Some planar embeddings of chainable continua can be expressed as inverse limit spaces
by Susan Pamela Schwartz

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics
Montana State University

Abstract:
It is well known that chainable continua can be expressed as inverse limit spaces and that chainable continua are embeddable in the plane. We give necessary and sufficient conditions for the planar embeddings of chainable continua to be realized as inverse limit spaces.

As an example, we consider the Knaster continuum. It has been shown that this continuum can be embedded in the plane in such a manner that any given composant is accessible. We give inverse limit expressions for embeddings of the Knaster continuum in which the accessible composant is specified. We then show that there are uncountably many non-equivalent inverse limit embeddings of this continuum.

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ABSTRACT

It is well known that chainable continua can be expressed as inverse limit spaces and that chainable continua are embeddable in the plane. We give necessary and sufficient conditions for the planar embeddings of chainable continua to be realized as inverse limit spaces.

As an example, we consider the Knaster continuum. It has been shown that this continuum can be embedded in the plane in such a manner that any given composant is accessible. We give inverse limit expressions for embeddings of the Knaster continuum in which the accessible composant is specified. We then show that there are uncountably many non-equivalent inverse limit embeddings of this continuum.

CHAPTER 1

INTRODUCTION

This thesis presents the necessary and sufficient conditions for a planar embedding of a chainable continuum to be expressed as an inverse limit space. The theorem is illustrated by giving inverse limit expressions for embeddings of the Knaster bucket handle continuum. In the process, results are obtained concerning the accessibility of composants and the equivalence of embeddings.

Chainable continua are essentially planar. Bing [1] has demonstrated that every chainable continuum can be embedded in the plane in such a manner that the defining sequence of chains are comprised of interiors of rectangles, i.e. topological disks. We wish to distinguish a similar type of chainability. We will say that a continuum in the plane is <u>disk-chainable</u> if it can be chained in such a manner that for every $\epsilon > 0$ there is an ϵ-chain whose links are topological disks and all intersections of those links are topological disks.

Furthermore, Bing [1] has shown that given a chainable continuum X, there are uncountably many mutually exclusive homeomorphic copies of X in the plane $\mathbb{R}^2$. If Y and Z are elements of this collection, there exists a homeomorphism Φ which takes Y onto Z. If, in addition, Φ extends to a homeomorphism of the plane then Y and Z are <u>equivalently</u> <u>embedded</u>. Bing [1] has given the following example to demonstrate that a chainable continuum may have non-equivalent embeddings. This example also serves as an illustration that there exist non-disk-chainable embeddings of a continuum.

> Let $f_1(x)$ be the function whose graph is the sum of (a) the graph A_1 of $y = 2\sin 3x/2 \quad (0 \le x \le 2\pi/3)$, (b) the graph A_2 of $y = 3\sin 6x \quad (2\pi/3 \le x \le 5\pi/6)$, (c) the graph A_3 of $y = -\cos 3x \quad (\pi/2 \le x \le 5\pi/6)$, and (d) the set $B_1 + B_2 + B_3$, where B_i is symmetric to A_i with respect to the point $(\pi/2, 0)$. Then $f_1(x)$ is a single valued function for some values, triple valued for others, and double valued for two values. The closure M_1 of the graph of $y = f_1(\pi/x \quad mod \quad \pi) \quad (0 < x \le 1)$ is a snake-like continuum but it does not have the property that for each positive number ϵ there is an ϵ-chain covering it whose links are connected.
>
> Let $f_2(x)$ be the single valued function whose graph is the sum of A_1, A_2, the graph A_4 of $y = \cos 3x \quad (5\pi/6 \le x \le 7\pi/6)$ and the set $C_1 + C_2 + C_4$, where C_i is symmetric to A_i with respect to the point $(7\pi/6, 0)$. The closure M_2 of the graph of $y = f_2(\pi/x \quad mod \quad 7\pi/3) \quad (0 < x \le 3/7)$ is homeomorphic with M_1 but M_2 can be covered by ϵ-chains with connected links.

It is well known [2] that a chainable continuum, being arc-like, is homeomorphic to an inverse limit of arcs with onto bonding maps. But what of the embedding of a chainable continuum? Must it always be expressable as an inverse limit space? In seeking to explain what we mean by an embedding expressed as an inverse limit we have considered three things: (*i*) the fact that chainable continua are inverse limits of arcs, (*ii*) a theorem of Brown [3] which says that the inverse limit of copies of the same compact metric space with bonding maps which are near-homeomorphisms is itself that compact metric space, and (*iii*) a technique of Martin and Barge [4] for constructing global attractors in the plane. We then make the following definition:

The bounded planar continuum X is <u>embeddable using inverse limits</u> provided

(i) there exists a sequence of near homeomorphisms $\{F_n\}_{n\geq 0}$ from the closed disk D onto itself where $F_n(I) = I$ for I, an arc in D; and

(ii) there exists a homeomorphism $\Phi : (D, F_n) \to \hat{D}$, a closed disk containing X, such that $\Phi\big((I, F_{n|I})\big) = X$.

We are concerned with the question: which embeddings of chainable continua are embeddable using inverse limits? In answer, we present our main theorem, whose proof is given in Chapter 3:

Theorem 3.1: A continuum in the plane is disk-chainable if and only if it is embeddable using inverse limits.

Rephrasing Bing's results in our language and applying our results, we have that every chainable continuum has a disk-chainable embedding. That is, every chainable continuum has an embedding which can be expressed as an inverse limit. In addition, a chainable continuum may have other, non-equivalent embeddings. These may not have an inverse limit expression, as in Bing's example M_1. Or it may be the case that there are non-equivalent disk-chainable embeddings. We demonstrate the latter in Chapter 4, where we embed the Knaster continuum, M, using inverse limits. In the process we obtain a second affirmative answer to a question of Martin and Barge as to whether M can be embedded in the plane with any composant accessible.

CHAPTER 2

TERMS AND NOTATION

A continuum is a nondegenerate, compact, connected metric space. A continuum, X, is chainable or snake-like, if for every $\epsilon > 0$ there exists an ϵ-chain of X; an ϵ-chain being a finite open cover of X whose elements, links, have the property that (i) the diameter of each link is smaller than ϵ and (ii) only adjacent links have non-empty intersection. That is, let $C = \{l_1, \ldots, l_n\}$ be an ϵ-chain of X. Then $diam \quad l_j < \epsilon \quad \forall j \in \{1, \ldots, n\}$ and $l_i \cap l_j \neq \emptyset$ iff $|i - j| \leq 1$. We note that the links need not be connected. Also, every chainable continuum has a defining sequence of chains [5]. That is, there exists $\{C^j\}_{j=0}^{\infty}$, a sequence of ϵ_j-chains of X where $\epsilon_{j+1} < \epsilon_j$ and $\lim_{j\to\infty} \epsilon_j = 0$. In addition, the closure of each link of C^{j+1} is contained in a link of C^j.

In its broadest definition, the inverse limit space X_∞ associated with the inverse limit sequence $\{X_\alpha, f_{\alpha\beta}\}$ is as follows: let $\mathcal{A}$ be a directed set ordered by "$>$"; X_α a topological space; $\alpha, \beta, \gamma \in \mathcal{A}$; $\quad \gamma \geq \beta \geq \alpha$; $\quad \{f_{\alpha\beta} : X_\beta \to X_\alpha\}$ a collection of bonding maps with the properties (i) $f_{\alpha\alpha} = Id_{|X_\alpha}$ and (ii) $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$. Then X_∞ is a subset of the product space $\prod_{\alpha\in\mathcal{A}} X_\alpha$. With the product topology and projection maps $\pi_\beta : \prod_{\alpha\in\mathcal{A}} X_\alpha \to X_\beta$ we describe the inverse limit space X_∞ as

$$X_\infty = \{\underline{x} \in \prod_{\alpha\in\mathcal{A}} X_\alpha | f_{\alpha\beta} \circ \pi_\beta(\underline{x}) = \pi_\alpha(\underline{x}) \text{ whenever } \beta \geq \alpha\}.$$

For our purpose we only consider the countable indexing set $\mathbb{N}$; our factor spaces $\{X_n\}_{n=0}^\infty$ are metric spaces; and the bonding maps are continuous. For simplicity of notation we denote the bonding maps by $\{f_n : X_{n+1} \to X_n\}_{n=0}^\infty$ so that for $m \geq n$, f_{nm} is written as $f_n \circ f_{n+1} \ldots \circ f_{m-1}$. We choose the standard notation (X_n, f_n) for the inverse limit space X_∞ and casually refer to it as the "inverse limit". By x_n we mean $\pi_n(\underline{x})$. Then

$$(X_n, f_n) = \{\underline{x} \in \prod_{n=0}^{\infty} X_n | f_n(x_{n+1}) = x_n\}$$

with the induced metric topology is a metric space which is compact (connected) whenever the X_n are compact (connected). In fact, if $\{X_n\}$ are $\mathcal{P}$-like and $\{f_n\}$ are surjections $\forall n \in \mathbb{Z}^+$ then (X_n, f_n) is $\mathcal{P}$-like [2]. This implies that the inverse limit of arc-like continua is arc-like. Since arc-like is synonymous with chainable [2], we see that the inverse limit of chainable continua with onto bonding maps is a chainable continuum.

In the case where we have an inverse limit (X_n, f_n) with a single bonding map, there is an induced homeomorphism $\hat{f}$ on (X_n, f_n) called the shift map, defined by $\hat{f}((x_0, x_1, \ldots,)) = (f(x_0), x_0, x_1, \ldots)$.

A continuous function is a near-homeomorphism if it can be uniformly approximated by a sequence of homeomorphisms.

In this paper we appeal to the following theorem of Morton Brown [3]:

> Let $S = (X_n, f_n)$ where the X_n are all homeomorphic to a compact metric space X, and for all n, f_n is a near-homeomorphism. Then S is homeomorphic to X.

CHAPTER 3

MAIN THEOREM

In this chapter we prove the necessary and sufficient conditions for our theorem:

> **Theorem 3.1** A continuum in the plane is disk-chainable if and only if it is embeddable using inverse limits.

Before proceeding with this proof, we are in need of some vocabulary, a technical proposition and two supporting lemmas.

