

Numerical solution of fourth-order time-dependent problems with applications to parameter identification
by Ralph Conover Smith

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics
Montana State University
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Abstract:
A fully Galerkin method in both space and time is developed for fourth-order time-dependent problems. Sipc basis functions are used with suitable alterations made so that the method will handle combinations of free, simply supported, or fixed boundary conditions as well as general initial conditions. A full error analysis is given which determines the exponential convergence rate of the method. This convergence rate is attained for both analytic problems and problems with boundary singularities. The method is then applied to both forward problems and inverse problems involving the identification of unknown parameters. Through the judicious choice of conformal maps, approximate solutions to the forward problems are obtained which are valid on the infinite time interval rather than only on a truncated time domain. Moreover, because the scheme yields a global approximate solution, no interpolation is necessary when evaluating at points throughout the domain. When the forward methods are applied to parameter identification problems, the resulting inverse problems are ill-posed (small perturbations in the data may give rise to large perturbations in the solution). The method of Tikhonov regularization is discussed, and the resulting unconstrained minimization problems are solved via a quasi-Newton / trust region algorithm. Finally, several numerical examples are given which highlight various aspects of the method when applied to forward and inverse problems.

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by

Ralph Conover Smith

A thesis submitted in partial fulfillment
of the requirements for the degree

of

Doctor of Philosophy

in

Mathematics

MONTANA STATE UNIVERSITY
Bozeman, Montana

May 1990

D378
Sm643

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of a thesis submitted by

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ACKNOWLEDGEMENTS

The author wishes to thank Rene' Tritz for her superb typing and her patience when explaining the mysteries of TEX. Through her, a never-ending set of hand written pages was transformed into a finished document. Special thanks are also extended to my family for their complete support throughout my academic career.

The author also thanks Dr. Marcy Barge, Dr. Gary Bogar, Dr. Norm Eggert, Dr. John Lund, and Dr. Curt Vogel for their support both as committee members and as instructors. Their guidance and encouragement throughout the years has been invaluable. Finally, the author thanks Dr. Ken Bowers for the help, support, and guidance that he has extended as an instructor, an advisor, and a friend. By teaching me how to teach myself when confronted with an unsolved problem, he started me on the road toward becoming a mathematician.

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ABSTRACT

A fully Galerkin method in both space and time is developed for fourth-order time-dependent problems. Sinc basis functions are used with suitable alterations made so that the method will handle combinations of free, simply supported, or fixed boundary conditions as well as general initial conditions. A full error analysis is given which determines the exponential convergence rate of the method. This convergence rate is attained for both analytic problems and problems with boundary singularities. The method is then applied to both forward problems and inverse problems involving the identification of unknown parameters. Through the judicious choice of conformal maps, approximate solutions to the forward problems are obtained which are valid on the infinite time interval rather than only on a truncated time domain. Moreover, because the scheme yields a global approximate solution, no interpolation is necessary when evaluating at points throughout the domain. When the forward methods are applied to parameter identification problems, the resulting inverse problems are ill-posed (small perturbations in the data may give rise to large perturbations in the solution). The method of Tikhonov regularization is discussed, and the resulting unconstrained minimization problems are solved via a quasi-Newton / trust region algorithm. Finally, several numerical examples are given which highlight various aspects of the method when applied to forward and inverse problems.

CHAPTER 1

INTRODUCTION

The subject of this thesis is the numerical solution of fourth-order time-dependent problems via a Sinc-Galerkin method. Such problems arise in the modeling of beam and plate-like structures and thus are significant in areas ranging from the control of large flexible space structures to the development of robotics designs.

A thorough development of the underlying equations of motion for such structures can be found in [32], [33], [34], [42], and [43]. In general, these equations are derived by considering the kinetic and static properties of the beam. A common assumption is that the beams have a constant cross-sectional area A and constant density ρ. It is further assumed that the effects of rotary inertia and shear deformation are negligible so that the equations of motion derive from the Euler-Bernoulli model (thin beam theory) rather than the Timoshenko model (thick beam theory).

If a transverse forcing function $f(x,t)$ is applied to the beam, then moment and force balancing yields the state equation

$$-\frac{\partial^2 M}{\partial x^2}(x,t) + f(x,t) = \rho A \frac{\partial^2 u}{\partial t^2}(x,t) \tag{1.1}$$

where $u(x,t)$ denotes the displacement of the beam and $M(x,t)$ is the bending moment.

A second equation relating the moment and the displacement can be derived by considering the static properties of the beam which depend not only on moment

and force balancing, but also on intrinsic properties of the material itself. If the material is purely elastic, then one can assume a constitutive law which posits that stress is proportional to strain. Hooke's law for isotropic materials states that

$$\sigma = \mathrm{E}\epsilon$$

where σ denotes nonzero stress, ϵ denotes linear strain, and E is Young's modulus. As shown in [32], this then leads to the bending moment

$$M(x,t) = \mathrm{EI}(x)\frac{\partial^2 u}{\partial x^2}(x,t) \quad . \tag{1.2}$$

Here I is the moment of inertia of the beam cross section about the centroid axis, and the strictly positive quantity $\mathrm{EI}(x)$ is often referred to as the flexural rigidity or stiffness coefficient.

From (1.1) and (1.2) it then follows that the displacement $u(x,t)$ can be determined by solving a coupled pair of second-order problems. Many numerical methods are developed around this format and actually consist of coupled second-order schemes. For the methods of this work however, it is advantageous to combine the coupled equations into the fourth-order time-dependent equation

$$\rho A\frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\partial^2}{\partial x^2}\left(\mathrm{EI}(x)\,\frac{\partial^2 u}{\partial x^2}(x,t)\right) = f(x,t) \quad . \tag{1.3}$$

Since the equation of motion, (1.3), involves a second-order derivative with respect to time and a fourth-order derivative with respect to space, two initial conditions and four boundary conditions are needed in order to find a unique solution $u(x,t)$. The initial conditions will be taken of the form

$$u(x,0) = \varphi_0(x) \tag{1.4}$$

and

$$\frac{\partial u}{\partial t}(x,0) = \varphi_1(x) \tag{1.5}$$

where $\varphi_0(x)$ and $\varphi_1(x)$ denote the lateral displacement and velocity at time $t = 0$. When considering the boundary conditions, note that if the end of a horizontal beam is fixed, then the displacement and slope are zero. For a pinned or simply supported end, the displacement and bending moment are zero and in the case of a free end, the bending moment and shear are zero. If the ends of the beam are at $x = a$ and $x = b$, this yields the following common boundary conditions:

(i) Fixed Ends

$$\begin{aligned} u(a,t) &= u(b,t) = 0 \\ \frac{\partial u}{\partial x}(a,t) &= \frac{\partial u}{\partial x}(b,t) = 0 \end{aligned} \tag{1.6}$$

(ii) Simply Supported Ends

$$\begin{aligned} u(a,t) &= u(b,t) = 0 \\ \left(\mathrm{EI}\frac{\partial^2 u}{\partial x^2}\right)(a,t) &= \left(\mathrm{EI}\frac{\partial^2 u}{\partial x^2}\right)(b,t) = 0 \end{aligned} \tag{1.7}$$

(iii) Free Ends

$$\begin{aligned} \left(\mathrm{EI}\frac{\partial^2 u}{\partial x^2}\right)(a,t) &= \left(\mathrm{EI}\frac{\partial^2 u}{\partial x^2}\right)(b,t) = 0 \\ \frac{\partial}{\partial x}\left(\mathrm{EI}\frac{\partial^2 u}{\partial x^2}\right)(a,t) &= \frac{\partial}{\partial x}\left(\mathrm{EI}\frac{\partial^2 u}{\partial x^2}\right)(b,t) = 0 \end{aligned} \tag{1.8}$$

The conditions in (1.6) and (1.8) can also be combined to form the cantilever boundary conditions

$$\begin{aligned} u(a,t) = 0 \quad &, \quad \left(\mathrm{EI}\frac{\partial^2 u}{\partial x^2}\right)(b,t) = 0 \\ \frac{\partial u}{\partial x}(a,t) = 0 \quad &, \quad \frac{\partial}{\partial x}\left(\mathrm{EI}\frac{\partial^2 u}{\partial x^2}\right)(b,t) = 0 \end{aligned} \tag{1.9}$$

which are used in modeling beams that are fixed at one end and free at the other. Although many other types of boundary conditions physically arise in fourth-order

problems (see [34, page 384]), the conditions in (1.6)-(1.9) are the ones that will be used throughout this work.

Let B and I denote the boundary and initial operators in (1.6)-(1.9) and (1.4), (1.5), respectively. Also, for the sake of clarity, take $\rho A = 1$. The construction of an approximate solution to problems of the form

$$\mathcal{L}u(x,t) \equiv \frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\partial^2}{\partial x^2}\left(\mathrm{EI}(x)\frac{\partial^2 u}{\partial x^2}(x,t)\right) = f(x,t) \quad , \quad \begin{matrix} a < x < b \\ t > 0 \end{matrix}$$

$$\mathcal{B}u(x,t) = g(x,t) \quad , \quad x = a, b \quad , \quad t > 0 \tag{1.10}$$

$$\mathcal{I}u(x,0) = w(x) \quad , \quad a \leq x \leq b$$

commonly begins with a Galerkin discretization of the spatial variable with time-dependent coefficients. This yields a system of ordinary differential equations which is solved via differencing techniques. Due to stability constraints on the discrete evolution operator, low order methods with small time steps are often required to obtain accurate approximations. In contrast, the method of this work implements a Galerkin scheme in time as well as space. The basis functions in both domains are the compositions of sinc functions with suitable conformal maps. A thorough review of sinc function properties is found in [39].

The fully Sinc-Galerkin method in space and time has many salient features due both to the properties of the basis functions and to the manner in which the problem is discretized. First, the judicious choice of a conformal map provides an approximate solution to (1.10) which is valid on the infinite time interval rather than only on a truncated time domain. Furthermore, the optimal order of convergence, $\mathcal{O}\left(e^{-\kappa\sqrt{N}}\right)$, $\kappa > 0$, is maintained even in the presence of boundary singularities. Finally, the discrete system requires no numerical integrations to fill either the coefficient matrix or the right-hand side vector. All three features prove to be advantageous when solving both forward and inverse fourth-order problems.

A drawback to the method is that it produces a full system in contrast to the banded matrices which are associated with finite difference and finite element methods. In part, the exponential convergence rate offsets this disadvantage.

Chapter 2 begins with a brief review of sinc functions and the associated sinc quadrature rule. These results are then applied to the general fourth-order problem

$$\begin{aligned} \mathcal{L}u(x) \equiv \eta(x)u''''(x) + \varrho(x)u'''(x) + \nu(x)u''(x) + \tau(x)u'(x) \\ + \sigma(x)u(x) = f(x) \quad , \quad a < x < b \end{aligned}$$

(1.11)

$$u(a) = u(b) = 0$$

$$u'(a) = u'(b) = 0 \quad .$$

The Galerkin approximation as well as the choice of inner product are discussed, and error bounds are presented. The implementation of the method is important and guidance is given as to the appropriate choice of weight to be used in the inner product. The resulting parameter selections are then clearly defined with care taken to describe formulations that will be more efficient when numerically solving the fourth-order time-dependent problems of interest. The chapter closes with a pair of examples which illustrate the ease of implementation of the method as well as its flexibility for fourth-order ordinary differential equaitons.

In Chapter 3, the analysis in Chapter 2 is used to develop methods for solving fourth-order ordinary differential equations with higher order boundary conditions and special types of variable coefficients. In particular, problems of the form

$$\mathcal{L}u(x) = f(x) \quad , \quad a < x < b$$

$$\mathcal{B}u(x) = g(x) \quad , \quad x = a, b$$

will be considered. Here $\mathcal{B}$ denotes the boundary operators corresponding to (1.7)-(1.9). Also, an alternative technique for discretizing problems of the type

$$(\mathrm{EI}(x)u''(x))'' = f(x) \quad , \quad a < x < b$$

$$\mathcal{B}u(x) = g(x) \quad , \quad x = a, b$$

is presented which will prove to be advantageous in the parameter identification problems of Chapter 5. In contrast to the methods discussed in [30] and [39], this procedure is carried out in a fashion which avoids the direct collocation or differentiation of the parameter $\mathrm{EI}(x)$. In each section, implementation issues are addressed and numerical examples are given which highlight various aspects of the method.

In Chapter 4, the techniques from Chapters 2 and 3 are combined with temporal results to produce methods which very accurately approximate the solutions to the fourth-order time-dependent problems

$$\begin{aligned} &\mathcal{L}u(x,t) \equiv \frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\partial^2}{\partial x^2}\left(\mathrm{EI}(x)\frac{\partial^2 u}{\partial x^2}(x,t)\right) = f(x,t) \quad , \quad \begin{matrix} a < x < b \\ t > 0 \end{matrix} \\ &\mathcal{B}u(x,t) = g(x,t) \quad , \quad x = a, b \ . \ t > 0 \\ &\mathcal{I}u(x,0) = w(x) \quad , \quad a \le x \le b \ . \end{aligned} \tag{1.12}$$

Special attention is given to the treatment of the variable coefficient $\mathrm{EI}(x)$, and the modifications that are necessary for handling various initial and boundary conditions are carefully noted. Three equivalent formulations for the resulting discrete system are presented, and the advantages and disadvantages of each are discussed. Those linear algebra topics which are necessary for this development are also included. As in the previous chapter, each section closes with a discussion about method implementation and a set of numerical examples.

Chapter 5 provides an introduction to the use of Sinc-Galerkin methods for the recovery of the variable parameter $\mathrm{EI}(x)$ in the fourth-order time-dependent problems

$$\mathcal{L}(\mathrm{EI})u \equiv \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(\mathrm{EI}(x)\frac{\partial^2 u}{\partial x^2}\right) = f(x,t) \quad , \quad \begin{matrix} a < x < b \\ t > 0 \end{matrix}$$

$$\mathcal{B}u(x,t) = g(x,t) \quad , \quad x = a, b \quad , \quad t > 0 \tag{1.13}$$

$$\mathcal{I}u(x,0) = w(x) \quad , \quad a \leq x \leq b \quad ,$$

given discrete measurements of the state variable $u(x,t)$ at points in $(a,b)\times(0,\infty)$. As noted in [3], parameter estimation problems of this type often arise in the modeling and control of large flexible space structures. The discussion in this chapter is by no means exhaustive but instead introduces an important application of the methodology from earlier chapters. The convergence analysis and finite element techniques that have been developed by [1], [2], [4], [5], and [23] hint at further directions of study in this area. In the first half of the chapter, the necessary operator and optimization theory is reviewed. The inverse problem that is associated with (1.13) is then described and a stable and efficient numerical scheme for solving the resulting regularization problem is developed. The chapter closes with numerical examples which demonstrate the viability of the method.

CHAPTER 2

THE FOURTH-ORDER SPATIAL PROBLEM

Introduction

The purpose of this chapter is to derive the discrete Sinc-Galerkin system necessary for constructing the approximate solution to the linear fourth-order problem

$$\mathcal{L}u(x) \equiv \eta(x)u''''(x) + \varrho(x)u'''(x) + \nu(x)u''(x) + \tau(x)u'(x)$$

$$+ \sigma(x)u(x) = f(x) \quad , \quad a < x < b$$

$$u(a) = u(b) = 0$$

$$u'(a) = u'(b) = 0 \quad .$$

In the second section of the chapter, an overview of sinc functions will be given and salient features will be highlighted. The derivation of the Sinc-Galerkin method will be presented in the third section with special attention given to the error analysis and structure of the resulting system. The fourth section addresses the implementation of the method, namely, the choice of system parameters and weights. An alternate formulation for the approximate solution is given in the fifth section. The analysis leading to the implementation of the alternate formulation will provide the framework necessary for extending the method to problems with higher order boundary conditions. The final section consists of numerical examples which demonstrate important aspects of the method.

Sinc Function Preliminaries

The sinc function is defined on the whole real line by

$$\text{(2.1)} \qquad \operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \quad , \quad -\infty < x < \infty \quad .$$

For $h > 0$, the translated sinc functions are defined by

$$\text{(2.2)} \qquad S(k,h)(x) = \operatorname{sinc}\left(\frac{x - kh}{h}\right) \quad , \quad k = 0, \pm 1, \pm 2, \ldots \quad .$$

For various values of h, the central function $S(0,h)(x)$ is illustrated in Figure 1.

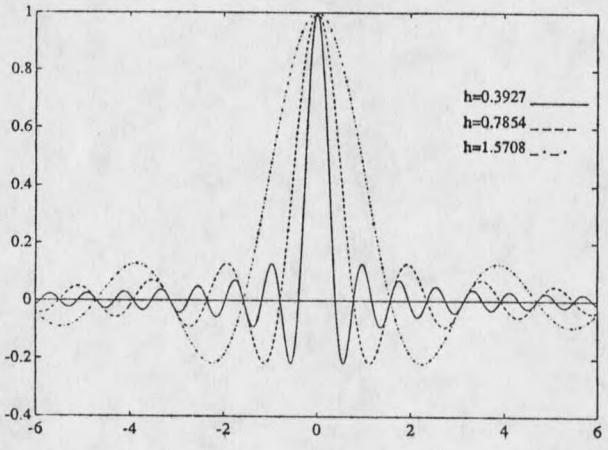


Figure 1. Central Sinc Basis Function for $h = \frac{\pi}{8}$, $\frac{\pi}{4}$, $\frac{\pi}{2}$.

If f is defined over the real line, then for $h > 0$ the series

$$\text{(2.3)} \qquad C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh)\operatorname{sinc}\left(\frac{x - kh}{h}\right)$$

is called the Whittaker cardinal expansion for f whenever the series converges. The properties of this series have been studied extensively and are thoroughly surveyed in [37]. To use (2.3) as an approximation on $\mathbb{R}^1$, define

$$\text{(2.4)} \qquad C_{M,N}(f,h)(x) = \sum_{k=-M}^{N} f(kh)\operatorname{sinc}\left(\frac{x - kh}{h}\right) \quad .$$

As a prelude to extending this approximation to arbitrary intervals and developing the necessary quadrature rules, the following definition is made.

Definition 2.1 *Let D be a simply connected domain in the complex plane with boundary points $a \neq b$. Let χ be a conformal map of D onto the strip*

$$D_S = \left\{ \xi = \varsigma + i\eta : |\eta| < d \leq \frac{\pi}{2} \right\} \tag{2.5}$$

with $\chi(a) = -\infty$ and $\chi(b) = \infty$ (see Figure 2). If the inverse of χ is denoted by ψ, define

$$\Gamma = \{\psi(t) \in D \ : -\infty < t < \infty\}$$

and let

$$z_k = \psi(kh) \quad , \quad k = 0, \pm 1, \pm 2, \ldots \quad . \tag{2.6}$$

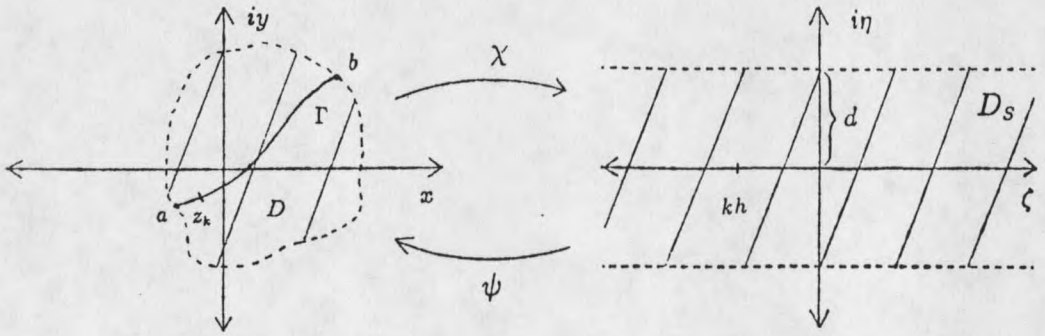


Figure 2. The Domains D and D_S.

The class of functions that is considered for sinc quadrature is denoted $B(D)$ and is defined as follows.

Definition 2.2 *Let $B(D)$ be the class of functions F which are analytic in D and satisfy*

$$\int_{\psi(t+L)} |F(z)|dz \to 0 \quad , \quad t \to \pm\infty \tag{2.7}$$

where $L = \{i\eta : |\eta| < d \leq \frac{\pi}{2}\}$ and

$$\mathcal{N}(F) \equiv \int_{\partial D} |F(z)dz| < \infty \quad . \tag{2.8}$$

A discussion of the properties of functions in $B(D)$ may be found in [39]. In particular, this class includes many functions with boundary singularities. The following quadrature theorems for functions in $B(D)$ are discussed in [31] and [37].

Theorem 2.3 *If $F \in B(D)$ then*

$$\int_\Gamma F(z)dz - h \sum_{k=-\infty}^{\infty} \frac{F(z_k)}{\chi'(z_k)} = \frac{i}{2} \int_{\partial D} \frac{F(z)\kappa(\chi,h)(z)}{\sin\left(\frac{\pi}{h}\,\chi(z)\right)}\, dz \equiv I_F \tag{2.9}$$

where

$$\kappa(\chi,h)(z) = \exp\left[\frac{i\pi\chi(z)}{h}\,\mathrm{sgn}(\mathrm{Im}(\chi(z)))\right] \quad . \tag{2.10}$$

Theorem 2.4 *If $F \in B(D)$ and h is sufficiently small, then*

$$\left|\int_\Gamma F(z)dz - h \sum_{k=-\infty}^{\infty} \frac{F(z_k)}{\chi'(z_k)}\right| \leq K_1 e^{-2\pi d/h} \quad . \tag{2.11}$$

Theorem 2.4 illustrates the exponential convergence which is a trademark of the sinc methods. It will be shown that in certain cases, the infinite series

appearing in (2.11) can be evaluated directly but in general it must be truncated. With additional hypotheses, the truncation need not be at the expense of the exponential decay.

Theorem 2.5 *Assume that $F \in B(D)$ and that there exist positive constants K, α and β such that*

$$\left|\frac{F(\tau)}{\chi'(\tau)}\right| \leq K \begin{cases} \exp(-\alpha|\chi(\tau)|) & , \ \tau \in \psi((-\infty, 0)) \\ \exp(-\beta|\chi(\tau)|) & , \ \tau \in \psi([0, \infty)) \end{cases} \quad . \tag{2.12}$$

Then for h sufficiently small

$$\left|\int_\Gamma F(z)dz - \sum_{k=-M}^{N} \frac{F(z_k)}{\chi'(z_k)}\right| \leq K_1 e^{-2\pi d/h} + \frac{K}{\alpha} e^{-\alpha M h} + \frac{K}{\beta} e^{-\beta N h} \quad . \tag{2.13}$$

The proof of Theorem 2.5 for centered sums ($M = N$) is given in [38]. In [27], the proof is expanded to handle noncentered sums ($M \neq N$). The results from Theorem 2.4 and Theorem 2.5 will be used to establish uniform error bounds when building the approximate solutions to the differential equations of interest.

In this work, the intervals (a, b) and $(0, \infty)$ will be of particular interest. The eye-shaped domain

$$D_E \equiv \left\{ z = x + iy : \left|\arg\left(\frac{z-a}{b-z}\right)\right| < d \leq \frac{\pi}{2} \right\} \tag{2.14}$$

is mapped conformally onto D_S via

$$\phi(z) = \ell n\left(\frac{z-a}{b-z}\right) \quad . \tag{2.15}$$

Similarly, the wedge-shaped domain

$$D_W \equiv \left\{ w = t + is : |\arg(w)| < d \leq \frac{\pi}{2} \right\} \tag{2.16}$$

is mapped conformally onto D_S via

$$\Upsilon(w) = \ell n(w) \quad . \tag{2.17}$$

The domains D_E, D_W and D_S are pictured in Figure 3.

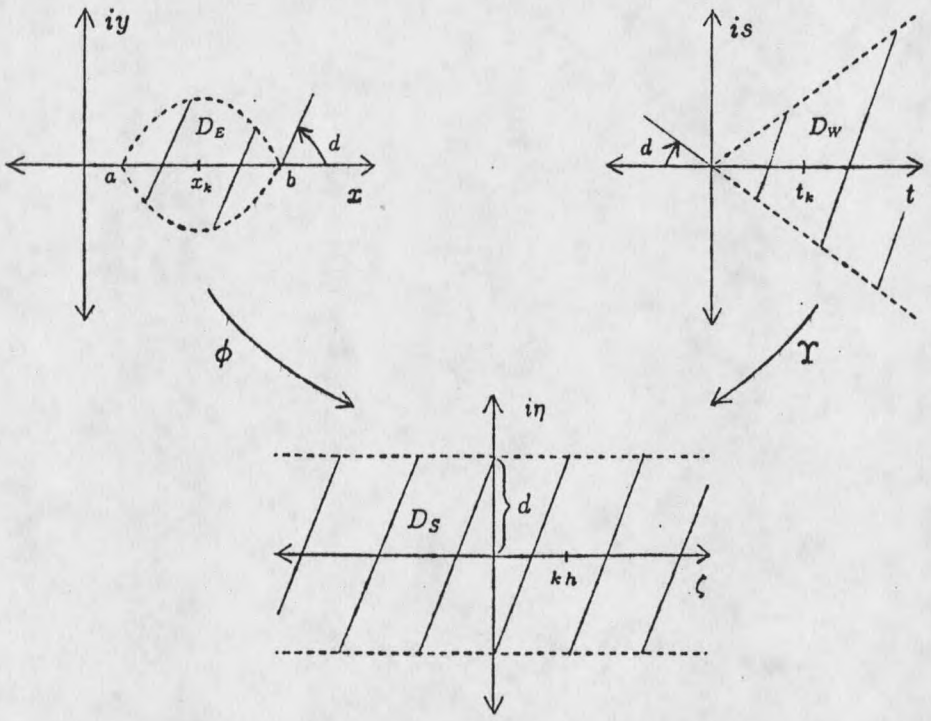


Figure 3. The Domains D_E, D_W and D_S.

The sinc gridpoints $z_k \in \Gamma = (a, b)$ in D_E will be denoted x_k since they are real. Similarly, the gridpoints $w_k \in \Gamma = (0, \infty)$ in D_W will be denoted t_k. Both are the inverse images of the equispaced grid in D_S; that is

$$x_k = \phi^{-1}(kh) = \frac{be^{kh} + a}{1 + e^{kh}} \tag{2.18}$$

and

$$t_k = \Upsilon^{-1}(kh) = e^{kh} \quad . \tag{2.19}$$

Note that if $F \in B(D_E)$ or $F \in B(D_W)$ then Theorem 2.3 provides quadrature rules on (a,b) and $(0,\infty)$ where χ and z_k are replaced by the respective conformal maps and sinc gridpoints.

The Fundamental Sinc System

The previous section gives the necessary background for formulating the numerical solution of linear fourth-order ordinary differential equations via the Sinc-Galerkin method. Consider now the general boundary value problem

$$\begin{aligned} \mathcal{L}u(x) \equiv \eta(x)u''''(x) + \varrho(x)u'''(x) + \nu(x)u''(x) + \tau(x)u'(x) \\ + \sigma(x)u(x) = f(x) \quad , \quad a < x < b \\ u(a) = u(b) = 0 \\ u'(a) = u'(b) = 0 \quad . \end{aligned} \tag{2.20}$$

This equation models a beam with fixed ends.

A Sinc-Galerkin method to approximate the solution of (2.20) can be derived as follows. Define the set of basis functions $\{S_i\}_{i=-M}^{N}$ by

$$S_i(x) \equiv S(i,h) \circ \phi(x) \tag{2.21}$$

where $\phi : D_E \to D_S$ is given in (2.15). Three adjacent members of this basis $(S(i,h) \circ \phi(x),\ i = -1, 0, 1)$ are shown in Figure 4 with $h = \pi/4$ and $(a,b) = (0,1)$.

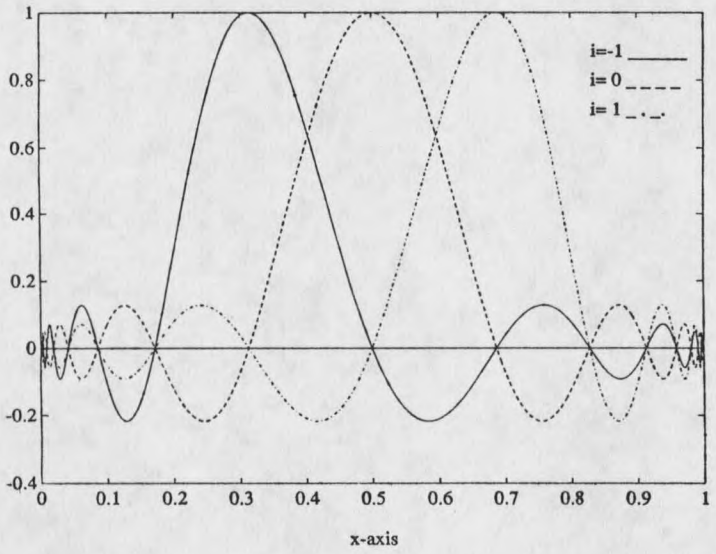


Figure 4. Three Adjacent Members $(S(i,h)\circ\phi(x),\ i=-1,0,1,\ h=\pi/4)$ of the Mapped Sinc Basis on the Interval $(0,1)$.

The approximate solution $u_m(x)$ is then defined by

$$u_m(x)=\sum_{i=-M}^{N} u_i S_i(x) \quad , \quad m=M+N+1 \tag{2.22}$$

and the unknown coefficients $\{u_i\}$ in (2.22) are determined by orthogonalizing the residual $\mathcal{L}u_m - f$ with respect to the set of sinc functions $\{S_p\}_{p=-M}^{N}$. This yields the discrete Galerkin system

$$\begin{aligned} 0 &= (\mathcal{L}u_m - f, S_p) \\ &= (\eta u_m'''', S_p) + (\varrho u_m''', S_p) + (\nu u_m'', S_p) + (\tau u_m', S_p) \\ &\quad + (\sigma u_m, S_p) - (f, S_p) \end{aligned} \tag{2.23}$$

where $-M \le p \le N$. The inner product $(\cdot,\cdot)$ is taken to be

$$(F,G) = \int_a^b F(x)G(x)w(x)dx \tag{2.24}$$

where $w(x)$ is a weight function which will be specified later.

The most direct development of the discrete system for (2.20) is obtained by substituting (2.22) into (2.23). The system can then be expressed in integral form via (2.24). This approach however, obscures the analysis which is necessary for applying sinc quadrature formulas to (2.23). An alternate approach is to analyze instead

$$(\mathcal{L}u - f, S_p) = 0 \quad , \quad -M \leq p \leq N \quad . \tag{2.25}$$

The discrete system resulting from (2.25) will be the same as that arising from (2.23) to within the accuracy of the method. The expansion of (2.25) yields

$$(\eta u'''', S_p) + (\varrho u''', S_p) + (\nu u'', S_p) + (\tau u', S_p) + (\sigma u, S_p) = (f, S_p) \tag{2.26}$$

for $-M \leq p \leq N$.

The approximation of the last four inner products in (2.26) has been thoroughly treated in [27] and [38]. For convenience those results will be recorded here along with the ones for $(\eta u'''', S_p)$ and $(\varrho u''', S_p)$. Rather than directly apply Theorem 2.3 to the inner products, integration by parts is repeatedly applied. After four integrations by parts

$$(\eta u'''', S_p) = B_T^{(4)} + \ < u, (S_p \eta w)'''' > \tag{2.27}$$

where $< \cdot, \cdot >$ denotes the standard $L^2(a,b)$ inner product (weight function $w(x) = 1$ in (2.24)) and the boundary term is

$$B_T^{(4)} = \{u'''(S_p\eta w) - u''(S_p\eta w)' + u'(S_p\eta w)'' - u(S_p\eta w)'''\}(x)\Big|_a^b \quad . \tag{2.28}$$

Similarly, after three integrations by parts

$$(\varrho u''', S_p) = B_T^{(3)} - \ < u, (\varrho S_p w)''' > \tag{2.29}$$

where

$$B_T^{(3)} = \{u''(\varrho S_p w) - u'(\varrho S_p w)' + u(\varrho S_p w)''\}(x)\Big|_a^b \quad . \tag{2.30}$$

The following five identities will be useful in formulating the discrete system. Let

$$\begin{aligned} \delta_{pi}^{(4)} &\equiv h^4 \frac{d^4}{d\phi^4} [S(p,h) \circ \phi(x)]\Big|_{x=x_i} \\ &= \begin{cases} \frac{\pi^4}{5} & , \ i = p \\ \frac{-4(-1)^{i-p}}{(i-p)^4} [6 - \pi^2 (i-p)^2] & , \ i \neq p \end{cases} , \end{aligned} \tag{2.31}$$

$$\begin{aligned} \delta_{pi}^{(3)} &\equiv h^3 \frac{d^3}{d\phi^3} [S(p,h) \circ \phi(x)]\Big|_{x=x_i} \\ &= \begin{cases} 0 & , \ i = p \\ \frac{(-1)^{i-p}}{(i-p)^3} [6 - \pi^2 (i-p)^2] & , \ i \neq p \end{cases} , \end{aligned} \tag{2.32}$$

$$\delta_{pi}^{(2)} \equiv h^2 \frac{d^2}{d\phi^2} [S(p,h) \circ \phi(x)]\Big|_{x=x_i} = \begin{cases} -\frac{\pi^2}{3} & , \ i = p \\ \frac{(-2)(-1)^{i-p}}{(i-p)^2} & , \ i \neq p \end{cases} , \tag{2.33}$$

$$\delta_{pi}^{(1)} \equiv h \frac{d}{d\phi} [S(p,h) \circ \phi(x)]\Big|_{x=x_i} = \begin{cases} 0 & , \ i = p \\ \frac{(-1)^{i-p}}{(i-p)} & , \ i \neq p \end{cases} \tag{2.34}$$

and

$$\delta_{pi}^{(0)} \equiv [S(p,h) \circ \phi(x)]\Big|_{x=x_i} = \begin{cases} 1 & , \ i = p \\ 0 & , \ i \neq p \end{cases} \tag{2.35}$$

denote the evaluation at the gridpoint x_i of the basis functions and their derivatives with respect to the map ϕ.

The application of Theorem 2.3 to (2.27) then yields

$$(2.36) \qquad (\eta u'''', S_p) = h \sum_{i=-\infty}^{\infty} \frac{u(x_i)}{\phi'(x_i)} (S_p \eta w)''''(x_i) + B_T^{(4)} + I_F^{(4)}$$

where the boundary term $B_T^{(4)}$ is given by (2.28) and $I_F^{(4)}$ is the contour integral error on the right-hand side of (2.9) with F replaced by $u(S_p \eta w)''''$. If the derivatives are expanded, then (2.36) may be written as

$$\begin{aligned}(\eta u'''', S_p) = h \sum_{i=-\infty}^{\infty} \frac{u(x_i)}{\phi'(x_i)} \Big[& \frac{1}{h^4} \delta_{pi}^{(4)} g_4^{(4)}(x_i) + \frac{1}{h^3} \delta_{pi}^{(3)} g_4^{(3)}(x_i) \\ & + \frac{1}{h^2} \delta_{pi}^{(2)} g_4^{(2)}(x_i) + \frac{1}{h} \delta_{pi}^{(1)} g_4^{(1)}(x_i) + \delta_{pi}^{(0)} g_4^{(0)}(x_i) \Big] \\ & + B_T^{(4)} + I_F^{(4)} \quad .\end{aligned}$$

Here

$$(2.37) \qquad g_4^{(4)}(z) = \eta(z) w(z) \big(\phi'(z)\big)^4 \quad ,$$

$$(2.38) \qquad g_4^{(3)}(z) = 4\big(\eta(z)w(z)\big)' \big(\phi'(z)\big)^3 + 6\eta(z)w(z)\big(\phi'(z)\big)^2 \phi''(z) \quad ,$$

$$\begin{aligned}(2.39) \qquad g_4^{(2)}(z) = {} & 6\big(\eta(z)w(z)\big)'' \big(\phi'(z)\big)^2 + 12\big(\eta(z)w(z)\big)' \phi'(z)\phi''(z) \\ & + 4\eta(z)w(z)\phi'(z)\phi'''(z) + 3\eta(z)w(z)\big(\phi''(z)\big)^2 \quad ,\end{aligned}$$

$$\begin{aligned}(2.40) \qquad g_4^{(1)}(z) = {} & 4\big(\eta(z)w(z)\big)''' \phi'(z) + 6\big(\eta(z)w(z)\big)'' \phi''(z) \\ & + 4\big(\eta(z)w(z)\big)' \phi'''(z) + \eta(z)w(z)\phi'''(z)\end{aligned}$$

and

$$(2.41) \qquad g_4^{(0)}(z) = \big(\eta(z)w(z)\big)'''' \quad .$$

Similarly, application of the quadrature theorem to (2.29) yields

$$(\varrho u''', S_p) = -h \sum_{i=-\infty}^{\infty} \frac{u(x_i)}{\phi'(x_i)} \left[\frac{1}{h^3}\, \delta_{pi}^{(3)} g_3^{(3)}(x_i) + \frac{1}{h^2}\, \delta_{pi}^{(2)} g_3^{(2)}(x_i) + \frac{1}{h}\, \delta_{pi}^{(1)} g_3^{(1)}(x_i) + \delta_{pi}^{(0)} g_3^{(0)}(x_i) \right] + B_T^{(3)} + I_F^{(3)}$$

where $B_T^{(3)}$ is defined in (2.30) and $I_F^{(3)}$ is the contour integral error on the right-hand side of (2.9) with F replaced by $u(\varrho S_p w)'''$. The terms $g_3^{(k)}, k = 0,1,2,3,$ are given by

$$g_3^{(3)}(z) = \varrho(z) w(z) \big(\phi'(z)\big)^3 \quad , \tag{2.42}$$

$$g_3^{(2)}(z) = 3\big(\varrho(z)w(z)\big)'\big(\phi'(z)\big)^2 + 3\varrho(z)w(z)\phi'(z)\phi''(z) \quad , \tag{2.43}$$

$$g_3^{(1)}(z) = 3\big(\varrho(z)w(z)\big)''\phi'(z) + 3\big(\varrho(z)w(z)\big)'\phi''(z) + \varrho(z)w(z)\phi'''(z) \tag{2.44}$$

and

$$g_3^{(0)}(z) = \big(\varrho(z)w(z)\big)''' \quad . \tag{2.45}$$

The results for the remaining four inner products in (2.26) are recorded here. Details can be found in [27]. These inner products may be written as

$$(\nu u'', S_p) = h \sum_{i=-\infty}^{\infty} \frac{u(x_i)}{\phi'(x_i)} \left[\frac{1}{h^2}\, \delta_{pi}^{(2)} g_2^{(2)}(x_i) + \frac{1}{h}\, \delta_{pi}^{(1)} g_2^{(1)}(x_i) + \delta_{pi}^{(0)} g_2^{(0)}(x_i) \right] + B_T^{(2)} + I_F^{(2)} \quad ,$$

$$(\tau u', S_p) = -h \sum_{i=-\infty}^{\infty} \frac{u(x_i)}{\phi'(x_i)} \left[\frac{1}{h}\, \delta_{pi}^{(1)} g_1^{(1)}(x_i) + \delta_{pi}^{(0)} g_1^{(0)}(x_i) \right] + B_T^{(1)} + I_F^{(1)} \quad ,$$

$$(\sigma u, S_p) = h\sigma(x_i)u(x_i)\,\frac{w(x_i)}{\phi'(x_i)} + I_F^{(0)}$$

and

$$(f, S_p) = hf(x_i)\,\frac{w(x_i)}{\phi'(x_i)} + I_R^{(0)} \quad .$$

Here

$$g_2^{(2)}(z) = \nu(z)w(z)\big(\phi'(z)\big)^2 \quad , \tag{2.46}$$

$$g_2^{(1)}(z) = \nu(z)w(z)\phi''(z) + 2\big(\nu(z)w(z)\big)'\phi'(z) \quad , \tag{2.47}$$

$$g_2^{(0)}(z) = \big(\nu(z)w(z)\big)'' \quad , \tag{2.48}$$

$$g_1^{(1)}(z) = \mp\,\tau(z)w(z)\phi'(z) \quad , \tag{2.49}$$

$$g_1^{(0)}(z) = \big(\tau(z)w(z)\big)' \quad , \tag{2.50}$$

$$B_T^{(2)} = \mp\,\{u'(\nu S_p w) - u(\nu S_p w)'\}\,(x)\Big|_a^b \quad , \tag{2.51}$$

and

$$B_T^{(1)} = \{u\tau S_p w\}\,(x)\Big|_a^b \quad . \tag{2.52}$$

Also, $I_F^{(2)}$, $I_F^{(1)}$, $I_F^{(0)}$ and $I_R^{(0)}$ are the contour integral errors on the right-hand side of (2.9) with F replaced by $u(\nu S_p w)''$, $u(\tau S_p w)'$, $u\sigma S_p w$ and $f S_p w$, respectively.

