helium) the excitations above the ground state will all involve one node somewhere among the configurations and hence a wavelength of order a (the atomic separation). This must mean an excess kinetic energy of the order $\hbar^2/a^2m$, or at least of order $\hbar^2/a^2mN$, where N is the number of atoms and m is the mass of each^{*}--leaving an appreciable gap from the ground state."

To sum up his discussion, the point is this: a mere interchange of

^{*} See, for example, Richtmeyer, Kennard and Lauritsen, Introduction to Modern Physics, Fifth edition, (McGraw-Hill 1955) p. 199.

the positions of any number of atoms in liquid helium will not change the energy of the liquid. Therefore an object moving through liquid helium will not excite any energy levels in the liquid (exhibit viscosity) unless it is moving at sufficient velocity to excite a compression wave (phonon excitation and only when $|\vec{v}| > c$), or excite vortex motion (roton excitation and only if $|\vec{v}| > \sqrt{\frac{2\mu}{\Delta}}$ --this is several times less than the velocity needed to excite a phonon). The vortex motion is postulated to involve only a few atoms in a restricted volume. His reasons for the restricted volume idea are given in a later article⁸.

The exact nature of the microscopic motion in helium is still largely unresolved as is indicated by Feynman in the same article:

"I am now brought to the same position as Casimir, who first told me about this problem. He said, 'There is only one way to go about working this out. It is simply to guess the quality of the answer'."

This completes our study of the state of quantum hydrodynamics up to the present time. Its only application has been to liquid helium II.

III. TURBULENCE AND THE BOUNDARY LAYER PROBLEM

A. Reynold's Early Experiments and the Character of Turbulent Flow.

It has long been known that the flow of fluids may be of two different kinds, laminar and turbulent, but Osborne Reynolds¹⁷ seems to have been the first to make a careful study of the two types of flow and of the transition from one type of flow to the other. He experimented with forcing water through glass tubes of various diameters at different velocities. To make the motion of the liquid observable, a small, steady stream of aniline dye was injected at the upstream end of the tube and flowed along the tube as a small filament. For very low velocities, the filament remained thin and unbroken throughout the tube*. As the velocity was increased, the filament appeared agitated in a somewhat random fashion at the downstream end of the tube. At still higher velocities, the motion became highly turbulent and the water became uniformly colored with the dye. Upon still further increase in velocity, the beginning of the turbulent part of the flow moved toward the upstream end of the tube.

What is significant is--that at sufficiently low velocities the flow is "laminar", that is, characterized by the fluid elements flowing in nearly straight lines down the tube and that at the higher velocities the fluid appears to be in a violent mixing process which is characteristic of "turbulent" flow.

It was later established that turbulence seems to set in at a certain

* A photograph of this phenomenon by Dr. Hunter Rouse of Columbia University may be found on the frontispiece of a book by Bakhmeteff, Boris A., The Mechanics of Turbulent Flow (Princeton University Press, 1936).

value of Reynold's number, a dimensionless parameter defined by

$$R = \frac{u D}{\eta / \rho}$$

where u is the average stream velocity, D is the diameter of the pipe, η is the viscosity coefficient, and ρ is the density of the fluid. Of importance is the observation that the value of Reynold's number where turbulence begins is independent of the type fluid used. Air, water, oil, glycerin, etc. all have the same critical Reynold's number (R_c). In general, for the flow to be laminar, Reynold's number must be below R_c which is about

$$R_c \approx 2300.$$

B. The Boundary Layer. We now consider the problem of fluid flow past a flat plate which is parallel to the direction of flow. The "boundary layer" is defined to be that portion of the fluid which has been appreciably slowed down by the viscous drag of the flat plate. This is illustrated in Figure 5.

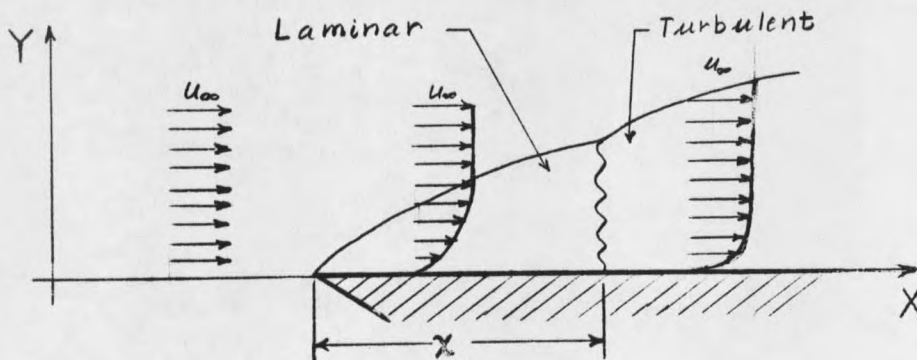


Figure 5. The boundary layer depicting both laminar and turbulent flow. (u_∞ is the free stream velocity).