Suppose (X, d) is a compact metric space and

$$\mathcal{C}(X) = \{C | C \text{ is a compact subset of } X\}.$$

Then $d_H : \mathcal{C}(X) \times \mathcal{C}(X) \to \mathbb{R}^+$, the Hausdorf metric on $\mathcal{C}(X)$ is given by

$$d_H(C_1, C_2) = \inf\{r | C_i \subseteq N_r(C_j) \qquad i \neq j \in \{0,1\}\}$$

where

$$N_r(C_j) = \cup_{x \in C_j} B_r(x)$$

and

$$B_r(x) = \{y \in X | d(x,y) < r\}.$$

That is, $N_r(C_j)$ is an open neighborhood about the set C_j in the d-metric topology on X.

An equivalent definition is useful:

$$d_H(C_1, C_2) = \max\{d(x_i, C_j) | x_i \in C_i, i \neq j \in \{0, 1\}\}$$

where $d(x_i, C_j)$ is the usual distance between a point and a set.

Lemma 3.2: If (X, d) is a compact metric space, then $(\mathcal{C}(X), d_H)$ is also a compact metric space.

Proof of lemma. It is known that $(\mathcal{C}(X), d_H)$ is a complete metric space [6]. Therefore it suffices to show that $(\mathcal{C}(X), d_H)$ is totally bounded.

In the metric topology on $(\mathcal{C}(X), d_H)$, let $B_\epsilon(C)$ be an open ϵ-ball about the point C, that is

$$B_\epsilon(C) = \{D \in \mathcal{C}(X) | d_H(D, C) < \epsilon\}.$$

Given $\epsilon > 0$, we will produce a finite number of elements of $\mathcal{C}(X)$, $\{C_i\}_{i=1}^s$, such that $\cup_{i=1}^s B_\epsilon(C_i) = \mathcal{C}(X)$.

With a given $\epsilon > 0$, cover X with $\epsilon/4$-balls, $\{X_i\}$, and consider a finite subcover $\{X_1, X_2, \ldots, X_n\}$ where $n = n(\epsilon)$. Let $x_i \in X_i$, $i \in \{1, 2, \ldots, n\}$. Consider the power set $\mathcal{P}(\{x_1, \ldots, x_n\})$. $\# \ \mathcal{P}(\{x_1, \ldots, x_n\}) = \sum_{k=0}^n \binom{n}{k} = 2^n$. Let $\{C_i\}_{i=1}^{2^n}$ be the 2^n elements of the power set. Claim:

$$\mathcal{C}(X) = \cup_{i=1}^{2^n} B_\epsilon(C_i).$$

The proof of the claim is as follows.

(*i*) Clearly $\cup_{i=1}^{2^n} B_\epsilon(C_i) \subseteq \mathcal{C}(X)$.

(*ii*) To show containment in the other direction, let $D \in \mathcal{C}(X)$ and then consider

D as a subset of X. $\exists \{i_0, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ such that $D \cap X_i \neq \emptyset$, $i \in \{i_0, \ldots, i_k\}$ and $D \cap X_i = \emptyset$ for $i \notin \{i_0, \ldots, i_k\}$.

Let $C_{\hat{i}} = \{x_{i_0}, \ldots, x_{i_k}\}$. We show that $D \in B_\epsilon(C_{\hat{i}})$ as follows:

(a) $D \subseteq \cup_{j=0}^{k} X_{i_j} \Rightarrow$

$$\forall x \in D \qquad \exists j \text{ such that } d(x, x_{i_j}) < \epsilon/2$$

which implies

$$D \subseteq N_{\epsilon/2}(C_{\hat{i}})$$

since $C_{\hat{i}} = \{x_{i_0}, \ldots, x_{i_k}\}$ and $N_{\epsilon/2}(C_{\hat{i}}) = \cup_{j=0}^{k} B_{\epsilon/2}(x_{i_j})$.

(b) On the other hand,

$$N_{\epsilon/2}(D) = \cup_{x \in D} B_{\epsilon/2}(x)$$

and we have that

$$D \cap X_{i_j} \neq \emptyset \qquad \forall j \in \{0, \ldots, k\}.$$

We recall that X_{i_j} is an open $\epsilon/4$-ball in (X, d). Thus

$$\exists x \in D \text{ such that } d(x, x_{i_j}) < \epsilon/2$$

implies that for each $j \in \{0, \ldots, k\}$

$$x_{i_j} \in B_{\epsilon/2}(x) \text{ for some } x \in D,$$

which implies that

$$C_{\hat{i}} \subseteq N_{\epsilon/2}(D).$$

Then we see that

$$d_H(D, C_{\hat{i}}) = \inf\{r | C_{\hat{i}} \subseteq N_r(D) \text{ and } D \subseteq N_r(C_{\hat{i}})\} \leq \epsilon/2 < \epsilon,$$

$\Rightarrow$

$$D \in B_\epsilon(C_{\hat{i}})$$

$\Rightarrow$

$$\mathcal{C}(X) \subseteq \cup_{i=1}^{2^n} B_\epsilon(C_i).$$

We have shown that $(\mathcal{C}(X), d_H)$ is totally bounded in addition to being complete, therefore $(\mathcal{C}(X), d_H)$ is compact. ∎

Our object here is this: $(\mathcal{C}(X), d_H)$ being a compact metric space is sequentially compact.

A function is called monotone if point inverses are connected. We next show that in a certain setting, near-homeomorphisms are monotone.

Proposition 3.3: If (X, d) is a compact, locally connected metric space and $H : X \to X$ is a near- homeomorphism, then H is monotone.

Proof. We wish to show that any point inverse is connected. We will do so by demonstrating that in $(\mathcal{C}(X), d_H)$ a point inverse of H is the limit of a sequence of (compact) connected sets in X, and is therefore itself (compact) connected [6].

Let $c \in X$ and $H = \lim_{n\to\infty} H_n$ where $H_n : X \to X$ are homeomorphisms. Given $\epsilon' > 0$, by local connectedness $\exists 0 < \epsilon \leq \epsilon'$ and a U_ϵ, open and connected with *diam* $U_\epsilon < \epsilon$ and $c \in U_\epsilon \subseteq B_{\epsilon'}(c)$. We note that $\bar{U}_\epsilon$ is compact and connected, as is $H_n^{-1}(\bar{U}_\epsilon)$. Also, $x \in H^{-1}(c)$ if and only if given any $\eta > 0$, there is an integer $N(\eta)$, dependent on η, such that $n \geq N \Rightarrow d(H_n(x), c) < \eta$.

Define

$$A_k = H_{n(k)}^{-1}(\bar{U}_{1/k}) \text{ for } k > 0 \text{ and } n(k) \geq N(1/k).$$

We see that $\{A_k\}_{k=1}^{\infty}$ has a convergent subsequence because in the previous lemma $(\mathcal{C}(X), d_H)$ was shown to be sequentially compact. For simplicity of notation, call this subsequence $\{A_k\}_{k=1}^{\infty}$. It is now possible to find a subsequence of this $\{A_k\}_{k=1}^{\infty}$ such that if $k > j$ then $n(k) > n(j)$. Again call this subsequence $\{A_k\}_{k=1}^{\infty}$ and let $A = \lim_{k\to\infty} A_k$.

Claim:

$$A = H^{-1}(c).$$

The claim is proven as follows. (*i*) Suppose $x \in H^{-1}(c)$. Then by definition, $x \in A_k \quad \forall k$. If $x \notin A$, then because A is compact we can say that $d(x, A) = r > 0$. $\exists k(r)$ such that $k \geq k(r) \Rightarrow d_H(A, A_k) < r/2$. This yields $A_k \subseteq N_{r/2}(A)$. But x cannot be in an $r/2$-neighborhood about A since the distance between x and A is r which is larger than $r/2$. Then it cannot be that x is in A_k, which by definition it is. Hence a contradiction. Thus $x \in H^{-1}(c)$ implies $x \in A$ which implies $H^{-1}(c) \subseteq A$.

(*ii*) Suppose $x \in A$. We wish to show that x must be in $H^{-1}(c)$. To show $x \in H^{-1}(c)$ it suffices to do the following: given $\epsilon > 0$, produce a p such that $d(H_n(x), c) < \epsilon \quad \forall n \geq n(p)$.

Let $\epsilon > 0$.

(a) Because $\{H_n\}$ is a uniformly Cauchy sequence,

$$\exists I \text{ such that } i_1, i_2 \geq I$$

$\Rightarrow$

$$d(H_{i_1}(y), H_{i_2}(y)) < \epsilon/6 \qquad \forall y \in X.$$

(b) Choose k such that

$$1/k < \epsilon/4$$

and

$$n(k) \geq I.$$

(c) Since $H_{n(k)}$ is continuous, we have the existence of a $\delta(k, \epsilon)$ such that

$$d(x, y) < \delta \Rightarrow d(H_{n(k)}(x), H_{n(k)}(y)) < \epsilon/4.$$

(d) Because $\{A_k\}$ converges we have a J such that

$$j \geq J \Rightarrow d_H(A, A_j) < \delta.$$

Then

$$\max\{d(y, A_j), d(y_j, A) | y \in A, y_j \in A_j\} < \delta$$

$\Rightarrow$

$$d(x, A_j) < \delta$$

$\Rightarrow$

$$\exists y_j \in A_j \text{ such that } d(x, y_j) < \delta.$$

Now let $p > \max\{J, k\}$. Then

$$p > k \Rightarrow 1/p < 1/k < \epsilon/4$$

and

$$n(p) > n(k) \geq I.$$

Further,

$$\exists y \in A_p \text{ such that } d(x, y) < \delta.$$

This implies

$$d(H_{n(k)}(x), H_{n(k)}(y)) < \epsilon/4$$

by part *(c)*. By the triangle inequality we obtain

$$d(H_{n(p)}(x), c) < 5\epsilon/6$$

as follows:

$$d(H_{n(p)}(x), c) \leq d(H_{n(p)}(x), H_{n(k)}(x))$$

$$+d(H_{n(k)}(x), H_{n(k)}(y)) + d(H_{n(k)}(y), H_{n(p)}(y)) + d(H_{n(p)}(y), c)$$

$$< \epsilon/6 + \epsilon/4 + \epsilon/6 + \epsilon/4 = 5\epsilon/6.$$

And by part *(a)*

$$d(H_n(x), H_{n(p)}(x)) < \epsilon/6 \qquad \forall n \geq n(p) > n(k) \geq I$$

so

$$d(H_n(x), c) \leq d(H_n(x), H_{n(p)}(x)) + d(H_{n(p)}(x), c) < 5\epsilon/6 + \epsilon/6 = \epsilon.$$

Hence we have show that given $\epsilon > 0$ and $x \in A$, $\exists p$ such that $n \geq n(p)$ implies

$$d(H_n(x), c) < \epsilon$$

$\Rightarrow$

$$x \in H^{-1}(c)$$

$\Rightarrow$

$$A \subseteq H^{-1}(c).$$

Parts (i) and (ii) prove the claim that $A = H^{-1}(c)$. But A being the limit of (compact) connected sets is (compact) connected. Thus $H^{-1}(c)$ is connected, and H is monotone. ■

Lemma 3.4: If H is a near-homeomorphism and U is an open disk, then $H^{-1}(U)$ is simply connected.

