A STUDY OF THE RELATIONSHIP BETWEEN INTRODUCTORY CALCULUS STUDENTS' UNDERSTANDING OF FUNCTION AND THEIR UNDERSTANDING OF LIMIT

by

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STATEMENT OF PERMISSION TO USE

DEDICATION

This dissertation is dedicated to my beautiful wife, Krista, for whom there are no words which can adequately describe my gratitude and love. Krista, tú eres mi mejor amiga y la bendición más maravillosa de mi vida. *Siempre* te amo.

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ABSTRACT

Introductory calculus students are often successful in doing procedural tasks in calculus even when their understanding of the underlying concepts is lacking, and these conceptual difficulties extend to the limit concept. Since the concept of limit in introductory calculus usually concerns a process applied to a single function, it seems reasonable to believe that a robust understanding of function is beneficial to and perhaps necessary for a meaningful understanding of limit. Therefore, the main goal of this dissertation is to quantitatively correlate students' understanding of function and their understanding of limit. In particular, the correlation between the two is examined in the context of an introductory calculus course for future scientists and engineers at a public, land grant research university in the west.

In order to measure the strength of the correlation between understanding of function and understanding of limit, two tests—the Precalculus Concept Assessment (PCA) to measure function understanding and the Limit Understanding Assessment (LUA) to measure limit understanding—were administered to students in all sections of the aforementioned introductory calculus course in the fall of 2008. A linear regression which included appropriate covariates was utilized in which students' scores on the PCA were correlated with their scores on the LUA. Nonparametric bivariate correlations between students' PCA scores and students' scores on particular subcategories of limit understanding measured by the LUA were also calculated. Moreover, a descriptive profile of students' understanding of limit was created which included possible explanations as to why students responded to LUA items the way they did.

There was a strong positive linear correlation between PCA and LUA scores, and this correlation was highly significant ($p < 0.001$). Furthermore, the nonparametric correlations between PCA scores and LUA subcategory scores were all statistically significant ($p < 0.001$). The descriptive profile of what the typical student understands about limit in each LUA subcategory supplied valuable context in which to interpret the quantitative results.

Based on these results, it is concluded that understanding of function is a significant predictor of future understanding of limit. Recommendations for practicing mathematics educators and indications for future research are provided.

CHAPTER 1

INTRODUCTORY OVERVIEW

A Case for the Present Research

Students' Struggles in Calculus

> The keynote of Western culture is the function concept, a notion not even remotely hinted at by any earlier culture. And the function concept is anything but an extension or elaboration of previous number concepts—it is rather a complete emancipation from such notions.
>
> (W. A. Schaaf, as quoted in Tall, 1992, p. 497)

More than any other mathematical concept, the concept of function ties the many diverse branches of modern mathematics together. "Having evolved with mathematics, [the function] now plays a central and unifying role" (Selden and Selden, 1992, p. 1). To facilitate success in higher mathematics, students need a deep and flexible mental conception of function. In particular, calculus students must possess a strong understanding of function—for calculus is the gateway course to higher mathematical thinking, and function "is the central underlying concept in calculus" (Vinner, 1992, p. 195).

Unfortunately, "for the overwhelming majority of students, the calculus is not a body of knowledge, but a repertoire of imitative behavior patterns" (E. E. Moise, as quoted in Tall, 1997, p. 2). Although many calculus students are successful in doing procedural tasks, their understanding of underlying concepts in calculus is lacking. This is true even in students who have been highly successful on exams (Cornu, 1991). For example, two-thirds of students earning an “A” or “B” grade in a traditional calculus class could not solve a single non-routine problem on a test presenting such problems (Selden, Selden, and Mason, 1994). As another example, given a picture model of an oddly shaped bottle to be filled with water, only two students in twenty who had just received an “A” grade in second-semester calculus could draw an appropriate graph of a function relating the water’s height to its volume (Carlson et al., 2002).

Students’ conceptual difficulties in calculus extend to the limit concept, which is foundational to all of modern standard analysis as well as to the traditional pedagogical treatment of introductory calculus. Problematically for mathematics educators trying to teach this difficult concept, “the mathematical approach to limit and the cognitive approach to limit are quite different” (Williams, 2001, p. 342). The conceptions of limit held by students are intuitive in nature (Mamona-Downs, 2001) and not necessarily mathematically rigorous. In particular, being able to state the formal definition of limit does not indicate understanding of it (Tall and Vinner, 1981). Clearly, the questions of how students regard the limit concept and how mathematics educators can teach this concept better so their students can more fully and deeply comprehend its meaning are questions worth answering.

Possible Connection between Understanding of Function and Understanding of Limit

Since the concept of limit in introductory calculus usually concerns a process applied to a single function, it seems reasonable to believe that a robust understanding of function is beneficial to and perhaps necessary for a meaningful understanding of limit.

This belief is supported by a genetic decomposition of how the limit concept in calculus can be learned (Cottrill et al., 1996). The result of a careful analysis of a particular mathematical concept, a *genetic decomposition* is “a description, in terms of [a] theory and based on empirical data, of the mathematics involved and how a subject might make the constructions that would lead to an understanding of it” (Dubinsky, 1991). Included in the genetic decomposition of the limit postulated by Cottrill and his team (1996) are seven specific steps, each step building upon the previous one, leading to a full understanding of the formal limit concept. Introductory calculus classes may provide an opportunity for students to progress through the first four steps of this genetic decomposition, which are presented in Figure 1.

---

(1) Evaluating the function f at a single point close to a or at a itself;

(2) Evaluating the function f at a few points that successively get closer to a;

(3) Constructing a domain process in which x approaches a, constructing a range process in which y approaches L, and coordinating these processes using the function f;

(4) Performing actions on the limit concept, thus treating it as a legitimate mathematical object.

(Modified from Cottrill et al., 1996, pp. 177–178)

Figure 1. Four steps in a genetic decomposition of limit understanding.

Helping students to build up their mental conceptions of limit to the degree necessary to accomplish these four steps is a desirable goal for mathematics educators teaching introductory calculus, since the limit concept is essential in allowing students to move beyond simple algebra to the more powerful methods of analysis. Eves and Newsom made this distinction when they declared:

> Generally the presence of the limit concept distinguishes analysis from algebra.
>
> (Eves and Newsom, 1958, p. 183)

Furthermore, a report written by the Committee on the Undergraduate Program in Mathematics (CUPM) of the Mathematical Association of America (MAA) contains a recommendation that "courses designed for mathematical science majors should ensure that students progress from a procedural/computational understanding of mathematics to a broad understanding encompassing logical reasoning, generalization, abstraction, and formal proof" (MAA, 2004, p. 44). This statement applies to the concept of limit. A broad understanding of the limit concept requires a student to use indirect proofs and deduction rather than just computation (Davis and Vinner, 1986). Thus, an understanding of limit helps students to begin their journey toward achieving the curricular goal advocated in the CUPM's report.

The first three steps of the genetic decomposition of limit as listed in Figure 1 appear to be directly related to a student's understanding of function. Hence, it may not be an exaggeration to state that understanding the function concept is necessary to grasping the limit concept. Function understanding could also be viewed as crucial to a conceptual comprehension of other calculus concepts (Carlson and Oehrtman, 2005). However, the connection between understanding of function and understanding of the many other concepts of calculus is too large a task to undertake in one research study.

Hence, the main goal of the present research is to quantitatively correlate students' understanding of function and their understanding of limit.

Categorizing Mathematical Understanding

Students' Concept Images

Although most mathematics educators are aware of the formal definitions of mathematical concepts like function or limit, of equal relevance to success in teaching mathematics is an awareness of a student's *concept image* of such mathematical objects. Tall and Vinner (1981) provided a working definition for this term:

> We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures.
>
> (p. 152; emphasis in original)

Importantly, it is entirely possible for a student to be "successful" in calculus, as measured by grades, but hold fundamentally flawed concept images of calculus concepts, as Vinner (1983) noted:

> Students can succeed in examinations even when having a wrong concept image. Hence if we do not care about the concept image we do not have to bother with it. On the other hand, if we do care about it then what we do is not enough. More time and pedagogical efforts are required.
>
> (p. 305)

The preceding quote is pertinent to calculus courses, especially if computation and mechanics are stressed above concepts (Tall, 1997). On the other hand, a student may fail a traditional calculus exam but have some genuine understanding of the subject matter, because understanding only increases the *tendency* of students to behave in a mathematically sensible way in any particular situation (Breidenbach et al., 1992; Dubinsky and Harel, 1992). Other factors may make a student's performance inaccurately reflect what he really knows. Caution must be taken, therefore, when discussing students and their level of understanding of calculus concepts.

What is "Understanding"?

As slippery and nebulous as the word "understanding" is, some kind of theory of what the term means is usually needed to pursue educational research. The theory of understanding that researchers choose to use is an important underlying assumption of the entire study. For example, Zandieh (2000) postulated that a true understanding of the concept of derivative involves a student's understanding all of the underlying processes that go into its definition, including those of ratio, limit, and function. This theory of understanding of derivative, in her estimation, superseded simple procedural prowess with derivatives. Often, a student may appear to understand the derivative when she computes its values or applies it to find the maximum of a function; however, if the underlying processes are not fully understood, then such a student is demonstrating a *pseudostructural conception*. This means that the "conception is thought of as a whole without parts, a single entity without any underlying structure" (Sfard as paraphrased by Zandieh, 2000, p. 108).

If computation problems were all that mathematics educators wanted calculus students to be able to solve, then they would not have to worry about the connections students make among calculus concepts like derivative, limit, and function. However, the authors of a standards document produced by the American Mathematical Association of

Two-Year Colleges (AMATYC) asserted that "students gain the power to solve meaningful problems through in-depth study of mathematics topics. [Hence] the meaning and use of mathematical ideas should be emphasized and attention to rote memorization deemphasized" (AMATYC, 2006, p. 5). In short, no mathematics educator should be satisfied with superficial understanding as measured by computational proficiency.

A Theory of Mathematical Understanding

In support of the belief that mathematical understanding goes beyond mechanics and rote memorization, Kaput (1989) postulated a theory delineating four sources of mathematical meaning, of which at least three go beyond the superficial scope of symbolic manipulation skills. The four sources of mathematical meaning Kaput described are displayed in Figure 2. The first two sources of meaning listed in Figure 2

---

(1) Via translations between mathematical representation systems;

(2) Via translations between mathematical representations and non-mathematical systems (including physical systems, as well as natural language, pictures, etc.);

(3) Via pattern and syntax learning through transformations within and operations on the notations of a particular representation system;

(4) Via mental entity building through the reification of actions, procedures, and concepts into phenomenological objects which can then serve as the basis for new actions, procedures, and concepts at a higher level of organization (often called "reflective abstraction").

(Kaput, 1989, p. 168)

Figure 2. Four sources of mathematical meaning.

were categorized by Kaput under the heading of *referential extension* while the latter two were placed into a *consolidation* category (Kaput, 1989, p. 168). A reasonable theory of mathematical understanding can be derived using these four sources of mathematical meaning, in the sense that a student with a well-developed understanding of some mathematical concept will cognitively draw on all four sources of meaning in his consideration of that concept. The names I give to these four sources of mathematical meaning are *translation*, *interpretation*, *familiarization*, and *reification*, respectively. Notably, these names are somewhat parallel to terms used by O'Callaghan (1998) and Sfard (1991).

The importance of the referential extension sources of meaning is clearly advocated by researchers from White and Mitchelmore (1996) to the Calculus Consortium based at Harvard University (Gleason and Hughes-Hallett, 1992). The former two researchers asserted that solving calculus application problems—that is, problems from "real life"—requires students to create their own functions to model the many different situations they might encounter (White and Mitchelmore, 1996). Modeling the natural world with functions falls under the purview of interpretation (using my terminology). In promoting a curriculum in which "three aspects of calculus—graphical, numerical, and analytical—[are] all … emphasized throughout" (Gleason and Hughes-Hallett, 1992, p. 1), the members of the Calculus Consortium based at Harvard University underscored a clear assumption of the value of translation in Kaput's theory. Connecting this "Rule of Three" (ibid.) to the function concept is natural, since functions can be represented graphically, numerically, and algebraically.

Familiarization includes computational aptitude, and although Kaput asserted that "school mathematics tends to be dominated by the third form of meaning building [familiarization]" (Kaput, 1989, p. 168), which he characterized as the "narrowest and most superficial form [of meaning]" (ibid.), some curricular approaches allow for students to make stronger connections among their mental constructions of mathematical concepts (see for example Heid, 1988; Schwarz et al., 1990). However, reification requires a very mature understanding of the concept to be reified and hence is a major obstacle for most students, even with thoughtfully planned pedagogy (Sfard, 1992). Because of the apparent gap in the level of understanding needed to perform familiarization tasks versus reification tasks, it would be justifiable and advantageous to add another source of meaning to Kaput's theory that resides somewhat between these two.

To be certain, "the idea of turning [a] process into an autonomous entity" (Sfard, 1991, p. 18), a stage in concept development called *condensation* (ibid.), must precede reification. In the case of function, condensation allows a student to be "more capable … of playing with a mapping as a whole, without actually looking into its specific values" (p. 19), so that she can, in time, "investigate functions, draw their graphs, combine couples of functions, (e.g., by composition), even … find the inverse of a given function" (ibid). Unfortunately, "consolidation" and "condensation" can be considered synonyms; hence, I refer to this new source of mathematical meaning in the consolidation category—which bridges the gap between familiarization and reification—as *instantiation*. The reason I chose this particular name will become apparent later on.

This modified theory of sources of mathematical meaning is briefly summarized in Table 1. Note that the first two sources of meaning in Table 1 are categorized under the heading of *referential extension* while the latter three are placed into a *consolidation* category, to use the terminology of Kaput (1989). As before, a theory of mathematical understanding is created by assuming that significant connections among these sources of

Table 1. Sources of Mathematical Meaning (REC Theory).

Category	Source of Meaning	Definition
referential extension	translation	via translations between mathematical representation systems
	interpretation	via translations between mathematical representations and non-mathematical systems (including physical systems, as well as natural language, pictures, etc.)
consolidation	familiarization	via pattern and syntax learning through transformations within and operations on the notations of a particular representation system
	instantiation	via investigation of the properties of specific examples of mathematical processes, allowing for the coordination, reversal, and combination of such processes
	reification	via mental entity building through the reification of actions, procedures, and concepts into phenomenological objects which can then serve as the basis for new actions, procedures, and concepts at a higher level of organization

(Modified from Kaput, 1989, p. 168)

mathematical meaning in the mind of a student constitute understanding. In referring to this theory and its components as defined in Table 1, I use the term REC theory (where REC is an acronym for referential extension and consolidation). It will be seen shortly that REC theory provides a practical scheme for organizing the research literature concerning understanding of function as well as the research literature about understanding of limit. In the present research study, the theory also supplies a rational foundation for an operational definition for understanding of limit.

A Model of Understanding of Function

A model of mathematical understanding, based on the theory of Kaput (1989), was created specifically to measure students' understanding of the function concept. This model, developed by O'Callaghan (1998), includes four component competencies for function along with an associated set of procedural skills. Each of the component competencies for function is rooted in one of the theoretical sources of mathematical meaning given in Figure 2. The names and corresponding definitions for these four competencies (along with a definition of procedural skills) as given by O'Callaghan (1998) are presented in Figure 3. Although O'Callaghan's model was constructed to measure college algebra students' understanding of functions, the model might be expanded to include elements of function understanding in the context of introductory calculus.

In attempting to expand and modify O'Callaghan's model to be suitable for introductory calculus students' level of understanding, at least two important viewpoints of researchers in calculus education warrant attention, as these viewpoints concern what

Modeling. Mathematical problem solving involves a transition from a problem situation to a mathematical representation of that situation, [which] entails the use of variables and functions.

Interpreting. The reverse procedure [of modeling] is the interpretation of functions … in their different representations in terms of real-life applications.

Translating. [This is] the ability to move from one representation of a function to another.

Reifying. [This is] defined as the creation of a mental object from what was initially perceived as a process or procedure.

Procedural skills. These skills consist of transformations and other procedures that allow students to operate within a mathematical representational system.

(From O'Callaghan, 1998, pp. 24–26)

Figure 3. Component competencies and procedural skills in O'Callaghan's model of function understanding.

it means to understand the function concept sufficiently well to succeed in introductory calculus. In illuminating students' work with models to assess their function understanding, Monk (1992) presented a dichotomy between a point-wise view of function versus an across-time view. A *point-wise* view of function does not permit a student to see beyond evaluating a function at a particular input value, whereas an *across-time* view allows a student to describe patterns of change in functional values as the input values vary across intervals (e.g., time intervals). An across-time view of functions is essential to conceptual understanding in calculus. For example, "the definition of the tangent line to a graph or of its slope would be utterly meaningless to someone who could only look at the graph of the function at a few specific points at a time" (Monk, 1992, p. 21).

Somewhat analogous to an across-time view functions is the notion of *covariational reasoning*. A crucial component to understanding calculus, covariational reasoning is the ability to "coordinat[e] two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354). For example, data from an experiment with nineteen senior undergraduate and graduate-level mathematics students led a researcher to conclude that students' troubles with the Fundamental Theorem of Calculus "stem from having impoverished concepts of rates of change" as well as "poorly coordinated images of functional covariation" (Thompson, 1994, p. 229).

Common to both of these theories—both the theory of a dichotomy of function "views" and the theory of covariational reasoning—is the assumption that an ability to analyze a function's properties holistically is a significant indicator of function understanding. In introductory calculus, many of these function properties are given names—e.g., continuity, differentiability, monotonicity, concavity, and so on. The definitions of these properties, when they are given, will not be understood by a student with insufficient understanding of function. To be more specific, a student must have mentally consolidated the function concept to an appropriate degree so she can, for example, investigate a single function over entire intervals without computing any of its specific values, or as another example, compose specific functions together as if they were legitimate objects. In the terminology of the theory of mathematical understanding developed for the present research, such a student can reliably *instantiate* the function concept in specific situations.

In contrast, *reifying* the function concept requires a student to be able to consider the properties of entire *sets* of functions as opposed to just one specific function. While an introductory calculus student who can instantiate specific examples of the function concept can identify a continuous function (from $\mathbb{R}$ to $\mathbb{R}$) versus a non-continuous function, a student who has reified the function concept can explain *why* the set of functions that are continuous on a closed, bounded interval I is exactly the set of functions to which the conclusion of the Intermediate Value Theorem holds on I (as opposed to the set of differentiable functions on I or the set of Riemann integrable functions on I). Another example of the contrast between instantiation and reification of mathematical concepts appears as part of the presentation, in the next section, of a model of understanding of limit.

An expanded and modified model of understanding of function for use in the present research is presented in Table 2. In order to be consistent with REC theory, I have changed the names and order of the component competencies of O'Callaghan's (1998) function model and included something akin to procedural skills as a legitimate component of the new model. Additionally, it must be noted that my definition for each of these component competencies represents a synthesis of similar definitions and descriptions found in the literature, and as such, does not exactly match O'Callaghan's characterization of each component competency. (A comparison of Figure 3 and Table 2 confirms this assertion. Note that I have merged O'Callaghan's *modeling* and

Table 2. Component Competencies in Function Understanding.

Component Competency	Definition
translation	consists of moving from one representation (e.g., symbolical, graphical, numerical) of a function to another
interpretation	involves modeling a real-life problem situation using an appropriate function *or* necessitates the reverse procedure, the interpretation of functions in meaningful, real-life terms
familiarization	consists of computations and other mechanical procedures involving functions
instantiation	involves an investigation of the properties of specific examples of functions, allowing for the coordination, reversal, and combination of whole functions
reification	necessitates the viewing of function as an entirely manipulatable object instead of just as a process or procedure, to the point that properties of entire *sets* of functions can be considered

Note: The italicized component competencies together comprise the TIRI function model.

interpreting component competencies, which both fall under the banner of *interpretation* to be consistent with REC theory.) A student displays understanding of function by correctly engaging in the activities listed in Table 2. This new model of understanding of function serves as a guide to account for findings from prior research, and it was employed in familiarizing research assistants with REC theory in the completion of the present research study. When referencing this model of function understanding *without* any consideration of the relatively narrow, procedural component competency of familiarization, I use the term TIRI function model (where TIRI is an acronym for translation, interpretation, reification, and instantiation).

A Model of Understanding of Limit

Since the present research entails measuring students' understanding of limit as well as their understanding of function, it follows that the development of a specific model of understanding of limit would be a desirable goal. For consistency's sake, this limit model should be based on the same theory of mathematical understanding as the model of understanding of function already developed—that is, it should be based on REC theory. Rewording the definitions given in Table 2 to reflect the mathematical concept of limit yields the five component competencies given in Table 3, which together constitute a model of understanding of limit. A student displays understanding of limit by correctly engaging in the activities listed in Table 3.

For convenience, when referring to the model of limit understanding that is the basis for an instrument measuring introductory calculus students' understanding of limit, I use the term TIFI limit model (where TIFI is an acronym for translation, interpretation, familiarization with the formal definition, and instantiation). As with the model of understanding of function, an example contrasting the instantiation of the limit concept and the reification of it would presently be informative. A student who can dependably instantiate the limit concept (as it is taught in introductory calculus) can identify a convergent sequence versus a sequence that does not converge. On the other hand, a student who has reified the limit concept can explain *why* the set of all bounded

Table 3. Component Competencies in Limit Understanding.

Component Competency	Definition
translation	consists of moving from one representation (e.g., symbolical or graphical) of a limit to another
interpretation	involves modeling a real-life problem situation using an appropriate limit *or* necessitates the reverse procedure, the interpretation of limits in meaningful, real-life terms
familiarization	consists of computations and other mechanical procedures involving limits (including procedures which utilize the formal definition of limit)
instantiation	involves an investigation of the properties of specific examples of limits, allowing for the mental creation of appropriate examples of limit (each of a suitably chosen individual function) demonstrating specific characteristics *and* for the evaluation of arguments involving specific examples of limit
reification	requires some understanding of the formal definition of limit so that it is viewed as an entirely manipulatable object instead of just as a process or procedure, to the point that properties of entire *categories* of limit (e.g. entire *sets* of infinite sequences or different *types* of convergence) can be considered

Note: The italicized component competencies together comprise the TIFI limit model, but with the *familiarization* component competency being considered only as it applies to the formal definition of limit.

monotonic sequences necessarily converges or how the traditional definition of the limit of a function (as x approaches a) can be couched in sequential terms (i.e., the consistent convergence of all sequences of functional values corresponding to sequences of x-values which approach a but do not coincide with it).

A careful examination of the types of homework exercises required of introductory calculus students in courses utilizing standard textbooks suggests that the TIFI limit model can be fruitfully applied in the selection of items for an instrument that operationalizes understanding of limit by reflecting four of the five component competencies of the model. In particular, the component competency of reification was *not* considered in designing the instrument of limit understanding, because other researchers (e.g., Cottrill et al., 1996) have argued that introductory calculus students are unlikely to be able to reify the limit concept at such an early stage in mathematical coursework, citing a thorough lack of data substantiating such reification occurring among students taking introductory calculus. On the other hand, the component competency of familiarization is included in the instrument measuring limit understanding, but only as it applies to the formal definition of limit. The reason for this qualification is because—to use the language of Bloom's taxonomy of cognitive objectives (Sax, 1974)—items measuring the component competency of familiarization as it pertains to simple limit computations are likely to necessitate only superficial knowledge of the limit concept rather than meaningful comprehension or application of it.

Statement of the Problem

Purposes of the Study

The basic objective of the present research study is to ascertain the strength of the relationship between understanding of the function concept and understanding of the limit concept. In particular, I examine the correlation between the two in the context of an introductory calculus course for future scientists and engineers. Although the research design does not allow a firm "cause and effect" relationship to be confirmed, it has the potential to quantitatively establish a connection between the understandings worthy of further investigation. There are two main purposes for the present research—both beneficial to a preliminary examination of the link between students' understanding of function and their understanding of limit. The first is to design and validate (via a pilot study) an instrument called the Limit Understanding Assessment (LUA). The second is to quantitatively establish a correlation between understanding of function and understanding of limit in the context of a traditional calculus course for future scientists and engineers.

Since many introductory calculus students' understanding were measured using the LUA in the course of implementing the present research study, the idea emerged that an informative profile of a typical introductory calculus student's understanding of limit could be created through careful analysis of students' responses to LUA items. Such a profile could prove beneficial to future research on students' understanding of limit by providing a descriptive "snapshot" of what students at present actually learn about limit in the course of studying the concept. Hence, a secondary purpose for the present research is to establish a profile of students' understanding of limit, interpreted through the framework lens of the TIFI limit model, in the context of a traditional calculus course for future scientists and engineers.

Pilot Study Questions

Before attempting to determine the scope of the connection between understanding of function and understanding of limit, I completed a pilot study to iteratively develop a new instrument, intended to measure limit understanding, called the Limit Understanding Assessment (LUA). Using both quantitative and qualitative data from that pilot study, I provide answers to the questions found in Figure 4 regarding the LUA. After creating a first version of the LUA, I administered the test to introductory

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(1) Does the Limit Understanding Assessment (LUA) provide a reliable measure of students' understanding of the limit concept? In particular, does the LUA demonstrate reliability via:

- (a) clear, justifiable standards in its scoring?
- (b) adequate discrimination among students with different levels of understanding of the limit concept?
- (c) internal consistency among the test items?

(2) Does the LUA provide a valid measure of students' understanding of the limit concept? In particular, does the LUA demonstrate validity via:

- (a) appropriate content for the test items?
- (b) reasonable interpretation of test items from students taking the test?
- (c) acceptable categorization of test items based on the component competencies of the TIFI limit model?

Figure 4. Research questions answered via the pilot study.

calculus students and then modified the test, based on quantitative and qualitative evidence, so as to increase the reliability and validity of subsequent versions of the LUA. The manner in which the pilot study data was collected and employed to accomplish this goal can be found in Chapter 3.

Major Research Questions

Establishing the strength of the correlation between understanding of function and understanding of limit required the administration of two instruments, one to measure function understanding, the other to measure limit understanding. The Precalculus Concept Assessment (PCA), an instrument which gauges student understanding of precalculus concepts and which was developed by Marilyn Carlson and colleagues at Arizona State University, was administered to all 10 sections ($n = 376$) of an introductory calculus course at a western university noted for science and engineering. The intention was to measure students' understanding of function at the beginning of introductory calculus. After the limit concept had been covered in the course, the Limit Understanding Assessment (LUA) was administered to determine each student's conceptual understanding of limit. I employed statistical tests on these quantitative data to answer the major research questions listed in Figure 5. In addition to those major research questions, I present a secondary question considered in the present research in Figure 6. A more detailed description of the manner in which the major data were collected and analyzed so as to answer all of these questions is presented in Chapter 3.

Before discussing the potential significance of the present research study, a quick word of caution about the extent of the possible conclusions to the study must be inserted.

(1) How strong is the correlation between students' scores on the PCA and their scores on the LUA?

(2) How strong is the correlation between students' scores on the PCA and their scores on the portion of the LUA that measures understanding of limit:

- (a) in the component competency of translation?
- (b) in the component competency of interpretation?
- (c) in the component competency of instantiation?
- (d) in the component competency of familiarization as it applies to the formal definition of limit?

Figure 5. Primary research questions answered via the major research study.

In the context of a traditional calculus course for future scientists and engineers, what is the profile of a typical student's understanding of limit when interpreted through the framework lens of the TIFI limit model?

Figure 6. Secondary research question answered via the major research study.

The operational definitions of function understanding and limit understanding as measured by the PCA and LUA, respectively, are not the only reasonable definitions for them. Furthermore, other factors in the implementation of the study—including the characteristics of the sample of students chosen for study, the pre-experimental one-group design of the study, and experimental attrition—create limitations in what conclusions from the research can be safely inferred. The assumptions and limitations of the present research study are discussed in more detail in Chapter 3.

Significance of the Study

The present research study establishes a valid instrument that can be used to measure students' understanding of limit in terms of a practical model based on a credible theory of mathematical understanding. Additionally, by quantitatively establishing a correlation between students' understanding of function and their understanding of limit, this study may quantify in some respects the conclusions of a research study in which a link between a student's "content beliefs" about function and his understanding of limit was discovered (Szydlik, 2000). In that paper involving 27 second-semester introductory calculus students, the researcher found a correlation between students' conceptions of what a function actually is and their ability to explain, in structured interviews, their own reasoning in making limit computations (for both algebraic and graphical limit problems). Students with more flexible notions of what a function is were better able to explain their own reasoning in doing limit calculations. However, the study was limited in its measurement of students' ability to use functions and in the types of questions asked of them about limit. The present research study utilizes an instrument differentiating among a broad range of abilities to use function as well as an instrument which measures limit understanding in multiple component competencies rather than in just limit computations. Hence, this paper should provide a fuller picture of the relationship between understanding of function and understanding of limit than previously afforded.

The dissertation also has the potential to corroborate the apparent importance of function in the genetic decomposition of how sound limit conceptions can be mentally constructed (Cottrill et al., 1996). As mentioned earlier, the first three steps of the genetic decomposition of limit as listed in Figure 1 appear to be directly related to a student's understanding of function. More generally, it is logical to believe understanding of function is necessary to understanding of limit, since in introducing limit calculus instructors must discuss the limit of a particular function in order to talk about it all. However, the paper by Cottrill and his colleagues does not address what level of understanding of function is necessary to construct a sound mental conception of limit. Nor does their work address understanding of limit using the full spectrum of the TIFI limit model with its component competencies of translation and interpretation. It appears the genetic decomposition has more to do with familiarization and eventual reification of the formal definition of limit than anything else. Therefore, the present research should fill a need in the research literature by providing a picture of what levels of understanding of function correlate with limit understanding as a whole as well as with different types of limit understanding as categorized in the TIFI limit model.

Furthermore, verifying the substantial strength of a correlation between understanding of function and understanding of limit would confirm the belief that the concept of function should be central to both precalculus and introductory calculus curricula, because it substantiates the assertion that the limit concept, foundational to all of modern standard analysis as well as to the traditional pedagogical treatment of introductory calculus, is meaningfully understood only when a mature understanding of function is already in place or well on its way to development in the mind of a student.

Finally, the dissertation has the potential of demonstrating the general value of referential extension and consolidation (REC) theory—an extension of a theory originally postulated by Kaput (1989)—to mathematics education research. It will be seen that the TIRI function model and TIFI limit model, both introduced in this chapter and both grounded in REC theory, can be employed to guide an important synopsis of past research work by providing suitable lenses with which to examine theories on function understanding and on limit understanding, respectively. This TIRI- and TIFI-driven illustration of how prior research justifies and points toward the present research study is the aim of Chapter 2. Part of the body of research examined in that chapter suggests possible connections between students' understanding of function and their understanding of limit. I also discuss the methodologies of research studies that together point out a constructive path to follow in the implementation of the present research study, so as to ultimately provide a valuable quantitative measurement of the important connection between students' understanding of function and their understanding of limit.