The following inequalities, the last two of which are new, will be used in bounding the contour integral errors $I_R^{(0)}$ and $I_F^{(\ell)}$, $\ell = 0, 1, 2, 3, 4$. Let

$$\left|\frac{S(p,h)\circ\phi(z)}{\sin\big(\pi\phi(z)/h\big)}\right|\Bigg|_{z\in\partial D_E} \le \frac{h}{\pi d} \equiv C_0(h,d) \quad , \tag{2.53}$$

$$(2.54)\qquad \left|\frac{\frac{d}{d\phi}\left[S(p,h)\circ\phi(z)\right]}{\sin\left(\pi\phi(z)/h\right)}\right|\Bigg|_{z\in\partial D_E} \leq \frac{1}{d}C_0(h,d)+\frac{1}{d\tanh(\pi d/h)}\equiv C_1(h,d) \quad ,$$

$$(2.55)\qquad \left|\frac{\frac{d^2}{d\phi^2}\left[S(p,h)\circ\phi(z)\right]}{\sin\left(\pi\phi(z)/h\right)}\right|\Bigg|_{z\in\partial D_E} \leq \frac{2}{d}\,C_1(h,d)+\frac{\pi}{hd}\equiv C_2(h,d) \quad ,$$

$$(2.56)\qquad \left|\frac{\frac{d^3}{d\phi^3}\left[S(p,h)\circ\phi(z)\right]}{\sin\left(\pi\phi(z)/h\right)}\right|\Bigg|_{z\in\partial D_E} \leq \frac{3}{d}\,C_2(h,d)+\frac{\pi^2}{h^2 d\tanh(\pi d/h)}\equiv C_3(h,d)$$

and

$$(2.57)\qquad \left|\frac{\frac{d^4}{d\phi^4}\left[S(p,h)\circ\phi(z)\right]}{\sin\left(\pi\phi(z)/h\right)}\right|\Bigg|_{z\in\partial D_E} \leq \frac{4}{d}\,C_3(h,d)+\frac{\pi^3}{h^3 d}\equiv C_4(h,d) \quad .$$

Of the many choices for the weight function $w(x)$, a common selection is one which results in $B_T^{(\ell)}=0$, $\ell=1,2,3,4$. This is used, along with the identity

$$(2.58)\qquad |\kappa(\phi,h)(z)|\Big|_{z\in\partial D_E} = e^{-\pi d/h}$$

where κ is defined in (2.10), to obtain the bound

$$
\begin{aligned}
\left|(\eta u'''', S_p) - h\sum_{i=-\infty}^{\infty} \frac{u(x_i)}{\phi'(x_i)}(\eta S_p w)''''(x_i)\right| &= |I_F^{(4)}| \\
&\le \frac{e^{-\pi d/h}}{2}\Bigg\{ \int_{\partial D_E} |u(z)|\,|g_4^{(0)}(z)| \left|\frac{S(p,h)\circ\phi(z)}{\sin(\pi\phi(z)/h)}\right| |dz| \\
&+ \int_{\partial D_E} |u(z)|\,|g_4^{(1)}(z)| \left|\frac{\frac{d}{d\phi}[S(p,h)\circ\phi(z)]}{\sin(\pi\phi(z)/h)}\right| |dz| \\
&+ \int_{\partial D_E} |u(z)|\,|g_4^{(2)}(z)| \left|\frac{\frac{d^2}{d\phi^2}[S(p,h)\circ\phi(z)]}{\sin(\pi\phi(z)/h)}\right| |dz| \\
&+ \int_{\partial D_E} |u(z)|\,|g_4^{(3)}(z)| \left|\frac{\frac{d^3}{d\phi^3}[S(p,h)\circ\phi(z)]}{\sin(\pi\phi(z)/h)}\right| |dz| \\
&+ \int_{\partial D_E} |u(z)|\,|g_4^{(4)}(z)| \left|\frac{\frac{d^4}{d\phi^4}[S(p,h)\circ\phi(z)]}{\sin(\pi\phi(z)/h)}\right| |dz|\Bigg\} \\
&\le \frac{1}{2}\Big[C_0(h,d)\mathcal{N}(ug_4^{(0)}) + C_1(h,d)\mathcal{N}(ug_4^{(1)}) \\
&\quad + C_2(h,d)\mathcal{N}(ug_4^{(2)}) + C_3(h,d)\mathcal{N}(ug_4^{(3)}) \\
&\quad + C_4(h,d)\mathcal{N}(ug_4^{(4)})\Big]\, e^{-\pi d/h} \\
&\equiv C_4\, e^{-\pi d/h}
\end{aligned}
\tag{2.59}
$$

where the bounds (2.53)–(2.57) have been applied. Similarly

$$
\begin{aligned}
&\left|(\varrho u''', S_p) + h \sum_{i=-\infty}^{\infty} \frac{u(x_i)}{\phi'(x_i)} (\varrho S_p w)'''(x_i)\right| = |I_F^{(3)}| \\
&\qquad \le \frac{e^{-\pi d/h}}{2} \Bigg\{ \int_{\partial D_E} |u(z)|\, |g_3^{(0)}(z)| \left|\frac{S(p,h)\circ\phi(z)}{\sin(\pi\phi(z)/h)}\right| |dz| \\
&\qquad + \int_{\partial D_E} |u(z)|\, |g_3^{(1)}(z)| \left|\frac{\frac{d}{d\phi}[S(p,h)\circ\phi(z)]}{\sin(\pi\phi(z)/h)}\right| |dz| \\
&\qquad + \int_{\partial D_E} |u(z)|\, |g_3^{(2)}(z)| \left|\frac{\frac{d^2}{d\phi^2}[S(p,h)\circ\phi(z)]}{\sin(\pi\phi(z)/h)}\right| |dz| \\
&\qquad + \int_{\partial D_E} |u(z)|\, |g_3^{(3)}(z)| \left|\frac{\frac{d^3}{d\phi^3}[S(p,h)\circ\phi(z)]}{\sin(\pi\phi(z)/h)}\right| |dz| \Bigg\} \\
&\qquad \le \frac{1}{2}\Big[C_0(h,d)\mathcal{N}(ug_3^{(0)}) + C_1(h,d)\mathcal{N}(ug_3^{(1)}) \\
&\qquad + C_2(h,d)\mathcal{N}(ug_3^{(2)}) + C_3(h,d)\mathcal{N}(ug_3^{(3)})\Big] e^{-\pi d/h} \\
&\qquad \equiv C_3\, e^{-\pi d/h} \quad .
\end{aligned} \tag{2.60}
$$

The remaining estimates are derived in [38] and recorded here as

$$
\begin{aligned}
&\left|(\nu u'', S_p) - h \sum_{i=-\infty}^{\infty} \frac{u(x_i)}{\phi'(x_i)} (\nu S_p w)''(x_i)\right| = |I_F^{(2)}| \\
&\qquad \le \frac{1}{2}\Big[C_0(h,d)\mathcal{N}(ug_2^{(0)}) + C_1(h,d)\mathcal{N}(ug_2^{(1)}) \\
&\qquad\qquad + C_2(h,d)\mathcal{N}(ug_2^{(2)})\Big] e^{-\pi d/h} \\
&\qquad \equiv C_2\, e^{-\pi d/h} \quad ,
\end{aligned} \tag{2.61}
$$

$$
\begin{aligned}
&\left|(\tau u', S_p) + h \sum_{i=-\infty}^{\infty} \frac{u(x_i)}{\phi'(x_i)} (\tau S_p w)'(x_i)\right| = |I_F^{(1)}| \\
&\qquad \le \frac{1}{2}\Big[C_0(h,d)\mathcal{N}(ug_1^{(0)}) + C_1(h,d)\mathcal{N}(ug_1^{(1)})\Big] e^{-\pi d/h} \\
&\qquad \equiv C_1\, e^{-\pi d/h} \quad ,
\end{aligned} \tag{2.62}
$$

$$\left|(\sigma u, S_p) - h\sigma(x_p)u(x_p)\,\frac{w(x_p)}{\phi'(x_p)}\right| = |I_F^{(0)}|$$

(2.63)

$$\leq \frac{1}{2}\,C_0(h,d)\mathcal{N}(\sigma u w)e^{-\pi d/h}$$

and

$$\left|(f, S_p) - hf(x_p)\,\frac{w(x_p)}{\phi'(x_p)}\right| = |I_R^{(0)}|$$

(2.64)

$$\leq \frac{1}{2}\,C_0(h,d)\mathcal{N}(f\dot{w})e^{-\pi d/h} \quad .$$

For the truncation of the sum in (2.59) assume that for $\ell = 1,2,3,4$ the terms satisfy the decay condition

$$\left|\frac{u(x)g_4^{(\ell)}(x)}{\phi'(x)}\right| \leq L\begin{cases}\exp(-\alpha|\phi(x)|) & , \ x \in \gamma_L \\ \exp(-\beta|\phi(x)|) & , \ x \in \gamma_R\end{cases} \tag{2.65}$$

where

$$\gamma_L = \left\{x : x \in \phi^{-1}\big((-\infty, 0)\big)\right\} = \left(a, \frac{a+b}{2}\right)$$

$$\gamma_R = \left\{x : x \in \phi^{-1}\big([0, \infty)\big)\right\} = \left[\frac{a+b}{2}, b\right) \quad .$$

This is used, along with the first of the inequalities

$$|\delta_{pi}^{(4)}| \leq \frac{\pi^4}{5} \quad , \tag{2.66}$$

$$|\delta_{pi}^{(3)}| \leq \frac{\pi^2}{2} - \frac{3}{4} \quad , \tag{2.67}$$

$$|\delta_{pi}^{(2)}| \leq \frac{\pi^2}{3} \tag{2.68}$$

and

$$|\delta_{pi}^{(\ell)}| \leq 1 \quad , \quad \ell = 0,1 \tag{2.69}$$

to obtain the bound

$$\frac{1}{h^3}\sum_{i=N+1}^{\infty}\left|\delta_{pi}^{(4)}\ \frac{u(x_i)g_4^{(4)}(x_i)}{\phi'(x_i)}\right| \le \frac{L\pi^4}{5h^3}\sum_{i=N+1}^{\infty} e^{-\beta ih}$$
$$= \frac{L\pi^4}{5h^3}\left|\frac{e^{-\beta(N+1)h}}{1-e^{-\beta h}}\right|$$
$$\le \frac{L\pi^4}{5\beta h^4}\ e^{-\beta N h} \quad .$$

Similarly,

$$\frac{1}{h^3}\sum_{i=-\infty}^{-M-1}\left|\delta_{pi}^{(4)}\ \frac{u(x_i)g_4^{(4)}(x_i)}{\phi'(x_i)}\right| \le \frac{L\pi^4}{5\alpha h^4}\ e^{-\alpha M h} \quad .$$

When the remaining pieces in (2.59) are bounded, the inequality

$$\left|(\eta u'''', S_p) - h\sum_{i=-M}^{N}\frac{u(x_i)}{\phi'(x_i)}\left[\frac{1}{h^4}\ \delta_{pi}^{(4)}g_4^{(4)}(x_i) + \frac{1}{h^3}\ \delta_{pi}^{(3)}g_4^{(3)}(x_i)\right.\right.$$
$$\left.\left. + \frac{1}{h^2}\ \delta_{pi}^{(2)}g_4^{(2)}(x_i) + \frac{1}{h}\ \delta_{pi}^{(1)}g_4^{(1)}(x_i) + \delta_{pi}^{(0)}g_4^{(0)}(x_i)\right]\right| \tag{2.70}$$
$$\le L\left[\frac{\pi^4}{5h^4} + \frac{2\pi^2-3}{4h^3} + \frac{\pi^2}{3h^2} + \frac{1}{h}\right]\left(\frac{e^{-\alpha M h}}{\alpha} + \frac{e^{-\beta N h}}{\beta}\right)$$
$$+ C_4 e^{-\pi d/h}$$

is obtained. The errors are balanced by making the selections

$$h = \left(\frac{\pi d}{\alpha M}\right)^{1/2} \tag{2.71}$$

and

$$N = \left[\left|\frac{\alpha}{\beta}\ M + 1\right|\right] \tag{2.72}$$

and by noting that since $C_j(h,d) = \mathcal{O}(h^{1-j})$, C_j is bounded by $K_\ell M^{(\ell-1)/2}$ where K_ℓ is constant and $\ell = 0,1,2,3,4$. The right-hand side of (2.70) can then

be bounded by

$$
\begin{aligned}
L\Bigg[\frac{\pi^4}{5}\left(\frac{\alpha}{\pi d}\right)^2 &+ \frac{(2\pi^2-3)}{4}\left(\frac{\alpha}{\pi d}\right)^{3/2} + \frac{\pi^2}{3}\left(\frac{\alpha}{\pi d}\right) \\
&+ \left(\frac{\alpha}{\pi d}\right)^{1/2}\Bigg]\left(\frac{1}{\alpha}+\frac{1}{\beta}\right) M^2 e^{-(\pi d\alpha M)^{1/2}} \\
&+ K_4 M^2 e^{-(\pi d\alpha M)^{1/2}} \\
&\equiv L_4 M^2 e^{-(\pi d\alpha M)^{1/2}} \quad .
\end{aligned} \tag{2.73}
$$

For convenience this and the remaining inner product approximations are referenced in the following theorem which is an extension of Theorem 2.11 of [38].

Theorem 2.6 *Let ϕ conformally map the simply connected domain D_E onto D_S (Figure 3). Let $x_i = \phi^{-1}(ih)$ and let $\mathcal{N}(\cdot)$ be defined by (2.8). Let $C_\ell(h,d)$ be given by (2.53)-(2.57), $\ell = 0,1,2,3,4$ and let C_ℓ be given by (2.59)-(2.62), $\ell = 1,2,3,4$. By selecting h and N as specified in (2.71) and (2.72), the following bounds are obtained.*

(a) If $fw \in B(D_E)$ then

$$
\begin{aligned}
\left|(f,S_p) - hf(x_p)\frac{w(x_p)}{\phi'(x_p)}\right| &\le \frac{1}{2}\,C_0(h,d)\,\mathcal{N}(fw)e^{-(\pi d\alpha M)^{1/2}} \\
&\le \frac{1}{2}\,K_0\,\mathcal{N}(fw)M^{-1/2}e^{-(\pi d\alpha M)^{1/2}} \quad .
\end{aligned}
$$

(b) If $\sigma u w \in B(D_E)$ then

$$
\begin{aligned}
&\left|(\sigma u,S_p) - h\sigma(x_p)u(x_p)\frac{w(x_p)}{\phi'(x_p)}\right| \\
&\qquad \le \frac{1}{2}\,K_0\,\mathcal{N}(\sigma u w)M^{-1/2}e^{-(\pi d\alpha M)^{1/2}} \quad .
\end{aligned}
$$

(c) If $B_T^{(1)}$ in (2.52) is zero, $ug_1^{(\ell)} \in B(D_E)$, $\ell = 0,1$ and $|ug_1^{(1)}/\phi'|$ is bounded by

the right-hand side of (2.65) then

$$\left|(\tau u', S_p) + h \sum_{i=-M}^{N} \frac{u(x_i)}{\phi'(x_i)} \left[\frac{1}{h}\, \delta_{pi}^{(1)} g_1^{(1)}(x_i) + \delta_{pi}^{(0)} g_1^{(0)}(x_i)\right]\right|$$

$$\leq L \left(\frac{\alpha}{\pi d}\right)^{1/2} \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) M^{1/2} e^{-(\pi d \alpha M)^{1/2}} + K_1 M^{1/2} e^{-(\pi d \alpha M)^{1/2}}$$

$$\equiv L_1 M^{1/2} e^{-(\pi d \alpha M)^{1/2}} \quad .$$

(d) If $B_T^{(2)}$ in (2.51) is zero, $ug_2^{(\ell)} \in B(D_E)$, $\ell = 0,1,2$ and $|ug_2^{(\ell)}/\phi'|$, $\ell = 1,2$ is bounded by the right-hand side of (2.65) then

$$\left|(\nu u'', S_p) - h \sum_{i=-M}^{N} \frac{u(x_i)}{\phi'(x_i)} \left[\frac{1}{h^2}\, \delta_{pi}^{(2)} g_2^{(2)}(x_i)\right.\right.$$

$$\left.\left. + \frac{1}{h} \delta_{pi}^{(1)} g_2^{(1)}(x_i) + \delta_{pi}^{(0)} g_2^{(0)}(x_i)\right]\right|$$

$$\leq L \left[\frac{\pi^2}{3} \left(\frac{\alpha}{\pi d}\right) + \left(\frac{\alpha}{\pi d}\right)^{1/2}\right] \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) M e^{-(\pi d \alpha M)^{1/2}}$$

$$+ K_2 M e^{-(\pi d \alpha M)^{1/2}}$$

$$\equiv L_2 M e^{-(\pi d \alpha M)^{1/2}} \quad .$$

(e) If $B_T^{(3)}$ in (2.30) is zero, $ug_3^{(\ell)} \in B(D_E)$, $\ell = 0,1,2,3$ and $|ug_3^{(\ell)}/\phi'|$, $\ell = 1,2,3$ is bounded by the right-hand side of (2.65) then

$$\left|(\varrho u''', S_p) + h \sum_{i=-M}^{N} \frac{u(x_i)}{\phi'(x_i)} \left[\frac{1}{h^3}\, \delta_{pi}^{(3)} g_3^{(3)}(x_i) + \frac{1}{h^2}\, \delta_{pi}^{(2)} g_3^{(2)}(x_i)\right.\right.$$

$$\left.\left. + \frac{1}{h}\, \delta_{pi}^{(1)} g_3^{(1)}(x_i) + \delta_{pi}^{(0)} g_3^{(0)}(x_i)\right]\right|$$

$$\leq L \left[\frac{2\pi^2 - 3}{4} \left(\frac{\alpha}{\pi d}\right)^{3/2} + \left(\frac{\alpha}{\pi d}\right)^{1/2}\right.$$

$$\left. + \left(\frac{\alpha}{\pi d}\right)^{1/2}\right] \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) M^{3/2} e^{-(\pi d \alpha M)^{1/2}}$$

$$+ K_3 M^{3/2} e^{-(\pi d \alpha M)^{1/2}}$$

$$\equiv L_3 M^{3/2} e^{-(\pi d \alpha M)^{1/2}} \quad .$$

(f) If $B_T^{(4)}$ *in (2.28) is zero,* $ug_4^{(\ell)} \in B(D_E)$, $\ell = 0,1,2,3,4$ *and* $|ug_4^{(\ell)}/\phi'|$, $\ell = 1,2,3,4$ *is bounded by the right-hand side of (2.65) then from (2.73)*

$$\left|(\eta u'''', S_p) - h \sum_{i=-M}^{N} \frac{u(x_i)}{\phi'(x_i)} \left[\frac{1}{h^4}\, \delta_{pi}^{(4)} g_4^{(4)}(x_i) + \frac{1}{h^3}\, \delta_{pi}^{(3)} g_4^{(3)}(x_i) \right.\right.$$

$$\left.\left. + \frac{1}{h^2}\, \delta_{pi}^{(2)} g_4^{(2)}(x_i) + \frac{1}{h}\, \delta_{pi}^{(1)} g_4^{(1)}(x_i) + \delta_{pi}^{(0)} g_4^{(0)}(x_i) \right] \right|$$

$$\leq L_4 M^2 e^{-(\pi d \alpha M)^{1/2}} \quad .$$

The inner product approximations in Theorem 2.6 in conjunction with (2.26) can be used to obtain the Sinc-Galerkin system for the boundary value problem (2.20). Letting the solution $u(x)$ of (2.20) be approximated by the finite sinc expansion

$$u_m(x) = \sum_{i=-M}^{N} u_i S_i(x) \quad , \quad m = M + N + 1 \quad , \tag{2.74}$$

the discrete system for the determination of the unknown coefficients $\{u_i\}$ is given by

$$A\vec{u} = D\left(f\, \frac{w}{\phi'}\right) \vec{1} \tag{2.75}$$

where

$$A = \left[\frac{1}{h^4}\, I^{(4)} D(a_4) + \frac{1}{h^3}\, I^{(3)} D(a_3) + \frac{1}{h^2}\, I^{(2)} D(a_2) \right.$$

$$\left. + \frac{1}{h}\, I^{(1)} D(a_1) + I^{(0)} D(a_0) \right] \quad . \tag{2.76}$$

Here $I^{(\ell)}$, $\ell = 0,1,2,3,4$ are $m \times m$ matrices whose pi-th entry is given by $\delta_{pi}^{(\ell)}$ from (2.31)–(2.35), $\vec{u}$ is an m-vector containing the unknown coefficients, $\vec{1}$ is an m-vector each of whose components is 1 and $D(g)$ is an $m \times m$ diagonal matrix

(with diagonal entries $a(x_{-M}), a(x_{-M+1}), \ldots, a(x_N)$). The functions $a_\ell(x)$, $\ell = 0,1,2,3,4$ are given by

$$a_0(x) = \left[g_4^{(0)}(x) - g_3^{(0)}(x) + g_2^{(0)}(x) - g_1^{(0)}(x) + \sigma(x)w(x)\right]/\phi'(x) \quad , \tag{2.77}$$

$$a_1(x) = \left[g_4^{(1)}(x) - g_3^{(1)}(x) + g_2^{(1)}(x) - g_1^{(1)}(x)\right]/\phi'(x) \quad , \tag{2.78}$$

$$a_2(x) = \left[g_4^{(2)}(x) - g_3^{(2)}(x) + g_2^{(2)}(x)\right]/\phi'(x) \quad , \tag{2.79}$$

$$a_3(x) = \left[g_4^{(3)}(x) - g_3^{(3)}(x)\right]/\phi'(x) \tag{2.80}$$

and

$$a_4(x) = g_4^{(4)}(x)/\phi'(x) \quad . \tag{2.81}$$

It should be pointed out that the discrete system obtained in (2.75) is the same as the one that is obtained if the analysis is continued with the inner product $(\mathcal{L}u_m - f, S_p)$ rather than $(\mathcal{L}u - f, S_p)$. This follows from the fact that an analogue of Theorem 2.6 can be proved once it has been noted that all infinite sums collapse to finite sums as a result of the property

$$u_m(x_p) = \sum_{i=-M}^{N} u_i S_i(x_p) = u_p$$

for $-M \leq p \leq N$. Because of this inherent truncation of sums, the decay conditions would not appear in that theorem and would therefore be dealt with separately. In the setting of this chapter, the analysis and approximation of $(\mathcal{L}u - f, S_p)$ is more direct than that of $(\mathcal{L}u_m - f, S_p)$, but in other problems such as those found in later chapters, the opposite is often true.

Since the $m \times m$ matrices $I^{(\ell)}$, $\ell = 1, 2, 3, 4$, are a fundamental component of the matrix A, their form and properties will be carefully documented. From their definition in (2.31)–(2.34), it is seen that they are Toeplitz with structure given by

$$I^{(1)} = \begin{bmatrix} 0 & -1 & \frac{1}{2} & \cdots & \frac{(-1)^{m-1}}{(m-1)} \\ 1 & & & & \vdots \\ -\frac{1}{2} & & \ddots & & \frac{1}{2} \\ \vdots & & & & -1 \\ \frac{(-1)^m}{(m-1)} & \cdots & -\frac{1}{2} & 1 & 0 \end{bmatrix} \tag{2.82}$$

$$I^{(2)} = \begin{bmatrix} -\frac{\pi^2}{3} & 2 & \frac{-2}{2^2} & \cdots & \frac{2(-1)^m}{(m-1)^2} \\ 2 & & & & \vdots \\ \frac{-2}{2^2} & & \ddots & & \frac{-2}{2^2} \\ \vdots & & & & 2 \\ \frac{2(-1)^m}{(m-1)^2} & \cdots & \frac{-2}{2^2} & 2 & -\frac{\pi^2}{3} \end{bmatrix} \tag{2.83}$$

$$(2.84)\qquad I^{(3)} = \begin{bmatrix} 0 & -(6-\pi^2) & \cdots & \frac{(-1)^{m-1}\left(6-(m-1)^2\pi^2\right)}{(m-1)^3} \\ (6-\pi^2) & & & \\ -(6-2^2\pi^2)/2^3 & & & \vdots \\ \vdots & & & \\ & & & -(6-\pi^2) \\ \frac{(-1)^m\left(6-(m-1)^2\pi^2\right)}{(m-1)^3} & \cdots & 6-\pi^2 & 0 \end{bmatrix}$$

and

$$(2.85)\qquad I^{(4)} = \begin{bmatrix} \pi^4/5 & 4(6-\pi^2) & \cdots & \frac{4(-1)^m\left(6-(m-1)^2\pi^2\right)}{(m-1)^4} \\ 4(6-\pi^2) & & & \\ -4(6-2^2\pi^2)/2^4 & & & \vdots \\ \vdots & & & 4(6-\pi^2) \\ \frac{4(-1)^m\left(6-(m-1)^2\pi^2\right)}{(m-1)^4} & \cdots & 4(6-\pi^2) & \pi^4/5 \end{bmatrix}$$

The matrices $I^{(1)}$ and $I^{(2)}$ have been discussed in [38]. Their significant properties have been recorded in the following theorem along with the derivation of the analogous properties for $I^{(3)}$ and $I^{(4)}$. The Toeplitz form for each $I^{(\ell)}$, $\ell = 1,2,3,4$, is obtainable from the Fourier series expansion of $(-it)^\ell$ on $[-\pi,\pi]$, [14].

Theorem 2.7 *Let $I^{(\ell)}$, $\ell = 1,2,3,4$, be the $m \times m$ matrices ($m = M+N+1$) given by (2.31)-(2.34).*

(a) $I^{(1)}$ is a skewsymmetric, skewcentrosymmetric, Toeplitz matrix with eigenvalues $\{i\lambda_k^{(1)}\}_{k=-M}^{N}$ satisfying $-\pi \le \lambda_{-M}^{(1)} \le \lambda_{-M+1}^{(1)} \le \cdots \le \lambda_N^{(1)} \le \pi$.

(b) $I^{(2)}$ is a symmetric, centrosymmetric, negative definite, Toeplitz matrix with eigenvalues $\{-\lambda_k^{(2)}\}_{k=-M}^{N}$ satisfying $0 < 4\sin^2\big(\pi/(2m+2)\big) \le \lambda_{-M}^{(2)} \le \lambda_{-M+1}^{(2)} \le \cdots \le \lambda_N^{(2)} \le \pi^2$. The ℓ^2-condition number of $I^{(2)}$ is bounded by $\pi^2/\{4\sin^2\big(\pi/(2m+2)\big)\} = \mathcal{O}\big((m+1)^2\big)$.

(c) $I^{(3)}$ is a skewsymmetric, skewcentrosymmetric, Toeplitz matrix with eigenvalues $\{i\lambda_k^{(3)}\}_{k=-M}^{N}$ satisfying $-\pi^3 \le \lambda_{-M}^{(3)} \le \lambda_{-M+1}^{(3)} \le \cdots \le \lambda_N^{(3)} \le \pi^3$.

(d) $I^{(4)}$ is a symmetric, centrosymmetric, positive definite, Toeplitz matrix with eigenvalues $\{\lambda_k^{(4)}\}_{k=-M}^{N}$ satisfying $0 < 16\sin^4\big(\pi/(2m+2)\big) \le \lambda_{-M}^{(4)} \le \lambda_{-M+1}^{(4)} \le \cdots \le \lambda_N^{(4)} \le \pi^4$. The ℓ^2-condition number of $I^{(4)}$ is bounded by $\pi^4/\{16\sin^4\big(\pi/(2m+2)\big)\} = \mathcal{O}\big((m+1)^4\big)$.

Proof

(a) and *(b)* are discussed in [38]. $I^{(3)}$ is a skewsymmetric, Toeplitz matrix by inspection. Using the results in [14, p.65], $\{\lambda_k^{(3)}\}_{k=-M}^{N}$ satisfy the given inequality. A matrix A is centrosymmetric if $JAJ = A$ where J is the counter-identity with $j_{r,m-r+1} = 1$, $r = 1,2,\ldots,m$ and all other entries zero. Thus A is centrosymmetric if $a_{rs} = a_{m+1-r,m+1-s}$ for $r,s = 1,\ldots,m$. Since $\delta_{rs}^{(3)} = -\delta_{m+1-r,m+1-s}^{(3)}$ $I^{(3)}$ is termed skewcentrosymmetric. Next, $I^{(4)}$ is a symmetric, centrosymmetric, Toeplitz matrix where this time from [14, pp. 64 and 67] $0 \le \lambda_{-M}^{(4)} \le \cdots \le \lambda_N^{(4)} \le \pi^4$. Since $t^4 \ge (2-2\cos t)^2 \equiv g(t)$ on $[-\pi,\pi]$, the eigenvalues $\{\lambda_k^{(4)}\}_{k=-M}^{N}$ are bounded below by those of the Toeplitz matrix W generated from the Fourier series expansion of $g(t)$ on $[-\pi,\pi]$. Here W is a symmetric, pentadiagonal matrix

given by

$$W = \begin{bmatrix} 6 & -4 & 1 & 0 & \cdots & \cdots & 0 \\ -4 & & & & & & \vdots \\ 1 & & & \ddots & & & 0 \\ 0 & & & & & & 1 \\ \vdots & & & & & & -4 \\ 0 & \cdots & \cdots & 0 & 1 & -4 & 6 \end{bmatrix} = T^2 + Y$$

where T is the symmetric tridiagonal matrix

$$T = \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & & & & & \vdots \\ 0 & & \ddots & & & \vdots \\ \vdots & & & & & 0 \\ \vdots & & & & & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}$$

and

$$Y = \begin{bmatrix} 1 & 0 & & \cdots & & 0 \\ 0 & 0 & & & & 0 \\ \vdots & & \ddots & & & \vdots \\ & & & & 0 & 0 \\ 0 & & \cdots & & 0 & 1 \end{bmatrix} .$$

The eigenvalues of T^2 are $\tau_\ell = 16\sin^4\big(\ell\pi/(2m+2)\big)$, $\ell = 1, 2, \ldots, m$. Therefore by Weyl's Monotonicity Theorem [44, pages 101–102] the minimum eigenvalue of W is bounded below by $16\sin^4\big(\pi/(2m+2)\big)$ and is thus a lower bound for $\lambda_{-M}^{(4)}$. Finally, the ℓ^2-condition number of $I^{(4)}$ is

$$\begin{aligned} \mathcal{K}_2(I^{(4)}) &= \frac{\lambda_N^{(4)}}{\lambda_{-M}^{(4)}} \\ &\leq \frac{\pi^4}{16\sin^4\big(\pi/(2m+2)\big)} \\ &= \mathcal{O}\big((m+1)^4\big) \quad . \end{aligned}$$

Implementation of the Method

The implementation of the Sinc-Galerkin method requires that a suitable weight function be selected for the inner product in (2.24), that the discrete system be formulated and that computational issues relating to matrix structure be addressed. The choice of weight function can be crucial to the performance of the method and is thus addressed first.

The weight function in the Sinc-Galerkin inner products may be chosen for a variety of reasons; for example, (1) to simplify the discrete system, (2) to introduce special structure in the discrete system such as symmetry where possible, (3) to help in the truncation of the infinite series, (4) to force the boundary terms $B_T^{(\ell)}$, $\ell = 1,2,3,4$ to be zero, (5) to broaden the applicability of the method to a wider class of problems, or (6) to control the conditioning of the matrix A. Although even more reasons exist, the choices made here will be governed by the ones just listed.

The weight $w(x)$ is chosen to have the general form

$$w(x) = \frac{1}{[\phi'(x)]^{\varsigma}} \tag{2.86}$$

where $\phi'(x) = \frac{b-a}{(x-a)(b-x)}$. Various possibilities for ς will result from consideration of the criteria (1)–(6). Once the weight has been specified, the decay conditions (as in (2.65)) that are necessary for the truncation of the infinite sums can be more clearly defined. For the general weight given in (2.86), it is required that

$$|u(x)| \leq L \begin{cases} (x-a)^{\alpha+3-\varsigma} & , \ x \in \gamma_L \\ (b-x)^{\beta+3-\varsigma} & , \ x \in \gamma_R \end{cases} . \tag{2.87}$$

Since the entire development of the previous section hinged on the elimination of the boundary terms $B_T^{(\ell)}$, criterion (4) will be discussed first. Consider a differential equation with a regular boundary singularity as defined in [6]. Focusing

on $B_T^{(1)}$ initially, it is seen from (2.52) that a suitable choice to enforce $B_T^{(1)} = 0$ is

$$w(x) = \frac{1}{[\phi'(x)]^2} \quad . \tag{2.88}$$

The choice (2.88) in each of $B_T^{(2)}$, $B_T^{(3)}$ and $B_T^{(4)}$ ensures that for a problem with regular boundary singularities, all boundary terms will vanish. From (2.87), it follows that the truncation requirement is

$$|u(x)| \leq L \begin{cases} (x-a)^{\alpha+1} & , \ x \in \gamma_L \\ (b-x)^{\beta+1} & , \ x \in \gamma_R \end{cases} \quad . \tag{2.89}$$

There is another choice of weight which provides direct computational benefits. As is the case with second-order problems, maintaining symmetry is advantageous in many settings [27]. Here, choosing

$$w(x) = \frac{1}{[\phi'(x)]^{3/2}} \tag{2.90}$$

will lead to a symmetric discrete system for the self-adjoint problem

$$\begin{aligned} u''''(x) + \sigma(x)u(x) &= f(x) \quad , \quad a < x < b \\ u(0) = u(1) &= 0 \\ u'(0) = u'(1) &= 0 \quad . \end{aligned} \tag{2.91}$$

The choice of $w(x) = \frac{1}{[\phi'(x)]^2}$ does not provide this symmetry. The computational gains provided by symmetry are not dramatic in the forward solution of fourth-order ordinary differential equations. However, the application of this method to fourth-order partial differential equations and time-dependent inverse problems leads to computational environments where the symmetry is important. Thus it warrants investigation here. It should be noted that the approximation based on the weight function (2.90) is applicable to a more restrictive class of problems

than the approximation based on the weight in (2.88). This is seen by comparing (2.89) with the decay condition for the weight $w(x) = \frac{1}{[\phi'(x)]^{3/2}}$ given by

$$|u(x)| \leq L \begin{cases} (x-a)^{\alpha+3/2} & , \ x \in \gamma_L \\ (b-x)^{\alpha+3/2} & , \ x \in \gamma_R \end{cases} \quad . \tag{2.92}$$

With the weight function (2.90), the Sinc-Galerkin system for the self-adjoint problem (2.91) is given by

$$\left[\frac{1}{h^4} I^{(4)} + \frac{1}{h^2} I^{(2)} D\left(\frac{g_4^{(2)}}{g_4^{(4)}}\right) + D\left(\frac{g_4^{(0)} + \sigma(\phi')^{-3/2}}{g_4^{(4)}}\right)\right] D\left(\frac{g_4^{(4)}}{\phi'}\right) \vec{u}$$
$$= D\left(\frac{f}{(\phi')^{5/2}}\right) \vec{1} \quad .$$

By using (2.15), (2.37), (2.39) and (2.41), this system can be rewritten as

$$\left[\frac{1}{h^4} I^{(4)} - \frac{5}{2h^2} I^{(2)} + D\left(\frac{[(\phi')^{-3/2}]'''' + \sigma(\phi')^{-3/2}}{(\phi')^{5/2}}\right)\right] \vec{v}$$
$$= D\left(\frac{f}{(\phi')^{5/2}}\right) \vec{1} \tag{2.93}$$

where

$$\vec{v} = D\left((\phi')^{3/2}\right) \vec{u} \quad .$$

Note that the diagonal matrix multipliers of $I^{(3)}$ and $I^{(1)}$ ($D(a_3)$ and $D(a_1)$, respectively) are zero with this choice of weight.

Once the choice of weight has been made, the decay parameters α and β are chosen. It should be pointed out that asymptotic techniques such as the method of Frobenius [35] can be used to find α and β when the true solution is not known. Furthermore, numerical tests show that a fairly wide range of parameters yield good results with optimal ones being obtained when the true parameters are used. Thus the method is robust even in problems in which the true parameters are unknown.

With the weight and decay parameters specified, h and N are determined via

$$h = \sqrt{\frac{\pi d}{\alpha M}} \tag{2.94}$$

and

$$N = \left[\left|\frac{\alpha}{\beta} M + 1\right|\right] \quad . \tag{2.95}$$

Note that if $\frac{\alpha}{\beta} M$ is an integer, it suffices to choose $N = \frac{\alpha}{\beta} M$. The system (2.75) (or (2.93) for the self-adjoint problem) is then solved for the unknown coefficients u_i of the expansion (2.74). The asymptotic convergence rate for the method is then $\mathcal{O}\left(e^{-(\pi d \alpha M)^{1/2}}\right)$. The computational performance of this method will be illustrated in the sixth section.

An Alternate Formulation

For the purpose of analysis, it is useful to consider an alternate formulation for the approximate solution to the problem

$$\begin{aligned} \mathcal{L}u(x) = \eta u''''(x) + \varrho(x)u'''(x) + \nu(x)u''(x) + \tau(x)u'(x) \\ + \sigma(x)u(x) = f(x) \quad , \quad a < x < b \end{aligned} \tag{2.96}$$

$$\begin{aligned} u(a) = u(b) = 0 \\ u'(a) = u'(b) = 0 \end{aligned} \tag{2.97}$$

when in addition it is known that the true solution has consecutive higher derivatives that are zero at the endpoints. If u satisfies (2.97) and furthermore has the property that

$$u''(a) = u''(b) = 0 \quad ,$$

then the alternate formulation

$$u_h(x) = \sum_{i=-M}^{N} u_i[(x-a)(b-x)]^2 S_i(x) \tag{2.98}$$

will be considered for the approximate solution. Similarly, if u satisfies (2.97) as well as

$$\begin{aligned} u''(a) = u''(b) = 0 \\ u'''(a) = u'''(b) = 0 \quad , \end{aligned} \tag{2.99}$$

then the alternate formulation

$$u_h(x) = \sum_{i=-M}^{N} u_i[(x-a)(b-x)]^3 S_i(x) \tag{2.100}$$

can be used. The functions

$$v(x) = [(x-a)(b-x)]^{\ell} \quad , \quad \ell = 2,3 \tag{2.101}$$

can be thought of as basis weights. The formulations (2.98) and (2.100) will be useful for analyzing the weights and decay parameters for approximate solutions to problems with higher order boundary conditions.

Since Theorem 2.6 is formulated in terms of the true solution rather than in terms of the approximation solution, those results will hold in this section just as they did in the second section. Thus the decay parameters α and β will be chosen in the same manner as before, regardless of whether the approximate solution is given by (2.22), (2.98), or (2.100). This observation will be especially important in the next chapter where problems with higher order boundary conditions will be considered. To facilitate the decay analysis in those problems, a change of variables will be introduced which reduces the problem to the form (2.96) with approximations given by either (2.98) or (2.100). The decay parameters will then be chosen accordingly.

As a result of Theorem 2.6 and the property

$$u_h(x_p) = u_p v(x_p) \quad ,$$

the discrete systems corresponding to (2.98) and (2.100) are

$$\begin{aligned} &\left[\frac{1}{h^4}\, I^{(4)} D(va_4) + \frac{1}{h^3}\, I^{(3)} D(va_3) + \frac{1}{h^2}\, I^{(2)} D(va_2)\right. \\ &\left. \quad + \frac{1}{h}\, I^{(1)} D(va_1) + I^{(0)} D(va_0)\right] \vec{u} = D\left(f\,\frac{w}{\phi'}\right)\vec{1} \end{aligned} \tag{2.102}$$

with ℓ in (2.101) taken to be two and three, respectively. The functions $a_\ell(x)$, $\ell = 0,1,2,3,4$ are defined in (2.77)–(2.81).

The final issue to be considered in this section is the correlation between the basis weight $v(x)$ and the inner product weight $w(x)$. In most settings, the basis weight $v(x)$ is dictated by the problem and the inner product weight $w(x)$ must then be chosen according to the criteria (1)–(6) from the previous section. Since the first five have been discussed in some detail, the analysis here will center on (6), the effect of the weight $w(x)$ on the conditioning of the matrix A. Since it is difficult to analyze the matrix derived from the general problem (2.96), two very simplified examples will be considered instead. Although the results determined in these examples are not general, they can be combined with numerical results to qualitatively indicate the effect of the weight function on matrices arising in more general problems.

Example 2.8 Consider the boundary value problem

$$\begin{aligned} &u''''(x) + \varrho(x)u'''(x) + \nu(x)u''(x) + \tau(x)u'(x) \\ &\quad + \sigma(x)u(x) = f(x) \quad , \quad 0 < x < 1 \\ &u(0) = u(1) = 0 \\ &u'(0) = u'(1) = 0 \quad . \end{aligned}$$

Let the coefficients be recursively defined as

$$\varrho(x) = 4\,\frac{w'(x)}{w(x)} + 6\,\frac{\phi''(x)}{\phi'(x)} \quad ,$$

$$\nu(x) = 3\,\frac{(\varrho(x)w(x))'}{w(x)} + 3\,\frac{\varrho(x)\phi''(x)}{\phi'(x)} - 6\,\frac{w''(x)}{w(x)} - 12\,\frac{\phi''(x)w'(x)}{\phi'(x)w(x)} - 4\,\frac{\phi'''(x)}{\phi'(x)} + 3\,\frac{(\phi''(x))^2}{(\phi'(x))^2} \quad ,$$

$$\tau(x) = 2\,\frac{(\nu(x)w(x))'}{w(x)} + \frac{\nu(x)\phi''(x)}{\phi'(x)} - 3\,\frac{(\varrho(x)w(x))''}{w(x)} - 3\,\frac{(\varrho(x)w(x))'\phi''(x)}{w(x)\phi'(x)} - \frac{\varrho(x)\phi'''(x)}{\phi'(x)} + \frac{w'''(x)}{w(x)} + 6\,\frac{\phi''(x)w''(x)}{\phi'(x)w(x)} + 4\,\frac{\phi'''(x)w'(x)}{\phi'(x)w(x)} + \frac{\phi''''(x)}{\phi'(x)}$$

and

$$\sigma(x) = -\frac{w''''(x)}{w(x)} + \frac{(\varrho(x)w(x))'''}{w(x)} - \frac{(\nu(x)w(x))''}{w(x)} + \frac{(\tau(x)w(x))'}{w(x)} \quad .$$

The weight, for the moment, is taken to be of the general form

$$w(x) = \frac{1}{[\phi'(x)]^{\varsigma}} \quad . \tag{2.103}$$

Finally, it is assumed that u satisfies $u''(0) = u''(1) = 0$ as well as (2.97) so the formulation

$$u_h(x) = \sum_{i=-M}^{N} u_i [x(1-x)]^2 S_i(x)$$

is appropriate for the approximate solution. By design, the diagonal matrices $D(va_3)$, $D(va_2)$, $D(va_1)$ and $D(va_0)$ in (2.102) are zero for any values of ς. By using (2.81) and (2.103), the discrete system for the problem may be written as

$$\frac{1}{h^4}\, I^{(4)} D\left([x(1-x)]^{\varsigma-1}\right) \vec{u} = D\left(f[x(1-x)]^{\varsigma-1}\right) \vec{1} \quad .$$

By varying ς in (2.103), one may note directly the effect of the weight $w(x)$ on the matrix

$$A = \frac{1}{h^4}\, I^{(4)}\, \mathcal{D}\left([x(1-x)]^{\varsigma-1}\right) \quad .$$

It follows immediately from the properties of norms that

$$\|A\|_2 \leq \frac{1}{h^4}\, \|I^{(4)}\|_2\, \|\mathcal{D}\left([x(1-x)]^{1-\varsigma}\right)\|_2$$

and

$$\|A^{-1}\|_2 \leq h^4 \|\mathcal{D}^{-1}\left([x(1-x)]^{\varsigma-1}\right)\|_2\, \|(I^{(4)})^{-1}\|_2 \quad .$$

Furthermore, Theorem 2.7 yields the bounds

$$\|I^{(4)}\|_2 \leq \pi^4$$

and

$$\|(I^{(4)})^{-1}\|_2 \leq \frac{1}{16\sin^4(\pi/(2m+2))} \quad .$$

Finally, the norms of the diagonal matrices can be calculated from (2.18) and the geometric properties of the function $g(x) = x(1-x)$. By considering the cases $\varsigma < 1$, $\varsigma = 1$ and $\varsigma > 1$, one obtains the bounds

$$K_2(A) \leq K_2(I^{(4)}) \cdot \left(\frac{1}{4}\right)^{|\varsigma-1|} \left[\frac{(1+e^{\mathcal{M}h})^2}{e^{\mathcal{M}h}}\right]^{|\varsigma-1|} \tag{2.104}$$

where $\mathcal{M} = \max(M, N)$ and

$$K_2(I^{(4)}) \leq \frac{\pi^4}{16\sin^4(\pi/(2m+2))} \quad . \tag{2.105}$$

As before, $K_2(A)$ denotes the ℓ^2-condition number of the matrix A. Although the inequality in (2.104) prevents direct comparisons, the asymptotic statement

$$K_2(I^{(4)}) \cdot \left(\frac{1}{4}\right)^{|\varsigma-1|} \left[\frac{(1+e^{\mathcal{M}h})^2}{e^{\mathcal{M}h}}\right]^{|\varsigma-1|} \sim K_2(I^{(4)}) \left(\frac{1}{4}\right)^{|\varsigma-1|} e^{\mathcal{M}h|\varsigma-1|} \tag{2.106}$$

for large M, indicates that the conditioning of A should improve as $\varsigma \to 1$. Numerical results verify this trend even for problems with more general coefficients ϱ, τ, ν and σ. Also, if $\varsigma \neq 1$ and M is fixed, the asymptotic expression in (2.106) is minimized when h is minimized. This can be combined with (2.104) and (2.105) to predict that for smaller choices of the decay parameter α, the conditioning of A should become worse. Numerical results indicate that in general this is also true.