Proof. Let $H = \lim_{t\to 0} H_t$ where H_t are homeomorphisms. Let S be a simple closed curve in the open set $H^{-1}(U)$. S is compact, thus $H(S)$ is a compact subset of $U = H(H^{-1}(U))$. There exist a $\delta > 0$ and a $T > 0$ such that for all $t \leq T$, $H_t(S) \subseteq U$; in fact, $d(H_t(S), \partial U) > \delta > 0$.

Now the region B bounded by S is an open disk. So $H_t(B)$ is an open disk contained in U.

$$H(B) = \lim_{t\to 0} H_t(B)$$

and

$$d(H_t(B), \partial U) = d(H_t(S), \partial U) > \delta > 0 \quad \text{for } t \leq T.$$

This implies that $H(B) \subseteq U$ which implies that

$$B \subseteq H^{-1}(H(B)) \subseteq H^{-1}(U).$$

Thus every simple closed curve in $H^{-1}(U)$ bounds a region which is entirely contained in $H^{-1}(U)$. This implies that $H^{-1}(U)$ is simply connected. ∎

We now prove the necessary conditions of Theorem 3.1:

If a continuum is embeddable using inverse limits then it is disk-chainable.

Proof. Let D be a compact disk and I be a closed arc in D; $D_n = D$, $I_n = I$ and $I_n \subseteq D_n \quad \forall n \in \mathbb{Z}^+$. Suppose there are near-homeomorphisms $F_n : D_{n+1} \to D_n$, where $F_{n|I_{n+1}} = f_n : I_{n+1} \to I_n$ is a surjection. We know that $(D_n, F_n) \cong D$. Let $\Phi\Big((D_n, F_n)\Big) = D$ where Φ is a homeomorphism. Then (I_n, f_n) and hence $\Phi\Big((I_n, f_n)\Big)$ is a chainable continuum [2]. $\Phi\Big((I_n, f_n)\Big)$ is embeddable using inverse limits. We will show that $\Phi\Big((I_n, f_n)\Big)$ is a disk-chainable embedding by chaining (I_n, f_n) in (D_n, F_n) with chains whose links are topological disks, and the intersections of the links are also disks.

We notice the following. Given any n, let $C = \{l_1, \ldots, l_m\}$ be a δ-chaining of I_n by disks where $l_i \cap l_j$ is a disk $\forall i, j \in \{1, \ldots, m\}$.

(*i*) $F_n^{-1}(l_j)$ is non-empty and open because F_n is continuous; $F_n^{-1}(l_j)$ is connected by proposition 3.3; and $F_n^{-1}(l_j)$ is simply connected by lemma 3.4. That is, $F_n^{-1}(l_j)$ is a topological disk.

(*ii*) $\cup_{j=1}^{m} F_n^{-1}(l_j)$ covers I_{n+1}.

(iii) $\left(l_i \cap l_j \neq \emptyset \text{ iff } F_n^{-1}(l_i) \cap F_n^{-1}(l_j) \neq \emptyset\right) \Rightarrow \{F_n^{-1}(l_1), \ldots, F_n^{-1}(l_m)\}$ is a chaining of I_{n+1}.

(iv) $F_n^{-1}(l_i) \cap F_n^{-1}(l_j) = F_n^{-1}(l_i \cap l_j)$ is a disk for the same reasons as given in (i).

Let $\epsilon > 0$ be given. Let n be large enough such that

$$\frac{1}{2^{n-3}} < \frac{\epsilon}{diam \quad D}.$$

Choose

$$\delta < \min\{(1 - \frac{1}{2^n}) \cdot \epsilon, \alpha\}$$

where $\alpha > 0$ is chosen in a manner such that if

$$C = \{l_1, \ldots, l_m\} \quad \text{is an } \alpha - \text{chain}$$

then

$$diam \quad F_{n-1-k} \circ \cdots \circ F_{n-2} \circ F_{n-1}(l_j) < \frac{2^{n-2}}{2^n - 1} \cdot \epsilon \qquad \text{for} \qquad k = 0, 1, \ldots, n-1.$$

For this n, $\exists \delta$-chain of I_n whose links are connected. Call this δ-chain of I_n

$$C_n = \{l_1, l_2, \ldots, l_{m_n}\}.$$

Let

$$\hat{l}_j = \{\underline{x} \in (D_k, F_k) | x_n \in l_j\} \qquad \text{for} \quad j \in \{1, 2, \ldots, m_n\}$$

and let

$$\hat{C}_n = \{\hat{l}_1, \hat{l}_2, \ldots, \hat{l}_{m_n}\}.$$

Claim: $\hat{C}_n$ is an ϵ-chain of (I_k, f_k) whose links, and the intersections of those links, are connected. This claim is established in the following manner.

(i) We first establish that

$$diam \ \hat{l}_j < \epsilon \qquad \forall j \in \{1, 2, \ldots, m_n\}.$$

Let $\underline{x}, \underline{y} \in \hat{l}_j$. Then

$$\begin{aligned} d(\underline{x}, \underline{y}) &= \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i} \\ &\leq \sum_{i=0}^{n-1} \frac{|x_i - y_i|}{2^i} + \sum_{i=n}^{\infty} \frac{diam \quad D}{2^i} \\ &\leq \sum_{i=0}^{n-1} \frac{diam \quad F_{n-1-i} \circ \cdots \circ F_{n-2} \circ F_{n-1}(l_j)}{2^{n-1-i}} + (diam \quad D)(\frac{1}{2^{n-1}}) \\ &< \frac{2^{n-2}}{2^n - 1} \cdot \epsilon \sum_{i=0}^{n-1} \frac{1}{2^i} + \frac{\epsilon}{4} \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{4} = 3\epsilon/4. \end{aligned}$$

Thus

$$diam \quad \hat{l}_j = \sup_{\underline{x}, \underline{y} \in \hat{l}_j} d(\underline{x}, \underline{y}) \leq 3\epsilon/4 < \epsilon.$$

(ii) We next observe the $\hat{C}_n$ covers (I_k, f_k). Let

$$\underline{x} = (x_0, x_1, \ldots) \in (I_k, f_k) \subseteq (D_k, F_k).$$

Then

$$\pi_n(\underline{x}) = x_n \in I_n \Rightarrow \exists j \in \{1, 2, \ldots, m_n\} \text{ such that } x_n \in l_j.$$

By definition

$$\hat{l}_j = \{\underline{y} \in (D_k, F_k) | y_n \in l_j\} \Rightarrow \underline{x} \in \hat{l}_j.$$

(iii) We also see that $\hat{l}_j \cap \hat{l}_i \neq \emptyset$ iff $|j - i| \leq 1$, as follows: $\hat{l}_j \cap \hat{l}_i \neq \emptyset$ iff $\exists \underline{x} \in (\hat{l}_j \cap \hat{l}_i)$ iff $\exists n$ such that $x_n \in (l_j \cap l_i)$ iff $(l_j \cap l_i) \neq \emptyset$ iff $|j - i| \leq 1$ since $C_n = \{l_1, \ldots, l_{m_n}\}$ is a δ-chain of I_n.

(iv) Finally we see that the elements of $\hat{C}_n$ are open and connected sets with connected intersections.

Let $\underline{x} \in \hat{l}_j$ for any $\hat{l}_j \in \hat{C}_n$. Consider the map $\underline{x} = (x_0, x_1, \ldots) \to (x_n, x_{n+1}, \ldots)$ i.e. $h : (D_k, F_k) \to (D_k, F_k)$ by $h(\underline{x}) = \underline{y}$ where $y_k = x_{n+k}$. Clearly h is a homeomorphism. For simplicity of notation, let $\hat{l}_j$ now denote $h(\hat{l}_j)$ and $\hat{C}_n$ denote $h(\hat{C}_n)$ as follows:

$$\hat{C}_n = \{\hat{l}_j\}_{j=1}^{m_n} \text{ where } \quad \hat{l}_j = \{\underline{y} \in h((D_k, F_k)) | y_0 \in l_j\}.$$

Then

$$\pi_k(\hat{l}_j) = \begin{cases} l_j & \text{for } k = 0 \\ F_{n+k-1}^{-1} \circ \cdots \circ F_{n+1}^{-1} \circ F_n^{-1}(l_j) & \text{for } k > 0 \end{cases}.$$

Thus $\pi_k(\hat{l}_j)$ is a disk. And we see that

$$\hat{l}_j = (\pi_k(\hat{l}_j), F_{n+k})$$

and

$$\overline{\hat{l}_j} = \overline{(\pi_k(\hat{l}_j), F_{n+k})} = (\overline{\pi_k(\hat{l}_j)}, F_{n+k}).$$

Invoking Brown's theorem, we have $\hat{l}_j = \text{int}\overline{\hat{l}_j}$ is an open disk. This makes $\hat{C}_n$ a chaining by disks. Furthermore, since $l_i \cap l_j$ is a disk, the same argument gives us $\hat{l}_i \cap \hat{l}_j = \widehat{l_i \cap l_j}$ is a disk.

In conclusion, $\forall \epsilon > 0 \quad \exists \hat{C}_n$, an ϵ-chain of (I_n, f_n) whose links and intersections of those links are disks. We have shown that the continuum $\Phi\big((I_n, f_n)\big)$ which is embeddable using inverse limits is disk-chainable. □

We proceed with the proof of the sufficient conditions for theorem 3.1:

If X is a disk chainable embedding of a chainable continuum, then X is embeddable using inverse limits.

Proof. Let C be a chainable continuum and X a disk chainable embedding of C in the closed disk D. We can make the following construction:

(i) $\{B^j\}_{j=0}^{j=\infty}$ is a defining sequence of chains for X where

(a) the links $\{B^j_m\}_{m=1}^{m(j)}$ of B^j are connected,

(b) the closures of the intersections of adjacent links in a chain are connected,

(c) $diam \quad B^j_m = \bar{\epsilon}_j \quad \forall m \in \{1, 2, \ldots, m(j)\}$ where $\lim_{j\to\infty} \bar{\epsilon}_j = 0$;

(ii) $K_j \subseteq \cup_{m=1}^{m(j)} B^j_m$ is a closed arc such that

(a) $K_j \cap (B^j_m \cap B^j_{m-1})$ is an arc $\forall m \in \{2, 3, \ldots, m(j)\}$ and

(b) the endpoints of K_j are in $X \cap B^j_1$ and $X \cap B^j_{m(j)}$ respectively.