For convenience, the definitions of five important terms used throughout the present research paper are given in Table 4 on the next page. The table also includes references to other tables where more detailed information about some of the terms can be found.

Table 4. Key Terms in the Present Research Paper.

Term	Definition
function	A function f from set A to set B is a subset of the Cartesian product $A \times B$ such that $(a, b_1) \in f$ and $(a, b_2) \in f$ implies $b_1 = b_2$ and such that $\forall a \in A, \exists\, b \in B$ such that $(a, b) \in f$.
limit	Let f be a function defined on an open interval that contains the number $c,$ except possibly at c itself. Then the limit of $f(x)$ as x approaches c is $L,$ written mathematically as $\lim_{x \to c} f(x) = L,$ if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that if $0 < \lvert x - c \rvert < \delta,$ then $\lvert f(x) - L \rvert < \varepsilon.$
REC theory	Referential extension and consolidation (REC) theory is a theory of mathematical understanding in which it is assumed that significant connections among five sources of mathematical meaning (listed in Table 1) in the mind of a student constitute understanding.
TIRI function model	The TIRI function model is a model of mathematical understanding based on REC theory. The letters are an acronym for translation, interpretation, reification, and instantiation—component competencies of the model which are defined in Table 2.
TIFI limit model	The TIFI limit model is a model of mathematical understanding based on REC theory. The letters are an acronym for translation, interpretation, familiarization with the formal, and instantiation—component competencies of the model which are defined in Table 3.

CHAPTER 2

REVIEW OF THE LITERATURE

Introduction

The aim of this chapter is to illustrate how prior research justifies and points toward the present research study. In particular, the three major sections of this chapter are dedicated to discussions on students' understanding of function, on students' understanding of limit, and on the connection between the two, in that order. Each of these major sections begins with a brief introduction declaring the purpose of that section and concludes with a summary as to how the research literature supports the theoretically based models underlying the present research. Following the three major sections, I also discuss the methodologies of research studies that together point out a constructive path to follow in the implementation of the present research study, so as to ultimately provide a valuable quantitative measurement of the important connection between students' understanding of function and their understanding of limit. This discussion includes principles of test construction found in the literature that guided me in the creation and validation of an instrument for measuring students' understanding of limit. The chapter concludes with an overall summary as to why prior research demonstrates the rationality for and potential value of the present research study.

Students' Understanding of the Concept of Function

The purpose of this section is to tie REC theory and the associated TIRI function model to previous research studies' findings about students' understanding of function. A summary of REC theory and of the TIRI function model can be found in Table 1 and in Table 2, respectively. A student displays understanding of function by correctly engaging in the activities listed in Table 2. I now consider students' personal concept definitions for function, theories on students' mental constructions of function, and epistemological theories on function understanding in light of REC theory and the associated TIRI function model.

Students' Personal Concept Definitions for Function

The modern set-theoretic definition of function as formulated in 1939 by Bourbaki can be stated as follows: A function f from set A to set B is a subset of the Cartesian product $A \times B$ such that $(a, b_1) \in f$ and $(a, b_2) \in f$ implies $b_1 = b_2$ and such that $\forall a \in A, \exists b \in B$ such that $(a, b) \in f$. Although most mathematicians are aware of this formal, ordered pair definition of function, of equal relevance to mathematics educators is the *personal concept definition* (Tall and Vinner, 1981) each individual student has for function—that is, how each student defines the term function in her own mind. A student's personal concept definition of function is part of that student's overall *concept image* of function, a term whose definition appears (in the form of a quotation) in Figure 7. Other terms related to Tall and Vinner's theory are also presented in that figure in the form of pertinent quotations from their research. While simply one piece of his

We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures.

We shall call the portion of the concept image which is activated at a particular time the *evoked concept image*. At different times, seemingly conflicting images may be evoked.

We shall regard the [*personal*] *concept definition* to be a form of words used [by the student] to specify that concept.

[The] *formal* concept definition [is] a concept definition which is accepted by the mathematical community at large.

We shall call a part of the concept image or concept definition which may conflict with another part of the concept image or concept definition, a *potential conflict factor*. Such factors need never be evoked in circumstances which cause actual cognitive conflict but if they are so evoked the factors concerned will then be called *cognitive conflict factors*.

(Quotations found in Tall and Vinner, 1981, pp. 152–154; emphasis in original)

Figure 7. Quotations summarizing the major terms of concept image theory.

overall concept image of function, a student's personal concept definition of function is very illuminating as to his understanding of function, as an examination of the research literature shows. When discussing students' personal concept definitions for function, I employ three new terms that relate to REC theory. These words and their definitions are given in Table 5.

Although many students correctly view a function as a correspondence between two sets (Vinner and Dreyfus, 1989), they often have incorrectly restricted the types of

Table 5. Three Technical Terms Related to REC Theory.

Term	Definition
one-sidedness	an unsuitable inclination to view a mathematical concept, such as function or limit, from only one particular perspective
misinterpretation	an incorrect interpretation of a mathematical concept, such as function or limit, in real-life terms
overgeneralization	a faulty attempt at instantiation (in one or more specific situations) wherein properties are assigned to a mathematical concept, such as function or limit, which are not consonant with sound mathematics

correspondences they consider to be functions. Failure to accept certain functions as legitimate may constitute a misinterpretation of the function concept or an overgeneralization of the function concept or both. Note that mathematical overgeneralizations can prevent conceptual understanding, as they preclude a student from consistently instantiating mathematical concepts in an appropriate way, or from correctly reifying those concepts.

One of these mental restrictions on function relates to a student's belief that a function must manipulate its input (Dubinsky and Harel, 1992; Vinner, 1992; Vinner and Dreyfus, 1989). In an extreme example of this belief, instead of writing the simple function mapping all real inputs to negative one as $f(x) = -1$, a student might denote it by $f(x) = \frac{-x}{x}$. Similar to this mental restriction on functions is a mental restriction involving quantities or involving equations—respectively, that a function must have numbers as its inputs and outputs or that a function must be defined explicitly by an algebraic equation to be a function (Dubinsky and Harel, 1992; Vinner and Dreyfus, 1989). Such one-sidedness in students may prevent correct mental transitions from one function representation to another, which in the TIRI function model, constitutes a lack of ability in the component competency of translation. In many instances, a student's concept image of function does not allow for discontinuities (Dubinsky and Harel, 1992; Vinner, 1992; Vinner and Dreyfus, 1989). Since the limit concept is foundational to a correct modern understanding of continuity, this concept image of function could make understanding limits more difficult. In discussing this mental restriction on functions, Vinner (1992) stated that some students, especially when viewing a graphical representation of a function, equate that function's continuity at x_0 with its being defined at x_0, and likewise associate a function's discontinuity at x_0 with its not being defined there.

Unfortunately, it is also the case that many students confuse the functional property of univaluedness—that is, the property that each input of a function has only one corresponding output—with the property only some functions possess, that of being one-to-one—wherein each image of the function has only one preimage (Dubinsky and Harel, 1992; Vinner, 1992; Vinner and Dreyfus, 1989). Hence, they might assume all functions are one-to-one, an overgeneralization of a property only held by some functions.

Overly simplistic *prototype examples* of functions in the mind of students can also be a problem. A description of this term is given by Tall and Bakar as follows:

> We hypothesize that students develop 'prototype examples' of the function concept in their mind, such as: a function is like $y = x^2$, or a polynomial, or $1/x$, or a sine function. When asked if a graph is a function, in the absence of an operative definition of function, the mind attempts to respond by resonating with these mental prototypes. If there is a resonance, the individual experiences the sensation and responds positively. If there is no resonance, the individual experiences confusion, searching in the mind for a meaning to the question, attempting to formulate the reason for failure to obtain a mental match. … Positive resonances may be erroneous because they evoke properties of prototypes which are not part of the formal definition. … Negative resonances may be equally incorrect [when a legitimate function] does not match any of the prototypes.
>
> (Tall and Bakar, 1992, p. 40)

Because most of the functions students encounter in algebra classes have recognizable formulas and graphical representations, many students believe such functions are the only ones that exist. This mental construction on the part of students manifests itself in multiple ways.

For example, a relatively mild mental hurdle for some students is a simple preference for linear functions above all others (Dreyfus and Eisenberg, 1983). However, in more severe cases, students move beyond just preference for some functions to outright rejection of others as being legitimate functions. Many students insist that a function only have a single rule (Vinner, 1992; Vinner and Dreyfus, 1989), so they do not view piecewise-defined functions as functions. Others modify this to allow piecewise-defined functions, but only if the domains of each piece are half-lines or intervals (Vinner, 1992). Another modification of the narrow, "single rule" view is that the algebraic formula defining the function must be "regular" or the graph of the function must be "regular." Supposing a change in the independent variable must systematically change the dependent variable characterizes the former belief, while insisting sudden changes in a graph indicate it does not represent a function distinguishes the latter (Vinner, 1992; Vinner and Dreyfus, 1989). Mental restrictions as to which correspondences represent real functions may indicate a lack of understanding of function in one or more component competencies of the TIRI function model—namely, in translation, in interpretation, in instantiation, or in some combination of all three.

Perhaps less common to most students' concept image of function, but no less damaging to flexible mathematical reasoning, is the partiality a few students show to one type of function representation—e.g., graphical, algebraic, or numeric—over all the others. This one-sidedness on the part of students may be a manifestation of a lack of understanding in the component competency of translation in the TIRI function model, in the sense that such students are demonstrating inflexibility in moving from one function representation to another. In the most debilitating cases, a student even believes that one particular representation *is* the function (rather than just a representation of it), while simultaneously rejecting all other representations as being somehow inferior or false (Dreyfus and Eisenberg, 1983; Eisenberg and Dreyfus, 1994; Vinner, 1992; Vinner and Dreyfus, 1989). This constitutes a serious overgeneralization of the abstract concept of function, which students should know does not coincide with any of its individual representations (as noted by Kaput, 1989, p. 170). More particularly, Dreyfus and Eisenberg (1983) have demonstrated the difficulty some students have in going from a graphical representation of a function to its algebraic formula and vice versa. The same researchers (Eisenberg and Dreyfus, 1994) have noted the dominance that the algebraic representation of function (i.e., the formula or equation) has over other representations in the minds of many students. In any case, students' inability to translate among representations of functions may produce future difficulties in translation among the various representations of limit, inhibiting their understanding of that concept.

Interestingly, some students only accept functions as valid if declared to be so by the mathematical community (Vinner, 1992; Vinner and Dreyfus, 1989). This may relate to the belief in some students that mathematics is strictly that which can be found in a mathematics textbook or that which is taught by their mathematics teacher. This is, at the very least, a misinterpretation of what mathematics actually is. According to the authors of a standards document produced by AMATYC,

> for many students, mathematics is viewed as a 'string of procedures to be memorized, where right answers count more than right thinking.' … [But] students [need] to recognize that mathematics is more than computation or getting the single right answer—it is a balance of process and product—a combination of good thinking and meaningful answers.
>
> (AMATYC, 2006, p. 22–23)

In other words, it is beneficial to students when mathematics educators promote a suitable level of autonomy in their students' thinking about mathematical concepts such as function.

Theories on Students' Mental Constructions of Function

Common to much of the research I analyzed for the present study was an underlying learning theory, namely, *constructivism*. A definition of this term was given by Selden and Selden (1992), and in Figure 8 this definition is presented in quotation form. Adhering to this theory necessarily involves the belief that effective teaching requires an awareness of what and how individual students think, not just a knowledge of the content students need to learn. Each of the theories presented in this section was

> [C]onstructivism is a *theory of knowing* which many researchers in mathematics education hold; according to this view, whose origins go back to Piaget, students necessarily construct (or reconstruct) their own mathematical knowledge within themselves. … Learning is active, not passive. Whatever we tell students, whatever activities we have them engage in, they will interpret these as best they can, based on whatever ideas are present in their minds. The act of cognition is adaptive, in the sense that it tends to organize experience so it "fits" with previously constructed knowledge.
>
> (Quotation found in Selden and Selden, 1992, p. 7; emphasis in original)

Figure 8. Quotation defining the term constructivism.

created by researchers who felt that a theory substantiated by observations and interviews of students would be likely to greatly assist educators to teach the function concept better, so that more students could make meaningful mental constructions of function. Importantly, the TIRI function model complements each of these theories.

One general theory of constructing mathematical concepts that appears to apply to the concept of function is the *process-object duality* theory given by Sfard (1991). She postulated that there are three distinguishable steps in mental concept formation. These three steps are presented in Figure 9. Sfard hypothesized that during the development of a sophisticated mental function conception, the steps listed in Figure 9 might proceed as follows: First, "the idea of variable is learned and the ability of using a formula to find values of the 'dependent' variable is acquired" (Sfard, 1991, p. 19). Second, the student

(1) Interiorization – A new concept (viewed mentally as a new process to be done) is performed on already familiar objects.

(2) Condensation – The idea of turning the new process into an autonomous entity emerges.

(3) Reification – The ability to see the new entity as an integrated, object-like whole is acquired.

(Modified from Sfard, 1991, p. 18)

Figure 9. Three steps in mental concept formation (a.k.a. process-object duality theory).

becomes "more capable … of playing with a mapping as a whole, without actually looking into its specific values" (ibid.), and can, in time, "investigate functions, draw their graphs, combine couples of functions (e.g., by composition), even … find the inverse of a given function" (ibid.). Last, the student evidences "proficiency in solving equations in which the 'unknowns' are functions," demonstrates "ability to talk about general properties of different processes performed on functions," and recognizes "that computability is not a necessary characteristic of the sets of ordered pairs which are to be regarded as functions" (p. 20). The steps to mental concept formation in the process-object duality theory—interiorization, condensation, and reification—are reflected somewhat congruously by the component competencies of familiarization, instantiation, and reification, respectively, in the TIRI function model.

Also somewhat parallel to the process-object duality theory is a theory suggested by Breidenbach and his colleagues (1992) and extended by Cottrill's team of researchers (1996) to become what is known as APOS theory. (APOS is an acronym for actions, processes, objects, and schemas). In the original theory (Breidenbach et al., 1992), the researchers presented a theory of mental construction of the function concept with five major components: action, interiorization, process, encapsulation, and object. Cottrill and his colleagues (1996) added another major component to the theory which they called schema. A summary of the connections among these elements in the theory appears in Figure 10, in the form of two concise quotations. In particular, Breidenbach

> An *action* is any repeatable physical or mental manipulation that transforms objects (e.g., numbers, geometric figures, sets) to obtain objects. When the total action can take place entirely in the mind of the subject, or just be imagined as taking place, without necessarily running through all of the specific steps, we say that the action has been *interiorized* to become a *process*. It is then possible for the subject to use the process to obtain new processes, for example by reversing it or coordinating it with other processes. … Finally, when it becomes possible for a process to be transformed by some action, then we say that it has been *encapsulated* to become an *object*.
>
> (Breidenbach et al., 1992, p. 249–50; emphasis in original)
>
> A *schema* is a coherent collection of actions, processes, objects, and other schemas that are linked in some way and brought to bear on a problem situation. … There are at least two ways of constructing objects—from processes and from schemas.
>
> (Cottrill et al., 1996, p. 172; emphasis in original)

Figure 10. Quotations summarizing the major components of APOS theory.

and his colleagues (1992) stated that "the main difference … between an action and a process is the need in the former for an explicit recipe or formula that describes the transformation, … [while the latter] represents a transformation that does not have to be very explicit, nor must a subject be absolutely certain that it exists" (p. 278). It appears that the interiorization phase in APOS theory most closely matches the condensation step in the process-object duality theory. It is also fairly similar to the component competency of instantiation in the TIRI function model.

Just as Sfard operationalized process-object duality theory into a model by specifically linking it to the formation and growth of the mental concept of function, Breidenbach and his colleagues applied APOS theory to create a rough hierarchy of function conceptions in students, as summarized in the quotation shown in Figure 11.

> For *prefunction* we consider that the subject really does not display very much of a function concept. … [An *action*] conception of function would involve, for example, the ability to plug numbers into an algebraic expression and calculate. It is a static conception in that the subject will tend to think about it one step at a time. … A *process* conception of function involves a dynamic transformation of objects according to some repeatable means that, given the same original object, will always produce the same transformed object. The subject is able to think about the transformation as a complete activity.
>
> (Breidenbach, et al., 1992, p. 251; emphasis in original)

Figure 11. Quotation defining certain function conceptions in students.

This hierarchy of function conceptions has proved useful in partially categorizing students' understanding of function. Researchers appear to agree that a process conception of function, developed via interiorization in the framework of Breidenbach and his colleagues (equivalently, via condensation using Sfard's terminology or via instantiation using my terminology), would obviate many of the poor personal concept definitions of function mentioned in the previous section. Interestingly, Breidenbach's research team was able to design an instructional treatment which helped some students' attain a process conception of function by allowing them to create their own functions, using a programming language called ISETL, to model simulated real-life situations. This illustrates a possible connection between the component competencies of interpretation and instantiation, from the TIRI function model, in students' understanding of function.

While both the model of function understanding given by Sfard and the model of function understanding given by Breidenbach's research team appear to describe an ideal mental construction of function as opposed to the messy reality of most students' mental constructions of it, the models have proven useful in providing desirable benchmarks in the process of constructing a strong function understanding. The measurement of students' ability in the component competencies of the TIRI function model may also provide a consequential categorization of students' function understanding. Since neither Sfard's model nor the model of Breidenbach's research team directly addresses the importance of translation or interpretation to a student's overall understanding of the function concept, both of which seemed constructive in interpreting students' incorrect views of function—as seen in the previous section—the TIRI function model formulated for the present research might be used to sharpen the classification of students' understanding of function by characterizing more of the elements of understanding needed by students to fully comprehend the function concept.

Of course, there is a tradeoff: the TIRI function model developed for the present research does not provide clear-cut benchmarks in the growth of students' understanding of function. Nevertheless, I do not think a model of understanding of function should overlook the aspects of translation and interpretation. Helping students to be more conscious of the multiple representations of function has been the goal of many researchers (e.g., Eisenberg and Dreyfus, 1994; Schwarz et al., 1990), illustrating the concern about students' ability to successfully translate among function representations. Moreover, Sierpinska affirmed the importance of being able to model the real world using functions—which is part of the component competency of interpretation in the TIRI function model—when she stated that

> [a] perception of functions as an appropriate tool for modeling or mathematizing relationships between physical (or other) magnitudes is a *sine qua non* condition for making sense of the concept of function at all.
> (Sierpinska, 1992, p. 42; emphasis in original)

Hence, the apparent loss of hierarchical explanatory capacity as to how students might build a mature understanding of function in the TIRI function model is more than offset by its increased descriptive power in elucidating more completely students' understanding of function.

Epistemological Theories of Function Understanding

This section is a further discussion of two research studies which were used in Chapter 1 to justify expanding O'Callaghan's model of function understanding to the present TIRI function model. In both of these research studies, the authors grappled with what it means to understand the function concept sufficiently well to succeed in introductory calculus. In illuminating students' work with models to assess their function understanding, Monk (1992; 1994) presented a dichotomy between a pointwise understanding of function versus an across-time understanding of it. A *point-wise* view does not permit a student to see beyond evaluating a function at a particular input value, whereas an *across-time* view allows a student to describe patterns in change in functional values as the input values vary across intervals (e.g., time intervals). Monk's across-time view apparently parallels the *process conception* of function in the framework of Breidenbach and his colleagues (1992). With an across-time view of function, a student is investigating the function holistically, which allows her to be more cognizant of properties of a given function such as continuity or monotonicity. Using the terminology of the TIRI function model, such a student can instantiate—at least sometimes—the concept of function.

Somewhat analogous to an across-time view of functions is the notion of *covariational reasoning*, which is the ability to "coordinat[e] two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354). By its very definition, covariational reasoning would appear to require a process conception of function (as defined in Figure 11) in the student's mind at the very least, which requires reliable, consistent instantiation of functions on the part of the student, to use the terminology of the TIRI function model. It has been argued (Zandieh, 2000) that it even necessitates the student's being in transition to an *object conception* of function, which happens via *encapsulation* according to Dubinsky and Harel:

> A function is conceived of as an *object* if it is possible to perform actions on it, in general actions that transform it. … It is important to point out that [such] an object conception is constructed by encapsulating a process.
> (Dubinsky and Harel, 1992, p. 85; emphasis in original)

Encapsulation is apparently congruous to reification, another component competency of the TIRI function model.

Notably, an analysis of data illustrating students' struggles with the Fundamental Theorem of Calculus, which relates differential and integral calculus, led Thompson (1994) to conclude that students' troubles with the Fundamental Theorem of Calculus "stem from having impoverished concepts of rates of change" as well as "poorly coordinated images of functional covariation" (p. 229). Hence, covariational reasoning may facilitate a meaningful understanding of the calculus concepts of derivative and integral and the connection between them. A conceptual understanding of derivative, in turn, may enhance students' ability in the TIRI function model component competency of translation, by providing students with a language in which to easily move from algebraic representations of a function and its derivatives to a graphical representation of the function and vice versa.

Summary of Function Understanding

Students' personal concept definitions for function, as described in prior research studies, might reflect the students' ability (or lack thereof) in component competencies of the TIRI function model—specifically, the competencies of translation, interpretation, and instantiation. Previous research models of students' mental constructions of function given by Sfard (1991) and Breidenbach's research team (1992) are essentially harmonious with the TIRI function model, with the exception that the former models are hierarchically oriented—that is, they provide desirable benchmarks in the process of constructing a strong function understanding—while the TIRI function model is not entirely hierarchical but is more descriptively complete, in that it includes aspects of translation and interpretation as part of the model. Furthermore, two epistemological theories of function understanding—created by Monk and by Carlson's research team, respectively—are consonant with the TIRI function model developed for the present research study. In all these cases, the TIRI function model complements and expands the frameworks of other research studies.

Students' Understanding of the Concept of Limit

The purpose of this section is to connect REC theory and the associated TIFI limit model to previous research studies' findings about students' understanding of limit. A summary both of REC theory and of the TIFI limit model can be found in Table 1 and in Table 3, respectively. (Note that both of these tables are found in Chapter 1.) A student displays understanding of limit by correctly engaging in the activities listed in Table 3. I now consider students' conceptions of limit and possible sources of those conceptions as well as constructivist theories of learning the limit concept in light of REC theory and the associated TIFI limit model. In particular, I endeavor to demonstrate the practicality of REC theory and the TIFI limit model in that the theory and model together afford a suitable language, via the theory's sources of meaning and the model's corresponding component competencies, to summarize and explain students' thinking about limits. This includes providing appropriate sources of understanding for two constructivist theories of learning limit by aptly describing what kinds of misunderstandings might lead to students' having certain mental images and espousing particular beliefs about limits.

Three terms that relate to REC theory, found in Table 5 of this chapter, are to be utilized again periodically throughout this entire section.

Students' Conceptions of Limit and
Possible Sources of those Conceptions

It would be beneficial to review what researchers have observed in regards to students' conceptions (and misconceptions) of the limit concept. Such researchers have tried to delineate, based on reasonable theories and observations of students, some of the possible sources of students' conceptions of limit. While explorations in this arena have been extensive and varied, at least five major categories can be detected in the research: (a) spontaneous conceptions of limit; (b) an overemphasis on procedural aspect of limit; (c) dynamic versus static conceptions of limit; (d) assertions that limits are unreachable; (e) limit conceptions related to notions of actual infinity. It should be noted that these categories are not mutually exclusive, with many studies containing elements belonging to multiple categories. A brief look at each of these categories is now in order.

Spontaneous Conceptions of Limit. Cornu introduced the notion of *spontaneous conceptions* as follows: "For most mathematical concepts, teaching does not begin on virgin territory. In the case of limits, before any teaching on this subject the student already has a certain number of ideas, intuitions, images, knowledge, which come from daily experience, such as the colloquial meaning of the terms being used. We describe these conceptions of an idea, which occur prior to formal teaching, as *spontaneous conceptions*" (Cornu, 1991, p. 154; emphasis in original). Previous to this description, Davis and Vinner (1986) portended such a notion as being important in research of students' learning of the limit concept. They found that students who had apparently understood the concept of a limit of an infinite sequence actually reverted, after a summer break, to what they termed *naive* conceptualizations of the limit concept. In the terminology of the TIFI limit model, students misinterpreted limits based on their experiences prior to their taking a course on limits. This is not to say the students had completely forgotten what they had learned; in fact, since "learning a new idea does not obliterate an earlier idea" (Davis and Vinner, 1986, p. 284), it is very possible the students knew the correct ideas behind the limit concept but "retrieved the old idea[s]" (p. 284).

Understandably, a student is more likely to be affected by his "primary intuitions" when "dealing with [novel] mathematical problems" (Tirosh, 1991, p. 213). Using the terminology of REC theory, a student's apparent relapse back to his primary intuitions may be due to a lack of familiarity with hitherto unconsidered mathematical challenges, making it difficult for him to bring apposite knowledge to bear on such problems. Students' reliance on a priori (i.e., spontaneous) conceptions when struggling with unexpectedly different mathematical tasks leads to failure in doing applied calculus problems, failure which has been documented in many studies (e.g., Carlson et al., 2002; White and Mitchelmore, 1996).

By their very nature, students' spontaneous conceptions of limit are highly idiosyncratic (Tall and Schwarzenberger, 1978). However, many of the likely sources of spontaneous conceptions are endemic to the very learning of the limit concept (Davis and Vinner, 1986). The non-technical meanings of many words used in describing limits—

the “colloquial meaning” (Cornu, 1991, p. 154) of terms like *approaching*, *close to*, *tends toward*, and *limit*—can be troublesome for students because the everyday meanings of the words are incongruous with formal mathematical definitions (Cornu, 1991; Davis and Vinner, 1986; Tall and Schwarzenberger, 1978). This can lead to misinterpretations, in reference to the TIFI limit model, of the meaning of mathematical limits. Teachers can exacerbate the problem by their teaching, often inadvertently, by failing to provide students with specific examples that illustrate the inconsistencies in their thinking in comparison to the correct mathematical ideas. This makes it more likely for students to overgeneralize the concept of limit. To be more specific, without proper correction of these overgeneralizations, a student cannot dependably instantiate the limit concept to create examples of limit for herself that satisfy a set of given conditions. Students are especially apt to overgeneralize early examples of the limit concept (Cornu, 1991; Davis and Vinner, 1986), so teachers must be willing to provide examples that do not fit exactly with previous cases.

Overemphasis on Procedural Aspect of Limit. The fact that 14 out of 36 students arriving at university claimed that $\lim_{n\to\infty}(1+\frac{9}{10}+\frac{9}{100}+\cdots+\frac{9}{10^n})=2$ but also simultaneously asserted $0.\overline{9}<1$ (Tall and Vinner, 1981, p. 158) demonstrates that students tend to compartmentalize their knowledge into procedural and conceptual categories (Williams, 1991). In particular, compartmentalization of any mathematical concept can lead to debilitating one-sidedness in students’ thinking. In contrast, true understanding is the glue that connects seemingly disparate mental ideas together. This is reflected in the four component competencies of the TIFI limit model, since a student must draw on all of these component competencies to fully understand the limit concept.

Unfortunately, many of the limit conceptions of students are solely algorithmic in nature, including a conception in which students think $\lim_{x \to a} f(x)$ literally means evaluating a function at successively closer *x*-values to the *x*-value of a or a conception in which students think $\lim_{x \to a} f(x)$ literally means imagining the graph of the function and moving along it to the limiting value (Przenioslo, 2004). These procedural conceptions would not be problematic if there were conceptual underpinnings behind them in the mind of the student, but often such supporting conceptions are absent. In espousing these algorithmic constructions of limit, students may be evidencing one or more of the following: a belief that limits are actually just approximations (Davis and Vinner, 1986); an overly strong faith in algebraic formulas (Williams, 1991); a conviction that graphical representations exist for all functions (Szydlik 2000; Williams, 1991). The first of these is an overgeneralization of the limit concept, the second and third overgeneralizations of the function concept. Additionally, the second and third misconceptions also illustrate the difficulty in comprehending limits when function understanding is weak.

Students' lack of motivation to learn the conceptual underpinnings to the limit concept is understandable, considering the algorithmic nature of many calculus courses (Cornu, 1991; Williams, 1991). When a teacher sees his students struggling with the conceptually difficult concept of limit, he may make the dual mistakes of focusing his instruction solely on the "symbolic routines of differentiation and integration" and of creating exam questions that are not "conceptually challenging" (Tall, 1997, p. 17). Of course, this only compounds the problem of one-sided, superficial understanding of the limit concept in students. It leads to what White and Mitchelmore (1996) call a *manipulation focus* where the underlying meanings for the written symbols are lost. Students have little or no hope of understanding limits if they focus solely on symbolic manipulations involving limits, because the algebraic rules used in symbolic manipulations do not in and of themselves provide the student with any context with which to properly interpret limits in real-life terms or to correctly instantiate the limit concept to examine its properties. It may also prevent flexible translation among the different representations of limit. If an experienced mathematician can be fooled by the "self-sufficiency of a formula" without considering the assumptions behind it (Ervynck, 1981), it follows that teachers must not allow their instruction to glide over these important assumptions, because it can be detrimental to students' conceptual understanding of limits.

In a traditional calculus course, students may come to believe that practical, expedient, and simple mental models of limit are the only ones that really matter, even if they are not always logical or correct (Williams, 1991), because the practical models seem to work most of the time. Rejection of the formal model of limit for one that is more convenient and seemingly more functional is understandable, especially considering many students' beliefs concerning the nature of mathematics itself and its relative importance. The following statement from a humanities student is typical of the beliefs of many students concerning mathematical truth: "Mathematics is completely abstract and far from reality … [so] with those mathematical transformations you can prove all kinds of nonsense" (Sierpinska, 1987, p. 375). Seemingly lacking the internal sources of conviction espoused by Szydlik (2000)—and hence failing to presuppose logic and consistency are crucial to mathematics—such students cannot interpret complex mathematical concepts such as limit in meaningful, real-life ways, and they will "not make sense of mathematics unless their sources of conviction are shifted" because "to them, mathematics is not *supposed* to make sense" (Szydlik, 2000, p. 273; emphasis in original).

Furthermore, many students do not adequately understand the importance of mathematics to the larger world. For example, students in a mathematics class for future elementary teachers indicated on a survey that mathematical knowledge is useful (Jensen, 2007a); however, in interviews, future elementary teachers were able to articulate only a very few particular circumstances in which they found mathematics to be useful in their own lives (Jensen, 2007b). Although these studies did not involve calculus students, it is reasonable to assume any student's conceptions of both why mathematics "works"—conceptions that come about by instantiation and reification of the appropriate mathematical concepts—and what mathematics can be used for—conceptions that arise from sound interpretations of the mathematical ideas—are crucial influences on that student's motivation to reflect sufficiently on new mathematics concepts with which he is less familiar in order to truly understand them—especially advanced concepts like limit.