Example 2.9 Consider the problem from Example 2.8 with the added assumptions

$$u''(0) = u''(1) = 0$$

$$u'''(0) = u'''(1) = 0 \quad .$$

Let the appropriate solution be given by

$$u_h(x) = \sum_{i=-M}^{N} u_i [x(1-x)]^3 S_i(x) \quad .$$

Analysis identical to that in Example 2.8 indicates that for the general weight

$$w(x) = \frac{1}{[\phi'(x)]^\varsigma} \quad ,$$

the conditioning of A should improve as $\varsigma \to 0$. Furthermore, for fixed M the conditioning should deteriorate as $\alpha \to 0$. Numerical results verify that in general both trends are true.

When coupled with numerical results, the last two examples show that the conditioning of A can be influenced by the choice of the inner product weight $w(x)$. In general, the conditioning improves as the power on the weight decreases. The use of the weight to control the conditioning of A will be especially important in Chapter 3 when problems with higher order boundary conditions are considered. Depending on the problem, this final criterion for weight choice must therefore

be balanced against those previously discussed. The example sections in this and later chapters will show ways in which this balancing is accomplished.

Numerical Examples

The two examples reported in this section were selected from a large collection of problems to which the Sinc-Galerkin method was applied. The results are representative of those obtained on other sample problems. The ones included here illustrate various features of the method, demonstrating the ease of implementation and assembly of the discrete system, the choice of parameters and the exponential convergence rate (even in the presence of boundary singularities).

In all examples $d = \frac{\pi}{2}$. The errors are reported on both the set of sinc gridpoints

$$S = \{x_{-M}, x_{-M+1}, \ldots, x_0, \ldots, x_N\} \quad ;$$
$$x_k = \frac{be^{kh} + a}{1 + e^{kh}} \quad , \quad k = -M, \ldots, N \tag{2.107}$$

and the set of uniform gridpoints with stepsize $\ell = \frac{b-a}{100}$

$$U = \{z_0, z_1, \ldots, z_{100}\} \quad ;$$
$$z_k = a + k\ell \quad , \quad k = 0, 1, \ldots, 100 \quad . \tag{2.108}$$

The errors on these grids are

$$\|E_S^M(h)\| = \max_{-M \le j \le N} |u(x_j) - u_m(x_j)| \tag{2.109}$$

and

$$\|E_U^M(\ell)\| = \max_{0 \le j \le 100} |u(z_j) - u_m(z_j)| \quad , \tag{2.110}$$

respectively. The dependence of both errors on the number of sinc basis functions is indicated with the superscript M. From Theorem 2.6 it follows that if

exponential convergence is realized, then

$$\left(\|E_U^{M_1}(\ell)\|\right)^{\sqrt{M_2/M_1}} \approx \|E_U^{M_2}(\ell)\| \tag{2.111}$$

where M_1 and M_2 denote the number of basis functions in the lower portion of the sum. In the examples of this section, $M_2 = 2M_1$, and the exponential convergence is verified by comparing $\left(\|E_U^{M_2/2}(\ell)\|\right)^{\sqrt{2}}$ and $\|E_U^{M_2}(\ell)\|$.

The error and convergence results are tabulated in the form $.aaa - \gamma$ which represents $.aaa \times 10^{-\gamma}$. All problems were run with sixteen place accuracy on a Vax 8550.

Example 2.10

$$\begin{aligned} u''''(x) + u(x) &= f(x) \quad , \quad 0 < x < 1 \\ u(0) = u(1) &= 0 \\ u'(0) = u'(1) &= 0 \quad . \end{aligned} \tag{2.112}$$

The function $f(x)$ is consistent with the solution $u(x) = [x(1-x)]^{7/2}$ which has algebraic singularities at $x = 0$ and $x = 1$. In this example the two weight functions given in (2.88) and (2.90) will be compared. The Sinc-Galerkin method with weight function given by $w(x) = (\phi'(x))^{-2}$ necessitates the parameter choices of $\alpha = \beta = \frac{5}{2}$ (see 2.89). Hence from (2.94) $h = \pi/\sqrt{5M}$ and the asymptotic error is $\mathcal{O}\left(e^{-\pi\sqrt{5M}/2}\right)$. Since $\alpha = \beta$, centered sums $(M = N)$ result. The errors are reported in Table 1. Note that the uniform errors asymptotically follow the predicted errors thus demonstrating the exponential convergence rate of the method.

M	N	h	$\|E_S^M(h)\|$	$\|E_U^M(\ell)\|$	$\left(\|E_U^{M/2}(\ell)\|\right)^{\sqrt{2}}$
2	2	0.9935	$.3042-3$	$.3739-3$	———
4	4	0.7025	$.9159-4$	$.9159-4$	$.1423-4$
8	8	0.4967	$.8735-5$	$.8735-5$	$.1946-5$
16	16	0.3512	$.2292-6$	$.2292-6$	$.7012-7$
32	32	0.2484	$.9455-9$	$.9455-9$	$.4073-10$

Table 1. Errors on the Sinc Grid S and the Uniform Grid U for Example 2.10 with $w(x) = (\phi'(x))^{-2}$.

The weight function $w(x) = (\phi'(x))^{-3/2}$ dictates the choices $\alpha = \beta = 2$ (see (2.92)). Hence $h = \pi/(2\sqrt{M})$ and the asymptotic error is $\mathcal{O}\left(e^{-\pi\sqrt{M}}\right)$. It is again noted that the method exhibits exponential convergence as demonstrated by the results of the last column of Table 2. By comparing the results from Table 1 and Table 2, it is seen that as M increases, the errors decrease more rapidly for this weight function than for the choice $w(x) = (\phi'(x))^{-2}$. This was borne out in all other examples. A discussion of this feature is found in [27]. This combined with the computational efficiency gained for self-adjoint problems (a symmetric discrete system arises as in (2.93)) makes the weight $w(x) = (\phi'(x))^{-3/2}$ advantageous in many problems.

The true and approximate solutions to (2.112) are shown in Figure 5. Included are the approximations for $M = 2, 4$. For larger values of M the approximate solutions are indistinguishable (on this scale) from the true solution due to the exponential convergence rate of the method.

M	N	h	$\|E_S^M(h)\|$	$\|E_U^M(\ell)\|$	$\left(\|E_U^{M/2}(\ell)\|\right)^{\sqrt{2}}$
2	2	1.1107	$.5165-3$	$.5348-3$	———
4	4	0.7854	$.2562-4$	$.5279-4$	$.2360-4$
8	8	0.5554	$.2247-5$	$.2630-5$	$.8928-6$
16	16	0.3927	$.3885-7$	$.3885-7$	$.1284-7$
32	32	0.2777	$.8299-10$	$.8299-10$	$.3310-10$

Table 2. Errors on the Sinc Grid S and the Uniform Grid U for Example 2.10 with $w(x) = (\phi'(x))^{-3/2}$.

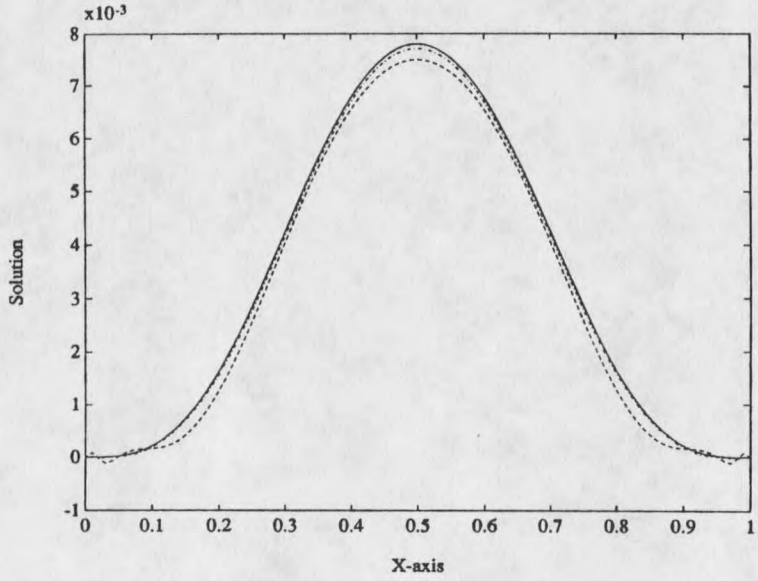


Figure 5. True and Approximate Solutions to Example 2.10
$- - -$ $(M = 2)$, $- \cdot -$ $(M = 4)$, — (True).

Example 2.11

$$u''''(x) + \frac{1}{x}\, u'''(x) + \frac{1}{x^2}\, u''(x) + \frac{1}{x^3}\, u'(x) + \frac{1}{x^4}\, u(x) = f(x) \quad , \quad 0 < x < 1$$

$$u(0) = u(1) = 0$$

$$u'(0) = u'(1) = 0 \quad .$$

Here $f(x)$ is consistent with the true solution $u(x) = x^{5/2}(\ell n\,(x))^3$. This solution has both an algebraic and a logarithmic singularity at $x = 0$. This example also illustrates the feasibility of the Sinc-Galerkin method for the full fourth-order variable coefficient problem. The discrete system which arises is (2.75) where the weight function is taken to be $w(x) = (\phi'(x))^{-2}$. The parameters $\alpha = \frac{3}{2}$ and $\beta = 2$ yield the stepsize $h = \pi/\sqrt{3M}$ and an asymptotic error of $\mathcal{O}\left(e^{-\pi\sqrt{3M}/2}\right)$. The upper limit N is given by (2.95) which leads to noncentered sums ($M \neq N$). The errors and solutions are reported in Table 3 and Figure 6, respectively. Again the exponential convergence rate quickly decreases the errors that are initially present for small M.

M	N	h	$\|E_S^M(h)\|$	$\|E_U^M(\ell)\|$	$\left(\|E_U^{M/2}(\ell)\|\right)^{\sqrt{2}}$
2	2	1.2825	$.2055-1$	$.2334-1$	———
4	3	0.9069	$.8739-2$	$.8756-2$	$.4922-2$
8	6	0.6413	$.1359-2$	$.1361-2$	$.1230-2$
16	12	0.4534	$.4228-4$	$.4228-4$	$.8853-4$
32	24	0.3206	$.1003-6$	$.1005-6$	$.6523-6$

Table 3. Errors on the Sinc Grid S and the Uniform Grid U for Example 2.11.

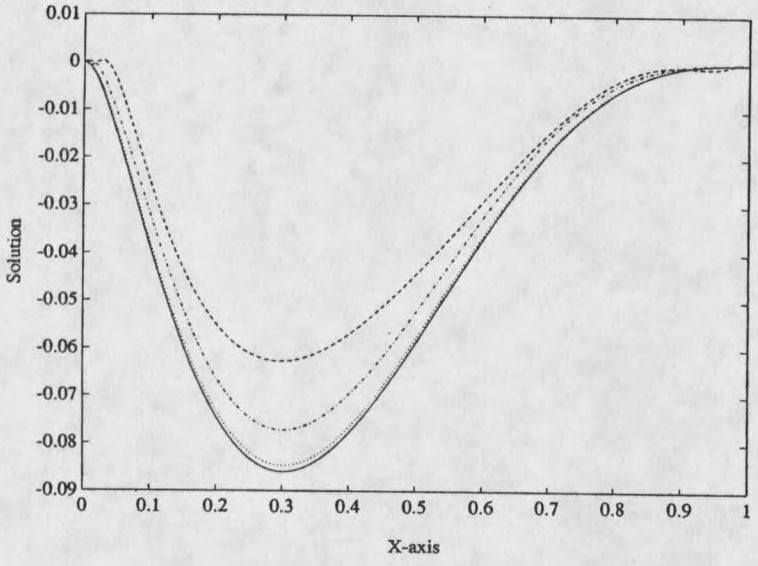


Figure 6. True and Approximate Solutions to Example 2.11
$- - -$ $(M = 2)$, $- \cdot -$ $(M = 4)$, $\cdots$ $(M = 8)$, — (True).

CHAPTER 3

EXTENSIONS TO THE SPATIAL PROBLEM

Introduction

The purpose of Chapter 3 is to use the analysis derived in Chapter 2 to develop methods for solving problems with higher order boundary conditions and special types of variable coefficients. In the second section of this chapter, problems of the form

$$\begin{aligned} \mathcal{L}u(x) \equiv \eta(x)u''''(x) + \varrho(x)u'''(x) + \nu(x)u''(x) + \tau(x)u'(x) \\ + \sigma(x)u(x) = f(x) \quad , \quad a < x < b \\ u(a) = \overline{\alpha} \quad , \quad u(b) = \overline{\gamma} \\ u''(a) = \overline{\beta} \quad , \quad u''(b) = \overline{\delta} \end{aligned} \tag{3.1}$$

will be considered. The derivation of the Sinc-Galerkin method will be presented along with numerical examples which demonstrate various aspects of the method. This analysis will be extended to problems with the boundary conditions

$$\begin{aligned} u''(a) = \overline{\alpha} \quad , \quad u''(b) = \overline{\gamma} \\ u'''(a) = \overline{\beta} \quad , \quad u'''(b) = \overline{\delta} \end{aligned} \tag{3.2}$$

in the third section. A fourth type of boundary conditions that will be considered in this work is

$$\begin{aligned} u(a) = \overline{\alpha} \quad , \quad u''(b) = \overline{\gamma} \\ u'(a) = \overline{\beta} \quad , \quad u'''(b) = \overline{\delta} \quad . \end{aligned} \tag{3.3}$$

The system corresponding to (3.3) is presented in the fourth section. In the final section of this chapter, an alternate method is developed for solving symmetric variable coefficient problems of the form

$$\mathcal{L}u(x) \equiv (\mathrm{EI}(x)u''(x))'' = f(x) \quad , \quad a < x < b \quad . \tag{3.4}$$

This method will be especially useful for solving inverse problems involving the identification of the parameter $\mathrm{EI}(x)$. The modifications needed for various types of boundary conditions will be noted and further numerical examples will be given to highlight properties of the method.

Simple Boundary Conditions

Consider the general fourth-order boundary value problem

$$\begin{aligned} \mathcal{L}u(x) \equiv \eta(x)u''''(x) + \varrho(x)u'''(x) + \nu(x)u''(x) + \tau(x)u'(x) \\ + \sigma(x)u(x) = f(x) \quad , \quad a < x < b \\ u(a) = \overline{\alpha} \quad , \quad u(b) = \overline{\gamma} \\ u''(a) = \overline{\beta} \quad , \quad u''(b) = \overline{\delta} \quad . \end{aligned} \tag{3.5}$$

As was noted in Chapter 1, (3.5) with $\overline{\alpha} = \overline{\beta} = \overline{\gamma} = \overline{\delta} = 0$ may be used to model a beam that is simply supported at both ends.

A Sinc-Galerkin method to approximate the solution of (3.5) can be derived as follows. Define the set of basis functions $\{\psi_i\}_{i=-M-3}^{N+3}$ by

$$\psi_i(x) = \begin{cases} b_{-M-3}(x) & , \ i = -M-3 \\ b_{-M-2}(x) & , \ i = -M-2 \\ b_{-M-1}(x) & , \ i = -M-1 \\ v(x)S_i(x) & , \ -M \le i \le N \\ b_{N+1}(x) & , \ i = N+1 \\ b_{N+2}(x) & , \ i = N+2 \\ b_{N+3}(x) & , \ i = N+3 \end{cases} \tag{3.6}$$

where $S_i(x) \equiv S(i,h) \circ \phi(x)$ and the basis weight $v(x)$ is taken to be

$$v(x) = [(x-a)(b-x)]^2 \quad . \tag{3.7}$$

The boundary basis functions are given by

$$b_{-M-1}(x) = \frac{(b-x)^3}{(b-a)^5}[6(x-a)^2 + 3(x-a)(b-x) + (b-a)^2] \quad , \tag{3.8}$$

$$b_{N+1}(x) = \frac{(x-a)^3}{(b-a)^5}[6(b-x)^2 + 3(b-x)(x-a) + (b-a)^2] \quad , \tag{3.9}$$

$$b_{-M-2}(x) = \frac{(b-x)^3}{(b-a)^4}[3(x-a)^2 + (x-a)(b-a)] \quad , \tag{3.10}$$

$$b_{N+2}(x) = -\frac{(x-a)^3}{(b-a)^4}[3(b-x)^2 + (b-x)(b-a)] \quad , \tag{3.11}$$

$$b_{-M-3}(x) = \frac{1}{2}\,\frac{(b-x)^3}{(b-a)^3}\,(x-a)^2 \tag{3.12}$$

and

$$b_{N+3}(x) = \frac{1}{2}\,\frac{(x-a)^3}{(b-a)^3}\,(b-x)^2 \quad . \tag{3.13}$$

The graphs of the boundary basis functions on the interval $[0,1]$ are found in Figure 7.

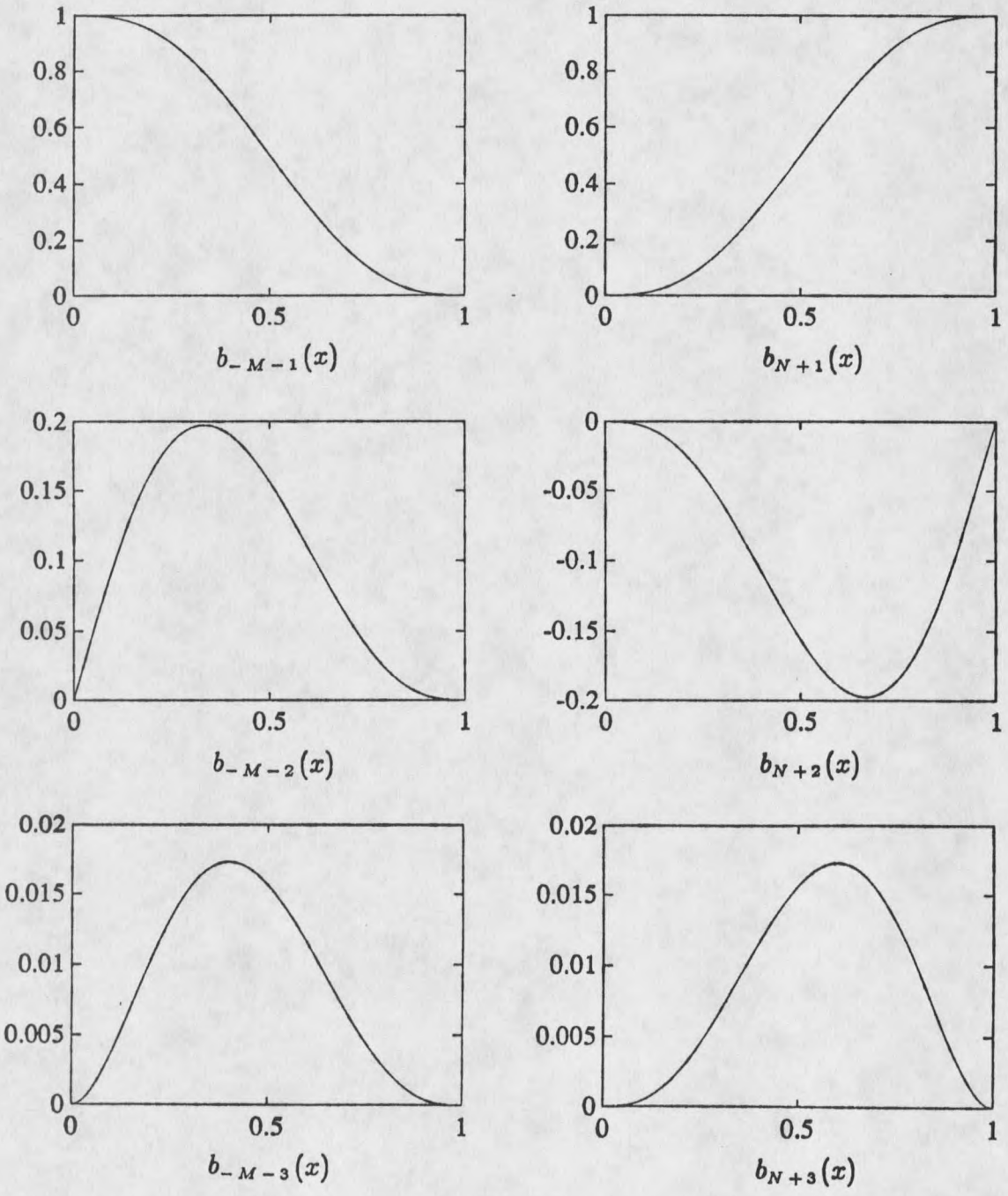


Figure 7. Boundary Basis Functions for Problems with Simple Boundaries.

By design the functions satisfy

$$b_{-M-1}(a) = 1$$
$$b_{-M-1}(b) = b^{(\ell)}_{-M-1}(a) = b^{(\ell)}_{-M-1}(b) = 0 \quad , \quad \ell = 1,2 \quad , \tag{3.14}$$

$$b_{N+1}(b) = 1$$
$$b_{N+1}(a) = b^{(\ell)}_{N+1}(a) = b^{(\ell)}_{N+1}(b) = 0 \quad , \quad \ell = 1,2 \quad , \tag{3.15}$$

$$b'_{-M-2}(a) = 1$$
$$b'_{-M-2}(b) = b^{(\ell)}_{-M-2}(a) = b^{(\ell)}_{-M-2}(b) = 0 \quad , \quad \ell = 0,2 \quad , \tag{3.16}$$

$$b'_{N+2}(b) = 1$$
$$b'_{N+2}(a) = b^{(\ell)}_{N+2}(a) = b^{(\ell)}_{N+2}(b) = 0 \quad , \quad \ell = 0,2 \quad , \tag{3.17}$$

$$b''_{-M-3}(a) = 1$$
$$b''_{-M-3}(b) = b^{(\ell)}_{-M-3}(a) = b^{(\ell)}_{-M-3}(b) = 0 \quad , \quad \ell = 0,1 \tag{3.18}$$

and

$$b''_{N+3}(b) = 1$$
$$b''_{N+3}(a) = b^{(\ell)}_{N+3}(a) = b^{(\ell)}_{N+3}(b) = 0 \quad , \quad \ell = 0,1 \quad . \tag{3.19}$$

For $m = M + N + 1$, the approximate solution $u_m(x)$ is then defined by

$$\begin{aligned} u_m(x) = {} & \overline{\alpha}\psi_{-M-1}(x) + \overline{\beta}\psi_{-M-3}(x) + \overline{\gamma}\psi_{N+1}(x) + \overline{\delta}\psi_{N+3}(x) \\ & + u_{-M-2}\psi_{-M-2}(x) + u_{N+2}\psi_{N+2}(x) + \sum_{i=-M}^{N} u_i\psi_i(x) \end{aligned} \tag{3.20}$$

where $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ and $\overline{\delta}$ are known and the coefficients $\{u_i\}$ are unknown. Note that $u_m(x)$ satisfies the boundary conditions; that is,

$$u_m(a) = \overline{\alpha} \quad , \quad u_m(b) = \overline{\gamma}$$

$$u''_m(a) = \overline{\beta} \quad , \quad u''_m(b) = \overline{\delta} \quad .$$

The $m+2$ unknown coefficients in (3.20) are determined by orthogonalizing the residual with respect to the set of sinc functions $\{S_p\}_{p=-M-1}^{N+1}$. This is in contrast with those Galerkin methods in which the residual is orthogonalized with respect to the basis and is done to take advantage of the exponential accuracy of point evaluation in the quadratures. This yields the discrete Galerkin system

$$\begin{aligned} 0 &= (\mathcal{L}u_m - f, S_p) \\ &= (\eta u_m'''', S_p) + (\varrho u_m''', S_p) + (\nu u_m'', S_p) + (\tau u_m', S_p) \\ &\quad + (\sigma u_m, S_p) - (f, S_p) \end{aligned} \tag{3.21}$$

where $-M-1 \le p \le N+1$. The inner product $(\cdot,\cdot)$ is again taken to be

$$(F,G) = \int_a^b F(x)G(x)w(x)dx \tag{3.22}$$

where $w(x)$ is a weight function which will be specified later in the development. If the definition

$$u_h(x) = \sum_{i=-M}^{N} u_i \psi_i(x) \tag{3.23}$$

is made, then (3.6), (3.20) and (3.21) can be combined to yield the system

$$\begin{aligned} 0 = \Big\{ &(\overline{\alpha}\mathcal{L}b_{-M-1}, S_p) + (\overline{\beta}\mathcal{L}b_{-M-3}, S_p) + (\overline{\gamma}\mathcal{L}b_{N+1}, S_p) + (\overline{\delta}\mathcal{L}b_{N+3}, S_p) \\ &+ (u_{-M-2}\mathcal{L}b_{-M-2}, S_p) + (u_{N+2}\mathcal{L}b_{N+2}, S_p) \\ &+ (\mathcal{L}u_h, S_p) \Big\} - (f, S_p) \quad , \quad -M-1 \le p \le N+1 \quad . \end{aligned}$$

Finally, the known terms are collected and the system is written as

$$(\mathcal{L}b_{-M-2}, S_p)u_{-M-2} + (\mathcal{L}b_{N+2}, S_p)u_{N+2} + (\mathcal{L}u_h, S_p) = (\overline{f}, S_p) \tag{3.24}$$

$(-M-1 \le p \le N+1)$ where

$$\overline{f}(x) \equiv f(x) - \overline{\alpha}\mathcal{L}b_{-M-1}(x) - \overline{\beta}\mathcal{L}b_{-M-3}(x) - \overline{\gamma}\mathcal{L}b_{N+1}(x) - \overline{\delta}\mathcal{L}b_{N+3}(x) \quad .$$

Approximations to the inner products in (3.24) are referenced in the following theorem. The proof of this theorem is very similar to that of Theorem 2.6 once it is noted that all infinite sums collapse to finite sums as a result of the property

$$u_h(x_j) = \sum_{i=-M}^{N} u_i v(x_j) S_i(x_j) = u_j v(x_j)$$

for $-M \le j \le N$. Because of the similarity and for the sake of brevity, the proof will not be repeated here. Because the sums have already been truncated, the criteria necessary for choosing the decay parameters α and β will not be discussed at this point but will instead be addressed in the next section.

Theorem 3.1 *Let ϕ conformally map the simply connected domain D_E onto D_S. Let $x_i = \phi^{-1}(ih)$ and define the functions $g_\ell^{(j)}(z)$, $\ell = 0,1,2,3,4$; $j = 0,1,\ldots,\ell$, as they are in (2.37)-(2.50). Finally, take*

$$h = \left(\frac{\pi d}{\alpha M}\right)^{1/2} \quad \text{and} \quad N = \left[\left|\frac{\alpha}{\beta} M + 1\right|\right] \quad . \tag{3.25}$$

(a) If $Gw \in B(D_E)$ then

$$\left|(G, S_p) - hG(x_p)\,\frac{w(x_p)}{\phi'(x_p)}\right| \le L_{00} M^{-1/2} e^{-(\pi d \alpha M)^{1/2}} \tag{3.26}$$

where L_{00} depends on G, w and d. The bound (3.26) holds for G defined to be $\mathcal{L}b_{-M-2}$, $\mathcal{L}b_{N+2}$ or $\overline{f}$.

(b) If $\sigma u_h w \in B(D_E)$ then

$$\left|(\sigma u_h, S_p) - h\sigma(x_p)u_h(x_p)\,\frac{w(x_p)}{\phi'(x_p)}\right| \le L_{01} M^{-1/2} e^{-(\pi d \alpha M)^{1/2}} \tag{3.27}$$

where L_{01} depends on σ, u_h, w and d.

(c) If $u_h g_1^{(\ell)} \in B(D_E)$, $\ell = 0,1$ and

$$B_T^{(1)} = \{u_h \tau S_p w\}(x)\Big|_a^b = 0 \tag{3.28}$$

then

$$\left|(\tau u_h', S_p) + h \sum_{i=-M}^{N} \frac{u_h(x_i)}{\phi'(x_i)} \left[\sum_{\ell=0}^{1} \frac{1}{h^\ell}\, \delta_{pi}^{(\ell)} g_1^{(\ell)}(x_i)\right]\right| \tag{3.29}$$
$$\leq L_1 M^{1/2} e^{-(\pi d \alpha M)^{1/2}}$$

where L_1 *depends on* τ, u_h, w *and* d.

(d) If $u_h g_2^{(\ell)} \in B(D_E)$, $\ell = 0,1,2$ *and*

$$B_T^{(2)} = \{u_h'(\nu S_p w) - u_h(\nu S_p w)'\}(x)\Big|_a^b = 0 \tag{3.30}$$

then

$$\left|(\nu u_h'', S_p) - h \sum_{i=-M}^{N} \frac{u_h(x_i)}{\phi'(x_i)} \left[\sum_{\ell=0}^{2} \frac{1}{h^\ell}\, \delta_{pi}^{(\ell)} g_2^{(\ell)}(x_i)\right]\right| \tag{3.31}$$
$$\leq L_2 M e^{-(\pi d \alpha M)^{1/2}}$$

where L_2 *depends on* ν, u_h, w *and* d.

(e) If $u_h g_3^{(\ell)} \in B(D_E)$, $\ell = 0,1,2,3$ *and*

$$B_T^{(3)} = \{u_h''(\varrho S_p w) - u_h'(\varrho S_p w)' + u_h(\varrho S_p w)''\}(x)\Big|_a^b = 0 \tag{3.32}$$

then

$$\left|(\varrho u_h''', S_p) + h \sum_{i=-M}^{N} \frac{u_h(x_i)}{\phi'(x_i)} \left[\sum_{\ell=0}^{3} \frac{1}{h^\ell}\, \delta_{pi}^{(\ell)} g_3^{(\ell)}(x_i)\right]\right| \tag{3.33}$$
$$\leq L_3 M^{3/2} e^{-(\pi d \alpha M)^{1/2}}$$

where L_3 *depends on* ϱ, u_h, w *and* d.

(f) If $u_h g_4^{(\ell)} \in B(D_E)$, $\ell = 0,1,2,3,4$ *and*

$$B_T^{(4)} = \{u_h'''(S_p w) - u_h''(S_p w)' + u_h'(S_p w)'' - u_h(S_p w)'''\}(x)\Big|_a^b = 0 \tag{3.34}$$

then

$$\left|(\eta u_h'''', S_p) - h \sum_{i=-M}^{N} \frac{u_h(x_i)}{\phi'(x_i)} \left[\sum_{\ell=0}^{4} \frac{1}{h^\ell} \, \delta_{pi}^{(\ell)} g_4^{(\ell)}(x_i) \right] \right| \leq L_4 M^2 e^{-(\pi d \alpha M)^{1/2}} \tag{3.35}$$

where L_4 depends on η, u_h, w and d.

The inner product approximations in Theorem 3.1 in conjunction with (3.24) can be used to obtain the Sinc-Galerkin system for the boundary value problem (3.5). Let $\vec{a}_{-M-2}$, $\vec{a}_{N+2}$ and $\vec{b}$ denote the $(m+2)\times 1$ vectors containing the product of $\frac{1}{h}$ and the approximations to the inner products $(\mathcal{L}b_{-M-2}, S_p)$, $(\mathcal{L}b_{N+2}, S_p)$ and $(\overline{f}, S_p)$, respectively. Thus the p-th entries of the respective vectors are

$$(\vec{a}_{-M-2})_p = \left[\eta(x) b'''_{-M-2}(x_p) + \varrho(x_p) b'''_{-M-2}(x_p) + \nu(x_p) b''_{-M-2}(x_p) + \tau(x_p) b'_{-M-2}(x_p) + \sigma(x_p) b_{-M-2}(x_p)\right] \frac{w(x_p)}{\phi'(x_p)} \quad , \tag{3.36}$$

$$(\vec{a}_{N+2})_p = \left[\eta(x) b''''_{N+2}(x_p) + \varrho(x_p) b'''_{N+2}(x_p) + \nu(x_p) b''_{N+2}(x_p) + \tau(x_p) b'_{N+2}(x_p) + \sigma(x_p) b_{N+2}(x_p)\right] \frac{w(x_p)}{\phi'(x_p)} \tag{3.37}$$

and

$$(\vec{b})_p = \overline{f}(x_p) \, \frac{w(x_p)}{\phi'(x_p)} \quad . \tag{3.38}$$

Furthermore, let A_m denote the $(m+2) \times m$ matrix which results from the expansion of the inner product $(\mathcal{L}u_h, S_p)$. The approximations from Theorem 3.1 and the notation from Chapter 2 can be used to express A_m as

$$A_m = \frac{1}{h^4} \, I^{(4)} D(a_4 v) + \frac{1}{h^3} \, I^{(3)} D(a_3 v) + \frac{1}{h^2} \, I^{(2)} D(a_2 v) + \frac{1}{h} \, I^{(1)} D(a_1 v) + I^{(0)} D(a_0 v) \quad . \tag{3.39}$$

Here $I^{(\ell)}$, $\ell = 0,1,2,3,4$ are $(m+2) \times m$ matrices whose pi-th entry is given by $\delta_{pi}^{(\ell)}$ from (2.31)–(2.35) and $D(\varsigma)$ is an $m \times m$ diagonal matrix (with diagonal entries $\varsigma(x_{-M})$, $\varsigma(x_{-M+1})$, $\ldots$, $\varsigma(x_N)$). The functions $a_\ell(x)$, $\ell = 0,1,2,3,4$ are defined in (2.77)–(2.81).

The discrete system for the determination of the unknown coefficients $\{u_i\}$ is given by

$$A\vec{u} = \vec{b} \tag{3.40}$$

where the $(m+2) \times (m+2)$ matrix A is defined to be

$$A \equiv [\vec{a}_{-M-2} \mid A_m \mid \vec{a}_{N+2}] \quad . \tag{3.41}$$

The above notation indicates that the first and last columns of A contain the entries from $\vec{a}_{-M-2}$ and $\vec{a}_{N+2}$ respectively whereas the middle m columns of A contain the components of the submatrix A_m. Finally, $\vec{u}$ is defined to be the $(m+2) \times 1$ vector

$$\vec{u} = [u_{-M-2}, u_{-M}, u_{-M+1}, \ldots, u_N, u_{N+2}]^T \tag{3.42}$$

containing the unknown coefficients.

Implementation for Simple Boundary Conditions

The implementation of the Sinc-Galerkin method for problems with simple boundary conditions requires that (1) decay parameters α and β be chosen and (2) a suitable weight function $w(x)$ be selected for the inner product (3.22). Since the choice of decay parameters directly influences the values of M and N that are used in the previous section, this topic is discussed first.

From Theorem 3.1, it is seen that the parameters α and β describe the decay of that part of the true solution which is approximated by $u_h(x)$. In order to

isolate that particular part of the true solution, consider the formal change of variables given by

$$\begin{aligned} U(x) = u(x) - \overline{\alpha} b_{-M-1}(x) - \overline{\beta} b_{-M-3}(x) - \overline{\gamma} b_{N+1}(x) - \overline{\delta} b_{N+3}(x) \\ - u'(a) b_{-M-2}(x) - u'(b) b_{N+2}(x) \quad . \end{aligned} \tag{3.43}$$

The new variable $U(x)$ satisfies the fourth-order boundary value problem

$$\begin{aligned} \mathcal{L}U(x) &= \tilde{f}(x) \\ U(a) = U(b) &= 0 \\ U'(a) = U'(b) &= 0 \end{aligned} \tag{3.44}$$

where

$$\begin{aligned} \tilde{f}(x) \equiv f(x) + \mathcal{L}\{\overline{\alpha} b_{-M-1}(x) + \overline{\beta} b_{-M-3}(x) + \overline{\gamma} b_{N+1}(x) + \overline{\delta} b_{N+3}(x) \\ + u'(a) b_{-M-2}(x) + u'(b) b_{N+2}(x)\} \quad . \end{aligned}$$

In addition to satisfying the boundary conditions in (3.44), $U(x)$ satisfies

$$U''(a) = U''(b) = 0 \quad .$$

Thus $U(x)$ is exactly that part of the true solution which is approximated by

$$u_h(x) = \sum_{i=-M}^{N} u_i \psi_i(x) = \sum_{i=-M}^{N} u_i [(x-a)(b-x)]^2 S_i(x) \quad .$$

The decay conditions α and β follow immediately by noting that (3.44) is precisely the problem (2.96) which was discussed in the fifth section of Chapter 2. As was demonstrated in that section, α and β are chosen so that the decay conditions in Theorem 2.6 (with u replaced by U) are satisfied. Thus the parameters are completely dictated by the decay of the "homogeneous" part of the true solution.

It should be pointed out that in many problems, it is very difficult to determine the true decay parameters even with the asymptotic methods that were

mentioned in Chapter 2. In other cases, the asymptotic methods indicate negative decay parameters which clearly will not work in this method. Repeated numerical tests have shown that in both situations, small positive choices for α and β yield very good results. Thus the method is fairly robust even in problems where the true decay parameters are unknown or cannot be used.

The second factor which must be considered when implementing the method is the choice of the inner product weight function. The weight can be chosen for many reasons, some of which were enumerated in Chapter 2. For general values of $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ and $\overline{\delta}$ in the boundary conditions, one would not expect to obtain a symmetric system so that criterion for choosing a weight will not be expanded on in this section. Instead, the discussion will center on choosing the weight $w(x)$ so that (1) the boundary terms $B_T^{(\ell)}$, $\ell = 1,2,3,4$, are forced to be zero, (2) the method is applicable to the widest possible class of problems, and (3) the conditioning of the matrix A in (3.41) is controlled.

Consider first a differential equation with a regular singular point on the boundary. Focusing on $B_T^{(1)}$ in (3.28), note that weights of the form

$$w(x) = \frac{1}{[\phi'(x)]^{\varsigma}} \quad , \quad \varsigma \geq 0 \tag{3.45}$$

will enforce $B_T^{(1)} = 0$. The choice (3.45) in each of $B_T^{(2)}$, $B_T^{(3)}$ and $B_T^{(4)}$ (as defined in (3.30), (3.32) and (3.34)) ensures that for regular singular problems, the boundary terms will all vanish.

For the weight given in (3.45), and $\mathcal{U}(x)$ from (3.43), the decay condition is

$$|\mathcal{U}(x)| \leq L \begin{cases} (x-a)^{\alpha+3-\varsigma} & , \ x \in \gamma_L \\ (b-x)^{\beta+3-\varsigma} & , \ x \in \gamma_R \end{cases} . \tag{3.46}$$

From (3.46), it can be seen that as ς decreases, the class of problems to which the approximations based on the weight in (3.45) can be applied becomes increasingly restricted. This must be considered when applying the method to problems in which α and β are small.

The final criterion that must be considered when choosing the weight is the effect it will have on the conditioning of A. For reasons similar to those expounded on in the fifth section of Chapter 2, the conditioning of the matrix generally decreases as ς decreases. The issue of conditioning is more crucial in this setting since the presence of the boundary vectors $\vec{a}_{-M-2}$ and $\vec{a}_{N+2}$ in A causes the conditioning to be worse than that of matrices encountered in the last chapter.

Thus the competing criteria of (2) and (3) must be balanced according to the problem. For problems with the boundary conditions of (3.5), good results have been obtained with the weight

$$w(x) = \frac{1}{\phi'(x)} \quad . \tag{3.47}$$

Numerical Examples for Simple Boundary Conditions

The two examples reported in this section illustrate various features of the method when applied to problems of the form (3.5). The notation developed in Chapter 2 will be used when reporting errors and convergence results.

Example 3.2

$$u''''(x) + u(x) = f(x) \quad , \quad 0 < x < 1$$

$$u(0) = 1 \quad , \quad u(1) = 2$$

$$u''(0) = -2 \quad , \quad u''(1) = -2$$

The function $f(x)$ is consistent with the true solution $u(x) = [x(1-x)]^{7/2} + 1 + 2x - x^2$. The discrete system is given by (3.40) where the weight function is taken to be $w(x) = (\phi(x))^{-1}$. The decay condition (3.46) dictates the parameter choices $\alpha = \beta = \frac{3}{2}$. In this case the asymptotic error is $\mathcal{O}\left(e^{-\pi\sqrt{3M}/2}\right)$ and the sums are centered sums as shown with the errors in Table 4. These results can be compared with those from Example 2.10 where a similar problem with fixed boundary conditions was considered. The true and approximate solutions are shown in Figure 8.

M	N	h	$\|E_S^M(h)\|$	$\|E_U^M(\ell)\|$	$\left(\|E_U^{M/2}(\ell)\|\right)^{\sqrt{2}}$
2	2	1.2825	$.6569-2$	$.6569-2$	———
4	4	0.9069	$.6500-3$	$.6500-3$	$.8194-3$
8	8	0.6413	$.9881-4$	$.9881-4$	$.3110-4$
16	16	0.4534	$.6414-5$	$.6414-5$	$.2167-5$
32	32	0.3206	$.9840-7$	$.9840-7$	$.4531-7$

Table 4. Errors on the Sinc Grid S and the Uniform Grid U for Example 3.2.

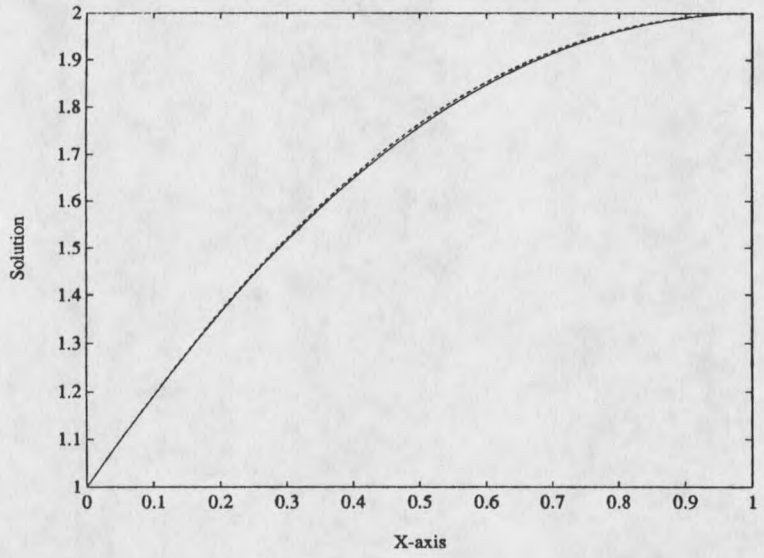


Figure 8. True and Approximate Solutions to Example 3.2
$- - -$ $(M = 2)$, — (True).

