A boundary value problem with singular data
by John Alan Brown

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Mathematics
Montana State University

Abstract:
The classical theory of Boundary value problems ensures that there is a solution of the boundary value problem if the given boundary data possesses certain smoothness properties. There are numerous well known results which enable one to use various representations, such as a Fourier series or a Poisson integral, to represent the solution. In many cases of an applied nature, boundary value problems are formulated in terms of boundary functions which have singularities for which the classical results do not apply. A problem of this type is the main consideration of this thesis.

The thesis has three parts. The first establishes a theorem concerning the Fourier series representation of a function which is not integrable but does have an integral in the sense of a Cauchy Principal Value. In the second part, a boundary value problem is solved, where the boundary data is given by a function of the type specified above.

Thirdly, a problem from hydrodynamics is analyzed, obtaining new results concerning performance characteristics of certain types of hydrofoils and airfoils.

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The author, John Alan Brown, was born in Hennepin County, Minnesota, March 26, 1930, the son of Mr. and Mrs. James L. Brown. He received his secondary education at Central High School, Minneapolis, Minnesota, graduating in 1947. He attended the University of Minnesota at Minneapolis and received a Bachelor of Arts degree in Bacteriology in 1951. In 1959 he received a Master of Science degree in Mathematics from the University of Minnesota.

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ABSTRACT

The classical theory of boundary value problems ensures that there is a solution of the boundary value problem if the given boundary data possesses certain smoothness properties. There are numerous well known results which enable one to use various representations, such as a Fourier series or a Poisson integral, to represent the solution. In many cases of an applied nature, boundary value problems are formulated in terms of boundary functions which have singularities for which the classical results do not apply. A problem of this type is the main consideration of this thesis.

The thesis has three parts. The first establishes a theorem concerning the Fourier series representation of a function which is not integrable but does have an integral in the sense of a Cauchy Principal Value. In the second part, a boundary value problem is solved, where the boundary data is given by a function of the type specified above. Thirdly, a problem from hydrodynamics is analyzed, obtaining new results concerning performance characteristics of certain types of hydrofoils and airfoils.

CHAPTER 0

INTRODUCTION

The use of Fourier Series or various integral representations to solve boundary value problems is subject to rather severe limitations. In the case of Fourier series, there is a multitude of sufficient conditions for convergence, all requiring a certain smoothness of the boundary data. It is stated in many texts that for most applications there is sufficient smoothness to ensure the desired convergence properties. However, there are many boundary value problems of considerable practical as well as pedagogical interest in which the boundary data does not possess the properties which enable the classical theorems to apply. An example of this is in analyzing the flow past a parabolic strut. This particular problem will be studied in a later chapter.

In the following, the (C,1) summability of

$\frac{a_o}{2} + \sum_{1} a_k \cos kx + b_k \sin kx$, where the a_k and b_k are defined

in terms of Cauchy Principal Value integrals, will be investigated. Subsequently, a boundary value problem with singular data will be solved and a uniqueness theorem established. Finally, a particular problem in linearized hydrodynamics will be analyzed.

CHAPTER I

AN EXAMPLE

In analyzing the free boundary flow past a parabolic strut an essential part of the analysis is the representation of a function which is to be analytic for $|z|>1$ and whose imaginary part is to be $\frac{1}{\cos\Theta}$ for $|z|=1$. Formally, such a representation would be

$$i\, C_0/2 + i \sum_{n=1}^{\infty} C_n z^{-n} \tag{1.1}$$

where C_n are the Fourier cosine coefficients of $\frac{1}{\cos\Theta}$. Unfortunately, an inspection of the integrals defining these coefficients shows that they do not exist except in the sense of a Cauchy principal value.

Interpreting these C_n as P.V. $\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos n\Theta\, d\Theta}{\cos\Theta}$, we shall investigate the series (1.1).

To calculate the C_n, let $z = e^{i\Theta}$. Then $\cos\Theta = z + \frac{1}{z}$, $\cos n\Theta = z^n + \frac{1}{z^n}$, and $C_n = \text{P.V. } \frac{1}{\pi}\int_C \frac{z^n + \frac{1}{z^n}}{z + \frac{1}{z}} \frac{dz}{iz}$, where C is the unit circle. Thus

$$\pi C_n = \text{P.V.} \int_C \frac{z^{2n}+1}{z^n(z^2+1)} \frac{dz}{i} = 2\pi i \text{ (residue at } z = 0)$$

$$+ \pi i \text{ (residue at } z = \pm i).$$

The residue at $z = i$ is given by $-\frac{1}{2}(i^n + i^{-n})$. (1.2)

If $n = 2k + 1$, we have $-\frac{i}{2}\left(\frac{i^{4k} - 1}{i^{2k}}\right) = 0$. If $n = 2k$, (1.2) becomes $-i^{-2k}$. For $k = 2j$, we have that the residue is -1, and for $k = 2j + 1$, the residue is 1.

The residue at $z = -i$ is given by $\frac{(-1)^n}{2} \; \frac{i^{2n} + 1}{i^n}$. (1.3)

If $n = 2k + 1$, $\frac{i^{2n} + 1}{i^n} = \frac{i^{4k+2} + 1}{i^{2k+1}} = 0$. If $n = 2k$, (1.3) becomes $\frac{1}{2} \; \frac{i^{4k} + 1}{i^{2k}}$. For $k = 2j$, this is 1 and for $k = 2j + 1$, it is -1.

Combining these results, we have that the sum of the residues at $z = \pm i$ is zero.

To calculate the residue at $z = 0$, let $G(z) = \frac{1}{i} \; \frac{z^{2n} + 1}{z^2 + 1}$, and compute $\frac{1}{(n-1)!} G^{(n-1)}(0)$.

$$i\,G(z) = (z^{2n} + 1)(1 - z^2 + z^4 - z^6 + \dots + (-1)^n z^{2n} + \dots).$$

$$= 1 - z^2 + z^4 - z^6 + \dots + \left((-1)^n + 1\right) z^{2n} + \left((-1)^{n+1} - 1\right) z^{2n+2} + \dots.$$

If n is odd,

$$i\,G(z) = 1 - z^2 + z^4 - \dots + (-1)^{n-1} z^{2n-2} = 1 - z^2 + z^4 - \dots (-1)^{\frac{n-1}{2}} z^{n-1}$$

$\pm \cdots + (-1)^{n-1} z^{2n-2}$ and $i\, G^{(n-1)}(z) \Big|_{z=0} = (-1)^{\frac{n-1}{2}} (n-1)!$. That is, if

$n = 2k + 1$, $\dfrac{G^{2k}(z)}{2k!} \Big|_{z=0} = \dfrac{(-1)^k}{i}$.

If n is even, the series expansion of $G(z)$ does not terminate and contains only even powers of z. Hence $G^{(n-1)}(z) \Big|_{z=0} = 0$.

Finally, we have that

$$C_n = \begin{cases} 0, & n \text{ even} \\ 2(-1)^k, & n = 2k + 1 \end{cases}$$

The formal Fourier series associated with $\dfrac{1}{\cos x}$ is

$$2(\cos x - \cos 3x + \cos 5x \ldots) \tag{1.4}$$

Suppose, for $x \neq \frac{\pi}{2}$, $\cos (2n - 1) x \to 0$ as $n \to \infty$. Then $\cos (2n - 1) x - \cos (2n + 1) x = 2 \sin 2n x \sin x \to 0$ and $\cos (2n - 1) x + \cos (2n + 1) x = 2 \cos 2n x \cos x \to 0$. Thus, $\sin 2n x \to 0$ and $\cos 2n x \to 0$. Since $\sin^2 2n x + \cos^2 2n x = 1$, this is a contradiction. Since the general term does not go to zero for any x, save $x = \frac{\pi}{2}$, the series does not converge.

Although the series does not converge it is summable (C,1) to the function $\frac{1}{\cos\theta}$. By an elementary identity (see [8], page 358).

$$\cos\theta - \cos 3\theta + \dots + (-1)^n \cos(2n+1)\theta = \frac{\sin^2 (n+1)\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)} .$$

Then, $S_{n+1} = 2\left(\cos\theta - \cos 3\theta + \dots + (-1)^n \cos(2n+1)\theta\right)$

$$= \frac{1}{\cos\theta}\left(1 - \cos 2(n+1)t\right), \text{ where } t = \frac{\pi}{2} - \theta.$$

$$\sum_{0}^{n-1} S_k = \frac{1}{\cos\theta}\left(1 - \cos 2t + 1 - \cos 4t + \dots + 1 - \cos 2n\, t\right)$$

$$= \frac{1}{\cos\theta}\left(n - \frac{\sin nt \cos(n+1)t}{\sin t}\right).$$

$$\sigma_n = \frac{1}{n}\sum_{0}^{n-1} S_k = \frac{1}{\cos\theta}\left(1 - \frac{1}{n}\,\frac{\sin(nt)\cos(n+1)t}{\sin t}\right).$$

Thus, $\sigma_n \to \frac{1}{\cos\theta}$.

It is interesting to note that while the process of summation is invalid for $\theta = \frac{\pi}{2}$, the series converges to zero for $\theta = \frac{\pi}{2}$. This is, in a sense, the average value of the function.