In general, the flow is found to be laminar up to some distance χ from the leading edge at which point a transition takes place from laminar to turbulent flow. The distance χ is not fixed but varies rather unpredictably with time. It is found that the average transition point³ seems to occur in the neighborhood of

$$R_c \approx 4 \times 10^5.$$

It must be remarked that R_c is also dependent upon some additional factors--such as roughness of the plate and disturbances in the free stream flow. Critical Reynold's numbers as high as 3×10^6 have been reached in particularly smooth streams of air²⁰.

The Navier-Stokes equations in hydrodynamics are widely believed capable of accurately representing the motion of a fluid under rather general conditions.

Even for nonviscous, incompressible flow, however, general solutions to those equations have not been obtained. When viscosity is taken into account, it is found that the character of the stress tensor depends upon whether the flow is laminar or turbulent. In laminar flow, the stresses are proportional to the velocity gradients while in turbulent flow they are proportional to products of the mean deviations from the average velocity components.

In spite of the obvious difficulties, solutions to these equations have been found which give very good results for laminar flow. For turbulent flow the results are much less satisfactory. However, semi-empirical equations have been developed which give fair results in many instances.

The treatment of transition from laminar to turbulent flow has been under intense investigation since the beginning of the century. It has not yet been solved. We give a short sketch of some of the more successful of these treatments.

C. Tollmien's and Schlichting's Method of Small Disturbances. The method of small disturbances was first used by Rayleigh, in which he assumed a non-viscous fluid. This method was later extended by Prandtl in 1921 to include the most important viscous terms in the boundary. Tollmien²⁴, and then Schlichting²⁰, further developed the treatment to the point where it is now the basis for stability analysis employing the small disturbance method. The instabilities are supposed, in this method, to arise from within the boundary layer. We outline the principle of this method, using a two dimensional analysis on the flat plate previously mentioned.

Let U , V be the mean flow velocities parallel and perpendicular, respectively, to the plate and let P be the mean pressure in the fluid. Assume that

$$U = U(y), \quad V = 0, \quad P = P(x, y) \quad (2.21)$$

may be taken to be the mean values of these quantities. Upon the mean flow we assume superimposed a two dimensional disturbance which is a function of time as well as space. Its velocity components and pressure are, respectively,

$$u' = u'(x, y, t), \quad v' = v'(x, y, t), \quad p' = p'(x, y, t) \quad (2.22)$$

The resultant motion, then, is

$$u = U + u', \quad v = v', \quad p = P + P'. \quad (2.23)$$

It is assumed (2.21) satisfied the Navier-Stokes equation and it is required that (2.23) also satisfy the Navier-Stokes equation. The superimposed velocities are assumed "small" in the sense that all quadratic terms in the fluctuating components may be neglected with respect to the linear terms. The problem is to determine whether a disturbance decays or is amplified with time. The mean flow will then be regarded as stable or unstable depending on whether the turbulent velocities decay or amplify with time.

We recall the Navier-Stokes equation for a viscous, incompressible fluid

$$\left. \begin{aligned} \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) &= X - \frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \\ \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) &= Y - \frac{\partial P}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \end{aligned} \right\} \quad (2.24)$$

and the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (2.25)$$

Substituting (2.23) into (2.24) and (2.25) and neglecting quadratic terms in the disturbance velocity components, we obtain

$$\left. \begin{aligned} \frac{\partial u'}{\partial t} + u \frac{\partial u'}{\partial x} + v' \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{1}{\rho} \frac{\partial P'}{\partial x} &= \frac{\mu}{\rho} \left(\frac{\partial^2 u}{\partial y^2} + \nabla^2 u' \right) \\ \frac{\partial v'}{\partial t} + u \frac{\partial v'}{\partial x} + \frac{1}{\rho} \frac{\partial P}{\partial y} + \frac{1}{\rho} \frac{\partial P'}{\partial y} &= \frac{\mu}{\rho} \nabla^2 v' \end{aligned} \right\} \quad (2.26)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0 \quad (2.27)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