Example 3.3

$$u''''(x) - 4\pi^2 u''(x) + 16\pi^4 u(x) = f(x) \quad , \quad 0 < x < 1$$

$$u(0) = u(1) = 0$$

$$u''(0) = u''(1) = 0$$

The forcing function $f(x)$ is consistent with the true solution $u(x) = \sin(2\pi x)$ and the weight is again taken to be $w(x) = (\phi'(x))^{-1}$. Since (3.46) dictates unrealistic decay parameters, we will simply set $\alpha = \beta = 1$. As was noted earlier, the robustness of the method allows for some flexibility in the choice of decay parameters. The errors and solutions are reported in Table 5 and Figure 9, respectively. Again, the method exhibits exponential convergence as demonstrated by the results in the last column.

M	N	h	$\|E_S^M(h)\|$	$\|E_U^M(\ell)\|$	$\left(\|E_U^{M/2}(\ell)\|\right)^{\sqrt{2}}$
2	2	1.5708	$.8206-1$	$.1210-0$	———
4	4	1.1107	$.1273-1$	$.1930-1$	$.5045-1$
8	8	0.7854	$.3797-3$	$.9128-3$	$.3762-2$
16	16	0.5554	$.3283-4$	$.3360-4$	$.5027-4$
32	32	0.3927	$.1281-5$	$.1282-5$	$.4713-6$

Table 5. Errors on the Sinc Grid S and the Uniform Grid U for Example 3.3.

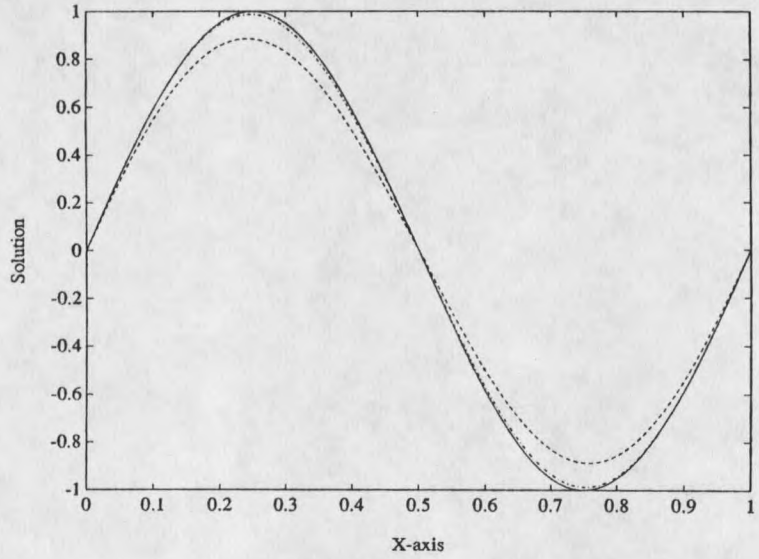


Figure 9. True and Approximate Solutions to Example 3.3
$- - -$ $(M=2)$, $- \cdot -$ $(M=4)$, — (True).

Free Boundary Conditions

In this section, a Sinc-Galerkin method is developed for fourth-order boundary value problems of the form

$$\begin{aligned} \mathcal{L}u(x) \equiv \eta(x)u''''(x) + \varrho(x)u'''(x) + \nu(x)u''(x) + \tau(x)u'(x) \\ + \sigma(x)u(x) = f(x) \quad , \quad a < x < b \end{aligned} \tag{3.48}$$

$$u''(a) = \overline{\alpha} \quad , \quad u''(b) = \overline{\gamma}$$

$$u'''(a) = \overline{\beta} \quad , \quad u'''(b) = \overline{\delta} \quad .$$

As was discussed in Chapter 1, (3.48) with $\overline{\alpha} = \overline{\beta} = \overline{\gamma} = \overline{\delta} = 0$ can be used to model a beam that is free at both ends.

The development here will parallel very closely that given in the previous section. The main differences will occur in the definition and manipulation of the algebraic boundary basis functions. The general basis functions $\{\psi_i\}_{i=-M-4}^{N+4}$ are

$$\psi_i(x) = \begin{cases} b_{-M-4}(x) & , \ i = -M-4 \\ b_{-M-3}(x) & , \ i = -M-3 \\ b_{-M-2}(x) & , \ i = -M-2 \\ b_{-M-1}(x) & , \ i = -M-1 \\ v(x)S_i(x) & , \ -M \le i \le N \\ b_{N+1}(x) & , \ i = N+1 \\ b_{N+2}(x) & , \ i = N+2 \\ b_{N+3}(x) & , \ i = N+3 \\ b_{N+4}(x) & , \ i = N+4 \end{cases} \quad . \tag{3.49}$$

Here $S_i(x) \equiv S(i,h) \circ \phi(x)$ and the basis weight $v(x)$ is taken to be

$$v(x) = [(x-a)(b-x)]^3 \quad . \tag{3.50}$$

The boundary basis functions are

$$\begin{aligned} b_{-M-1}(x) = \frac{(b-x)^4}{(b-a)^7} [20(x-a)^3 + 10(x-a)^2(b-a) \\ + 4(x-a)(b-a)^2 + (b-a)^3] \quad , \end{aligned} \tag{3.51}$$

$$b_{N+1}(x) = \frac{(x-a)^4}{(b-a)^7}\,[20(b-x)^3 + 10(b-x)^2(b-a) + 4(b-x)(b-a)^2 + (b-a)^3] \quad , \tag{3.52}$$

$$b_{-M-2}(x) = \frac{(b-x)^4(x-a)}{(b-a)^6}\,[10(x-a)^2 + 4(x-a)(b-a) + (b-a)^2] \quad , \tag{3.53}$$

$$b_{N+2}(x) = -\frac{(x-a)^4(b-x)}{(b-a)^6}\,[10(b-x)^2 + 4(b-x)(b-a) + (b-a)^2] \quad , \tag{3.54}$$

$$b_{-M-3}(x) = \frac{(b-x)^4(x-a)^2}{(b-a)^5}\,\left[2(x-a) + \frac{1}{2}(b-a)\right] \quad , \tag{3.55}$$

$$b_{N+3}(x) = \frac{(x-a)^4(b-x)^2}{(b-a)^5}\,\left[2(b-x) + \frac{1}{2}(b-a)\right] \quad , \tag{3.56}$$

$$b_{-M-4}(x) = \frac{1}{6}\,\frac{(b-x)^4(x-a)^3}{(b-a)^4} \tag{3.57}$$

and

$$b_{N+4}(x) = -\frac{1}{6}\,\frac{(x-a)^4(b-x)^3}{(b-a)^4} \quad . \tag{3.58}$$

The graphs of the boundary basis functions on the interval $[0,1]$ are found in Figure 10. By design the functions satisfy

$$\begin{aligned} &b_{-M-1}(a) = 1 \\ &b_{-M-1}(b) = b^{(\ell)}_{-M-1}(a) = b^{(\ell)}_{-M-1}(b) = 0 \quad , \quad \ell = 1,2,3 \end{aligned} \quad , \tag{3.59}$$

$$\begin{aligned} &b_{N+1}(b) = 1 \\ &b_{N+1}(a) = b^{(\ell)}_{N+1}(a) = b^{(\ell)}_{N+1}(b) = 0 \quad , \quad \ell = 1,2,3 \end{aligned} \quad , \tag{3.60}$$

$$
\begin{aligned}
&b'_{-M-2}(a) = 1 \\
&b'_{-M-2}(b) = b^{(\ell)}_{-M-2}(a) = b^{(\ell)}_{-M-2}(b) = 0 \quad , \quad \ell = 0,2,3 \quad ,
\end{aligned} \tag{3.61}
$$

$$
\begin{aligned}
&b'_{N+2}(b) = 1 \\
&b'_{N+2}(a) = b^{(\ell)}_{N+2}(a) = b^{(\ell)}_{N+2}(b) = 0 \quad , \quad \ell = 0,2,3 \quad ,
\end{aligned} \tag{3.62}
$$

$$
\begin{aligned}
&b''_{-M-3}(a) = 1 \\
&b''_{-M-3}(b) = b^{(\ell)}_{-M-3}(a) = b^{(\ell)}_{-M-3}(b) = 0 \quad , \quad \ell = 0,1,3 \quad ,
\end{aligned} \tag{3.63}
$$

$$
\begin{aligned}
&b''_{N+3}(b) = 1 \\
&b''_{N+3}(a) = b^{(\ell)}_{N+3}(a) = b^{(\ell)}_{N+3}(b) = 0 \quad , \quad \ell = 0,1,3 \quad ,
\end{aligned} \tag{3.64}
$$

$$
\begin{aligned}
&b'''_{-M-4}(a) = 0 \\
&b'''_{-M-4}(b) = b^{(\ell)}_{-M-4}(a) = b^{(\ell)}_{-M-4}(b) = 0 \quad , \quad \ell = 0,1,2
\end{aligned} \tag{3.65}
$$

and

$$
\begin{aligned}
&b'''_{N+4}(b) = 1 \\
&b'''_{N+4}(a) = b^{(\ell)}_{N+4}(a) = b^{(\ell)}_{N+4}(b) = 0 \quad , \quad \ell = 0,1,2 \quad .
\end{aligned} \tag{3.66}
$$

For $m = M + N + 1$, the approximate solution is then defined to be

$$
\begin{aligned}
u_m(x) = {} & \overline{\alpha}\psi_{-M-3}(x) + \overline{\beta}\psi_{-M-4}(x) + \overline{\gamma}\psi_{N+3}(x) + \overline{\delta}\psi_{N+4}(x) \\
& + u_{-M-1}\psi_{-M-1}(x) + u_{-M-2}\psi_{-M-2}(x) \\
& + u_{N+1}\psi_{N+1}(x) + u_{N+2}\psi_{N+2}(x) \\
& + \sum_{i=-M}^{N} u_i \psi_i(x)
\end{aligned} \tag{3.67}
$$

where $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ and $\overline{\delta}$ are known and the coefficients $\{u_i\}$ are unknown.

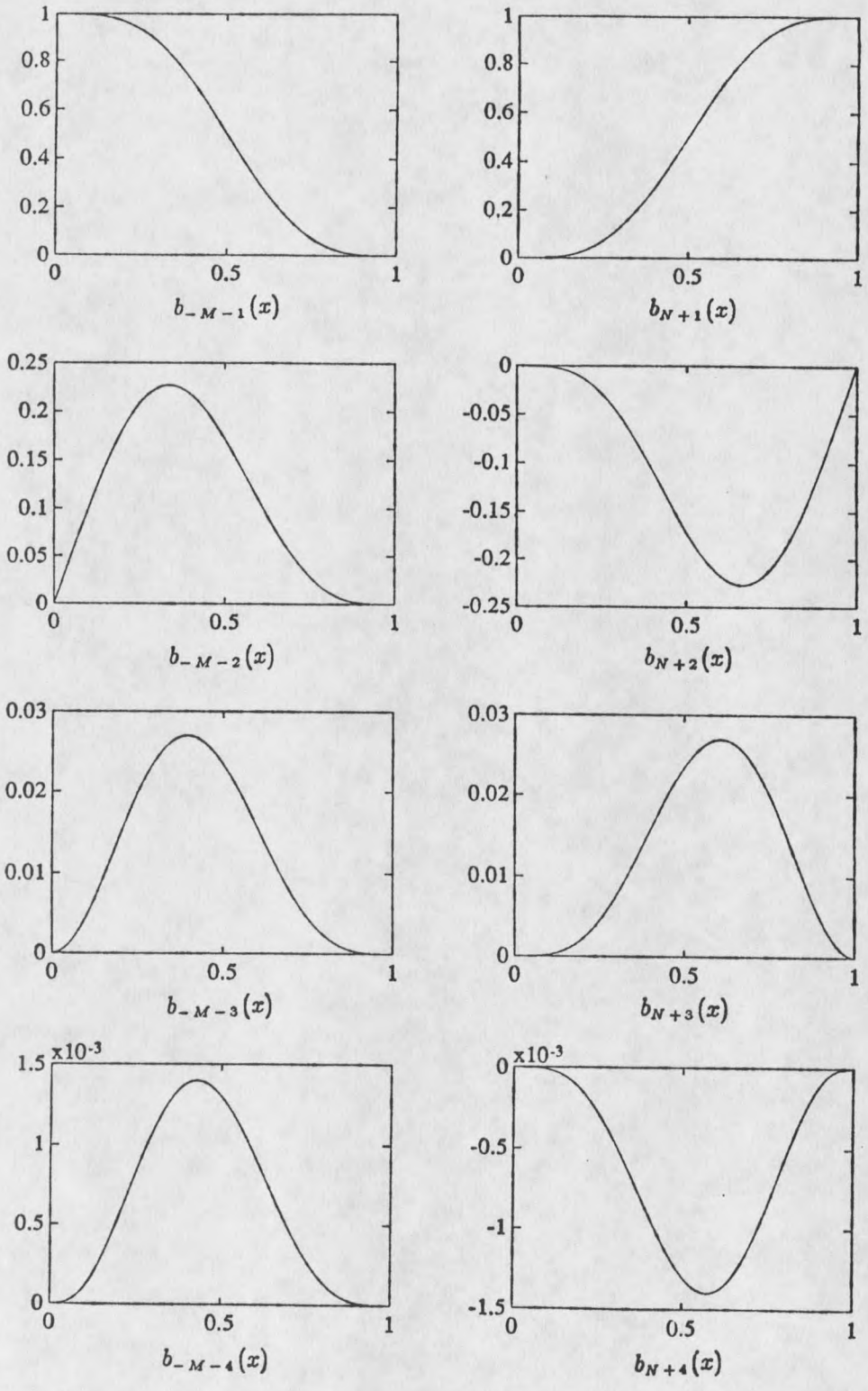


Figure 10. Boundary Basis Functions for Problems with Free Boundaries.

Again, $u_m(x)$ satisfies the boundary conditions; that is

$$u_m''(a) = \overline{\alpha} \quad , \quad u_m''(b) = \overline{\gamma}$$

$$u_m'''(a) = \overline{\beta} \quad , \quad u_m'''(b) = \overline{\delta} \quad .$$

The $m+4$ coefficients in (3.67) are determined by orthogonalizing the residual with respect to the set of sinc functions $\{S_p\}_{p=-M-2}^{N+2}$. This yields the discrete system

$$(\mathcal{L}u_m - f, S_p) = 0 \tag{3.68}$$

for $-M-2 \leq p \leq N+2$. The inner product is defined in (3.22). Expressions (3.48), (3.49) and (3.68) are combined to yield the system

$$\begin{aligned}
&(\overline{\alpha}\mathcal{L}b_{-M-3}, S_p) + (\overline{\beta}\mathcal{L}b_{-M-4}, S_p) + (\overline{\gamma}\mathcal{L}b_{N+3}, S_p) \\
&+ (\overline{\delta}\mathcal{L}b_{N+4}, S_p) + (u_{-M-1}\mathcal{L}b_{-M-1}, S_p) + (u_{-M-2}\mathcal{L}b_{-M-2}, S_p) \\
&+ (u_{N+1}\mathcal{L}b_{N+1}, S_p) + (u_{N+2}\mathcal{L}b_{N+2}, S_p) + (\mathcal{L}u_h, S_p) \\
&= (f, S_p) \quad , \quad -M-2 \leq p \leq N+2 \quad .
\end{aligned}$$

The "homogeneous" approximation $u_h(x)$ is defined in (3.23) with $v(x)$ given by (3.50). If the known terms are collected on the right-hand side, then this system may be written as

$$\begin{aligned}
&(\mathcal{L}b_{-M-1}, S_p)u_{-M-1} + (\mathcal{L}b_{-M-2}, S_p)u_{-M-2} + (\mathcal{L}b_{N+1}, S_p)u_{N+1} \\
&+ (\mathcal{L}b_{N+2}, S_p)u_{N+2} + (\mathcal{L}u_h, S_p) \\
&= (\overline{f}, S_p) \quad , \quad -M-2 \leq p \leq N+2
\end{aligned} \tag{3.69}$$

with

$$\begin{aligned}
\overline{f}(x) \equiv f(x) &- \overline{\alpha}\mathcal{L}b_{-M-3}(x) - \overline{\beta}\mathcal{L}b_{-M-4}(x) \\
&+ \overline{\gamma}\mathcal{L}b_{N+3}(x) + \overline{\delta}\mathcal{L}b_{N+4}(x) \quad .
\end{aligned}$$

The approximations to the inner products in (3.69) follow directly from Theorem 3.1 if G in (3.26) is defined to be $\mathcal{L}b_{-M-1}$, $\mathcal{L}b_{N+1}$, $\mathcal{L}b_{-M-2}$, $\mathcal{L}b_{N+2}$ or $\overline{f}$.

These approximations in conjunction with (3.69) can be used to obtain the Sinc-Galerkin system for the boundary value problem (3.48). Let $\vec{a}_{-M-1}$, $\vec{a}_{-M-2}$, $\vec{a}_{N+1}$, $\vec{a}_{N+2}$ and $\vec{b}$ denote the $(m+4) \times 1$ vectors containing the product of $\frac{1}{h}$ and the approximations to the inner products $(\mathcal{L}b_{-M-1}, S_p)$, $(\mathcal{L}b_{-M-2}, S_p)$, $(\mathcal{L}b_{N+1}, S_p)$, $(\mathcal{L}b_{N+2}, S_p)$ and $(\overline{f}, S_p)$, respectively (see (3.36), (3.37) and (3.38)). Furthermore, let the $(m+4) \times m$ matrix A_m be defined as it was in (3.39) with v given by (3.50).

The discrete system for the determination of the unknown coefficients $\{u_i\}$ is then given by

$$A\vec{u} = \vec{b} \tag{3.70}$$

where the $(m+4) \times (m+4)$ matrix A is defined to be

$$A \equiv [\vec{a}_{-M-2} \mid \vec{a}_{-M-1} \mid A_m \mid \vec{a}_{N+1} \mid \vec{a}_{N+2}] \quad . \tag{3.71}$$

The above notation indicates that the first, second, next to last and last columns of A contain the entries $\vec{a}_{-M-2}$, $\vec{a}_{-M-1}$, $\vec{a}_{N+1}$ and $\vec{a}_{N+2}$, respectively. The middle m columns of A contain the components of the submatrix A_m. Finally, $\vec{u}$ is defined to be the $(m+4) \times 1$ vector

$$\vec{u} = [u_{-M-2}, u_{-M-1}, u_{-M}, \ldots, u_N, u_{N+1}, u_{N+2}]^T \tag{3.72}$$

containing the unknown coefficients. It is noted that the matrices A as defined in (3.41) and (3.71) differ only on the borders. The construction of the interior is identical.

Implementation for Free Boundary Conditions

As noted in previous sections, implementation of the Sinc-Galerkin method requires that (1) decay parameters α and β be chosen and (2) a suitable weight

function $w(x)$ be selected. Since α and β depend on that part of the true solution which is approximated by $u_h(x)$, the formal change of variables

$$\begin{aligned} \mathcal{U}(x) = u(x) &- \overline{\alpha} b_{-M-3}(x) - \overline{\beta} b_{-M-4}(x) - \overline{\gamma} b_{N+3}(x) \\ &- \overline{\delta} b_{N+4}(x) - u(a) b_{-M-1}(x) - u'(a) b_{-M-2}(x) \\ &- u(b) b_{N+1}(x) - u'(b) b_{N+2}(x) \end{aligned} \tag{3.73}$$

is introduced. The new variable $\mathcal{U}(x)$ satisfies (2.96) and (2.99) so the analysis from the fifth section of Chapter 2 then applies. As was demonstrated in that section, the parameters are chosen according to the decay of the "homogeneous" part of the true solution.

The weight $w(x)$ is again chosen to balance the criteria (1)–(3) that were discussed in the implementation part of the last section. From (3.28), (3.30), (3.32) and (3.34) it is seen that weights of the form

$$w(x) = \frac{1}{[\phi'(x)]^{\varsigma}} \quad , \quad \varsigma \geq -1 \tag{3.74}$$

will ensure that for regular singular problems, the boundary terms all vanish. For the weight given in (3.74) and $\mathcal{U}(x)$ from (3.73), the decay condition is

$$|\mathcal{U}(x)| \leq L \begin{cases} (x-a)^{\alpha+3-\varsigma} & , \ x \in \gamma_L \\ (b-x)^{\beta+3-\varsigma} & , \ x \in \gamma_R \end{cases} . \tag{3.75}$$

Again, as ς decreases, the class of problems to which the approximations based on the weight in (3.74) can be applied becomes increasingly restricted. Finally, as demonstrated in Example 2.9, the conditioning of the matrix A generally decreases as ς decreases. The competing second and third criteria can be balanced by the choices

$$w(x) = \frac{1}{[\phi'(x)]^{1/2}} \tag{3.76}$$

or

$$w(x) = 1 \quad . \tag{3.77}$$

Depending on the problems, both choices give reasonable results.

Numerical Examples for Free Boundary Conditions

The following examples illustrate various features of the methods developed in this section. The results from the first example can be compared with those from Example 2.10 and Example 3.2 where similar differential equations with lower order boundary conditions were considered.

Example 3.4

$$u''''(x) + u(x) = f(x) \quad , \quad 0 < x < 1$$

$$u''(0) = u''(1) = -2$$

$$u'''(0) = u'''(1) = 0 \quad .$$

Here $f(x)$ is consistent with the true solution $u(x) = [x(1-x)]^{7/2} + 1 + 2x - x^2$. The weight function is taken to be $w(x) = 1$ which yields the parameter choices $\alpha = \beta = \frac{1}{2}$ (see (3.75)). Hence $h = \pi/\sqrt{M}$ and the asymptotic error is $\mathcal{O}\left(e^{-\pi\sqrt{M}/2}\right)$. The errors and solutions are reported in Table 6 and Figure 11, respectively. It is noted that the method converges exponentially even though the errors are much larger than those obtained in Example 2.10 and Example 3.2. The larger errors may be attributed both to the smaller parameters and to the presence of the high order algebraic boundary basis functions in the system.

M	N	h	$\|E_S^M(h)\|$	$\|E_U^M(\ell)\|$	$\left(\|E_U^{M/2}(\ell)\|\right)^{\sqrt{2}}$
8	8	1.1107	$.9824-1$	$.9824-1$	———
16	16	0.7854	$.2131-1$	$.2131-1$	$.3757-1$
32	32	0.5554	$.2004-2$	$.2004-2$	$.4328-2$
64	64	0.3927	$.6012-4$	$.6012-4$	$.1529-3$

Table 6. Errors on the Sinc Grid S and the Uniform Grid U for Example 3.4.

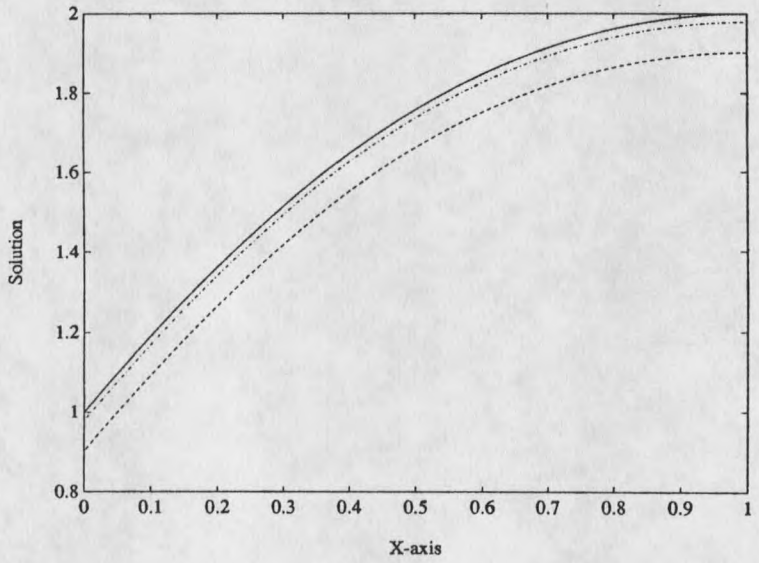


Figure 11. True and Approximate Solutions to Example 3.4
$- - -$ $(M=8)$, $- \cdot -$ $(M=16)$, $\cdots$ $(M=32)$, — (True).

Example 3.5

$$(\mathrm{EI}(x)u''(x))'' + u(x) = f(x) \quad , \quad 0 < x < 1$$

$$(\mathrm{EI}\,u'')(0) = 8\pi \quad , \quad (\mathrm{EI}\,u'')(1) = -8\pi$$

$$(\mathrm{EI}\,u'')'(0) = 8\pi^2 - 24\pi \quad , \quad (\mathrm{EI}\,u'')'(1) = 8\pi^2 - 24\pi \quad .$$

The forcing function $f(x)$ is consistent with the parameter $\mathrm{EI}(x) = 1 + \sin \pi x$ and the true solution $u(x) = x(1-x)\sin(4\pi x)$. The weight function is again taken to be $w(x) = 1$. Since (3.75) yields unrealistic decay parameters, set $\alpha = \beta = 1$. This latitude in parameter choice hints at the robustness of the method. The errors and solutions are reported in Table 7 and Figure 12, respectively. It is noted that a large number of basis functions are needed for good accuracy even though the method converges exponentially. This again may be attributed to the small values of α and β as well as the presence of high order algebraic boundary basis functions in the system.

M	N	h	$\|E_S^M(h)\|$	$\|E_U^M(\ell)\|$	$\left(\|E_U^{M/2}(\ell)\|\right)^{\sqrt{2}}$
16	16	0.5554	$1.3141 - 0$	$1.3141 - 0$	———
32	32	0.3927	$0.1794 - 1$	$0.1794 - 1$	———
64	64	0.2777	$0.2600 - 3$	$0.2600 - 3$	$.3393 - 2$

Table 7. Errors on the Sinc Grid S and the Uniform Grid U for Example 3.5.

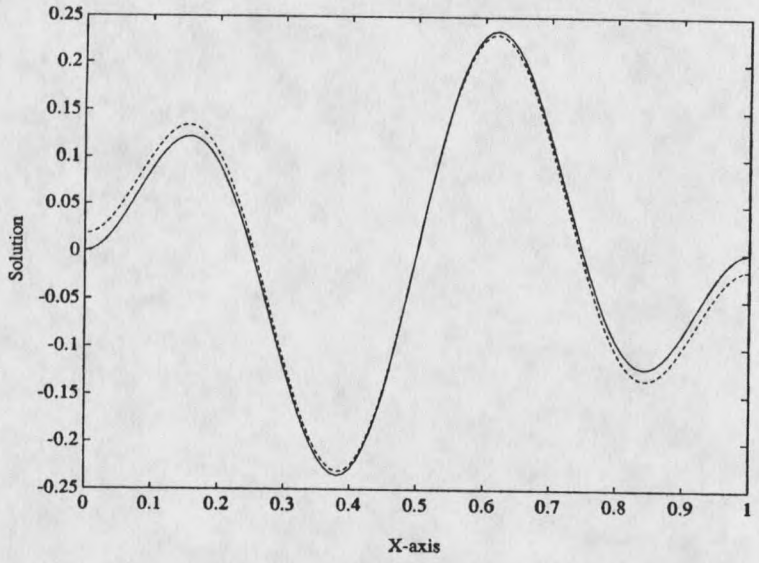


Figure 12. True and Approximate Solutions to Example 3.5
$- - -$ $(M = 32)$, — (True).

Cantilever Boundary Conditions

In addition to the three types of boundary conditions that have been discussed in previous sections, many hybrid conditions are commonly found in physical applications. Consider, for example, the problem

$$\begin{aligned} \mathcal{L}u(x) \equiv \eta(x)u''''(x) + \varrho(x)u'''(x) + \nu(x)u''(x) + \tau(x)u'(x) \\ + \sigma(x)u(x) = f(x) \quad , \quad a < x < b \end{aligned}$$

(3.78)

$$u(a) = \overline{\alpha} \quad , \quad u''(b) = \overline{\gamma}$$

$$u'(a) = \overline{\beta} \quad , \quad u'''(b) = \overline{\delta} \quad .$$

In Chapter 1 it was shown that (3.78) with $\overline{\gamma} = \overline{\delta} = 0$ can be used to model a cantilever beam. This physical application makes it worthwhile to present the Sinc-

Galerkin method for the problem. Furthermore, certain pitfalls will be pointed out which make the formulation more than a mere application of methods from the sections on fixed and free boundary conditions.

The basis $\{\psi_i\}_{i=-M-4}^{N+4}$ is taken to be that defined in (3.49). It is tempting to use fewer algebraic boundary basis functions and the basis weight

$$v(x) = (x-a)(b-x)^3 \quad ,$$

but in many problems this results in nonzero boundary terms when integrating by parts. By using the basis weight $v(x) = [(x-a)(b-x)]^3$ and a full complement of algebraic terms, this pitfall can be avoided.

The approximate solution is then defined to be

$$\begin{aligned} u_m(x) = {} & \overline{\alpha}\psi_{-M-1}(x) + \overline{\beta}\psi_{-M-2}(x) + \overline{\gamma}\psi_{N+3}(x) + \overline{\delta}\psi_{N+4}(x) \\ & + u_{-M-3}\psi_{-M-3}(x) + u_{-M-4}\psi_{-M-4}(x) \\ & + u_{N+1}\psi_{N+1}(x) + u_{N+2}\psi_{N+2}(x) \\ & + \sum_{i=-M}^{N} u_i\psi_i(x) \quad . \end{aligned} \tag{3.79}$$

As before, $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ and $\overline{\delta}$ are known and the coefficients $\{u_i\}$ are unknown. Again $u_m(x)$ satisfies the boundary conditions. The $m+4$ coefficients in (3.79) are determined by orthogonalizing the residual with respect to the set of sinc functions $\{S_p\}_{p=-M-2}^{N+2}$. This yields the discrete system

$$(\mathcal{L}u_m - f, S_p) \doteq 0 \tag{3.80}$$

for $-M-2 \le p \le N+2$. If (3.80) is expanded and the known terms are collected on the right-hand side, then the system may be written as

$$\begin{aligned} & (\mathcal{L}b_{-M-3}, S_p)u_{-M-3} + (\mathcal{L}b_{-M-4}, S_p)u_{-M-4} + (\mathcal{L}b_{N+1}, S_p)u_{N+1} \\ & + (\mathcal{L}b_{N+2}, S_p)u_{N+2} + (\mathcal{L}u_h, S_p) \\ & = (\overline{f}, S_p) \quad , \quad -M-2 \le p \le N+2 \quad . \end{aligned} \tag{3.81}$$

The "homogeneous" approximation $u_h(x)$ is defined in (3.23) and

$$\overline{f}(x) \equiv f(x) - \overline{\alpha}\mathcal{L}b_{-M-1}(x) - \overline{\beta}\mathcal{L}b_{-M-2}(x)$$

$$- \overline{\gamma}\mathcal{L}b_{N+3}(x) - \overline{\delta}\mathcal{L}b_{N+4}(x) \quad .$$

The approximations to the inner products in (3.81) follow directly from Theorem 3.1 if G in (3.26) is defined to be $\mathcal{L}b_{-M-3}$, $\mathcal{L}b_{-M-4}$, $\mathcal{L}b_{N+1}$, $\mathcal{L}b_{N+2}$ or $\overline{f}$. These approximations can then be used to obtain the Sinc-Galerkin system for the boundary value problem (3.78). Let $\vec{a}_{-M-3}$, $\vec{a}_{-M-4}$, $\vec{a}_{N+1}$, $\vec{a}_{N+2}$ and $\vec{b}$ denote the $(m+4) \times 1$ vectors containing the product of $\frac{1}{h}$ and the approximations to the inner products $(\mathcal{L}b_{-M-3}, S_p)$, $(\mathcal{L}b_{-M-4}, S_p)$, $(\mathcal{L}b_{N+1}, S_p)$, $(\mathcal{L}b_{N+2}, S_p)$ and $\left(\overline{f}, S_p\right)$, respectively. Also, let the $(m+4) \times m$ matrix A_m be defined as it was in (3.39) with v given by (3.50).

The discrete system for the determination of the unknown coefficients $\{u_i\}$ is then given by

$$A\vec{u} = \vec{b} \tag{3.82}$$

where the $(m+4) \times (m+4)$ matrix A is defined to be

$$A \equiv [\vec{a}_{-M-4} \mid \vec{a}_{-M-3} \mid A_m \mid \vec{a}_{N+1} \mid \vec{a}_{N+2}] \quad . \tag{3.83}$$

Here $\vec{u}$ is defined to be the $(m+4) \times 1$ vector

$$\vec{u} = [u_{-M-4}, u_{-M-3}, u_{-M}, \ldots, u_N, u_{N+1}, u_{N+2}]^T \tag{3.84}$$

containing the unknowns.

The decay parameters and inner product weight are chosen in the same manner as those for the problem (3.48) discussed in the third section. Numerical examples demonstrating the effectiveness of the Sinc-Galerkin method for problems

with cantilever boundary conditions will be given in the chapter on fourth-order time-dependent problems.

Alternate Formulation for Symmetric Problems

In this section an alternate Sinc-Galerkin formulation is developed for problems of the form

$$\mathcal{L}u(x) \equiv (\mathrm{EI}(x)u''(x))'' = f(x) \quad , \quad a < x < b$$

$$u(a) = u(b) = 0 \tag{3.85}$$

$$u'(a) = u'(b) = 0$$

and

$$\mathcal{L}u(x) \equiv (\mathrm{EI}(x)u''(x))'' = f(x) \quad , \quad a < x < b$$

$$u(a) = \overline{\alpha} \quad , \quad (\mathrm{EI}\,u'')(b) = \overline{\gamma} \tag{3.86}$$

$$u'(a) = \overline{\beta} \quad , \quad (\mathrm{EI}\,u'')'(b) = \overline{\delta} \quad .$$

The emphasis in this section is on the discretization of (3.85) and (3.86) since it proceeds somewhat differently than the procedures developed earlier in this work or in [30]. Here the procedure is carried out in a fashion which avoids the differentiation or collocation of the parameter $\mathrm{EI}(x)$. This formulation will be especially useful for solving inverse problems involving the identification of the parameter $\mathrm{EI}(x)$.

Fixed Boundary Conditions in Symmetric Problems

Consider the fixed beam model

$$\mathcal{L}u(x) \equiv (\mathrm{EI}(x)u''(x))'' = f(x) \quad , \quad a < x < b$$

$$u(a) = u(b) = 0 \tag{3.87}$$

$$u'(a) = u'(b) = 0 \quad .$$

In the method to be presented, both the parameter EI(x) and the solution $u(x)$ will be expanded as the linear combination of sinc and algebraic basis functions $\{\tilde{\psi}_k\}_{k=-M_E-2}^{N_E+2}$ and $\{\psi_i\}_{i=-M}^{N}$, respectively. These two sets of basis functions are defined to be

$$\psi_i(x) = S_i(x) \tag{3.88}$$

and

$$\tilde{\psi}_k(x) = \begin{cases} \tilde{b}_{-M_E-2}(x) & , \ k = -M_E - 2 \\ \tilde{b}_{-M_E-1}(x) & , \ k = -M_E - 1 \\ v_E(x)\tilde{S}_k(x) & , \ -M_E \le k \le N_E \\ \tilde{b}_{N_E+1}(x) & , \ k = N_E + 1 \\ \tilde{b}_{N_E+2}(x) & , \ k = N_E + 2 \end{cases} . \tag{3.89}$$

Here $S_i(x) \equiv S(i,h) \circ \phi(x)$, $\tilde{S}_k(x) \equiv S(k,h_E) \circ \phi(x)$, and the basis weight is taken to be

$$v_E(x) = [(x-a)(b-x)]^{3/2} . \tag{3.90}$$

The algebraic boundary basis functions are given by

$$\tilde{b}_{-M_E-1}(x) = \frac{(b-x)^2}{(b-a)^3}[2(x-a)+(b-a)] , \tag{3.91}$$

$$\tilde{b}_{N_E+1}(x) = \frac{(x-a)^2}{(b-a)^3}[2(b-x)+(b-a)] , \tag{3.92}$$

$$\tilde{b}_{-M_E-2}(x) = \frac{(x-a)(b-x)^2}{(b-a)^2} \tag{3.93}$$

and

$$\tilde{b}_{N_E+2}(x) = -\frac{(b-x)(x-a)^2}{(b-a)^2} . \tag{3.94}$$

By design these functions satisfy

$$\tilde{b}_{-M_E-1}(a) = 1$$
$$\tilde{b}_{-M_E-1}(b) = \tilde{b}'_{-M_E-1}(a) = \tilde{b}'_{-M_E-1}(b) = 0 \quad , \tag{3.95}$$

$$\tilde{b}_{N_E+1}(b) = 1$$
$$\tilde{b}_{N_E+1}(a) = \tilde{b}'_{N_E+1}(a) = \tilde{b}'_{N_E+1}(b) = 0 \quad , \tag{3.96}$$

$$\tilde{b}'_{-M_E-2}(a) = 1$$
$$\tilde{b}'_{-M_E-2}(b) = \tilde{b}_{-M_E-2}(a) = \tilde{b}_{-M_E-2}(b) = 0 \tag{3.97}$$

and

$$\tilde{b}'_{N_E+2}(b) = 1$$
$$\tilde{b}'_{N_E+2}(a) = \tilde{b}_{N_E+2}(a) = \tilde{b}_{N_E+2}(b) = 0 \quad . \tag{3.98}$$

The parameter approximation $\mathrm{EI}_{m_E}(x)$ is then defined to be

$$\mathrm{EI}_{m_E}(x) = \mathrm{EI}_h(x) + \mathrm{EI}_c(x) \quad , \quad m_E = M_E + N_E + 1 \tag{3.99}$$

where

$$\mathrm{EI}_h(x) = \sum_{k=-M_E}^{N_E} c_k \tilde{\psi}_k(x) \tag{3.100}$$

and

$$\mathrm{EI}_c(x) = c_{-M_E-1}\tilde{\psi}_{-M_E-1}(x) + c_{-M_E-2}\tilde{\psi}_{-M_E-2}(x)$$
$$+ c_{N_E+1}\tilde{\psi}_{N_E+1}(x) + c_{N_E+2}\tilde{\psi}_{N_E+2}(x) \quad . \tag{3.101}$$

In the forward problem the coefficients $\{c_k\}_{k=-M_E-2}^{N_E+2}$ are known and $\mathrm{EI}_{m_E}(x)$ is merely a finite dimensional approximation to the parameter $\mathrm{EI}(x)$. In the corresponding inverse problem, the coefficients are unknown and are determined

via methods to be discussed in Chapter 5. It should be noted that for the forward problem, an approximation containing only two algebraic functions and the basis weight $v_E(x) \equiv 1$ would be sufficient. The expansion (3.99) however, is needed for the inverse problem and is thus developed here. Similarly, the choice of $v_E(x) = [(x-a)(b-x)]^{3/2}$ over $v_E(x) = [(x-a)(b-x)]$ is motivated by analysis to come in Chapter 5.

The approximate solution $u_m(x)$ is defined by

$$u_m(x) = \sum_{i=-M}^{N} u_i \psi_i(x) = \sum_{i=-M}^{N} u_i S_i(x) \quad , \quad m = M+N+1 \quad . \tag{3.102}$$

The m unknowns $\{u_i\}$ in (3.102) are determined by orthogonalizing the residual with respect to the set of sinc functions $\{S_p\}_{p=-M}^{N}$. This yields the discrete Galerkin system

$$((\mathrm{EI}_{m_E} u_m'')'', S_p) = (f, S_p) \tag{3.103}$$

for $-M \le p \le N$. As usual $(\cdot,\cdot)$ denotes the weighted inner product that is defined in (3.22). After four integrations by parts

$$\begin{aligned} ((\mathrm{EI}_{m_E} u_m'')'', S_p) &= ((\mathrm{EI}_h u_m'')'', S_p) + ((\mathrm{EI}_c u_m'')'', S_p) \\ &= < u_m, (\mathrm{EI}_h (S_p w)'')'' > \; + B_T^{(1)} \\ &\quad + \; < u_m, (\mathrm{EI}_c (S_p w)'')'' > \; + B_T^{(2)} \end{aligned} \tag{3.104}$$

where $< \cdot,\cdot >$ denotes the standard $L^2(a,b)$ inner product and

$$\begin{aligned} B_T^{(1)} = \big\{ &(\mathrm{EI}_h u_m'')'(S_p w) - (\mathrm{EI}_h u_m'')(S_p w)' \\ &+ u_m' (\mathrm{EI}_h (S_p w)'') - u_m (\mathrm{EI}_h (S_p w)'')' \big\}(x) \Big|_a^b \quad , \end{aligned} \tag{3.105}$$

$$\begin{aligned} B_T^{(2)} = \big\{ &(\mathrm{EI}_c u_m'')'(S_p w) - (\mathrm{EI}_c u_m'')(S_p w)' \\ &+ u_m' (\mathrm{EI}_c (S_p w)'') - u_m (\mathrm{EI}_c (S_p w)'')' \big\}(x) \Big|_a^b \quad . \end{aligned} \tag{3.106}$$

Approximations to the above inner products are referenced in the following theorem. The proof follows directly from Theorem 2.5 once it is noted that all infinite sums collapse to finite sums as a result of the property

$$u_m(x_j) = u_j$$

for $-M \le j \le N$. Because the sums have already been truncated, the criteria necessary for choosing the decay parameters α and β will not be discussed at this point but will instead be addressed in the subsection on method implementation.

Theorem 3.6 *Let ϕ conformally map the simply connected domain D_E onto D_S. Let $x_p = \phi^{-1}(ph)$, $x_i = \phi^{-1}(ih)$, and take*

$$h = \left(\frac{\pi d}{\alpha M}\right)^{1/2} \quad \text{and} \quad N = \left[\left|\frac{\alpha}{\beta} M + 1\right|\right] \quad . \tag{3.107}$$

(a) If $fw \in B(D_E)$ then

$$\left|(f, S_p) - hf(x_p)\frac{w(x_p)}{\phi'(x_p)}\right| \le L_0 M^{-1/2} E^{-(\pi d \alpha M)^{1/2}} \tag{3.108}$$

where L_0 depends on f, w and d.

(b) If $u_m\,(\mathrm{EI}_c(S_p w)'')'' \in B(D_E)$ and $B_T^{(2)} = 0$ then

$$\left|((\mathrm{EI}_c u_m'')'', S_p) - h\sum_{i=-M}^{N} u_i \frac{1}{\phi'(x_i)}\,(\mathrm{EI}_c(S_p w)'')''(x_i)\right| \le L_{41} M^2 e^{-(\pi d \alpha M)^{1/2}} \tag{3.109}$$

where L_{41} depends on u_m, EI_c, w and d.

(c) If $u_m\,(\mathrm{EI}_h\,(S_p w)'')'' \in B(D_E)$ *and* $B_T^{(1)} = 0$ *then*

$$\left| ((\mathrm{EI}_h u_m'')'', S_p) - h \sum_{i=-M}^{N} u_i \, \frac{1}{\phi'(x_i)} \, (\mathrm{EI}_h\,(S_p w)'')''\,(x_i) \right| \leq L_{42} M^2 e^{-(\pi d \alpha M)^{1/2}} \tag{3.110}$$

where L_{42} *depends on* u_m, EI_h, w *and* d.