CHAPTER II

A FOURIER THEOREM

In order to establish conditions under which a certain type of trigonometric series is (C,1) summable, we shall prove some preliminary lemmas.

It is known (see [2], page 115) that if $f(x)$ satisfies a Lipschitz condition for $-\pi \leq x \leq \pi$, then

$$\text{P.V.} \int_{-\pi}^{\pi} \frac{f(x)}{x}\, dx$$ exists. We shall find it convenient to assume

$f(x)$ has a continuous and bounded derivative for $-\pi < x < \pi$. Since $f(x) - f(0) = xf'(\bar{x})$, $0 < \bar{x} < x$, or $x \leqslant \bar{x} < 0$ $f(x)$ satisfies a Lipschitz

condition and thus $\text{P.V.} \int_{-\pi}^{\pi} \frac{f(x)}{x}\, dx$ exists. The analyticity of $\cos kx$

and $\sin kx$ ensure the existence of

$$\text{P.V.} \int_{-\pi}^{\pi} \frac{f(x)}{x} \cos kx\, dx \text{ and } \text{P.V.} \int_{-\pi}^{\pi} \frac{f(x)}{x} \sin kx\, dx.$$ We also note

that if $f(x)$ has a continuous derivative the Mean Value theorem shows

that $f'(\bar{x})$, where $f'(\bar{x}) = \dfrac{f(x) - f(y)}{x - y}$, is a continuous function of x.

Lemma 2.1 Let x be such that $0 < x < \pi$, then if $f(x) \in C^1$

$$\lim_{n\to\infty} \frac{1}{n\pi} \text{P.V.} \int_{0}^{\frac{\pi}{2}} \frac{f(x - 2\alpha)}{x - 2\alpha} \frac{\sin^2 n\alpha}{\sin^2 \alpha}\, d\alpha = \frac{f(x)}{2x}.$$

Proof. We shall follow the usual proof of Fejer's Theorem. Given any $\epsilon > 0$, there exists an η such that

$$\left| \frac{f(x-2\alpha)}{x-2\alpha} - \frac{f(x)}{x} \right| < \epsilon, \text{ for } 0 < \alpha \leq \eta < \frac{x}{2} .$$

Write P.V. $\frac{1}{n\pi} \int_0^{\frac{\pi}{2}} \frac{f(x-2\alpha)}{x-2\alpha} \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha$ as

$$\frac{1}{n\pi} \int_0^{\eta} \left(\frac{f(x-2\alpha)}{x-2\alpha} - \frac{f(x)}{x} \right) \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha$$

$$+ \text{P.V. } \frac{1}{n\pi} \int_{\eta}^{\frac{\pi}{2}} \frac{f(x-2\alpha)}{x-2\alpha} \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha$$

$$+ \frac{1}{n\pi} \frac{f(x)}{x} \int_0^{\frac{\pi}{2}} \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha$$

$$- \frac{1}{n\pi} \frac{f(x)}{x} \int_{\eta}^{\frac{\pi}{2}} \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha.$$

Call these integrals I_1, I_2, I_3, I_4.

Since $I_3 = \frac{1}{n\pi} \frac{f(x)}{x} \int_0^{\frac{\pi}{2}} \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha = \frac{f(x)}{2x}$, it must be shown that given any $\epsilon > 0$, there exists an N such that for $n > N$, $|I_1 + I_2 + I_4| < \epsilon$. We have that

$$|I_1| \leqq \frac{1}{n\pi} \int_0^{\eta} \left| \frac{f(x - 2\alpha)}{x - 2\alpha} - \frac{f(x)}{x} \right| \frac{\sin^2 n\alpha}{\sin^2 \alpha} < \frac{\epsilon}{n\pi} \int_0^{\eta} \frac{\sin^2 n\alpha}{\sin^2 \alpha}\, d\alpha$$

$$< \frac{\epsilon}{n\pi} \int_0^{\frac{\pi}{2}} \frac{\sin^2 n\alpha}{\sin^2 \alpha}\, d\alpha = \frac{\epsilon}{2}$$

$$|I_4| \leqq \left| \frac{f(x)}{\eta} \right| \frac{1}{\sin^2 \eta} \frac{(\frac{\pi}{2} - \eta)}{n\pi} < \frac{M_o}{n},$$ say. We note that M_o depends on x and thus uniform convergence will not be established.

Choose a and b such that

(i) $\eta < a < \frac{x}{2} < b < \frac{\pi}{2}$

(ii) if m is the maximum of $|f(x - 2\alpha)|$ for $0 \leqq \alpha \leqq \frac{\pi}{2}$, $\frac{m}{\pi \sin^2 \eta}(b - a) < \frac{\epsilon}{3}$. Now

$$I_2 = \text{P.V.} \frac{1}{n\pi} \int_{\eta}^{\frac{\pi}{2}} \frac{f(x - 2\alpha)}{x - 2\alpha} \frac{\sin^2 n\alpha}{\sin^2 \alpha}\, d\alpha = \frac{1}{n\pi} \int_{\eta}^{a} \frac{f(x - 2\alpha)}{x - 2\alpha} \frac{\sin^2 n\alpha}{\sin^2 \alpha}\, d\alpha$$

$$+ \text{P.V.} \frac{1}{n\pi} \int_a^b \frac{f(x - 2\alpha)}{x - 2\alpha} \frac{\sin^2 n\alpha}{\sin^2 \alpha}\, d\alpha + \frac{1}{n\pi} \int_b^{\frac{\pi}{2}} \frac{f(x - 2\alpha)}{x - 2\alpha} \frac{\sin^2 n\alpha}{\sin^2 \alpha}\, d\alpha.$$

$$\left| \frac{1}{n\pi} \int_{\eta}^{a} \frac{f(x-2\alpha)}{x-2\alpha} \; \frac{\sin^2 n\alpha}{\sin^2 \alpha} \, d\alpha \right| \leqq \frac{M_1}{n},$$ where M_1 is the

maximum of $\left| \frac{f(x-2\alpha)}{x-2\alpha} \; \frac{1}{\sin^2 \alpha} \right|$ for $\eta \leqq \alpha \leq a$. Similarly

$$\left| \frac{1}{n\pi} \int_{b}^{\frac{\pi}{2}} \frac{f(x-2\alpha)}{x-2\alpha} \; \frac{\sin^2 n\alpha}{\sin^2 \alpha} \, d\alpha \right| \leqq \frac{M_2}{n}.$$ It remains to estimate

$$\text{P.V.} \; \frac{1}{n\pi} \int_{a}^{b} \frac{f(x-2\alpha)}{x-2\alpha} \; \frac{\sin^2 n\alpha}{\sin^2 \alpha} \, d\alpha. \qquad (2.1)$$

Using the Mean Value theorem to express $\sin^2 n\alpha$ as

$\sin^2 n\left(\frac{x}{2}\right) + 2n\left(\alpha - \frac{x}{2}\right)\sin n\bar{\alpha}\cos n\bar{\alpha}$, $\alpha < \bar{\alpha} < \frac{x}{2}$ or $\frac{x}{2} < \bar{\alpha} < \alpha$, (2.1) becomes

$$\frac{1}{n\pi} \, \text{P.V.} \int_{a}^{b} \frac{f(x-2\alpha)}{x-2\alpha} \; \frac{\sin^2 n(x/2)}{\sin^2 \alpha} \, d\alpha - \frac{1}{\pi} \int_{a}^{b} \frac{f(x-2\alpha)}{\sin^2 \alpha} \sin n\bar{\alpha} \cos n\bar{\alpha} \, d\alpha.$$

Since $\text{P.V.} \; \sin^2 n(x/2) \int_{a}^{b} \frac{f(x-2\alpha)}{x-2\alpha} \; \frac{1}{\sin^2 \alpha} \, d\alpha$ exists,

$$\left| \frac{1}{n\pi} \; \text{P.V.} \int_{a}^{b} \frac{f(x-2\alpha)}{x-2\alpha} \; \frac{\sin^2 n(x/2)}{\sin^2 \alpha} \, d\alpha \right| < \frac{M_3}{n}.$$ Finally,

$$\left| \frac{1}{\pi} \int_a^b \frac{f(x - 2\alpha)}{\sin^2 \alpha} \sin n\bar{\alpha} \cos n\bar{\alpha} \, d\alpha \right| \leqq \frac{m}{\pi} \frac{1}{\sin^2 \eta} (b - a) < \frac{\epsilon}{3} .$$

Combining the previous estimates, we have

$|I_1 + I_2' + I_4| < \frac{\epsilon}{2} + \frac{1}{n}(M_1 + M_2 + M_3) + \frac{\epsilon}{3}$. Thus we can find an N such that for $n > N$, $|I_1 + I_2 + I_3| < \epsilon$.

Lemma 2.2

If $0 < x < \pi$, then $\lim_{n\to\infty} \frac{1}{n\pi} \int_0^{\frac{\pi-x}{2}} \frac{f(x + 2\alpha)}{x + 2\alpha} \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha = \frac{f(x)}{2x}$.