As the mean flow itself satisfies the Navier-Stokes equations, (2.26) and (2.27) simplify to

$$\left. \begin{aligned} \frac{\partial u'}{\partial t} + u \frac{\partial u'}{\partial x} + v' \frac{du}{dy} + \frac{1}{\rho} \frac{\partial P'}{\partial x} &= \frac{\mu}{\rho} \nabla^2 u' \\ \frac{\partial v'}{\partial t} + u \frac{\partial v'}{\partial x} + \frac{1}{\rho} \frac{\partial P'}{\partial y} &= \frac{\mu}{\rho} \nabla^2 v' \end{aligned} \right\} \quad (2.28)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0, \quad (2.29)$$

we specify as boundary conditions that u' and v' vanish on the walls (no slip condition). We may easily eliminate the pressure P' from the equations (2.28) which leaves us two equations for u' and v' (after elimination of P' , equations (2.28) become a single equation of third order in u' and v').

We now assume that the disturbances consist of partial fluctuations or waves propagated in the x-direction and assume a solution of (2.29) to be of the form

$$\psi(x, y, t) = \phi(y) e^{i(\alpha x - \beta t)} \quad (2.30)$$

where α is assumed to be real and β is taken to be complex,

$$\beta = \beta_r + i\beta_i$$

β_r is the circular frequency of the partial oscillation and β_i determines the degree of amplification or damping. The laminar mean flow described by these equations is stable when $\beta_i < 0$, whereas for $\beta_i > 0$ instability sets in. It is convenient to introduce

$$C = \frac{\beta}{\alpha} = C_r + i C_i. \quad (2.31)$$

Remembering that α is positive, the stability conditions for C_i are the same as for β_i . The amplitude function, ϕ , of the fluctuation is assumed to depend upon y alone only because the mean flow depends on y alone. We notice that, as (2.30) must satisfy the equation of continuity (2.29), we may write

$$\left. \begin{aligned} u' &= \frac{\partial \psi}{\partial y} = \phi'(y) e^{i(\alpha x - \beta t)} \\ v' &= -\frac{\partial \psi}{\partial x} = -i\alpha \phi(y) e^{i(\alpha x - \beta t)} \end{aligned} \right\} \quad (2.32)$$

We introduce these expressions for u' and v' into the equations (2.28) after the pressure P' has been eliminated. We obtain

$$(u - C)(\phi'' - \alpha^2 \phi) - u''\phi = -\frac{i}{2} \frac{u}{R} (\phi'''' - 2\alpha^2 \phi'' + \alpha^4 \phi).$$

It is further convenient to introduce $\bar{u} = \frac{u}{u_\infty}$ and $\bar{y} = \frac{y}{\delta}$, where u_∞ is the free stream velocity and δ is the boundary layer height. The equation may now be written

$$(u - C)(\phi'' - \alpha^2 \phi) - u''\phi = -\frac{i}{\alpha R} (\phi'''' - 2\alpha^2 \phi'' + \alpha^4 \phi) \quad (2.33)$$

where the bars are left off and ϕ' is taken to be $\phi' = \frac{d\phi}{d(\frac{y}{\delta})}$ in terms of the old coordinates. R is the Reynold's number referred to the boundary layer

height.

(2.33) is the fundamental "stability equation". The problem of stability has now been reduced to an eigenvalue problem. The asymptotic solutions for very small and for very large R are usually considered along with appropriate boundary conditions. As we are concerned only with the methods used and not with particular solutions that may be obtained we terminate the discussion of this treatment with one additional remark. According to a general theorem by Lord Rayleigh, the existence of a point of inflection in the velocity profile of the mean flow ($u''(y) = 0$) is a necessary condition for the amplification of disturbances ($C_i > 0$). Tollmien later proved that it was also a sufficient condition. Hence we have the simple theorem: velocity profiles with a point of inflection are unstable.

D. Dryden's and Taylor's Approach to Turbulence. This approach was first proposed by Dryden⁴ and developed theoretically by Taylor²¹. They view the breakdown of a laminar boundary layer to be the results of local pressure gradients initiating in the free stream. We only mention their treatment to illustrate the difference in principle between their theory and Tollmien's and Schlichting's theory. The difference is: Tollmien and Schlichting supposed the turbulence to be nearly independent of the free stream conditions while Dryden and Taylor supposed the free stream conditions to be the determining ones. There is considerable experimental backing for either theory.

E. Emmons' Theory of Turbulence. Emmons visualizes transition from laminar to turbulent flow as follows: Initially the viscosity of a fluid causes a laminar boundary layer to develop completely covering the object.