<u>Dynamic Versus Static Conceptions of Limit.</u> A serious hindrance to the development of a formal limit concept in the minds of students is the "difficulty of forming an appropriate concept image" (Tall and Vinner, 1981)—the difficulty, that is, of

reliably instantiating the limit concept for different examples of limit. For informal conceptions of limit, the most appealingly intuitive concept image for students to choose seems to be a dynamic one (Tall and Vinner, 1981; Williams, 1991; Williams, 2001)—that is, one with a "definite feeling of motion" (Tall and Vinner, 1981, p. 161). The informal language used by mathematicians in describing limits—terms like *approaching*, *goes to*, or *tends toward*—are likely sources for this concept image. The reason mathematicians use such imagery of motion is probably a matter of mathematical history, since Cauchy (who devised the formal definition of limit used today) envisioned the real continuum not as a set of "*actual* points" but as a set of "*moving* points" (Lakatos, 1978, p. 57; emphasis in original). Such intuition in the mind of a mathematical giant like Cauchy may be an indication for the reasonableness of allowing for dynamic conceptions of limit. However, like the modern set-theoretic concept of function as a "certain subset of [a] Cartesian product" (Kleiner, 1989), the formal definition of limit is usually interpreted by mathematicians to be completely static—in the sense that when asserting $\lim_{x \to a} f(x) = L,$ one is not, in fact, saying that x is moving whatsoever.

A few students may actually construct an *intuitive static* conception of limit, which can be characterized by the belief that "the limit of a function is L if whenever x is close to the limiting value $s,$ the function is close to L" (Szydlik, 2000, p. 268). One researcher found that conceptions of limit based on neighborhoods—which are roughly equivalent to Szydlik's intuitive static conception of limit—were the "most efficient in solving … problems" (Przenioslo, 2004, p. 113). On the other hand, Szydlik cautioned that a dual teaching approach to limits might be needed:

> Some students are not ready to hear proofs or arguments. … [T]hey must experience that mathematics makes sense. Another group of students is frustrated when they are not provided with formal structure. … For them, instructors must be cautious not to downplay rigor and the importance of formal definitions.
>
> (Szydlik, 2000, p. 274)

These conclusions suggest that there are advantages and disadvantages to both the formal (and more static) conceptions of limit and the informal (and more dynamic) ones, so both types of conceptions may lead to appropriate instantiations by students of the limit concept. It is debatable as to which type of conception is more facilitative to an eventual reification of the limit concept. When comparing Greek and English calculus students, with the former taught a more formal limit concept than the latter, one researcher found that while “the English [were] deprived of insight about the mainstream modern model … of the real continuum,” some Greek students “did show some conflicts between the dynamic and static approaches, suggesting that the first is more natural to their original intuition” (Mamona-Downs, 1990, p. 75). These findings typify the potential debate as to the comparative disadvantages of the respective conceptions (dynamic and static) of limit.

Why is it that most students conceive of limits as a dynamic process rather than as a static object? According to the process-object duality theory of Sfard (1991), an operational conception of any mathematical concept necessarily precedes a structural conception of it, both historically and psychologically. This theory is echoed by Dubinsky and Harel (1992) in the specific case of the function concept, who state a process conception of function temporally goes before an object conception of function. When applied to the concept of limit, this theory suggests that a dynamic (operational) conception of limit will come first, with a static (structural) conception appearing later. A deep understanding of the formal definition of limit presumes a static conception of limit, with or without the help of a complementary dynamic conception. Because the formal definition of limit is difficult for many students to grasp, they may not see a static conception of limit as an adequate *competing scheme* that can appropriately complement their dynamic view of limit (Williams, 2001). Indeed, a "lack [of] sense of the role of definitions in mathematics in general" (Przenioslo, 2004, p. 129), indicative of an inability to interpret mathematical definitions and properly value their explanatory power, could persuade students who have seen the formal definition of limit to ignore it and hence never build a static conception of limit.

<u>Assertions that Limits are Unreachable.</u> A "compelling metaphor" (Williams, 1991, p. 230) for limit which involves walking to a wall appeals to many students:

> The paradigm picture seems to be the classical geometric progression involved in walking halfway to a wall, then half the remaining distance, and so forth; students seem willing to accept the fact that we never reach the wall even though we may know exactly where the wall is.
>
> (Williams, 1991, p. 230)

As this metaphor illustrates, many students interpret limits to be fundamentally *unreachable*. In trying to explain this phenomenon, Szydlik (2000) stated that "if a student imagines infinitesimally small distances, he may believe that function values can get infinitely close to a limit but never reach it" (p. 260). While observing English calculus students, Mamona-Downs (1990) discovered they often thought of limits in terms of a "Leibniz-Cauchy model, where the numbers on the real line have infinitesimal neighborhoods" (p. 75). Imagining such infinitesimal magnitudes is inconsistent with standard analysis, an overgeneralization of the concept of the real number line, as students are assigning to the real line a property (that of the existence of infinitesimal distances) that it does not actually have. However, such conceptions of limit are consistent with non-standard analysis, an alternative foundation to introductory calculus that has been successfully used in the calculus classroom (Sullivan, 1976). Nevertheless, the adoption of non-standard analysis in the teaching of introductory calculus has never become widespread.

The conception of limit as unreachable is located within the context of limits of functions of a real variable, as the independent variable approaches some finite value. The mental hurdle for students in that case may involve believing in infinitesimally small distances. On the other hand, limits at infinity refer to limits of infinite sequences or limits of functions as the independent variable becomes arbitrarily large, and students' reasons for claiming limits at infinity cannot be reached are different than in the previous case. This is because limits at infinity invoke in students' minds some infinite process that goes on without end. This notion of infinite processes may explain why students claim $0.\overline{9} < 1$, for example, because "the process of appending 9s never ends" (Szydlik, 2000, p. 260). Apparently, students may incorrectly interpret the mathematical concept of infinity as having a temporal aspect, since time is a ubiquitous aspect of real life. Since no less a mathematical giant than Gauss protested against the "use of an infinite quantity as a completed one" and argued that "the infinite is only a *façon de parler* [figure of speech] in which one properly speaks of limits" (Tirosh, 1991, p. 200;

emphasis in original), it is quite understandable that many calculus students are philosophically opposed to infinity being realized.

Speaking of limits at infinity, it is worth mentioning that some students have a conception of limit as some sort of bound, perhaps evident when a student refuses to admit the graph of a function that crosses its own horizontal asymptote. In a variation of this conception, some students believe that "within a certain tolerance of the limiting value, the limit acts as a boundary; however, they [do] not think of the limit as a global boundary" (Szydlik, 2000, p. 271). In other words, these students believe the function must not assume the limiting value when the independent variable values are sufficiently close to the point of interest. For either boundary conception (global or local), students are misinterpreting the word limit (which can mean "boundary" in regular English) or simply overgeneralizing the limit concept from the first examples of limits at infinity they encounter.

Conceptions of Limit Related to Notions of "Actual" Infinity. In teaching students about the limits of infinite sequences, Davis and Vinner (1986) found that some students assume infinite sequences have a "last term, a sort of a_∞" (p. 294). Possibly related to this misconception is the supposition that one can somehow "go through infinitely many terms of the sequence" (p. 294). The source of such notions could be the *generic limit concept* hypothesized by Tall, cited in Cornu (1991), in which the "[limiting] object is believed to have the properties of the objects in the process" (p. 162). This generic limit concept is a mathematical overgeneralization, and it proves fallacious in many instances. For example, it is tempting to assume the pointwise limit of a sequence of continuous functions is continuous, but in fact, that is not necessarily true. Nevertheless, the generic limit concept misconception is very appealing, and cognitively "to be expected" (ibid.).

Although a metaphorical mental jump to "actual" infinity may allow a student to consider a limit as reachable, it may be "the most important cognitive obstacle to learning the formal definition" (Williams, 2001, p. 364), because "the ε - δ definition, and indeed most of modern mathematics, categorically rejects the concept of actual infinity" (Tirosh as paraphrased by Williams, 2001, p. 364). Considering the static nature of the formal definition of limit (and its reliance on neighborhoods of real numbers with no allowance for hyperreal elements), it is reasonable to believe that a student may stumble when trying to reconcile her intuition of actual infinity with this definition and hence fail to instantiate suitable examples of limits which she can adequately examine in the light of the formal definition of limit.

Constructivist Theories of Learning Limit

In this section, two constructivist theories about how students learn the concept of limit are analyzed. (A working definition of constructivism can be found in Figure 8 of this chapter.) The first constructivist theory to be discussed is that of students' concept images, which first appeared in Chapter 1, while the second theory is that of cognitive obstacles to learning limit. A third theory about how students might learn the concept of limit, the genetic decomposition of limit given by Cottrill and his colleagues (1996), was discussed previously and will be touched on again, though not in this particular section.

A Closer Look at Students' Concept Images. While mathematics is "usually regarded as a subject of great precision … the psychological realities are somewhat different" (Tall and Vinner, 1981, p. 151). Tall and Vinner used the term *concept image* to mean "the total cognitive structure that is associated with [a] concept, which includes all the mental pictures and associated properties and processes" (p. 152). The definitions of other terms in Tall and Vinner's concept image theory can be found in Figure 7 of this chapter. In the sense of the TIFI limit model, the concept image can include elements from all four of the component competencies of the model. However, for many students, one or more of the four component competencies may be a little part, or no part at all, of their concept images for limit. If a student is unreliable in demonstrating a strong understanding of the limit concept, it may be because he cannot always evoke the expedient parts of his concept image. This may be evidence of insufficient mental connections among the four component competencies of the TIFI limit model.

One part of a calculus student's concept image of limit might be his personal concept definition, i.e., how he would define limit in words. This most closely relates to the component competency of interpretation in the TIFI limit model. However, not all students' concept images of limit include personal concept definitions. Besides, an introductory calculus student's personal concept definition of limit is unlikely to be perfectly compatible with the formal concept definition of limit. It is this formal definition of limit that mathematics educators hope at least a few of their students can eventually comprehend and actually use in a flexible way. In other words, they hope a few students can instantiate their own apposite examples of formal limits and determine what information the formal definition conveys about those examples.

According to Tall and Vinner's theory, modifications to a student's concept image can only result when a potential conflict factor is evoked in such a way as to cause actual cognitive conflict. Researchers of students' understanding of limit have used this theory in studies where the main thrust of the methodology is to actively promote cognitive conflict, allowing for students to modify inconsistent concept images in order to resolve the cognitive conflict (e.g., Mamona-Downs, 2001; Przenioslo, 2004; Williams, 1991). In the sense of the TIFI limit model, the researchers are trying to actively combat one-sidedness in students' thinking about limits and to undo any misinterpretations or overgeneralizations of the limit concept students make so as to allow a proper interpretation of the concept to be formed and meaningful mental instantiations of the concept to take place.

<u>Cognitive Obstacles.</u> A cognitive obstacle is literally a mental hurdle that a person must overcome in order to learn (Cornu, 1991). Cognitive obstacles can be divided into three major types: genetic and psychological obstacles, which "occur as a result of the personal development of the student"; epistemological obstacles, which "occur because of the nature of the mathematical concepts themselves"; and didactical obstacles, which "occur because of the nature of the teaching and the teacher" (p. 158). The first two types are to be considered in this section, while didactical considerations were examined earlier, at least indirectly, in the sections entitled "Spontaneous Conceptions of Limit" and "Dynamic Versus Static Conceptions of Limit."

Psychological obstacles in the learning of the limit concept are likely related to students' beliefs about the nature of mathematics itself as well as their beliefs about concepts directly underlying the limit concept. Szydlik (2000) categorized students' general beliefs about mathematics using a dichotomy of *external* sources of conviction versus *internal* sources of conviction. The former sources are defined as "appeals to authority for the determination of mathematical truth," while the latter sources are "appeals to empirical evidence, intuition, logic, or consistency" (p. 258). Szydlik also divided *content beliefs* about the real numbers, infinity, and functions—concepts that provide a foundation for limits—into two categories: beliefs that are consonant with correct mathematical definitions and axioms and those that are not.

In reporting her findings, Szydlik noted that students tending to have external sources of conviction had poorer conceptions of limit than those with internal sources of conviction. In the terminology of REC theory, the former sources are a misinterpretation of what constitutes mathematical truth, whereas the latter are an accurate interpretation of how the truth of mathematical statements is assessed. Content beliefs about functions also had an effect on students' performance in Szydlik's study, which directly supports the thrust of the current research study. Incorrect content beliefs about the function concept could be the result of one-sided thinking, misinterpretations, overgeneralizations, or some combination of all three.

Epistemological obstacles are also of interest to mathematics educators, because such obstacles are "unavoidable and essential constituents of the knowledge to be acquired" and are "found, at least in part, in the historical development of the concept"

(Bachelard as paraphrased by Cornu, 1991, p. 158). In one study, Sierpinska (1987) attempted to categorize in detail, for humanities students, the epistemological obstacles blocking their understanding of limits of sequences. These obstacles, in turn, led to eight different “models of limit” (for sequences) espoused by students. Since the context of the study involved humanities students as opposed to calculus students, and the discussions among students were fundamentally philosophical in nature rather than mathematically rigorous (see Sierpinska, 1987), it is difficult to decipher the researcher’s “models of limit” as anything other than a possible categorization of students’ interpretations of the mathematical concepts of infinity and of limits of sequences. However, the theory of epistemological obstacles is an important one in the research literature, and it has been utilized by other researchers (e.g., Davis and Vinner, 1986; Ely, 2007; Williams, 1991) to study students’ understanding of limit in other experimental contexts.

Summary of Limit Understanding

The results of prior research into students’ conceptions of limit and possible sources of those conceptions can be rationally described in terms of REC theory and in terms of the component competencies of the TIFI limit model. This is true in the case of research studies in all five examined categories of sources of limit conceptions as well as in the case of research studies in which the authors postulated constructivist theories of learning limit. For the aforementioned constructivist theories, REC theory provides possible sources for students’ mental images and beliefs about limits by aptly describing what kinds of misunderstandings might lead to these mental images and beliefs. Overall, previous research on students’ understanding of limit supports and is complemented by

REC theory as well as the associated TIFI limit model. It is worth noting here that I utilized the TIFI model extensively in creating an instrument measuring limit understanding for use in the present research study.

Connection between Understanding of Function and Understanding of Limit

The objective of this section is to present the results of prior research which imply a link between students' understanding of function and their understanding of limit. Using the genetic decomposition of limit given by Cottrill's research team (1996) as a suitable springboard, I build a strong theoretical case for a link in understanding between functions and limits by tying in other research studies to this genetic decomposition.

Research Directly Linking Function and Limit

The first four steps in the genetic decomposition of how a student might learn the limit concept are shown in Figure 1 of Chapter 1. I now present those steps again, but in a slightly different form, in Figure 12. In particular, the term *schema*, which did not appear in Figure 1, appears in Figure 12. An introductory calculus student can demonstrate increasing understanding of limits by progressing through the steps shown in Figure 12.

To paraphrase Dubinsky (1991), the two constructivist theories regarding students' learning of limits which were described previously—namely, the theory of student concept images and the theory of cognitive obstacles—can be characterized as explanations as to why students do not easily develop meaningful understanding of the limit concept, while the theory of reflective abstraction—which is the basis for the

(1) Evaluating the function f at a single point close to a or at a itself;

(2) Evaluating the function f at a few points that successively get closer to a;

(3) Constructing a coordinated schema by: (a) constructing a domain process in which x approaches a; (b) constructing a range process in which y approaches L; (c) coordinating these processes using the function f;

(4) Performing actions on the limit concept by encapsulating the schema from step three to become an object.

(Modified from Cottrill et al., 1996, p. 177–178)

Figure 12. Four steps in a genetic decomposition of limit understanding (second version).

genetic decomposition in Figure 12—focuses on what actually needs to happen in order for students to understand limits.

The first three steps of the genetic decomposition of limit as listed in Figure 12 appear to be directly related to a student's understanding of function. Hence, it may not be an exaggeration to state that understanding the function concept is necessary to grasping the limit concept. Szydlik (2000) affirmed the data she gathered from 577 second-semester calculus students "suggest a connection between understanding of function and understanding of limit" (p. 272). But what kind of understanding of function is necessary for a mathematically sound understanding of limit?

One description of an apparently insufficient understanding of function for understanding limits involves "[using] its graph or formula as if it were only a table" (Monk, 1994, p. 21), which "[constitutes] a pointwise understanding of functions" (ibid.).

Contrasting this conception of function, Monk defined an across-time understanding of function as an ability to answer questions like “how does change in one variable lead to change in others?” or “how is the behavior in output variables influenced by variation in the input variable?” (ibid.). An examination of the genetic decomposition of how limit may be learned, postulated by Cottrill and his colleagues (1996), seems to indicate that an across-time understanding of function may be necessary in order to accomplish step #3 of the decomposition. Furthermore, covariational reasoning, which is the ability to “coordinat[e] two varying quantities while attending to the ways in which they change in relation to each other” (Carlson et al., 2002, p. 354), appears to be exactly what is required of students in order for them to accomplish step #3(c) of the genetic decomposition. Hence, it might be beneficial for students’ understanding of limit to promote across-time understanding of functions as well as covariational reasoning in students, both in precalculus courses and in calculus courses themselves.

The characterizations of function understanding given by Monk and by Carlson’s research team coincide somewhat with the process conception of function given by Breidenbach’s research team (1992). This process conception of function—in which functions are regarded as complete processes that can be composed together or reversed—is seemingly required to mentally regard the limit as a process on functions. Viewing limit as a process on functions allows a student to investigate the properties of specific examples of limit—to engage in instantiation to use the terminology of the TIFI limit model—rather than to assume limit simply means the evaluation of a function at few points or at a single point.

Among the aforementioned prior research models for function are reflected all four component competencies of the TIRI function model—translation, interpretation, reification, and instantiation. Modifying the process-object duality theory as defined by Sfard (1991) to fit the present research terminology, the reification of the function concept may well coincide temporally with consistent, reliable instantiations of the limit concept. This dual, simultaneous increase in understanding of the two concepts appears to put the student in position to perhaps begin attempting step #4 of the genetic decomposition.

In fact, researchers observed very few students that "gave any indication of passing very far beyond the first four steps of [the] genetic decomposition" (Cottrill, et al., p. 186). Although not presented here, steps five through seven of the genetic decomposition are all related to gaining an understanding of the formal ε - δ definition of limit, which precedes (or perhaps coincides with, in some cases) reification of the full limit concept. This suggests that a process conception of limit—one that allows the student to view the limit process in its totality without any explicit computation—might be what mathematics educators can reasonably expect their students to mentally construct in introductory calculus, rather than a fully reified and formal conception of limit. Deliberately stimulating students' abilities in the limit component competencies of translation and interpretation—so as to enrich students' overall understanding of the concept of limit by giving them the ability to consider multiple representations for limit and to place the limit concept in appropriate real-life contexts—may accelerate their progress in attaining a process conception of limit.

Unfortunately, many students confuse the limit process with simple function evaluation (Przenioslo, 2004), which could prevent them from completing more than one or two steps in the genetic decomposition delineated by Cottrill's research team. On the other hand, some critics may contend that computational proficiency with limits is all that most introductory calculus students need to learn. Although I do not agree with such an argument, a case for the importance of function understanding can even be made as it relates to simple procedural proficiency in computing limits. In other words, misconceptions about simple limit computation can often be attributed to a student's lack of a conception of function that is consonant with the accepted mathematical definition. For example, in the case of limits of sequences, Tall and Vinner brought up an interesting example of limit computation issues that had to do with lack of understanding of function, which is reproduced in Figure 13.

When a small group of … students [was] shown the example

$$s_n = \begin{cases} 0 & (n \text{ odd}) \\ \frac{1}{2n} & (n \text{ even}), \end{cases}$$

they insisted that it was not one sequence, but *two*. The even terms *tended* to [zero] and the odd terms were *equal* to [zero]. So this was not a *genuine* example of a sequence some of whose terms equaled the limit!

(Tall and Vinner, 1981, p. 160; emphasis in original)

Figure 13. An example of weak function understanding affecting limit understanding.

In another study, Davis and Vinner (1986) found that some students insisted that sequences must go toward a limit—e.g., "the sequence 1, 1, 1, … would be rejected as divergent, since the [terms] ‘aren’t *going toward*’ anything” (p. 294; emphasis in original). For functions of a real variable, some students think a limit at x_0 exists if and only if the function is “continuous enough” (Przenioslo, 2004, p. 119). In both of these cases, a student’s inflexibility in what constitutes a legitimate function (or sequence) hinders his capacity to accept sound mathematical reasoning in computing limits.

Summary of Function-Limit Link

Reinforcing the link between function understanding and limit understanding that is suggested in the early steps of the genetic decomposition of limit given by Cottrill and his colleagues (1996), prior research studies (viz., Carson et al., 2002; Monk, 1994) have illustrated the ways in which a student’s understanding of function could contribute to his being able to perform the critical third step in the genetic decomposition, thus allowing for reliable instantiations of a meaningful if informal conception of limit. Notably, the reification of the function concept may well coincide temporally with consistent, reliable instantiations of the limit concept. Consistent instantiations of the limit concept are consonant with a process conception of limit—one that allows the student to view the limit process in its totality without any explicit computation. Such a limit conception might be what mathematics educators can reasonably expect their students to mentally construct in introductory calculus, rather than a fully reified and formal conception of limit. Additionally, it is worth noting that even students’ procedural misconceptions in regards to limit—that is, misconceptions in doing computations and other mechanical procedures using limits—can often be attributed to a weak understanding of function.

A Plan of Action Based on the Literature

The purpose of this section is to outline the methodological strategies of important research studies mentioned previously in order to formulate a rational plan, grounded in the research literature, of how to credibly attempt to answer the major research questions of the present study, which can be found in Chapter 1.

In order to measure students' understanding of function or their understanding of limit, researchers have often used quantitative instruments to meet the needs of their respective studies (e.g., Breidenbach et al., 1992; Davis and Vinner, 1986; Monk, 1994; Vinner and Dreyfus, 1989; Williams, 1991). For other research designs, interviews with individual students were desirable because of the added depth in the data obtained (e.g., Cottrill et al., 1996). However, most relevant to the first goal of the present research study—namely, to create an instrument measuring students' understanding of limit in terms of the TIRI limit model—appears to be a methodology which combines the administration of a researcher-designed quantitative test along with follow-up interviews of a few students. The "testing followed by interviews" design has been employed by researchers looking into understanding of function (e.g., Carlson, 1998; Carlson et al., 2002) as well as researchers examining understanding of limit (e.g., Przenioslo, 2004; Szydlik, 2000). For the present research, this design would provide both quantitative and qualitative data that would facilitate recognition as to what portions or items of the instrument are acceptable based on principles of good educational test design and what portions or items need adjustment in order to fit those principles.

What principles are both relevant to the needed instrument and of importance when it comes to educational testing in general? In his classic book on this particular subject, Sax (1974) identifies a number of important principles in constructing reliable and valid educational tests. I have singled out six of those principles, each of which is reflected in one of the pilot study questions (found in Chapter 1). These principles, as well as pertinent quotes from the book by Sax (1974), appear in Figure 14. A repeated

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(1) Scoring reliability – "To the extent that chance or random conditions have been reduced, reliability will be high. … Chance factors include … disregard or lack of clear standards for scoring." (p. 172)

(2) Variability (for adequate discrimination) – "The classical concept of reliability assumes variability in student performance; if that variability is lacking, reliability estimates will be low or [zero]." (p. 185)

(3) Internal consistency – Reliability may be thought of as a special type of correlation that measures consistency of observations or scores." (p. 175)

(4) Suitable content – "Tests have content validity if the behavior and subject matter called for in the items correspond to the behavior and subject matter identified in the [test] objective." (p. 207–8)

(5) Item clarity – "Empirical evidence is required to demonstrate that test [items] do in fact measure [the desired] attribute." (p. 209)

(6) Correct construct categorization – "[One] purpose for determining construct validity concerns the development and refinement of educational and psychological theory. Empirical observations … provide input data to help clarify and define the theory." (p. 212)

(Quotations found in Sax, 1974)

Figure 14. Important principles in creating a test of limit understanding.

cycle of test administration followed by interviews with a few students followed by test modification based on the quantitative and qualitative data obtained followed by another test administration (and so on) was used during the pilot study in order to help realize the six principles outlined above in regards to my instrument on understanding of limit. More specific strategies employed in order to achieve each of the above test construction principles are discussed in Chapter 3.

In the major portion of the present research study, regression was used to quantitatively correlate introductory calculus students' scores on a function test with their scores on the limit test of my design. As with a study correlating students' understanding of function with their understanding of derivative (Pinzka, 1999), appropriate steps were taken to ensure the assumptions for the statistical tests were satisfied, so that the conclusions made from those tests could be trusted with a reasonable amount of confidence. More details about both the pilot study and the major study can be found in Chapter 3.

Chapter Summary

In reviewing the research literature, REC theory has proven to be invaluable. In particular, the TIRI function model complements and expands the frameworks of research studies concerned with students' personal concept definitions for function, studies presenting models of students' mental constructions of function, and studies expounding epistemological theories of function understanding. While the TIRI function model is not entirely hierarchical, it is more descriptively complete than previous models, in that it includes aspects of translation and interpretation as part of the model. Overall, the research literature supports the theoretically based TIRI function model, which in turn helps to clarify previous research.

Additionally, REC theory and the component competencies of the associated TIFI limit model can be reasonably used to explain the results of prior research into students' conceptions of limit and possible sources of those conceptions. This is true in the case of studies in which researchers examined sources of limit conceptions as well as in the case of studies in which researchers postulated constructivist theories about the learning of limit. For the aforementioned constructivist theories, REC theory provides possible sources for students' mental images and beliefs about limits by aptly describing what kinds of misunderstandings might lead to these mental images and beliefs. In general, previous research on students' understanding of limit supports and is complemented by REC theory, including the TIFI limit model. Notably, the research literature justifies the use of the theoretically based TIFI limit model in the present research study.

Researchers have considered function and limit separately, for the most part, in research studies found in the literature. However, a few prior research studies have shown how a student's understanding of function could contribute to her being able to perform the critical third step in the genetic decomposition given by Cottrill and his colleagues (1996) of how students might learn limit. Hence, a student's understanding of function might play a vital role in allowing her to consistently instantiate meaningful examples of limit. On the other hand, it is worth noting that even students' procedural misconceptions in regards to limit—that is, misconceptions in doing computations and other mechanical procedures using limits—can often be attributed to a weak understanding of function.

The link between understanding of function and understanding of limit is strongly intimated by previous research studies, enough so that engaging in the present research study is reasonable and may prove very valuable. It stands to reason that the confirmation of a strong positive correlation between students' understanding of function and their understanding of limit reinforces the conclusions of research studies linking understanding of function to understanding of limit (e.g., Cottrill et al., 1996; Szydlik, 2000). Furthermore, it helps to confirm that the concept of function should be central to both precalculus and introductory calculus curricula, because the correlation substantiates the assertion that a meaningful understanding of the limit concept is dependent on an adequately mature understanding of function.

CHAPTER 3

DESIGN AND METHODS

Research Design

The basic objective of the present research study is to ascertain, using quantitative methods, the strength of the relationship between understanding of the function concept and understanding of the limit concept. I caution that the implemented research design does not allow for the possibility of decisively proving or discrediting a "cause and effect" relationship between the two understandings. However, the results suggested by the data collected may corroborate the results of previous research which suggest such a relationship and thus strengthen the overall case for it. In order to accomplish the stated objective, the research study can be characterized as having two main purposes: (1) the creation and validation of an instrument, the Limit Understanding Assessment (LUA), along with an appropriate scoring rubric, in order to measure introductory calculus students' understanding of limit; and (2) the collection and suitable interpretation of quantitative data relating introductory calculus students' understanding of function and their understanding of limit.

To ensure there is no misunderstanding, I must stress that while both the TIRI function model and TIFI limit models are grounded in REC theory, there is no logical basis to believe there is a direct correlation between students' understanding in one component competency of the TIRI function model and their understanding in the corresponding component competency of the TIFI limit model. Although REC theory provides a reasonable description of the components of a robust understanding of both function and of limit, it is not intended to describe the manner in which the two mathematical concepts are related in the mind of a student. Hence, although the instrument used to measure understanding of function, the Precalculus Concept Assessment (PCA), was not created with the TIRI function model in mind, this fact is not overly important because finding a "component to component" correlation is not feasible or even desirable. For the purposes of the present research, it is sufficient to note that while individual items on the PCA were not designed with the TIRI function model in mind, students' abilities in the component competencies of the TIRI function model are well measured, as a group, by the PCA. Moreover, the instrument employed to measure understanding of limit, the LUA, is fully grounded in the TIFI limit model, as was my intent. In gathering the data for the major portion of the research study, I utilized the PCA and LUA in tandem to operationalize the cognitive constructs of understanding of function and understanding of limit, respectively.

The first part of the research study, the pilot study, required the use of both quantitative and qualitative methods in order to adequately establish the appropriateness of using the LUA in the second part of the study. For the second part—the major portion of the research study—the link between understanding of function and understanding of limit was examined using a one-group, pre-experimental, posttest only design, in the sense that students' understanding of function and students' understanding of limit were both done after students had those concepts presented to them in a science-based calculus course. In other words, the growth in mathematical knowledge of students over time was not directly measured in this study.

In addition to the two main purposes of the present research, a secondary purpose emerged relating only to introductory calculus students' understanding of limit. In particular, large subgroups of students' responses on each individual LUA item (gathered during the major portion of the research study) were analyzed in order to establish a profile of students' understanding of limit, interpreted through the framework lens of the TIFI limit model, in the context of a traditional calculus course for future scientists and engineers. This descriptive "snapshot" of what students actually learn about limit in the course of studying the concept might further strengthen the conclusions of previous research studies on limit as well as provide a springboard from which future research studies about understanding of limit can be conducted. Details about the specific methods used for this secondary research purpose appear later on.

Research Questions

The present research study involves five research questions, and these questions are shown in Figures 4, 5, and 6 of Chapter 1. Of these, the four questions in Figures 4 and 5 are related to the two main purposes mentioned in the previous section. Specifically, these four questions concern the development and implementation of a quantitative instrument, the Limit Understanding Assessment (LUA), in order to measure introductory calculus students' understanding of limit, as well as a correlation between students' understanding of function and their understanding of limit. To that end, a quantitative instrument—designed by researchers at Arizona State University (Carlson et al., under review) and called the Precalculus Concept Assessment (PCA)—was administered to students along with the LUA. The two research questions appearing in Figure 4 were attended to via a pilot study, while research questions presented in Figure 5 were considered during the major portion of the research study.