(c) $B^j_1 \cap X$ is not a subset of B^j_2 and $B^j_{m(j)} \cap X$ is not a subset of $B^j_{m(j)-1}$.

We now have "nice" disks with arcs running through them and can construct maps which "collapse" the disks onto the arcs in the following manner:

(iii) $G_j : D \to D$ are Lipschitz near-homeomorphisms with Lipschitz constants $\bar{\delta}_j$; and there are open neighborhoods V_j of $\cup_{m=1}^{m(j)} B^j_m$ where

(a)

$$V_0 \subseteq D, \qquad \cup_{m=1}^{m(j)} B^j_m \subseteq V_j \subseteq \cup_{m=1}^{m(j-1)} B^{j-1}_m \quad \text{for} \quad j = 1, 2, \ldots$$

(b)

$$G_j(V_j) = K_j \quad \forall j = 0, 1, 2, \ldots$$

(c)

$$G_j(B^j_m) \subseteq B^j_m \qquad \forall m \in \{1, 2, \ldots, m(j)\}; \qquad j \in \{0, 1, 2, \ldots, \}$$

and

$$G_j(B^{j-1}_m) \subseteq B^{j-1}_m$$

(d) G_j restricted to the complement of $\cup_{m=1}^{m(j-1)} B^{j-1}_m$ is the identity, $j = 1, 2, \ldots$ and G_0 is the identity outside some neighborhood of V_0.

(a) through *(d)* gives us

$$\|G_j - id\| \leq \bar{\epsilon}_{j-1}$$

and

$$d(g_j(x), x) \leq \bar{\epsilon}_j \quad \text{where} \quad g_j = G_{j|K_{j+1}}.$$

Now we will choose a subsequence $\{C^j_m\}$ of $\{B^j_m\}$. Find a j_0 such that $\bar{\epsilon}_{j_0} < 1/2$. Let

$$\epsilon_0 = \bar{\epsilon}_{j_0} < 1/2; \qquad C^0_m = B^{j_0}_m; \qquad I_0 = K_{j_0}; \qquad U_0 = V_{j_0}.$$

Rename G_{j_0} as F_0 and define

$$\delta_0 = \max\{\bar{\delta}_{j_0}, 1\}.$$

Next choose $j_1 > j_0$ with $\bar{\epsilon}_{j_1} < \frac{1}{2\delta_0}$. Let $\epsilon_1 = \bar{\epsilon}_{j_1}$ and rename $B^{j_1}_m, K_{j_1}, V_{j_1}$ and G_{j_1} as C^1_m, I_1, U_1, F_1, respectively. Define $\delta_1 = \max\{\bar{\delta}_{j_1}, 1\}$.

Choose $j_2 > j_1$ with $\bar{\epsilon}_{j_2} < \frac{1}{2\delta_0\delta_1 2^2}$ and proceed to reindex the chain, arc, open set and function, and define a new δ as above.

Continue thus, so that

$$\epsilon_k < \frac{1}{2\delta_0\delta_1 \cdots \delta_{k-1} \cdot k^k} \quad \text{and} \quad \delta_k = \max\{\bar{\delta}_{j_k}, 1\}.$$

Let $f_j = F_{j|I_{j+1}}$ and define

$$\Lambda = (J_j, f_{j|J_{j+1}}) \quad \text{where} \quad J_j = \cap_{k \geq j} f_j \circ \cdots \circ f_k(I_{k+1}).$$

We may also think of J_j as

$$J_j = \cap_{k \geq 0} J_j^k \text{ where } J_j^k = f_j \circ \cdots \circ f_{j+k}(I_{j+k+1}).$$

Given a j, we will show that

$$J_j^k \cap C_1^j \neq \emptyset \quad \forall k \geq 0.$$

From our construction, we have that

$$I_{j+1} \cap C_m^{j+1} \neq \emptyset \quad \forall m \in \{1, 2, \ldots, m(j+1)\},$$

and that

$$\exists m_1 \in \{1, 2, \ldots, m(j+1)\} \text{ such that } cl\ C_{m_1}^{j+1} \subseteq C_1^j.$$

Similarly,

$$I_{j+2} \cap C_m^{j+2} \neq \emptyset \quad \forall m \in \{1, 2, \ldots, m(j+2)\}$$

and $\exists m_2$ in this set with $cl\ C_{m_2}^{j+2} \subseteq C_{m_1}^{j+1}$. Thus given an m_k, there is an $m_{k+1} \in \{1, 2, \ldots, m(j+k+1)\}$ with

$$cl\ C_{m_{k+1}}^{j+k+1} \subseteq C_{m_k}^{j+k} \text{ and } I_{j+k+1} \cap C_{m_{k+1}}^{j+k+1} \neq \emptyset.$$

Consider $J_j^0 = f_j(I_{j+1})$. There is an $a_0 \in I_{j+1} \cap C_{m_1}^{j+1}$ and $b_0 = f_j(a_0) \in f_j(I_{j+1} \cap C_{m_1}^{j+1}) \subseteq f_j(I_{j+1}) \cap f_j(C_{m_1}^{j+1}) \subseteq J_j^0 \cap f_j(C_1^j) \subseteq J_j^0 \cap C_1^j$. Thus $J_j^0 \cap C_1^j \neq \emptyset$, since $b_0 \in J_j^0 \cap C_1^j$.

Suppose we have $m_1, m_2, \ldots, m_{k+1}$ where $cl\ C_{m_{k+1}}^{j+k+1} \subseteq \cdots \subseteq cl\ C_{m_1}^{j+1} \subseteq C_1^j$. Then there is an $a_k \in I_{j+k+1} \cap C_{m_k}^{j+k+1}$ with

$$f_j \circ \cdots \circ f_{j+k}(a_k) \in f_j \circ \cdots \circ f_{j+k}(I_{j+k+1} \cap C_{m_k}^{j+k+1})$$

$$\subseteq f_j \circ \cdots \circ f_{j+k}(I_{j+k+1}) \cap f_j \circ \cdots \circ f_{j+k}(C_{m_{k+1}}^{j+k+1})$$

$$\subseteq J_j^k \cap f_j \circ \cdots \circ f_{j+k-1}(C_{m_k}^{j+k}).$$

The key step here is that

$$f_{j+k}(C_{m_{k+1}}^{j+k+1}) \subseteq f_{j+k}(C_{m_k}^{j+k}) \subseteq C_{m_k}^{j+k}.$$

So we see that

$$b_k = f_j \circ \cdots \circ f_{j+k}(a_k) \in J_j^k \cap C_1^j.$$

Thus

$$J_j^k \cap C_1^j \neq \emptyset \quad \forall k \geq 0;$$

and

$$\lim_{k\to\infty} b_k \in cl\ C_1^j,$$

$$\lim_{k\to\infty} b_k \in \cap_{k\geq 0} J_j^k$$

implies

$$J_j \cap cl\ C_1^j \neq \emptyset.$$

We now see that our choice of the first link C_1^j of C^j was not crucial to the argument. We could in fact have shown

$$J_j \cap cl\ C_m^j \neq \emptyset \quad \forall m \in \{1, 2, \ldots, m(j)\}.$$

All told, J_j is not only a compact, connected, non-empty (nested intersection of non-empty compact sets) subset of an arc, but has interior as well (i.e. is not a point). Thus Λ is a non-degenerate planar continuum.

To complete the proof, it suffices to show that Λ and X are equivalently embedded, and that this makes X embeddable using inverse limits. We define

$$H : (D, F_j) \to D \text{ by } H(\underline{z}) = \lim_{n \to \infty} z_n$$

where $z_n = \pi_n(\underline{z})$. We now establish that H is the homeomorphism we seek by verifying that

(i) H is well-defined,

(ii) H is continuous,

(iii) $H(\Lambda) = X$,

(iv) H is one to one,

(v) H is onto.

(i) *Proof that H is well defined.* We wish to show that $\{z_n\}$ has a unique limit point. Suppose not. Suppose w and y are both cluster points for $\{z_n\}$; and $d(w, y) = \delta > 0$. Then $\exists$ subsequences $\{z_{n_j}\}$ and $\{z_{n_k}\}$ such that $z_{n_j} \to w$, $z_{n_k} \to y$. Thus, given $\delta/3$, $\exists J, K$ such that

$$\{z_{n_j}\} \subseteq B_{\delta/3}(w) \quad \text{for} \quad j \geq J,$$

and

$$\{z_{n_k}\} \subset B_{\delta/3}(y) \quad \text{for} \quad k \geq K.$$

Choose $n_j > n_k$ such that

$$j \geq J, \quad k \geq K \quad \text{and} \quad \sum_{i=n_k}^{\infty} \epsilon_i < \delta/3.$$

We have

$$z_{n_k} = f_{n_k} \circ \cdots \circ f_{n_j-2} \circ f_{n_j-1}(z_{n_j}).$$

In addition,

$$d(f_n(x), x) \leq \epsilon_n,$$

so

$$d(z_{n_j}, z_{n_k}) \leq d(z_{n_j}, z_{n_j-1}) + d(z_{n_j-1}, z_{n_j-2}) + \cdots + d(z_{n_k+1}, z_{n_k})$$

$$= d(z_{n_j}, f_{n_j-1}(z_{n_j})) + d(z_{n_j-1}, f_{n_j-2}(z_{n_j-1})) + \cdots + d(z_{n_k+1}, f_{n_k}(z_{n_k+1}))$$

$$\leq \epsilon_{n_j-1} + \epsilon_{n_j-2} + \cdots + \epsilon_{n_k}$$

$$\leq \sum_{i=n_k}^{\infty} \epsilon_i < \delta/3.$$

Now

$$d(w, y) \leq d(w, z_{n_j}) + d(z_{n_j}, z_{n_k}) + d(z_{n_k}, y)$$

$\Rightarrow$

$$d(w, y) < \delta/3 + \delta/3 + \delta/3 = \delta.$$

This is a contradiction. Thus there is not more than one cluster point for $\{z_n\}$ and $H(\underline{z}) = \lim_{n\to\infty} z_n$ is well defined. ∎

(*ii*) *Proof that H is continuous.* Let $\epsilon > 0$ be given, and choose $\underline{z} \in (D, F_j)$. We will produce a $\delta > 0$ such that if $d(\underline{z}, \underline{y}) < \delta$ then $d(H(\underline{z}), H(\underline{y})) < \epsilon$.