This laminar boundary layer is disturbed by random motions carried in by the fluid from the free stream. The perturbations are random in position, time, frequency, and amplitude, and they are amplified or damped as they move along the surface. Under critical conditions the perturbations cause the flow to become locally turbulent forming a turbulent spot. Emmons considers these spots as breaks in the laminar flow. The turbulence grows in size as it progresses downstream with the flow. Transition is thus pictured as a statistical phenomenon which is described by a probability function specifying the fraction of the time that the flow is turbulent at a given point. He finds that this probability function is given by

$$f(P) = 1 - e^{-\int_A g(P_c) dV_c} \quad (2.34)$$

where g is the turbulent source spot strength per unit wetted area, (i.e. the probability of n turbulent spots being generated per second in a given area of the plate). The integral is taken over the upstream region of influence of the point P . The differential of volume dV_c has the units of area x seconds.

Unfortunately g is not known. It therefore must be determined experimentally in terms of other physical parameters of the flow. An argument in its favor is that it is a statistical theory and predicts a dynamic transition point. An expression for g will be derived in Section IV.

IV. A QUANTUM THEORY OF FLUID MOTION

A. An Extension of the Present Theory. Landau postulated that the cooperative behavior of the entire liquid should be quantized rather than the behavior of the individual atoms. There is no suggestion, however, in his quantum hydrodynamical development, that liquid helium is the only fluid to which the theory is applicable. The operators that were developed were quite independent of the properties of liquid helium. We attempt, in this part of the paper, to extend the present interpretation of the theory such that it will apply to the problem of viscous flow.

To illustrate the following discussion, consider a stream of air flowing parallel to a flat plate such as that depicted in figure 5. We restrict our attention, for the moment to the laminar portion of the flow in the boundary layer. Immediately obvious is the fact that the velocity gradient $\frac{\partial u}{\partial y} = |\nabla \times \vec{u}|$ is different from zero throughout the boundary layer. $\frac{\partial u}{\partial y}$ is, therefore, a measure of the strength of the vortex motion of the air in the boundary layer.

As in liquid helium, we suppose that these vortex motions are quantized, that is, that there are finite energy levels between the different states of vortex motion. We regard each level of vortex motion as being composed of a number of elementary excitations or "rotons". The physical properties of a roton are not well understood. We postpone a discussion of their probable nature until later.

In the treatment of liquid helium, it was noted that $\text{rot } \vec{v}$ does not commute with the Hamiltonian and hence is not conserved. Let us consider the following experiment: Let some air be enclosed in a thermally insulated

cylinder which is rotating about its axis with a constant angular velocity. Assume that the air is in thermal equilibrium and is rotating with the cylinder. Let the cylinder be suddenly stopped. Obviously, the air will also stop rotating in time due to the viscous friction between the air and the walls of the cylinder. The air now has the same energy as it had before but now at a higher temperature--the additional thermal energy being gained from the energy of its original vortex motion. Again consider the air rotating in thermal equilibrium with the cylinder. Let the cylinder suddenly vanish, leaving the cylindrical "chunk of air" rotating in the surrounding atmosphere which we take to be at the same temperature as the air originally in the cylinder. Due to the viscous action of the surrounding atmosphere, the rotational motion is again lost to the cylindrical chunk of air and, as before, it experiences a rise in temperature. Angular momentum is not lost, of course, it is only that it is rather rapidly distributed to the surrounding atmosphere. The heat generated is not as rapidly distributed and the original cylindrical chunk becomes the center of a temporary higher temperature region.

The results of the foregoing experiment are quite in agreement with the expectations of classical hydrodynamics. They are mentioned only to illustrate the following quantum analysis of viscous motion.

Again consider air flowing past a flat plate as in figure 5. It is supposed that the viscous action of the flat plate in generating the vortex motion of the air is in reality exciting the vortex spectrum, i.e., creating rotons. The rotons then are assumed to behave somewhat as an ideal gas--moving with temperature dependent velocities, colliding with each other, etc.

As $\text{rot } \vec{v}$ is not generally conserved in any finite volume of the gas, we may expect the rotons to be annihilated as well as created. At the point where a roton is annihilated, we would expect a small, slightly higher temperature region to develop, somewhat analogously to the manner in which the rotating chunk of air previously mentioned developed one. Associated with the high temperature region would be a high pressure area.