Proof: As before, $f(x)$ has a continuous derivative for $|x| \leq \pi$. Then

$$\frac{1}{n\pi} \int_0^{\frac{\pi-x}{2}} \frac{f(x + 2\alpha)}{x + 2\alpha} \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha = \frac{1}{n\pi} \int_0^{\eta} \left(\frac{f(x + 2\alpha)}{x + 2\alpha} - \frac{f(x)}{x} \right) \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha$$

$$+ \frac{1}{n\pi} \int_0^{\frac{\pi}{2}} \frac{f(x)}{x} \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha + \frac{1}{2\pi} \int_{\eta}^{\frac{\pi-x}{2}} \frac{f(x + 2\alpha)}{x + 2\alpha} \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha$$

$$- \frac{1}{n\pi} \int_{\eta}^{\frac{\pi}{2}} \frac{f(x)}{x} \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha ,$$ where η is chosen such that

$\left| \frac{f(x + 2\alpha)}{(x + 2\alpha)} - \frac{f(x)}{x} \right| < \epsilon$ for $0 < \alpha \leq \eta < \frac{\pi - x}{2}$. Then as in Lemma 1

$$\left| \frac{1}{n\pi} \int_0^{\eta} \left(\frac{f(x + 2\alpha)}{x + 2\alpha} - \frac{f(x)}{x} \right) \frac{\sin^2 n\alpha}{\sin^2 \alpha} \, d\alpha \right| < \epsilon ;$$

$$\left| \frac{1}{n\pi} \int_{\eta}^{\frac{\pi - x}{2}} \frac{f(x + 2\alpha)}{x + 2\alpha} \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha \right| < \frac{M_1}{n} \quad ; \quad \left| \frac{1}{n\pi} \int_{\eta}^{\frac{\pi}{2}} \frac{f(x)}{x} \frac{\sin^2 n\alpha}{\sin^2 \alpha} d\alpha \right| < \frac{M_2}{n} .$$

Thus, $\left| \frac{1}{n\pi} \int_0^{\frac{\pi - x}{2}} \frac{f(x + 2\alpha)}{x + 2\alpha} \frac{\sin^2 n\alpha}{\sin^2 \alpha} - \frac{f(x)}{x} d\alpha \right| < \epsilon + \frac{M_1}{n} + \frac{M_2}{n}$ and the lemma is proved. The ease of proving Lemma 2.2 as compared with Lemma 2.1 is due to the integrand in Lemma 2.2 being nonsingular. We now turn to the main theorem.

Theorem 2.1. Let $f(x)$ have a continuous derivative for $-\pi \leq x \leq \pi$. Then the partial sums of the series

$$\frac{a_o}{2} + \sum_1^{\infty} (a_k \cos kx + b_k \sin kx), \text{ where}$$

$$a_k = \text{P.V. } \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x)}{x} \cos kx \, dx \text{ and } b_k = \text{P.V. } \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x)}{x} \sin kx \, dx, \text{ are}$$

summable $(C,1)$ to $\frac{f(x)}{x}$ for $0 < |x| < \pi$.

Proceeding as in the usual proof of Fejer's theorem (see [1]), we let

$$\sigma_n(x) = \frac{1}{n}\sum_{0}^{n-1} S_k(x), \text{ where } S_k = \frac{a_o}{2} + \sum_{1}^{k}(a_j \cos jx + b_j \sin jx).$$

Then $\sigma_n = \dfrac{1}{2n\pi}\,\text{P.V.}\displaystyle\int_{-\pi}^{\pi} \frac{f(x')}{x'} \frac{\sin^2 n(\frac{x'-x}{2})}{\sin^2(\frac{x'-x}{2})}\,dx'.$

For definiteness let x be such that $0 < x < \pi$. Then

$$\text{P.V.}\ \frac{1}{2n\pi}\int_{-\pi}^{\pi} \frac{f(x')}{x'} \frac{\sin^2 n(\frac{x'-x}{2})}{\sin^2(\frac{x'-x}{2})}\,dx' = \text{P.V.}\ \frac{1}{2n\pi}\int_{-\pi+x}^{x} \frac{f(x')}{x'} \frac{\sin^2 n(\frac{x'-x}{2})}{\sin^2(\frac{x'-x}{2})}\,dx'$$

$$+ \frac{1}{2n\pi}\int_{x}^{\pi} \frac{f(x')}{x'} \frac{\sin^2 n(\frac{x'-x}{2})}{\sin^2(\frac{x'-x}{2})}\,dx'$$

$$+ \frac{1}{2n\pi}\int_{\pi}^{\pi+x} \frac{f(x'-2\pi)}{x'-2\pi} \frac{\sin^2 n(\frac{x'-x}{2})}{\sin^2(\frac{x'-x}{2})}\,dx'$$

Consider the last integral. Since $\pi \leq x' \leq \pi + x$, $0 < \frac{\pi-x}{2} \leqq \frac{x'-x}{2} \leqq \frac{\pi}{2}$,

the integrand is continuous for $\pi \leqq x' \leq \pi + x$.

Thus $\left| \dfrac{1}{2n\pi}\displaystyle\int_{\pi}^{\pi+x} \frac{f(x'-2\pi)}{x'-2\pi} \frac{\sin^2 n(\frac{x'-x}{2})}{\sin^2(\frac{x'-x}{2})}\,dx' \right| < \dfrac{M}{n}$. Substituting

$x' = x - 2\alpha$ in the first integral and $x' = x + 2\alpha$ in the second we obtain

$$\sigma_n(x) = \frac{1}{n\pi}\ \text{P.V.} \int_0^{\frac{\pi}{2}} \frac{f(x-2\alpha)}{x-2\alpha}\ \frac{\sin^2 n\alpha}{\sin^2 \alpha}\, d\alpha + \frac{1}{n\pi} \int_0^{\frac{\pi-x}{2}} \frac{f(x+2\alpha)}{x+2\alpha}\ \frac{\sin^2 n\alpha}{\sin^2 \alpha}\, d\alpha$$

$$+ \frac{1}{2n\pi} \int_{\pi}^{\pi+x} \frac{f(x'-2\pi)}{x'-2\pi}\ \frac{\sin^2 n[(x'-x)/2]}{\sin^2[(x'-x)/2]}\, dx'$$

Using Lemma 2.1, Lemma 2.2, and the above estimate establishes the theorem.

Since the Fourier Series of $f(x)/x$ can be (C,1) summed to $f(x)/x$, a Fourier Series, and in turn a Poisson Integral approach to the Dirichlet problem is suggested. That is, we seek a function $U(r,\phi)$ which is harmonic for $|r| < 1$ and $U(r,\phi) \to f(\theta_o)/\theta_o$, $(\theta_o \neq 0)$, as $r \to 1$ and $\phi \to \theta_o$. We shall explore this problem in the next chapter.

CHAPTER III

A BOUNDARY VALUE PROBLEM

The solution of the Dirichlet problem being well known, the case where the boundary data is given as $f(\theta)/\theta$, $f'(\theta)$ continuous, may be reduced to the case of the boundary data given as $1/\theta$. Since $f(\theta)/\theta = f(0)/\theta + f'(\bar{\theta})$, we solve the problem for $1/\theta$ (or $f(0)/\theta$) and add the solution for the case of the boundary data given as $f'(\bar{\theta})$.

Let $$K(r,\theta,\phi) = \frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2}, \quad 0 \leq r < 1, \ \pi \leq \theta \leq \pi.$$

Let $$G(r,\phi) = \text{P.V.}\ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\theta} K(r,\theta,\phi)\,d\theta,$$

$G(r,\phi)$ exists as $K(r,\theta,\phi)$ has a continuous derivative with respect to θ. We shall show that $G(r,\phi)$ does assume the prescribed boundary values and later that $G(r,\phi)$ is harmonic.

Lemma 3.1 If ϕ is such that $0 < A < |\phi| \leq \pi$, then

$$\lim_{r\to 1} \text{P.V.} \int_{-A}^{A} \frac{1}{\theta} K(r,\theta,\phi)\,d\theta = 0$$

Proof: We have that

$$\text{P.V.} \int_{-A}^{A} \frac{1}{\theta} K(r,\phi,\phi)\,d\theta = \lim_{\delta\to 0} \left(\int_{-A}^{-\delta} \frac{1}{\theta} K(r,\theta,\phi)\,d\theta + \int_{\delta}^{A} \frac{1}{\theta} K(r,\theta,\phi)\,d\theta \right).$$

In the first integral substitute $\theta = -\alpha$ and after some simplifcation

obtain $\text{P.V.} \int_{-A}^{A} \frac{1}{\theta} K(r,\theta,\phi)d\theta$

$$= \lim_{\delta \to 0} (1-r^2)4r \sin\phi \int_{\delta}^{A} \frac{\sin\theta}{\theta} \frac{1}{(1-2r\cos(\theta-\phi)+r^2)\ (1-2r\cos(\theta+\phi)+r^2)}\ d\theta.$$

Since ϕ is not in the range of integration, the integrand is continuous,

properly defined for $\theta=0$, for $0 \leq r \leq 1$, $0 \leq \theta \leq A$, and $\text{P.V.}\int_{-A}^{A} \frac{1}{\theta} K(r,\theta,\phi)d\theta$

$$= (1-r^2)4r \sin\phi \int_{0}^{A} \frac{\sin\theta}{\theta} \frac{1}{(1-2r\cos(\theta-\phi)+r^2)\ (1-2r\cos(\theta+\phi)+r^2)}\ d\theta\ .$$

Again, since the integrand is continuous for $0 \leq r \leq 1$ and $0 \leq \theta \leq A$, we

have $\lim_{r\to 1} \text{P.V.} \int_{-A}^{A} \frac{1}{\theta} K(r,\theta,\phi)d\theta = 0.$

It is clear that if $g(\theta)$ is continuous, and $0 < A < |\phi| \leq \pi$

$$\lim_{r\to 1} \int_{-A}^{A} g(\theta)\ K(r,\theta,\phi)d\theta = 0.$$

Theorem 3.1. If $G(r,\phi) = \text{P.V.}\ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\theta} K(r,\theta,\phi)d\theta$, then

$G(r,\phi) \to \frac{1}{\theta_0}$ as $r \to 1$, $\phi \to \theta_0 \neq 0$.