Let N be sufficiently large, such that

$$n \geq N \Rightarrow d(z_n, H(\underline{z})) < \epsilon/3$$

and

$$\sum_{k=N-1}^{\infty} \epsilon_k < \epsilon/6.$$

Choose

$$\delta < \frac{\epsilon}{3 \cdot 2^N}.$$

Find a $\underline{y}$ with

$$d(\underline{z}, \underline{y}) < \delta.$$

Then

$$d(\underline{z}, \underline{y}) = \sum_{k=0}^{\infty} \frac{d(z_k, y_k)}{2^k} < \delta$$

$\Rightarrow$

$$\frac{d(z_k, y_k)}{2^k} < \delta \qquad \forall k = 0, 1, 2, \ldots$$

$\Rightarrow$

$$\frac{d(z_N, y_N)}{2^N} < \delta < \frac{\epsilon}{3 \cdot 2^N}$$

$\Rightarrow$

$$d(z_N, y_N) < \epsilon/3.$$

Now choose $m > N$ large enough such that

$$d(y_m, H(\underline{y})) < \epsilon/6.$$

Then

$$d(H(\underline{z})), H(\underline{y})) < d(H(\underline{z}), z_N) + d(z_N, y_N) + d(y_N, y_m) + d(y_m, H(\underline{y}))$$

$$< \epsilon/3 + \epsilon/3 + \sum_{k=N-1}^{\infty} \epsilon_k + \epsilon/6$$

$$\leq \epsilon/3 + \epsilon/3 + \epsilon/6 + \epsilon/6 = \epsilon. \blacksquare$$

(*iii*) *Proof that* $H(\Lambda) = X$. We will first show that $H(\Lambda) \subseteq X$. Suppose $\underline{x} \in \Lambda$ but, for the purposes of contradiction, $H(\underline{x}) = \lim_{n\to\infty} x_n \notin X$. Then $d(H(\underline{x}), X) = \eta > 0$ because both $\{H(\underline{x})\}$ and X are compact sets. Choose N large enough so that (a) $\epsilon_n < \eta/3 \quad \forall n \geq N$ and (b) $d(x_n, H(\underline{x})) < \eta/3 \quad \forall n \geq N$. Now

$$x_n \in I_n \subseteq C^n_m \quad \text{for some} \quad m \in \{1, 2, \ldots, m_n\}$$

and

$$X \cap C^n_m \neq \emptyset \quad \text{for that} \quad m.$$

In other words, both x_n and points of X are in a set of diameter smaller than ϵ_n. So

$$d(x_n, X) = \inf_{x \in X} d(x_n, x) \leq \epsilon_n.$$

Thus

$$\eta = d(H(\underline{x}), X) \leq d(H(\underline{x}), x_n) + d(x_n, X)$$

$$< \eta/3 + \eta/3 < \eta$$

and this is a contradiction. We conclude that

$$\underline{x} \in \Lambda \Rightarrow H(\underline{x}) \subseteq X$$

which implies that

$$H(\Lambda) \subseteq X.$$

In order to show that $X \subseteq H(\Lambda)$, observe that Λ is a closed subset of a compact space, hence compact. As H is a continuous function, we have that $H(\Lambda)$ is a compact subset of a closed disk in $\mathbb{R}^2$, hence closed. Thus, given $x \in X$, it suffices to show

$$x \in \overline{H(\Lambda)} = H(\Lambda).$$

That is, $\forall \epsilon > 0$ we must produce a $\underline{y}(\epsilon) \in \Lambda$ such that

$$H(\underline{y}(\epsilon)) \in B_\epsilon(x).$$

Let $x \in X$ and $\epsilon > 0$. Choose n large enough so that

$$\sum_{k=n}^{\infty} \epsilon_k < \epsilon/2,$$

and note that this implies $\epsilon_n < \epsilon/2$. Since $x \in X$,

$$\exists m_n \text{ such that } x \in C^n_{m_n}.$$

We have already shown that

$$C^n_k \cap J_n \neq \emptyset \quad k \in \{1, 2, \cdots, m(n)\}.$$

Let

$$y_n \in C^n_{m_n} \cap J_n.$$

Then

$$d(x, y_n) \leq \epsilon_n$$

because both $y_n \in C^n_{m_n}$ and $x \in C^n_{m_n}$. And

$$\exists \underline{y}(\epsilon) \equiv \underline{y} \in \Lambda \text{ such that } \pi_n(\underline{y}) = y_n$$

because $y_n \in J_n$.

We now see that

$$d(x, H(\underline{y})) \leq d(x, y_n) + d(y_n, H(\underline{y}))$$

$$= d(x, y_n) + d(y_n, \lim_{k \to \infty} y_k)$$

$$\leq d(x, y_n) + d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots$$

$$\leq \epsilon_n + \sum_{k=n}^{\infty} \epsilon_k$$

$$< \epsilon/2 + \epsilon/2 = \epsilon,$$

which is equivalent to

$$H(\underline{y}) \in B_\epsilon(x). \blacksquare$$

(*iv*) *Proof that H is one to one.* We first prove that H is one to one on Λ. Suppose

$$\exists \quad \underline{x}, \underline{y} \in \Lambda \text{ such that } H(\underline{x}) = H(\underline{y}) = x.$$

We will show that this implies $\underline{x} = \underline{y}$ by demonstrating that given any $\epsilon > 0$,

$$d(x_j, y_j) < \epsilon \quad \forall j = 0, 1, 2, \cdots.$$

Let $\epsilon > 0$ be given. Choose n large enough so that

$$\sum_{k=n}^{\infty} \frac{1}{k^k} < \epsilon.$$

Now

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y_n)$$

$$\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots$$

$$= d(f_n(x_{n+1}), x_{n+1}) + \cdots + d(f_n(y_{n+1}), y_{n+1}) + \cdots$$

$$\leq 2 \cdot \sum_{k=n}^{\infty} \epsilon_k.$$

And

$$d(x_{n-1}, y_{n-1}) = d(f_{n-1}(x_n), f_{n-1}(y_n)) \leq \delta_{n-1} \cdot d(x_n, y_n).$$

This implies that $\forall j < n$

$$d(x_j, y_j) \leq 2 \cdot \delta_j \cdot \delta_{j+1} \cdots \delta_{n-1} \cdot \sum_{k=n}^{\infty} \epsilon_k$$

$$< 2 \cdot \delta_j \cdot \delta_{j+1} \cdots \delta_{n-1} \cdot \sum_{k=n}^{\infty} \frac{1}{2 \cdot \delta_0 \cdot \delta_1 \cdots \delta_{k-1} \cdot k^k}$$

$$< 2 \cdot \delta_0 \cdot \delta_1 \cdots \delta_j \cdots \delta_{n-1} \cdot \sum_{n=k}^{\infty} \frac{1}{2 \cdot \delta_0 \cdot \delta_1 \cdots \delta_{k-1} \cdot k^k}$$

$$= \frac{1}{n^n} + \frac{1}{\delta_n \cdot (n+1)^{(n+1)}} + \frac{1}{\delta_n \cdot \delta_{n+1} \cdot (n+2)^{(n+2)}} + \cdots$$

$$\leq \sum_{k=n}^{\infty} \frac{1}{k^k} < \epsilon.$$

Thus for $j < n$, we see immediately that

$$d(x_j, y_j) < \epsilon$$

and for $j \geq n$ we see that

$$d(x_j, y_j) \leq 2 \cdot \sum_{k=j}^{\infty} \epsilon_k$$

$$\leq 2 \cdot \sum_{k=j}^{\infty} \frac{1}{2 \cdot \delta_0 \delta_1 \cdots \delta_{k-1} \cdot k^k}$$

$$\leq \sum_{k=j}^{\infty} \frac{1}{k^k}$$

and if

$$\sum_{k=n}^{\infty} \frac{1}{k^k} < \epsilon$$

then

$$\sum_{k=j}^{\infty} \frac{1}{k^k} < \epsilon$$

for $j \geq n$.

We now prove that H is one-to-one off Λ. Suppose

$$\exists \underline{x}, \underline{y} \in (D, F_j) - \Lambda \text{ such that } H(\underline{x}) = H(\underline{y}) = x.$$

We claim that

$$\underline{x} \notin \Lambda \Rightarrow \exists M \text{ such that } n \geq M \Rightarrow x_n = x_M.$$

The proof of the claim proceeds as follows.

$$\underline{x} \notin \Lambda \Rightarrow \exists N \text{ such that } x_N \notin J_N$$

which means that for m infinitely often, $x_m \notin I_m$. Choose one such $m > 0$. If $x_m \notin I_m$, then $x_{m+1} \notin U_m$, since $F_m(U_m) = I_m$ and $F_m(x_{m+1}) = x_m$. This implies that

$$x_{m+1} \in (D - \cup_{k=1}^{k(m-1)} C_k^{m-1}) \cup (\cup_{k=1}^{k(m-1)} C_k^{m-1} - U_m).$$

If

$$x_{m+1} \in D - \cup_{k=1}^{k(m-1)} C_k^{m-1}$$

then

$$F_m(x_{m+1}) = x_{m+1} = x_m$$

and in fact

$$F^{-1}_{m+1}(x_{m+1}) = x_{m+1} = x_{m+2}, = \cdots = F^{-1}_{m+j}(x_{m+j}) = x_{m+j} = x_{m+j+1} \qquad \forall j \geq 0$$

so

$$n \geq m \Rightarrow x_n = x_m.$$

If

$$x_{m+1} \in \cup_{k=1}^{k(m-1)} C_k^{m-1} - U_m,$$

knowing that $x_{m+1} = F_{m+1}(x_{m+2})$ and that F_{m+1} is the identity on $D - \cup_{k=1}^{k(m)} C_k^m$ then either

(a) $x_{m+2} \notin \cup_{k=1}^{k(m)} C_k^m$ and $x_n = x_{m+1} \quad \forall n \geq m+1$ or

(b) $x_{m+2} \in \cup_{k=1}^{k(m)} C_k^m$. And because $F_{m+1}(U_{m+1}) = I_{m+1}$ we have more precisely that $x_{m+2} \in \cup_{k=1}^{k(m)} C_k^m - U_{m+1}$, otherwise x_{m+1} would be in I_{m+1}, which would contradict our choice of m. This means that $x_{m+1} = F_{m+1}(x_{m+2}) = x_{m+2}$. But $x_{m+1} \notin U_m$ therefore $x_{m+1} \notin \cup_{k=1}^{k(m)} C_k^m$ and cannot equal x_{m+2}.