The application of Theorem 3.6 to (3.103) yields the discrete system

$$\sum_{k=-M_E-2}^{N_E+2} c_k \sum_{i=-M}^{N} u_i \, \frac{1}{\phi'(x_i)} \left[\tilde{\psi}_k\,(S_p w)''\right]''(x_i) = f(x_p) \, \frac{w(x_p)}{\phi'(x_p)} \tag{3.111}$$

for $-M \leq p \leq N$. This may be written in the matrix formulation

$$A\vec{u} = \vec{b} \tag{3.112}$$

where

$$A = \left[\Phi^{(2)} D\,(\vec{p}_{\Psi(2)}) + 2\Phi^{(3)} D\,(\vec{p}_{\Psi(1)}) + \Phi^{(4)} D\,(\vec{p}_{\Psi(0)})\right] \quad , \tag{3.113}$$

$$\vec{u} = [u_{-M}, \ldots, u_N]^T \quad , \tag{3.114}$$

and the p-th component of $\vec{b}$ is

$$\left[\vec{b}\right]_p = f(x_p) \, \frac{w(x_p)}{\phi'(x_p)} \quad . \tag{3.115}$$

The notation $D\,(\vec{p}_{\Psi(\ell)})$, $\ell = 0,1,2$ denotes the diagonal matrices containing the components of the vectors

$$\vec{p}_{\Psi(\ell)} = \Psi^{(\ell)} \vec{c} \quad , \quad \ell = 0,1,2 \tag{3.116}$$

where $\vec{c} = [c_{-M_E-2}, c_{-M_E-1}, c_{-M_E}, \ldots, c_{N_E}, c_{N_E+1}, c_{N_E+2}]^T$. The matrices $\Phi^{(j)}$, $j = 2,3,4$ and $\Psi^{(\ell)}$, $\ell = 0,1,2$ are defined componentwise by

$$[\Phi^{(j)}]_{pi} = \frac{1}{\phi'(x_i)} \, (S_p w)^{(j)}(x_i) \tag{3.117}$$

and

$$[\Psi^{(\ell)}]_{ik} = \tilde{\psi}_k^{(\ell)}(x_i) \quad . \tag{3.118}$$

The notation on the right-hand sides of (3.117) and (3.118) indicates the j-th and ℓ-th derivatives, respectively.

Note that for general values of M and M_E, the matrix A will not be square. In such cases, the system (3.112) could be solved via a least squares method. Although this is feasible in the time-independent problem as stated, it will not be useful when extending the formulation to time-dependent problems. Thus to guarantee a square matrix and to simplify its structure, take

$$M_E = M - 2 \tag{3.119}$$

and

$$h_E = h \quad . \tag{3.120}$$

In general, the computational gains compensate for the loss of accuracy caused by this restriction.

To illustrate the dependence of $\Phi^{(j)}$, $j = 2, 3, 4$, and $\Psi^{(\ell)}$, $\ell = 0, 1, 2$, on previously defined matrices, the respective expansions are listed below. The diagonal matrices D and the matrices $I^{(\ell)}$, $\ell = 0, 1, 2, 3, 4$, have sizes consistent with the range of the indices i, p and k. The functions $g_i^{(\ell)}(z)$ are defined in (2.37)–(2.48) with $\eta(z) = \varrho(z) = \nu(z) \equiv 1$. From (3.117) it follows that

$$\Phi^{(2)} = \frac{1}{h^2} I^{(2)} D\left(\frac{g_2^{(2)}}{\phi'}\right) + \frac{1}{h} I^{(1)} D\left(\frac{g_2^{(1)}}{\phi'}\right) + I^{(0)} D\left(\frac{g_2^{(0)}}{\phi'}\right) \quad , \tag{3.121}$$

$$\Phi^{(3)} = \frac{1}{h^3}\, I^{(3)} \mathcal{D}\left(\frac{g_3^{(3)}}{\phi'}\right) + \frac{1}{h^2}\, I^{(2)} \mathcal{D}\left(\frac{g_3^{(2)}}{\phi'}\right) + \frac{1}{h}\, I^{(1)} \mathcal{D}\left(\frac{g_3^{(1)}}{\phi'}\right) + I^{(0)} \mathcal{D}\left(\frac{g_3^{(0)}}{\phi'}\right) \tag{3.122}$$

and

$$\Phi^{(4)} = \frac{1}{h^4}\, I^{(4)} \mathcal{D}\left(\frac{g_4^{(4)}}{\phi'}\right) + \frac{1}{h^3}\, I^{(3)} \mathcal{D}\left(\frac{g_4^{(3)}}{\phi'}\right) + \frac{1}{h^2}\, I^{(2)} \mathcal{D}\left(\frac{g_4^{(2)}}{\phi'}\right) + \frac{1}{h}\, I^{(1)} \mathcal{D}\left(\frac{g_4^{(1)}}{\phi'}\right) + I^{(0)} \mathcal{D}\left(\frac{g_4^{(0)}}{\phi'}\right) . \tag{3.123}$$

For $\ell = 0, 1, 2$ the $m \times m$ matrices in (3.118) are given by

$$\Psi^{(\ell)} = \left[\hat{b}^{(\ell)}_{-M_E-2} \mid \hat{b}^{(\ell)}_{-M_E-1} \mid B^{(\ell)} \mid \hat{b}^{(\ell)}_{N_E+1} \mid \hat{b}^{(\ell)}_{N_E+2}\right] \tag{3.124}$$

where $\hat{b}^{(\ell)}_k = \left[\tilde{b}^{(\ell)}_k(x_{-M}), \ldots, \tilde{b}^{(\ell)}_k(x_N)\right]^T$, for $k = -M_E - 2, -M_E - 1, N_E + 1$ and $N_E + 2$. Again, the superscript ℓ indicates the ℓ-th derivative. The $m \times (m-4)$ matrices $B^{(\ell)}$ are

$$B^{(0)} = \mathcal{D}(v_E) I^{(0)} \quad , \tag{3.125}$$

$$B^{(1)} = -\frac{1}{h}\, \mathcal{D}(v_E \phi') I^{(1)} + \mathcal{D}(v_E) I^{(0)} \tag{3.126}$$

and

$$B^{(2)} = \frac{1}{h^2}\, \mathcal{D}\left(v_E (\phi')^2\right) I^{(2)} - \frac{1}{h}\, \mathcal{D}(v_E \phi'' + 2v'_E \phi') I^{(1)} + \mathcal{D}(v''_E) I^{(0)} \quad . \tag{3.127}$$

The negative signs that appear in the definitions of $B^{(1)}$ and $B^{(2)}$ result from the fact that $(I^{(1)})^T = -I^{(1)}$. The implementation of the method is discussed in the next subsection.

Implementation of Symmetric Problems with Fixed Boundaries

The implementation of the Sinc-Galerkin method for the problem (3.87) is similar to that of previous problems in that (1) decay parameters α and β must be chosen and (2) an inner product weight $w(x)$ must be selected. In contrast to previous cases however, the parameters here depend not only on the decay of the state solution $u(x)$ but also on the decay of that part of $\mathrm{EI}(x)$ which is approximated by $\mathrm{EI}_h(x)$. That particular component of $\mathrm{EI}(x)$ is isolated via the formal change of variables

$$\begin{aligned}\mathcal{E}\,I(x) = \mathrm{EI}(x) &- \mathrm{EI}(a)\tilde{b}_{-M_E-1}(x) - \mathrm{EI}(b)\tilde{b}_{N_E+1}(x) \\ &- \mathrm{EI}'(a)\tilde{b}_{-M_E-2}(x) - \mathrm{EI}'(b)\tilde{b}_{N_E+2}(x) \quad .\end{aligned} \tag{3.128}$$

The first decay condition

$$|\mathcal{E}\,I(x)| \le L \begin{cases} (x-a)^{\alpha} & , \ x \in \gamma_L \\ (b-x)^{\beta} & , \ x \in \gamma_R \end{cases} \tag{3.129}$$

is an interpolation requirement for $\mathcal{E}\,I(x)$ (see [39]). Since $u_m(x)$ directly approximates $u(x)$, it follows from (3.109), (3.110) and Theorem 2.5 that other decay conditions are

$$\left| \frac{u(x)}{\phi'(x)} \left(\mathcal{E}\,I(S_p w)''\right)''(x) \right| \le L \begin{cases} (x-a)^{\alpha} & , \ x \in \gamma_L \\ (b-x)^{\beta} & , \ x \in \gamma_R \end{cases} \tag{3.130}$$

and

$$\left| \frac{u(x)}{\phi'(x)} \left(\tilde{b}_k (S_p w)''\right)''(x) \right| \le L \begin{cases} (x-a)^{\alpha} & , \ x \in \gamma_L \\ (b-x)^{\beta} & , \ x \in \gamma_R \end{cases} \tag{3.131}$$

for $\tilde{b}_k$, $k = -M_E - 1$, $-M_E - 2$, $N_E + 1$ and $N_E + 2$, as defined in (3.91)–(3.94).

The decay parameters α and β are ideally chosen so that (3.129)–(3.131) are satisfied. If the three conditions dictate very different parameters, the method may give results that are less than optimal. Numerical results have indicated

however, that in most cases there is a range of parameter choices that give good results. This flexibility accommodates the fact that the true $u(x)$ and $\mathrm{EI}(x)$ are often unknown.

The weight function $w(x)$ is chosen for reasons similar to those expounded on in the previous section. For the problem (3.87), the weight is chosen to be

$$w(x) = \frac{1}{[\phi'(x)]^{3/2}} \quad . \tag{3.132}$$

From (3.105) and (3.106) this choice enforces $B_T^{(1)} = B_T^{(2)} = 0$. With this weight, the decay requirements for the inner product approximations are

$$|u(x)\mathcal{E}\mathcal{I}(x)| \leq L \begin{cases} (x-a)^{\alpha+3/2} & , \ x \in \gamma_L \\ (b-x)^{\beta+3/2} & , \ x \in \gamma_R \end{cases} \tag{3.133}$$

and

$$|u(x)\tilde{b_k}(x)| \leq L \begin{cases} (x-a)^{\alpha+3/2} & , \ x \in \gamma_L \\ (b-x)^{\beta+3/2} & , \ x \in \gamma_R \end{cases} \tag{3.134}$$

with $\tilde{b}_k$ defined as in (3.91)–(3.94).

Numerical Examples of Symmetric Problems with Fixed Boundaries

The following examples illustrate various features of the methods that were developed in this section. In these examples, the parameter $\mathrm{EI}(x)$ is known and the state solution $u(x)$ is unknown. The results from the first example can be compared with those from Example 3.5.

Example 3.7 Consider the self-adjoint problem

$$(\mathrm{EI}(x)u''(x))'' = f(x) \quad , \quad 0 < x < 1$$

$$u(0) = u(1) = 0$$

$$u'(0) = u'(1) = 0$$

where $\mathrm{EI}(x) = 1 + \sin(\pi x)$ and $f(x)$ is consistent with the true solution $u(x) = x(1-x)\sin(4\pi x)$. The weight $w(x) = (\phi'(x))^{-3/2}$ dictates the parameter choices $\alpha = \beta = 1$ (see (3.129), (3.133) and (3.134)). Hence $h = \pi/\sqrt{2M}$ and the asymptotic error is $\mathcal{O}\left(e^{-\pi\sqrt{M/2}}\right)$. The errors and solutions are reported in Table 8 and Figure 13, respectively.

M	N	h	$\|E_S^M(h)\|$	$\|E_U^M(\ell)\|$	$\left(\|E_U^{M/2}(\ell)\|\right)^{\sqrt{2}}$
4	4	1.1107	$1.3821-0$	$1.4558-0$	———
8	8	0.7854	$0.2173-0$	$0.2500-0$	———
16	16	0.5554	$0.3910-2$	$0.1025-1$	$.1155-0$
32	32	0.3927	$0.9721-5$	$0.1378-3$	$.1537-2$
64	64	0.2777	$0.3217-7$	$0.2155-6$	$.3468-5$

Table 8. Errors on the Sinc Grid S and the Uniform Grid U for Example 3.7.

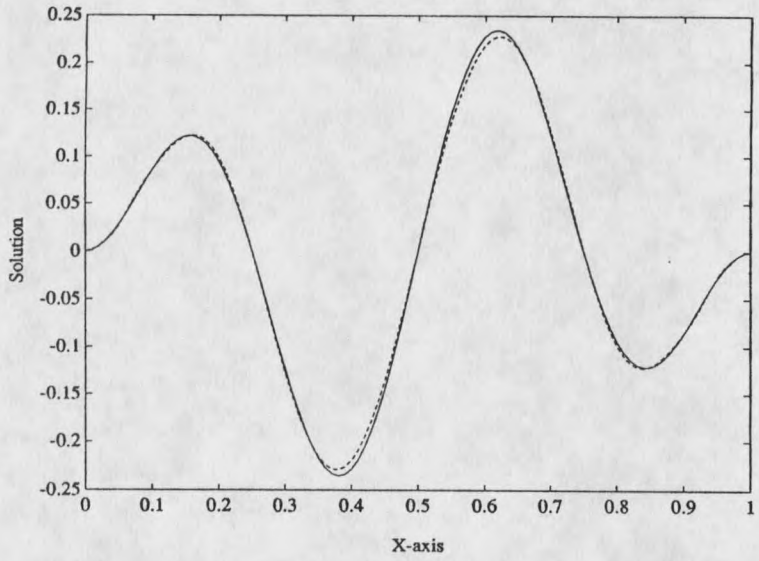


Figure 13. True and Approximate Solutions to Example 3.7
$- - -$ $(M = 16)$, — (True).

Example 3.8 Consider the self-adjoint problem

$$(\mathrm{EI}(x)u''(x))'' = f(x) \quad , \quad 0 < x < 1$$

$$u(0) = u(1) = 0$$

$$u'(0) = u'(1) = 0$$

where $\mathrm{EI}(x) = 1 + \frac{1}{2}\sin(4\pi x)$. The forcing function $f(x)$ is consistent with the true solution $u(x) = (x\,\ell n(x))^2$. The weight $w(x) = (\phi'(x))^{-3/2}$ again dictates the parameter choices $\alpha = \beta = 1$. This in turn yields $h = \pi/\sqrt{2M}$ and the asymptotic error $\mathcal{O}\left(e^{-\pi\sqrt{M/2}}\right)$. The errors and solutions are reported in Table 9 and Figure 14, respectively.

M	N	h	$\|E_S^M(h)\|$	$\|E_U^M(\ell)\|$	$\left(\|E_U^{M/2}(\ell)\|\right)^{\sqrt{2}}$
4	4	1.1107	$.4175-0$	$.4610-0$	———
8	8	0.7854	$.3853-0$	$.4038-0$	$.3345-0$
16	16	0.5554	$.1079-1$	$.1105-1$	$.2774-0$
32	32	0.3927	$.9411-4$	$.1252-3$	$.1710-2$
64	64	0.2777	$.1670-5$	$.1835-5$	$.3028-5$

Table 9. Errors on the Sinc Grid S and the Uniform Grid U for Example 3.8.

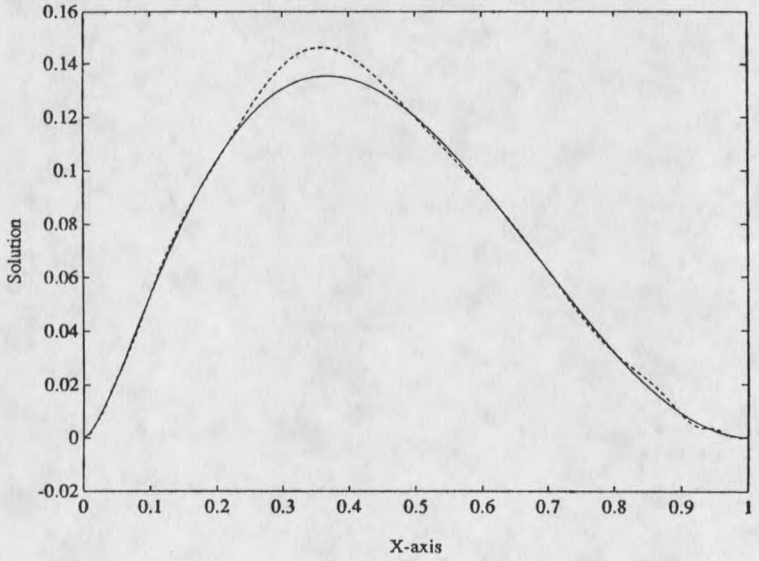


Figure 14. True and Approximate Solutions to Example 3.8
$- - -$ $(M = 16)$, — (True).

Cantilever Boundary Conditions for Symmetric Problems

Consider the variable coefficient problem

$$\mathcal{L}u(x) \equiv (\mathrm{EI}(x)u''(x))'' = f(x) \quad , \quad a < x < b$$

(3.135)
$$u(a) = \overline{\alpha} \quad , \quad (\mathrm{EI}\,u'')(b) = \overline{\gamma}$$

$$u'(a) = \overline{\beta} \quad , \quad (\mathrm{EI}\,u'')'(b) = \overline{\delta} \quad .$$

As was pointed out in Chapter 1, (3.135) with $\overline{\gamma} = \overline{\delta} = 0$ can be used to model a beam with cantilever boundary conditions.

The method which follows is similar to that used on the variable coefficient fixed beam model with adjustments made for the higher order boundary conditions. The basis functions for this problem are $\{\psi_i\}_{i=-M-4}^{N+4}$ and $\{\tilde{\psi}_k\}_{k=-M_E-2}^{N_E+2}$ as defined in (3.49) and (3.89), respectively. The definition (3.99) is taken for $\mathrm{EI}_{m_E}(x)$, and the state approximation $u_m(x)$ is defined to be

(3.136)
$$u_m(x) = u_h(x) + u_U(x) + u_R(x)$$

where

$$u_h(x) = \sum_{i=-M}^{N} u_i v(x) S_i(x) \quad ,$$

$$u_U(x) = u_{-M-3}b_{-M-3}(x) + u_{-M-4}b_{-M-4}(x)$$

$$+ u_{N+1}b_{N+1}(x) + u_{N+2}b_{N+2}(x)$$

and

$$u_R(x) = \overline{\alpha}b_{-M-1}(x) + \overline{\beta}b_{-M-2}(x) + \tilde{\gamma}b_{N+3}(x) + \tilde{\delta}b_{N+4}(x) \quad .$$

Here

(3.137)
$$v(x) = [(x-a)(b-x)]^3$$

and

$$\tilde{\gamma} \equiv \frac{1}{\mathrm{EI}(b)}\,\overline{\gamma} \tag{3.138}$$

$$\tilde{\delta} \equiv \frac{1}{\mathrm{EI}(b)}\,\overline{\delta} - \frac{\mathrm{EI}'(b)}{[\mathrm{EI}(b)]^2}\,\overline{\gamma} \quad . \tag{3.139}$$

From the discussion of beam physics in Chapter 1, it follows that the requirement $\mathrm{EI}(b) \neq 0$ is not physically restrictive. Furthermore, $u_m(x)$ satisfies the boundary conditions; that is

$$u_m(a) = \overline{\alpha} \quad , \quad (\mathrm{EI}\,u_m'')(b) = \overline{\gamma}$$

$$u_m(b) = \overline{\beta} \quad , \quad (\mathrm{EI}\,u_m'')(b) = \overline{\delta} \quad .$$

The $m+4$ unknowns $\{u_i\}$ in (3.136) are determined by orthogonalizing the residual with respect to the set of sinc functions $\{S_p\}_{p=-M-2}^{N+2}$. This yields the discrete Galerkin system

$$((\mathrm{EI}_{m_B}\,u_m'')'', S_p) = (f, S_p) \tag{3.140}$$

for $-M-2 \leq p \leq N+2$. Expansion and integration by parts yields

$$\begin{aligned}((\mathrm{EI}_{m_B}\,u_m'')'', S_p) = &\left\{< u_h, (\mathrm{EI}_h\,(S_p w)'')'' > \; + B_T^{(1)}\right\} \\ &+ \left\{< u_h, (\mathrm{EI}_c\,(S_p w)'')'' > \; + B_T^{(2)}\right\} \\ &+ \left\{< \mathrm{EI}_h\,(u_R + u_U)'', (S_p w)'' > \; + B_T^{(3)}\right\} \\ &+ < (\mathrm{EI}_c\,(u_R + u_U)'')'', (S_p w) >\end{aligned} \tag{3.141}$$

where $< \cdot, \cdot >$ again denotes the $L^2(a,b)$ inner product. The boundary terms are

$$\begin{aligned}B_T^{(1)} = &\left\{(\mathrm{EI}_h\,u_h'')'(S_p w) - (\mathrm{EI}_h\,u_h'')(S_p w)' + u_h'\,(\mathrm{EI}_h\,(S_p w)')'\right. \\ &\left. - u_h\,(\mathrm{EI}_h\,(S_p w)')''\right\}(x)\Big|_a^b \quad ,\end{aligned} \tag{3.142}$$

$$B_T^{(2)} = \left\{(\mathrm{EI}_c u_h'')'(S_p w) - (\mathrm{EI}_c u_h'')(S_p w)' + u_h' \left(\mathrm{EI}_c (S_p w)'\right)'\right.$$
$$\left. - u_h \left(\mathrm{EI}_c (S_p w)'\right)''\right\}(x)\Big|_a^b \tag{3.143}$$

and

$$B_T^{(3)} = \left\{\left(\mathrm{EI}_h (u_R + u_U)''\right)' (S_p w) - \left(\mathrm{EI}_h (u_R + u_U)''\right) (S_p w)'\right\}(x)\Big|_a^b \quad . \tag{3.144}$$

The following theorem lists approximations with error bounds to the inner products in (3.140) and (3.141). The criteria necessary for choosing the decay parameters α and β will again be discussed in a later subsection.

Theorem 3.9 *Let ϕ conformally map the simply connected domain D_E onto D_S. Let $x_i = \phi^{-1}(ih)$, $x_p = \phi^{-1}(ph)$, and $x_k = \phi^{-1}(kh_E)$ denote the quadrature points. Furthermore, take*

$$h = \left(\frac{\pi d}{\alpha M}\right)^{1/2} \quad , \tag{3.145}$$

$$h_E = \left(\frac{\pi d}{\alpha M_E}\right)^{1/2} \quad , \tag{3.146}$$

$$N = \left[\left|\frac{\alpha}{\beta} M + 1\right|\right] \tag{3.147}$$

and

$$N_E = \left[\left|\frac{\alpha}{\beta} M_E + 1\right|\right] \quad . \tag{3.148}$$

(a) If $fw \in B(D_E)$ then

$$\left|(f, S_p) - hf(x_p) \frac{w(x_p)}{\phi'(x_p)}\right| \leq L_{01} M^{-1/2} e^{-(\pi d \alpha M)^{1/2}} \tag{3.149}$$

where L_{01} *depends on* f, w *and* d.

(b) If $(\mathrm{EI}_c(u_R+u_U)'')''\, w \in B(D_E)$ *then*

$$\left|\left((\mathrm{EI}_c(u_R+u_U)'')'', S_p\right) - h\,\frac{w(x_p)}{\phi'(x_p)}\,(\mathrm{EI}_c(u_R+u_U)'')''(x_p)\right| \leq L_{02}M^{-1/2}e^{-(\pi d\alpha M)^{1/2}} \tag{3.150}$$

where L_{02} *depends on* EI_c, u_R, u_U, w *and* d.

(c) If $u_h\,(\mathrm{EI}_c(S_p w)'')'' \in B(D_E)$ *and* $B_T^{(2)} = 0$ *then*

$$\left|((\mathrm{EI}_c u_h'')'', S_p) - h\sum_{i=-M}^{N} u_i\,\frac{v(x_i)}{\phi'(x_i)}\,(\mathrm{EI}_c(S_p w)'')''(x_i)\right| \leq L_{41}M^2 e^{-(\pi d\alpha M)^{1/2}} \tag{3.151}$$

where L_{41} *depends on* u_h, EI_c, w *and* d.

(d) If $u_h\,(\mathrm{EI}_h(S_p w)'')'' \in B(D_E)$ *and* $B_T^{(1)} = 0$ *then*

$$\left|((\mathrm{EI}_h u_h'')'', S_p) - h\sum_{i=-M}^{N} u_i\,\frac{v(x_i)}{\phi'(x_i)}\,(\mathrm{EI}_c(S_p w)'')''(x_i)\right| \leq L_{42}M^2 e^{-(\pi d\alpha M)^{1/2}} \tag{3.152}$$

where L_{42} *depends on* u_h, EI_h, w *and* d.

(e) If $(\mathrm{EI}_h(u_R+u_U)'')\,(S_p w)'' \in B(D_E)$ *and* $B_T^{(3)} = 0$ *then*

$$\left|\left((\mathrm{EI}_h(u_R+u_U)'')'', S_p\right) - h_E\sum_{k=-M_E}^{N_E} c_k\,\frac{v_E(x_k)}{\phi'(x_k)}\,(u_R+u_U)''(S_p w)''(x_k)\right| \leq L_2 M e^{-(\pi d\alpha M_E)^{1/2}} \tag{3.153}$$

where L_2 *depends on* EI_h, u_R, u_U, w *and* d.

The application of Theorem 3.9 to (3.140) yields the discrete system

$$
\begin{aligned}
& h \sum_{k=-M_E-2}^{N_E+2} c_k \sum_{i=-M}^{N} u_i \, \frac{v(x_i)}{\phi'(x_i)} \left(\tilde{\psi}_k (S_p w)''\right)''(x_i) \\
& + h_E \sum_{k=-M_E}^{N_E} c_k \, \frac{v_E(x_k)}{\phi'(x_k)} \, u_U''(x_k)(S_p w)''(x_k) \\
& + h \, \frac{w(x_p)}{\phi'(x_p)} \, (\mathrm{EI}_c u_U'')''(x_p) \\
& = h \, \frac{w(x_p)}{\phi'(x_p)} \, f(x_p) \\
& - h_E \sum_{k=-M_E}^{N_E} c_k \, \frac{v_E(x_k)}{\phi'(x_k)} \, u_R''(x_k)(S_p w)''(x_k) \\
& - h \, \frac{w(x_p)}{\phi'(x_p)} \, (\mathrm{EI}_c u_R'')''(x_p)
\end{aligned}
\tag{3.154}
$$

for $-M-2 \le p \le N+2$. The matrix system corresponding to (3.154) may be formulated as follows. Let $\Phi^{(j)}$, $\Psi^{(\ell)}$ and $\vec{p}_{\Psi(\ell)}$ be defined as they were in (3.117), (3.118) and (3.116), respectively (with $\ell = 0,1,2$ and $j = 2,3,4$). Furthermore, let $\hat{\Phi}''$ denote the matrix which is defined componentwise by

$$
\left[\hat{\Phi}''\right]_{pk} = \frac{1}{\phi'(x_k)} \, (S_p w)''(x_k) \tag{3.155}
$$

and let $\tilde{c} = [c_{-M_E}, \ldots, c_{N_E}]^T$. Finally, for $i = -M-4, \ldots, -M-1$ and $i = N+1, \ldots, N+4$, let $\vec{a}_i$ denote the $(m+4) \times 1$ vectors

$$
\vec{a}_i = \left[h_E \, \hat{\Phi}'' \mathcal{D}(v_E b_i'') \tilde{c} + h \mathcal{D} \left(\frac{w}{\phi'} \, (\mathrm{EI}_c b_i)'' \right) \vec{1} \right] \quad . \tag{3.156}
$$

The discrete system (3.154) can then be written as

$$
A\vec{u} = \vec{b} \tag{3.157}
$$

where

$$
A = [\vec{a}_{-M-4} \mid \vec{a}_{-M-3} \mid A_m \mid \vec{a}_{N+1} \mid \vec{a}_{N+2}] \quad , \tag{3.158}
$$

$$\vec{u} = [u_{-M-4}, u_{-M-3}, u_{-M}, \ldots, u_N, u_{N+1}, u_{N+2}]^T \tag{3.159}$$

and

$$\vec{b} = h\mathcal{D}\left(\frac{w}{\phi'} f\right)\vec{1} - \overline{\alpha}\vec{a}_{-M-1} - \overline{\beta}\vec{a}_{-M-2} - \tilde{\gamma}\vec{a}_{N+3} - \tilde{\delta}\vec{a}_{N+4} \quad . \tag{3.160}$$

The $(m+4) \times m$ submatrix A_m is given by

$$A_m = h\left[\Phi^{(2)}\mathcal{D}(v\vec{p}_{\Psi(2)}) + 2\Phi^{(3)}\mathcal{D}(v\vec{p}_{\Psi(1)}) + \Phi^{(4)}\mathcal{D}(v\vec{p}_{\Psi(0)})\right] \quad . \tag{3.161}$$

As noted before, the matrix A will not be square for general values of M and M_E. To guarantee a square matrix and to simplify its structure, take

$$M_E = M \tag{3.162}$$

and

$$h_E = h \quad . \tag{3.163}$$

With these definitions, the matrices $\Phi^{(j)}$, $\hat{\Phi}^{(j)}$ and $\Psi^{(\ell)}$ can be expanded in terms of fundamental matrices in a manner similar to that in (3.121)–(3.127). This again simplifies the implementation of the method. Further implementation issues will be discussed in the next subsection.

Implementation of Symmetric Problems with Cantilever Boundaries

The final issues to be addressed in this section are the criteria necessary for choosing the decay parameters and inner product weight. In this case, the decay parameters α and β depend both on $\mathcal{E}\,I(x)$, as defined in (3.128), and on that part of the state solution which is approximated by $u_h(x)$. That particular component of $u(x)$ is isolated via the formal change of variables

$$\begin{aligned} \mathcal{U}(x) = u(x) &- \overline{\alpha}b_{-M-1}(x) - \overline{\beta}b_{-M-2}(x) - \tilde{\gamma}b_{N+3}(x) \\ &- \tilde{\delta}b_{N+4}(x) - u''(a)b_{-M-3}(x) - u'''(a)b_{-M-4}(x) \\ &- u(b)b_{N+1}(x) - u'(b)b_{N+2}(x) \quad . \end{aligned} \tag{3.164}$$

The first decay condition is again the interpolation requirement given in (3.129). From (3.151)–(3.153) and Theorem 2.5 it follows that other decay conditions are

$$\left|\frac{\mathcal{U}(x)}{\phi'(x)}\left(\mathcal{E}\,\mathcal{I}(S_p w)''\right)''(x)\right| \le L\begin{cases}(x-a)^{\alpha} & , \ x\in\gamma_L\\ (b-x)^{\beta} & , \ x\in\gamma_R\end{cases}, \tag{3.165}$$

$$\left|\frac{\mathcal{U}(x)}{\phi'(x)}\left(\tilde{b}_k(S_p w)''\right)''(x)\right| \le L\begin{cases}(x-a)^{\alpha} & , \ x\in\gamma_L\\ (b-x)^{\beta} & , \ x\in\gamma_R\end{cases} \tag{3.166}$$

and

$$\left|\frac{\mathcal{E}\,\mathcal{I}(x)}{\phi'(x)}\, b_i''(S_p w)''(x)\right| \le L\begin{cases}(x-a)^{\alpha} & , \ x\in\gamma_L\\ (b-x)^{\beta} & , \ x\in\gamma_R\end{cases} \tag{3.167}$$

where $k = -M_E - 1,\ -M_E - 2,\ N_E + 1,\ N_E + 2$ and $i = -M - 1,\ -M - 2,\ -M - 3,\ -M - 4,\ N + 1,\ N + 2,\ N + 3$ and $N + 4$.

The decay parameters α and β are ideally chosen so that (3.129) and (3.165)–(3.167) are satisfied. As was mentioned before, the results may be less than optimal if the four conditions dictate very different decay parameters.

For reasons that were elaborated on in the fourth section, the weight for this problem is chosen to be

$$w(x) = \frac{1}{[\phi'(x)]^{\varsigma}} \tag{3.168}$$

where $\varsigma = \frac{1}{2}$ or $\varsigma = 0$. It is noted that in (3.142)–(3.144) both weights enforce $B_T^{(1)} = B_T^{(2)} = B_T^{(3)} = 0$. With this weight, the decay requirements for the inner product approximations are

$$|\mathcal{U}(x)\mathcal{E}\,\mathcal{I}(x)| \le L\begin{cases}(x-a)^{\alpha+3-\varsigma} & , \ x\in\gamma_L\\ (b-x)^{\beta+3-\varsigma} & , \ x\in\gamma_R\end{cases}, \tag{3.169}$$

$$|\mathcal{U}(x)\tilde{b}_k(x)| \le L\begin{cases}(x-a)^{\alpha+3-\varsigma} & , \ x\in\gamma_L\\ (b-x)^{\beta+3-\varsigma} & , \ x\in\gamma_R\end{cases} \tag{3.170}$$

and

$$|\mathcal{E}\, I(x) b_i''(x)| \leq L \begin{cases} (x-a)^{\alpha+3-\varsigma} & , \ x \in \gamma_L \\ (b-a)^{\beta+3-\varsigma} & , \ x \in \gamma_R \end{cases} \tag{3.171}$$

where i and k range as they did in (3.166) and (3.167).

Numerical Example of a Symmetric Problem with Cantilever Boundary Conditions

The following example illustrates the alternative coefficient formulation for problems with cantilever boundary conditions. The results from this example can be compared to those from Example 3.5 and Example 3.7.

Example 3.10 Consider the symmetric problem

$$(\mathrm{EI}(x)u''(x))'' = f(x) \ , \ 0 < x < 1$$

$$u(0) = 0 \ , \ (\mathrm{EI}\, u'')(1) = -8\pi$$

$$u'(0) = 0 \ , \ (\mathrm{EI}\, u'')'(1) = 8\pi^2 - 24\pi$$

where $\mathrm{EI}(x) = 1 + \sin(\pi x)$ and $f(x)$ is consistent with the true solution $u(x) = x(1-x)\sin(4\pi x)$. The discrete system is given by (3.157) where the weight function is taken to be $w(x) = 1$. Since the decay conditions (3.169)–(3.171) yield unrealistic parameters, set $\alpha = \beta = 1$. This choice does satisfy the interpolation requirement (3.129). The errors and solutions are reported in Table 10 and Figure 15. As was the case in Example 3.5, a large number of basis functions are needed for good accuracy even though the method converges exponentially. This again may be attributed to the discrepancy in the decay conditions as well as the presence of high order algebraic boundary basis functions in the system.

M	N	h	$\|E_S^M(h)\|$	$\|E_U^M(\ell)\|$	$\left(\|E_U^{M/2}(\ell)\|\right)^{\sqrt{2}}$
8	8	0.7854	$9.2668-0$	$9.2881-0$	———
16	16	0.5554	$0.3824-1$	$0.3825-1$	———
32	32	0.3927	$0.2669-3$	$0.2669-3$	$.9898-2$
64	64	0.2777	$0{,}3039-5$	$0.3039-5$	$.8832-5$

Table 10. Errors on the Sinc Grid S and the Uniform Grid U for Example 3.10.

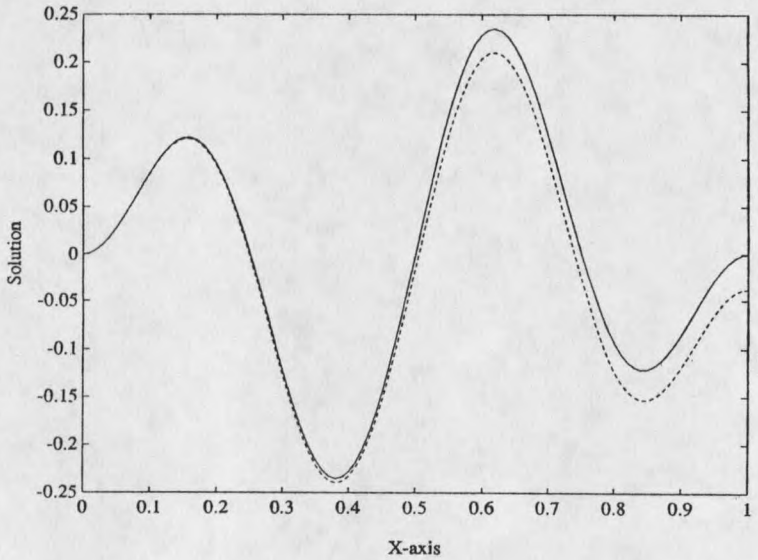


Figure 15. True and Approximate Solutions to Example 3.10
$- - -$ $(M = 16)$, — (True).

CHAPTER 4

FOURTH-ORDER TIME-DEPENDENT PROBLEMS

Introduction

In this chapter, the approximations from Chapters 2 and 3 are combined with temporal results to produce a method which very accurately approximates the solution to many fourth-order time-dependent partial differential equations. In preparation for this, the one-dimensional temporal problems

$$\begin{aligned} &\ddot{u}(t) = f(t) \quad , \quad 0 < t < \infty \\ &u(0) = \dot{u}(0) = 0 \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &\ddot{u}(t) = f(t) \quad , \quad 0 < t < \infty \\ &u(0) = \varphi_0 \quad , \quad \dot{u}(0) = \varphi_1 \end{aligned} \tag{4.2}$$

are considered in the second section with emphasis placed on the error analysis and system formulation. In the next section, the temporal results are combined with spatial results in the development of a Sinc-Galerkin method for problems of the form

$$\begin{aligned} &\mathcal{L}u(x,t) \equiv \frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\partial^2}{\partial x^2}\left(\mathrm{EI}(x)\,\frac{\partial^2 u}{\partial x^2}(x,t)\right) = f(x,t) \quad , \quad \begin{matrix} a < x < b \\ t > 0 \end{matrix} \\ &u(a,t) = u(b,t) = 0 \quad , \quad t > 0 \\ &\frac{\partial u}{\partial x}(a,t) = \frac{\partial u}{\partial x}(b,t) = 0 \quad , \quad t > 0 \\ &u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0 \quad , \quad a \le x \le b \quad . \end{aligned} \tag{4.3}$$

Special attention will be given to the treatment of the variable coefficient $\mathrm{EI}(x)$, and methods for solving the discrete system will be discussed. This section includes several numerical examples which demonstrate various aspects of the method. In the fourth section, the earlier analysis is extended to the general problems

$$
\begin{aligned}
&\mathcal{L}u(x,t) \equiv \frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\partial^2}{\partial x^2}\left(\mathrm{EI}(x)\frac{\partial^2 u}{\partial x^2}(x,t)\right) = f(x,t) \quad , \quad \begin{matrix} a<x<b \\ t>0 \end{matrix} \\
&u(a,t) = \overline{\alpha}(t) \quad , \quad \left(\mathrm{EI}\frac{\partial^2 u}{\partial x^2}\right)(b,t) = \overline{\gamma}(t) \quad , \quad t>0 \\
&\frac{\partial u}{\partial x}(a,t) = \overline{\beta}(t) \quad , \quad \frac{\partial}{\partial x}\left(\mathrm{EI}\frac{\partial^2 u}{\partial x^2}\right)(b,t) = \overline{\delta}(t) \quad , \quad t>0 \\
&u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0 \quad , \quad a \le x \le b
\end{aligned}
\tag{4.4}
$$

and

$$
\begin{aligned}
&\mathcal{L}u(x,t) \equiv \frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\partial^2}{\partial x^2}\left(\mathrm{EI}(x)\frac{\partial^2 u}{\partial x^2}(x,t)\right) = f(x,t) \quad , \quad \begin{matrix} a<x<b \\ t>0 \end{matrix} \\
&u(a,t) = \overline{\alpha}(t) \quad , \quad \left(\mathrm{EI}\frac{\partial^2 u}{\partial x^2}\right)(b,t) = \overline{\gamma}(t) \quad , \quad t>0 \\
&\frac{\partial u}{\partial x}(a,t) = \overline{\beta}(t) \quad , \quad \frac{\partial}{\partial x}\left(\mathrm{EI}\frac{\partial^2 u}{\partial x}\right)(b,t) = \overline{\delta}(t) \quad , \quad t>0 \\
&u(x,0) = \varphi_0(x) \quad , \quad \frac{\partial u}{\partial t}(x,0) = \varphi_1(x) \quad , \quad a \le x \le b \quad .
\end{aligned}
\tag{4.5}
$$

Again, method implementation will be addressed and further numerical examples will be given to highlight properties of the method.

The Discrete System for the Temporal Problems

The fundamental matrices for the partial differential equations in the later sections contain contributions from both the spatial and temporal parts of the

operator $\mathcal{L}$. The first set of temporal matrices derives from the one-dimensional problem

$$\mathcal{L}u(t) \equiv \ddot{u}(t) = f(t) \quad , \quad 0 < t < \infty$$
$$u(0) = \dot{u}(0) = 0 \quad . \tag{4.6}$$

A Sinc-Galerkin method to approximate the solution of (4.6) can be developed as follows. Define the set of basis functions $\{S_j^*\}_{j=-M}^{N}$ by

$$S_j^*(t) = S(j,h) \circ \Upsilon(t) \tag{4.7}$$

where $\Upsilon : D_W \rightarrow D_S$ is given in (2.17). The approximate solution $u_m(t)$ is then defined by

$$u_m(t) = \sum_{j=-M}^{N} u_j S_j^*(t) \quad , \quad m + M + N + 1 \quad . \tag{4.8}$$

The m unknown coefficients $\{u_j\}$ in (4.8) are determined by orthogonalizing the residual with respect to the set of sinc functions $\{S_q^*\}_{q=-M}^{N}$. This leads to the analysis of

$$(\mathcal{L}u, S_q^*) = (f, S_q^*) \tag{4.9}$$

for $-M \leq q \leq N$. The inner product for (4.9) is defined to be

$$(F,G) = \int_0^\infty F(t)G(t)w^*(t)dt \quad . \tag{4.10}$$

and the weight is taken to be

$$w^*(t) = \frac{1}{\sqrt{\dot{\Upsilon}(t)}} \tag{4.11}$$

for reasons that are discussed in [31]. Approximations to the inner products in (4.9) are referenced in the following theorem. The proof of this theorem may be found in [31].

Theorem 4.1 *Let Υ conformally map the simply connected domain D_W onto D_S. Let $t_j = \Upsilon^{-1}(jh)$ and define the identities $\delta_{qj}^{(\ell)}$, $\ell = 0, 2$, as they are in (2.33)–(2.35) with ϕ replaced by Υ. Let N be given by (4.17) or (4.18) and take*

$$h = \left(\frac{\pi d}{\gamma M}\right)^{1/2} \tag{4.12}$$

where γ is a decay condition to be specified.

(a) If $fw^* \in B(D_W)$ then

$$\left| (f, S_q^*) - h \, \frac{f(t_q)}{\left(\dot{\Upsilon}(t_q)\right)^{3/2}} \right| \leq L_0 M^{-1/2} e^{-(\pi d \gamma M)^{1/2}}$$

where L_0 depends on f, w^* and d.