As with the main research purposes, the secondary research purpose needed to be reformulated so as to be stated as a question, rather than as a declarative statement. This secondary question is given in Figure 6 of Chapter 1. In answering this question, only data gathered during the major portion of the study was considered, because the reliability and validity of the LUA had to be established before the question could be answered with a reasonable level of confidence.

Sample

The Research Setting

Montana State University (MSU) is a public, land grant research university located in the northern Rocky Mountains. The university offers baccalaureate degrees in 51 different fields as well as many master's and doctoral degree programs. It is accredited by the Northwest Commission on Colleges and Universities. In the fall of 2008, there were 12,369 students attending MSU, of which 77% were full-time students. Approximately 85% of the students attending MSU in the fall of 2008 were undergraduates (Montana State University Office of Planning and Analysis, 2009).

Of particular interest in the context of the present research are the engineering and science programs at MSU, since students majoring—or at least considering majoring—in these fields made up the majority of the participants in the present research study. There are 10 different bachelor degree programs in engineering at MSU. Furthermore, MSU offers bachelor's degrees in mathematics, physics, chemistry, and earth sciences, as well as several different biology or ecology related baccalaureate degrees. In total, about 548 bachelor's degrees were awarded during the 2007–08 school year at MSU to students who majored in one of the degree programs just named (MSU Office of Planning and Analysis, 2008).

The Math 181 Curriculum

Math 181 is a four credit, first-semester introductory calculus course for future engineers, scientists, mathematicians, and high school mathematics teachers. It is taught using Stewart's (2008) *Single Variable Calculus: Early Transcendentals*, Sixth Edition. In covering calculus concepts such as limit, derivative, and integral, instructors follow a common online syllabus, and the material is presented in a traditional way—that is, the material is ordered in a mathematically logical manner so that new material builds on previously presented definitions, theorems, and examples. Note that such a curricular approach may not be very logical in a cognitive sense: for example, Williams (2001) asserts that "the mathematical approach to limit and the cognitive approach to limit are quite different" (p. 342). It is also worth mentioning that lectures are the dominant pedagogical choice for most Math 181 instructors.

As explained in the online syllabus (Montana State University Department of Mathematical Sciences, 2008), most sections in the first five chapters of Stewart's (2008) textbook are covered in Math 181. The titles of the chapters and sections covered in the course, exactly as they are worded in the book's table of contents (ibid., pp. v–vii), appear in Figure 15.

The focus of the course is more procedural than conceptual. This claim is supported by the following statement taken from the online syllabus: "Our examination problems will focus on the basic formulas and problem solving techniques which every student of calculus must know" (MSU Department of Mathematical Sciences, 2008, p. 1). Since 450 points out of a total possible 550 points that can be earned by a student in the course are points on the aforementioned examinations, it is fairly safe to say the course stresses computation and the mechanical techniques of calculus. During the fall and spring semesters, instructors in Math 181 administer "common hour" examinations, written by the course supervisor, so that all Math 181 students, regardless of who their teachers are, take the same midterm exams and the same final exam at the same time. In contrast, during each of the two summer semesters, only one section of Math 181 is offered, so the instructor of that section is solely in charge of writing and administering examinations, though she may still ultimately answer to a course supervisor.

In the fall and spring semesters, many sections of Math 181 are taught by graduate students in mathematics, with adjunct instructors also teaching some sections, depending on the semester. Oftentimes the course supervisor, a professor of mathematics, teaches one section as well. On the other hand, the one section of Math 181

Chapter 1 – Functions and Models
Section 1.1 – Four ways to represent a function
1.2 – Mathematical models: A catalog of essential functions
1.3 – New functions from old functions
1.5 – Exponential functions
1.6 – Inverse functions and logarithms
Chapter 2 – Limits and Derivatives
Section 2.1 – The tangent and velocity problems
2.2 – The limit of a function
2.3 – Calculating limits using the limit laws
2.4 – The precise definition of a limit
2.5 – Continuity
2.6 – Limits at infinity; horizontal asymptotes
2.7 – Derivatives and rates of change
2.8 – The derivative as a function
Chapter 3 – Differentiation Rules
Section 3.1 – Derivatives of polynomials and exponential functions
3.2 – The product and quotient rules
3.3 – Derivatives of trigonometric functions
3.4 – The chain rule
3.5 – Implicit differentiation
3.6 – Derivatives of logarithmic functions
3.7 – Rates of change in the natural and social sciences
3.8 – Exponential growth and decay
3.9 – Related rates
3.10 – Linear approximations and differentials
3.11 – Hyperbolic functions
Chapter 4 – Applications of Differentiation
Section 4.1 – Maximum and minimum values
4.2 – The mean value theorem
4.3 – How derivatives affect the shape of a graph
4.4 – Indeterminate forms and l'Hospital's rule
4.5 – Summary of curve sketching
4.7 – Optimization problems
4.8 – Newton's method
4.9 – Antiderivatives
Chapter 5 – Integrals
Section 5.1 – Areas and distances
5.2 – The definite integral
5.3 – The fundamental theorem of calculus
5.4 – Indefinite integrals and the net change theorem
5.5 – The substitution rule

Figure 15. Sections of Stewart's (2008) *Single Variable Calculus: Early Transcendentals* (6th Edition) covered in Math 181.

offered during each of the two summer semesters is taught exclusively by a mathematics graduate student. With generally less supervision in the summer, and with examinations written by the graduate student teaching the course, it is possible that a Math 181 instructor's pedagogical focus in a summer semester may be different than it is for instructors in the fall and spring semesters. However, since a summer Math 181 instructor has usually taught the class at least one time during a traditional fall or spring semester and must still follow the basic online syllabus for the course, it is unlikely that the summer course is radically different from those taught in the fall and spring semesters. The examinations administered by the summer instructor may be somewhat different, but they are likely to be modeled after exams written previously by Math 181 course supervisors.

Students in the Study

For the pilot study, I administered LUA Version A in the spring of 2008 to students in one section of Math 181 for which I was the instructor. From the group of students who took the LUA, four were selected to participate in follow-up interviews. Based on the results of this round of data gathering, the LUA was appositely adjusted and administered (LUA Version B) to students taking Math 181 from a different instructor in the first 2008 summer session. Once again, four students were picked to provide detailed verbal explanations of their written responses, and the LUA was suitably fine-tuned for another test administration. A shortened form of LUA Version C was administered to students taking Math 181 in the second summer session of 2008. Follow-up interviews with four students were again employed, and from the results of this last wave of pilot data, LUA Version D was created for use in the major portion of the study. All of the items from this final version of the LUA appear in an appendix. More details about the pilot study are provided later on in this chapter.

Both the PCA and the LUA were administered in the fall of 2008 to students in all sections of Math 181 taught during the semester. In those test administrations, 376 students took the PCA, while 332 students took the LUA. More details about the major portion of the research study—in particular, its purposes and the methods used for its implementation—can also be found later on in this chapter.

Instruments

In order to correlate students' understanding of function to their understanding of limit, these two constructs had to be operationalized using appropriate quantitative instruments, so as to validly and reliably measure the correlation between them. The next two sections contain a discussion of these instruments, namely, the Precalculus Concept Assessment (PCA) and the Limit Understanding Assessment (LUA). After a data collection period utilizing students in the Math 181 course, students' scores on the PCA and LUA were compared to quantitatively substantiate the correlation between understanding of function and understanding of limit.

The Precalculus Concept Assessment (PCA)

Researchers at Arizona State University (Carlson et al., under review) have created an instrument called the Precalculus Concept Assessment (PCA) which measures students' understanding of precalculus concepts. The underlying taxonomy of its 25 multiple-choice question items is two-dimensional: the first dimension concerns students' reasoning abilities, while the second involves students' understandings of function. The researchers classified all 25 items into this taxonomy on both dimensions, but because the distinctions among types of reasoning abilities and types of understandings were not clear cut, the researchers placed many items into more than one position on the taxonomy's grid. As part of the present research study, the PCA was employed as a means to measure understanding of function.

Admittedly, the PCA was not originally designed for the same purpose as the present research purpose, so I am attempting to interpret the PCA, post hoc, solely in terms of understanding of function. However, an examination of the second dimension of the PCA Taxonomy reveals that all three types of understandings—namely, understanding meaning of function concepts, understanding growth rate of function types, and understanding function representations (Carlson et al., under review)—concern understanding of function. Importantly, all 25 items on the PCA are placed into one or more of the above categories of understanding. Hence, it appears that the PCA demonstrates sufficient content validity with regards to measuring students' understanding of function. It is also noteworthy that the categories of understanding in the PCA Taxonomy are, holistically, entirely complementary to the component competencies of the TIRI function model.

Developed over the course of ten years—with several versions of the test administered to students during that time and subsequent improvements made to it—the PCA was viewed by researchers (Carlson et al., under review) as a reliable measure of understanding of precalculus concepts in this sense: an individual student's score on the entire test is a broad indicator of her understanding relative to the PCA taxonomy, though scores on subportions of the test can be compared only in the context of groups of students. Therefore, measuring function understanding using the PCA in order to try to correlate it with limit understanding appears to be a valid utilization of the test, based on researchers' conclusions (ibid.). Although using the PCA as a predictor of success in an introductory calculus course (as measured by final grades) is not directly pertinent to the methodology of the present research, it is worth mentioning that the PCA demonstrated predictive validity in that sense (ibid.).

The Limit Understanding Assessment (LUA)

In order to measure students' understanding of limit, I designed an instrument called the Limit Understanding Assessment (LUA), using introductory calculus textbook problems as models for the questions on the test. This was done to help ensure content validity of the instrument, which will be discussed in more detail later on. Each of the textbooks from which model problems were taken was published no earlier than 1981, and each was a standard, traditional introductory calculus textbook. The complete list of textbooks I examined can be found in an appendix.

To begin the process of creating the LUA, I selected about 200 homework exercises concerning the limit concept, some dealing with the related concept of continuity, from these textbooks. The selection was based on the level of thinking about limits, above and beyond simple computation, that was required by those exercises. To use the language of Bloom's taxonomy of cognitive objectives (discussed in Sax, 1974, pp. 57–59), I mostly rejected items that required just knowledge (the simplest cognitive objective) in favor of items necessitating comprehension, application, and perhaps even analysis. Using the TIFI limit model, I categorized all of the chosen textbook exercises into the model's component competencies of translation, interpretation, familiarization with the formal definition, and instantiation. In order to ensure I picked the appropriate component competency for the items which were ultimately used in the major portion of the study, I had two professional colleagues also categorize these items in order to verify my own categorizations. I discuss this "construct validity check" in more detail later on.

<u>Paring Down the Item List.</u> To get an accurate accounting of the variety of topics among the items selected previously, I created subcategories (e.g., translating from symbolical to graphical, interpreting infinite limits, algebraically finding a specific δ for a given ε as part of the formal limit definition, instantiating limit laws, etc.). Based on the effort required of me to correctly complete each item, I also assigned a difficulty level of easy, moderate, or hard to each one. In Figure 16, examples of easy, moderate, and hard items related to "limit computation laws" are presented. As can be seen in that figure, simple selection of the appropriate limit law is all that is necessary to correctly answer the easy item, while the moderate and hard items require much more in depth understanding of a suitable law. Furthermore, answering the hard item demands constructing a rigorous proof, which means the student must analyze the structure of the limit law, not just comprehend its use.

By employing the aforementioned subcategories and difficulty levels, I chose about 40 "best" items which reflected nearly the entire range of topics originally found

Easy Item. If $\lim_{x \to 0} f(x) = 4$ and $\lim_{x \to 0} g(x) = 0$, then $\lim_{x \to 0} f(x)g(x) = 0$. Is this statement true or false? What law applies in answering this question?

Moderate Item. When asked to evaluate $\lim_{x \to 3} \frac{x^2 + x - 12}{x - 3}$, I say that the limit does not exist since $\lim_{x \to 3}(x - 3) = 0$, citing the limit law for quotients and the fact that division by zero is undefined. What error have I committed?

Hard Item. Prove that if $\lim_{x \to c} f(x)$ exists, then $\lim_{x \to c}[f(x) + g(x)]$ does not exist if and only if $\lim_{x \to c} g(x)$ does not exist.

Figure 16. Easy, moderate, and hard items related to limit computation laws.

and which were mostly of a moderate difficulty level. Since a reliable test requires adequate variability in the responses of the students participating, I wanted questions that would require meaningful understanding of limit to answer, yet not be completely beyond the reach of introductory calculus students. Of course, since all of the items were initially taken from introductory calculus textbooks, it follows that for each item, some mathematics educator thought the item was accessible to these students. On the other hand, I have teaching experience with Math 181 students at Montana State University, giving me insight into what can be reasonably expected from students who are enrolled in that course—most of whom are freshmen majoring in engineering or science.

Upon examining the list of 40 or so “best” items, I narrowed the list even further by eliminating all but what I deemed to be the most important subcategories of each of the TIFI limit model’s component categories of translation, interpretation, familiarization with the formal, and instantiation. This was done for feasibility purposes, to ensure that I would end up with an instrument which students could complete in the time period allotted to one class meeting (i.e., in 50 minutes). In narrowing the list, I tried to select especially representative items so as to maintain construct validity for the component competencies of the TIFI limit model. Notably, no continuity items were chosen as LUA questions, because the TIFI limit model was not designed to incorporate understanding of the concept of continuity, only understanding of limit itself. After this final paring of introductory calculus textbook items, I created 11 of my own questions, some with multiple parts, to be included in a preliminary version of the LUA (known as Version A). Many of the questions I made were appreciably changed from the items on which they were modeled, but these changes were more or less necessary in order for the questions to have sufficient clarity so as to yield reliable and valid data from the students' responses. However, these changes were done while keeping in mind the importance of maintaining content validity in the items.

<u>Preliminary Rubric.</u> To make sense of the quantitative data gathered via the administration of LUA Version A, I made a preliminary rubric for each of the 11 questions on that version of the test. For consistency's sake, each individual rubric was built on the foundation of a whole number scoring scale, from 0 (lowest) to 3 (highest). In other words, each student taking LUA Version A received a score from 0 points to 3 points on each item, based on the preliminary rubric I created. That scoring system also forms the basis for the other administered versions of the LUA, including Version D. I made changes to the individual rubrics developed for each LUA question based on the

findings of the pilot study, as will be seen hereafter. However, the general overall rubric is still the basis for the individual question rubrics—with the notable exception that with regards to the TIFI limit model's component competency of familiarization with the formal, LUA Version D contains just a single item measuring that component worth up to 4 points, and that item includes the possibility of half points.

The Pilot Study

The two research questions answered via the pilot study are shown in Figure 4 of Chapter 1. The main goal of the pilot study was to validate and refine an instrument called the Limit Understanding Assessment (LUA), used to measure limit understanding in light of the TIFI limit model. To that end, three successive versions of the LUA were administered to students in sections of Math 181 in the spring of 2008 and in the summer of 2008. Each version was given to a different section of students. Follow-up interviews with four students were also conducted after each successive administration of the test. The specific process for selecting each group of four students for interviews will be discussed later. Following data gathering from these administrations of the LUA and from interviews, the pilot study closed with the construction and subsequent validation of LUA Version D, which was also used to provide data measuring students' understanding of limit in the major portion of the research study. In the next section, I provide details as to the procedures employed to gather pilot study data.

Procedures

Test Administration. Although the manner in which LUA Versions A, B, and C were administered was similar from one version to the next, there were enough differences in each test administration to necessitate providing a separate description for each successive administration of the LUA. For LUA Version A, I gave the test to students in a section of Math 181 that I was teaching during the spring of 2008. Unfortunately, the test was administered near the end of the semester, so the limit concept—covered in Chapter 2 in Stewart (2008)—was not necessarily very fresh in the students' minds. However, it should be noted that students were given advance warning about the LUA in the days before it was administered, so some students might have studied the limit concept before the test date in order to do better on the exam. Fourteen students took LUA Version A, and they were allowed 50 minutes to complete the exam, which consisted of 11 items. Of the students taking the test, there were 4 students still working when time was called.

In the first summer session of 2008, I administered LUA Version B to 27 students taking Math 181 from another teacher. The test was administered after Chapter 2 in Stewart (2008) had been covered, and since the summer session was only 6 weeks long rather than 15 weeks long (the length of the spring 2008 semester), it is probable that the limit concept was fresher in the minds of these students than for the previous administration. Again, students were given advance warning about the LUA before it was given, so some students may have studied the limit concept in preparation for the exam. The time allotted for the completion of LUA Version B, which consisted of only 8 items, was still 50 minutes. Therefore, only one student worked through the entire testing period—all the others finished early.

The instructor of Math 181 for the second summer session of 2008 imposed a time limit of just 25 minutes for students to complete LUA Version C. For that reason, I created a LUA Short Version C consisting of the 5 items from the full length LUA Version C that still needed the most scrutiny before major data gathering began, and this short form was what was given to 13 students for the last pilot study administration of the LUA. Most of these students were still working on the exam when time was called. In all other material respects, however, the administration of LUA Short Version C mirrored that of the LUA Version A and Version B administrations.

Follow-up Interviews. While quantitative analyses of students' scores on LUA Versions A, B, and C helped to determine the basic direction in which modifications were made to the LUA for subsequent versions, most important in assisting the LUA modification process were the follow-up interviews conducted with individual students. Although the manner in which follow-up interviews were done after the administration of LUA Versions A, B, and C was similar from one version to the next, there were enough differences in each round of interviews to necessitate providing a separate description for each successive round. After LUA Version A was administered, I chose 4 students to interview about their exam experience. In doing so, I made it a point to select students who were all active participants in class and with whom I had an adequate rapport, because I felt the interview data would be enhanced if students were sufficiently comfortable with me, their instructor as well as interviewer. Additionally, I picked students of differing levels of calculus understanding (based on their class grades to that point) to increase the breadth in the interview data obtained.

The basic interview protocol followed with students who took LUA Version A appears in Figure 17. Each item on the LUA was considered separately in the discussion I had with each student. The major intent of each interview was to obtain a verbal record

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Read problem <problem number> aloud. [Student reads the problem aloud.] Now, walk me through how you answered this problem.

Follow-up Questions (asked for further clarification into student's thinking, if needed):

- What do you think this problem is asking you to do?
- After you reached this point in answering the problem, what did you try to do in order to proceed further?
- What additional information do you think would have been helpful in order to complete this problem?
- How confident are you that your answer to this problem is correct? Why do you feel that way?
- If you had a chance to answer this problem again, would you do it differently? How?

Figure 17. Basic interview protocol for each LUA item.

of what the student thought each question was asking and of how the student remembered the steps he took in trying to answer the question. With each participant's permission, I took handwritten notes of the student's verbal responses, which were the sole means for interview data recording.

The selection of students to be interviewed after the administration of LUA Version B was similar to the selection of interviewees after LUA Version A was given.

To be specific, the instructor of Math 181 during the first summer session of 2008 invited 4 students to be interviewed based on their participation level in class and such that the students' understanding of calculus material was different from one student to the next. Once again, I employed the basic interview protocol shown in Figure 17, and handwritten notes were taken to record the interview data.

With fewer questions to discuss with interviewees who attempted LUA Short Version C, it was desirable to add additional questions to the interview protocol to make the most out of the five items to which those students responded. These additional interview questions, which appear in Figure 18, provided a more in-depth understanding of each student's way of thinking about the items on LUA Short Version C than that afforded in previous interviews. Of course, with fewer items to discuss, there was a decrease in the breadth of data gathered in each interview in comparison to that found in interview data gathered after LUA Versions A and B were administered. Unlike for the first two rounds of LUA follow-up interviews, the selection process for obtaining interview participants to discuss LUA Short Version C involved me deciding on students to invite to be interviewed based on their written responses to items on the test. In particular, I chose students whose answers constituted data that I felt could best be qualitatively enhanced through interviews. In other words, it seemed some answers would benefit from additional information as to what the student was thinking rather than just what was written, and I selected the students who gave such answers to be interviewed.

Let's look at problem #1 again. I've changed the numbers and listed the conditions up there on the chalkboard. Now, pretend I'm a classmate who is having a tough time translating these statements into graphical form. On the chalkboard, draw a graph that reflects these conditions and explain to me what you're doing as you're doing it, your classmate. [Student responds to this prompt.]

Let's look at problem #2. Once again, I'm a classmate that needs help in doing this problem. I've changed the conditions a bit, as you can see on the chalkboard. Explain to me, your classmate, what each of the numbers and symbols in the statements mean in "real-life" terms so I understand what is going on here. [Student responds to this prompt.]

Skipping ahead to problem #4. You've said that this/these statement(s) is/are false. Can you think of a specific example of a function (or pair of functions) that show that the statement(s) is/are false? You've said that this/these statement(s) is/are true. What is it about the statement(s) that leads you to that conclusion? [Student responds to these questions.]

Pretend I'm a classmate who does not understand the "epsilon-delta" definition of limit. Explain to me what delta is for in the context of problem #5a. Now, tell me what epsilon is actually for in the context of problem #5b. You told me before how you computed delta for problem #5c. What if I, your classmate, were to ask you why the answer is not _____ [depends on interviewee]. What would you say? Finally, look at these numbers for problem #5d [point to the different numbers in turn]. Explain to me, your classmate, what each of these numbers means in terms of picking the right graph. [Student responds to this prompt.]

Figure 18. Additional questions asked about LUA Short Version C.

Now that I have gone over pilot study procedures, I wish to look specifically at the two pilot study questions (see Figure 4 in Chapter 1) and discuss the process of ensuring reliability and validity of the LUA.

Reliability Questions

Scoring Reliability. During the course of the pilot study, it became apparent that the rubric used to score individual items had both the potential to obfuscate or to elucidate what students do and do not understand about limits. In adjusting the rubric from one version of the LUA to the next, my aim was to minimize the chance a student demonstrating some genuine understanding of limit would be penalized too harshly on a particular item because of rigid rubric standards, yet simultaneously to grade consistently so as to produce reliable data and to be fair to all students. Since reliability can be defined as “the ratio of the variance of true scores to the variance of observed scores” (Gliner and Morgan, 2000, p. 312), and increasing that ratio depends on the discriminative ability of items in both the positive and negative directions, the aforementioned goals for the rubric seemed desirable.

After gathering data for the major portion of the study, I asked a professional colleague who has substantial experience in teaching introductory college calculus to use the rubric developed for LUA Version D to score all items on 30 randomly chosen tests, so as to provide an interrater reliability coefficient measuring the effect of scorer subjectivity. (This rubric, in its entirety, appears in an appendix.) Before scoring the 30 chosen tests, however, this colleague was trained regarding the intended interpretation of the LUA Version D rubric. In particular, I gave 10 randomly chosen tests to be graded to my colleague along with a copy of the rubric. After grading all items on these 10 tests, this colleague met with me so we could discuss each item’s scoring from each test. We especially focused our discussion on those items for which there was disagreement in our scoring on any individual test. This training allowed for improved consistency and understanding in scoring with regards to the 30 randomly chosen tests for which correlation coefficients were computed.

In Table 6, I present each item from LUA Version D in the same order in which the items appeared on the actual test Math 181 students took. (These items, in a different order, are shown in an appendix). The "discrepancy" columns of Table 6 contain the

Table 6. Interrater Reliability Data for LUA Version D.

		Discrepancy (%)		
Item	Correlation*	0	±1	±2
1	0.987	97	3	0
2	0.818	63	37	0
3	1.000	100	0	0
4	0.956	90	10	0
5	0.845	87	10	3
6	0.817	70	23	7
7	0.963	90	10	0
8**	0.965	87	13	0

***** Bivariate Pearson correlation coefficient. These correlations are significant at the 0.01 level.
****** For this item, discrepancies listed the ±1 column are actually discrepancies of ±0.5.
Note: Percentages are rounded.

percentage of tests for which my colleague and I assigned scores which were different by exactly 0, 1, or 2 points. One salient point in the table is that the level of disagreement on the scoring of items 2 and 6 was relatively high in comparison to the other items. This is likely due to the fact that both items 2 and 6 were designed to measure the TIFI limit model component competency of interpretation, a component competency whose measurement seems naturally more subjective because it often requires the grader to try and infer what a student meant to write rather than what he actually wrote. In other words, students' general ability in writing the English language and in composing mathematical statements seems to be a confounding factor when scoring interpretation items.

That being said, based on the Pearson correlation coefficients calculated for each individual item, it can be stated the effect of scorer subjectivity on LUA Version D scores was relatively small. In particular, it may be safely assumed each item's scores can be meaningfully interpreted, at least when comparing an entire group of students receiving one possible score on an item to another group receiving a different score on the same item.

Before moving on, it should be noted that the use of Pearson coefficients—like any parametric test—assumes that the data are normally distributed, among other things (Fields, 2000). While the Pearson coefficient is "an extremely robust statistic" (ibid., p. 87)—meaning that only marked violations of its assumptions make it unfeasible—I computed Spearman correlation coefficients for the items in Table 6 in case the assumption of normality did not hold. Notably, all Spearman coefficients were significant at the $\alpha = 0.01$ level.

<u>Scoring Variability.</u> Adequate variability in test data is paramount to having any faith in that data's reliability. For this reason, any LUA question on Versions A, B, or C for which score variance was low was considered suspect and was either appropriately

modified or thrown out altogether. For example, consider the question in Figure 19 which appeared on LUA Version A. In scoring students' responses to that question, I

Let f and g be functions that are defined for all real numbers. Give a specific example of such functions such that the three conditions below (or as many of the three as you can manage) simultaneously hold.

(i) $\lim_{x \to 0}[f(x) + g(x)]$ exists;

(ii) neither $\lim_{x \to 0} f(x)$ nor $\lim_{x \to 0} g(x)$ exists;

(iii) $\lim_{x \to 0}[f(x) + g(x)] \neq f(0) + g(0)$.

Figure 19. Sample item on LUA Version A.

found that not one student responded with a function satisfying two or more of the stated conditions simultaneously. Out of the fourteen students who took LUA Version A, in fact, only three answered this item with a function that satisfied condition (ii), which constituted a major hurdle in solving the problem (according to students who were interviewed about this item). Hence, subsequent versions of this item were made appreciably easier to work out. Based on data gathered using LUA Version D—data which is presented in Chapter 4—all items from LUA Version D appear to have evoked sufficient variation in students' responses and in the associated scores, so a coefficient of internal consistency can be rightfully computed and meaningfully interpreted.

<u>Internal Consistency.</u> In order to gauge whether or not the LUA is internally consistent—in particular, whether or not the test as a whole measures understanding of

limit as defined in the TIFI limit model and not something else, I used a split-half method, which involves correlating two halves of the same test (Gliner and Morgan, 2000, pp. 314–315). In particular, LUA Version D consists of eight items: two measure the component competency of translation (from the TIFI limit model), two measure the component competency of interpretation, three measure the component competency of instantiation, and one measures the component competency of familiarization with the formal. Hence, in dividing LUA Version D in half, I picked one problem from each component competency in the TIFI limit model to put on one half and the other to put on the other half, with the following exception: one item from the component competency of instantiation was paired with the single item from the component competency of familiarization with the formal. These latter two items were paired up because both required students to analyze logical "if-then" statements in order to successfully answer them.

Since testing internal consistency in the previously described manner tends to underestimate reliability, I adjusted the computed correlation coefficient between students' scores on the two halves of the test using the Spearman-Brown formula (Gliner and Morgan, 2000, p. 315). The adjusted correlation coefficient obtained was $r = 0.65$. While this coefficient appears to be only moderately high, it should be noted that because LUA Version D consists of only eight items, a higher correlation coefficient could hardly be expected. At any rate, the unadjusted coefficient, as computed by Predictive Analytics Software (PASW) Statistics Gradpack 17.0, was found to be statistically significant at the

$\alpha = 0.01$ level, so it is reasonable to believe that an acceptable degree of internal consistency is present in the LUA.

Validity Questions

Content Validity. Paraphrasing Lissitz' and Samuelsen's admonition about validity in educational research, Carlson and her colleagues (under review) asserted that "the most critical issue in instrument development is the process used to establish the content validity of the tool." In designing the LUA, I used problems from standard, traditional introductory calculus textbooks as my source material for the items on the test, including problems from Stewart (2008), the textbook used in Math 181, the course from which students in the research sample were taken. This protocol at least partially ensures content validity of the LUA, since a proper evaluation of content validity "begin[s] at the planning stage" (Sax, 1974, p. 208). Additionally, I shared LUA Version B with the supervisor of Math 181, who found it a sufficiently valid instrument for measuring students' understanding of limit so as to include it (in Version D form) as part of the graded course curriculum in the fall of 2008. While it cannot be said that this supervisor's approbation of the LUA constitutes an unqualified confirmation of its content validity, it does represent evidence for it, which is the way validity is ultimately established (Sax, 1974).

Students' Interpretation of Test Items. When designing a test, it is entirely possible for the test maker to include an item that is not interpreted by the test takers in the way in which she intended. When this happens, the item can be said to be lacking validity, because it does not measure the desired objective, if indeed it measures anything at all. In order to avoid this dilemma when gathering data in the major portion of the research study, I modified the LUA using the results of pilot study interviews of students in which they explained verbally how they interpreted and answered LUA questions. These interviews revealed unintended interpretations and hurdles for students in responding to LUA items. For example, consider the question in Figure 20 which appeared on LUA Version B. Despite the parenthetical note on part (ii) of that question,

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Bob discovers that if he spends x dollars annually to advertise his company's only product, then the number of items N his company sells annually can be modeled using a function $N(x)$.

(i) Suppose $N(x)$ is an increasing function and that $\lim_{x\to+\infty} N(x) = 2000$. Explain in one sentence what this limit represents in real-life context.

(ii) Considering that in real life N must be an integer, explain how the limit computed in part (i) could help Bob set a practical limit on how much money should be spent per year on advertising in order to still maximize the number of items sold. (*Note:* Do *not* consider profit, revenue, or cost—only consider the number of items sold and the amount of money spent on advertising.)

Figure 20. Sample item on LUA Version B.

several students insisted on bringing their knowledge of economics into their responses to it, taking them away from the mathematics of limit they were asked to consider in responding to the question. For this reason, part (ii) of this item was replaced with something more straightforward for subsequent versions of the LUA. Through the pilot study, significant efforts were made to ensure correct interpretation of items by the students, so as to produce a valid LUA instrument for the major portion of the study.

<u>Categorization of Test Items (Construct Validity).</u> Another concern with regards to validity has to do with the categorization of the items into the component competencies of the TIFI limit model. A student's understanding of a mathematical concept combines multiple sources of meaning (e.g., component competencies of the TIFI limit model) for that concept. The implication is that the TIFI limit model's component competencies do not have clear-cut boundaries. Hence, in order to establish construct validity in the context of the LUA instrument, in which items were selected as though more definitive margins existed for each component competency of the TIFI model, I asked two professional colleagues who have substantial experience in teaching introductory college calculus to independently categorize items from LUA Version D according to definitions of the component competencies provided to them. Before doing so, however, I trained these colleagues in the language of REC theory and how the theory related to both the TIRI function model and the TIFI limit model. I conclude this section by first describing this training process in more detail, then presenting the results of my colleagues' categorizations of LUA items and how they compared to my own categorizations of those items.