We are now able to visualize a possible process of transition from laminar to turbulent flow in the boundary layer. "Elementary vortex excitations" or rotons are created at the flat plate and travel through the free stream, colliding with each other and with rotons already in the free stream. As the rotons are annihilated, small high pressure areas develop in the boundary layers. These increase in number as the distance from the leading edge of the plate is increased. The last condition is supposed true because the rotons farthest from the leading edge have had, in general, more time to reach equilibrium conditions with the surrounding free stream. We note, in passing, that the small disturbances assumed to be formed in the boundary layer by Tollmien and Schlichting follow as a result of our quantum picture. That is to say, the finite energy release accompanying the annihilation of one or more rotons would be expected to set up the pressure fluctuations which they had assumed to exist.

B. A Derivation of the Velocity Dependence of Emmons' "g" Factor.

Although a quantitative mathematical theory of the behavior of rotons and phonons in air is still lacking, we wish to illustrate the value of the concept.

We recall that Emmons' g factor is the turbulent source spot strength

per unit wetted area. From the discussion immediately preceding this, it is seen that turbulence in the boundary layer is attributed to the annihilation of rotons in the boundary layer. Let us assume that rotons are annihilated only in collisions with other rotons.

We take the rotons to be of small, finite dimensions and also associate a collision cross-section σ and an effective mass μ with each one. Suppose further that the rotons are identical and that their number density is n . We assume that the roton density in the boundary layer is sufficiently rare so that we may use, to a rough approximation, the theory of ideal gases.

For an ideal gas in statistical equilibrium, it is known that the average number of collisions that a particle will suffer in time dt is

$$Z = \sigma n \bar{v} dt$$

where $\bar{v}$ is the average velocity of the particle. The total number of collisions suffered by all of the molecules in a volume V in time dt is then

$$\frac{V}{2} n \sigma n \bar{v} dt = \frac{V}{2} \sigma n^2 \bar{v} dt .$$

The $1/2$ factor enters in to avoid counting each collision twice. The total number of collisions per unit volume per unit time is

$$f = \frac{1}{2} \sigma n^2 \bar{v} . \quad (2.35)$$

The number of rotons being annihilated per unit time per unit volume we take to be proportional to f . It is then reasonable to expect that Emmons' g factor should be proportional to f , i.e.

$$g = C f = \frac{C}{2} \sigma n^2 \bar{v} . \quad (2.36)$$

The average velocity of the particles in an ideal gas is nearly equal to the root-mean-square velocity. We take $\bar{v}$ to be the root mean square velocity for convenience. We suppose that $\bar{v}$ is temperature dependent only, i.e. that the rotons travel about with a mean velocity that depends only upon the temperature. We note that for velocities well below that of sound the temperature in the laminar portion of the boundary layer differs little from the free stream temperature. Hence $\bar{v}$ is expected not to vary much with the free stream velocity. The cross-section σ and the effective mass μ we also assume not to be strongly temperature dependent.

In the boundary layer, the original translational kinetic energy of the free stream is partly converted into rotational and thermal energy. The average amount which appears as rotational energy may be taken to be about one-third of the original free stream energy of translation. We assume that the amount of rotational energy that appears in the boundary layer is proportional to the number of rotons in the boundary layer. That is

$$n = K \left(\frac{1}{2} \rho u_{\infty}^2 \right). \quad (2.37)$$

Substituting n in equation (2.37) into equation (2.36) we get

$$g = \frac{c}{2} \sigma \bar{v} \left(\frac{K}{2} \rho \right)^2 u_{\infty}^4.$$

As u_{∞} is the only variable we may rewrite this as

$$g = C_1 u_{\infty}^4. \quad (3.28)$$

Experimental evidence indicates that this is the correct dependence of g on the free stream velocity. The theory is, as yet, too qualitative to

evaluate the proportionality constants that we have used. The results are significant, however, in view of the fourth power dependence on the free stream velocity.

C. The Properties of Rotons and Phonons. It must be emphasized that the rotons and phonons are not to be thought of as real particles. They are "elementary excitations" and do not seem to be associated with any particular group of particles. This is illustrated for rotons by the boundary layer in figure 5. We must suppose that "elementary vortex excitations" travel outward from the plate to form the boundary layer. Actual particle flow is known to be approximately parallel to the plate. In the case of phonons it appears possible to associate two or more phonons with a single air particle. We illustrate this as follows: according to Landau, the phonons are the quanta of the "sound excitations". A sound excitation has an associated frequency and wavelength. It is well known that sounds of different frequencies may simultaneously be propagated through the same medium. The waves add by the familiar principle of superposition.. The inference to be drawn is obvious. With due reserve, then, we speak of rotons and phonons as obeying ideal gas laws and having an "effective mass". We do so only for convenience and do not mean to imply that they are of the same nature as ordinary particles.