(b) Assume that u satisfies either

$$|u(t)[\dot{\Upsilon}(t)]^{1/2}| \leq L \begin{cases} t^\gamma & , \ t \in (0,1) \\ t^{-\delta} & , \ t \in [1, \infty) \end{cases} \tag{4.13}$$

or

$$|u(t)[\dot{\Upsilon}(t)]^{1/2}| \leq L \begin{cases} t^\gamma & , \ t \in (0,1) \\ e^{-\delta t} & , \ t \in [1, \infty) \end{cases} \tag{4.14}$$

If in addition $u \, \frac{d^2}{dt^2} (S_q^* w^*) \in B(D_W)$ and

$$B_T^{(2)} = \left\{ \dot{u}(S^* w^*) - u \, \frac{d}{dt} (S^* w^*) \right\} (t) \Big|_0^\infty = 0$$

then

$$\left| (\ddot{u}, S_q^*) - h \sum_{j=-M}^{N} u(t_j) \left[\frac{1}{h^2} \, \delta_{qj}^{(2)} (\dot{\Upsilon}(t_j))^{1/2} + (\dot{\Upsilon}(t_j))^{-1} \left\{ (\dot{\Upsilon}(t_j))^{-1/2} \right\}'' \right] \right|$$

$$\leq L_2 M e^{-(\pi d \gamma M)^{1/2}}$$

where L_2 depends on u, w^* and d.

The inner product approximations in Theorem 4.1 can then be used to obtain the Sinc-Galerkin system for the problem (4.6). Let $I^{(\ell)}$, $\ell = 0, 2$ denote the $m \times m$ matrices whose qj-th entry is $\delta_{qj}^{(\ell)}$ and let $\mathcal{D}(\varsigma)$ be the diagonal matrix with entries $\varsigma(x_{-M}), \ldots, \varsigma(x_N)$. With the usual definitions for $\vec{u}$ and $\vec{1}$ and the identity

$$\left(\dot{\Upsilon}(t)\right)^{-3/2} \left\{ \left(\dot{\Upsilon}(t)\right)^{-1/2} \right\}'' = -\frac{1}{4} \quad , \tag{4.15}$$

the system for the determination of the unknown coefficients $\{u_j\}$ is given by

$$A_t \vec{u} = \mathcal{D}\left(\frac{f}{(\dot{\Upsilon})^{3/2}} \right) \vec{1}$$

where

$$A_t = \left[\frac{1}{h^2} I^{(2)} - \frac{1}{4} I \right] \mathcal{D}\left((\dot{\Upsilon})^{1/2} \right) \quad . \tag{4.16}$$

With the system thus specified, the implementation of the method follows directly. The decay parameter γ is dictated by (4.13) and h is given by (4.12). If u decays algebraically as $t \to \infty$, then δ is given by (4.13) and N is chosen to be

$$N = \left[\left| \frac{\gamma}{\delta} M + 1 \right| \right] \quad . \tag{4.17}$$

If, on the other hand, the solution decays exponentially as in (4.14), then the truncation terms are asymptotically balanced by choosing

$$N = \left[\left| \frac{1}{h} \ell n \left(\frac{\gamma}{\delta} Mh \right) + 1 \right| \right] \tag{4.18}$$

(see [31]). The selection N in (4.18) significantly reduces the size of the discrete system with no loss of accuracy. Examples involving the choice of temporal parameters will be given in the later sections of this chapter.

Further Temporal Results

A second temporal problem of interest is

$$\mathcal{L}u(t) \equiv \ddot{u}(t) = f(t) \quad , \quad 0 < t < \infty$$

$$u(0) = \varphi_0 \quad , \quad \dot{u}(0) = \varphi_1 \quad . \tag{4.19}$$

The discussion of this problem will be brief since it parallels the development for (4.6) and mimics the techniques of Chapter 3. Here the basis functions $\{\psi_j^*\}_{j=-M-2}^{N}$ are defined by

$$\psi_j^*(t) = \begin{cases} b_{-M-2}^*(t) & , \ j = -M-2 \\ v^*(t) S_j^*(t) & , \ -M \le j \le N \\ b_{-M-1}^*(t) & , \ j = -M-1 \end{cases} \tag{4.20}$$

where $S_j^*(t) \equiv S(j,h) \circ \Upsilon(t)$ and the basis weight $v^*(t)$ is taken to be

$$v^*(t) = \frac{t}{(1+t)^2} \quad . \tag{4.21}$$

The temporal boundary basis functions are given by

$$b_{-M-1}^*(t) = -\frac{1}{(1+t)^2} + \frac{2}{1+t} \tag{4.22}$$

and

$$b_{-M-2}^*(t) = -\frac{1}{(1+t)^2} + \frac{1}{1+t} \quad . \tag{4.23}$$

The graphs of the boundary basis functions are found in Figure 16. By design the functions satisfy

$$b_{-M-1}^*(0) = 1$$

$$\dot{b}_{-M-1}^*(0) = \lim_{t \to \infty} b_{-M-1}^*(t) = 0 \tag{4.24}$$

and

$$\dot{b}_{-M-2}^*(0) = 1$$

$$b_{-M-2}^*(0) = \lim_{t \to \infty} b_{-M-2}^*(t) = 0 \quad . \tag{4.25}$$

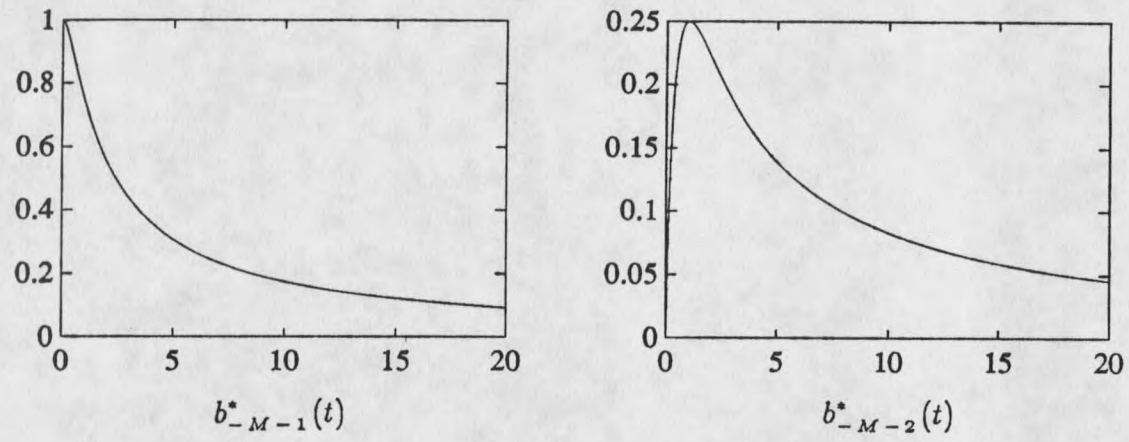


Figure 16. The Temporal Boundary Basis Functions.

The approximate solution $u_m(t)$ is taken to be

$$u_m(t) = \varphi_0 b^*_{-M-1}(t) + \varphi_1 b^*_{-M-2}(t) + \sum_{j=-M}^{N} u_j \psi^*_j(t) \tag{4.26}$$

where φ_0 and φ_1 are known and the coefficients $\{u_j\}$ are unknown. Note that $u_m(t)$ satisfies the initial conditions. The m unknown coefficients in (4.26) are determined by orthogonalizing the residual with respect to the set of sinc functions $\{S^*_q\}_{q=-M}^{N}$ thus yielding the discrete Galerkin system

$$\begin{aligned} 0 &= (\mathcal{L}u_m - f, S^*_q) \\ &= (\ddot{u}_m, S^*_q) - (f, S^*_q) \end{aligned} \tag{4.27}$$

where $-M \le q \le N$. The weighted inner product $(\cdot,\cdot)$ and weight $w^*(t)$ are defined in (4.10) and (4.11), respectively. If the definition

$$u_h(t) = \sum_{j=-M}^{N} u_j \psi^*_j(t) \tag{4.28}$$

is made, then (4.26) and (4.27) can be combined to yield the system

$$(\ddot{u}_h, S_q^*) = (\overline{f}, S_q^*) \quad , \quad -M \leq q \leq N \tag{4.29}$$

where

$$\overline{f}(t) \equiv f(t) - \varphi_0 \ddot{b}^*_{-M-1}(t) - \varphi_1 \ddot{b}^*_{-M-2}(t) \quad .$$

In order to use Theorem 4.1 to approximate the inner products in (4.29), that particular part of the true solution which is approximated by $u_h(t)$ must be isolated. This is accomplished via the formal change of variables

$$\mathcal{U}(t) = u(t) - \varphi_0 b^*_{-M-1}(t) - \varphi_1 b^*_{-M-2}(t) \quad . \tag{4.30}$$

The application of Theorem 4.1 (with $u(t)$ and $f(t)$ replaced by $\mathcal{U}(t)$ and $\overline{f}(t)$, respectively) to (4.29) yields the matrix system

$$A_t \vec{u} = \mathcal{D}\left(\frac{\overline{f}}{(\dot{\Upsilon})^{3/2}}\right)\vec{1} \tag{4.31}$$

where

$$A_t = \left[\frac{1}{h^2}\, I^{(2)} - \frac{1}{4}\, I\right] \mathcal{D}\left(v^*(\dot{\Upsilon})^{1/2}\right) \quad . \tag{4.32}$$

From (4.13) and (4.14) it follows that the decay conditions are either

$$\left|\mathcal{U}(t)[\dot{\Upsilon}(t)]^{1/2}\right| \leq L \begin{cases} t^\gamma & , \ t \in (0,1 \\ t^{-\delta} & , \ t \in [1,\infty) \end{cases} \tag{4.33}$$

or

$$\left|\mathcal{U}(t)[\dot{\Upsilon}(t)]^{1/2}\right| \leq L \begin{cases} t^\gamma & , \ t \in (0,1) \\ e^{-\delta t} & , \ t \in [1,\infty) \end{cases} \quad . \tag{4.34}$$

Once γ and δ have been specified, h and N are given by (4.12) and (4.17) or (4.18), respectively. Again, numerical examples involving the choice of temporal parameters will be given in the later sections of this chapters.

The Time-Dependent Problem with Fixed Boundary Conditions

In this section, a Sinc-Galerkin method is developed for fourth-order time-dependent problems of the form

$$
\begin{aligned}
\mathcal{L}u(x,t) &\equiv \frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\partial^2}{\partial x^2}\left(\mathrm{EI}(x)\frac{\partial^2 u}{\partial x^2}(x,t)\right) = f(x,t) \quad , \quad \begin{matrix} a < x < b \\ t > 0 \end{matrix} \\
u(a,t) &= u(b,t) = 0 \quad , \quad t > 0 \\
\frac{\partial u}{\partial x}(a,t) &= \frac{\partial u}{\partial x}(b,t) = 0 \quad , \quad t > 0 \\
u(x,0) &= \frac{\partial u}{\partial t}(x,0) = 0 \quad , \quad a \le x \le b \quad .
\end{aligned} \tag{4.35}
$$

Recall that (4.35) can be used to model a beam with fixed supports at $x = a$ and $x = b$.

A Sinc-Galerkin method to approximate the solution to (4.35) can be formulated as follows. Select the set of basis functions

$$\{S_i(x)S_j^*(t)\}$$

($S_i(x)$ and $S_j^*(t)$ are defined in (2.21) and (4.7), respectively) and define an approximate solution to (4.35) by way of the tensor product expansion

$$
u_{m_x m_t}(x,t) = \sum_{i=-M_x}^{N_x}\sum_{j=-M_t}^{N_t} u_{ij}S_i(x)S_j^*(t) \quad , \quad \begin{matrix} m_x = M_x + N_x + 1 \\ m_t = M_t + N_t + 1 \end{matrix} \quad . \tag{4.36}
$$

The $m_x \cdot m_t$ unknown coefficients $\{u_{ij}\}$ are determined by orthogonalizing the residual with respect to the set of sinc functions $\{S_p(x)S_q^*(t)\}_{p=-M_x,\ldots,N_x}^{q=-M_t,\ldots,N_t}$. This yields the discrete Galerkin system

$$\left(\mathcal{L}u_{m_x m_t} - f, S_p S_q^*\right) = 0 \tag{4.37}$$

where $-M_x \le p \le N_x$ and $-M_t \le q \le N_t$. The inner product $(\cdot,\cdot)$ is taken to be

$$(F,G) = \int_0^\infty \int_a^b F(x,t)G(x,t)w(x,t)dxdt \tag{4.38}$$

where

$$w(x,t) = w(x)w^*(t) \quad . \tag{4.39}$$

The one-dimensional weights $w(x)$ and $w^*(t)$ are chosen to satisfy the criteria that were presented in Chapters 2 and 3 and in the second section of this chapter.

The expressions (4.36)-(4.38) are combined to give the discrete system

$$\begin{aligned} &\int_0^\infty \int_a^b \frac{\partial^2}{\partial t^2} \left(u_{m_x m_t}(x,t)\right) S_p(x) S_q^*(t) w(x) w^*(t) dx dt \\ &+ \int_0^\infty \int_a^b \frac{\partial^2}{\partial x^2} \left(\mathrm{EI}(x) \frac{\partial^2}{\partial x^2} u_{m_x m_t}(x,t) \right) S_p(x) S_q^*(t) w(x) w^*(t) dx dt \\ &= \int_0^\infty \int_a^b f(x,t) S_p(x) S_q^*(t) w(x) w^*(t) dx dt \end{aligned} \tag{4.40}$$

for $-M_x \leq p \leq N_x$ and $-M_t \leq q \leq N_t$. The one-dimensional results in Theorems 2.6, 3.6 and 4.1 can then be used to transform (4.40) into a finite dimensional matrix equation. Appropriate integration by parts and the application of the theorems yields the discrete Sinc-Galerkin system

$$A(\mathrm{EI}) U C_t^T + C_x U A_t^T = G \tag{4.41}$$

where

$$C_t \equiv \mathcal{D}\left(\frac{w^*}{\dot{\Upsilon}}\right) \quad , \tag{4.42}$$

$$C_x \equiv \mathcal{D}\left(\frac{w}{\phi'}\right) \tag{4.43}$$

and

$$G \equiv \mathcal{D}\left(\frac{w}{\phi'}\right) F \mathcal{D}\left(\frac{w^*}{\dot{\Upsilon}}\right) \quad . \tag{4.44}$$

If the temporal weight is taken to be $w^*(t) = \frac{1}{\sqrt{\dot{\Upsilon}(t)}}$ (see (4.11)), then the $m_t \times m_t$ matrix A_t is given by

$$A_t = \left[\frac{1}{h_t^2}\, I^{(2)} - \frac{1}{4}\, I\right] \mathcal{D}\left((\dot{\Upsilon})^{1/2}\right) \tag{4.45}$$

as shown in (4.16). Furthermore, the $m_x \times m_t$ matrices U and F are defined componentwise by

$$[U]_{ij} = u_{ij}$$

and

$$[F]_{pq} = f(x_p, t_q) \quad .$$

It should be noted that the ordering of the coefficients u_{ij} in U mimics that used in most standard time-differencing schemes. This is a matter of convenience since the Sinc-Galerkin method is not bound by any specific ordering of the grid.

The structure of the $m_x \times m_x$ matrix $A(\mathrm{EI})$ depends on the scheme that is used to discretize $\mathrm{EI}(x)$. If the parameter is fully differentiated as was done in Chapter 2, then $A(\mathrm{EI})$ is given by (2.76). In this case, one would take $\eta(x) = \mathrm{EI}(x)$, $\rho(x) = 2\mathrm{EI}'(x)$, $\nu(x) = \mathrm{EI}''(x)$, and $\tau(x) = \sigma(x) = 0$ when defining the diagonal matrices in (2.76). If, on the other hand, $\mathrm{EI}(x)$ is approximated by a linear combination of sinc and algebraic basis functions as done in the sixth section of Chapter 3, then $A(\mathrm{EI})$ is given by (3.113).

Various methods exist for solving the equation (4.41). As shown in [24], (4.41) is algebraically equivalent to the system

$$\mathcal{A}\vec{u} \equiv \{C_t \otimes A(\mathrm{EI}) + A_t \otimes C_x\} \overrightarrow{\mathrm{co}(U)} = \overrightarrow{\mathrm{co}(G)} \tag{4.46}$$

where the tensor or Kronecker product of an $m \times n$ matrix E with a $p \times q$ matrix G is defined by

$$E \otimes G \equiv [e_{ij} G]_{mp \times nq} \quad . \tag{4.47}$$

The vector $\vec{u} \equiv \overrightarrow{\mathrm{co}(U)}$ is the concatenation of the $m_x \times m_t$ matrix U obtained by successively "stacking" the columns of U, one upon another, to obtain an $m_x m_t \times 1$ vector.

The system (4.46) can be solved directly via any of the decomposition methods that are available for linear systems. Although this system is easily formulated, the fact that $\mathcal{A}$ is very large ($m_x \cdot m_t \times m_x \cdot m_t$) and not banded causes this method to be impractical in many problems.

A second method for solving the system (4.41) is the diagonalization procedure that is described in [31], [39]. Although this method works very well in many problems, it has the following disadvantages here; (i) it requires that both of the matrices $E = \mathcal{D}\left(\frac{\phi'}{w}\right) A(\mathrm{EI})$ and $H = A_t^T \mathcal{D}\left(\frac{\dot{\Upsilon}}{w^*}\right)$ be diagonalized, and (ii) it requires that the eigenvalues of E and H be of comparable magnitudes so that they can be added without loss of accuracy. Problem (i) is primarily one of computational complexity. The second disadvantage results from the fact that in fourth-order problems with high order boundary conditions, the eigenvalues of E and H will differ in size by as much as 25 orders of magnitude. In these cases, Fortran quadruple precision is needed to maintain the method's accuracy. Because of these disadvantages, this method of solution is not used in this work.

The third procedure depends on the following theorem from [13, page 396].

Theorem 4.2 (Generalized Schur Decomposition) *Let A and B be $n \times n$ matrices. Then there exist unitary matrices Q and Z such that*

$$Q^* A Z = P \quad , \quad Q^* B Z = R$$

where P and R are upper triangular.

The problem (4.41) can then be reduced as follows. As guaranteed by Theorem 4.2, there exist unitary matrices Q_1, Z_1, Q_2 and Z_2 such that

$$Q_1^* A(\mathrm{EI}) Z_1 = P \quad , \quad Q_1^* C_x Z_1 = R \qquad (P, R \text{ upper triangular})$$

and

$$Q_2^* C_t Z_2 = S \quad , \quad Q_2^* A_t Z_2 = T \qquad (S, T \text{ upper triangular}) \quad .$$

If $Y = Z_1^* U Z_2$ and $C = Q_1^* G Q_2$, then (4.41) transforms to

$$PYT^* + RYS^* = C \quad .$$

By comparing the k-th columns, one finds that

$$P \sum_{j=k}^{n} t_{kj} y_j + R \sum_{j=k}^{n} s_{kj} y_j = c_k \tag{4.48}$$

which yields

$$(t_{kk} P + s_{kk} R) y_k = c_k - P \sum_{j=k+1}^{n} t_{kj} y_j - R \sum_{j=k+1}^{n} s_{kj} y_j \tag{4.49}$$

(for convenience, it is assumed that all matrices are $n \times n$ and indexed from 1 to n). With the assumption that the matrix $(t_{kk} P + s_{kk} R)$ is nonsingular, the problem (4.49) is easily solved by recursively solving triangular systems.

Because this algorithm is both robust and efficient, it will be used throughout this chapter when solving problems of the form (4.41). It should be noted that a "real" version of this algorithm also exists (see [12], [13]). In that case, Q_1, R_1, Q_2 and R_2 are orthogonal with P, S quasi-upper triangular and R, T upper triangular.

Numerical Implementation and Examples (Fixed Boundaries)

The three examples reported in this section illustrate various features of the Sinc-Galerkin method when applied to problems of the form (4.35). In the first example, $\mathrm{EI}(x)$ is constant and the methods of Chapter 2 are used to obtain $A(\mathrm{EI})$. The weight is taken to be $w(x) = (\phi'(x))^{-3/2}$ and the decay conditions (2.92) and (4.14) translate to the bound

$$|u(x,t)| \leq L(x-a)^{\alpha+3/2}(b-x)^{\beta+3/2}t^{\gamma+1/2}e^{-\delta t} \quad . \tag{4.50}$$

In the other examples, $\mathrm{EI}(x)$ is variable and the methods of the sixth section of Chapter 3 are used to discretize the spatial part of the operator $\mathcal{L}$. The weight is again taken to be $w(x) = (\phi'(x))^{-3/2}$ and the decay conditions are

$$|\mathcal{E}\mathcal{I}(x)| \leq L(x-a)^{\alpha}(b-x)^{\beta} \quad , \tag{4.51}$$

$$|\mathcal{E}\mathcal{I}(x)u(x,t)| \leq L(x-a)^{\alpha+3/2}(b-x)^{\beta+3/2}t^{\gamma+1/2}e^{-\delta t} \tag{4.52}$$

and

$$|\tilde{b}_k(x)u(x,t)| \leq L(x-a)^{\alpha+3/2}(b-x)^{\beta+3/2}t^{\gamma+1/2}e^{-\delta t} \tag{4.53}$$

where $\mathcal{E}\mathcal{I}(x)$ and $\tilde{b}_k(x)$ are defined in (3.128) and (3.91)–(3.94), respectively (see also (3.129), (3.133), and (3.134)).

For the problems considered, the angle d shown in Figure 3 is taken to be $\frac{\pi}{2}$. Once M_x is chosen, the asymptotic errors that are given in the third section of Chapter 2 and the sixth section of Chapter 3 are balanced by choosing the remaining stepsizes and summation limits as follows:

$$h_x = \frac{\pi}{\sqrt{2\alpha M_x}} \quad , \tag{4.54}$$

(4.55) $$h = h_t = h_x \quad ,$$

(4.56) $$N_x = \left[\left|\frac{\alpha}{\beta}\, M_x + 1\right|\right] \quad ,$$

(4.57) $$M_t = \left[\left|\frac{\alpha}{\gamma}\, M_x + 1\right|\right]$$

and

(4.58) $$N_t = \left[\left|\frac{1}{h_t}\, \ell n\left(\frac{\gamma}{\delta}\, M_t h_t\right) + 1\right|\right] \quad .$$

Again it is noted that if $\frac{\alpha}{\beta}\, M_x$ and/or $\frac{\alpha}{\gamma}\, M_x$ are integers, then (4.56) and/or (4.57) can be replaced by these integer values.

In the examples, both the maximum absolute error over the set of sinc gridpoints and the maximum absolute error on a uniform grid will be reported. The errors over the set of sinc gridpoints

(4.59) $$S = \{(x_i, t_j)\}_{i=-M_x,\ldots,N_x}^{j=-M_t,\ldots,N_t} \quad ; \quad x_i = \frac{be^{ih_x} + a}{1 + e^{ih_x}} \quad , \quad t_j = e^{jh_t}$$

are reported as

(4.60) $$\|E_S^{M_x}(h_x, h_t)\| = \max_{\substack{-M_x \le i \le N_x \\ -M_t \le j \le N_t}} |u(x_i, t_j) - u_{m_x m_t}(x_i, t_j)|$$

and the maximum errors over the set of uniform gridpoints $\left(\ell_x = \frac{b-a}{50} \ , \ \ell_t = \frac{1}{5}\right)$

(4.61) $$U = \{(z_p, s_q)\}_{p=1,\ldots,50}^{q=1,\ldots,50} \quad ; \quad z_p = a + \ell_x p \quad , \quad s_q = \ell_t q$$

are reported as

(4.62) $$\|E_U^{M_x}(\ell_x, \ell_t)\| = \max_{\substack{0 \le p \le 50 \\ 0 \le q \le 50}} |u(z_p, s_q) - u_{m_x m_t}(z_p, s_q)| \quad .$$

The dependence of both errors on the number of sinc basis functions is indicated with the superscript M_x. The exponential convergence is again verified by comparing $\left(\|E_U^{M_x/2}(\ell_x,\ell_t)\|\right)^{\sqrt{2}}$ and $\|E_U^{M_x}(\ell_x,\ell_t)\|$ (see (2.111)).

Example 4.3

$$\frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\partial^4 u}{\partial x^4}(x,t) = f(x,t) \quad , \quad \begin{matrix} 0 < x < 1 \\ t > 0 \end{matrix}$$

$$u(0,t) = u(1,t) = 0 \quad , \quad t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0 \quad , \quad t > 0$$

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0 \quad , \quad 0 \le x \le 1$$

The function $f(x,t)$ is consistent with the solution $u(x,t) = [x(1-x)]^{7/2} t^{5/2} e^{-t}$ which has algebraic singularities at $x = 0$ and 1. The decay condition (4.50) dictates the parameter choices $\alpha = \beta = \gamma = 2$ and $\delta = 1$. The asymptotic error rate $\mathcal{O}\left(e^{-\pi\sqrt{M_x}}\right)$ is maintained as shown in Table 11. Figure 17 shows the true solution $u(x,t)$ while Figure 18 shows the true and approximate solutions at the time slice $t = 2$. The results from this example can be compared with those in Example 2.10.

M_x	N_x	M_t	N_t	$\|E_S^{M_x}(h_x,h_t)\|$	$\|E_U^{M_x}(\ell_x,\ell_t)\|$	$\left(\|E_U^{M_x/2}(\ell_x,\ell_t)\|\right)^{\sqrt{2}}$
2	2	2	2	$.3984-3$	$.1090-2$	———
4	4	4	3	$.2040-4$	$.3184-3$	$.6461-4$
8	8	8	4	$.9361-5$	$.3618-4$	$.1134-4$
16	16	16	7	$.1372-6$	$.1788-5$	$.5233-6$
32	32	32	11	$.1742-9$	$.1338-7$	$.7441-8$

Table 11. Errors on the Sinc Grid S and the Uniform Grid U for Example 4.3.

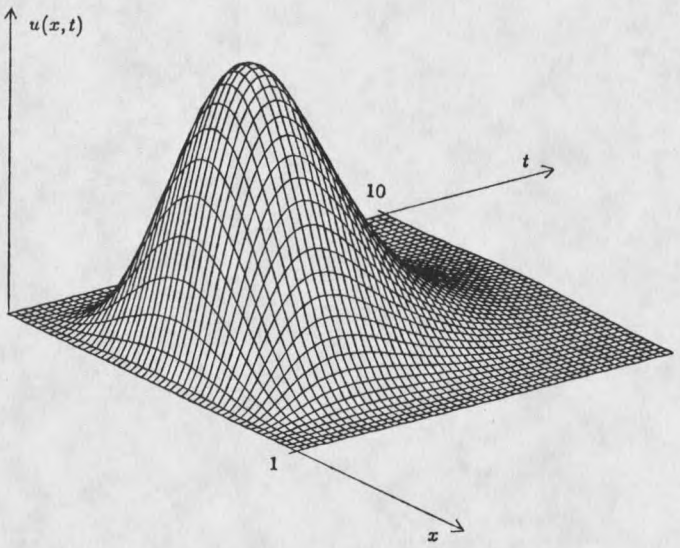


Figure 17. True Solution to Example 4.3.

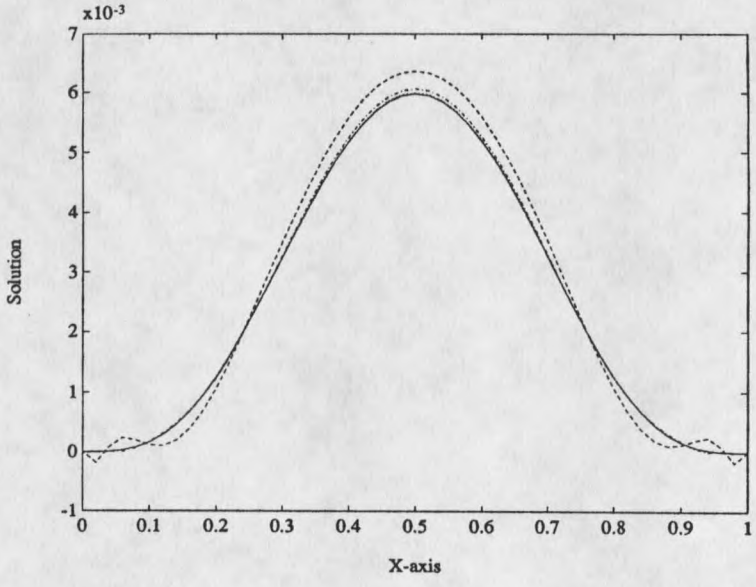


Figure 18. True and Approximate Solutions to Example 4.3 at Time $t = 2$ $- - -$ $(M_x = 2)$, $- \cdot -$ $(M_x = 4)$, $\cdots$ $(M_x = 8)$, — (True).

Example 4.4 Consider the problem

$$\mathcal{L}u(x,t) = f(x,t) \quad , \quad 0 < x < 1 \ , \ t > 0$$

$$u(0,t) = u(1,t) = 0 \quad , \quad t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0 \quad , \quad t > 0$$

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0 \quad , \quad 0 \leq x \leq 1$$

where $\mathrm{EI}(x) = 1 + \sin(\pi x)$ and $f(x,t)$ is consistent with the true solution $u(x,t) = x(1-x)\sin(4\pi x)t^2 e^{-t}$. The decay conditions (4.51)–(4.53) dictate the parameter choices $\alpha = \beta = 1$, $\gamma = 3/2$, $\delta = 1$. Hence the asymptotic error is $\mathcal{O}\left(e^{-\pi\sqrt{M_x/2}}\right)$. The errors and solutions are reported in Table 12 and Figures 19 and 20, respectively. The results from this example can be compared with those in Example 3.7.

M_x	N_x	M_t	N_t	$\|E_S^{M_x}(h_x,h_t)\|$	$\|E_U^{M_x}(\ell_x,\ell_t)\|$	$\left(\|E_U^{M_x/2}(\ell_x,\ell_t)\|\right)^{\sqrt{2}}$
4	4	3	2	$.4302-0$	$.4875-0$	———
8	8	6	3	$.3131-0$	$.3539-0$	$.3620-0$
16	16	11	4	$.5479-2$	$.1109-1$	$.2302-0$
32	32	22	7	$.7838-5$	$.1288-3$	$.1718-2$

Table 12. Errors on the Sinc Grid S and the Uniform Grid U for Example 4.4.

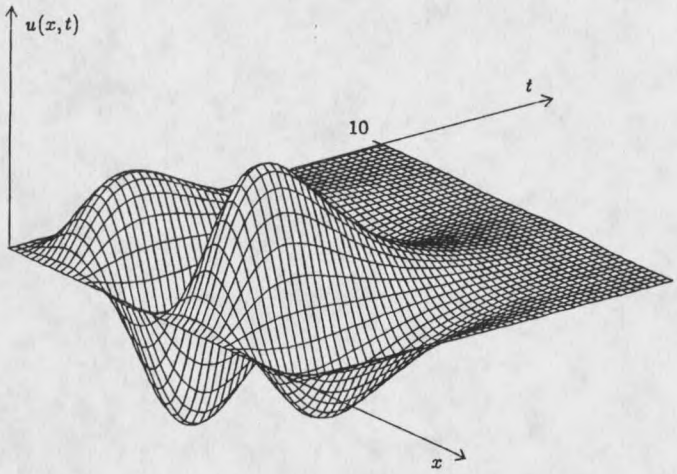


Figure 19. True Solution to Example 4.4.

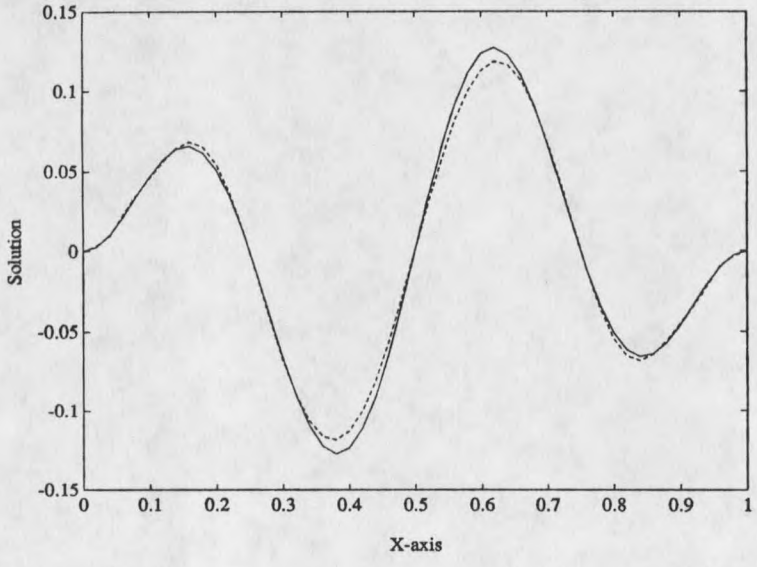


Figure 20. True and Approximate Solutions to Example 4.4 at Time $t = 2$ $- - -$ ($M_x = 16$), — (True).

Example 4.5 This example differs from Example 4.4 only in the choice of the parameter $\mathrm{EI}(x)$. Both examples will be reconsidered in Chapter 5 at which point the parameter $\mathrm{EI}(x)$ will be unknown. Thus let

$$\mathcal{L}u(x,t) = f(x,t) \quad , \quad 0 < x < 1 \, , \; t > 0$$

$$u(0,t) = u(1,t) = 0 \quad , \quad t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0 \quad , \quad t > 0$$

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0 \quad , \quad 0 \le x \le 1$$

where $\mathrm{EI}(x) = 1 + \frac{1}{4} e^{-40(x-1/3)^2}$ and $f(x,t)$ is consistent with the true solution $u(x,t) = x(1-x)\sin(4\pi x)t^2 e^{-t}$. In this problem, it is more difficult to analytically determine the spatial decay parameters. Good results are obtained however with the choices $\alpha = \beta = \gamma = \frac{3}{2}$, $\delta = 1$. The asymptotic error rate $\mathcal{O}\left(e^{-\pi\sqrt{3M_x/2}}\right)$ is maintained as shown in Table 13. The true solution is again shown in Figure 19.

M_x	N_x	M_t	N_t	$\|E_S^{M_x}(h_x,h_t)\|$	$\|E_U^{M_x}(\ell_x,\ell_t)\|$	$\left(\|E_U^{M_x/2}(\ell_x,\ell_t)\|\right)^{\sqrt{2}}$
4	4	4	2	$1.1838-0$	$1.8103-0$	———
8	8	8	4	$.2700-0$	$.2980-0$	———
16	16	16	6	$.1359-1$	$.1457-1$	$.1805-0$
32	32	32	9	$.1950-3$	$.1950-3$	$.2528-2$

Table 13. Errors on the Sinc Grid S and the Uniform Grid U for Example 4.5.

The Time-Dependent Problem with Cantilever Boundary Conditions

In this section, a Sinc-Galerkin method is developed for time-dependent problems of the form

$$
\begin{aligned}
&\mathcal{L}u(x,t) \equiv \frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\partial^2}{\partial x^2}\left(\mathrm{EI}(x)\frac{\partial^2 u}{\partial x^2}(x,t)\right) = f(x,t) \quad , \quad \begin{matrix} a<x<b \\ t>0 \end{matrix} \\
&u(a,t) = \overline{\alpha}(t) \quad , \quad \left(\mathrm{EI}\frac{\partial^2 u}{\partial x^2}\right)(b,t) = \overline{\gamma}(t) \quad , \quad t>0 \\
&\frac{\partial u}{\partial x}(a,t) = \overline{\beta}(t) \quad , \quad \frac{\partial}{\partial x}\left(\mathrm{EI}\frac{\partial^2 u}{\partial x^2}\right)(b,t) = \overline{\delta}(t) \quad , \quad t>0 \\
&u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0 \quad , \quad a \le x \le b
\end{aligned}
\tag{4.63}
$$

and

$$
\begin{aligned}
&\mathcal{L}u(x,t) = f(x,t) \quad , \quad a<x<b \ , \ t>0 \\
&u(a,t) = \overline{\alpha}(t) \quad , \quad \left(\mathrm{EI}\frac{\partial^2 u}{\partial x^2}\right)(b,t) = \overline{\gamma}(t) \quad , \quad t>0 \\
&\frac{\partial u}{\partial x}(a,t) = \overline{\beta}(t) \quad , \quad \frac{\partial}{\partial x}\left(\mathrm{EI}\frac{\partial^2 u}{\partial x^2}\right)(b,t) = \overline{\delta}(t) \quad , \quad t>0 \\
&u(x,0) = \varphi_0(x) \quad , \quad \frac{\partial u}{\partial x}(x,0) = \varphi_1(x) \quad , \quad a \le x \le b \quad .
\end{aligned}
\tag{4.64}
$$

As shown in Chapter 1, (4.63) and (4.64) with $\overline{\gamma}(t) = \overline{\delta}(t) = 0$ can be used to model a beam with cantilever boundary conditions.

Consider first the problem (4.63). The basis for this problem is taken to be

$$\{\psi_i(x)S_j^*(t)\}$$

where $\psi_i(x)$ and $S_j^*(t)$ are defined in (3.49) and (4.7), respectively. Here the

approximate solution is taken to be

$$
\begin{aligned}
u_{m_x m_t}(x,t) &= \sum_{i=-M_x}^{N_x} \sum_{j=-M_t}^{N_t} u_{ij}\psi_i(x)S_j^*(t) \\
&+ \sum_{j=-M_t}^{N_t} S_j^*(t)\{u_{-M_x-3,j}\psi_{-M_x-3}(x) + u_{-M_x-4,j}\psi_{-M_x-4}(x) \\
&\qquad + u_{N_x+1,j}\psi_{N_x+1}(x) + u_{N_x+2,j}\psi_{N_x+2}(x)\} \\
&+ \{\overline{\alpha}(t)\psi_{-M_x-1}(x) + \overline{\beta}(t)\psi_{-M_x-2}(x) \\
&+ \tilde{\gamma}(t)\psi_{N_x+3}(x) + \tilde{\delta}(t)\psi_{N_x+4}(x)\}
\end{aligned}
\tag{4.65}
$$

where

$$
\tilde{\gamma}(t) = \frac{1}{\mathrm{EI}(b)}\,\overline{\gamma}(t) \tag{4.66}
$$

and

$$
\tilde{\delta}(t) = \frac{1}{\mathrm{EI}(b)}\,\overline{\delta}(t) - \frac{\mathrm{EI}'(b)}{[\mathrm{EI}(b)]^2}\,\overline{\gamma}(t) \quad . \tag{4.67}
$$

It should be noted that the approximate solution does satisfy the boundary conditions in (4.63).

The $(m_x+4)\cdot m_t$ unknowns $\{u_{ij}\}$ in (4.65) are determined by orthogonalizing the residual with respect to the sinc functions $\{S_p(x)S_q^*(t)\}_{p=-M_x-2,\ldots,N_x+2}^{q=-M_t,\ldots,N_t}$. This yields the discrete system

$$
(\mathcal{L}u_{m_x m_t} - f, S_p S_q^*) = 0 \quad , \quad \begin{matrix} -M_x - 2 \le p \le N_x + 2 \\ -M_t \le q \le N_t \end{matrix} \tag{4.68}
$$

where $(\cdot,\cdot)$ is defined in (4.38). If (4.68) is expanded and the known terms are

collected on the right-hand side, then the system may be written as

$$
\begin{aligned}
&\left(\mathcal{L} \sum_{i=-M_x}^{N_x} \sum_{j=-M_t}^{N_t} u_{ij} v S_i S_j^*, S_p S_q^* \right) \\
&+ \left(\mathcal{L} \sum_{j=-M_t}^{N_t} S_j^* \left\{ u_{-M_x-3,j} b_{-M_x-3} + u_{-M_x-4,j} b_{-M_x-4} \right. \right. \\
&\qquad \left. \left. + u_{N_x+1,j} b_{N_x+1} + u_{N_x+2,j} b_{N_x+2} \right\}, S_p S_q^* \right) \\
&= \left(\overline{f}, S_p S_q^* \right)
\end{aligned} \tag{4.69}
$$

where $-M_x - 2 \leq p \leq N_x + 2$ and $-M_t \leq q \leq N_t$. Here

$$
\begin{aligned}
\overline{f}(x,t) \equiv f(x,t) &- \mathcal{L}\left(\overline{\alpha}(t) b_{-M_x-1}(x)\right) - \mathcal{L}\left(\overline{\beta}(t) b_{-M_x-2}(x)\right) \\
&- \mathcal{L}\left(\tilde{\gamma}(t) b_{N_x+3}(x)\right) - \mathcal{L}\left(\tilde{\delta}(t) b_{N_x+4}(x)\right) .
\end{aligned}
$$

The one-dimensional results from Chapters 2, 3 and 4 can then be used to transform (4.69) into a finite dimensional matrix equation. Appropriate integration by parts and the application of earlier theorems yields the discrete Sinc-Galerkin system

$$
A(\mathrm{EI}) U C_t^T + C_x M U A_t^T = G \tag{4.70}
$$

where C_t, C_x and A_t are defined in (4.42), (4.43) and (4.45), respectively, and

$$
G = \mathcal{D}\left(\frac{w}{\phi'}\right) \overline{F} \mathcal{D}\left(\frac{w^*}{\dot{\Upsilon}}\right) . \tag{4.71}
$$

The $(m_x + 4) \times m_t$ matrices U and $\overline{F}$ are defined componentwise by

$$
[U]_{ij} = u_{ij} \tag{4.72}
$$

and

$$
[\overline{F}]_{ij} = \overline{f}(x_i, t_j) . \tag{4.73}
$$

The $(m_x+4)\times(m_x+4)$ matrix $A(\mathrm{EI})$ is given by either (3.83) or (3.158) depending on which scheme is used to discretize EI. Finally, the $(m_x+4)\times(m_x+4)$ matrix M has the form

$$M = \left[\begin{array}{c|c|c|c|c} & & 0 & & \\ & & \text{---} & & \\ \hat{b}_{L2} & \hat{b}_{L1} & D_M & \hat{b}_{R1} & \hat{b}_{R2} \\ & & \text{---} & & \\ & & 0 & & \end{array}\right] \tag{4.74}$$

where the $m_x \times m_x$ submatrix D_M and the $(m_x+4)\times 1$ vectors are given by

$$D_M = \mathcal{D}(v) \quad ,$$

$$\hat{b}_{L2} = \mathcal{D}\left(b_{-M_x-4}\right)\vec{1} \quad ,$$

$$\hat{b}_{L1} = \mathcal{D}\left(b_{-M_x-3}\right)\vec{1} \quad ,$$

$$\hat{b}_{R1} = \mathcal{D}\left(b_{N_x+1}\right)\vec{1}$$

and

$$\hat{b}_{R2} = \mathcal{D}\left(b_{N_x+2}\right)\vec{1} \quad .$$

As before, the system (4.70) is solved via the generalized Schur algorithm (4.49) as discussed in the last section.