Proof: Let $\overline{U}(\theta) = \begin{cases} \frac{1}{\theta}, & 0 < \delta \leq |\theta| \leq \pi \\ U_\delta(\theta), & |\theta| \leq \delta \end{cases}$, where $U_\delta(\theta)$ is a continuous function that coincides with $\frac{1}{\theta}$ at $\theta = \pm\delta$. For example, $U_\delta(\theta)$ could be a linear function that assumes the values $\pm\frac{1}{\delta}$ at $\theta = \pm\delta$. Take δ such that $0 < \delta < |\theta_0|$. We know that since $\overline{U}(\theta)$ is continuous

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{U}(\theta)\ K(r,\theta,\phi)d\theta \to \overline{U}(\theta_0) = \frac{1}{\theta_0}$$ as $r \to 1$ and $\phi \to \theta_0$. Now

$$\left|G(r,\phi) - \frac{1}{\theta_0}\right| \leq \left|G(r,\phi) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{U}(\theta)\ K(r,\theta,\phi)d\theta\right|$$

$$+ \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{U}(\theta)\ K(r,\theta,\phi)d\theta - \frac{1}{\theta_0}\right|.$$ Since $\overline{U}(\theta) = \frac{1}{\theta}$ for $\delta \leq |\theta| \leq \pi$,

$$G(r,\phi) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{U}(\theta)\ K(r,\theta,\phi)d\theta$$

$$= \text{P.V.}\ \frac{1}{2\pi}\int_{-\delta}^{\delta}\frac{1}{\theta}K(r,\theta,\phi)d\theta - \frac{1}{2\pi}\int_{-\delta}^{\delta}U_\delta(\theta)\ K(r,\theta,\phi)d\theta$$ we have by Lemma 3.1 that

$$\left|G(r,\phi) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{U}(\theta)\ K(r,\theta,\phi)d\theta\right| \to 0 \text{ as } r \to 1.$$

Since $\frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{U}(\Theta)\ K(r,\Theta,\phi)d\Theta \to \frac{1}{\Theta_o}$ as $r \to 1$ and $\phi \to \Theta_o$, we have that

$G(r,\phi) \to \frac{1}{\Theta_o}$ as $r \to 1$ and $\phi \to \Theta_o$.

The next step in the solution of the boundary value problem will be to show that if $f(\Theta)$ has a continuous derivative for $-\pi \leq \Theta \leq \pi$, then

$$\text{P. V.}\ \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(\Theta)}{\Theta}K(r,\Theta,\phi)d\Theta = \frac{a_o}{2} + \sum_{1}^{\infty} r^k\,(a_k \cos k\phi + b_k \sin k\phi)$$

where $a_k = \frac{1}{\pi}$ P.V. $\int_{-\pi}^{\pi}\frac{f(\Theta)}{(\Theta)}\cos k\Theta\, d\Theta$ and $b_k = \frac{1}{\pi}$ P.V. $\int_{-\pi}^{\pi}\frac{f(\Theta)}{\Theta}\sin k\Theta\, d\Theta$.

For a crude estimate of a_k and b_k note that

$\frac{f(x)}{x}\cos kx = \frac{f(x)}{x}\left(\cos k0 + x\,(-k \sin k\overline{x})\right) = \frac{f(x)}{x} -$

$f(x)\ k \sin k\overline{x}$ and $\frac{f(x)}{x}\sin kx = \frac{f(x)}{x}(\sin k0 + xk \cos k\overline{x}) = kf(x)\cos k\overline{x}$.

$$\left|b_k\right| = \left|\text{P.V.}\ \frac{1}{\pi}\int_{-\pi}^{\pi} k\ f(x)\ \cos k\overline{x}\ dx\ \right| < M_1 k \text{ and}$$

$$\left|a_k\right| = \left|\text{P.V.}\ \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{f(x)}{x} - k\ f(x)\ \sin k\overline{x}\right)dx\ \right| < M_2 + M_3 k, \text{ so}$$

$a_k = O(k)$ and $b_k = O(k)$.

Let $I = \text{P.V.}\ \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(\theta)}{\theta}K(r,\theta,\phi)\,d\theta$. We know that given any $\epsilon > 0$, there exists a Δ such that

$$\left| I - \frac{1}{2\pi}\int_{-\pi}^{-\alpha}\frac{f(\theta)}{\theta}K(r,\theta,\phi)\,d\theta - \frac{1}{2\pi}\int_{\alpha}^{\pi}\frac{f(\theta)}{\theta}K(r,\theta,\phi)\,d\theta \right| < \epsilon \text{ for } 0 < \alpha < \Delta.$$

Consider
$$\left| I - \left(\frac{a_o}{2} + \sum_1^\infty r^k\,(a_k \cos k\phi + b_k \sin k\phi)\right)\right|. \tag{3.1}$$

The series is dominated in magnitude by $\sum_1^\infty Mkr_1^{\ k}$, provided $0 \leq r \leq r_1$.

(3.1) can be written as
$$\left| I - \left(\frac{a_o}{2} + \sum_1 \frac{r^k}{\pi}\left(\int_{-\pi}^{-\alpha}\frac{f(\theta)}{\theta}\cos k\,(\theta - \phi)\,d\theta + \int_{\alpha}^{\pi}\frac{f(\theta)}{\theta}\cos k\,(\theta - \phi)\,d\theta\right)\right) + \frac{a_o}{2} + \sum_1 \frac{r^k}{\pi}\left(\int_{-\pi}^{-\alpha}\frac{f(\theta)}{\theta}\cos k(\theta - \phi)\,d\theta + \int_{\alpha}^{\pi}\frac{f(\theta)}{\theta}\cos k\,(\theta - \phi)\,d\theta\right) - \left(\frac{a_o}{2} + \sum_1 r^k\,(a_k \cos k\phi + b_k \sin k\phi)\right)\right|.$$

We use the triangle inequality and examine each term. That is, we consider

$$\left| I - \left(\frac{a_o}{2} + \sum_1 \frac{r^k}{\pi} \left(\int_{-\pi}^{\alpha} \frac{f(\theta)}{\theta} \cos k(\theta - \phi) d\theta + \int_{\alpha}^{\pi} \frac{f(\theta)}{\theta} \cos k(\theta - \phi) d\theta \right) \right) \right| \tag{3.2}$$

and
$$\left| \sum_1 \frac{r^k}{\pi} \left(\int_{-\pi}^{-\alpha} \frac{f(\theta)}{\theta} \cos k(\theta - \phi) d\theta + \int_{\alpha}^{\pi} \frac{f(\theta)}{\theta} \cos k(\theta - \phi) d\theta \right) \right.$$

$$\left. - \sum_1 r^k (a_k \cos k\phi + b_k \sin k\phi) \right| . \tag{3.3}$$

In (3.2) the order of integration and summation may be interchanged, as the resulting series' are uniformly convergent with respect to θ, giving

$$\left| I - \left(\frac{1}{\pi} \int_{-\pi}^{-\alpha} \frac{f(\theta)}{\theta} \left(\frac{1}{2} + \sum_1 r^k \cos k (\theta - \phi) \right) d\theta + \text{P.V.} \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \frac{f(\theta)}{\theta} d\theta + \right. \right.$$

$$\left. \left. \frac{1}{\pi} \int_{\alpha}^{\pi} \frac{f(\theta)}{\theta} \left(\frac{1}{2} + \sum_1 r^k \cos k (\theta - \phi) \right) d\theta \right) \right| \text{ (see } |1|\text{, page 251).} \tag{3.4}$$

By the definition of P.V. $\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\theta)}{\theta} K(r,\theta,\phi)\, d\theta$, (3.4) is less than ϵ for sufficiently small α. (3.3) may be rearranged to yield

$$\sum_1 \frac{r^k}{\pi} (A_k \cos k\phi + B_k \sin k\phi) \tag{3.5}$$

where $A_k = \text{P.V.} \int_{-\alpha}^{\alpha} \frac{f(\theta)}{\theta} \cos k\theta\, d\theta$ and $B_k = \text{P.V.} \int_{-\alpha}^{\alpha} \frac{f(\theta)}{\theta} \sin k\theta\, d\theta$. As before $A_k = O(k)$ and $B_k = O(k)$. Hence (3.5) is dominated by $\sum Mkr^k$ and is uniformly convergent in α. It is clear that the A_k's and B_k's are continuous functions of α. Taking the limit as α goes to zero, term by term, shows that (3.3) can be made less than ϵ for α sufficiently small. Thus

$$I = \frac{a_o}{2} + \sum_{1} r^k (a_k \cos k\phi + b_k \sin k\phi).$$

At this point we have established the following:

(a) P.V. $\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\theta)}{\theta} K(4,\theta,\phi) d\theta$ exists and assumes the values $\frac{f(\theta)}{\theta}$ as $r \to 1$.