Now if

$$\exists M \text{ such that } x_n = x_M \quad \forall n \geq M$$

then

$$H(\underline{x}) = \lim_{n\to\infty} x_n = x_M.$$

Since we have assumed

$$H(\underline{y}) = H(\underline{x})$$

we also have

$$\lim_{n\to\infty} y_n = x_M.$$

In addition,

$$\underline{y} \notin \Lambda \Rightarrow \exists N \text{ such that } n \geq N \Rightarrow y_n = y_N$$

so

$$\lim_{n\to\infty} y_n = y_N \Rightarrow x_M = y_N.$$

Thus if $k = \max\{M, N\}$ we have $x_n = x_k$ and $y_n = y_k \quad \forall n \geq k$. But $x_k = y_k$. Then $x_n = y_n \quad \forall n \geq k$ which also implies that $x_n = y_n \quad \forall n < k$; and we conclude that $\underline{x} = \underline{y}$. Thus H is one-to-one off Λ.

In addition, we have shown that if $\underline{x} \notin \Lambda$ there will be an N with $n \geq N \Rightarrow x_n \notin \cup_{m=1}^{m(n)} C_m^n \Rightarrow \lim_{n\to\infty} x_n \notin \cap_{n\geq N} \cup_{m=1}^{m(n)} C_m^n \Rightarrow x_n \notin \cap_{n\geq 0} \cup_{m=1}^{m(n)} C_m^n$. But $X \subseteq \cap_{n\geq 0} \cup_{m=1}^{m(n)} C_m^n$. So we may conclude that

$$\underline{x} \notin \Lambda \Rightarrow \lim_{n\to\infty} x_n = H(\underline{x}) \notin X.$$

We know that $H(\Lambda) = X$. Thus if $\underline{x} \notin \Lambda$ and $\underline{y} \in \Lambda$ then $H(\underline{x}) \neq H(\underline{y})$. ■

(v) *Proof that H is onto.* Let $x \in D$.

(a) If

$$x \in X$$

then by (iii)

$$\exists \underline{x} \in \Lambda \text{ such that } H(\underline{x}) = x.$$

(b) If

$$x \in D - X$$

then

$$\exists N \text{ such that } n \geq N \Rightarrow x \notin \cup_{m=1}^{m(n)} C_m^n.$$

Let

$$\underline{x} \in (D, F_j) \text{ such that } \pi_{N+1}(\underline{x}) = x.$$

Then referring to the definition of the F_j we see that $H(\underline{x}) = x$. ■

We have shown that H is a continuous bijection from the compact space (D, F_j) to the Hausdorf space D, hence a homeomorphism. It remains to show that X is embeddable using inverse limits.

Let $\{\varphi_n\}_{n\geq 0}$ be homeomorphisms of the disk D that take J_n onto I. Define $\{G_n\}_{n\geq 0}$ to be the near-homeomorphisms

$$G_n = \varphi_n \circ F_n \circ \varphi_{n+1}^{-1}.$$

This gives us

$$G_n(I) = I.$$

The $\{\varphi_n\}_{n\geq 0}$ induce a homeomorphism φ from (D, G_n) onto (D, F_n) [5]. Then

$$\Phi \equiv H \circ \varphi : (D, G_n) \to D$$

is a homeomorphism such that

$$\Phi((I, G_{n_{|I}})) = H(\Lambda) = X.$$

We have assumed X to be disk-chainable and shown that it then must be embeddable using inverse limits, completing the proof of the sufficient conditions for theorem 3.1. ■

CHAPTER 4

AN EXAMPLE

In this chapter we embed the Knaster continuum, M, using inverse limits. We first give some definitions and describe two functions which we use as bonding maps. We then present a proposition concerning the accessible points of these embeddings. In a corollary to the proposition, we answer a question of Barge and Martin concerning which composants can be made accessible in embeddings of the Knaster continuum. This question has been addressed by Mahavier [7], and our results complement his. We conclude this chapter with a proposition which relates the equivalence of inverse limit embeddings to an equivalence relations on the composants of M.

If X is a continuum in a space Z, we say $x \in X$ is accessible if there exists an arc $\alpha \subseteq Z$ such that (i) $\alpha = \text{ Im } \hat{\alpha}, \quad \hat{\alpha} : [0,1] \to Z$; (ii) $\hat{\alpha}(0) = x$; (iii) $\hat{\alpha}(t) \cap X = \emptyset \quad \forall t \in (0,1]$. Equivalently, we say, α accesses x (from $Z - X$).

Given a continuum X and a point $x \in X$, the composant of X determined by x is

$$C_x = \{y \in X | \exists \text{ a proper subcontinuum containing } x \text{ and } y\}.$$

The Knaster continuum is an example of an indecomposable continuum, one which cannot be expressed as the union of two proper subcontinua. For an indecomposable continuum, we know [2] that the composants are uncountable in number, the composants partition the continuum, each composant is dense in the continuum, and "$x \in C_y$" is an equivalence relation. In the case of the Knaster continuum the composants are arc components [2]. Furthermore, M is the full attracting set of the standard Smale horseshoe [8].

If $\underline{\alpha}$ is an arc in an inverse limit space, we assume it is the image of $[0, 1]$ under a map called $\hat{\alpha}$. By α_n we mean $\pi_n(\underline{\alpha})$, which is an arc in the factor space.

The tent map $T : I \to I$ is given by

$$T = \begin{cases} 2x & \text{for } 0 \le x \le 1/2 \\ 2 - 2x & \text{for } 1/2 \le x \le 1 \end{cases}.$$

We define the Knaster continuum M in a usual manner as the inverse limit of the tent map, i.e. $M = (I, T)$.

Let $I_0 = [0, 1/2]$ and $I_1 = [1/2, 1]$. Define the relation

$$\theta_n : (I, T) \to \{0, 1\}$$

by

$$\theta_n(\underline{x}) = \begin{cases} k & \text{if } x_n \in I_k, \quad x_n \ne 1/2 \\ 0 \text{ or } 1 & \text{if } x_n = 1/2 \end{cases}.$$

In the case of (I, T) it is known that $\underline{x}$ and $\underline{y}$ are in the same composant, that is $\underline{x} \in C_{\underline{y}}$, equivalently, $\underline{y} \in C_{\underline{x}}$ if and only if there exists an N such that $\theta_n(\underline{x}) = \theta_n(\underline{y}) \quad \forall n \ge N$. We notice that if $\underline{x} \in (I, T)$ and $x_n = 1/2$ for some n, then that n is unique. This is because $x_{n-1} = 1$ and $x_k = 0 \quad \forall 0 \le k < n - 1$. So if we were to have $x_n = 1/2$ and $x_k = 1/2$ for some $k > n$ then $x_n \in \{0, 1\}$ which would contradict $x_n = 1/2$. Therefore the ambiguity in the definition of $\theta_n(\underline{x})$ causes no confusion in determining which points would be in $C_{\underline{x}}$, nor does it cause a problem in any of the subsequent proofs regardless as to whether 0 or 1 is chosen for $\theta_n(\underline{x})$ when $x_n = 1/2$.

Let $\underline{0}$ be the point in (I, T) such that $\pi_n(\underline{0}) = 0 \quad \forall n$. $\underline{0}$ is a topologically distinguished point. If any two subcontinua in (I, T) contain $\underline{0}$, then one is contained in the other. This makes $\underline{0}$ an endpoint of (I, T). It is in fact the only endpoint.

Let

$$D = \{z \in \mathbf{C} | z = 1/2 + re^{i\theta} \text{ for } 0 \leq r < 1/2\}$$

$$D_u = \{z \in \mathbf{C} | z \in D \text{ and } Im(z) > 0\}$$

$$D_l = \{z \in \mathbf{C} | z \in D \text{ and } Im(z) < 0\}$$

$$I_0 = \{z \in \mathbf{C} | z = r, 0 \leq r \leq 1/2\}$$

and similarly

$$I_1 = \{z \in \mathbf{C} | z = r, 1/2 \leq r \leq 1\}.$$

Let

$$I = I_0 \cup I_1.$$

Define

$$g_t : \overline{D_u} \to \overline{D}$$

as the map which takes

$$1/2 + re^{i\theta} \to 1/2 + re^{i[t\theta + (1-t)\pi]} \text{ for } 0 < t \leq 1, \quad 0 \leq \theta \leq \pi.$$

Define

$$h_t : \overline{D_l} \to \overline{D}$$

as the map which takes

$$1/2 + re^{i\theta} \to 1/2 + re^{i[(2-t)\theta+(t-1)\pi]} \text{ for } 0 < t \leq 1, \quad \pi \leq \theta \leq 2\pi.$$

Define

$$s_t : \overline{D} \to \overline{D}$$

as a homeomorphic extension of the map which takes I_0 into I by

$$r \to (2-t)r \text{ for } 0 < t \leq 1$$

where the s_t converge uniformly as $t \to 0$ to a near-homeomorphism s_0; and where (i) $s_0(\overline{D_u}) = \overline{D_u}$, (ii) $s_0^{-1}(I) = I$ and (iii) $s_0^{-1}(1) = I_1$. Define

$$F_t : \overline{D} \to \overline{D}$$

as

$$F_t = \begin{cases} s_t \circ g_t & \text{for } z \in \overline{D_u} \\ s_t \circ h_t & \text{for } z \in \overline{D_l} \end{cases}.$$

Note that for $t \neq 0$, F_t is a homeomorphism, and

$$F_0 = \lim_{t \to 0} F_t$$

is a near-homeomorphism. Further, F_0 takes D_l homeomorphically onto $D - I_0$. By F_1 we denote F_0 followed by a reflection in the cartesian x-axis. Now continuously extend both F_0 and F_1 to B, a closed disk of radius 1 centered at $z = 1/2$ in such a manner that they map $B - \overline{D}$ homeomorphically onto itself. For simplicity of notation continue to call them F_0 and F_1. Let f_0 and f_1 denote $F_{0|I}$ and $F_{1|I}$ respectively. Observe that f_0 and f_1 are merely the tent map on the interval. Therefore, for $j_n \in \{0,1\}$, (I, f_{j_n}) is the Knaster continuum embedded in (B, F_{j_n}), which by the theorem of Brown, is homeomorphic to a closed disk. When we say "(B, F_{j_n}) embeds M" we will be referring to this embedding. Furthermore, the choice of the j_n determine the unique accessible composant of the embedding. This is the content of the lemmas, propositions and corollaries which now follow.