The basic training protocol I used to prepare two expert colleagues to be able to categorize LUA Version D items into the component categories of the TIFI limit model can be found in Figure 21. Note that the protocol appears as a series of email communications I sent to these colleagues as they worked through this project. In each of the face-to-face discussions I had with these colleagues—discussions referred to in the protocol in Figure 21—I prompted them to share with me their categorization of function

Excerpts from email #1:

There are three basic steps which I need you to do in order for this project to be completed:

(1) Reading – Read relevant excerpts from my doctoral dissertation in order to familiarize yourself with the theoretical framework (and corresponding models) of my thesis;

(2) Training – Train yourself on the framework by categorizing test items about the function concept to match the framework and by discussing with me your categorizations, to ensure you and I are on the same page when it comes to the framework's categories;

(3) Confirming – Categorize test items about the limit concept to match the framework and provide me with your results.

____

Excerpts from email #2:

I have attached two files to this email. They are needed for steps (1) and (2) of the dissertation project. The file called "Proposal Excerpts.doc" is a Word file I would like you to read first. It contains the necessary background information you'll need to do the other steps. If you have questions as you're reading the file, please email those questions to me so I can assist you. The most important thing to try to understand, in order to complete step (2), is the list of "component competencies" on page 7.

After you feel comfortable with my framework through your reading, print out a copy of the file "Function Test.pdf". Categorize each circled problem using one of three "component competencies": translation, interpretation, instantiation. Do not use the component competencies of familiarization or reification, as I have determined that nearly all the problems require some of the former, making it not useful, while on the other hand, none of the problems really require the latter. If you feel that one or more of the circled problems fits more than one "component competency" category, write down all the categories to which you think the problem fits. This function test, by the way, is not of my making—it is authored by O'Callaghan (1998).

If you have any questions about the above paragraph, please let me know so I can assist you. Once you have completed your categorizations, please email me so we can make an appointment to sit down together and discuss your categorizations. After our interview, I will send you another file so you can do step (3).

____

Excerpts from email #3:

I have attached a file called "Limit Test.pdf" to this email. It is needed for step (3) of the dissertation project.

The most important thing to understand, in order to complete step (3), is the list of "component competencies" found on pp. 8–9 of the reading. The basis of the limit test attached to this email is the TIFI limit model (where TIFI is an acronym for translation, interpretation, familiarization with the formal, and instantiation). To do step (3), please categorize each of the eight test items from the "Limit Test.pdf" file using one of the aforementioned component competencies. Do not use the component competency of reification, as none of the test items requires a student to have reified the limit concept. If you feel that one or more of the eight problems fits more than one "component competency" category, write down all the categories to which you think the problem fits. This limit test, by the way, is of my own making.

If you have any questions about the above paragraph, please let me know so I can assist you. Once you have completed your categorizations, please email me. I will want to meet with you to discuss your categorizations.

Modified from personal communications to two colleagues (2009).

Figure 21. Training protocol for experts' categorization of LUA items.

test items or LUA items as well as their reasoning for choosing the component competencies they did.

The upshot of this project to establish construct validity for LUA Version D is that my expert colleagues and I agreed for the most part as to which limit items should be categorized under translation, interpretation, familiarization with the formal, or instantiation. However, I instructed my colleagues to place items into more than one component competency if they wished, and since the component competencies have somewhat fuzzy boundaries, it turned out that these colleagues placed some items into two categories rather than just one. Importantly, on such items, the component competency from the TIFI limit model which I myself had chosen was the primary category to which my colleagues placed the items as well—with one notable exception, to which I now attend.

One item from LUA Version D, which I reproduce in Figure 22, was categorized by both expert colleagues as primarily a translation item rather than an instantiation item, which is what I intended for it. Both colleagues felt the former component competency

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Recall that a piecewise function is a function like $g(x) = \begin{cases} x+1 & x \le 10 \\ \sqrt{x} & x > 10 \end{cases}$, where the formula to use for finding the function's value depends on the piece of the domain in which the x-value is located. Now, find the formula for a single piecewise function f that satisfies all of the conditions below. (Drawing a possible graph for f first may help, but the graph will not be scored.)

$f(0) = 0$ $\qquad \lim_{x \to 0^-} f(x) = 3$ $\qquad \lim_{x \to -\infty} f(x) = +\infty$ $\qquad \lim_{x \to +\infty} f(x) = 2$

Figure 22. Instantiation item which was categorized as a translation item by other experts.

was appropriate because students had to convert the given limit conditions into a function formula on this item, which is true. On the other hand, students did not have to mentally move from one representation of limit to the other to answer the item, which is how I defined translation in the TIFI limit model. It may be that the apparent similarity of this item to another item which required students to graph a function satisfying certain limit conditions—an intended translation item—instigated the categorization of the item in Figure 22 as a translation item as well. At any rate, this discrepancy among my colleagues and I does not constitute significant evidence against the construct validity of LUA Version D. Overall, the high level of agreement among us as to which component competency best suits each item ensures the component competency constructs are meaningfully represented by the LUA.

Methods of Major Study

In order to ascertain the strength of the relationship between understanding of the function concept and understanding of the limit concept, two tests were administered to students in all sections of Math 181 in the fall of 2008. The Precalculus Concept Assessment (PCA) was administered to students near the beginning of the academic term, after the students received a review of precalculus concepts contained in Chapter 1 of Stewart (2008). After the students learned about limits as contained in Chapter 2 of Stewart (2008), Version D of the Limit Understanding Assessment (LUA) was administered to them.

The consideration of two main research questions as well as a secondary question made up the major portion of the present research study. These three research questions appear in Figures 5 and 6 of Chapter 1, with the two main research questions appearing in Figure 5. In the next three sections, I discuss in detail the procedures and methods used to answer these three research questions.

Major Procedures

The test administrations completed in the fall of 2008, both of the PCA and of the LUA, followed a basic protocol so as to increase the uniformity of testing conditions for all students taking the tests. Furthermore, because there were no more than two sections of Math 181 meeting at any one time, it was only necessary to employ two different people—namely, Math 181 course supervisor and myself—to handle the test administrations for all sections of Math 181. This relatively small number of test administrators increased the uniformity of testing conditions for all students. If a student asked the administrator a question about a specific item on the test, the administrator declined to answer the question. The two basic protocols followed by the test administrators for the PCA and for the LUA appear in an appendix. The manner in which the statistical analysis of the data proceeded is the subject of the next section.

Statistical Methods

In order to answer the first major research question, a linear regression was utilized in which students' scores on the PCA were correlated with students' scores on the LUA (Version D) using PASW Statistics Gradpack 17.0. Assuming understanding of function truly correlates well with limit understanding, a student's score on the PCA should be statistically significant in predicting that student's LUA score even after controlling for the basic mathematics skills the student gained in high school. For that reason, the additional explanatory variable of students' scores on the mathematics portion of the ACT exam was included. (In the case of some students, this score was actually a "converted" SAT Mathematics score. More details about how this variable's values were recorded are provided in Chapter 4.) Clearly, different sections of Math 181 have different characteristics among students and instructors, and it may be possible that those characteristics are what contribute to measurable differences in students' understanding of limit, rather than those students' understanding of function. Hence, dummy variables were also included in the regression model to control for these sectional differences. The use of such variables is fully explained in Chapter 4.

Simple bivariate correlations were used to investigate the possible relationship between students' understanding of function and students' understanding of limit in the specific component competencies of the TIFI limit model. Students' scores in each of these component competencies were based on responses to only one, two, or three items, and since understanding only increases the tendency of students to behave in a mathematically sensible way in any particular situation (Breidenbach et al., 1992; Dubinsky and Harel, 1992), it is reasonable to conclude a higher relative amount of variance in students' scores on any one component competency was due to random "noise" than the relative amount of variance in their scores on the LUA as a whole, which consisted of eight items. Hence, the previous covariates (viz., ACT Mathematics score and class section) were not included in these correlational analyses, to avoid excessive masking of the importance of students' PCA scores on their scores in each component competency. Of course, this lack of controlling variables decreases the conclusive strength of any significant correlation found between PCA scores and individual component competency scores, but that can be regarded as a reasonable limitation of these analyses.

Secondary Procedures

As a preliminary step in scoring students' responses to each item from LUA Version D, I used cluster sampling to select a subsample of all students who took the test, and I graded this subsample first. Each subsample consisted of all the students in a randomly chosen section of Math 181 along with half of the students in another randomly chosen section of Math 181. In the spirit of equity, no section of Math 181 was chosen more than once as the "full section" for a subsample, nor was it chosen more than once as the "half section" for a subsample. When scoring students' responses in a particular subsample related to one item of the LUA, careful notes were taken for each student's response to that item.

These notes served two useful purposes. First, they provided me with the opportunity to refine, one last time, the rubric used to score each item on the LUA. This ultimate version of the item's rubric was then used to score all other students' responses to that specific item. (The final version of the LUA items' rubrics can be found in an appendix.) Second, the notes afforded a suitably large, yet manageable, sample of responses from which to answer the secondary "limit profile" question (Figure 6 of

Chapter 1) of the research study. In answering the secondary question, I provide descriptive statistics (e.g., counts, percentages) along with possible explanations as to why students responded the way they did. These results are given in Chapter 4.

Assumptions and Limitations

Assumptions

In an attempt to appropriately operationalize understanding of function and understanding of limit, the Precalculus Concept Assessment (PCA) and Limit Understanding Assessment (LUA) were employed to measure students' understandings of function and of limit, respectively. It is not assumed that these operational definitions for understanding of function and for understanding of limit are the only rational operational definitions, or that a correlation between PCA scores and LUA scores constitutes a correlation of understandings beyond what is measured by the tests. Hence, any correlation found between the two aforementioned understandings is assumed to be a reflection of those understandings as they are measured on the PCA and LUA.

Limitations

Incoming Knowledge about Limit. In theory, students who enroll in Math 181 with prior experience involving the calculus concept of limit may be more successful in comprehending limit in Math 181, in completing specific problems on the LUA, or in both endeavors (which are assumed to be related). However, since I only tested the claim that understanding of function at a specific moment in time is correlated with understanding of limit at another specific moment in time, such knowledge should not invalidate possible conclusions for the study as long as these conclusions are not overstated. In fact, it can be sensibly assumed, given the ideas driving the present research, that prior experience with limit could increase both a student's understanding of limit and of function, thus strengthening the argument there is a strong link between function understanding and limit understanding.

Generalizing the Research Results. The standards students must reach academically in order to enroll in Math 181 may not be the same standards as at other universities, imposing limitations with regards to generalizations to other populations. The traditional approach to calculus employed by most Math 181 instructors at this university must also be considered in attempts to generalize to other college groups.

Pre-experimental One-Group Design. With no experimental treatment and no control group, it was not possible to irrefutably determine if a link found between students' understanding of function and their understanding of limit constitutes a genuine cause-effect relationship rather than simply a confirmation of a quantitative correlation between the two.

Experimental Attrition. Only those students who took either the ACT Mathematics exam or the SAT Mathematics exam and who took both the PCA and the LUA were included in the statistical analysis used to identify any correlation between students' understanding of function and their overall understanding of limit. Only students who took both the PCA and the LUA were included in the statistical analyses used to indentify any correlation between students' understanding of function and their understanding of specific component competencies of the TIFI limit model. Because many students dropped out of Math 181 before the administration of the LUA, or simply missed class on one or both days when the PCA and LUA were given, it is not possible to know if there is any linkage between understanding of function and understanding of limit in these students. It may also be that the inclusion of such students would have strengthened or weakened the results suggested by the gathered data.

CHAPTER 4

RESULTS

Introduction

The purpose of this chapter is to present the results of the data gathered for the present research study. The design of the research study, along with the methodology utilized to implement that design and gather the necessary data, is discussed in Chapter 3. At this point, it suffices to mention that the experimental context for all of the data gathered for the present research study is Math 181, a first-semester introductory calculus course intended primarily for students majoring in science or engineering.

Of the four main sections of this chapter, each of the first three is dedicated to exactly one of the questions of the major research study, which are listed in Figures 5 and 6 of Chapter 1. The first of these sections is concerned with establishing how strong the relationship is between Math 181 students' scores on the Precalculus Concept Assessment (PCA)—which measures understanding of the mathematical concept of function—and their scores on the Limit Understanding Assessment (LUA)—which measures understanding of the introductory calculus concept of limit.

In the second main section, I impart the results of nonparametric bivariate correlation analyses, each of which relates Math 181 students' scores on the PCA with their scores on one of four component competencies contained in the LUA. (These four component competencies of limit understanding together constitute the TIFI limit model, found in Table 3 of Chapter 1.) The reason I chose nonparametric tests for these analyses is also established in that section.

Although a descriptive profile of a typical Math 181 student's understanding of limit is of secondary importance to the thrust of the overall research study, it supports the main purpose of the research by providing valuable insight as to what different responses on LUA items might actually signify with regards to Math 181 students' mental constructions of the limit concept. For this reason, the third main section of this chapter is entirely dedicated to the setting forth of such a student profile.

To conclude the chapter, a summary of all important results is provided.

Linear Regression of LUA Scores on PCA Scores

Using PASW Statistics Gradpack 17.0, a linear regression analysis was performed on data gathered from students taking Math 181 in the fall of 2008 in order to answer the following major research question:

> How strong is the correlation between students' scores on the PCA and their scores on the LUA?

It is important to understand that the results of the computed linear regression only apply to that population of students for which the sample of Math 181 students who actually participated is representative.

Variables

Main Variables. The explanatory variable of interest consists of Math 181 students' scores on the Precalculus Concept Assessment, which is henceforth denoted

PCA in this chapter. With 25 multiple-choice items on the PCA test, the maximum score a person can earn on the PCA is 25 points. For the dependent variable, values are assigned via students' scores on the Limit Understanding Assessment, hereafter denoted LUA in this chapter. With 7 items worth up to 3 points each and 1 item worth up to 4 points (including half-points), the maximum score a person can earn on the LUA is 25 points. Because the explanatory variable PCA might be correlated with other explanatory variables which could lead to the spurious conclusion that PCA scores affect LUA scores when, in fact, a third variable "causes both … and hence makes them appear related" (Hamilton, 1992, p. 29), it was desirable to include such explanatory variables in the regression model in order to control for their influence and thus appreciably strengthen the conclusion of the analysis. The specific reason for including each of the two additional explanatory variables discussed in next few paragraphs is spelled out in Chapter 3.

<u>Controlling Variable: ACT Mathematics Score.</u> As stated by its publisher, the ACT Mathematics test is "designed to assess the mathematical skills that students have typically acquired in courses taken up to the beginning of grade 12[,] … includ[ing] Algebra 1, Geometry, and Algebra 2 (which covers beginning trigonometry concepts)" (American College Testing, 2005, p. 7). Because many Math 181 students took the SAT Mathematics test in lieu of or in addition to the ACT mathematics test—and the SAT Mathematics test covers similar topics to the ACT Mathematics test, as attested by an examination of its publisher's website (The College Board, 2009)—such students' SAT Mathematics test scores were "converted" to ACT Mathematics test scores using a table

provided by the University of Texas (Lavergne and Walker, 2006, table 6 of webpage). One minor modification was made to this conversion table in order to match Montana State University's "prerequisite flowchart" for enrolling in different mathematics courses (Montana State University Department of Mathematical Sciences, 2009). The ACT Mathematics scores or "converted" SAT Mathematics scores constitute the values of the explanatory variable, which is hereafter denoted ACT. If a student took both the ACT Mathematics test and the SAT Mathematics test, then the better of his ACT score and converted SAT score was recorded under this variable. It should also be noted that this variable's values consist of each student's best ACT Mathematics test scores and/or his best SAT Mathematics score, which is important to know since many students take the ACT and/or SAT multiple times.

Controlling Variable: Section Number. A total of ten different sections of the Math 181 course were taught in the fall of 2008, all of which were included in the major research study. Nine dummy variables were included in the regression model to account for the explanatory variable of section number, and these dummy variables are henceforth referred to as SEC1 to SEC9, respectively. For example, if a student was enrolled in Section 1 of Math 181, her section number was coded by assigning a value of "1" for that student to variable SEC1 and a value of "0" for that student to variables SEC2 through SEC9. In contrast, if a student was enrolled in Section 10 of Math 181, her section number was coded by assigning a value of "0" for that student to all nine dummy variables, SEC1 through SEC9.

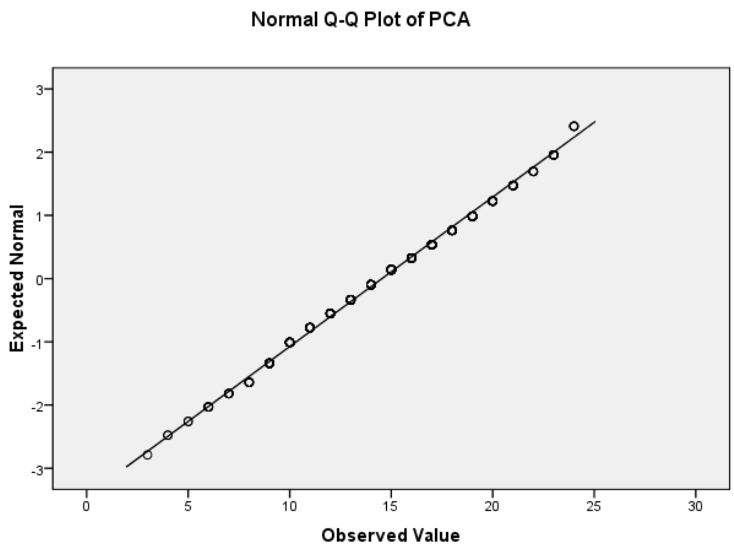
Assumptions of Regression Model

In order to have any confidence in the results of a linear regression analysis, the statistical assumptions of the linear regression model must be carefully examined in order to verify that the assumptions are satisfactorily met. The importance of substantiating these assumptions when performing regression analysis was asserted by Fields (2000). The most fruitful way to begin the obligatory substantiation of model assumptions is by "carefully studying each variable first before proceeding to more complex analyses" (Hamilton, 1992, p. 2). Following that recommendation necessitates a presentation of descriptive statistics for each of the non-nominal variables introduced in the previous section, and such a presentation appears in Table 7. It is important in linear regression

Table 7. Descriptive Statistics for PCA, ACT, and LUA.

Variable	n	Mean	SD	Max.	Min.
PCA	376	14.55	4.23	24	3
ACT	340	27.90	3.47	36	18
LUA	332	12.61	4.21	22.5	0

that all of the non-nominal variables have distributions which are not markedly non-normal. For that reason, normal quantile-quantile plots for each of the three variables in Table 7 were created, and these plots are shown in Figures 23 and 24. The graph points in each of the plots in Figures 23 and 24 do not depart overly much from the line representing a perfect normal distribution, and this constitutes sufficient evidence that the assumption of variables whose distribution is "normal enough" is not contradicted.



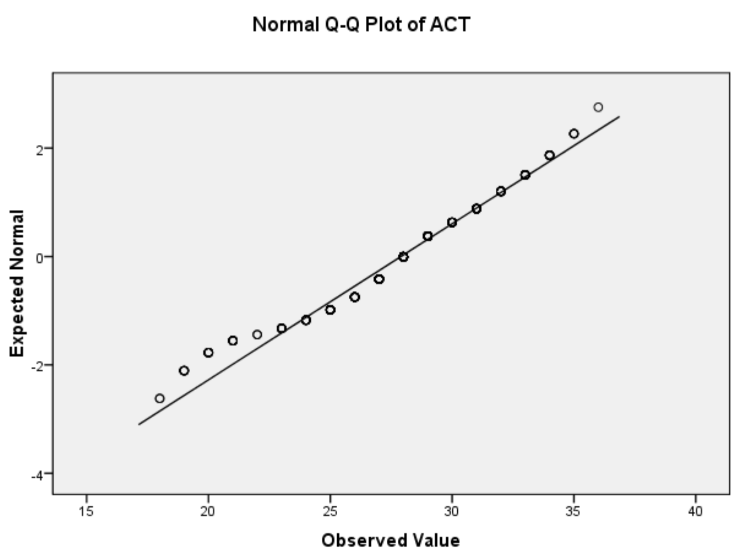


Figure 23. Normal quantile-quantile plots for non-nominal explanatory variables.

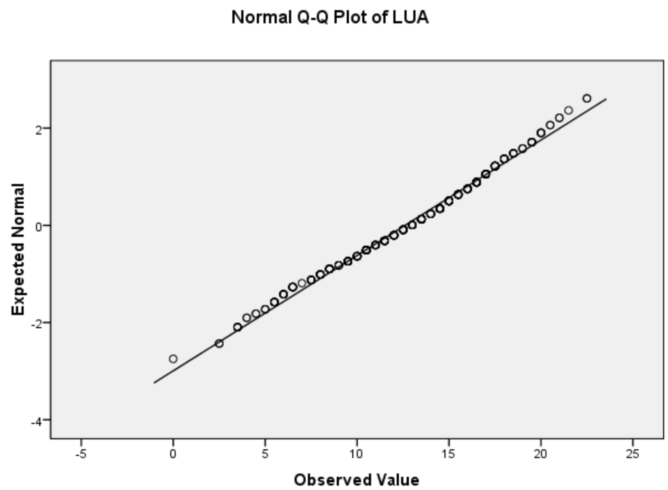


Figure 24. Normal quantile-quantile plot for LUA.

A closer look at more complex assumptions for the linear regression model is now in order. For convenience, five linear regression model assumptions and their definitions—as given by Fields (2000, pp. 128–129)—are shown in Figure 25. In the successive paragraphs of this section, I provide suitable evidence that these five important assumptions of the regression model are fulfilled. It should be noted that all accompanying figures in this section are associated with the full regression model.

<u>Linearity.</u> One way to verify the assumption of linearity is via use of a scatterplot matrix in which each scatterplot in the matrix relates two model variables. Such matrices provide "a quick overview of zero-order relationships" (Hamilton, 1992, p. 114). A scatterplot matrix relating LUA, PCA, and ACT appears in Figure 26. It is seen in this figure that the assumption of linearity can be safely inferred to hold.

(1) Linearity – The mean values of the outcome variable for each increment of the predictor(s) lie along a straight line.

(2) No perfect multicollinearity – There should be no perfect linear relationship between two or more of the predictors. So, the predictor variables should not correlate [too] highly.

(3) Independent errors – For any two observations the residual terms should be uncorrelated (or independent).

(4) Normally distributed errors – It is assumed that the residuals in the model are random [and] normally distributed … with a mean of zero.

(5) Homoscedasticity – At each level of the predictor variable(s), the variance of the residual terms should be constant.

(Fields, 2000, pp. 128–129)

Figure 25. Important assumptions of linear regression models.

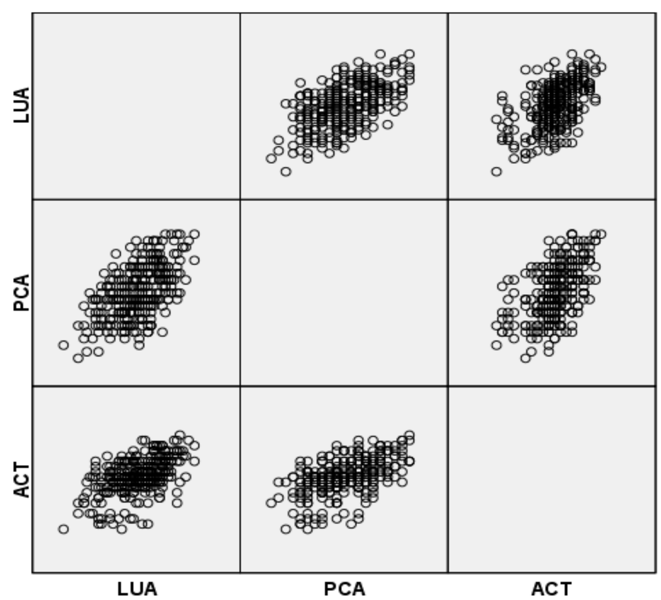


Figure 26. Scatterplot matrix relating non-nominal variables.

No Perfect Multicollinearity. The proportion of variance in each explanatory variable which is not attributable to the other explanatory variables is known as tolerance, and it is a statistic provided by PASW. Low tolerance values (i.e., values close to 0) are indicative of a multicollinearity problem in a regression model. For the regression model created for the present research study, the tolerance values for PCA and ACT were 0.692 and 0.686, respectively, which is completely acceptable. Furthermore, these two numbers indicate that a large proportion of the variance in PCA is not attributable to ACT or vice versa, which means PCA and ACT, while correlated, do not measure the same overall construct.

The tolerance values for the nine dummy variables were all below 0.5, which is to be expected—after all, they were perfectly correlated with each other, because in combination they actually only represented one nominal variable. Hence, in order to confirm that section number was not overly correlated with PCA or ACT, a process known as singular value decomposition (SVD) was performed as described in Belsley and Klema (1974). Since the regression model coefficients for SEC1 through SEC9 did not have high variance proportions on the same small eigenvalue from the SVD as did PCA or ACT, it follows that section number is not overly correlated with PCA or ACT.

Independent Errors. The assumption of independent errors can be assessed using the Durbin-Watson statistic, which "tests for correlations between errors" (Fields, 2000, p. 137). The statistic has a range from 0 to 4; values very near 2 indicate that the assumption of independent errors is satisfied. Since the Durbin-Watson statistic computed for the present research study's linear regression model was 2.009, I assert that this assumption is met for my regression model.

Normally Distributed Errors. In assessing the distribution of the residuals, non-normality can be detected via use of a normal probability-probability plot of the standardized residuals (Fields, 2000, p. 159). For the present research study, such a plot was produced, and the result is shown in Figure 27. The graph points in Figure 27 follow the line representing a perfect normal distribution rather closely, which indicates that the assumption of normally distributed errors is not violated.

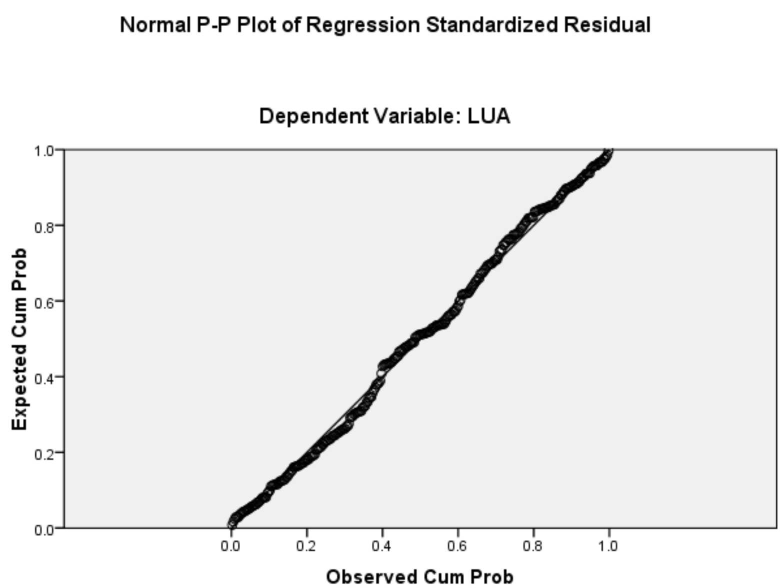


Figure 27. Normal probability-probability plot of standardized residuals.

Homoscedasticity. One way to assess the assumption of equal variances of the residuals for different levels of the non-nominal explanatory variables involves using Levene's test for equality of variances as described in Pryce (2002). To be specific, I divided the data into two groups: the first group consisted of students with a PCA score of 14 or less, while the second was comprised of students with a PCA score of 15 or more. (This cutoff point was chosen because the median PCA score among all 376 students in Math 181 taking the test was 14.) Then the variance of the standardized residuals—residuals obtained from the linear regression computation—were compared for these two groups using Levene's test. The test statistic was computed as $F = 1.055$, which had a corresponding p-value of 0.305, signifying that the assumption of homoscedasticity (for different levels of the variable PCA) can be inferred to hold.

Similarly, I again divided the data into two groups: the first group consisted of students with a recorded ACT Mathematics score of 28 or less, while the second was comprised of students with recorded ACT Mathematics scores of 29 or more. (This cutoff point was chosen because the median recorded ACT Mathematics score among the cohort of 340 students in Math 181 for which scores were recorded was 28.) Once again, the variance of the standardized residuals—residuals obtained from the linear regression computation—were compared for the two groups using Levene's test. In this case, the test statistic was computed as $F = 1.778$, which had a corresponding p-value of 0.183, indicating that the assumption of homoscedasticity (for different levels of the variable ACT) apparently holds.

Regression Results

Data from 301 students enrolled in Math 181 were included to create a linear regression model of LUA on PCA, ACT, and the SEC variables (1 through 9). Using PASW, the unstandardized coefficients were computed, along with the corresponding standardized coefficients, *t*-statistics and *p*-values. These calculated values can be found in Table 8. Included also in the table is the R^2 statistic for the regression model, which

Table 8. Statistics from Full Linear Regression Model $(n = 301)$.

Variable	Unstandardized Coefficient	Standardized Coefficient	t	*p*-value of *t*-test
PCA	0.406	0.395	7.381	0.000*
ACT	0.371	0.305	5.672	0.000*
SEC1	–0.321	–0.023	–0.359	0.720
SEC2	0.703	0.052	0.797	0.426
SEC3	–1.768	–0.128	–1.992	0.047
SEC4	0.171	0.012	0.189	0.851
SEC5	0.166	0.013	0.192	0.848
SEC6	0.828	0.064	0.954	0.341
SEC7	–0.126	–0.009	–0.142	0.888
SEC8	1.019	0.073	1.139	0.255
SEC9	0.358	0.026	0.406	0.658
Constant	–3.955			

For this model, $R^2 = 0.426$.

***** Note: PASW automatically rounded these values down, which implies that these *p*-values are both less than 0.0005.

measures the amount of variance in LUA which was explained by the variance in the explanatory variables. The reported value of $R^2 = 0.426$ is quite large for a study in education like the present study.

Because each of the SEC variables with the exception of SEC3 were statistically insignificant, a simpler model was created that only included the variables of PCA, ACT, and SEC3 as explanatory variables. Notably, the assumptions necessary to linear regression were also satisfactorily fulfilled for this simpler model. The statistics computed for this model appear in Table 9, which has the same format as Table 8 but with an additional column for the bivariate correlation coefficient between each explanatory variable which was kept and LUA. The overall $R^2 = 0.416$ for the simpler model is not noticeably smaller than that of the full model.

Table 9. Statistics from Simpler Linear Regression Model $(n = 301)$.