Numerical Implementation and Examples (Zero Initial Conditions)

The following two examples illustrate the use of the Sinc-Galerkin method for problems with cantilever boundary conditions and zero initial conditions. In the first example, $\mathrm{EI}(x)$ is constant and the methods of the fifth section of Chapter 3 are used to obtain $A(\mathrm{EI})$. The spatial weight $w(x) = (\phi'(x))^{-1/2}$ and the decay conditions (3.75) and (4.14) yield the decay requirement

$$|u(x,t)| \leq L(x-a)^{\alpha+5/2}(b-x)^{\beta+5/2}t^{\gamma+1/2}e^{-\delta t} \quad . \tag{4.75}$$

Here $U(x,t)$ denotes that part of the true solution which is approximated by

$$u_h(x,t) = \sum_{i=-M_x}^{N_x} \sum_{j=-M_t}^{N_t} u_{ij}\psi_i(x)S_j^*(t)$$

(see also (3.164)).

In the second example, $\mathrm{EI}(x)$ is variable and the methods of the sixth section of Chapter 3 are used to discretize the spatial part of the operator $\mathcal{L}$. The weight $w(x) = 1$ yields the decay condition (4.51) as well as

$$|\mathcal{E}I(x)U(x,t)| \leq L(x-a)^{\alpha+3}(b-x)^{\beta+3}t^{\gamma+1/2}e^{-\delta t} \quad , \tag{4.76}$$

$$|\tilde{b}_k(x)U(x,t)| \leq L(x-a)^{\alpha+3}(b-x)^{\beta+3}t^{\gamma+1/2}e^{-\delta t} \tag{4.77}$$

and

$$|b_i''(x)\mathcal{E}I(x)| \leq L(x-a)^{\alpha+3}(b-x)^{\beta+3}t^{\gamma+1/2}e^{-\delta t} \quad . \tag{4.78}$$

Here $\mathcal{E}I(x)$, $\tilde{b}_k(x)$ and $b_i(x)$ are defined in (3.128), (3.91)–(3.94), and (3.51)–(3.58), respectively.

The stepsizes and summation limits are chosen in the same manner as in the last section. The reporting of errors is also the same.

Example 4.6

$$\frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\partial^4 u}{\partial x^4}(x,t) = f(x,t) \quad , \quad 0 < x < 1 \, , \; t > 0$$

$$u(0,t) = 0 \quad , \quad \frac{\partial^2 u}{\partial x^2}(1,t) = 2t^2e^{-t} \quad , \quad t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad , \quad \frac{\partial^3 u}{\partial x^3}(1,t) = 0 \quad , \quad t > 0$$

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0 \quad , \quad 0 \leq x \leq 1 \quad .$$

The true solution $u(x,t) = \left[(x\,\ell n(x))^4 + x^2\right] t^2 e^{-t}$ dictates the forcing function $f(x,t)$. The decay condition (4.75) yields the parameter choices $\alpha = \beta = \gamma = \frac{3}{2}$, $\delta = 1$. Hence the asymptotic error is $\mathcal{O}\left(e^{-\pi\sqrt{3M_x}/2}\right)$. The errors and solutions are reported in Table 14 and Figures 21 and 22, respectively.

M_x	N_x	M_t	N_t	$\|E_S^{M_x}(h_x,h_t)\|$	$\|E_U^{M_x}(\ell_x,\ell_t)\|$	$\left(\|E_U^{M_x/2}(\ell_x,\ell_t)\|\right)^{\sqrt{2}}$
4	4	4	2	$.1005-0$	$.1096-0$	———
8	8	8	4	$.3165-1$	$.3202-1$	$.4386-1$
16	16	16	6	$.2809-2$	$.2878-2$	$.7697-2$
32	32	32	9	$.3533-4$	$.3567-4$	$.2550-3$

Table 14. Errors on the Sinc Grid S and the Uniform Grid U for Example 4.6.

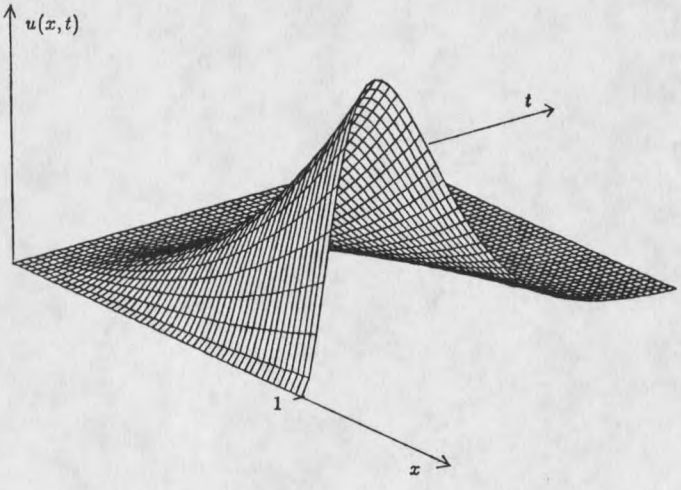


Figure 21. True Solution to Example 4.6.

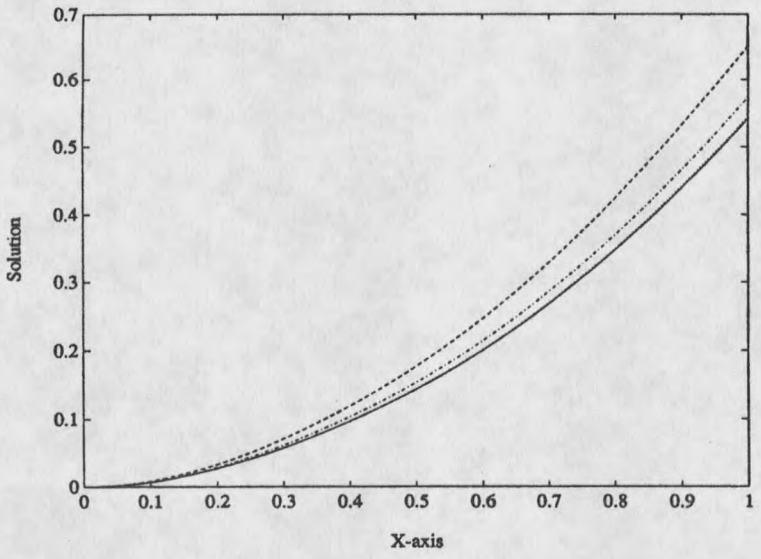


Figure 22. True and Approximate Solutions to Example 4.6 at Time $t = 2$ $- - -$ $(M_x = 4)$, $- \cdot -$ $(M_x = 8)$, $\cdots$ $(M_x = 16)$, — (True).

Example 4.7 Consider the problem

$$\mathcal{L}u(x,t) = f(x,t) \quad , \quad 0 < x < 1 \ , \ t > 0$$

$$u(0,t) = 0 \quad , \quad \left(\mathrm{EI}\frac{\partial^2 u}{\partial x^2}\right)(1,t) = -8\pi t^2 e^{-t} \quad , \quad t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = t^2 e^{-t} \quad , \quad \frac{\partial}{\partial x}\left(\mathrm{EI}\frac{\partial^2 u}{\partial x^2}\right)(1,t) = 8\pi(\pi - 3)t^2 e^{-t} \quad , \quad t > 0$$

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0 \quad , \quad 0 \le x \le 1$$

where $\mathrm{EI}(x) = 1 + \sin(\pi x)$ and $f(x,t)$ is consistent with the true solution $u(x,t) = [x(1-x)\sin(4\pi x) + x]t^2 e^{-t}$. The parameter choices $\alpha = \beta = 1$, $\gamma = \frac{3}{2}$, $\delta = 1$ yield the asymptotic error rate $\mathcal{O}\left(e^{-\pi\sqrt{M_x/2}}\right)$ (see the discussion in Example 3.10). The errors and solutions are reported in Table 15 and Figures 23 and 24, respectively. The results from this example can also be compared to those in Example 4.4.

M_x	N_x	M_t	N_t	$\|E_S^{M_x}(h_x,h_t)\|$	$\|E_U^{M_x}(\ell_x,\ell_t)\|$	$\left(\|E_U^{M_x/2}(\ell_x,\ell_t)\|\right)^{\sqrt{2}}$
8	8	6	3	18.8593	23.6535	———
16	16	11	4	$.1144-0$	$.1144-0$	———
32	32	22	7	$.2846-3$	$.3249-3$	$.4660-1$

Table 15. Errors on the Sinc Grid S and the Uniform Grid U for Example 4.7.

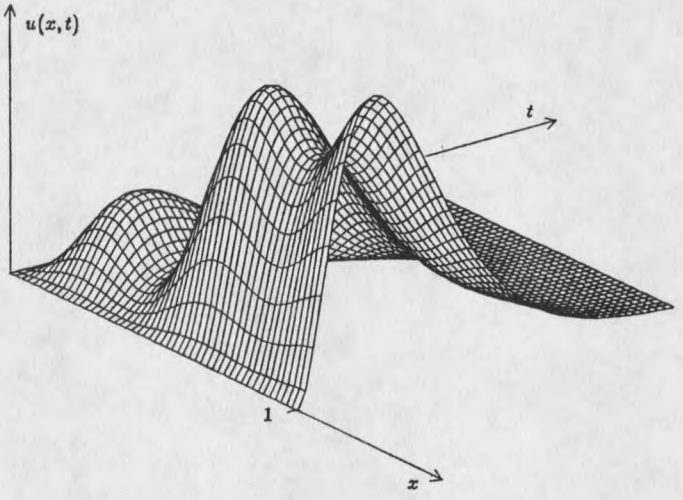


Figure 23. True Solution to Example 4.7.

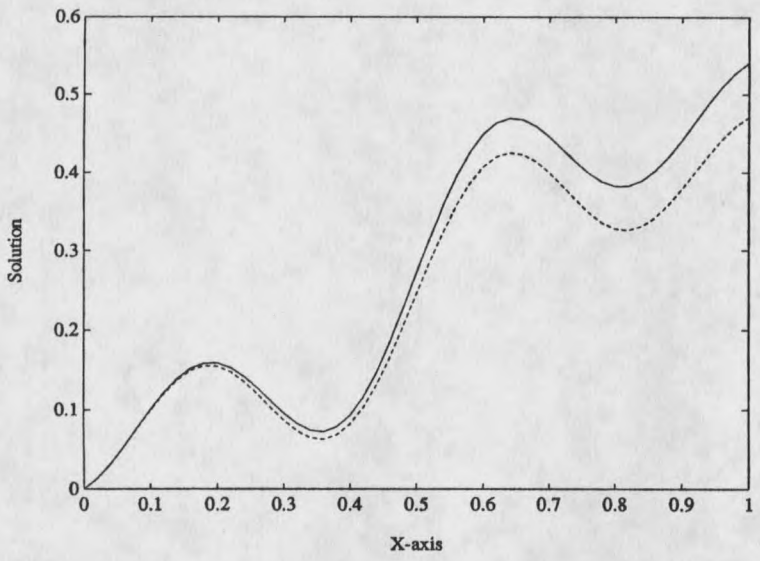


Figure 24. True and Approximate Solutions to Example 4.7 at Time $t = 2$ $- - -$ $(M_x = 16)$, — (True).

Nonzero Initial Conditions

The techniques and analysis from the previous sections can now be used to develop a Sinc-Galerkin method for the general problem

$$
\begin{aligned}
&\mathcal{L}u(x,t) = f(x,t) \quad , \quad a < x < b \ , \ t > 0 \\
&u(a,t) = \overline{\alpha}(t) \quad , \quad \left(\mathrm{EI}\,\frac{\partial^2 u}{\partial x^2}\right)(b,t) = \overline{\gamma}(t) \quad , \quad t > 0 \\
&\frac{\partial u}{\partial x}(a,t) = \overline{\beta}(t) \quad , \quad \frac{\partial}{\partial x}\left(\mathrm{EI}\,\frac{\partial^2 u}{\partial x^2}\right)(b,t) = \overline{\delta}(t) \quad , \quad t > 0 \\
&u(x,0) = \varphi_0(x) \quad , \quad \frac{\partial u}{\partial t}(x,0) = \varphi_1(x) \quad , \quad a \le x \le b
\end{aligned}
\tag{4.79}
$$

where $\mathcal{L}$ is defined in (4.63). The basis is taken to be

$$\{\psi_i(x)\psi_j^*(t)\}$$

where $\psi_i(x)$ and $\psi_j^*(t)$ are defined in (3.49) and (4.20), respectively. The approximate solution is taken to be

$$
\begin{aligned}
u_{m_x m_t}(x,t) = & \sum_{i=-M_x}^{N_x} \sum_{j=-M_t}^{N_t} u_{ij}\psi_i(x)\psi_j^*(t) \\
& + \sum_{j=-M_t}^{N_t} \psi_j^*(t) \{u_{-M_x-3,j}\psi_{-M_x-3}(x) + u_{-M_x-4,j}\psi_{-M_x-4}(x) \\
& \qquad + u_{N_x+1,j}\psi_{N_x+1}(x) + u_{N_x+2,j}\psi_{N_x+2}(x)\} \\
& + \Big\{\hat{\alpha}(t)\psi_{-M_x-1}(x) + \hat{\beta}(t)\psi_{-M_x-2}(x) \\
& \qquad + \hat{\gamma}(t)\psi_{N_x+3}(x) + \hat{\delta}(t)\psi_{N_x+4}(x)\Big\} \\
& + \varphi_0(x)\psi_{-M_t-1}^*(t) + \varphi_1(x)\psi_{-M_t-2}^*(t) \quad .
\end{aligned}
\tag{4.80}
$$

Here

$$
\hat{\alpha}(t) = \overline{\alpha}(t) - \varphi_0(a)\psi_{-M_t-1}^*(t) - \varphi_1(a)\psi_{-M_t-2}^*(t) \quad ,
$$

$$
\hat{\beta}(t) = \overline{\beta}(t) - \varphi_0'(a)\psi_{-M_t-1}^*(t) - \varphi_1'(a)\psi_{-M_t-2}^*(t) \quad ,
$$

$$
\hat{\gamma}(t) = \tilde{\gamma}(t) - \varphi_0''(b)\psi_{-M_t-1}^*(t) - \varphi_1''(b)\psi_{-M_t-2}^*(t)
$$

and

$$
\hat{\delta}(t) = \tilde{\delta}(t) - \varphi_0'''(b)\psi_{-M_t-1}^*(t) - \varphi_1'''(b)\psi_{-M_t-2}^*(t)
$$

where

$$
\tilde{\gamma}(t) = \frac{1}{\mathrm{EI}(b)}\,\overline{\gamma}(t)
$$

and

$$
\tilde{\delta}(t) = \frac{1}{\mathrm{EI}(b)}\,\overline{\delta}(t) - \frac{\mathrm{EI}'(b)}{[\mathrm{EI}(b)]^2}\,\overline{\gamma}(t) \quad .
$$

Note that the approximate solution as defined in (4.80) does satisfy the initial and boundary conditions in (4.79).

The $(m_x+4)\cdot m_t$ unknowns $\{u_{ij}\}$ in (4.80) are again found by orthogonalizing the residual with respect to the sinc functions $\{S_p(x)S_q^*(t)\}_{p=-M_x-2,\ldots,N_x+2}^{q=-M_t,\ldots,N_t}$. This yields the discrete system

$$(4.81) \qquad \left(\mathcal{L}u_{m_x m_t} - f, S_p S_q^*\right) = 0 \quad , \quad \begin{matrix} -M_x - 2 \le p \le N_x + 2 \\ -M_t \le q \le N_t \end{matrix}$$

where $(\cdot\,,\cdot)$ is defined in (4.38). If (4.81) is expanded and the known terms are collected, then the system may be written as it was in (4.69) with $\overline{f}(x,t)$ now given by

$$\begin{aligned} \overline{f}(x,t) \equiv f(x,t) &- \mathcal{L}\left(\hat{\alpha}(t)b_{-M_x-1}(x)\right) - \mathcal{L}\left(\hat{\beta}(t)b_{-M_x-2}(x)\right) \\ &- \mathcal{L}\left(\hat{\gamma}(t)b_{N_x+3}(x)\right) - \mathcal{L}\left(\hat{\delta}(t)b_{N_x+4}(x)\right) \\ &- \mathcal{L}\left(\varphi_0(x)b^*_{-M_t-1}(t)\right) - \mathcal{L}\left(\varphi_1(x)b^*_{-M_t-2}(t)\right) \quad . \end{aligned}$$

Integration by parts and the application of the one-dimensional results from Chapters 2, 3 and 4 yields the Sinc-Galerkin matrix equation

$$(4.82) \qquad A(\mathrm{EI})U\hat{M}C_t^T + C_x M U A_t^T = G$$

where C_t, C_x, M, U and G are defined in (4.42), (4.43), (4.74), (4.72) and (4.71), respectively. The $m_t \times m_t$ matrix A_t is given by (4.32) and $A(\mathrm{EI})$ is given by either (3.83) or (3.158) depending on which scheme is used to discretize EI. Finally, the $m_t \times m_t$ matrix $\hat{M}$ is defined by

$$(4.83) \qquad \hat{M} = \mathcal{D}(v^*) \quad .$$

The Schur algorithm (4.49) is used to solve for the matrix U containing the unknown coefficients $\{u_{ij}\}$.

Numerical Implementation and Examples (General Problem)

In both examples of this section, $\mathrm{EI}(x)$ is constant and the methods of the fifth section of Chapter 3 are used to obtain $A(\mathrm{EI})$. The weight choice $w(x) = (\phi'(x))^{-1/2}$ and the decay conditions (3.75) and (4.34) yield the requirement

$$|\mathcal{U}(x,t)| \leq L(x-a)^{\alpha+5/2}(b-x)^{\beta+5/2}t^{\gamma+1/2}e^{-\delta t} \quad . \tag{4.84}$$

In this case, $\mathcal{U}(x,t)$ denotes that part of the true solution which is approximated by

$$u_h(x,t) = \sum_{i=-M_x}^{N_x} \sum_{j=-M_t}^{N_t} u_{ij}\psi_i(x)\psi_j^*(t)$$

(see also (3.73) and (4.30)).

Example 4.8

$$\frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\partial^4 u}{\partial x^4}(x,t) = f(x,t) \quad , \quad \begin{matrix} 0 < x < 1 \\ t > 0 \end{matrix}$$

$$u(0,t) = 0 \quad , \quad \frac{\partial^2 u}{\partial x^2}(1,t) = 2t^2 e^{-t} \quad , \quad t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{1}{4}\, e^{-t} \quad , \quad \frac{\partial^3 u}{\partial x^3}(1,t) = 0 \quad , \quad t > 0$$

$$u(x,0) = \frac{1}{4}\, x \quad , \quad \frac{\partial u}{\partial t}(x,0) = -\frac{1}{4}\, x \quad , \quad 0 \leq x \leq 1 \quad .$$

The true solution $u(x,t) = \frac{1}{4}\, xe^{-t} + [(x\,\ell n(x))^4 + x^2]t^2 e^{-t}$ dictates the forcing function $f(x,t)$. The decay condition (4.84) yields the parameter choices $\alpha = \beta = \gamma = \frac{3}{2}$, $\delta = 1$. Hence the asymptotic error is $\mathcal{O}\left(e^{-\pi\sqrt{3M_x}/2}\right)$. The errors and solutions are reported in Table 16 and Figures 25 and 26, respectively. The results in this example can be compared to those in Example 4.6.

M_x	N_x	M_t	N_t	$\|E_S^{M_x}(h_x,h_t)\|$	$\|E_U^{M_x}(\ell_x,\ell_t)\|$	$\left(\|E_U^{M_x/2}(\ell_x,\ell_t)\|\right)^{\sqrt{2}}$
4	4	4	2	$.1005-0$	$.1395-0$	———
8	8	8	4	$.3166-1$	$.3205-1$	$.6170-1$
16	16	16	6	$.2829-2$	$.4264-2$	$.7708-2$
32	32	32	9	$.7756-4$	$.1334-2$	$.4447-3$

Table 16. Errors on the Sinc Grid S and the Uniform Grid U for Example 4.8.

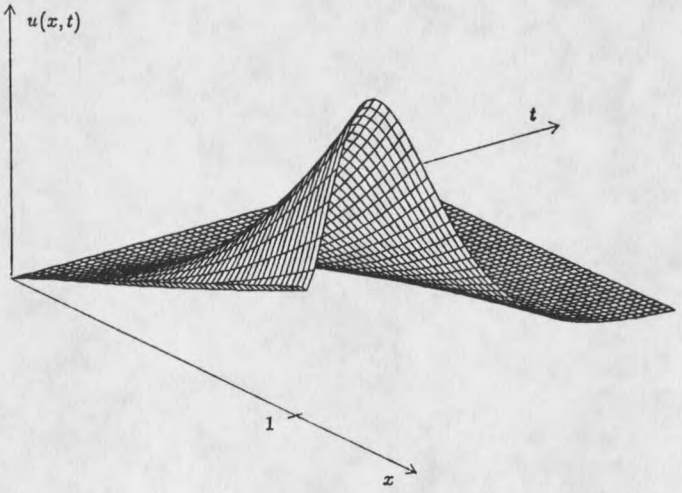


Figure 25. True Solution to Example 4.8.

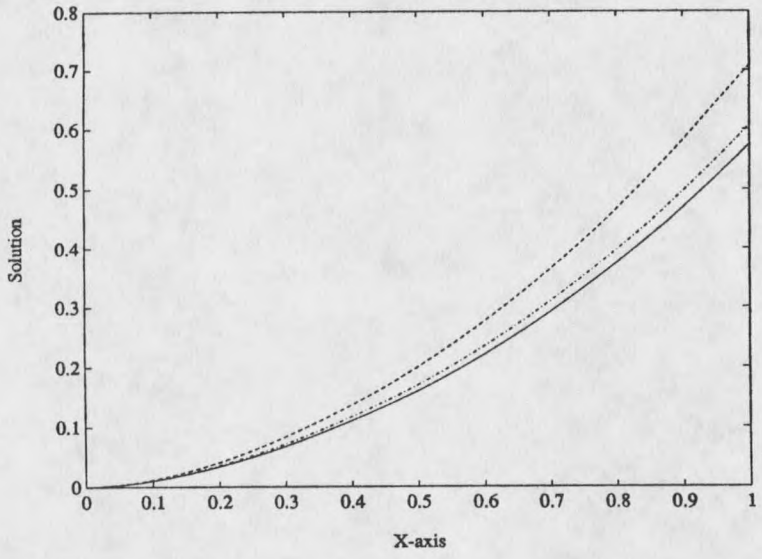


Figure 26. True and Approximate Solutions to Example 4.8 at Time $t = 2$
$- - -$ $(M_x = 4)$, $- \cdot -$ $(M_x = 8)$, — (True).

Example 4.9

$$\frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\partial^4 u}{\partial x^4}(x,t) = f(x,t) \quad , \quad \begin{matrix} 0 < x < 1 \\ t > 0 \end{matrix}$$

$$u(0,t) = 0 \quad , \quad \frac{\partial^2 u}{\partial x^2}(1,t) = e^{-t} \quad , \quad t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 5\pi \sin(te^{-t}) \quad , \quad \frac{\partial^3 u}{\partial x^3}(1,t) = 125\pi^3 \sin(te^{-t})$$

$$u(x,0) = \frac{1}{2}\, x^2 \quad , \quad \frac{\partial u}{\partial t}(x,0) = -\frac{1}{2}\, x^2 + \sin(5\pi x) \quad .$$

The true solution $u(x,t) = \sin(5\pi x)\sin(te^{-t}) + \frac{1}{2}\, x^2 e^{-t}$ dictates the forcing function $f(x,t)$. Since the decay condition (4.84) yields unrealistic spatial parameters, we will simply set $\alpha = \beta = \frac{3}{2}$. The choices $\gamma = \frac{1}{2}$, $\delta = 1$ do however follow from (4.84). The errors and solutions are reported in Table 17 and Figures 27 and 28, respectively.

M_x	N_x	M_t	N_t	$\|E_S^{M_x}(h_x,h_t)\|$	$\|E_U^{M_x}(\ell_x,\ell_t)\|$	$\left(\|E_U^{M_x/2}(\ell_x,\ell_t)\|\right)^{\sqrt{2}}$
8	8	24	4	$3.8179-0$	$4.0902-0$	———
16	16	48	6	$.7694-0$	$.7760-0$	———
32	32	96	6	$.5168-1$	$.5268-1$	$.6986-0$

Table 17. Errors on the Sinc Grid S and the Uniform Grid U for Example 4.9.

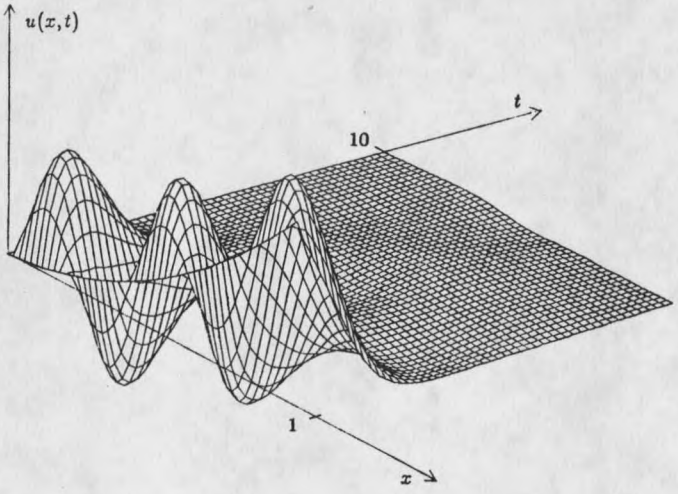


Figure 27. True Solution to Example 4.9.

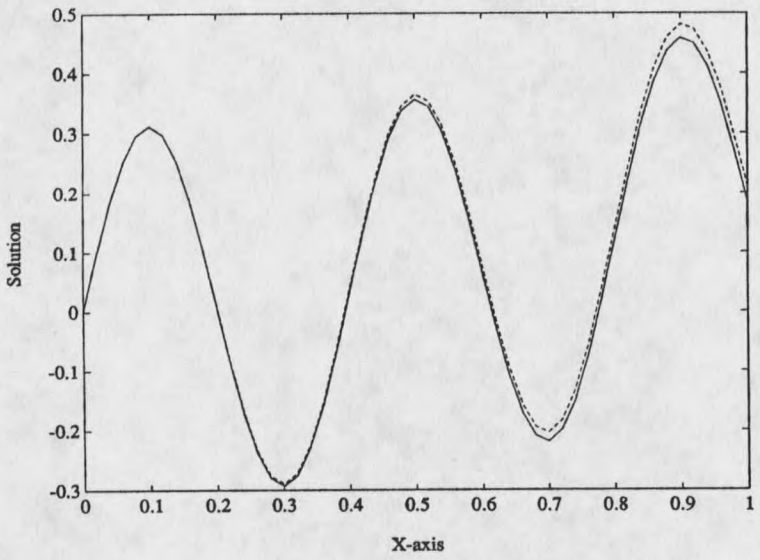


Figure 28. True and Approximate Solutions to Example 4.9 at Time $t = 2$ $- - -$ ($M_x = 32$), — (True).

CHAPTER 5

FOURTH-ORDER PARAMETER IDENTIFICATION PROBLEMS

Introduction

This chapter provides an introduction to the use of Sinc-Galerkin methods for the identification of variable parameters in fourth-order time-dependent problems. Its purpose is two-fold; (1) it provides an introduction to the operator theory that is used to solve the corresponding inverse problems, and (2) it demonstrates an important application of the forward methods that were developed in the previous chapters. The necessary background material is presented in the second section with the discussion centering around two principal topics. The first includes relevant results from the theory of compact operators and ideas such as weak continuity, ill-posedness, and regularization are reviewed. The section will conclude with an overview of the necessary optimization results including an outline of the model-trust region approach to minimization. The third section describes in detail the inverse problem associated with the identification of the parameter $\mathrm{EI}(x)$ in problems of the form

$$
\begin{aligned}
&\mathcal{L}(\mathrm{EI})u \equiv \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(\mathrm{EI}(x)\,\frac{\partial^2 u}{\partial x^2}\right) = f(x,t)\ , \quad \begin{matrix} a < x < b \\ t > 0 \end{matrix} \\
&u(a,t) = u(b,t) = 0\ ,\quad t > 0 \\
&\frac{\partial u}{\partial x}(a,t) = \frac{\partial u}{\partial x}(b,t) = 0\ ,\quad t > 0 \\
&u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0\ ,\quad a \le x \le b
\end{aligned}
\tag{5.1}
$$

and

$$
\begin{aligned}
&\mathcal{L}(\mathrm{EI})u = f(x,t) \quad , \quad a < x < b \ , \ t > 0 \\
&u(a,t) = \overline{\alpha}(t) \quad , \quad \left(\mathrm{EI}\frac{\partial^2 u}{\partial x^2}\right)(b,t) = \overline{\gamma}(t) \quad , \quad t > 0 \\
&\frac{\partial u}{\partial x}(a,t) = \overline{\beta}(t) \quad , \quad \frac{\partial}{\partial x}\left(\mathrm{EI}\frac{\partial^2 u}{\partial x^2}\right)(b,t) = \overline{\delta}(t) \quad , \quad t > 0 \\
&u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0 \quad , \quad a \le x \le b \quad .
\end{aligned}
\tag{5.2}
$$

The discussion in this section culminates with the formulation of a stable and efficient numerical method for solving the resulting regularization problems. The chapter closes with a section containing several model examples which demonstrate various aspects of the method.

Theoretical Background

Compact Operator Theory

A natural setting for studying many aspects of parameter identification is in a Hilbert space. The definitions and results which follow catalogue those properties of Hilbert spaces which will be needed throughout this chapter. Good references for this material are [17], [21] and [22]. Since the concepts of convergence and continuity are crucial to all that follows, these results are summarized first. The notation $\mathcal{X}$ and $\|\cdot\|$ denotes a general Hilbert space and its induced norm.

Definition 5.1 *The set of all continuous linear functionals on $\mathcal{X}$, that is, the set of all $\mathcal{F} : \mathcal{X} \to \mathbb{R}$ such that $\mathcal{F}$ is linear and continuous, is called the dual of $\mathcal{X}$ and is denoted $\mathcal{X}^*$.*

It is noted that in a Hilbert space $\mathcal{X}$, the dual space $\mathcal{X}^*$ is isometrically isomorphic to $\mathcal{X}$.

Definition 5.2 *A sequence $\{f_n\}$ in $\mathcal{X}$ is said to be strongly convergent if there is an $f \in \mathcal{X}$ such that*

$$\lim_{n \to \infty} \|f_n - f\| = 0 \quad .$$

This convergence is also referred to as norm convergence and it is denoted by

$$f_n \to f \quad .$$

Definition 5.3 *A sequence $\{f_n\}$ in $\mathcal{X}$ is said to be weakly convergent if there is an $f \in \mathcal{X}$ such that for any $\mathcal{F} \in \mathcal{X}^*$,*

$$\lim_{n \to \infty} \mathcal{F}(f_n) = \mathcal{F}(f) \quad .$$

This is also denoted

$$f_n \xrightarrow{w} f \quad .$$

It should be noted that if $\dim \mathcal{X} < \infty$, the concepts of weak and strong convergence are equivalent. In an infinite dimensional space however, a sequence may converge weakly but fail to converge strongly. An example demonstrating this can be found in [22, page 259].

As might be expected, the concept of weak continuity is intimately related to that of weak convergence.

Definition 5.4 *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces. The operator $K : \mathcal{X} \to \mathcal{Y}$ is said to be weakly continuous if*

$$Kf_n \to Kf$$

for all $f_n \xrightarrow{w} f$.

The following example establishes the weak continuity of a particular integral operator which is derived from the inverse problems in the third section.

Example 5.5 Let $\mathcal{X} = H^1(a,b)$ and $\mathcal{Y} = L^2(a,b)$. Furthermore, let $K : \mathcal{X} \to \mathcal{Y}$ be given by

$$[K(\mathrm{EI})](x) = C \int_a^b G(x,s,\mathrm{EI}(s)) f(s) ds$$

where $G : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous and has a continuous partial derivative with respect to the third argument and f is bounded. The continuously sampling observation operator $C : C(a,b) \to L^2(a,b)$ is defined by $C\psi = \{\psi(\cdot)\}$ (see [3, pages 56–57]). Then K is a weakly continuous operator.

Proof: Let $\mathrm{EI}_n \xrightarrow{w} \mathrm{EI}$. We then need to show that $K(\mathrm{EI}_n) \to K(\mathrm{EI})$ in the L^2 norm. For every $x \in (a,b)$ it follows that

$$\begin{aligned}
&\left| C \int_a^b G(x,s,\mathrm{EI}_n(s)) f(s) ds - C \int_a^b G(x,s,\mathrm{EI}(s)) f(s) ds \right| \\
&\leq \int_a^b |G(x,s,\mathrm{EI}_n(s)) - G(x,s,\mathrm{EI}(s))| \, |f(s)| ds \\
&\leq M \int_a^b |G(x,s,\mathrm{EI}_n(s)) - G(x,s,\mathrm{EI}(s))| \, ds \\
&= M \int_a^b \left| \frac{\partial G}{\partial \mathrm{EI}} (x,s,\xi_n(s)) \right| |\mathrm{EI}_n(s) - \mathrm{EI}(s)| \, ds \\
&\leq \mathcal{M} \int_a^b |\mathrm{EI}_n(s) - \mathrm{EI}(s)| \, ds \\
&\leq \mathcal{M}(b-a) \sup_{s \in [a,b]} |\mathrm{EI}_n(s) - \mathrm{EI}(s)| \quad .
\end{aligned}$$

Since weak convergence in $H^1(a,b)$ implies uniform convergence, it follows that $K(\mathrm{EI}_n)$ converges uniformly to $K(\mathrm{EI})$. The desired result follows from the fact that the uniform norm is stronger than the L^2 norm.

The following definitions characterize the operators of interest in this section.

Definition 5.6 *Let X be a metric space. A set $M \subset X$ is said to be compact if every sequence $\{f_n\}$ in M has a subsequence converging to an element $f \in M$. A set M is said to be relatively compact if its closure $\overline{M}$ is compact.*

Definition 5.7 *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces. The operator $K : \mathcal{X} \to \mathcal{Y}$ is said to be a compact operator if K is continuous, and for every bounded subset M of $\mathcal{X}$, the image $K(M)$ is relatively compact.*

The next theorem provides a link between weakly continuous and compact operators.

Theorem 5.8 *If K is a weakly continuous operator, then K is compact.*

Proof: See [19].

The utility of this theorem is demonstrated in the next example.

Example 5.9 Let $\mathcal{X} = H^1(a,b)$, $\mathcal{Y} = L^2(a,b)$ and let $K : \mathcal{X} \to \mathcal{Y}$ by

$$[K(\mathrm{EI})](x) = C \int_a^b G(x, s, \mathrm{EI}(s)) f(s) ds$$

where $\frac{\partial G}{\partial \mathrm{EI}}$ is continuous and f is bounded. The observation operator is again defined by $C\psi = \{\psi(\cdot)\}$. It follows immediately from Example 5.5 and Theorem 5.8 that K is a compact operator.

One can often learn a great deal about the behavior of compact operators by considering what is called their singular value representation. This representation is developed as follows.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces and let $K : \mathcal{X} \to \mathcal{Y}$ be a compact linear operator. Let K^* denote the Hilbert adjoint operator of K. It then follows that K^*K is both compact and self-adjoint. Denote by $\{v_j\}$ the orthonormal eigenvectors of K^*K, that is

$$K^*Kv_j = \lambda_j v_j \quad , \quad (v_i, v_j) = \delta_{ij} \quad .$$

All eigenvalues λ_j are nonnegative because

$$\lambda_j = \lambda_j(v_j, v_j) = (K^*Kv_j, v_j) = (Kv_j, Kv_j) \geq 0 \quad .$$

The singular values σ_j of the operator K are defined by

$$\sigma_j = \sqrt{\lambda_j} \quad , \quad \lambda_j \neq 0 \quad .$$

Throughout the remainder of this chapter, it will be assumed that the singular values are ordered so that $\sigma_1 \geq \sigma_2 \geq \sigma_3 \cdots$. Furthermore, if the definition

$$u_j = \frac{1}{\sigma_j} Kv_j$$

is made, it can be shown that

$$K^*Ku_j = \lambda_j u_j \quad ,$$

$$K^*u_j = \sigma_j u_j$$

and

$$(u_i, u_j) = \delta_{ij} \quad .$$

The triple $\{u_j, \sigma_j, v_j\}_{j \in J}$ is called a singular system for K. From the theory of compact operators, it follows that the index set J is countable. Finally, as proven in [19], the operator K may then be represented as

$$Kf = \sum_{j \in J} \sigma_j (f, v_j) u_j \quad .$$

It should be noted that if K is an $m \times n$ matrix, then the singular value decomposition (SVD) is

$$K = UDV^T \quad . \tag{5.3}$$

The columns of U and V are the singular vectors u_j and v_j, respectively. Both matrices are unitary. The singular values σ_j lie on the main diagonal of the matrix D (see also [13, pages 70–74]).

Two final topics will be addressed concerning the solution of the operator equation

$$Kf = g \tag{5.4}$$

where $K : \mathcal{X} \to \mathcal{Y}$ is a compact operator between two Hilbert spaces. These are (1) the formulation of a least squares solution and (2) the ill-posedness of (5.4).

It is first noted that a classical solution to (5.4) exists if and only if $g \in \mathcal{R}(K)$ where $\mathcal{R}(K)$ denotes the range of K. Instead of insisting on such a solution however, it is more appropriate in many cases to look instead for a least squares solution as defined below.

Definition 5.10 *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces and let $K : \mathcal{X} \to \mathcal{Y}$ be a bounded linear operator. The vector u is said to be a least squares solution of (5.4) if Ku is a best approximation to g from $\mathcal{R}(K)$, that is,*

$$\|Ku - g\| = \inf\{\|Kf - g\| : f \in \mathcal{X}\}$$

The following theorem provides some equivalent characterizations of least squares solutions. A proof may be found in [15, page 11].

Theorem 5.11 *Let P be the orthogonal projection of $\mathcal{Y}$ onto $\overline{\mathcal{R}(K)}$. Then the following conditions are equivalent:*

(i) u *is a least squares solution to* $Kf = g$;

(ii) $Ku = Pg$;

(iii) $K^*Ku = K^*g$.

From Theorem 5.11 it follows that a least squares solution to (5.4) exists if and only if $Pg \in \mathcal{R}(K)$. To this end, define the set

$$W = \{g \in \mathcal{Y} : Pg \in \mathcal{R}(K)\} \quad .$$

The Moore-Penrose generalized inverse of K, denoted $K^\dagger$, is then defined to be the operator with domain W which assigns to each $g \in W$ the least squares solution to (5.4). As proven in [15, pages 12–13], if K is compact and $g \in W$, then the Moore-Penrose inverse has the representation

$$K^\dagger g = \sum_{j \in J} \frac{(g, u_j)}{\sigma_j} v_j \quad .$$

The next two theorems show that except in simple cases, the operator $K^\dagger$ is not bounded.

Theorem 5.12 *Let $K : \mathcal{X} \to \mathcal{Y}$ be a compact operator. Then $K^\dagger$ is continuous if and only if $\mathcal{R}(K)$ is closed in $\mathcal{Y}$.*

Proof: See [15, page 12].

Theorem 5.13 *Let $K : \mathcal{X} \to \mathcal{Y}$ be a compact operator. Then $K^\dagger$ is continuous if and only if* $\dim \mathcal{R}(K) < \infty$.

Proof: See [15, page 12].

Those problems for which $K^\dagger$ is discontinuous are said to be ill-posed. The next definition more carefully describes what is meant by this concept.

Definition 5.14 *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces with $K : \mathcal{X} \to \mathcal{Y}$. The problem $Kf = g$ is well-posed in the sense of Hadamard if the following three conditions hold:*

(i) for all $g \in \mathcal{Y}$, there exists a solution $f \in \mathcal{X}$ such that $Kf = g$;

(ii) the solution f is unique;

(iii) the solution f depends continuously on the data.

The problem is said to be ill-posed if it is not well-posed.

It follows from Theorem 5.13 that except in the cases of finite rank operators, the problem $Kf = g$ with K compact is ill-posed even when the solution is taken to be the least squares solution. The degree of ill-posedness depends on the choice of the Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$. Since the solution to $Kf = g$ cannot be found directly, a procedure called Tikhonov regularization is applied [41]. This consists of solving the minimization problem

$$\min_{f \in \mathcal{X}} T_\alpha(f) \tag{5.5}$$

where the Tikhonov functional $T_\alpha(f)$ is defined to be

$$T_\alpha(f) \equiv \frac{1}{2}\left\{\|Kf - g\|^2 + \alpha J(f)\right\} \quad . \tag{5.6}$$

The nonnegative functional J is termed the penalty functional and the positive scalar α is called the regularization parameter. A solution f_α to (5.5) is called a regularized solution to the problem $Kf = g$.

Example 5.15 Let $\mathcal{X} = H^1(a,b)$. The penalty functional that will be used throughout the rest of this chapter is

$$J(f) = \|f\|_{\mathcal{X}}^2 \equiv \int_a^b [f''(x)]^2\,dx + \varepsilon \int_a^b [f(x)]^2\,dx \quad . \tag{5.7}$$

The parameter ε will be of the order 10^{-5}. The reasons for including the second term and enforcing J to be strictly positive will be discussed in the next section.

The next two theorems present existence and convergence criteria for the regularized solutions. The proofs are similar to those given in [19].

Theorem 5.16 *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces and let $K : \mathcal{X} \to \mathcal{Y}$ be weakly continuous. If J is coercive, i.e.,*

$$\lim_{\|f\| \to \infty} J(f) = \infty \quad ,$$

and weakly lower semicontinuous (see [17] for definition and properties), then there exists a solution f_α which minimizes the functional (5.6).

Theorem 5.17 *Let $g_n \to \overline{g}$ as $n \to \infty$ and assume that $Kf = \overline{g}$ has the unique solution $\overline{f}$, where K is weakly continuous. Furthermore, let $\alpha = \alpha(n)$ be chosen so that*

$$\alpha(n) \to 0 \quad \text{and} \quad \frac{\|Kf - g_n\|^2}{\alpha(n)} \to 0 \quad .$$

Then $f_n \to \overline{f}$ for f_n minimizing the functional (5.6) with $J(f)$ defined in (5.7).

An equivalent and quite intuitive way to think of Tikhonov regularization is in terms of the Tikhonov spectral filter. The development of this filter is outlined in the next example.