Lemma 4.1 Suppose $\underline{\alpha}$ is an arc in (D, F_{j_n}), that is

$$\exists \hat{\alpha} : [0,1] \to (D, F_{j_n}) \text{ with } Im(\hat{\alpha}) = \underline{\alpha}.$$

Define

$$\hat{\alpha}_n : [0,1] \to D$$

by

$$\hat{\alpha}_n = \pi_n \circ \hat{\alpha}.$$

Let $t \in [0,1]$ and $\underline{x} \in (D, F_{j_n})$. Then

$$\hat{\alpha}(t) = \underline{x} \text{ iff } \hat{\alpha}_n(t) = x_n \quad \forall n.$$

Proof of the sufficient conditions. $\hat{\alpha}(t) = \underline{x} \Rightarrow \hat{\alpha}_n(t) = \pi_n \circ \hat{\alpha}(t) = \pi_n(\underline{x}) = x_n \quad \forall n.$ ■

Proof of the necessary conditions. We have $\underline{x} \in (D, F_{j_n})$ and $\hat{\alpha}_n(t) = x_n$ $\forall n \geq 0 \Rightarrow \pi_n \circ \hat{\alpha}(t) = x_n$. $\hat{\alpha}(t) = \underline{y}$ for some $\underline{y} \in (D, F_{j_n})$. But $\pi_n \circ \hat{\alpha}(t) = \pi_n(\underline{y}) = x_n \quad \forall n \geq 0 \Rightarrow \underline{y} = \underline{x}$ and $\hat{\alpha}(t) = \underline{x}$. ■

Lemma 4.2 Suppose $\underline{x} \in (I, f_{j_n})$ and α_n accesses $x_n \in I$ from D_l. If $\theta_{n+1}(\underline{x}) = j_{n+1}$, then there exists an arc α_{n+1} which accesses x_{n+1} from D_l such that $F_{j_n}(\alpha_{n+1}) = \alpha_n$.

Proof. F_{j_n} is a homeomorphism from D_l onto $D - I$ for both $j_{n+1} = 0$ and $j_{n+1} = 1$. Thus for $0 < t \leq 1$, $F^{-1}_{j_{n+1}}(\hat{\alpha}_n(t))$ is well defined. In fact, $F^{-1}_{j_{n+1}}(\hat{\alpha}_n((0,1]))$ is a half open arc in D_l. Observe that the function $F_{k|D_l \cup I_k}$ has a continuous inverse. We have that

$$\hat{\alpha}_n(t) \subseteq D_l \cup I \qquad \forall t \in [0,1]$$

and that for $t \neq 0$

$$(F_{j_{n+1}|D_l \cup I_{j_{n+1}}})^{-1}(\hat{\alpha}_n(t)) = F^{-1}_{j_{n+1}}(\hat{\alpha}_n(t))$$

is a half open arc in D_l. In addition

$$\lim_{t\to 0}(F_{j_{n+1}|D_l \cup I_{j_{n+1}}})^{-1}(\hat{\alpha}_n(t))$$

$$= (F_{j_{n+1}|D_l \cup I_{j_{n+1}}})^{-1}(\lim_{t\to 0}\hat{\alpha}_n(t))$$

$$= (F_{j_{n+1}|D_l \cup I_{j_{n+1}}})^{-1}(\hat{\alpha}_n(0))$$

$$= (F_{j_{n+1}|D_l \cup I_{j_{n+1}}})^{-1}(x_n)$$

which must be a subset of $D_l \cup I_{j_{n+1}}$. But $(F_{j_{n+1}})^{-1}(x_n)$ is a semicircle in $\overline{D_u}$ with center $z = 1/2$, one endpoint in I_0, one in I_1 (or the point $z = 1/2$ if $x_n = 1$). Therefore $(F_{j_{n+1}|D_l \cup I_{j_{n+1}}})^{-1}(x_n)$ which is a subset of $F^{-1}_{j_{n+1}}(x_n)$ must be the endpoint in $I_{j_{n+1}}$. Furthermore $x_{n+1} \in F_{j_{n+1}}^{-1}(x_n) \cap I$. So x_{n+1} is one of the semicircle's endpoints. If $\theta_{n+1}(\underline{x}) = j_{n+1}$ then $x_{n+1} \in I_{j_{n+1}}$. Thus

$$(F_{j_{n+1}|D_l \cup I_{j_{n+1}}})^{-1}(x_n) = x_{n+1}.$$

We have shown that

$$\alpha_{n+1} = \{x_{n+1}\} \cup \{F_{j_{n+1}}^{-1}(\hat{\alpha}_n(t)), \quad t \neq 0\}$$

is an arc in $\overline{D_l}$ which accesses x_{n+1} from D_l. And clearly $F_{j_{n+1}}(\alpha_{n+1}) = \alpha_n$. ∎

Proposition 4.3 Suppose $\underline{x} \in (I, f_{j_n})$ and $\underline{x} \neq \underline{0}$.

$$\underline{x} \text{ is accessible}$$

iff

$$\exists N \text{ such that } j_n = \theta_n(\underline{x}) \qquad \forall n > N.$$

Proof of the sufficient condition. We will prove the contrapositive: if $j_n \neq \theta_n(\underline{x})$ infinitely often then $\underline{x}$ is not accessible. We are assuming that $j_n \neq \theta_n(\underline{x})$ infinitely often, and for the purposes of contradiction also suppose that $\underline{\alpha}$ accesses $\underline{x}$ from (B, F_{j_n}). We seek to show that there exists a $t \neq 0$ such that $\hat{\alpha}(t) \cap (I, f_{j_n}) \neq \emptyset$ and thereby obtain a contradiction.

$\underline{\alpha}$ is accessing $\underline{x} \neq \underline{0}$. This means that $\underline{0} \notin Im(\hat{\alpha})$. This implies the existence of an N_1 such that for $n \geq N_1$ we have $\hat{\alpha}_n(t) \neq 0 \quad \forall t \in [0,1]$.

Notice that if

$$\{\hat{\alpha}_n(t) | t \neq 0\} \subseteq \overline{D_u}$$

then

$$\alpha_k \subseteq I \qquad \forall k < n.$$

If this were to occur infinitely often, $\underline{\alpha}$ would lie in the continuum. In fact, there can be no $t \neq 0$ such that $\hat{\alpha}_n(t) \in \overline{D_u}$ infinitely often for much the same reason. For then $\hat{\alpha}(t)$ would be in the continuum and $\underline{\alpha}$ would not be an accessing arc. In particular, $\hat{\alpha}_n(1) \notin \overline{D_u}$ infinitely often. So there must be an N_2 such that for each $n \geq N_2$

$$T_n \doteq \{t \in [0,1] | \hat{\alpha}_n((t,1]) \cap \overline{D_u} = \emptyset\} \neq \emptyset.$$

Let $t_n = \inf T_n$.

We will show that $t_n = t_{n+1}$ $\forall n \geq N > \max\{N_1, N_2\}$. First we observe two facts: (i) that $\forall n \geq N$, $\hat{\alpha}_n(t_n) \in \partial\overline{D_u}$, but $t_n \in T_n$; that is, $\hat{\alpha}_n(t) \notin \overline{D_u}$ if $t > t_n$; and that (ii) $\hat{\alpha}_{n+1}(t) \in \partial\overline{D_u} \Rightarrow \hat{\alpha}_n(t) \in I$.

Now if $t_{n+1} > t_n$ then $\hat{\alpha}_n(t_{n+1}) \notin \overline{D_u}$. But $\hat{\alpha}_{n+1}(t_{n+1}) \in \partial\overline{D_u} \Rightarrow \hat{\alpha}_n(t_{n+1}) \in I$. This is a contradiction.

If $t_n > t_{n+1}$ then $t_n \in T_{n+1}$ and also $\hat{\alpha}_{n+1}(t_n) \notin \overline{D_u}$ which implies $\hat{\alpha}_{n+1}([t_n, 1]) \cap \overline{D_u} = \emptyset$. This means

$$\exists \epsilon < 0 \text{ such that } \hat{\alpha}_{n+1}([t_n, t_n + \epsilon)) \subseteq D_l \cup (B - \overline{D}).$$

If

$$\hat{\alpha}_{n+1}([t_n, t_n + \epsilon)) \subseteq D_l$$

then

$$\hat{\alpha}_n([t_n, t_n + \epsilon)) \subseteq D - I$$

hence

$$\hat{\alpha}_n(t_n) \in D - I.$$

If

$$\hat{\alpha}_{n+1}([t_n, t_n + \epsilon)) \subseteq B - \overline{D}$$

then

$$\hat{\alpha}_n(t_n) \in B - \overline{D}.$$

In either case, $\hat{\alpha}_n(t_n) \notin \partial\overline{D_u}$, which is a contradiction.

Since neither $t_{n+1} > t_n$ nor $t_n > t_{n+1}$ is true, it must be that $t_n = t_{n+1}$ for arbitrary $n \geq N$. Call this value of t, t'. Consider $\underline{y} = \hat{\alpha}(t')$. $\pi_n(\underline{y}) = \hat{\alpha}_n(t')$ which is an element of I for n infinitely often. Hence $\underline{y} \in (I, f_{j_n})$.