Variable	Raw Coefficient	Standardized Coefficient	t	p-value of t-test	Correlation with LUA*
PCA	0.399	0.388	7.384	0.000**	0.566
ACT	0.383	0.315	5.987	0.000**	0.532
SEC3	–2.092	–0.152	–3.413	0.001	–0.190
Constant	–3.848				

For this model, $R^2 = 0.416$.

***** Bivariate Pearson correlation coefficient. These correlations are significant at the 0.001 level.
****** Note: PASW automatically rounded these values down, which implies that these p-values are both less than 0.0005.

It was mentioned earlier that a large proportion of the variance in PCA is not attributable to ACT or vice versa. Therefore, it is unlikely that the highly significant coefficient on PCA seen in Table 9 is entirely due to students' mathematical understanding which is measured by both the PCA and ACT. This is important, because it suggests that the PCA is a significant explanatory variable for LUA even when taking the variable ACT into account.

A comparison of the relative importance of the explanatory variables PCA, ACT, and SEC3 on the dependent variable LUA can be done by a comparison of standardized coefficients. Examining the third column in Table 9, it is seen that PCA provides the most explanatory power in predicting LUA scores, ACT is next in explanatory power, and SEC3 contributes the least to the overall prediction model (although all three variables are highly significant).

While the unstandardized coefficients do not allow for a comparison of the three explanatory variables, they are of interest in that they provide a measure of the expected difference in LUA scores for two students who differ by a single "point" in just one of the explanatory variables, assuming both students have a mean score on the other two explanatory variables. (The mean scores for PCA and ACT are in Table 7.) Now, the "mean" SEC3 score, which is equal to 0.10, is a nonsensical measure of that variable's central tendency since SEC3 is binary. However, since the mode of SEC3 is 0, it makes sense to assume the value of SEC3 is 0 when considering the unstandardized coefficients on PCA and ACT. In particular, two students who both have an ACT Mathematics score of 28, neither of whom was enrolled in Section 3 of Math 181, but whose PCA scores differ by 5 points, could be expected to have LUA scores that differ by 2 points. In contrast, two students who both have a PCA score of 14 (or who both have a PCA score of 15), neither of whom was enrolled in Section 3 of Math 181, but whose ACT Mathematics scores differ by 8 points, could be expected to have LUA scores that differ by about 3 points. Finally, two students who both have an ACT Mathematics score of 28 and who both have a PCA score of 14 (or who both have a PCA score of 15), but one of them is enrolled in Section 3 while the other is not, could be expected to have LUA scores that differ by about 2 points.

For the purposes of the present research, PCA scores constitute the only explanatory variable of real interest; the others are simply controlling variables to increase confidence that the statistical significance of the correlation between PCA scores and LUA scores is genuine. As seen in Table 9, there is a strong positive linear correlation $(r = 0.566)$ between the variables PCA and LUA, and this correlation is highly significant. Since 32% of the variance in LUA scores is explained by the variance in PCA scores alone, the explanatory power of PCA scores on LUA scores is substantial, especially considering that the relatively small number of items on the LUA likely led to more unexplained variance in the scores than could be expected on more extensive tests. (Note that the statistical value of 32% explained variance was computed by squaring r.) In short, I can answer the first question of the major research study, given in Figure 5 of Chapter 1, by declaring that the positive linear correlation between PCA and LUA scores is very strong, even after accounting for ACT and section number.

Correlation of PCA Scores with LUA Component Competency Scores

Using PASW Statistics Gradpack 17.0, a correlational analysis was performed on data gathered from students taking Math 181 in the fall of 2008 in order to answer the following major research question:

> How strong is the correlation between students' scores on the PCA and their scores on the portion of the LUA that measures understanding of limit:
>
> (a) in the component competency of translation?
> (b) in the component competency of interpretation?
> (c) in the component competency of instantiation?
> (d) in the component competency of familiarization as it applies to the formal definition of limit?

As with the analysis presented in the previous section, it is important to keep in mind that the results of the correlation tests in this section only apply to that population of students for which the sample of Math 181 students who actually participated is representative.

Before proceeding with the presentation of the variables involved in the correlational analysis and the statistical results of that analysis, I wish to explain why nonparametric correlation tests were performed in order to answer the research question of interest. The primary reason nonparametric tests were chosen rather than linear regression tests is because the assumptions undergirding linear regression were not satisfied for the models relating PCA scores to students' component competency scores in translation, interpretation, or familiarization with the formal. Specifically, the variable representing each of these component competency scores demonstrated a non-normal distribution, or the distribution in the residuals computed from linear regression of that variable on PCA scores was non-normal or heteroscedastic, or both. For that reason, I chose to use nonparametric correlations for the sake of reliability in the results. In the case of students' component competency scores in instantiation, the assumptions underlying linear regression were satisfactorily met for the model relating PCA scores to instantiation scores. However, the goal of the present research is not to establish a significant positive linear correlation between variables, rather to verify a significant positive correlation of any kind between them. Nonparametric tests can accomplish this task, and so I selected nonparametric correlation analysis in this case for the sake of consistency in the results so as to allow comparisons among all the statistical models involved in answering the research question.

Variables

The explanatory variable of interest for each of the four correlation models is PCA, a variable described in detail previously. Each correlation model has one dependent variable, whose values are assigned via students' scores on a specific component competency of the Limit Understanding Assessment. With 2 items worth up to 3 points each, the maximum score a person can earn in the component competency of translation—denoted hereafter with its variable name TRA—is 6 points. The component competency of interpretation also follows this scoring scale, and I refer to this variable as IPR. In regards to familiarization with the formal—denoted hereafter with the variable name FML—the Limit Understanding Assessment contains just 1 item worth up to 4 points (including half-points) in this component category. Finally, the component competency of instantiation is represented by 3 items worth up to 3 points each, for a maximum possible score of 9 points. I refer to this last variable as IST. For reasons explained in Chapter 3, the controlling explanatory variables of ACT Mathematics score and class section were not included in these four correlation models.

Nonparametric Bivariate Correlation Results

As with the previous section, I begin by presenting some descriptive statistics for the variables involved in the correlational analyses. These variables' statistics are given in Table 10, with the exception of those for the variable PCA (whose descriptive statistics appear in Table 7). Based solely on the relative scores (i.e., points earned divided by points possible) for each variable, Math 181 students were most successful on items in

Table 10. Descriptive Statistics for TRA, IPR, FML, and IST.

Variable	n	Mean	SD	Max.	Min.
TRA	332	3.80	1.60	6	0
IPR	332	3.45	1.67	6	0
FML*	332	1.41	0.84	4	0
IST	332	3.95	1.94	9	0

***** Half-points are possible for this variable.

the component competency of translation (TRA), followed by interpretation (IPR), instantiation (IST), and familiarization with the formal (FML), in that order.

The basic nature of the relationship between PCA scores and component competency scores can be examined via use of a scatterplot matrix in which each scatterplot in the matrix relates two variables, as seen in Figure 28. In the figure, the scatterplots in the first column (or in the first row, but rotated) of the matrix are important, while the others are not of interest in the present research study. In order to decide which of the relationships seen in the first column scatterplots represent significant positive correlations, nonparametric correlation statistics were computed between PCA and each component competency variable. The results appear in Table 11. In that table, it is seen that the most statistically significant monotonically positive correlation is between PCA and IST, followed in order by the correlation between PCA and TRA, the correlation between PCA and FML, and the correlation between PCA and IPR. In the next few paragraphs, a short discussion of each correlation is imparted.

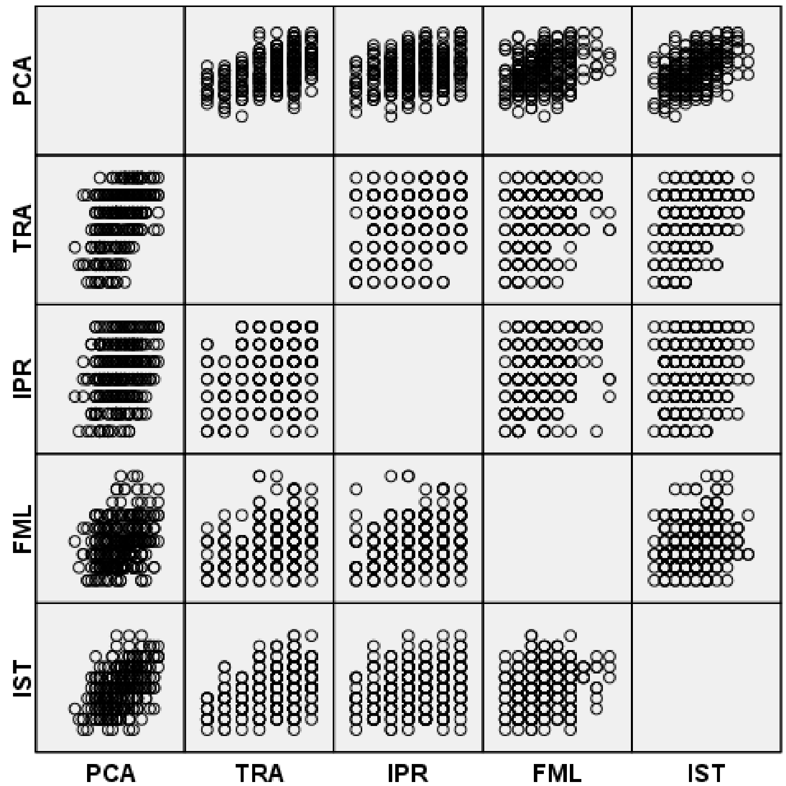


Figure 28. Scatterplot matrix relating PCA and component competency variables.

Table 11. Correlations between PCA and Each Component Competency ($n = 329$).

Variable	Bivariate Correlation with PCA*
TRA	0.459
IPR	0.257
FML	0.308
IST	0.490

***** Spearman correlation coefficient. All correlations are significant at the 0.001 level.

Translation and Instantiation. For the component competencies of translation and instantiation, the bivariate correlation between each variable and PCA scores is relatively strong. While nonparametric tests do not impose a preset structure on the relationship between PCA and TRA or between PCA and IST, the scatterplots reveal some idea of what may be true about the respective relationships. In the case of translation, students earning a high PCA score seemed to have a better chance of obtaining an above average TRA score than their lower-scoring peers. On the other hand, some students with high TRA scores had lower than average PCA scores. In contrast, students earning a below average PCA score appeared to have a lesser chance of receiving a high IST score than their higher-scoring peers, but even some students with above average PCA scores still did not do well on IST.

Interpretation and Familiarization with the Formal. For the component competencies of interpretation and familiarization with the formal, the bivariate correlation between each variable and PCA scores is somewhat weaker than with the previous two. While nonparametric tests do not impose a preset structure on the relationship between PCA and IPR or between PCA and FML, the scatterplots reveal some idea of what may be true about the respective relationships. In the case of interpretation, students earning a high PCA score appeared to have a better chance of getting an above average IPR score than their lower-scoring peers. This also seems to be the case with the variable FML. However, it should be noted that relatively few students (even those with high PCA scores) scored better than 2.5 points out of 4 points possible on FML.

In summary, scores on all four component competencies tend upward as PCA scores tend upward, but the nature of that upward trend is different for each component competency. In all cases, however, the bivariate correlations are at least moderately strong.

Profile of Typical Student's Limit Understanding

The setting forth of a typical Math 181 student's understanding of limit as measured on the LUA, after that concept has been presented to her, supplies a helpful descriptive context in which to situate the results from the previous sections of this chapter. With that purpose in mind, I analyzed notes which were carefully taken during the scoring of students' responses on each LUA item in order to answer the following secondary research question:

> In the context of a traditional calculus course for future scientists and engineers, what is the profile of a typical student's understanding of limit when interpreted through the framework lens of the TIFI limit model?

Specific details about the manner in which the analyzed notes were first created can be found in Chapter 3. At present, I only mention the fact that the notes (and hence, the profile results) for each LUA item were based on the responses given by no less than 45 but by no more than 58 students to that particular item. The rubrics used to score each LUA item can be found in an appendix. In the succeeding paragraphs, I present the results related to the secondary research question, organized by the component competencies of the TIFI limit model which were measured on the LUA.

Translation

Two items of the LUA were designed to measure the component competency of translation from the TIFI limit model. These items, in their LUA Version D form, appear in an appendix. Hereafter, I will refer to them as item T_1 and item T_2, respectively. In trying to answer T_1, each student faced the challenge of trying to compute limits in which the functions involved were not "well behaved," making the limit evaluations not necessarily obvious. In contrast, a student had to create a graphical representation of his own function when tackling T_2, and hence the typical student kept his chosen graphed function simple. I now examine each of the two items separately.

First Item. For item T_1, students tried to evaluate four graphical limits each in two separate graphs. Table 12 is a matrix of the relative frequencies of correct versus incorrect answers students gave in response to the limit statements for the first graph (graph A) while Table 13 is a matrix containing the same information only for the second graph (graph B). Since approximately 74% of all responses overall were correct, I postulate that the typical student is fairly competent at graphical limit evaluations, even "unusual" ones like those in T_1. However, a certain level of inconsistency was found in many students' answers. For example, a student might have indicated on part (i) of T_1 that $\lim_{x \to -3} f(x) = 2$ in graph A, yet declared on part (ii) that $\lim_{x \to -3} f(x)$ does not exist in

Table 12. Percentages of Responses to Graph A of Item T_1.

Limit	Correct	Incorrect; consistent reasoning	Incorrect; inconsistent reasoning
	Type of Response		
$x \to -3$	71	15	15
$x \to 0$	77	4	19
$x \to 4$	56	35	8
$x \to +\infty$	90	6	4
Overall	73	15	11

Note: Percentages are rounded.

Table 13. Percentages of Responses to Graph B of Item T_1.

Limit	Correct	Incorrect; consistent reasoning	Incorrect; inconsistent reasoning
	Type of Response		
$x \to -3$	65	10	25
$x \to 0$	79	8	13
$x \to 4$	75	13	13
$x \to +\infty$	83	6	10
Overall	76	9	15

Note: Percentages are rounded.

graph A. While the exact reason for such inconsistencies is not entirely clear, there are at least three distinct possibilities worth mentioning. First, a student who gave inconsistent responses may not be aware that limits are unique. For example, the student may think that a limit can be simultaneously equal to both 1 and 2, or even that a limit can be equal to 2 and yet still not exist. Second, a student whose answers were not consistent could be completely unaware of the inconsistency in her own reasoning. Using the terminology of Tall and Vinner (1981), such a student may have evoked a different part of her concept image of limit when doing part (i) of T_1 than the part of it she evoked when completing part (ii). Third, a student may have wanted to hedge his bet, so to speak, on what a particular limit's value was in the hopes of receiving partial credit.

It is interesting to note that the limit evaluation which tripped up the most students was the limit as $x \to 4$ in graph A. I suspect that few students had been previously exposed to a limit situation in which the function is not defined in a deleted neighborhood of a. Likely tempted by the single point $(4, 1)$ appearing in graph A, many decided that the limit existed and was equal to 1. Such students might have been mentally "filling in the gap" in the graph, or they might have been falling back on a naive conceptualization of limit (as defined by Davis and Vinner, 1986) they had had before instruction and hence just evaluated $f(4)$.

<u>Second Item.</u> Relatively few mistakes were made by students on T_2, as can be seen in Table 14. Furthermore, of the incorrect responses recorded, just over two-thirds of them involved wrong numerical values. For example, the student's graphed function satisfied the condition that $f(x) \to -\infty$ as $x \to 3^+$ rather than the required condition—namely, that $f(x) \to -\infty$ as $x \to -3^+$. As another example, the student's graphed function contained the point $(-1, 0)$ rather than the point $(0, -1)$. It is probable that in

Table 14. Percentages of Responses to Item T_2.

	Type of Response			
Condition*	Correct**	Wrong numeric value(s)	Multiple functions drawn	Graph partially missing
1	92	6	2	0
2 and 3	81	12	8	0
4 and 5	75	19	4	2
6 and 7	92	4	0	4
Overall	85	10	3	1

***** Conditions are listed in the same order in which they appear in T_2.

****** Students in this column responded to both conditions correctly (assuming two are listed).

Note: Percentages are rounded.

many of these cases, the student simply misread what the condition was saying and could demonstrate correct understanding if given another chance.

I am unsure as to why a few graphs were incomplete (i.e., parts of the graph were missing). On the other hand, there is a reasonable explanation as to why some students drew multiple functions on the same graph. While there is a small chance one or more of these students do not understand the function property of univaluedness (i.e. there is only one y-value for every x in the domain), it is much more plausible that these students simply tackled the listed conditions one or two at time rather than all seven conditions at once, not realizing that that was the required task. Whether or not they could have been successful had they realized this is unknown.

For the group of profiled students for item T_2, the mean number of condition groups listed in Table 14 that each student's response satisfied was about 3.4. Additionally, it is worth mentioning that every student in this profile group gave a response to T_2 satisfying at least one of the condition groups listed in Table 14.

Translation Summary. Translating symbolic limit statements into suitable graphical form is well within the grasp of the typical Math 181 student. On the other hand, matching seemingly unusual graphical limits with corresponding symbolic limit statements is more difficult for many students, and inconsistent reasoning is extant in quite a few students' matching attempts. The average student's lack of a mathematically sound personal concept definition for limit may be the reason it is easier for him to create an uncomplicated graph of a function satisfying limit statements than it is to evaluate strange looking limits.

Interpretation

Two items of the LUA were designed to measure the component competency of interpretation from the TIFI limit model. These items, in their LUA Version D form, appear in an appendix. I will refer to them in the succeeding paragraphs as, respectively, item P_1 and item P_2. Students were asked to summarize a real-life situation using a suitable limit statement in P_1, while they had to do the reverse in order to answer P_2. A separate examination of each of the two items is now in order.

First Item. Appearing in Table 15 are the relative frequencies of students' responses (correct or incorrect) to both parts (i) and part (ii) of P_1. It should be noted that part of the sample of students chosen to create the profile for this item was Section 3, a section of students with lower mean LUA scores than other sections. In particular,

Table 15. Percentages of Responses to Item P_1.

	Type of Response		
Part	Correct*	Incorrect limit statement given	No limit statement given
(i)	13	73	13
(ii)	18	62	20

* Students in this column may have made minor mistakes that were not scored as incorrect.
Note: Percentages are rounded.

students enrolled in Section 3 scored (on average) approximately one-half point lower than their peers on item P_1. This has an influence on the percentages in Table 15 and in the subsequent tables to be presented for this item; however, the typical mistakes made by students in Section 3 did not differ significantly from those made by students in other sections.

Because the manner in which students wrote incorrect limit statements varied, and these variations are of interest in making a detailed profile of what Math 181 students understand about limit, I created Tables 16 and 17 to delineate the differences in students' incorrect limit statement responses on part (i) and on part (ii), respectively, for those students who fell into that category of Table 15.

Table 16. Percentages Related Only to Incorrect Limit Statements on Part (i) of Item P_1.

Type of response	Percentage
Value of a incorrect	85
Value of L incorrect	52
Phrase "lim" missing	9
Equals sign missing	3

Note: Percentages are rounded. Furthermore, types of response are *not* mutually exclusive.

Table 17. Percentages Related Only to Incorrect Limit Statements on Part (ii) of Item P_1.

Type of response	Percentage
Value of a incorrect	21
Value of L incorrect	71
Phrase "lim" missing	18
Equals sign missing	21
Other	7

Note: Percentages are rounded. Furthermore, types of response are *not* mutually exclusive.

One salient point in the results is that not very many students could correctly create limit statements which encapsulated the given information. In particular, quite a few students wrote $\lim_{p\to 0}$ as part of their answer to part (i) rather than $\lim_{p\to 100}$, but then proceeded to write $\lim_{p\to 0}$ again as part of their answer to part (ii). I submit that one possible explanation for this seeming inconsistency is that students interpreted the variable p as representing the percentage of contaminant remaining in the well water for part (i), but then interpreted the same variable p as representing the contaminant removed from the well water for part (ii). Also noteworthy, many students who responded to part (ii) correctly wrote 5 (or 5,000,000) as the limiting value L, but then added a "ghost" term to L like ∞ or $C(p)$ itself (e.g., a student's answer to part (ii) looked like $\lim_{p\to 0} C(p) = 5 + \infty$). This might have been due to the students' confusion as to what p represents and the fact that $L = \infty$ in part (i). On the other hand, for students including an extra $C(p)$ term, the mistake might have entailed their not believing p actually reaches 0, and so they included a term representing some "leftover" additional cost which is not actually there.

Although writing a two-sided limit was not scored as incorrect on P_1, it is somewhat surprising how few students' responses actually were one-sided limits. In fact, of the responses which contained the correct value for a, whether it be for part (i) or for part (ii), only about 15% correctly involved a one-sided limit. Also of note, though again not scored as incorrect, is the number of students who substituted x in the place of p in their responses, or even $f(x)$ in the place of $C(p)$. Specifically, of the correct responses listed in Table 15, approximately 21% contained this type of error.

Second Item. Comparatively, students fared better on P_2 than they did on P_1. This fact can be verified by looking at Tables 18 and 19, which classify students' responses on part (i) and on part (ii) of P_2, respectively. Among students' responses containing mistakes, the most common error was vagueness. In other words, while most students mentioned both time and money (sales) in their answers to parts (i) and (ii), a large number did not mention the specific numeric value(s) of the limit statement which were related to time and/or money. Based on my own experience with the Math 181 curriculum, students are not generally given many opportunities to answer questions like

Table 18. Percentages of Responses to Part (i) of Item P_2.

Type of response	Percentage
Correct	63
Time not fully explained	31
Money not fully explained	13
Confused time and money	2
Other	2

Note: Percentages are rounded. Furthermore, types of response are *not* mutually exclusive (other than correct responses versus incorrect ones).

Table 19. Percentages of Responses to Part (ii) of Item P_2.

Type of response	Percentage
Correct	40
Time not fully explained	25
Money not fully explained	31
Confused time and money	13
Penalty of $30 rather than sale	10
Other	4

Note: Percentages are rounded. Furthermore, types of response are *not* mutually exclusive (other than correct responses versus incorrect ones).

those in P_2. With the majority of respondents answering at least one of part (i) or part (ii) correctly, I conjecture that with just a little more practice, students could be highly successful on this item, and answering this type of problem might come more naturally for Math 181 students than answering problems like P_1.

One interesting tidbit I would like to add about part (i) of P_2 concerns the validity of models. Clearly, there are not an infinite amount of minutes in a month, and yet many students' responses to part (i) implied this to be the case. When designing this LUA item, I purposely let $t \to +\infty$ on part (i) rather than computing how many minutes there are in a month, because I wanted to illustrate the point that while functions are very useful in modeling real-life situations, such modeling functions have limitations. In this case, the apparent "problem" with the limit statement is actually just a limitation of the modeling function $S(t)$. In the entire cohort of students who took the LUA, there were at least a few whose responses to part (i) of P_2 implied they were very much aware of that fact.

Interpretation Summary. While the typical Math 181 student can at least partially interpret contextually founded limit statements in real-life terms, she is not nearly as adept at modeling real-life situations using appropriate limit statements. Furthermore, the average student has trouble in consistently understanding the role each variable plays when creating her own model limit statements. She may also harbor the belief that limits are unreachable, hindering her ability to appropriately create limit statements.

Familiarization with the Formal

Just one item of the LUA was designed to measure the component competency of familiarization with the formal from the TIFI limit model—although this item did include four parts. The item as shown in an appendix is in its LUA Version D form. I will refer to the item in its entirety as item F_1. In parts (i) and (ii) of F_1, students were asked to graphically relate a given ε to several given δ's or to do the reverse, relating a given δ to several given ε's. For part (iii), students attempted to numerically compute a sufficiently small δ for a given ε, again with the help of a graph. Finally, in part (iv), students were prompted to circle the graph of each function for which a single statement akin to the formal definition of limit, but with all parameters specified, is true. Because the required manipulative skills with the formal definition were not uniform from part to part in F_1, it is logical to discuss the different parts of the item separately.

First and Second Parts. Based on the relative frequencies of responses given in Table 20, it appears that the typical Math 181 student does not have a good understanding of what how ε and δ are "supposed" to relate to one another. In fact, no one answered

Table 20. Percentages of Responses to Parts (i) and (ii) of Item F_1.

		Circled in part (i)		
		δ_1	δ_1 and δ_2	Other
Circled in part (ii)	ε_1	22	2	4
	ε_1 and ε_2	22	9	0
	ε_2 without lines drawn in	4	2	0
	ε_2 with lines drawn in	15	11	0
	ε_2 and ε_3	0	2	0
	Other	2	2	2

Note: Percentages are rounded.

both parts (i) and (ii) correctly in this group. The question then becomes what exactly were students thinking when they answered parts (i) and (ii) of F_1.

It is possible, and perhaps even probable, that students who circled δ_1 who also circled ε_1 believe that in general, smaller is better when it comes to both δ *and* ε. If they also circled either δ_2 or ε_2 or both, their instinct to select small was likely tempered by some understanding that δ and ε are somehow related. On the other hand, students who circled only ε_2 in part (ii) who also drew in lines on the corresponding graph were probably more influenced in their selection by the relationship between δ and ε more than they were by the size of the two parameters. This may also be true for students who circled only ε_2 in part (ii) who did not draw in lines on the corresponding graph, though such students' thinking is less clear in that case.

Third Part. It may be that responses to part (iii) were skewed by the inclusion of the word "largest" in the problem. In my defense, I included the word to ensure that students did not assume they could receive credit for guessing a small number like 0.001 that just happens to work. (It is likely that many Math 181 students are aware that if a particular δ works for a given ε, any smaller δ also works, which could logically lead to this guessing scenario.) However, the inclusion of the phrase "justify your answer by showing all work" is reasonably sufficient to prevent this type of guessing, so the word "largest" could have been left out.

At any rate, the relative frequencies of answers students gave to part (iii) of F_1 are shown in Table 21. The large percentage of students answering $1/9$ or $10/9$ leads me to believe at least some students were confused by the word "largest." Of course, some of the students writing $1/9$ or $10/9$ as their answer might not have bothered to compute a value on both sides of the $x = 1$, which is an error that has nothing to do with the directions. In all likelihood, most students who wrote $1/11$ on part (iii) realized that they

Table 21. Percentages of Numeric Answers to Part (iii) of Item F_1.

Answer	Percentage
Correct* ($1/11$)	20
$1/9$	28
$10/9$	4
$10/11$	2
0.1, 0.9, or 1.1	15
Other	30

***** Responses of $-1/11$, while technically incorrect, were included in this category.
Note: Percentages are rounded.

needed to pick the smaller distance between $1/9$ and $1/11$. Students answering $10/11$ were probably aware, for the most part, of this fact as well—although such students demonstrated an erroneous belief that δ refers to a particular *x*-value rather than a specific distance on the *x*-axis. (This mistaken belief is also indicated in answers of $10/9$). If a student responded with 0.1, 0.9, or 1.1 on part (iii), I submit that such a student either confused δ with ε or confused *x*-values with *y*-values. Last but not least, a large number of students provided answers which did not follow a systematic pattern I felt I could safely analyze, and so their responses are categorized under “other” in Table 21.

Fourth Part. Less than half the group whose responses to part (iv) of F_1 are included in Table 22 demonstrated a detectable level of understanding of this problem. While about 61% of responses to part (iv) of F_1 involved the circling of two or three of the graphs B, C, and E or the circling of graph E alone—which I considered to be

Table 22. Percentages of Responses to Part (iv) of Item F_1.

Circled graphs	Percentage
Two or three of B, C, E *or* E alone *with some* correct reasoning	43
Two or three of B, C, E *or* E alone *without any* correct reasoning	17
Any other combination of graphs	37
Blank	2

Note: Percentages are rounded.

necessary to prove at least a minimal understanding of the problem—about 29% of the students who gave one of these looked-for responses failed to show any logical reasoning as to *why* they circled the graphs they circled. Hence, I only considered the 43% of responses corresponding to row 1 of Table 22—i.e., those responses which did reveal some degree of understanding of the problem—for further analysis.

As a springboard for that analysis, I present Table 23, which further categorizes the responses corresponding to the first row of Table 22. While the if-then statement of part (iv) was not intended to imply that the function involved needed to satisfy the limit statement $\lim_{x \to 5} f(x) = 3$, many students inferred that was indeed the case, as shown by the

Table 23. Percentages Related Only to Responses Demonstrating Some Understanding of Part (iv) of Item F_1.

Circled graphs	Percentage
C & E	60
E alone	10
B, C, & E	15
Correct (B & E)	15

Note: Percentages are rounded.

relatively large number of responses in Table 23 which had graphs C and E circled (but not graph B). Although it should be noted that several of these responses included the drawing in of the correct vertical lines and horizontal lines—namely, the lines $x = 3$, $x = 7$, $y = 2$, and $y = 4$—the failure to select graph B while still choosing graph C is an indicator that the inferred limit statement was of primary importance in the thinking of these students rather than what the drawn-in lines actually signified. On the other hand, students who circled E alone may very well have not only inferred the limit statement but also have been cognizant of the significance of the drawn-in lines, and hence refused to pick graph B or graph C.

Of course, a seasoned mathematician reading the if-then statement would have realized that the statement can be true even if $\lim_{x \to 5} f(x)$ does not exist, and so the correct graphs to circle are B and E. Nevertheless, it is hardly to be expected that Math 181 students would, in general, be aware of that subtlety. A typical student is probably not keenly aware of the difference between an if-then statement with a specified ε and δ

versus the actual formal definition of limit, which contains (of necessity) an unspecified ε and δ. Regardless, some students did in fact select B and E, and such students might possibly have completely understood the if-then statement. It is impossible to know whether or not students who chose B, C, and E were more mindful of the inferred limit statement or the correctly drawn-in lines, though my instinct is to assume the former is true. If I am right with that conjecture, it could be an instance where the students involved thought $\lim_{x \to 5} f(x)$ was "close enough" to equaling 3 in graph B to warrant its selection. However, I admit this hypothesis is on somewhat shaky ground, with no hard evidence to verify its accuracy.

Familiarization with the Formal Summary. Although the typical Math 181 student understands that δ and ε are inextricably connected in the formal definition of limit, he is not entirely aware of the exact nature of the parameters' relationship. For the student scoring in the top half on part (iv) of LUA item F_1, a holistic understanding of the formal limit definition is probably present in his mind, but he would likely fail in trying to give a detailed analysis of the if-then statement of the formal limit definition.

Instantiation

Three items of the LUA were designed to measure the component competency of instantiation from the TIFI limit model. In an appendix, these items are shown in their LUA Version D form, and I will refer to them hereafter as S_1, S_2, and S_3, respectively. For item S_1, students were asked to decide whether limit statements involving a function or functions satisfying certain conditions were either true or false. Item S_2, on the other hand, was a short answer question in which students were prompted to analyze a limit computation "argument" given by a hypothetical student to determine its veracity or fallaciousness. Finally, students were asked to create a formula for a piecewise-defined function satisfying a few limit statement conditions in item S_3. I now examine each of the three items separately.