(b) P.V. $\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\theta)}{\theta} K(r,\theta,\phi) d\theta = \frac{a_o}{2} + \sum_{1} (a_k \cos k\phi + b_k \sin \phi) r^k, 0 \leqq r \leqq 1.$

It is now immediate that $U(r,\phi) =$ P.V. $2\pi \int_{-\pi}^{\pi} \frac{f(\theta)}{\theta} K(r,\theta,\phi) d\theta$ solves the boundary value problem. The fact that $U(r,\phi)$ is harmonic is immediate, since the derived series is uniformly convergent. Using the integral representation of $U(r,\phi)$ we have shown that $U(r,\phi)$ assumes the given boundary values.

CHAPTER IV

THE NATURE OF THE SOLUTION

It is well known that if the boundary data is continuous then the solution of the problem is unique. In the case of the boundary values given as $\frac{f(\Theta)}{\Theta}$ there is no unique solution. Consider the function $g(z) = \frac{1}{z-1} + \frac{1}{2}$. For $|z| = 1$, $z \neq 1$, $g(z) = -\frac{i \sin\Theta}{2(1-\cos\Theta)}$.

That is $\mathrm{Re}(g(z)) = 0$, for $|z| = 1$, $z \neq 1$.

Now, $\mathrm{Im}(g(z)^2) = 2\mathrm{Re}(g(z))\ \mathrm{Im}(g(z)) = 0$

$$\mathrm{Re}(\ i(g(z)^2) = \mathrm{Re}\ (i)\ \mathrm{Re}(\ g(z)^2) - \mathrm{Im}(i)\ \mathrm{Im}(\ g(z)^2) = 0.$$

$$\mathrm{Re}(i^2(g(z)^3)) = \mathrm{Re}(i^2)\ \mathrm{Re}(g(z)^3) - \mathrm{Im}(i^2)\ \mathrm{Im}(g(z)^3)$$

$$= -\mathrm{Re}(g(z)\ (g(z)^2))$$

$$= -\mathrm{Re}(g(z))\ \mathrm{Re}(g(z)^2) + \mathrm{Im}(g(z))\ \mathrm{Im}(g(z)^2) = 0.$$

Continuing in this fashion we can construct a countable number of harmonic functions with boundary values of zero. Thus, for this problem there is no unique solution. However, we shall establish that among all functions exhibiting a certain type of growth in a neighborhood of the singularity, the solution is unique.

A classical result is that if $U(\Theta)$ is continuous and bounded for $0 < \Theta < 2\pi$, then there is at most one bounded function $G(z)$, such that $G(z)$ is harmonic for $|z| < 1$ and $G(z) \to U(\Theta)$ as $|z| \to 1$, see [7], page 92.

Lemma 4.1.

$$\mathrm{P.V.}\ \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\ \frac{\sin\Theta}{1-\cos\Theta}\ K(r,\Theta,\phi)\ d\Theta = \frac{r\ \sin\phi}{r^2 - 2r\cos\phi + 1} = \mathrm{Im}\ (\frac{1}{1-z}).$$

Proof: Another form of the Poisson integral is

$$U(a) = \text{Re}\left[\frac{1}{2\pi i}\int_C \frac{z+a}{z-a}\, U(\theta)\,\frac{dz}{z}\right],$$ where C is the

unit circle. We want to compute $\text{Re}\left(\text{P.V.}\frac{1}{2\pi i}\int_C \frac{z+a}{z-a}\ \text{Im}\left(\frac{1}{z-1}\right)\frac{dz}{z}\right)$.

Consider $\frac{1}{2\pi i}\text{P.V.}\int_C \frac{z+a}{z-a}\left(\text{Re}\,\frac{1}{z-1} + i\ \text{Im}\,\frac{1}{z-1}\right)\frac{dz}{z}$ (4.1)

The residue at $z = a$ is $\frac{2}{a-1}$. The residue at $z = 0$ is 1 and at $z = 1$ it is $-\frac{a+1}{a-1}$. Thus we have that

$$\frac{1}{2\pi i}\text{P.V.}\int_C \frac{z+a}{z-a}\ \frac{1}{z-1}\ \frac{dz}{z} = \frac{2}{a-1} + 1 - \frac{1}{2}\left(\frac{a+1}{a-1}\right).$$ Letting

$a = x + iy$ we can write (4.1) as $\frac{2(x-1-iy)}{(x-1)^2+y^2} + 1 - \frac{1}{2}\left(\frac{x^2-1+y^2+i(-2y)}{(x-1)^2+y^2}\right)$

and the imaginary part of (4.1) is $\frac{-y}{(x-1)^2+y^2} = \text{Im}\left(\frac{1}{z-1}\right)$.

But, the imaginary part of the integral of (4.1) is

$$\text{Im}\left(\frac{1}{2\pi i}\int_C \frac{z+a}{z-a}\ \text{Re}\left(\frac{1}{z-1}\right)\frac{dz}{z}\right) + \text{Re}\left(\frac{1}{2\pi i}\int_C \frac{z+a}{z-a}\ \text{Im}\left(\frac{1}{z-1}\right)\frac{dz}{z}\right).$$

$$\frac{1}{2\pi i}\int_C \frac{z+a}{z-a}\ \text{Re}\left(\frac{1}{z-1}\right)\frac{dz}{z} = \frac{1}{2\pi i}\int_C \frac{z+a}{z-a}\left(-\frac{1}{2}\right)\frac{dz}{z} = -\frac{1}{2}\left[-1+2\right]$$

which is real. Hence, Re $\frac{1}{2\pi i} \int_C \frac{z+a}{z-a} \operatorname{Im} \frac{1}{z-1} \frac{dz}{z} = \operatorname{Im} \left(\frac{1}{z-1}\right)$.

Lemma 4.2. $\left| \text{P.V. } \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\theta)}{\theta} K(r,\theta,\phi)\, d\theta - f(0) \operatorname{Im} \left(\frac{1}{1-z}\right) \right|$

is bounded for $0 \leq r < 1$.

Proof: Let $U(\theta) = \frac{f(\theta)}{\theta}$, $U_1(\theta) = \frac{f(0)}{2} \frac{\sin\theta}{1-\cos\theta} = f(0) \operatorname{Im} \left(\left.\frac{1}{1-z}\right|_{z=e^{i\theta}}\right)$.,

and $U_2(\theta) = U(\theta) - U_1(\theta)$. By properly defining $U_2(0)$, $U_2(\theta)$ will be

continuous for $0 \leq \theta \leq 2\pi$. Let $\overline{U}_2(r,\phi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} U_2(\theta)\, K(r,\theta,\phi) d\theta$. All

the standard theorems on Poisson's integral apply, as $U_2(\theta)$ is continuous.

$$\overline{U}_2(r,\phi) = \text{P.V. } \left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} U(\theta)\, K(r,\theta,\phi) d\theta - \text{P.V. } \left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} U_1(\theta)\, K(r,\theta,\phi) d\theta.$$

By lemma 4.1 P.V. $\left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} U_1(\theta)\, K(r,\theta,\phi) d\theta = f(0) \operatorname{Im} \left(\frac{1}{1-z}\right)$

$\overline{U}_2(r,\phi) \to U_2(\theta)$ as $z \to e^{i\theta}$ and $U_2(\theta)$ is continuous for $0 \leq \theta \leq 2\pi$. Hence

$$\left| \text{P.V. } \left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} U(\theta)\, K(r,\theta,\phi) d\theta - f(0) \operatorname{Im} \left(\frac{1}{1-z}\right) \right| < M.$$

Theorem 4.1. Let H denote the class of all functions $g(r,\theta)$ such that $g(1,\theta) = \frac{f(\theta)}{\theta}$, and $g(r,\theta)$ is harmonic for $|z| < 1$. Then if

$$\left| g(r,\phi) - \mathrm{Im}\left(\frac{f(0)}{z-1}\right) \right| < M \text{ for } |z| < 1$$

$$g(r,\phi) = \text{P.V.}\left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} \frac{f(\theta)}{\theta} K(r,\theta,\phi)\, d\theta = U(r,\phi)$$

Proof: We have that $U(r,\phi)$ is a solution. Let $g(r,\phi)$ be another solution such that $\left| g(r,\phi) - \mathrm{Im}\, \frac{f(0)}{z-1} \right| < M$. Let $F(r,\phi) = U(r,\phi) - g(r,\phi)$. $F(1,\phi) = 0$ for all $\phi \neq 0$. $\left|F(r,\phi)\right| = \left|U(r,\phi) - \mathrm{Im}\,\frac{f(0)}{z-1} + \mathrm{Im}\,\frac{f(0)}{z-1} - g(r,\phi)\right|$, $\left|F(r,\phi)\right| \leq \left| U(r,\phi) - \mathrm{Im}\,\frac{f(0)}{z-1} \right| + \left| \mathrm{Im}\,\frac{f(0)}{z-1} - g(r,\phi)\right| < 2M$. Thus $F(r,\phi)$ is bounded, harmonic, and has the value zero for $z = e^{i\theta}$, $\theta \neq 0$. Since there is at most one function with these properties and 0 has these; $F(r,\phi) \equiv 0$. That is, $U(r,\phi) \equiv g(r,\phi)$.