Example 5.18 Let T_α be defined as in (5.6). Using Frechet differentiation (see [29, pages 33–34]), standard arguments yield

$$T'_\alpha(f)h = (K^*(Kf - g) + \alpha f, h) \quad .$$

Now, a necessary condition for T_α to have a minimum at f_α is

$$T'_\alpha(f_\alpha)h = 0$$

for all $h \in \mathcal{X}$ so it follows that

$$(K^*K + \alpha I)f_\alpha = K^*g \quad .$$

Since the spectrum of $K^*K + \alpha I$ lies in $[\alpha, \infty)$, $\alpha > 0$, the operator is invertible and

$$f_\alpha = (K^*K + \alpha I)^{-1}K^*g \quad .$$

Moreover, from the Spectral Mapping Theorem (see [15] and [45]) it follows that

$$\begin{aligned} f_\alpha &= \sum_{j \in J} \frac{1}{\alpha + \lambda_j} (K^*g, v_j)v_j \\ &= \sum_{j \in J} w_j(\alpha) \frac{(g, u_j)}{\sigma_j} v_j \end{aligned}$$

where

$$w_j(\alpha) = \frac{\sigma_j^2}{\alpha + \sigma_j^2} \quad .$$

The sequence $\{w_j(\alpha)\}_{j \in J}$ is called the Tikhonov spectral filter and it has the effect of filtering out those singular components (g, u_j) which correspond to small

singular values σ_j. This demonstrates the importance of the choice of α in the success of the regularization scheme. If α is too large, then information about the solution is lost. On the other hand, if α becomes too small, one obtains excessive amplification of error through small singular values. Finally, if one defines

$$K_\alpha^\dagger g = \sum_{j \in J} w_j(\alpha) \frac{(g, u_j)}{\sigma_j} v_j \quad ,$$

then it can be shown (see [15, pages 16–17]) that

$$\lim_{\alpha \to 0} K_\alpha^\dagger g = K^\dagger g \quad .$$

This concludes the discussion of the background material on compact operators. The remainder of this section will consist of those topics from optimization theory which are necessary for solving the minimization problem (5.5).

Numerical Optimization

The discussion in the last subsection ended with the formulation of the Tikhonov functional (5.6) and the corresponding infinite dimensional minimization problem (5.5). To numerically approximate the solution to (5.5) however, one must work in finite dimensional spaces. As will be shown in the next section, a finite dimensional analogue to (5.5) is the minimization problem

$$\min_{\mathbf{c} \in \mathbb{R}^n} \mathbf{T}_\alpha(\mathbf{c}) \tag{5.8}$$

where

$$\mathbf{T}_\alpha(\mathbf{c}) \equiv \frac{1}{2} \left\{ \|\mathbf{K}(\mathbf{c}) - \mathbf{g}\|^2 + \alpha \|R\mathbf{c}\|^2 \right\} \quad . \tag{5.9}$$

Here $\mathbf{K} : \mathbb{R}^n \to \mathbb{R}^m$ and R is an $n \times n$ upper triangular matrix. Throughout this subsection, the notation $\|\cdot\|$ will be used to denote the Euclidean norm in $\mathbb{R}^n$ or $\mathbb{R}^m$.

The functional $\mathbf{T}_\alpha(\mathbf{c})$ in (5.9) will be minimized via a quasi-Newton method in conjunction with a trust region scheme. The quasi-Newton step is calculated first at each iteration and criteria are checked to determine whether the steplength is greater than the size of the region in which the functional is accurately described by its quadratic model. If the step is not appropriate, the geometry of the problem is used to formulate a constrained minimization problem whose solution determines the new step. The method thus combines the fast local convergence of Newton's method with the global convergence of the trust region scheme.

From Taylor's theorem, it follows that the quadratic model to $\mathbf{T}_\alpha(\mathbf{c})$ is given by

$$\mathcal{M}(\mathbf{s}) = \frac{1}{2}\left\{\|\mathbb{K}(\mathbf{c}) + \mathbb{K}'(\mathbf{c})\mathbf{s} - \mathbf{g}\|^2 + \alpha\|R(\mathbf{c}+\mathbf{s})\|^2\right\} \quad .$$

One then considers the iteration

$$\mathbf{c}_{K+1} = \mathbf{c}_K + \mathbf{s}_K \tag{5.10}$$

where $\mathbf{c}_K$ currently minimizes $\mathbf{T}_\alpha$ and $\mathbf{s}_K$ minimizes the quadratic model $\mathcal{M}$. Now, a necessary condition for $\mathcal{M}$ to have a minimum at $\mathbf{s}_K$ is

$$\begin{aligned} 0 &= \mathcal{M}'(\mathbf{s}_K) \\ &= \mathbb{K}'(\mathbf{c}_K)^T\left[\mathbb{K}(\mathbf{c}_K) + \mathbb{K}'(\mathbf{c}_K)\mathbf{s}_K - \mathbf{g}\right] + \alpha R^T R(\mathbf{c}_K + \mathbf{s}_K) \quad . \end{aligned}$$

This yields

$$\mathbf{s}_K = -\hat{H}(\mathbf{c}_K)^{-1}G(\mathbf{c}_K)$$

where

$$\hat{H}(\mathbf{c}_K) = \mathbb{K}'(\mathbf{c}_K)^T\mathbb{K}'(\mathbf{c}_K) + \alpha R^T R$$

and

$$G(\mathbf{c}_K) = \mathbb{K}'(\mathbf{c}_K)^T\left[\mathbb{K}(\mathbf{c}_K) - \mathbf{g}\right] + \alpha R^T R\mathbf{c}_K \quad .$$

As indicated by the notation, $G(\mathbf{c}_K)$ is the gradient of $\mathbf{T}_\alpha$ and $\hat{H}(\mathbf{c}_K)$ is a first order approximation to the Hessian.

It should be noted that the quasi-Newton step $\mathbf{s}_K = -\hat{H}(\mathbf{c}_K)^{-1} G(\mathbf{c}_K)$ is in a descent direction of the functional $\mathbf{T}_\alpha$. Recall that $\mathbf{s}_K$ is a descent direction if

$$\mathbf{T}_\alpha(\mathbf{c}_K + \lambda \mathbf{s}_K) \leq \mathbf{T}_\alpha(\mathbf{c}_K)$$

for sufficiently small positive λ. This is equivalent to

$$\nabla \mathbf{T}_\alpha(\mathbf{c}_K)^T \mathbf{s}_K < 0 \quad .$$

Here

$$\nabla \mathbf{T}_\alpha(\mathbf{c}_K)^T \mathbf{s}_K = -G(\mathbf{c}_K)^T \hat{H}(\mathbf{c}_K)^{-1} G(\mathbf{c}_K) < 0$$

since $\hat{H}(\mathbf{c}_K)$ and hence $\hat{H}(\mathbf{c}_K)^{-1}$ are positive definite. This verifies the assertion.

The quasi-Newton method (5.10) will converge to the local minimizer $\mathbf{c}^*$ of (5.8) provided the initial guess $\mathbf{c}_0$ is good enough and the Hessian at $\mathbf{c}^*$ is positive definite. In many instances however, the iteration will not converge or it will converge to a solution $G(\mathbf{c}) = 0$ which is not a local minimizer. To guarantee convergence under weaker conditions, replace the iteration (5.10) by

$$\mathbf{c}_{K+1} = \mathbf{c}_K + \mathbf{s}_K \tag{5.11}$$

where $\mathbf{s}_K$ solves the constrained minimization problem

$$\min_{\mathbf{s} \in \mathbb{R}^n} \frac{1}{2} \left\{ \|\mathbf{K}(\mathbf{c}_K) + \mathbf{K}'(\mathbf{c}_K)\mathbf{s} - \mathbf{g}\|^2 + \alpha \|R(\mathbf{c}_K + \mathbf{s})\|^2 \right\} \tag{5.12}$$

subject to $\|\mathbf{s}\| \leq \delta_K$.

Problem (5.12) is the basis for the "model-trust region" approach to minimization (see [9, pages 129–147]). The name comes from viewing δ_K as a region in which we "trust" the quadratic model to adequately reflect the behavior of $\mathbf{T}_\alpha$. That is, δ_K is chosen so that

$$\mathbf{T}_\alpha(\mathbf{c}_K + \mathbf{s}) \approx \frac{1}{2}\left\{\|\mathbf{K}(\mathbf{c}_K) + \mathbf{K}'(\mathbf{c}_K)\mathbf{s} - \mathbf{g}\|^2 + \alpha\|R(\mathbf{c}_K + \mathbf{s})\|^2\right\}$$

whenever $\|\mathbf{s}\| \leq \delta_K$.

The minimization problem (5.12) will be solved using an approach similar to that in [20]. The problem is first diagonalized using the Singular Value Decomposition (SVD) (see (5.3)). With the change of variables

$$\bar{\mathbf{s}} = R\mathbf{s} \quad ,$$

the objective functional in (5.12) becomes

$$\frac{1}{2}\left\{\|A\bar{\mathbf{s}} - \mathbf{b}\|^2 + \alpha\|\bar{\mathbf{c}} + \bar{\mathbf{s}}\|^2\right\}$$

where $A \equiv \mathbf{K}'(\mathbf{c}_K)R^{-1}$, $\mathbf{b} \equiv \mathbf{g} - \mathbf{K}(\mathbf{c}_K)$, and $\bar{\mathbf{c}} \equiv R\mathbf{c}_K$. Let A have the SVD

$$A = UDV^T$$

where $U_{m\times m}$, $V_{n\times n}$ are orthogonal and

$$[D_{m\times n}]_{i,j} = \begin{cases} \sigma_i & , \text{ if } i = j \text{ and } i \leq n \\ 0 & , \text{ otherwise} \end{cases} .$$

The second change of variables

$$\hat{\mathbf{s}} = V^T\bar{\mathbf{s}} \quad , \quad \hat{\mathbf{b}} = U^T\mathbf{b} \quad , \quad \hat{\mathbf{c}} = V^T\bar{\mathbf{c}}$$

yields the diagonalized problem

$$\min_{\hat{\mathbf{s}} \in \mathbb{R}^n} \frac{1}{2} \left\{ \|D\hat{\mathbf{s}} - \hat{\mathbf{b}}\|^2 + \alpha \|\hat{\mathbf{c}} + \hat{\mathbf{s}}\|^2 \right\} \tag{5.13}$$

subject to $\|\hat{\mathbf{s}}\| \leq \delta_K$.

The theory of constrained optimization is used to solve (5.13). By the Kuhn-Tucker criterion [11] there exists a Lagrange multiplier $\mu \geq 0$ such that

$$D^T (D\hat{\mathbf{s}} - \hat{\mathbf{b}}) + \alpha(\hat{\mathbf{c}} + \hat{\mathbf{s}}) + \mu\hat{\mathbf{s}} = 0 \quad . \tag{5.14}$$

From (5.14) it follows that (5.13) has a unique solution of the form

$$\hat{\mathbf{s}} = \hat{\mathbf{s}}(\mu) = \left\{ D^T D + (\alpha + \mu) I \right\}^{-1} (D^T \hat{\mathbf{b}} - \alpha\hat{\mathbf{c}}) \quad .$$

If $\|\hat{\mathbf{s}}(0)\| < \delta_K$, then the constraint in (5.13) is not active and $\mathbf{s} = R^{-1} V \hat{\mathbf{s}}(0)$ solves (5.12). Otherwise the constraint is active and $\mathbf{s} = R^{-1} V \hat{\mathbf{s}}(\mu)$ solves (5.12). Here $\mu \geq 0$ is the unique solution to

$$g(\mu) \equiv \|\hat{\mathbf{s}}(\mu)\| - \delta_K = 0 \quad . \tag{5.15}$$

The hook step algorithm in [9, page 134] is used to find an approximate solution to the scalar equation (5.15). This algorithm requires both $g(\mu)$ and $g'(\mu)$. As shown in [20], the function $g(\mu)$ can be expanded as

$$g(\mu) = \left[\sum_{j=1}^{n} \left(\frac{\sigma_j \hat{b}_j - \alpha \hat{c}_j}{\sigma_j^2 + \alpha + \mu} \right)^2 \right]^{1/2} - \delta_K \tag{5.16}$$

where σ_j denotes a singular value of A and $\hat{b}_j$, $\hat{c}_j$ are components of $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$, respectively. The derivatives $g'(\mu)$ are easily obtained from the form (5.16).

The trust region radius δ_K in (5.13) is determined via the updating algorithm in [9, page 143]. The basic idea is to choose δ_K so that $\mathbf{T}_\alpha(\mathbf{c})$ has sufficient decrease at each iteration to guarantee convergence to a local minimizer of $\mathbf{T}_\alpha$. As was done in [19], the decay requirement was taken to be

$$\mathbf{T}_\alpha(\mathbf{c}_K + \mathbf{s}_K) \leq \mathbf{T}_\alpha(\mathbf{c}_K) + \tilde{\varepsilon}\nabla\mathbf{T}_\alpha(\mathbf{c}_K)^T\mathbf{s}_K$$

where $\tilde{\varepsilon} = 10^{-4}$. Thus this condition is not very stringent.

The previous discussion is summarized in the following algorithm from [19, page 43]. Further details concerning this algorithm and the trust region method in general can be found in [9], [11] and [19].

For $K = 1, 2, \ldots$:

1. Find the quasi-Newton step $\hat{s}(0)$
2. If $\|\hat{\mathbf{s}}(0)\| < \delta_K$ then go to 5, else
3. Approximately solve $\|\hat{\mathbf{s}}(\mu)\| - \delta_K = 0$ for μ
4. Find $\hat{\mathbf{s}}(\mu)$ and compute $\mathbf{s}(\mu) = R^{-1}V\hat{\mathbf{s}}(\mu)$
5. Decide whether $\mathbf{c}_{K+1} = \mathbf{c}_K + \mathbf{s}(\mu)$ is an acceptable new approximation.

If yes go to 7, else:

6. Decrease the trust region radius, go to 3
7. If the stopping criteria are satisfied, then end, else
8. Find δ_{K+1}, i.e., retain, increase or reduce δ_K, go to 1.

The Fourth-Order Parameter Identification Problem

In this section, Sinc-Galerkin methods are developed for numerically solving parameter identification problems of the form

$$
\begin{aligned}
&\mathcal{L}(\mathrm{EI})u \equiv \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(\mathrm{EI}(x)\,\frac{\partial^2 u}{\partial x^2}\right) = f(x,t)\ , \quad \begin{matrix} a<x<b \\ t>0 \end{matrix} \\
&u(a,t) = u(b,t) = 0\ , \quad t>0 \\
&\frac{\partial u}{\partial x}(a,t) = \frac{\partial u}{\partial x}(b,t) = 0\ , \quad t>0 \\
&u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0\ , \quad a \le x \le b
\end{aligned}
\tag{5.17}
$$

and

$$
\begin{aligned}
&\mathcal{L}(\mathrm{EI})u = f(x,t)\ , \quad a<x<b\ ,\ t>0 \\
&u(a,t) = \overline{\alpha}(t)\ , \quad \left(\mathrm{EI}\,\frac{\partial^2 u}{\partial x^2}\right)(b,t) = \overline{\gamma}(t)\ , \quad t>0 \\
&\frac{\partial u}{\partial x}(a,t) = \overline{\beta}(t)\ , \quad \frac{\partial}{\partial x}\left(\mathrm{EI}\,\frac{\partial^2 u}{\partial x^2}\right)(b,t) = \overline{\delta}(t)\ , \quad t>0 \\
&u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0\ , \quad a \le x \le b\ .
\end{aligned}
\tag{5.18}
$$

In both cases, the parameter $\mathrm{EI}(x)$ is considered to be unknown. The data for the problem will consist of measurements of the solution u at the points $\{(x_p,t_q)\}_{p=1,\ldots,n_p}^{q=1,\ldots,n_q}$ in $(a,b)\times(0,\infty)$.

In the next subsection, the problem is formulated as a nonlinear operator equation

$$K(\mathrm{EI}) = g\ .$$

It will then be shown that solutions to this problem are unstable with respect to perturbations in the data g and the operator K. To achieve stable approximate solutions, Tikhonov regularization and other techniques from the last section will

be used. The result is a stable and efficient numerical method for solving the problems (5.17) and (5.18).

Operator Formulation

In order to use the results and theory from the last section, the inverse problems corresponding to (5.17) and (5.18) must be formulated as operator equations of the form $K(\mathrm{EI}) = g$. The procedure for problems with fixed boundary conditions is outlined below. The formulation for problems with cantilever boundary conditions follows similarly. Further theory for this latter case can be found in [7].

In formulating the operator equation corresponding to (5.17), it is easiest to first consider the spatial problem

$$\mathcal{L}(\mathrm{EI})u \equiv (\mathrm{EI}(x)u'')'' - q(x)u = f(x) \quad , \quad a < x < b$$

$$u(a) = u(b) = 0 \tag{5.19}$$

$$u'(a) = u'(b) = 0$$

where $\mathrm{EI}(x) > 0$ and $q(x) > 0$ (the reason for the choice of sign on $q(x)$ will become apparent when the problem (5.17) is considered). From the theory of ordinary differential equations, the existence of a Green's function for (5.19) will be verified and a corresponding integral representation developed. The formulation and properties of the spatial inverse problem will follow immediately. These results can then be extended to the time-dependent problem once a separation of variables has been used to split the operator $\mathcal{L}$ into temporal and spatial components.

The existence of the Green's function for (5.19) follows from the theory of [25] as noted in [36] and [40]. The Green's function will be denoted by $G(x, s, \mathrm{EI}(s))$

to emphasize its dependence on EI. From [8], [36] it follows that G and $\frac{\partial G}{\partial \mathrm{EI}}$ are continuous and that the state solution u is given by

$$u(x) = \int_a^b G(x, s, \mathrm{EI}(s)) f(s) ds \quad .$$

The spatial inverse problem can then be formulated as the operator equation

$$K(\mathrm{EI}) = g$$

where $K : H^1(a,b) \to L^2(a,b)$ by

$$K(\mathrm{EI}) = C \int_a^b G(\cdot, s, \mathrm{EI}(s)) f(s) ds$$

and g denotes the data. Here the observation operator C is defined by

$$C\psi = \{\psi(\cdot)\} \quad .$$

From Examples 5.5 and 5.9 it follows that the operator K is compact.

The results in the previous discussion can be extended to the time-dependent case via a separation of variables. The substitution of $u(x,t) = X(x)T(t)$ into the homogeneous problem corresponding to (5.17) yields

$$T''(t) + \lambda T(t) = 0$$

and the eigenvalue problem

$$[\mathrm{EI}(x)X''(x)]'' - \lambda X(x) = 0 \quad , \quad a < x < b$$

$$X(a) = X(b) = 0$$

$$X'(a) = X'(b) = 0$$

where $\lambda > 0$. From the discussion of (5.19) and arguments similar to those in [8], [16] and [36], it follows that if $\mathrm{EI}(x) > 0$, then the state solution to (5.17) can be represented by

$$u(x,t) = \int_a^b \mathcal{G}(x, t, s, \mathrm{EI}(s)) ds \quad .$$

The function $\mathcal{G}$ is continuously differentiable and depends on a Green's function, the eigenfunctions of T, and the expansion of the forcing function f.

The inverse problem can then be written as the operator equation

$$K(\mathrm{EI}) = g \tag{5.20}$$

where g again denotes the data. The nonlinear operator K is defined by

$$K(\mathrm{EI}) = C \int_a^b \mathcal{G}(\cdot, \cdot, s, \mathrm{EI}(s)) ds \tag{5.21}$$

where the observation operator C maps the state solution into the infinite dimensional observation space $L^2\left((a,b) \times \mathbb{R}^+\right)$ by

$$Cu = \{u(\cdot, \cdot)\} \quad . \tag{5.22}$$

Arguments similar to those in Examples 5.5 and 5.9 can be used to show that K is a compact operator.

It is noted that the inverse problem corresponding to (5.18) can also be written as $K(\mathrm{EI}) = g$ with K given by (5.21). The function $\mathcal{G}$ will obviously be different in this latter problem since the Green's function is constructed so as to satisfy the boundary conditions. Further theory on problems of this type can be found in [7], [25].

Consider now the well-posedness of the problem

$$K(\mathrm{EI}) = g \tag{5.23}$$

with $K : H^1(a,b) \to L^2\left((a,b) \times \mathbb{R}^+\right)$ given by (5.21) (see also Definition 5.14). It is first noted that this choice of spaces is physically reasonable since the solution EI is assumed "smooth" whereas the observation space may not even be continuous. With these choices of spaces however, the problem is ill-posed. First, since $\mathcal{G}$ is

continuous, the operator K will be continuous for any $\mathrm{EI} \in H^1(a,b)$ (see [10], p. 159). Thus there exist elements of $g \in L^2\left((a,b) \times \mathbb{R}^+\right)$ for which (5.23) has no solution. Furthermore, since K is a compact operator with an infinite dimensional range, it follows from Theorem 5.13 that the operator $K^\dagger$ is discontinuous. This in turn indicates that small perturbations in the data g may give rise to arbitrarily large perturbations in the solution $\mathrm{EI} \in H^1(a,b)$. This means that any stable procedure for finding a solution to (5.23) must involve regularization.

Regularization and Operator Approximations

The regularization technique that is used is Tikhonov regularization as discussed in the second section. Thus the problem (5.23) is replaced by the minimization problem

$$\min_{\mathrm{EI} \in \mathcal{X}} T_\alpha(\mathrm{EI}) \tag{5.24}$$

where $\mathcal{X} = H^1(a,b)$ and

$$T_\alpha(\mathrm{EI}) = \frac{1}{2}\left\{\|K(\mathrm{EI}) - g\|^2 + \alpha\|\mathrm{EI}\|_{\mathcal{X}}^2\right\} \quad . \tag{5.25}$$

The norm $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^s$, $s = n_p \cdot n_q$, and $\|\cdot\|_{\mathcal{X}}$ is defined in (5.7).

In order to numerically approximate the solution to (5.25), one must work in the finite dimensional spaces which derive from the underlying differential equations (5.17) and (5.18). For the sake of brevity, the following discussion will be limited to the problem that is associated with (5.17). It is then noted that the changes which are needed to extend the method to (5.18) follow directly from the analysis in Chapters 2 and 3. Numerical results for both problems are included in the next section.

First, since g comes from measurements at the points $\{(x_p,t_q)\}_{p=1,\dots,n_p}^{q=1,\dots,n_q}$, the discretized version g of g is the s-vector containing the data $\{g(x_p,t_q)\}_{p=1,\dots,n_p}^{q=1,\dots,n_q}$.

Consider now the two sets of basis functions $\{S_i\}_{i=-M_x}^{N_x}$ and $\{S_j^*\}_{j=-M_t}^{N_t}$ as defined in (2.21) and (4.7), respectively. The solution $u(x,t)$ to (5.17) is then approximated by

$$(5.26) \qquad u_{m_x m_t}(x,t) = \sum_{i=-M_x}^{N_x} \sum_{j=-M_t}^{N_t} u_{ij} S_i(x) S_j^*(t) \quad , \quad \begin{matrix} m_x = M_x + N_x + 1 \\ m_t = M_t + N_t + 1 \end{matrix} \quad .$$

The $m_x \cdot m_t$ unknowns $\{u_{ij}\}$ are found by orthogonalizing the residual with respect to the basis. As shown in Chapter 4, this yields the system

$$(5.27) \qquad A(\mathrm{EI})U(\mathrm{EI})C_t^T + C_x U(\mathrm{EI})A_t^T = G$$

where $A(\mathrm{EI})$, A_t, C_x, C_t and G are defined in (3.113), (4.45), (4.43), (4.42) and (4.44), respectively. A generalized Schur algorithm is again used to solve the system (5.27) for the unknown matrix $U(\mathrm{EI})$ (see the third section of Chapter 4 for details). As indicated by the notation, the approximate solution depends on the parameter $\mathrm{EI}(x)$.

An approximation $\hat{K}(\mathrm{EI})$ to $K(\mathrm{EI})$ in (5.25) is obtained by applying a discrete version of the observation operator C in (5.22) to $u_{m_x m_t}$ in (5.26). If the set of evaluation points $\{(x_p,t_q)\}_{p=1,\dots,n_p}^{q=1,\dots,n_q}$ can be represented as a tensor product of spatial and temporal points, then $\hat{K}(\mathrm{EI})$ has the representation

$$(5.28) \qquad \hat{K}(\mathrm{EI}) = C\,\mathrm{co}(U(\mathrm{EI})) \quad .$$

The notation $\mathrm{co}(U(\mathrm{EI}))$ again denotes the concatenation of $U(\mathrm{EI})$ and C is an $s \times n$ ($s = n_p \cdot n_q$, $n = m_x \cdot m_t$) evaluation matrix which can be formulated as follows. Define the $n_p \times m_x$ spatial evaluation matrix E_x to have components

$$[E_x]_{p,i} = S_i(x_p) \quad , \quad 1 \le p \le n_p \ , \ -M_x \le i \le N_x$$

and define the $n_q \times m_t$ temporal evaluation matrix E_t to have components

$$[E_t]_{q,j} = S_j(t_q) \quad , \quad 1 \leq q \leq n_q \ , \ -M_t \leq j \leq N_t \quad .$$

Then

$$C = E_t \otimes E_x \quad .$$

It is noted that if the set of evaluation points is not rectangular as described above, then point evaluation can be done directly via (5.26). This latter option is less efficient however, than that defined in (5.28).

Since the parameter $\mathrm{EI}(x)$ is variable, it too must be projected into a finite dimensional space before a numerical method can be developed. With the basis $\{\tilde{\psi}_k\}_{k=-M_E-2}^{N_E+2}$ that is defined in (3.89), the parameter can be approximated by

$$\widehat{\mathrm{EI}}(x) = \left[\widehat{\mathrm{EI}}(\mathbf{c})\right](x) = \sum_{k=-M_E-2}^{N_E+2} c_k \tilde{\psi}_k(x) \tag{5.29}$$

where $\mathbf{c} = [c_{-M_E-2}, c_{-M_E-1}, \ldots, c_{N_E+2}]^T \in \mathbb{R}^{m_E+4}$, $m_E = M_E + N_E + 1$. The restriction of $\hat{K}$ to span $\{\tilde{\psi}_k\}_{k=-M_E-2}^{N_E+2}$ then yields the operator $\mathbf{K} : \mathbb{R}^{m_E+4} \to \mathbb{R}^s$ by

$$\mathbf{K}(\mathbf{c}) \equiv \hat{K}\left(\widehat{\mathrm{EI}}(\mathbf{c})\right) = C\,\mathrm{co}\left(U\left(\widehat{\mathrm{EI}}(\mathbf{c})\right)\right) \quad .$$

From the optimization results in the second section, it follows that in addition to the operator $\mathbf{K}$, the Jacobian $\mathbf{K}'(\mathbf{c})$ is needed at various places in the trust region scheme. Recall that the Jacobian is an $s \times (m_E + 4)$ matrix whose ν-th column is given by

$$[\mathbf{K}'(\mathbf{c})]_\nu = \lim_{\tau \to 0} \frac{1}{\tau}\left[\mathbf{K}(\mathbf{c} + \tau \mathbf{e}_\nu) - \mathbf{K}(\mathbf{c})\right]$$

where the standard unit vector $\mathbf{e}_\nu$ has components

$$[\mathbf{e}_\nu]_k = \delta_{\nu k} = \begin{cases} 1 & \text{if } k = \nu \\ 0 & \text{otherwise} \end{cases} \quad , \quad -M_E - 2 \leq k \leq N_E + 2 \quad .$$

The Jacobian can also be calculated via a directional derivative scheme such as that presented in [28]. This latter scheme takes advantage of the matrix structures resulting from the Schur algorithm and is thus quite efficient.

The final part of the functional T_α that needs to be discretized is the penalty term $\|\mathrm{EI}\|_\chi$. Given the expansion (5.29) and the definition (5.7), it follows that

$$\begin{aligned}
\|\widehat{\mathrm{EI}}\|_\chi^2 &= \int_a^b \left[\widehat{\mathrm{EI}}''(x)\right]^2 dx + \varepsilon \int_a^b [\mathrm{EI}(x)]^2 dx \\
&= \int_a^b \left(\sum_{k=-M_E-2}^{N_E+2} c_k \tilde{\psi}_k''(x) \right) \left(\sum_{\ell=-M_E-2}^{N_E+2} c_\ell \tilde{\psi}_\ell''(x) \right) dx \\
&\qquad + \varepsilon \int_a^b \left(\sum_{k=-M_E-2}^{N_E+2} c_k \tilde{\psi}_k(x) \right) \left(\sum_{\ell=-M_E-2}^{N_E+2} c_\ell \tilde{\psi}_\ell(x) \right) dx \\
&= \sum_{k=-M_E-2}^{N_E+2} \sum_{\ell=-M_E-2}^{N_E+2} c_k \left(\int_a^b \tilde{\psi}_k''(x)\tilde{\psi}_\ell''(x)dx + \varepsilon \int_a^b \tilde{\psi}_k(x)\tilde{\psi}_\ell(x)dx \right) c_\ell \\
&\equiv \mathbf{c}^T B \mathbf{c} \quad .
\end{aligned}$$

Here B is a symmetric and positive definite $(m_E + 4) \times (m_E + 4)$ matrix with entries

$$[B]_{k\ell} = \int_a^b \tilde{\psi}_k''(x)\tilde{\psi}_\ell''(x)dx + \varepsilon \int_a^b \tilde{\psi}_k(x)\tilde{\psi}_\ell(x)dx \quad . \tag{5.30}$$

Appropriate integration by parts and sinc quadrature (see Theorem 2.5) are used to evaluate those components of B for which k and ℓ are not both $M_E - 1$, $M_E - 2$, $N_E + 1$ or $N_E + 2$. It is noted that the basis weight $v(x) = [(b-x)(x-a)]^{3/2}$ ensures symmetric systems in both cases. From (3.89), it follows that the remaining 32 "corner" integrals have integrands consisting of third degree polynomials and can thus be evaluated exactly with Simpson's rule.

Since B is symmetric and positive definite, it has a Cholesky decomposition $B = R^T R$ where R is an upper triangular matrix. As seen in the optimization

section, this factorization admits a particularly useful diagonalization of the constrained optimization problem which results from (5.24). It should be noted that if one takes $\varepsilon = 0$, then B will be symmetric and nonnegative definite. The underlying optimization theory still holds but the numerical benefits of system diagonalization will be lost. For $\varepsilon > 0$, the penalty term $\|\mathrm{EI}\|_\chi^2$ in (5.25) yields the quadratic form

$$\|\widehat{\mathrm{EI}}\|_\chi^2 = \mathbf{c}^T R^T R\mathbf{c} = \|R\mathbf{c}\|^2$$

where $\|\cdot\|$ denotes the Euclidean norm and $B = R^T R$ is defined in (5.30).

The finite dimensional analogue of (5.24) can then be formulated as the minimization problem

$$\min_{\mathbf{c}\in\mathbb{R}^{m_E+4}} \mathbf{T}_\alpha(\mathbf{c}) \tag{5.31}$$

where

$$\mathbf{T}_\alpha(\mathbf{c}) \equiv \frac{1}{2}\left\{\|\mathbf{K}(\mathbf{c}) - \mathbf{g}\|^2 + \alpha\|R\mathbf{c}\|^2\right\} \quad . \tag{5.32}$$

This is exactly the problem which is addressed in the optimization results of the last section. As demonstrated there, the functional $\mathbf{T}_\alpha(\mathbf{c})$ is minimized via a quasi-Newton method in conjunction with a trust region scheme. Numerical examples demonstrating these procedures are given in the next section.

Numerical Examples

The examples in this section demonstrate the results that are obtained when the Sinc-Galerkin methods from the third section are applied to parameter identification problems of the form (5.1) and (5.2) posed on $(0,1)\times(0,\infty)$. The choice of decay values and weights for such problems was thoroughly discussed in Chapter 4 and will only be briefly mentioned in these examples. The errors for the

methods are reported on the set of uniform gridpoints with stepsize $\ell = \frac{1}{100}$

$$U = \{z_0, z_1, \ldots, z_{100}\}$$

(5.33)

$$z_k = k\ell \quad , \quad \ell = 0, 1, \ldots, 100 \quad .$$

If the true and approximate parameters are denoted by EI and $\widehat{\text{EI}}$ respectively, then the errors are reported as

$$\|\text{EI}_U^{M_z}(\ell)\| = \max_{0 \le k \le 100} |\text{EI}(z_k) - \widehat{\text{EI}}(z_k)| \quad . \tag{5.34}$$

From (5.34), the convergence rate of the methods can be checked in the same manner as it was in previous chapters. The dependence of the errors on M_E can be noted in (3.119) and (3.162).

The first two examples report results for problems with fixed boundary conditions while the third example gives results for a problem with cantilever boundary conditions. In all three examples, the data was sampled on a regular grid $\{(x_p, t_q)\} \subset (0,1) \times (0,3]$. Four equally spaced points $x_p = p\Delta x$, $\Delta x = .2$, were taken in space and six equally spaced temporal points $t_q = q\Delta t$, $\Delta t = .5$, were taken for a total of $s = 24$ data points. *A priori* information was used to get general forms for the initial guesses $\mathbf{c}_0$.

Example 5.19 Consider the parameter identification problem

$$\mathcal{L}(\text{EI})u \equiv \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(\text{EI}(x)\,\frac{\partial^2 u}{\partial x^2}\right) = f(x,t) \quad , \quad \begin{matrix} 0 < x < 1 \\ t > 0 \end{matrix}$$

$$u(0,t) = u(1,t) = 0 \quad , \quad t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0 \quad , \quad t > 0$$

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0 \quad , \quad 0 \le x \le 1 \quad .$$

The forcing function $f(x,t)$ is consistent with the true parameter $\text{EI}(x) = 1 + \sin(\pi x)$ and the true solution $u(x,t) = x(1-x)\sin(4\pi x)t^2 e^{-t}$. The analysis from

the forward problem (see Example 4.4) yields the decay parameters $\alpha = \beta = 1$, $\gamma = \frac{3}{2}$, and $\delta = 1$. The regularization parameter was taken to be $\alpha = 10^{-10}$ and the initial vector $\mathbf{c}_0$ was of the form $\mathbf{c}_0 = [2, .5, .5, \ldots, .5, .5, -2]^T$. The errors and solutions are reported in Table 18 and Figure 29, respectively. The number of basis functions $(m_E + 4)$ that are used in the expansion of $\widehat{\mathrm{EI}}$ are also reported in the table. These values follow from (3.119). As seen from the results in the last column of the table, the errors are decreasing exponentially. The method was also run with the regularization parameters $\alpha = 10^{-8}$ and $\alpha = 10^{-12}$ and the results were similar.

M_x	N_x	M_t	N_t	$(m_E + 4)$	$\|\mathrm{EI}_U^{M_x}(\ell)\|$	$\left(\|\mathrm{EI}_U^{M_x/2}(\ell)\|\right)^{\sqrt{2}}$
8	8	6	3	17	$1.7280 - 0$	———
16	16	11	4	33	$.6263 - 1$	———
32	32	22	7	65	$.2529 - 1$	$.1988 - 1$

Table 18. Errors on the Uniform Grid U for Example 5.19.

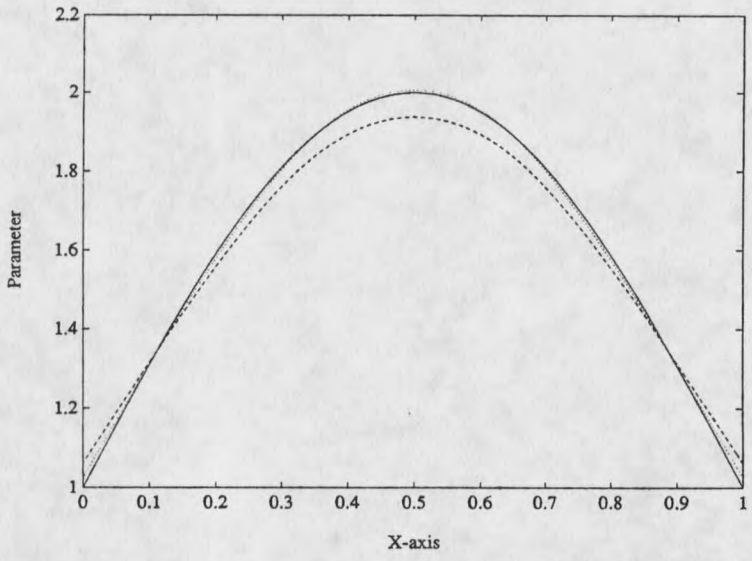


Figure 29. True and Approximate Parameters for Example 5.19
$- - -$ $(m_E + 4 = 33)$, $\cdots$ $(m_E + 4 = 65)$, — (True).

Example 5.20 Consider

$$\mathcal{L}(\mathrm{EI})u = f(x,t) \quad , \quad 0 < x < 1 \ , \ t > 0$$

$$u(0,t) = u(1,t) = 0 \quad , \quad t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0 \quad , \quad t > 0$$

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0 \quad , \quad 0 \le x \le 1 \quad .$$

Here $f(x,t)$ is consistent with the true parameter $\mathrm{EI}(x) = 1 + \frac{1}{4}\, e^{-40(x-1/3)^2}$ and the true solution $u(x,t) = x(1-x)\sin(4\pi x)t^2 e^{-t}$. The choices $\alpha = \beta = \gamma = \frac{3}{2}$, $\delta = 1$ were used for the decay parameters (see Example 4.5). The regularization parameter was taken to be $\alpha = 10^{-12}$, and the initial vector $\mathbf{c}_0$ was of the form $\mathbf{c}_0 = [0, .5, .5, \ldots, .5, .5, 0]^T$. The errors and solutions are reported in Table 19 and Figure 30. Here it is seen that the errors are decreasing, but not at an exponential rate. This may be due in part to the difficulties in choosing the

decay parameters (see again Example 4.5). The method was also run with the regularization parameter $\alpha = 10^{-10}$ and the results were similar.

M_x	N_x	M_t	N_t	$(m_E + 4)$	$\|\mathrm{EI}_U^{M_x}(\ell)\|$	$\left(\|\mathrm{EI}_U^{M_x/2}(\ell)\|\right)^{\sqrt{2}}$
8	8	8	4	17	$.4130 - 0$	———
16	16	16	6	33	$.8966 - 1$	$.2863 - 0$
32	32	32	9	65	$.6361 - 1$	$.3302 - 1$

Table 19. Errors on the Uniform Grid U for Example 5.20.

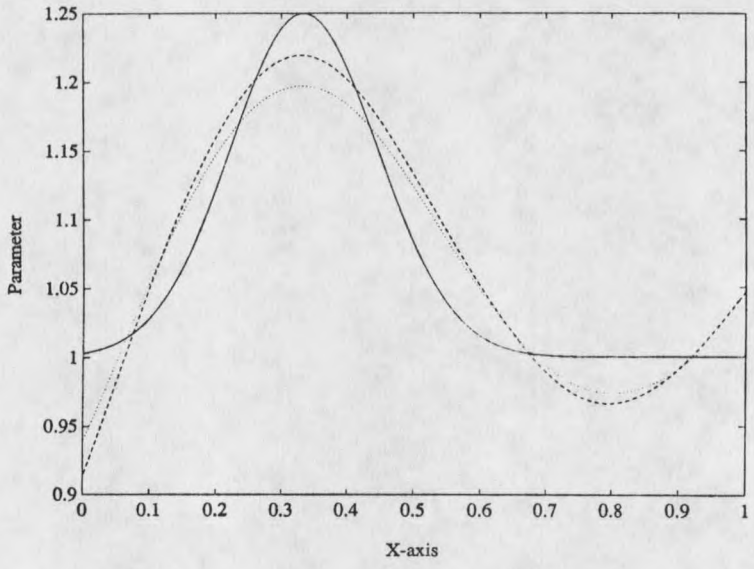


Figure 30. True and Approximate Parameters for Example 5.20
$- - -$ $(m_E + 4 = 33)$, $\cdots$ $(m_E + 4 = 65)$, — (True).

Example 5.21 Consider

$$\mathcal{L}(\mathrm{EI})u = f(x,t) \ , \ 0 < x < 1 \ , \ t > 0$$

$$u(0,t) = 0 \ , \ \left(\mathrm{EI}\,\frac{\partial^2 u}{\partial x^2}\right)(1,t) = -8\pi t^2 e^{-t} \ , \ t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = t^2 e^{-t} \ , \ \frac{\partial}{\partial x}\left(\mathrm{EI}\,\frac{\partial^2 u}{\partial x}\right)(1,t) = 8\pi(\pi - 3)t^2 e^{-t} \ , \ t > 0$$

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0 \ , \ 0 \le x \le 1 \ .$$

As a result of the cantilever boundary conditions, the unknown parameter $\mathrm{EI}(x)$ appears both in the differential operator $\mathcal{L}$ and in the boundary conditions. The forcing function $f(x,t)$ is consistent with the true parameter $\mathrm{EI}(x) = 1 + \sin(\pi x)$ and the true state solution $u(x,t) = x(1-x)\sin(4\pi x)t^2 e^{-t}$. The choices $\alpha = \beta = 1$, $\gamma = \frac{3}{2}$, $\delta = 1$ were used for the decay parameters (see Example 4.7), the regularization parameter was taken to be $\alpha = 10^{-8}$, and the initial guess was $\mathbf{c}_0 = [2, .5, .5, \ldots, .5, .5, -2]^T$. The errors and solutions are reported in Table 20 and Figure 31. The number of basis functions $(m_E + 4)$ that are used in the expansion of $\widehat{\mathrm{EI}}$ are also reported in the table. These values follow from (3.162).

M_x	N_x	M_t	N_t	$(m_E + 4)$	$\|\mathrm{EI}_U^{M_x}(\ell)\|$	$\left(\|\mathrm{EI}_U^{3M_x/4}(\ell)\|\right)^{2/\sqrt{3}}$
24	24	16	6	53	.9804 − 1	———
32	32	22	7	69	.6952 − 1	.6845 − 1

Table 20. Errors on the Uniform Grid U for Example 5.21.

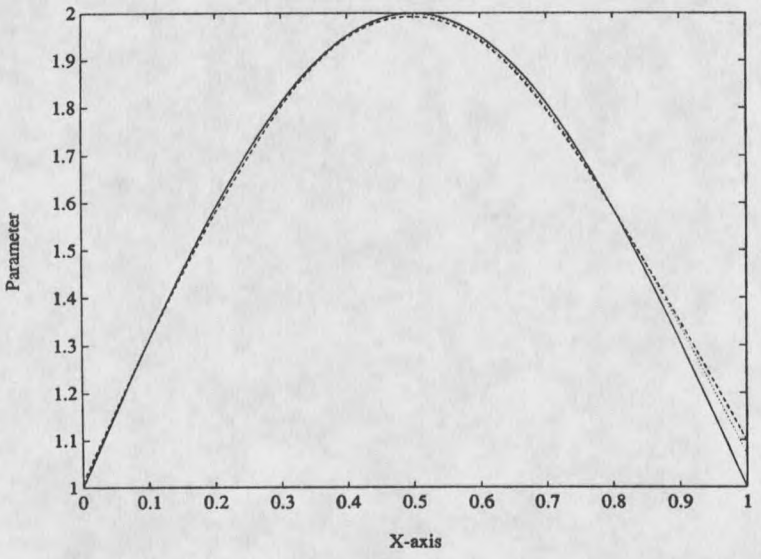


Figure 31. True and Approximate Parameters for Example 5.21
$- - -$ $(m_E + 4 = 53)$, $\cdots$ $(m_E + 4 = 69)$, — (True).

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