First Item. Four groups of true/false statements make up item S_1, each consisting of two statements related to the same definition of or theorem about limits. In particular, the first group, known hereafter as group A, has to do with an informal definition of limit; the second, group B, concerns the multiplication rule of limits; the third, group C, tests students' aptitude with so-called indeterminate quotients; and the fourth, group D, is associated with the squeeze theorem. The groups are placed in order in the problem itself (e.g., the first two true/false statements make up group A, the third and fourth true/false statements constitute group B, and so on). The relative success students had with each of these groups can be found in Table 24. For the sample of students whose responses are in the table, the mean number of correct responses per statement group is 1.04. Since the expected value of correct answers based on random guessing for any one group is 1 (out of 2), it is safe to say that students, in general, fared somewhat poorly on S_1.

Since Math 181 students have many opportunities to compute indeterminate quotients during their study of limits, I find it doubtful that students were mentally instantiating specific examples of function with the necessary limit properties when responding to the statements in group C, considering the low percentage of students that judged both statements correctly. In fact, I am inclined to wonder if some students had

Table 24. Percentages of Responses to Item S_1.

Group of true/false limit statements		Statements correctly circled			
		Both	First only	Second only	Neither
	A	39	22	22	16
	B	53	24	14	8
	C	8	18	27	47
	D	14	51	6	29

Note: Percentages are rounded.

specific examples in mind when they responded to any of these statements. It is possible that the seeming relative success students had with the true/false statements in groups A and B could have come about without the students' necessarily having particular limit examples in mind—though they ostensibly would have had to have some understanding of an informal (yet appositely rigorous) definition of limit and of the multiplication rule to rightly respond to group A and group B statements, respectively.

On the other hand, most of the squeeze theorem homework exercises in Stewart's (2008) textbook involved periodic functions like sine and cosine, which might explain students' strong showing on the first statement of group D. In this case, many students may have been thinking about the sine or cosine function in particular, which would constitute an appropriate instantiation of a specific function for the limit situation described in the statement. This agreeable possibility aside, either the parenthetical note in the directions of item S_1 did not, in general, produce the desired effect of convincing students to try to come up with fitting examples from which to decide whether a given limit statement was true or false, or the students were simply largely unsuccessful in attempting to do so.

Second Item. The remarkably wide variety of responses on S_2 led me to create broad categories of students' responses from which to judge the relative understanding of limit demonstrated in each individual student's response to the item. In other words, these categories were the foundation for the S_2 rubric developed for LUA Version D. The categories and their corresponding definitions are reproduced in Figure 29. Following that, Table 25 shows the relative frequencies of categorized responses for students in the profile group for S_2. Approximately 66% of these students' responses corresponded to categories A through E, and since I consider responses in these categories to be demonstrative of some genuine understanding of limit, I submit that most Math 181 students had an easier time establishing their conceptual limit understanding in S_2 than they did in S_1.

When it comes to revealing either a strong dynamic conception of limit—one with a "definite feeling of motion" (Tall and Vinner, 1981, p. 161)—or a comparably effective intuitive static conception of limit based on the idea of "closeness," it appears that students' responses corresponding to categories A through C, as well as some students' responses corresponding to category E perhaps, show that these students know the limit in question concerns what happens as one approaches zero (or equally valid, concerns the deleted neighborhood of zero) rather than what happens at zero. This is an encouraging

Category A: A student whose response falls under this category does these two things: (a) he disagrees with Chad's answer; (b) he claims that the exponent is unbounded as $x \to 0$. Note that although a student may couch this idea in technically incorrect terms like "the limit equals 1^{∞}," his basic understanding of the limit is arguably sound, so his answer falls under the best category.

Category B: A student whose response falls under this category does these two things: (a) he agrees with Chad's answer; (b) he claims that the exponent is unbounded as $x \to 0$. A student whose response fits this category understands the exponent is getting absolutely large, but supposes this does not change the fact that the limit is still just 1.

Category C: A student whose response falls under this category does these two things: (a) he disagrees with Chad's answer; (b) he claims that Chad cannot "plug in" zero for x because x is approaching zero, not equal to it. A student whose response fits this category understands that this limit is about what happens near $x = 0$ rather than what happens at $x = 0$, but he does not recognize the importance of the exponent's unbounded behavior to the limit evaluation.

Category D: A student whose response falls under this category does these two things: (a) he disagrees with Chad's answer; (b) he claims that Chad cannot "plug in" zero for x because $\frac{1}{0}$ is undefined. Note that a student may state this in a different way (e.g., $\sqrt[0]{1}$ does not exist, the exponent is undefined, etc.) A student whose response fits this category understands that Chad cannot "plug in" zero in for x in the base and not do so for x in the exponent, but he is not apparently cognizant of the fact that the limit is not actually about what happens at $x = 0$.

Category E: A student whose response falls under this category does these two things: (a) he agrees with Chad's answer; (b) he admits that the $\frac{1}{0}$ "problem" exists but does not ultimately matter. It is probable that such a student believes that the x in the base somehow goes to zero before the x in the exponent does.

Category F: A student whose response falls under this category disagrees with Chad's answer and does one of the following things: (a) he argues that Chad must simplify the expression before evaluating the limit; (b) he makes one correct computation of the expression for some *x*-value near zero and concludes the limit is not 1 based on that computation. A student whose response fits this category demonstrates some "knowledge" of algebra, but no real understanding of the limit concept of calculus.

Category G: A student whose response falls under this category either agrees or disagrees with Chad's answer, but his reasoning is nonsensical, nonexistent, or it does not fit one of the basic "logical" themes from the previous categories. With such a response, the student demonstrates no real understanding of the limit concept.

Category H: A student who fails to write any response down whatsoever has that "response" placed in this category.

Figure 29. Definitions of categories used to score item S_2.

Table 25. Percentages of Responses to Item S_2.

Category	Percentage
A	9
B	7
C	12
D	28
E	10
F	5
G	26
H	3

Note: Percentages are rounded.

development in the mental construction of the limit concept which is being demonstrated by these students. In contrast, students whose responses fell into category D as well as some students whose responses fell into category E probably have not have developed as sophisticated a limit conception as their category A through C counterparts, while students whose responses corresponded to categories F through H may not have much of a useful limit conception whatsoever at this point.

Third Item. About half of the students from Section 3, a section of students with lower mean LUA scores than other sections, were chosen as part of the sample of students used to create the profile for item S_3. However, students enrolled in Section 3 scored (on average) only negligibly lower than their peers on S_3. Hence, the percentages in the subsequent tables to be presented for this item are not likely to be very different than what would have been found had Section 3 not been included in the profile created for S_3.

Unlike T_2, an item in which students were prompted to come up with their own graphed function satisfying a list of limit conditions, students were only somewhat successful in coming up with a function formula satisfying a list, albeit a shorter one, of limit conditions in item S_3. The relative frequencies of profiled students' responses to S_3 are shown in Table 26. Interestingly, nearly all of the students whose responses

Table 26. Percentages of Responses to Item S_3.

	Type of Response				
Condition*	Correct	Wrong numeric value	Multiple functions defined	Bad or missing definition	Other**
1	62	9	9	9	
2	56	27	2	4	
3	67	16	2	4	
4	40	36	4	9	
Overall	56	22	4	7	11

***** Conditions are listed in the same order in which they appear in S_3.
****** Students in this column made a mistake which left all four conditions unsatisfied.
Note: Percentages are rounded.

satisfied both the second and third conditions used a formula of "$-x+3$" or "x^2+3" for $x<0$ in their piecewise-defined function. Virtually all of the student responses satisfying the fourth condition consisted of a formula of an aptly chosen rational function like "$2-\frac{1}{x}$" for $x>0$. It would not be a great stretch to surmise that students are more comfortable with algebraic functions than with transcendental functions, especially considering the relative time spent on algebraic functions versus the time spent working with transcendental functions in high school.

Speaking of algebraic functions, many students tried a formula like "$\sqrt{x}$" for the piece corresponding to $x > 0$, and there is some evidence in the responses to suggest some of these students believed that the graph of $y = \sqrt{x}$ somehow has a horizontal asymptote as $x \to +\infty$. It may be just speculation on my part, but I theorize that some Math 181 students equate an increasing function having downward concavity with the existence of a horizontal asymptote. Such an erroneous belief might be due to lack of experience analyzing the end behavior of function graphs, but then again, another unknown phenomenon may be the culprit in this case.

For the group of profiled students for item S_3, the mean number of conditions displayed in Table 26 that each student's response fulfilled was about 2.3. Furthermore, about 24% of all profiled students wrote down a piecewise-defined formula which satisfied all the listed conditions, while only 13% of these students gave an answer which satisfied none of the listed conditions.

<u>Instantiation Summary.</u> An inability to instantiate apposite examples of limits (of suitably chosen functions) inhibits the typical Math 181 student's capability to decide whether or not general limit statements are true or false. While the average student can create a formula for a function which satisfies some or all of a short list of limit statement requirements, her chosen formula is likely to demonstrate a decided preference for

algebraic functions over transcendental ones. Finally, about one-third of Math 181 students are well on the way to having a mature dynamic or static conception of limit which allows them to correctly critique someone else's limit arguments, another one-third are somewhat behind these but are making reasonable progress in constructing such a mental conception of limit, and another one-third do not appear to have made measurable development of such a limit conception.

Profile Summary

The Limit Understanding Assessment (LUA) was designed to measure students' understanding of limit from the perspective of the TIFI limit model, which includes four component competency areas of translation, interpretation, familiarization with the formal, and instantiation. Using the responses of various subsamples of students enrolled in Math 181 in the fall of 2008 for each LUA item, a descriptive profile of what the typical student understands about limit in each component competency was formulated. For convenience, I now replicate the summary paragraphs, in order, of the profile of students' understanding of limit in each component competency from the TIFI limit model.

Translating symbolic limit statements into suitable graphical form is well within the grasp of the typical Math 181 student. On the other hand, matching seemingly unusual graphical limits with corresponding symbolic limit statements is more difficult for many students, and inconsistent reasoning is extant in quite a few students' matching attempts. The average student's lack of a mathematically sound personal concept definition for limit may be the reason it is easier for him to create an uncomplicated graph of a function satisfying limit statements than it is to evaluate strange looking limits.

While the typical Math 181 student can at least partially interpret contextually founded limit statements in real-life terms, she is not nearly as adept at modeling real-life situations using appropriate limit statements. Furthermore, the average student has trouble in consistently understanding the role each variable plays when creating her own model limit statements. She may also harbor the belief that limits are unreachable, hindering her ability to appropriately create limit statements.

Although the typical Math 181 student understands that δ and ε are inextricably connected in the formal definition of limit, he is not entirely aware of the exact nature of the parameters' relationship. For the student scoring in the top half on part (iv) of LUA item F_1, a holistic understanding of the formal limit definition is probably present in his mind, but he would likely fail in trying to give a detailed analysis of the if-then statement of the formal limit definition.

An inability to instantiate apposite examples of limits (of suitably chosen functions) inhibits the typical Math 181 student's capability to decide whether or not general limit statements are true or false. While the average student can create a formula for a function which satisfies some or all of a short list of limit statement requirements, her chosen formula is likely to demonstrate a decided preference for algebraic functions over transcendental ones. Finally, about one-third of Math 181 students are well on the way to having a mature dynamic or static conception of limit which allows them to correctly critique someone else's limit arguments, another one-third are somewhat behind these but are making reasonable progress in constructing such a mental conception of limit, and another one-third do not appear to have made measurable development of such a limit conception.

Summary of Important Results

A list of the research questions answered via the major research study can be found in Figures 5 and 6 of Chapter 1, but for convenience, I reproduce those figures again along with synoptic answers to each question in Figure 30. There is a strong positive linear correlation between Math 181 students' understanding of function as measured by the Precalculus Concept Assessment (PCA) and their understanding of limit as measured by the Limit Understanding Assessment (LUA), even after accounting for students' ACT mathematics scores and the section number in which each student was enrolled. A full 32% of the variance in LUA scores is explained by the variance in PCA scores alone, which proves the statistical explanatory power of PCA scores on LUA scores is substantial. However, I caution that the present research methodology does not allow for the conclusion that greater understanding of function actually causes greater understanding of limit.

There is a monotonically increasing relationship between scores on each of the four component competencies from the TIFI limit model and PCA scores, but the exact nature of that upward trend can only be partially inferred. It does appear that the relationship with PCA scores is different for each component competency. Again, no cause-and-effect conclusions can be established given the research methodology.

(1) How strong is the correlation between students' scores on the PCA and their scores on the LUA?

The positive linear correlation between the two is very strong, even after accounting for ACT and section number.

(2) How strong is the correlation between students' scores on the PCA and their scores on the portion of the LUA that measures understanding of limit:

- (a) in the component competency of translation?
- (b) in the component competency of interpretation?
- (c) in the component competency of instantiation?
- (d) in the component competency of familiarization as it applies to the formal definition of limit?

The bivariate correlations are at least moderately strong in each case.

(3) In the context of a traditional calculus course for future scientists and engineers, what is the profile of a typical student's understanding of limit when interpreted through the framework lens of the TIFI limit model?

A quick answer to this question is not possible to put here, but the section entitled "Profile Summary" provides an economical synopsis of the desired profile.

Figure 30. Major research study questions along with synoptic answers.

A useful description of what students' responses to individual LUA items might actually mean in the context of limit understanding has been developed. Instead of rehashing the most salient points in this profile of a typical introductory calculus student's understanding of limit, I encourage the reading of the section entitled "Profile Summary" in order to get a general idea of what the created profile suggests about students' limit understanding.

CHAPTER 5

CONCLUSIONS

Introduction

In this final chapter, I present a brief summary of the objective of the research study. This is followed by a setting forth of how the present research results support and extend the findings of previous research. This study has important implications for practice and provides a foundation for future research work, which is discussed subsequently. A concluding section integrates the constituent threads of the chapter together.

Basic Research Objective

Fundamental Ideas

Introductory calculus students are often successful in doing procedural tasks in calculus even when their understanding of the underlying concepts is lacking (Cornu, 1991; Tall, 1997). These conceptual difficulties extend to the limit concept (Tall and Vinner, 1981), which is foundational to all of modern standard analysis as well as to the traditional pedagogical treatment of introductory calculus. The questions of how students regard the limit concept and how mathematics educators can teach this concept better so their students can more fully and deeply comprehend its meaning are questions worth answering.

Since the concept of limit in introductory calculus usually concerns a process applied to a single function, it seems reasonable to believe that a robust understanding of function is beneficial to and perhaps necessary for a meaningful understanding of limit. The logicality of this belief is supported by a genetic decomposition of how the limit concept in calculus can be learned—proposed by Cottrill and his colleagues (1996)—which is summarized in Figure 1 of Chapter 1. In particular, the first three steps of the genetic decomposition of limit as listed in Figure 1 appear to be directly related to a student's understanding of function, so it may not be an exaggeration to state that understanding the function concept is necessary to grasping the limit concept. With that in mind, the main goal of the present research is to quantitatively correlate students' understanding of function and their understanding of limit.

Research Questions

The basic objective of the present research study is to ascertain the strength of the relationship between understanding of the function concept and understanding of the limit concept. There are five specific research questions related to this objective, the first two which were answered via a pilot study, the last three which were answered via the major study. These questions are as follows:

1. Does the Limit Understanding Assessment (LUA) provide a reliable measure of students' understanding of the limit concept? In particular, does the LUA demonstrate reliability via:
    (a) clear, justifiable standards in its scoring?
    (b) adequate discrimination among students with different levels of understanding of the limit concept?
    (c) internal consistency among the test items?

2. Does the LUA provide a valid measure of students' understanding of the limit concept? In particular, does the LUA demonstrate validity via:
   (a) appropriate content for the test items?
   (b) reasonable interpretation of test items from students taking the test?
   (c) acceptable categorization of test items based on the component competencies of the TIFI limit model?

3. How strong is the correlation between students' scores on the PCA and their scores on the LUA?

4. How strong is the correlation between students' scores on the PCA and their scores on the portion of the LUA that measures understanding of limit:
   (a) in the component competency of translation?
   (b) in the component competency of interpretation?
   (c) in the component competency of instantiation?
   (d) in the component competency of familiarization as it applies to the formal definition of limit?

5. In the context of a traditional calculus course for future scientists and engineers, what is the profile of a typical student's understanding of limit when interpreted through the framework lens of the TIFI limit model?

Discussion of Results

This section is divided into three major subsections, each dedicated to an explanation of how the results of the present research study support and extend the findings of previous research studies. In the first major subsection, I discuss the value of REC theory and its associated models in mathematics education research. In the subsequent two major subsections, I consider the ways in which the results of the present research contribute to knowledge about the connection between students' understanding of function and their understanding of limit and also to additional knowledge which exclusively concerns introductory calculus students' understanding of limit.

Substantiation of a Theory of
Mathematical Understanding

The present research has substantiated the general value of referential extension and consolidation (REC) theory—an extension of a theory originally postulated by Kaput (1989)—to mathematics education research. Conceptual understanding models built on the foundation of REC theory are able to provide more complete descriptions of students' understanding of particular mathematical concepts than those employed in previous research work. The models of understanding founded upon REC theory which were built for this paper—namely, the TIRI function model and TIFI limit model—have increased the clarity with which previous research on understanding of function and on understanding of limit can be viewed. Furthermore, the TIFI limit model has supplied a sufficiently broad yet still effectively sharp focus for gauging introductory calculus students' understanding of limit, as illustrated by the Limit Understanding Assessment (LUA) created in the light of the model's framework. Hence, it seems desirable to utilize REC theory in mathematics education research involving function, limit, or other mathematical concepts for which researchers need data as to how well particular groups of students truly understand those concepts.

In applying the TIFI limit model to future projects, researchers should be aware that I experienced much more difficulty in devising reliable and valid questions in the component competency of interpretation than in the other three component competencies when designing the LUA. As implied in Table 11 of Chapter 4, there appeared to be more "noise"—i.e., unexplained variance—in students' interpretation scores than in their scores on the rest of the LUA. At least two factors are probably at work here. First, as mentioned in the section on interrater reliability of Chapter 3, the scoring of interpretation items seems naturally more subjective because it often requires the grader to try and infer what a student meant to write rather than what he actually wrote. In other words, students' general ability in writing the English language and in composing mathematical statements seems to be a confounding factor when scoring interpretation items. Second, the creation of a relatively simple real-life situation involving limits that does not require a priori knowledge in other areas is not straightforward. The item from LUA Version B which is displayed in Figure 20 of Chapter 3 is just one example of how a problem about limits can be muddled by context (in this case, economics). It might be argued that the basic idea of limits arises most naturally in the context of physics, as this is in fact how the concept arose historically. However, many introductory calculus students do not have the background in physics to attempt to solve the same problems tackled by Newton and his contemporaries. In short, measuring the component competency of interpretation from the TIFI limit model does pose some problems, which is something to be aware of when employing the TIFI limit model.

Contribution to Research on Connection between Function Understanding and Limit Understanding

Periodically in the next two sections, I make reference to individual items on LUA Version D using the same names given to them back in Chapter 4. For convenience, all LUA Version D items appear in an appendix.

In a paper discussing the creation and validation of the PCA, Carlson and her colleagues (under review) stated that “77% of the students scoring 13 [out of 25] or higher on the [PCA] passed the first-semester calculus course with a grade of C or better while 60% of the students scoring 12 or lower failed with a D or F or withdrew from the course” (p. 26). Notably, among the 234 students enrolled in Math 181 in the fall of 2008 who scored 13 or higher on the PCA, the mean LUA score was about 13.8 (out of 25), while among the 95 students who scored 12 or lower on the PCA, the mean LUA score was only about 9.8, a difference in scores of almost one full standard deviation $(\sigma = 4.21)$. Hence, the PCA not only discriminates well among students who successfully finish first-semester calculus (during a particular semester) versus those who do not, it also provides a benchmark in predicting whether or not an individual introductory calculus student can mentally construct a meaningful conception of limit. This supports the idea that understanding of function is a significant predictor of future understanding of limit. The question then becomes the following: in exactly what manner are students’ understanding of function and their understanding of limit related?

The required elements in reifying a mathematical concept are described in detail by Sfard in her work on process-object duality theory. In regards to students’ understanding of function, consider the statement in Figure 31 given by Sfard (1991), which seems particularly appropriate to the present research. Although the present research is only a preliminary step in establishing the important relationship between the concepts of function and limit in the mind of a student, it seems reasonable to declare that the limit concept is in fact a useful “higher-level process” which can be “interiorized”

(Sfard, 1991, p. 31)—or instantiated, to use TIFI limit model terminology—as the function concept is simultaneously “reified” in the mind of an introductory calculus student.

> In order to see a function as an object, one must try to manipulate it as a whole: there is no reason to turn process into object unless we have some higher-level processes performed on this simpler process. But here is a vicious circle: on one hand, without an attempt at the higher-level interiorization, the reification will not occur; on the other hand, existence of objects on which the higher-level processes are performed seems indispensable for the interiorization—without such objects the processes must appear quite meaningless. In other words: *the lower-level reification and the higher-level interiorization are prerequisite for each other*!
>
> (Quotation found in Sfard, 1991, p. 31; emphasis in original)

Figure 31. Quotation about understanding function in light of process-object duality theory.

As mentioned in Chapter 2, the first three steps in the genetic decomposition of limit as postulated by Cottrill’s (1996) research team—shown in Figure 12 of that chapter—might be directly related to a student’s understanding of function. This argument is confirmed by the present research study’s data in at least two important ways. First of all, the finding of a significant positive linear correlation $(r = 0.566)$ between PCA scores and LUA scores—even after accounting for ACT mathematics scores and the section number in which each student was enrolled—indicates that a student’s understanding of function or lack thereof apparently influences his ability to meaningfully understand limits, including the three steps of the decomposition just mentioned. Second, since the present research data showed an increased chance in obtaining high instantiation scores on the LUA for those students earning moderately high PCA scores, and the aforementioned genetic decomposition steps could be considered to be a detailed description of how a student learns to appropriately and consistently instantiate the limit concept, it is probable that function understanding—specifically, understanding at the beginning stages of reification from the TIRI function model—is important to progressing through these steps.

Moreover, a student's understanding of function seems to play a role in her skill in translating from one representation of limit to another as well as in her ability to write limit statements which correctly summarize a given situation. Tall and Vinner (1981) found that misconceptions about limit computations can often be attributed to a student's lack of a conception of function that is consonant with the accepted mathematical definition, and the translation item T_1 of the LUA reflected this reality. To be specific, a large number of Math 181 students decided, upon seeing graph A in T_1, that $\lim_{x \to 4} f(x) = 1$ even though f is not defined near 4. At least some of these students likely have not acquired a process conception of function as described by Breidenbach and his colleagues (1992), which could allow them to overcome a proclivity to view limit as simply the evaluation of a function at a few points or at a single point. (A process conception of function is defined in Figure 11 of Chapter 2.)

Based on the apparent confusion students felt as to the roles of the variables p and C on item P_1 of the LUA, writing appropriate limit statements which encapsulate a short description of a real-life situation clearly requires covariational reasoning, which is the ability to "coordinat[e] two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354). Covariational reasoning is clearly a characterization of function understanding, as it relates to the process conception of function given by Breidenbach's research team (1992). In short, a student needs an adequately strong understanding of function in order to meaningfully model the real world with apposite limit statements. Furthermore, some students who had trouble doing the reverse—that is, interpreting a contextually given limit statement using plain English sentences, as prompted in item P_2 of the LUA—did so incorrectly because of a noticeable lack of understanding of the role and/or natural limitations of the given modeling functions. In these several ways, students' understanding of function or lack thereof greatly affects their ability to genuinely understand the limit concept.

Additional Contributed Knowledge about Students' Understanding of Limit

When discussing students' spontaneous conceptions of limit as defined by Cornu (1991), I postulated that students need exposure to specific examples of the limit concept that illustrate the inconsistencies in their thinking in comparison to the correct mathematical ideas. Based on many self-contradictory responses students gave to individual LUA items, it appears that Math 181 students are not fully aware of the inconsistencies in their thinking about limit.

Students' inconsistency in responding to LUA items was especially noticeable on items T_1, F_1, and S_1, as seen in the summary data provided in Tables 12, 13, 20, and 24 of Chapter 4. It should be noted that much additional inconsistency on item F_1 was between parts—for example, a student might have correctly circled the right things in parts (i) and (ii) but given completely nonsensical answers to parts (iii) and (iv) of that item. While some incongruity in a student's responses to limit items is to be expected because of the epistemological obstacles confronting him—that is, because of obstacles which are "unavoidable and essential constituents of the knowledge to be acquired" (Bachelard as paraphrased by Cornu, 1991, p. 158)—based on the present research data, it seems that with more carefully selected instruction, some of these problems could be shored up and the level of self-disagreement decreased in students' reasoning about limit.

Previous research suggests that many of the limit conceptions of students are solely algorithmic in nature, including a conception in which students think $\lim_{x \to a} f(x)$ literally means evaluating a function at successively closer *x*-values to the *x*-value of a or a conception in which students think $\lim_{x \to a} f(x)$ literally means imagining the graph of the function and moving along it to the limiting value (Przenioslo, 2004). The present research confirms this finding, with a few students' responses to item S_2 reflecting the former conception and many students' responses to item T_1 revealing the latter. In my experience with Math 181, the chapter on limits in Stewart (2008) is taught in a computational way using mostly graphs and formulas, with some tabular representations of limit included in the earliest examples. As Williams (1991) hypothesized, in a traditional course like Math 181, students quickly find that simple metaphors about the nature of limits work most of the time for doing computations. Therefore, students stop worrying about the logicality or correctness of those metaphors, which may explain the evidence of these two limit conceptions being held by some of them.

In Chapter 4, I noted that many students responding to part (ii) of item P_1—in which students were prompted to represent cost using a limit statement—included a term representing some "leftover" additional cost that is not actually there. I also stated that the reason such students committed this error might be due to a belief that limits are fundamentally unreachable, wherein they "imagine infinitesimally small distances, … [so] that function values can get infinitely close to a limit but never reach it" (Szydlik, 2000, p. 260). Although I have not formally tried to measure how many Math 181 students might espouse a Leibniz-Cauchy model of the real line, which allows for "infinitesimal neighborhoods" (Mamona-Downs, 1990, p. 75), I have taken informal surveys of students enrolled in my own Math 181 classes concerning their belief in infinitesimally small distances and have found that the belief is common, at least among these students. Kurt Gödel once proclaimed "there are good reasons to believe that non-standard analysis, in some version or another, will be the analysis of the future" (quoted by Hermoso, date unknown), and I wonder, in this post-calculus reform movement era, if it might not be worthwhile to revisit non-standard analysis again as a foundation for introductory calculus courses (Sullivan, 1976), to the benefit of students in those courses.

In the next section, I proffer some recommendations which are primarily intended for mathematics educators teaching either precalculus or calculus courses, based on the results of the present research study.

Recommendations for Practice

Because students' understanding of function apparently impacts their understanding of limit, and since the limit concept is foundational to all of modern standard analysis as well as to the traditional pedagogical treatment of introductory calculus, it may be that a sufficiently high score on a test measuring function understanding should be used as a prerequisite to introductory calculus courses. Even if such a requirement is not feasible, such a test would provide valuable data to the introductory calculus instructor as to what his or her students will reasonably be able to learn about limit, based on their understanding as revealed on the function test. One such test of function understanding already exists, and the researchers who designed that test found it constituted a good general gauge of future success in an introductory calculus course (Carlson et al., under review). Hence, the use of a function test prior to calculus instruction is recommended.

While the development of algebraic manipulation skills is important, this development should not be overemphasized, especially at the expense of student growth in understanding of the function concept. This claim is supported by the recommendation of the Mathematical Association of America (MAA) that "courses designed for mathematical science majors should ensure that students progress from a procedural/computational understanding of mathematics to a broad understanding encompassing logical reasoning, generalization, abstraction, and formal proof" (MAA, 2004, p. 44), a broad understanding which is most easily facilitated by a strong understanding of function. The present research finding of the importance of students' understanding of function to their understanding of limit directly supports this assertion. Clearly, the concept of function should be central to both precalculus and introductory calculus curricula.

Based on a perusal of current college algebra and precalculus textbooks' tables of contents, one might argue that the curricular focus is on functions already. However, "school mathematics tends to be dominated by … the narrowest and most superficial form [of meaning]" (Kaput, 1989, p. 168)—namely, familiarization, which is largely comprised of "pattern and syntax learning" (ibid.), or manipulation skills. This can lead to a manipulation focus where the underlying meanings for the written symbols are lost (White and Mitchelmore, 1996), hindering a mature understanding of function, because function understanding requires a definite physical, contextual basis as pointed out by Sierpinska (1992). In short, by using current mathematics textbooks, mathematics educators may be paying lip service to the concept of function while not really providing a curricular environment in which growth in understanding of that crucial concept can be expected from students.

Therefore, it is recommended that mathematics educators develop and employ curricula that are specifically designed to promote greater function understanding, in the spirit of REC theory, in students. One such curriculum, used in at least three previous research studies, gives students repeated opportunities to create their own functions in a computer programming environment (Breidenbach et al., 1992; Dubinsky and Harel, 1992; Schwingendorf et al., 1992). The results of this experimental curriculum are highly encouraging, as many of the participant students in these research studies acquired a process conception of function—defined in Figure 11 of Chapter 2—via that curriculum. As another example, researchers were able to significantly increase the ability of students to mentally move from one function representation to another—which concerns the translation component of the TIRI function model—through computer-based activities involving the simultaneous display of a graph, formula, and table related to a given function (Schwarz et al., 1990). Such curriculum development efforts must continue, especially as advanced technology allows for more increasingly sophisticated curriculum possibilities.

When it comes to guiding students' development of the limit concept, all of the component competencies of the TIFI limit model should be incorporated into introductory calculus curricula. Although homework exercises from traditional textbooks often reflect these component competencies, many introductory calculus courses are algorithmic in nature (Cornu, 1991; Williams, 1991), and important exercises which evoke active *cognitive conflict* (Tall and Vinner, 1981) in a student's mind and hence promote a broader, more balanced understanding of limit may be avoided by instructors of traditional calculus, to the detriment of their students' limit understanding. Moreover, integrating all aspects of the TIFI limit model into calculus instruction may very well improve students' understanding of both the limit and function concepts, as implied in the statement by Sfard (1991) seen in Figure 31.

One final recommendation should be made with regards to the formal definition of limit: either more time needs to be spent with it, or it needs to be scrapped altogether.