Since $U(r,\phi) - \mathrm{Im}\left(\frac{f(0)}{z-1}\right)$ is bounded, one is tempted to say that the analytic function of which $U(r,\phi)$ is the real part, has a simple pole at $z = 1$. As $U(r,\phi)$ is not defined in a full neighborhood of $z = 1$, this would be incorrect. We shall show, however, that the singularity is of this nature. In the following, all limits will be one-sided limits.

For simplicity let $U_1(\theta) = \frac{1}{\theta}$, $U_2(\theta) = \mathrm{Im}\left(\frac{1}{1-z}\right)\Big|_{z = e^{i\theta}}$ and $U_3 = U_1 - U_2$. As before, by properly defining $U_3(0)$, $U_3(\theta)$ will be continuous for all θ.

Let $F(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(U_1 - U_2)\frac{\zeta + z}{\zeta - z}\, d\theta$ where $\zeta = e^{i\theta}$. Now by [9], page 181, we have $\mathrm{Re}(F(e^{i\theta})) = U_3(\theta)$. $F(z) = \mathrm{P.V.}\frac{1}{2\pi}\int_{-\pi}^{\pi} U_1 \frac{\zeta+z}{\zeta-z}\, d\theta - \mathrm{P.V.}\frac{1}{2\pi}\int_{-\pi}^{\pi} U_2 \frac{\zeta+z}{\zeta-z}\, d\theta$.

Let $f_1(z) = \mathrm{P.V.}\frac{1}{2\pi}\int_{-\pi}^{\pi} U_1 \frac{\zeta+z}{\zeta-z}\, d\theta$ and $f_2(z) = \mathrm{P.V.}\frac{1}{2\pi}\int_{-\pi}^{\pi} U_2 \frac{\zeta+z}{\zeta-z}\, d\theta$.

We have that $f_1(z) = F(z) + f_2(z)$, where $\mathrm{Re}(f_1(e^{i\theta})) = U_1(\theta)$ and $\mathrm{Re}(f_2(e^{i\theta})) = U_2(\theta)$. To exhibit the nature of the singularity consider $\lim_{z\to 1} (z-1)\, f_1(z)$. If $(z-1)f_1(z)$ or $(z-1)f_2(z)$ has a limit then we have $\lim_{z\to 1} (z-1)\, f_1(z) = \lim_{z\to 1} (z-1)\, F(z) + \lim_{z\to 1} (z-1)\, f_2(z)$. Since $(z-1)\, F(z) \to 0$ as $z \to 1$, $\lim_{z\to 1} (z-1)\, f_1(z) = \lim_{z\to 1} (z-1)\, f_2(z)$.

$$f_2(z) = \mathrm{P.V.}\ \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{\sin\theta}{1 - \cos\theta}\ \frac{\zeta+z}{\zeta-z}\, d\theta = \mathrm{P.V.}\ \frac{1}{4\pi}\int_C \frac{\zeta+1}{\zeta-1}\ \frac{\zeta+z}{\zeta-z}\ \frac{1}{\zeta}\, d\zeta.$$

The residue at $\zeta = 0$ is 1. The residue at $\zeta = 1$ is $2(\frac{1+z}{1-z})$, and at $\zeta = z$ is $2(\frac{z+1}{z-1})$. Thus $\mathrm{P.V.}\ \frac{1}{4\pi}\int_C \frac{\zeta+1}{\zeta-1}\ \frac{\zeta+z}{\zeta-z}\ \frac{d\zeta}{\zeta} = \frac{i}{2}\,(1 + \frac{z+1}{z-1})$.

Then, $\lim_{z\to 1} (z-1)\, f_2(z) = \lim_{z\to 1} (\ i\ (z-1) + i(z+1)) = 2i \neq 0$. Therefore

$f_1(z)$, where $\mathrm{Re}(f_1(e^{i\theta}))=\frac{1}{\theta}$, has a singularity at $z = 1$, and

$\lim_{z\to 1} (z-1)\, f_1(z) = K \neq 0$.

It should be noted that the Fourier series method of representing the boundary values is not valid for functions having singularities of a higher order. For example, if $f(\theta) = \frac{1}{\theta^3}$, the integrals

$$\text{P.V.} \int_{-\pi}^{\pi} \frac{1}{\theta^3} \sin k\theta \, d\theta$$

do not exist. This suggests a problem for future investigation.

CHAPTER 5

AN APPLICATION

As an application of the previous results we shall examine a problem in linearized hydrodynamics.

A detailed account of the use of linearized hydrodynamics in the calculation of performance characteristics of hydrofoils placed in a uniform stream and the analogous problem in aerodynamics is given in [3], [4], [5], and [6]. We shall concern ourselves with the computation of the lift and drag coefficients for hydrofoils with profiles of a more general nature than those given in [6] and while so doing, correct some serious errors in [3] and [4].

It is pointed out in [6] that after a mapping of the (z) plane onto the (ζ) plane, the lift and drag coefficients may be represented as

$$C_L = \int_C W(\zeta) \frac{(\zeta^2 - 1)(\zeta^2 + 1)}{2\zeta^3} d\zeta \tag{5.1}$$

$$C_D = \frac{1}{2i} \int_C W^2(\zeta) \frac{(\zeta^2 - 1)(\zeta^2 + 1)\, d\zeta}{2\zeta^3} \tag{5.2}$$

where C is the unit circle traversed in the clockwise sense. In the above expressions for C_L and C_D, $W(\zeta)$ is given as

$$W(\zeta) = -i(\alpha - c_o)\left(\frac{e^{i\theta'}}{\zeta - e^{i\theta'}} + \frac{e^{-i\theta'}}{\zeta - e^{-i\theta'}}\right) - i \sum_1 c_n \zeta^{-n}. \tag{5.3}$$

α is a real number called "the angle of attack".

The numbers c_n are the Fourier coefficients of $G^*(\Theta)$, (to be defined below), and Θ' is, for the fully wetted case, equal to $\frac{\pi}{2}$.

$G^*(\Theta)$ is defined by

$$G^*(\Theta) = \begin{cases} G(\Theta) & 0 \leq \Theta \leq \pi \\ G(-\Theta) & -\pi < \Theta \leq 0 \end{cases}$$

where, for each ζ, with $|\zeta| = 1$ and $\arg \zeta = \Theta$, $(0 \leq \Theta \leq \pi)$, $G(\Theta)$ is the hydrofoil slope at that point of the (z) plane which is mapped to ζ.

When considering a hydrofoil which has a profile such that the slope becomes infinite certain modifications become necessary. We now define H as the class of hydrofoils for which $G(\Theta)$ may be written as $\dfrac{f(\Theta)}{\Theta - \frac{\pi}{2}}$, where $f(\Theta)$ has a continuous derivative. Then, in the definition of $W(\zeta)$, we replace $\sum_{1} c_n \zeta^{-n}$ by the (C,1) sum:

$$\sum_{(C,1)} c_n \zeta^{-n}, \text{ where now the } c_n \text{ will be}$$

given by P.V. $\frac{1}{\pi} \int_{-\pi}^{\pi} G^*(\Theta) \cos n\Theta \, d\Theta$.

It has been observed by Nickel in [6], that since $W^2(\zeta)$ has a pole of order 2 at $\zeta = \pm i$ and $\zeta^2 + 1$ has a zero of order 1, the integral representation of C_D exists only in the sense of a Cauchy Principal Value. With our choice of the class H, the representation of C_L as well as that of C_D must be interpreted as a Cauchy Principal Value.

Before proceeding further, we need some preliminary lemmas.

Lemma 5.1.

Let $a_k = \text{P.V.}\ \frac{1}{\pi} \int_{-\pi}^{\pi} G^*(x) \cos kx\, dx$, where $G^*(x)$ is in the class H.