It is well known from elementary quantum mechanics that if a particle (we treat one for convenience) is confined to a finite volume of space, then its energy spectrum must be discrete. This is the familiar "potential well" problem. A new feature is added when the potential well has walls much higher than the total energy of the particle but the walls are of a finite

thickness. The potential wells associated with the nuclear binding forces are of this type. The particle, in this case, has a finite probability of escaping from the well but must still assume discrete energy values.

This leads to a rather interesting speculation. Let us focus our attention, momentarily, on a single particle which is but one member of a great number of particles forming a fluid. In a rough sense, the particle may be thought of as confined to the small volume of space determined by its surrounding neighbors. It would, of course, have a certain probability of escaping from this volume. Nevertheless, it seems reasonable to expect that the particle would exhibit a discrete energy spectrum which would depend upon how tightly and closely the surrounding particles confine this particular one. We consider the case for liquid helium. The particle separation is about 3×10^{-8} cm. If we suppose, for simplicity, that a helium atom is rather tightly confined by its neighbors then its rest or uncertainty energy must be roughly of the order of magnitude

$$E = \frac{\hbar^2}{2ma^2} \approx 10^{-16} \text{ ergs}$$

where $a = 3 \times 10^{-8}$ cm is the atomic separation and m is the mass of a helium atom.

Landau obtained for the energy needed to excite a roton

$$\Delta \approx 10^{-16} \text{ ergs.}$$

The coincidence is striking. F. London has already made a similar observation.

Feynman has extended the idea to the effect that rotons are motions

involving a few particles in a small volume while phonons involve large numbers of particles in large volumes.

Because of the small volume the roton occupies, its rest energy must be large compared to that of the phonon's rest energy. The thought suggests itself, however, that intermediate size particles or "excitations" may exist and in fact that only the relative number of particles involved in an excitation determines whether the excitation is a roton or a phonon.

An additional thought is that, analogous to the particle in a box concept, one might expect that the energy needed to raise a roton to its first excited level is comparable to its uncertainty energy. Let us extend our previous considerations to a gas, namely, air.

If we assume that an air particle appears to be enclosed in a spherical volume the diameter of which is taken to be the average mean free path of an air particle ($\lambda \approx 10^{-5}$ cm) we find for its uncertainty energy

$$E = \frac{\hbar^2}{2m\lambda^2} \approx 10^{-22} \text{ ergs.}$$

On the other hand if we substitute for m and ρ the values for air at S. T. P., we find for the roton excitation energy given by Landau's formula

$$\Delta = \hbar^2 \left(\frac{\rho^2}{m^5} \right)^{1/3} \approx 10^{-19} \text{ ergs.}$$

The number of particles that are enclosed by a sphere of diameter $\lambda = 10^{-5}$ cm is roughly in the neighborhood of

$$10^2 < N < 10^4.$$

If we suppose that all of these particles share equally in the roton's

energy, we find that the average energy per particle is roughly

$$10^{-21} - 10^{-23} \text{ ergs}$$

or about the same energy as that derived from a consideration of the uncertainty principle. Again the similarity is striking. Further speculation on the nature of these elementary excitations, though tempting, is not the purpose of this paper.

V. CONCLUSION

Perhaps the most significant indication that a quantum mechanical approach is the correct approach to the boundary layer problem is the fact that instabilities in the boundary layer follow as a consequence of our quantum treatment. That is, using the roton concept we have developed, it follows easily that turbulent spots should appear in the boundary layer. Then, following Emmons', we should also expect to find the entire boundary layer turbulent at a distance sufficiently far from the leading edge.

It was mentioned in the introduction that quantum statistics gives the same results as classical statistics when the density of points in phase space is small. For air in equilibrium at ordinary temperatures and pressures, this condition is satisfied.

Although the problem that we have treated is a non-equilibrium problem, our approach suggests that quantum phenomena may still be playing an important role in determining the macroscopic characteristics of the gas, as it certainly determines the microscopic characteristics.

It is felt that the preceding study should provide a good, if perhaps temporary, basis for a more quantitative treatment of the boundary layer problem.

In conclusion, I gratefully acknowledge my indebtedness to Dr. Frank Woods who not only suggested this study but whose continued interest, encouragement, and helpful suggestions have made its completion possible. I am also indebted to the Physics Department at Montana State College for financial assistance and for the opportunity to carry out this study.

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