Finally, we argue that $t' \neq 0$; that is, $\underline{y} \neq \underline{x}$. For if t' were equal to 0, then we would have $t_n = 0$ for all $n \geq N$. This, by definition would mean that

$$\hat{\alpha}_n((0,1]) \cap \overline{D_u} = \emptyset \quad \forall n \geq N.$$

Choose some $n + 1 > N$ where $\theta_{n+1}(\underline{x}) \neq j_{n+1}$. Then $\hat{\alpha}_n((0,1]) \cap \overline{D_u} \neq \emptyset$. This contradicts the implication we derived from supposing that $t' = 0$. Therefore we see that t' cannot equal 0. Hence $\underline{y} \neq \underline{x}$, $\underline{y} \in \underline{\alpha}$, and $\underline{y} \in (I, f_{j_n})$. We conclude that $\underline{\alpha}$ is not an accessing arc. ∎

Proof of the necessary condition. Suppose there exists an N such that $\forall n > N, \quad j_n = \theta_n(\underline{x})$. Let α_N access x_N from D_l. Let α_{N+1} be the arc given by lemma 4.2 where $F_{\theta_{N+1}(\underline{x})}(\alpha_{N+1}) = \alpha_N$, and α_{N+1} accesses x_{N+1} from D_l. Suppose we have defined α_{N+k} for $k \geq 1$. Then lemma 4.2 gives an α_{N+k+1} such that

$F_{\theta_{N+k+1}(\underline{x})}(\alpha_{N+k+1}) = \alpha_{N+k}$; and this arc α_{N+k+1} accesses x_{N+k+1} from D_l. This inductively demonstrates the existence, $\forall n > N$, of an α_n which accesses x_n from D_l where $F_{\theta_n(\underline{x})}(\alpha_n) = \alpha_{n-1}$. For $n < N$, let $\alpha_n = F_{j_{n+1}} \circ \cdots \circ F_{j_N}(\alpha_N)$. Define

$$\underline{\alpha} = (\alpha_n, F_{j_n|\alpha_n}).$$

We claim that $\underline{\alpha}$ accesses $\underline{x}$. The proof of the claim is as follows. α_n accesses x_n for $n > N$ from lemma 4.2. Thus for $n > N$,

$$\pi_n(\underline{\alpha}) \cap I = \{x_n\},$$

specifically,

$$\pi_n(\hat{\alpha}(0)) \cap \pi_n((I, f_{j_n})) = \{x_n\}$$

and

$$\pi_n(\hat{\alpha}(t)) \cap \pi_n((I, f_{j_n})) = \emptyset \text{ for } t \in (0, 1];$$

$\Rightarrow$

$$\underline{\alpha} \cap (I, f_{j_n}) = \{\underline{x}\}.$$

And clearly $\underline{\alpha}$ is an arc, implying that $\underline{\alpha}$ accesses $\underline{x}$, hence $\underline{x}$ is accessible. ■

If $\underline{x}, \underline{y} \in (I, f_{j_n})$, $\quad \underline{x} \neq \underline{0}$ and $\underline{y} \in C_{\underline{x}}$ then $\underline{x} \in C_{\underline{y}}$ and $\exists$ an N_1 such that $\theta_n(\underline{x}) = \theta_n(\underline{y}) \quad \forall n \geq N_1$. Thus if $\underline{x}$ is accessible $\exists N_2$ such that $\theta_n(\underline{x}) = j_n$ $\forall n \geq N_2$. Then $\forall n > N = \max\{N_1, N_2\}$ we have $\theta_n(\underline{x}) = j_n = \theta_n(\underline{y})$ which implies $\underline{y}$ is also accessible. We have just proven the next corollary.

Corollary 4.4 Suppose $\underline{x} \in (I, f_{j_n})$ and $\underline{x} \neq \underline{0}$. If $\underline{x}$ is accessible then $C_{\underline{x}}$ is accessible.

Furthermore, if $\underline{x}, \underline{y} \in (I, f_{j_n}) - \{\underline{0}\}$ and $C_{\underline{x}} \neq C_{\underline{y}}$, then $\theta_n(\underline{x}) \neq \theta_n(\underline{y})$ infinitely often. So if $\underline{x}$ and hence its composant $C_{\underline{x}}$ is accessible then $\exists N$ such that $n > N \Rightarrow j_n = \theta_n(\underline{x}) \Rightarrow j_n \neq \theta_n(\underline{y})$ infinitely often. This implies that $\underline{y}$ and hence $C_{\underline{y}}$ is not accessible. As a conclusion we have this corollary.

Corollary 4.5 (B, F_{j_n}) embeds M in such a way that there is a unique accessible composant.

Notice that the endpoint $\underline{0}$ is always accessible from the complement of the continuum because $\underline{\alpha}$ can be chosen with $\alpha_n \subseteq (B - \overline{D}) \cup \{z = 0\} \quad \forall n$.

As there are uncountably many sequences of $0's$ and $1's$, there are uncountably many embeddings of M by (B, F_{j_n}). We would like to sort these embeddings into collections of equivalent embeddings. Our approach is this. Let us establish an equivalence relation on the set of composants of (I, T). This will induce an equivalence relation on the set $\{(B, F_{j_n})\}$ of inverse limit embeddings of M which tells us precisely which of the $\{(B, F_{j_n})\}$ are equivalent embeddings.

First, consider this lemma concerning isotopic maps on (I, T) and composants of (I, T).

Lemma 4.6 If $f, g : (I, T) \to (I, T)$ are isotopic and K is a composant of (I, T), then $f(K) = g(K)$.

Proof. Since f and g are isotopic, there exists a homotopy $H : (I, T) \times [0, 1] \to (I, T)$ such that $H(\underline{x}, 0) = f(\underline{x})$ and $H(\underline{x}, 1) = g(\underline{x})$. Both $H(K, 0) = f(K)$ and $H(K, 1) = g(K)$ are composants of (I, T). We wish to show that $f(K) = g(K)$.

Let $\underline{y} \in f(K)$. Then $\exists \underline{x} \in K$ with $f(\underline{x}) = \underline{y}$. Let $\underline{z} = g(\underline{x})$. Consider

$$L = \{H(\underline{x}, t) | t \in [0,1]\}.$$

L is the continuous image of a compact, connected set; thus L is a subcontinuum of (I,T). L is arc connected, therefore a proper subcontinuum. In addition, both $\underline{z} = H(\underline{x}, 1)$ and $\underline{y} = H(\underline{x}, 0)$ are elements of L. This means that $\underline{z}$ and $\underline{y}$ are in the same composant and $g(K) = f(K)$. ■

Let K_1 and K_2 be composants of (I,T). Define an equivalence relation on the set of composants of (I,T) by $K_1 \sim K_2$ if and only if there exists a homeomorphism $\phi : (I,T) \to (I,T)$ such that $\phi(K_1) = K_2$. Watkins [9] has shown that every homeomorphism on (I,T) is isotopic to some power of the shift map $\hat{T} : (I,T) \to (I,T)$. Therefore if $\phi(K_1) = K_2$, our lemma and Watkins' result says that $\exists n \in \mathbb{N}$ such that $\hat{T}^n(K_1) = K_2$. This gives us a countable number of elements in each equivalence class $[K]$. As there are an uncountable number of composants, we have an uncountable number of equivalence classes.

Let us examine more carefully what is meant by the phrase "(B, F_{j_n}) embeds M." We have that $M = (I,T) = (I, f_{j_n}) \subseteq (B, F_{j_n})$. By the theorem of Morton Brown, $(B, F_{j_n}) \cong B$, which means that there exists a homeomorphism φ_j which takes (B, F_{j_n}) onto B. It is $\varphi_j((I, f_{j_n}))$ which is the planar embedding of M.

Define an equivalence relation on the set of embeddings $\{(B, F_{j_n})\}$ of M by $(B, F_{j_n}) \sim (B, F_{i_n})$ if and only if (B, F_{j_n}) and (B, F_{i_n}) are equivalent embeddings. To discuss the equivalence of two embeddings we must have a homeomorphism of

B which takes $\varphi_j((I, f_{j_n}))$ onto $\varphi_i((I, f_{i_n}))$. In our last proposition we see that this homeomorphism exists if and only if the composants which are accessible in B are equivalent in (I, T).

Proposition 4.7: Suppose K_j and K_i are the composants of (I, T) such that $\varphi_j(K_j)$ and $\varphi_i(K_i)$ are the accessible composants of $\varphi_j((I, f_{j_n}))$ and $\varphi_i((I, f_{i_n}))$ respectively. Then, using the equivalence relations established above,

$$K_j \sim K_i \text{ if and only if } (B, F_{j_n}) \sim (B, F_{i_n}).$$

Proof of the sufficienct condition. Suppose $K_j \sim K_i$. Then $\exists m$ such that $K_j = \hat{T}^m(K_i)$. Let $\underline{x} \in K_j$ and $\underline{y} \in K_i$. Since $\underline{x}$ and $\hat{T}^m(\underline{y})$ are in the same composant, $\exists N$ such that $\theta_n(\underline{x}) = \theta_n(\hat{T}^m(\underline{y})) \quad \forall n \geq N$. Suppose $N > m$ and consider the homeomorphism $h_N : (B, F_{j_n}) \to (B, F_{k_n})$ defined by $\pi_n(h_n(\underline{x})) = \pi_{n+N}(\underline{x})$. Note that h_{N-m}, defined similarly, is a homeomorphism form (B, F_{i_n}) to (B, F_{k_n}). Now define $\Phi : B \to B$ by

$$\Phi = \varphi_i \circ h_{N-m}^{-1} \circ h_N \circ \varphi_j^{-1}.$$

Φ is a homeomorphism of B which takes $\varphi_j((I, f_{j_n}))$ onto $\varphi_i((I, f_{i_n}))$, thus making $(B, F_{j_n}) \sim (B, F_{i_n})$. In the case that $N \leq m$, let $\Phi = \varphi_i \circ h_m \circ \varphi_j^{-1}$. ■

Proof of the necessary condition. Suppose that $(B, F_{j_n}) \sim (B, F_{i_n})$. Then $\exists \Phi : B \to B$, a homeomorphism such that

$$\Phi((I, f_{j_n})) = (I, f_{i_n}).$$

Then, because in equivalent embeddings accessible composants are taken to accessible composants,

$$\varphi_i^{-1} \circ \Phi \circ \varphi_j(K_j) = K_i.$$

Thus $\varphi_i^{-1} \circ \Phi \circ \varphi_j : (I,T) \to (I,T)$ is a homeomorphism which takes K_j onto K_i and $K_j \sim K_i$. ■

Let us take a look at the implications of this proof. An embedding (B, F_{j_n}) of M has a unique accessible composant $\varphi_j(K_j)$. Now K_j in (I, f_{j_n}) is really distinguished by a sequence of $0's$ and $1's$. Given $\underline{x}_1$ and $\underline{x}_2$ in K_j, there is an N such that $\theta_n(\underline{x}_1) = \theta_n(\underline{x}_2)$ forall $n \geq N$. In other words $\theta_n(\underline{x}_1)$ and $\theta_n(\underline{x}_2)$ "eventually agree." Furthermore, by proposition 4.3, given $\underline{x} \in K_j$, there is an L such that $\theta_n(\underline{x}) = j_n$ forall $n \geq L$. That is, the composant of M which will be accessible in the embedding (B, F_{j_n}) is distinguished by the sequence $\{j_n\}$; it contains those points whose image under $\{\theta_n\}$ eventually agrees with $\{j_n\}$. Proposition 4.7 says that two embeddings (B, F_{j_n}) and (B, F_{i_n}) are equivalently embedded if and only if there exists an N and an m such that $j_n = i_{n-m} \quad \forall n \geq N$; that is, if by disregarding a finite number of initial $0's$ and $1's$, the "realigned" sequences $\{j_n\}$ and $\{i_n\}$ eventually agree.

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