Then, for $|\theta| \neq \frac{\pi}{2}$

$$\sum_{(C,1)} a_k \sin k\theta = \text{P.V.}\ \frac{1}{2\pi} \int_{-\pi}^{\pi} G^*(x) \frac{\sin\theta}{\cos x - \cos\theta}\, dx.$$

Proof:

$$\text{Let } S_n = \sum_{1}^{n} a_k \sin k\theta = \sum_{1}^{n} \text{P.V.} \int_{-\pi}^{\pi} \frac{1}{\pi} G^*(x) \cos kx \sin k\theta\, dx$$

$$= \text{P.V.} \frac{1}{2\pi} \int_{-\pi}^{\pi} G^*(x) \left(\sum_{1}^{n} \sin k(\theta+x) + \sin k(\theta-x) \right) dx.$$

By some elementary identities we have

$$S_n = \text{P.V.} \frac{1}{2\pi} \int_{-\pi}^{\pi} G^*(x) \left(\frac{\sin n(\frac{\theta+x}{2}) \sin(n+1)(\frac{\theta+x}{2})}{\sin(\frac{\theta+x}{2})} + \frac{\sin n(\frac{\theta-x}{2}) \sin(n+1)(\frac{\theta-x}{2})}{\sin(\frac{\theta-x}{2})} \right) dx$$

and

$$\sum_{1}^{n} S_n = \text{P.V.}\ \frac{1}{2\pi} \int_{0}^{\pi} \frac{G(x)}{\sin(\frac{\theta+x}{2})} \left(n \cos(\frac{\theta+x}{2}) - \frac{\sin(n+1)(\theta+x)}{2 \sin(\frac{\theta+x}{2})} - \cos(\frac{\theta+x}{2}) \right) dx$$

$$+ \text{P.V.}\ \frac{1}{2\pi} \int_{0}^{\pi} \frac{G(x)}{\sin(\frac{\theta-x}{2})} \left(n \cos(\frac{\theta-x}{2}) - \frac{\sin(n+1)(\theta-x)}{2 \sin(\frac{\theta-x}{2})} - \cos(\frac{\theta-x}{2}) \right) dx$$

Using exactly the same technique as in the proof of the Fourier theorem of Chapter II, it is a straightforward but tedious matter to show that

$$\frac{1}{n}\,\text{P.V.}\,\frac{1}{2\pi}\int_0^{\pi}\frac{G(x)}{\sin\left(\frac{\theta+x}{2}\right)}\left(\frac{\sin(n+1)\left(\frac{\theta\pm x}{2}\right)}{2\sin\left(\frac{\theta\pm x}{2}\right)}-\cos\left(\frac{\theta\pm x}{2}\right)\right)dx\to 0 \text{ as } n\to\infty,$$

and thus

$$\sigma_n=\frac{1}{n}S_n\to \text{P.V.}\,\frac{1}{2\pi}\int_0^{\pi}G(x)\left(\frac{\cos\left(\frac{\theta+x}{2}\right)}{\sin\left(\frac{\theta+x}{2}\right)}+\frac{\cos\left(\frac{\theta-x}{2}\right)}{\sin\left(\frac{\theta-x}{2}\right)}\right)dx,$$

or that

$$\sum_{(C,1)} a_k \sin k\theta = \text{P.V.}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}G^*(x)\,\frac{\sin\theta}{\cos x-\cos\theta}\,dx.$$

Lemma 5.2.

Let $H(\theta)=\sum_{(C,1)} a_k \sin k\theta = \text{P.V.}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}G^*(x)\,\frac{\sin\theta}{\cos x-\cos\theta}\,dx,\ |\theta|\neq\frac{\pi}{2}$.

Then, P.V. $\int_{-\pi}^{\pi}H(\theta)d\theta$ exists.

Proof: For convenience, let us consider P.V. $\int_0^{\pi}H(\theta)d\theta$. The existence of P.V. $\int_{-\pi}^{0}H(\theta)d\theta$ and P.V. $\int_{-\pi}^{\pi}H(\theta)d\theta$ will then be immediate.

In view of the nature of $G^*(x)$,

$$\pi H(\Theta) = \text{P.V.} \int_0^{\pi} \frac{\overline{G}(x)}{x - \frac{\pi}{2}} \frac{\sin\Theta}{\cos x - \cos\Theta}\, dx,$$ where $\overline{G}(x)$ has a continuous derivative. Since $\sin\Theta \frac{x - \Theta}{\cos x - \cos\Theta} \to -1$ as $x \to \Theta$, we may express $\frac{\sin\Theta}{\cos x - \cos\Theta}$ as $-\frac{1}{x - \Theta} + A(x,\Theta)$, where $A(x,\Theta)$ is an analytic function of x (and of Θ). Then

$$\frac{\overline{G}(x)}{x - \frac{\pi}{2}} \frac{\sin\Theta}{\cos x - \cos\Theta} = \frac{\overline{G}(x)}{x - \frac{\pi}{2}} \left[-\frac{1}{x - \Theta} + A(x,\Theta) \right]$$

$$= \frac{1}{\Theta - \frac{\pi}{2}} \left(\frac{1}{x - \frac{\pi}{2}} - \frac{1}{x - \Theta} \right) \overline{G}(x) + \frac{\overline{G}(x)\, A(x,\Theta)}{x - \frac{\pi}{2}} .$$

Thus,

$$\pi H(\Theta) = \text{P.V.} \int_0^{\pi} \frac{\overline{G}(x)}{x-\frac{\pi}{2}} A(x,\Theta)dx + \frac{1}{\Theta-\frac{\pi}{2}} \text{P.V.} \int_0^{\pi} \frac{\overline{G}(x)}{x-\frac{\pi}{2}} dx - \text{P.V.} \frac{1}{\Theta-\frac{\pi}{2}} \int_0^{\pi} \frac{\overline{G}(x)}{x-\Theta} dx.$$

Call the integrals $H_1(\Theta)$, $H_2(\Theta)$, and $H_3(\Theta)$, respectively. A routine calculation shows that $H_1(\Theta)$ is a continuous function of Θ. It is obvious that $\text{P.V.} \int_0^{\pi} H_2(\Theta)d\Theta$ exists and it is well known (see [2], page 117) that $\text{P.V.} \int_0^{\pi} H_3(\Theta)d\Theta$ exists. Hence $\text{P.V.} \int_{-\pi}^{\pi} H(\Theta)d\Theta$ exists.

We now return to the calculation of C_L and C_D. Let K_g be a constant such that $G_K(\theta) = G^*(\theta) - \frac{K_g}{\cos\theta}$ is (by properly defining $G_K(\frac{\pi}{2})$) continuous. We shall now specialize the previously mentioned class H by restricting its members to be those $G(\theta)$ such that $G_K(\theta)$ has Fourier coefficients, a_k, where $\sum a_k \sin k\theta$ is uniformly convergent for $|\theta| < \pi$. Clearly there is a host of sufficient conditions for this, such as $G_K(\theta)$ and $G_K'(\theta)$ continuous and satisfy Dirichlet's conditions (see [1], page 270). We shall use the following notation:

$$G_K(\theta) = \sum_1 a_k \cos k\theta = \sum_1 (c_k - c_k') \cos k\theta, \text{ where}$$

$$c_k = \text{P.V.}\ \frac{1}{\pi}\int_{-\pi}^{\pi} G^*(\theta)\cos k\theta\, d\theta \text{ and } c_k' = \text{P.V.}\int_{-\pi}^{\pi} \frac{K_g}{\cos\theta} \cos k\theta\, d\theta.$$

$$C_L = \text{P.V.} \int_C \left(F_1(\zeta) + F_2(\zeta)\right) d\zeta, \text{ where}$$

$$F_1(\zeta) = \left(-i(\alpha - c_o)\left(\frac{i}{\zeta - i} - \frac{i}{\zeta + i}\right)\right)\left(\frac{(\zeta^2 + 1)(\zeta^2 - 1)}{2\zeta^3}\right)$$

$$\text{and } F_2(\zeta) = \left(-i \sum_{(C,1)} c_n \zeta^{-n}\right)\left(\frac{(\zeta^2 + 1)(\zeta^2 - 1)}{2\zeta^3}\right) .$$

A routine calculation shows that

$$\int_C F_1(\zeta)d\zeta = i(\alpha-c_o)\int_C \left(\frac{1}{\zeta} - \frac{1}{\zeta^3}\right) d\zeta = 2\pi(\alpha-c_o).$$

$$\text{P.V.} \int_C F_2(\zeta)d\zeta = -i \text{ P.V.} \int_{-\pi}^{\pi} \Big(G^*(\theta) - iH(\theta)\Big) \sin 2\theta \, d\theta.$$ Since $G^*(\theta)$

is even, $$\text{P.V.} \int_C F_2(\zeta)d\zeta = - \text{ P.V.} \int_{-\pi}^{\pi} H(\theta) \sin 2\theta \, d\theta.$$

We know (by our specification of the class H, and in turn, the coefficients a_k) that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \sum_1 a_k \sin k\theta \sin 2\theta \, d\theta = a_2.$$ Thus

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \sum_1 (c_k - c_k') \sin k\theta \sin 2\theta \, d\theta$$

$$= \frac{1}{\pi} \text{ P.V.} \int_{-\pi}^{\pi} \sum_{(C,1)} c_k \sin k\theta \sin 2\theta \, d\theta - \frac{1}{\pi} \text{ P.V.} \int_{-\pi}^{\pi} \sum_{(C,1)} c_k' \sin k\theta \sin 2\theta \, d\theta = c_2 - c_2'.$$

The c_k' have been computed in Chapter I, and it is easy to show that

$$\sum_{(C,1)} c_k' \sin k\theta = 0.$$ Thus

$$\text{P.V.} \int_{-\pi}^{\pi} \sum_{(C,1)} c_k \sin k\theta \sin 2\theta \, d\theta = \text{P.V.} \int_{-\pi}^{\pi} H(\theta) \sin 2\theta \, d\theta = \pi(c_2 - c_2') = \pi c_2.$$

Combining the above results, we have

$$C_L = \int_C \Big(F_1(\zeta) + F_2(\zeta)\Big) d\zeta = 2\pi(\alpha - c_o) - \pi c_2 .$$

It is convenient to divide the drag integral into three parts as follows:

$$C_D = \text{P.V.}\ \frac{1}{2i}\int_C W^2(\zeta)\,\frac{(\zeta^2-1)(\zeta^2+1)}{2\zeta^3}\,d\zeta = \frac{1}{2i}\,(I_1 + I_2 + I_3)$$

where $I_1 = -(\alpha-c_o)^2\ \text{P.V.}\int_C\left(\frac{i}{\zeta-i}-\frac{i}{\zeta+i}\right)^2\ \frac{(\zeta^2+1)(\zeta^2-1)}{2\zeta^3}\ d\zeta,$

$$I_2 = -2(\alpha-c_o)\ \text{P.V.}\int_C\left(\frac{i}{\zeta-i}-\frac{i}{\zeta+i}\right)\left(\sum_{(C,1)} c_n\,\zeta^{-n}\right)\left(\frac{(\zeta^2+1)(\zeta^2-1)}{2\zeta^3}\right)d\zeta,$$

and $I_3 = \text{P.V.}\int_C\left(\sum_{(C,1)} c_n\,\zeta^{-n}\right)^2\ \frac{(\zeta^2+1)(\zeta^2-1)}{2\zeta^3}\,d\zeta$, providing I_1, I_2 and I_3 exist.

Using the calculus of residues (for the details, see [6]), we find that

$$I_1 = 4\pi i\,(\alpha - c_o)^2.$$

After some changes in notation

$$I_2 = 4(\alpha-c_o)\ \text{P.V.}\int_{-\pi}^{\pi}\Big(G^*(\theta) - iH(\theta)\Big)(\cos\theta - i\sin\theta)\sin\theta\,d\theta.$$

Since $G^*(\theta)$ is even and $H(\theta)$ odd,

$$I_2 = -4i(\alpha-c_o)\ \text{P.V.}\int_{-\pi}^{\pi} H(\theta)\cos\theta\sin\theta\,d\theta - 4i(\alpha-c_o)\ \text{P.V.}\int_{-\pi}^{\pi} G^*(\theta)\sin^2\theta\,d\theta$$

$$= -2i(\alpha-c_o)\ \text{P.V.}\int_{-\pi}^{\pi} H(\theta)\sin 2\theta\,d\theta - 2i(\alpha-c_o)\ \text{P.V.}\int_{-\pi}^{\pi} G^*(\theta)(1-\cos 2\theta)\,d\theta.$$

We have already calculated P.V. $\int_{-\pi}^{\pi} H(\theta)\sin 2\theta\, d\theta$ and by the definition of c_o and c_2, we have

$$I_2 = -4\pi i(\alpha - c_o)c_o.$$

$$I_3 = \frac{1}{2}\,\text{P.V.}\int_{-\pi}^{\pi}\Big(G^*(\theta) - iH(\theta)\Big)^2 (-2\sin 2\theta)\, d\theta$$

$$= 2i\ \text{P.V.}\int_{-\pi}^{\pi} G^*(\theta)\, H(\theta)\, \sin 2\theta\, d\theta, \Big((G^*(\theta))^2 \text{ and } H^2(\theta) \text{ being even} \Big).$$

The following shows that I_3 exists. We note that (see [1], page 288)

$$\frac{\pi a_1^2}{2} = \int_{-\pi}^{\pi}\Big(\sum_1 a_k \cos k\theta\Big)\Big(\sum_1 a_k \sin k\theta\Big) \sin 2\theta\, d\theta$$

$$= \text{P.V.}\int_{-\pi}^{\pi}\Big(\sum_1 (c_k - c_k') \cos k\theta\Big)\Big(\sum_1 (c_k - c_k') \sin k\theta\Big) \sin 2\theta\, d\theta$$

$$= \text{P.V.}\int_{-\pi}^{\pi}\Big(\sum_{(C,1)} c_k \cos k\theta - \sum_{(C,1)} c_k' \cos k\theta\Big)\Big(\sum_{(C,1)} c_k \sin k\theta - \sum_{(C,1)} c_k' \sin k\theta\Big)\sin 2\theta\, d\theta$$

$$= \text{P.V.}\int_{-\pi}^{\pi}\Big(G^*(\theta) - \frac{K_g}{\cos\theta}\Big)\Big(\sum_{(C,1)} c_k \sin k\theta - 0\Big)\sin 2\theta\, d\theta$$

$$= \text{P.V.}\int_{-\pi}^{\pi} G^*(\theta)\, H(\theta)\, \sin 2\theta\, d\theta - \text{P.V.}\,2K_g\int_{-\pi}^{\pi} H(\theta)\, \sin\theta\, d\theta.$$

Thus, $\text{P.V.} \int_{-\pi}^{\pi} G^*(\theta)\, H(\theta) \sin 2\theta \, d\theta = \frac{\pi a_1^2}{2} + 2K_g \; \text{P.V.} \int_{-\pi}^{\pi} H(\theta) \sin\theta \, d\theta.$

$\text{P.V.} \int_{-\pi}^{\pi} H(\theta) \sin\theta \, d\theta$ is computed by noting that

$\int_{-\pi}^{\pi} \sum_{1} a_k \sin k\theta \sin\theta \, d\theta = \pi a_1 = \pi(c_1 - c_1')$. Then

$$\frac{1}{\pi} \text{P. V.} \int_{-\pi}^{\pi} \sum_{1} (c_k - c_k') \sin k\theta \sin\theta \, d\theta = \frac{1}{\pi} \text{P.V.} \int_{-\pi}^{\pi} \sum_{(C,1)} c_k \sin k\theta \sin\theta \, d\theta$$

$$- \frac{1}{\pi} \text{P.V.} \int_{-\pi}^{\pi} \sum_{(C,1)} c_k' \sin k\theta \sin\theta \, d\theta = \text{P.V.} \frac{1}{\pi} \int_{-\pi}^{\pi} H(\theta) \sin\theta \, d\theta = c_1 - c_1' .$$

Combining the previous results we have

$$\text{P.V.} \int_{-\pi}^{\pi} G^*(\theta)\, H(\theta) \sin 2\theta \, d\theta = \frac{\pi a_1^2}{2} + 2\pi K_g (c_1 - c_1') = \pi \left(\frac{(c_1 - c_1')^2}{2} + 2K_g (c_1 - c_1') \right)$$

and that

$$I_3 = 2i \; \text{P.V.} \int_{-\pi}^{\pi} G^*(\theta)\, H(\theta) \sin 2\theta \, d\theta = 2\pi i \, (c_1 - c_1') \left(\frac{c_1 - c_1'}{2} + 2K_g \right) .$$

Finally,

$$C_D = \frac{1}{2i} (I_1 + I_2 + I_3) = 2\pi(\alpha - c_o)^2 + \pi(c_1 - c_1') \left(\frac{c_1 - c_1'}{2} + 2K_g \right) - 2\pi c_o (\alpha - c_o).$$

It should be noted that in the evaluation of C_L and C_D, although there is a great temptation to perform term by term integration, (as Fabula does in [3] and [4]), it cannot be justified and by considering the example of Chapter I, it leads to incorrect results.

LITERATURE CITED

1. Carslaw, H. S., Fourier's Series and Integrals, Dover Publications, 1930.

2. Mikhlin, S. G., Integral Equations, The Macmillan Company, 1964.

3. Fabula, Andrew G., "Theoretical Lift and Drag on Vented Hydrofoils for Zero Cavity Number and Steady Two Dimensional Flow," NAVORD Report 7005, November, 1959.

4. Fabula, Andrew G., "Thin Airfoil Theory Applied to Hydrofoils with a Single Finite Cavity and Arbitrary Free-Streamline Detachment," Journal of Fluid Mechanics, (12), 1962.

5. Stewart, H. J., "A Simplified Two-Dimensional Theory of Thin Airfoils," Journal of the Aeronautical Sciences, (9), 1942.

6. Nickel, Paul A., "A Boundary Value Problem Approach to Linearized Hydrodynamics," The Boeing Company, Aero-Space Division, Applied Mathematics Section Mathematical Analysis Unit, Mathematical Analysis Note No. 28, September 20, 1962.

7. Heins, Maurice, Selected Topics in the Classical Theory of Functions of a Complex Variable, Holt, Rinehart and Winston, Inc., 1962.

8. Knopp, K., Theory and Application of Infinite Series, Hafner Publishing Company, 1947.

9. Ahlfors, L. V., Complex Analysis, McGraw-Hill, Inc., 1953.