The data gathered in the present research study do not support the continuation of a curriculum in which the ε-δ definition is covered in one or two class days and then ignored thereafter (until reaching advanced calculus, which most introductory students never take). A comparison of curricula including of the formal definition and not including it has been made (e.g., Mamona-Downs, 1990), and new strategies for its presentation—assuming it is included in the curriculum—have been postulated (Bokhari and Yushau, 2006). However, no matter what one believes about the formal definition of limit and its role in introductory calculus, it seems a brief overview of the formal definition with no substantial follow-up exercises and problems for students to work through is of little or no value to their overall understanding of limit.

Indications for Future Research

The present research study did not include an experimental treatment, nor was there any kind of control group. Therefore, it was not possible to irrefutably determine if a link found between students' understanding of function and their understanding of limit constitutes a genuine cause-effect relationship. Future research needs to be conducted in which the methodology includes an experimental comparison of students' limit understanding based on what type of instruction they receive in the understanding of function. In order to set up such an experiment, either an existing precalculus curriculum specifically designed for the enhancement of students' growth in function understanding could be compared to the traditional pedagogical approach in precalculus, or an entirely new precalculus curriculum stressing function understanding could be created prior to making a comparison. Whichever of these two possibilities is used, a methodology which includes a true experiment has the prospect to begin building the case of the existence of a cause-effect relationship between understanding of function and understanding of limit.

Of course, a strictly unidirectional cause-effect relationship may not be an entirely accurate assessment of the relationship between function understanding and limit understanding. Previously, I argued that students' acquisition of the ability to instantiate appropriate examples of limit may temporally coincide with their mental reification of the function concept. One important question related to this claim is the following: If an introductory calculus instructor followed the recommendation to administer a test of function understanding at the beginning of an introductory calculus course and found that some of her students were lacking somewhat in function understanding, would remediation for these students be possible in the course of calculus instruction itself?

To answer this question, longitudinal research studies examining students' understanding of function both before and after taking an introductory calculus course specifically designed to improve students' understanding of function are in order. One such study has been done with promising results (Schwingendorf et al., 1992); however, that study measured students' growth in function understanding over two semesters of calculus, not just one, and the treatment included extensive use of computer programming in a language called ISETL. Hence, more studies are needed which employ simpler pedagogical approaches and which are constrained to the time period of just one semester. In this respect, the methodologies utilized by Heid (1988) and Palmiter (1991) to increase conceptual understanding of calculus concepts via computer algebra systems—which are now readily available in the form of easy-to-use graphing calculators familiar to many college students—may be worth replicating, only with the particular goal of increasing function understanding in students.

Another worthwhile question related to the claim of mutually dependent growth in understanding of function and growth in understanding of limit is whether or not an informal concept of limit should appear in mathematics courses *preceding* calculus instruction. The computational procedures introduced in Chapter 2 of Stewart (2008) and in other traditional introductory calculus textbooks could be successfully taught to students in precalculus courses, allowing for a more conceptual pedagogical approach to limits in actual calculus courses themselves. In particular, more time could be spent in introductory calculus classes on the formal ε-δ definition of limit if students were already familiar with informal limits from the start of calculus instruction, which is apparently necessary to make presentation of the ε-δ definition worth doing at all. To verify the idea that limits can and should be introduced earlier in the college mathematics curriculum, research must be done to study the long-term effects of introducing the limit concept in college precalculus courses followed by a more conceptually rigorous consideration of it in calculus courses.

The research advocated here may appear irrelevant to someone who does not believe, as I do, that the concepts of derivative, integral, and infinite series are only well understood if a sufficiently mature understanding of limit is present in the mind of the introductory calculus student. Since mathematicians employed, even if in more primitive form, derivatives, integrals, and infinite series for at least 200 years before Weierstrass provided a rigorous limit concept which is the foundation for modern standard analysis, this criticism is not wholly without merit. Researchers have made a strong case for the importance of understanding function in the understanding of derivative (Pinzka, 1999; Zandieh, 2000) as well as its importance in understanding calculus concepts in general (Monk, 1994; Tall, 1997), and the results of the present research study indicate that understanding of function and understanding of limit are strongly correlated. Hence, all of this research together indirectly suggests understanding of limit may be beneficial, and even necessary, to understanding other calculus concepts. Nevertheless, the establishment of a direct link between students' understanding of limit and their understanding of derivative (or of integral, or of infinite series) would be a welcome addition to calculus education research.

In making links among understanding of limit and understandings of various other calculus concepts, the TIFI limit model may prove to be invaluable. It provides a rich spectrum of different types of understanding of the concept of limit, so as to allow future researchers to be able to correlate specific types of understanding of limit in students to their understandings of other concepts. Pursuing this line of research may have the additional benefit of uncovering other important explanatory variables—variables not included in the present research—which might correlate directly with limit understanding itself. After all, while a good portion of the variance in LUA scores was explained by the linear regression model shown in Table 9 of Chapter 4, there is on balance even more unexplained variance to be accounted for when it comes to students' understanding of limit (as measured by the LUA, at least). The question becomes even more relevant when one considers that the relatively low LUA scores earned by students in Section 3 is not explainable using the explanatory variables of PCA or ACT, because students in Section 3 were equivalent to other sections in regards to these two variables. While the reason for the difference in mean LUA scores between Section 3 and the group of all other sections may never be known, it is possible that future research will be able to shed light on other factors affecting limit understanding which might have solved this mystery had they been considered in this paper.

Conclusion

The present research study established a valid instrument, the Limit Understanding Assessment (LUA), which was used to measure students' understanding of limit in terms of a practical model—the TIFI limit model—based on a credible theory of mathematical understanding from Kaput (1989) which was extended and called REC theory in this paper. This instrument was used in tandem with another already existing instrument, the Precalculus Concept Assessment (PCA), which was employed to measure students' understanding of function so as to correlate students' understanding of function with their understanding of limit.

Among the data gathered for the major portion of the research study, there was a strong positive linear correlation between PCA scores and LUA scores, even after accounting for ACT mathematics scores and the section number in which each student was enrolled, and this correlation was highly significant. Nonparametric, bivariate correlation statistics computed between PCA scores and each component competency variable of the TIFI limit model—namely, translation scores, interpretation scores, familiarization with the formal scores, and instantiation scores—were also found to be significant. A descriptive profile of what students' responses to individual LUA items might actually mean in the context of limit understanding was developed, and this profile was referred to often in the interpretative portion of this chapter.

Based on the results shown in Chapter 4, several important conclusions were made as to the meaning of these results, conclusions which appear in this chapter. Clearly, understanding of function is a significant predictor of future understanding of limit. In fact, the limit concept is the desired "higher-level process" which can be "interiorized" (i.e., instantiated, using TIFI limit model terminology) as the function concept is simultaneously "reified" in the mind of an introductory calculus student (Sfard, 1991, p. 31). The significant positive linear correlation between PCA scores and LUA scores in general as well as the increased chance in obtaining high instantiation scores on the LUA for those students earning moderately high PCA scores both confirm the importance of function in the genetic decomposition of limit theorized by Cottrill and his colleagues (1996). The crucial role of function understanding to completing the first four steps of the decomposition is not simply a matter of logicality but a data-supported theory of the essential mental bond between the two concepts in order to meaningfully instantiate limits. Additionally, a student's understanding of function seems to play a role in her skill in translating from one representation of limit to another as well as in her ability to write limit statements which correctly summarize a given situation.

Recommendations for practicing mathematics educators were made in light of the findings of the present research study. Because students' understanding of function directly impacts their understanding of limit, the use of a function test prior to calculus instruction is recommended. While the development of algebraic manipulation skills is important, this development should not be overemphasized, especially at the expense of student growth in understanding of the function concept. Curriculum development efforts that enhance the understanding of function in students must continue, especially as advanced technology allows for more increasingly sophisticated curriculum possibilities.

Integrating all aspects of the TIFI limit model into calculus instruction may very well improve students' understanding of both the limit and function concepts, as implied by Sfard (1991). However, a strong understanding of the formal definition of limit appears to take a significant amount of time and effort on the part of students, so in introductory calculus curricula, either more time needs to be spent with the formal definition of limit, or it needs to be scrapped altogether.

Indications for future research opportunities were provided, so as to point out some fruitful directions in which the work of the present research might be continued. The present research study did not include an experimental treatment, nor was there any kind of control group, but a methodology which includes a true experiment has the prospect to begin building the case of the existence of a cause-effect relationship between understanding of function and understanding of limit. Since a strictly unidirectional cause-effect relationship may not be an entirely accurate assessment of the relationship between function understanding and limit understanding, longitudinal research studies examining students' understanding of function both before and after taking an introductory calculus course specifically designed to improve students' understanding of function are also in order.

Another worthwhile question related to the claim of mutually dependent growth in understanding of function and growth in understanding of limit is whether or not an informal concept of limit should appear in mathematics courses *preceding* calculus instruction. To verify the notion that limits can and should be introduced earlier in the college mathematics curriculum, research must be done to study the long-term effects of introducing the limit concept in college precalculus courses followed by a more conceptually rigorous consideration of it in calculus courses. Also, the establishment of a direct link between students' understanding of limit and their understanding of derivative (or of integral, or of infinite series) would be a welcome addition to calculus education research. In making links among understanding of limit and understandings of various other calculus concepts, the TIFI limit model may prove to be invaluable. It provides a rich spectrum of different types of understanding of the concept of limit, so as to allow future researchers to be able to correlate specific types of understanding of limit in students to their understandings of other concepts.

More generally, it seems desirable to utilize REC theory in mathematics education research involving function, limit, or other mathematical concepts for which researchers need data as to how well particular groups of students truly understand those concepts. Kaput's (1989) framework, extended in this paper, provides an excellent foundation on which many mathematics education research projects can be based.

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APPENDICES

APPENDIX A:

LUA VERSION D:
ITEMS AND SUGGESTED ANSWERS

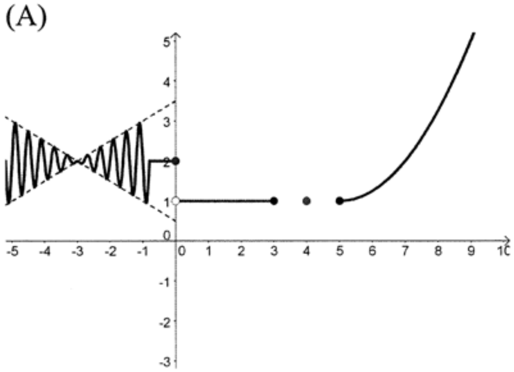
Translation Items.

1. Complete the tasks below.

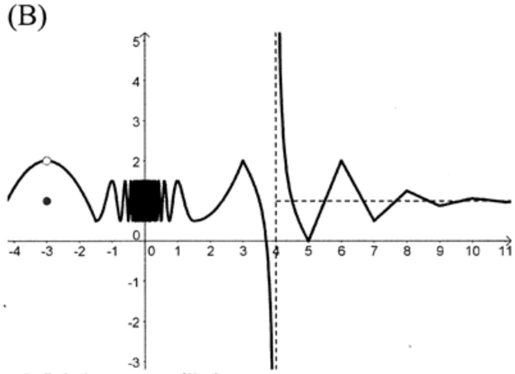
(i) For each of the limit statements on the right, write either A, B, AB, or N to indicate for which graph(s) the limit statement is true. Use the legend below in selecting your answer.

A = Limit statement is true for graph A only
B = Limit statement is true for graph B only
AB = Limit statement is true for *both* graph A and for graph B
N = Limit statement is true for *neither* graph A nor for graph B

(A)



(B)



Infinitely many oscillations near x = 0

N $\lim_{x\to -3} f(x) = 1$

AB $\lim_{x\to -3} f(x) = 2$

N $\lim_{x\to 0} f(x) = 1$

N $\lim_{x\to 0} f(x) = 2$

N $\lim_{x\to 4} f(x) = 1$

N $\lim_{x\to 4} f(x) = +\infty$

B $\lim_{x\to +\infty} f(x) = 1$

A $\lim_{x\to +\infty} f(x) = +\infty$

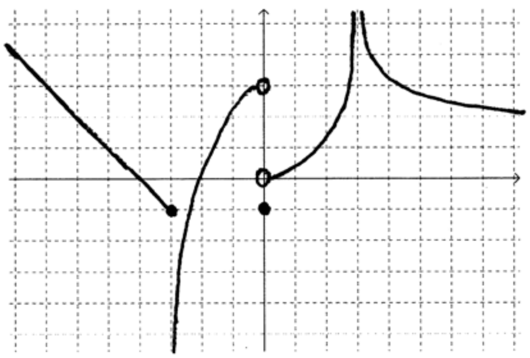
(ii) Circle the values of a for which $\lim_{x\to a} f(x)$ does not exist in graph A above.

(a) -3 (b) 0 (c) 4 (d) $+\infty$

(iii) Circle the values of a for which $\lim_{x\to a} f(x)$ does not exist in graph B above.

(a) -3 (b) 0 (c) 4 (d) $+\infty$

2. Using the coordinate system provided, sketch the graph of a single function f that satisfies all of the conditions listed on the right.



$f(0) = -1$

$\lim_{x \to 0^-} f(x) = 3$

$\lim_{x \to 0^+} f(x) = 0$

$\lim_{x \to 3^-} f(x) = +\infty$

$\lim_{x \to -3^+} f(x) = -\infty$

$\lim_{x \to -\infty} f(x) = +\infty$

$\lim_{x \to +\infty} f(x) = 2$

Interpretation Items.

1. A city's main well was recently found to be contaminated with trichloroethylene, a cancer-causing chemical. In order to submit a suitable proposal to the city council members addressing this problem, Alice models the cost C (in millions of dollars) of removing $p\%$ of the toxic pollutant by a suitable function $C(p)$.

(i) Alice determines the cost increases without bound as the amount of trichloroethylene to be removed approaches all of it being removed. Model this real-life finding using an appropriate mathematical limit statement involving $C(p)$.

$$\lim_{p \to 100^-} C(p) = +\infty$$

(ii) Alice determines there is a fixed cost of five million dollars in order to remove *any* of the trichloroethylene from the city's well. Envision the amount of chemical to be removed approaching zero and model Alice's finding using an appropriate mathematical limit statement involving $C(p)$.

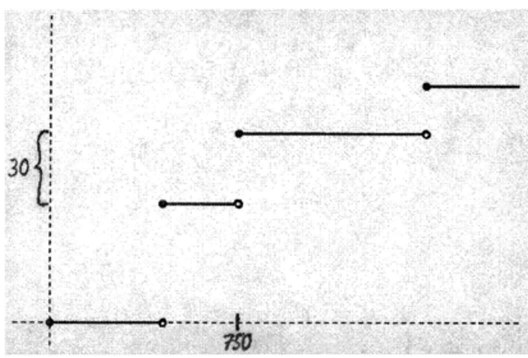
$$\lim_{p \to 0^+} C(p) = 5$$

2. Bob discovers that if he spends t minutes a month selling his product door-to-door, then the amount of money S he collects (in dollars and cents) through sales each month can be reasonably modeled using a function $S(t)$.

(i) Suppose that $S(t)$ is an increasing function and that $\lim_{t\to+\infty} S(t) = 2000$. Explain in one sentence what this limit represents in real-life context.

Bob cannot collect more than $2000 no month no matter how much time he spends selling his product.

(ii) To analyze his sales from *last* month, Bob creates a graph relating his accumulated sales A (in dollars and cents) to the amount of time t (in minutes) he had spent selling his product door-to-door up to that point in the month. A portion of Bob's graph, which indicates that $\lim_{t\to 750^-} A(t) = A(750) - 30$, appears below. Explain briefly what this limit statement indicates in real-life context.



At precisely the 750th minute of selling door-to-door, Bob made a sale of $30.

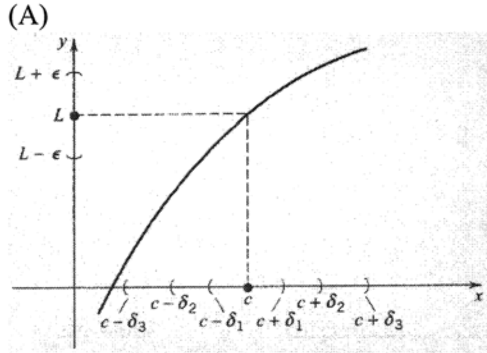
Familiarization with the Formal Item.

1. This problem is worth four points toward the total test score, and half-points are possible. Complete the tasks below, which are related to understanding the formal definition of limit. (The formal definition of limit is on the accompanying handout.)

(i) Consider graph A shown to the right, which illustrates $\lim_{x\to c} f(x) = L$. In the list below, circle all of the δ's that "work" for the given ε.

δ_1 $\quad\delta_2\quad$ δ_3

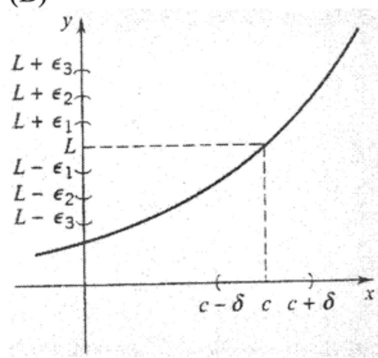
(A)



(ii) Consider graph B shown to the right, which illustrates $\lim_{x \to c} f(x) = L$. In the list below, circle all of the ε's for which the given δ "works".

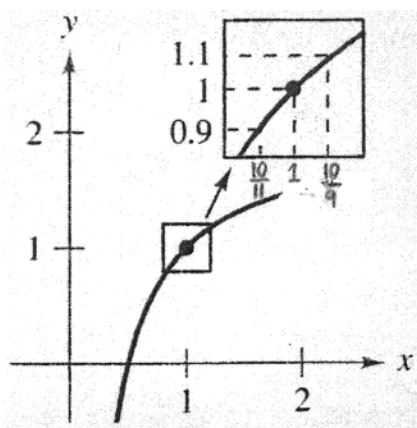
ε_1 $\quad$ ε_2 $\quad$ ε_3

(B)



(iii) Consider the graph shown below, which illustrates the function $y = f(x) = 2 - \frac{1}{x}$.

Find the largest $\delta > 0$ such that if $0 < |x - 1| < \delta$, then $|y - 1| < 0.1$. Justify your answer by showing all work in the space below; then write your answer here: $\delta = \underline{\frac{1}{11}}$



$1 - \frac{10}{11} = \frac{1}{11}$

$\frac{10}{9} - 1 = \frac{1}{9}$

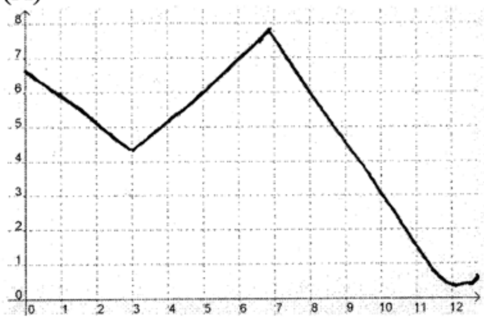
(iv) Suppose we have a function $f(x)$ for which we know that

$$\text{if } 0 < |x - 5| < 2, \text{ then } |f(x) - 3| < 1.$$

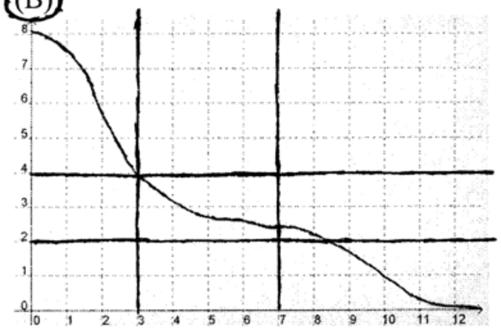
Which of the following graphs *could* be the graph of $y = f(x)$? Circle all of the letters which correspond to graphs that could represent f. (*Note:* There may be more than one correct answer.) Then draw in appropriate horizontal and vertical lines to illustrate the correctness of your answer(s).

(Graphs appear on next page.)

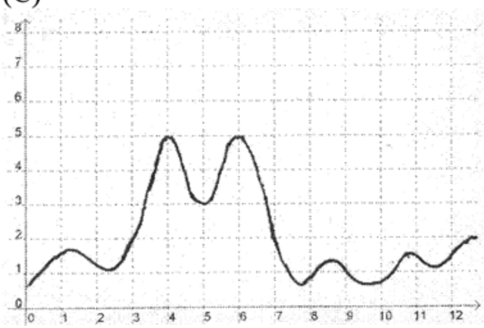
(A)



(B)

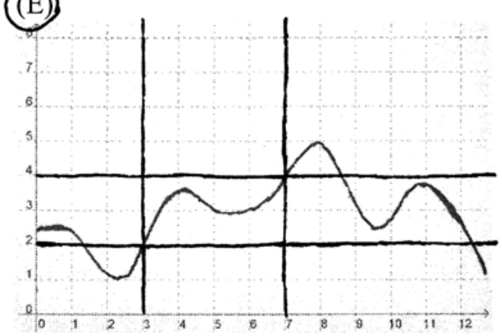


(C)



(D)

(E)



Instantiation Items.

1. Mark each of the following statements as either true or false. (*Note:* If there is a function f (or a pair of functions f and g) for which the statement is not true, circle the word "False". If the statement is true for all functions f (or for all pairs of functions f and g), circle "True".)

(Statements appear on next page.)

True False If $f(4) = 2$, then $\lim_{x \to 4} f(x) = 2$.

True False If $\lim_{x \to 4} f(x)$ exists, then f is defined at $x = 4$.

True False If $\lim_{x \to 1} f(x) = 5$ and $\lim_{x \to 1} g(x) = 0$, then $\lim_{x \to 1} f(x)g(x) = 0$.

True False If $\lim_{x \to 1} f(x) = 5$ and $g(1) = 0$, then $\lim_{x \to 1} f(x)g(x) = 0$.

True False If $\lim_{x \to -3} f(x)$ does not exist as a real number, then $\lim_{x \to -3} 1/f(x)$ does not exist.

True False If $\lim_{x \to -3} f(x) = 0$ and $\lim_{x \to -3} g(x) = 0$, then $\lim_{x \to -3} f(x)/g(x)$ does not exist.

True False If $\lim_{x \to 0} f(x) = 0$ and $-1 \le g(x) \le 1$ for all $x \ne 0$, then $\lim_{x \to 0} f(x)g(x) = 0$.

True False If $f(x) < g(x)$ for all $x \ne 0$, then $\lim_{x \to 0} f(x) \ne \lim_{x \to 0} g(x)$, assuming these limits exist.

2. In computing $\lim_{x \to 0} (1 + x)^{1/x}$ for a homework problem, Chad writes the following:

> As $x \to 0$, it is clear that $(1 + x) \to 1$. Since 1 raised to any power is 1, $\lim_{x \to 0} (1 + x)^{1/x} = \lim_{x \to 0} (1)^{1/x} = 1$.

Is this logic correct? Making use of appropriate properties for the given limit, write a short paragraph explaining either why you agree with Chad's answer or why you think Chad is wrong. This logic is incorrect. While it's true that $1+x \to 1$ as $x \to 0$, it is also true that $\frac{1}{x}$ is unbounded in any deleted neighborhood of $x=0$, and the exponent tends to increase the size (absolutely) of the expression $(1+x)^{1/x}$ as $x \to 0$. So the limit might not be 1.

3. Recall that a piecewise function is a function like $g(x)=\begin{cases} x+1 & x\le 10 \\ \sqrt{x} & x>10 \end{cases}$, where the formula to use for finding the function's value depends on the piece of the domain in which the x-value is located. Now, find the formula for a single piecewise function f that satisfies all of the conditions below. (Drawing a possible graph for f first may help, but the graph will not be scored.)

$f(0)=0$ $\quad\quad$ $\lim_{x\to 0^-} f(x)=3$ $\quad\quad$ $\lim_{x\to -\infty} f(x)=+\infty$ $\quad\quad$ $\lim_{x\to +\infty} f(x)=2$

$$f(x)=\begin{cases} -x+3 & x<0 \\ 2x/(x+1) & x\ge 0 \end{cases}$$

Notes on LUA Version D.

1. Students were provided with the following definition to help them with F_1.

Definition. Let f be a function defined on an open interval that contains the number c, except possibly at c itself. Then the limit of $f(x)$ as x approaches c is L, written mathematically as

$$\lim_{x\to c} f(x)=L$$

if for every number $\varepsilon>0$ there is a number $\delta>0$ such that

$$\text{if} \quad 0<|x-c|<\delta \quad \text{then} \quad |f(x)-L|<\varepsilon$$

2. The test items did not appear in this order when given to Math 181 students in the fall of 2008. Instead, items appeared in the following order (numbered 1 through 8): T_1, P_1, S_1, T_2, S_2, P_2, S_3, F_1. To receive a copy of the original LUA Version D, contact Taylor Jensen via email at tajknight@gmail.com.

APPENDIX B:

SOURCE MATERIAL FOR THE LUA:
CALCULUS TEXTBOOK REFERENCES

Armstrong, W. A. & Davis, D. E. (2000). *Brief Calculus: The Study of Rates of Change*. Upper Saddle River, NJ: Prentice-Hall.

Barnett, R. A. & Ziegler, M. R. (1993). *Calculus for Business, Economics, Life Sciences, and Social Sciences* (6th ed.). New York: Macmillan.

Berkey, D. D. & Blanchard, P. (1992). *Calculus* (3rd ed.). Fort Worth, TX: Harcourt Brace Jovanovich College Publishers.

Bittinger, M. L. (1992). *Calculus* (5th ed.). Reading, MA: Addison-Wesley.

Bradley, G. L. & Smith, K. J. (1999). *Single Variable Calculus* (2nd ed.). Upper Saddle River, NJ: Prentice-Hall.

Edwards, C. H. & Penney, D. E. (1998). *Calculus: Early Transcendentals with Analytic Geometry* (5th ed. / instructor's edition). Upper Saddle River, NJ: Prentice-Hall

Ellis, R. & Gulick, D. (1990). *Calculus with Analytic Geometry* (4th ed.). San Diego, CA: Harcourt Brace Jovanovich.

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Faires, J. D. & Faires, B. T. (1983). *Calculus and Analytic Geometry*. Boston, MA: Prindle, Weber, & Schmidt.

Flanders, H. (1985). *Calculus*. New York: W. H. Freeman & Company.

Gersting, J. L. (1984). *Technical Calculus with Analytic Geometry*. Belmont, CA: Wadsworth Publishing Company.

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Giordano, F. R., Weir, M. D., & Finney, R. L. (1998). *Calculus for Engineers and Scientists, Volume 1* (preliminary edition). Reading, MA: Addison-Wesley.

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Gottlieb, R. J. (2002). *Calculus: An Integrated Approach to Functions and their Rates of Change* (preliminary edition). Boston, MA: Addison-Wesley.

Hurley, J. F. (1987). *Calculus*. Belmont, CA: Wadsworth Publishing Company.

Johnston, E. H. & Mathews, J. C. (1996). *Calculus for Engineering and the Sciences, Volume 1* (preliminary version). New York: HarperCollins College Publishers.

Larson, R. E. & Edwards, B. H. (1999). *Brief Calculus: An Applied Approach* (5th ed.). Boston, MA: Houghton Mifflin Company.

Larson, R. E. & Edwards, B. H. (2003). *Calculus: An Applied Approach* (6th ed.). Boston, MA: Houghton Mifflin Company.

Larson, R. E. & Hostetler, R. P. (1986). *Calculus with Analytic Geometry* (3rd ed.). Lexington, MA: D. C. Heath & Company.

Larson, R. E., Hostetler, R. P., & Edwards, B. H. (1995). *Brief Calculus with Applications* (4th ed.). Lexington, MA: D. C. Heath & Company.

Larson, R. E., Hostetler, R. P., & Edwards, B. H. (2003). *Calculus: Early Transcendental Functions* (3rd ed.). Boston, MA: Houghton Mifflin Company.

Lial, M. L. & Miller, C. D. (1989). *Calculus with Applications* (4th ed. / brief version). Glenview, IL: Scott, Foresman & Company.

Mizrahi, A. & Sullivan, M. (1984). *Calculus with Applications to Business and Life Sciences* (2nd ed.). New York: John Wiley & Sons.

Paul, R. S. & Haeussler Jr., E. F. (1984). *Calculus for Business*. Reston, VA: Prentice-Hall.

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Stein, S. K. (1987). *Calculus and Analytic Geometry* (4th ed.). New York: McGraw-Hill.

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Strauss, M. J., Bradley, G. L., & Smith, K. J. (2002). *Calculus* (3rd ed. / instructor's edition). Upper Saddle River, NJ: Prentice-Hall.

Tan, S. T. (2005). *Applied Calculus for the Managerial, Life, and Social Sciences* (6th ed.). Belmont, CA: Thomson Learning, Inc.

Varberg, D. E., Purcell, E. J., & Rigdon, S. E. (2000). *Calculus* (8th ed.). Upper Saddle River, NJ: Prentice-Hall.

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APPENDIX C:

RUBRIC FOR LUA VERSION D

Translation Item 1.

Students are asked to evaluate eight different graphical limits. Note that for questions (ii) and (iii), students are simply providing more complete answers about the eight graphical limits of question (i). Hence, any inconsistent reasoning about a particular graphical limit (i.e., a response about a limit which is inconsistent with the response about the same limit elsewhere) is counted as incorrect. For example, a student's responses to the graphical limit as $x \to -3$ on graph A are consistently correct only if he puts "N" or "B" in the blank next to $\lim_{x \to -3} f(x) = 1$, writes an "A" or an "AB" in the blank next to $\lim_{x \to -3} f(x) = 2$, and has *not* circled choice (a) in part (ii). The table below summarizes the consistently correct responses to each of the eight graphical limits presented in this problem. The blanks in question (i) are numbered on the answer key for the grader's convenience.

Graphical Limit	**Consistently Correct Responses**
Graph A as $x \to -3$	"N" or "B" in blank 1 "A" or "AB" in blank 2 Question (ii) choice (a) *not* circled
Graph B as $x \to -3$	"A" or "N" in blank 1 "B" or "AB" in blank 2 Question (iii) choice (a) *not* circled
Graph A as $x \to 0$	"N" or "B" in blank 3 "N" or "B" in blank 4 Question (ii) choice (b) circled
Graph B as $x \to 0$	"N" or "A" in blank 3 "N" or "A" in blank 4 Question (iii) choice (b) circled
Graph A as $x \to 4$	"N" or "B" in blank 5 "N" or "B" in blank 6 Question (ii) choice (c) circled
Graph B as $x \to 4$	"N" or "A" in blank 5 "N" or "A" in blank 6 Question (iii) choice (c) circled
Graph A as $x \to +\infty$	"N" or "B" in blank 7 "A" or "AB" in blank 8 Note: Technically, question (ii) choice (d) should be circled, but the item is *not* counted wrong if it is not.
Graph B as $x \to +\infty$	"B" or "AB" in blank 7 "N" or "A" in blank 8 Question (iii) choice (d) *not* circled

If a student fails to write *anything* on a particular blank in part (i), the corresponding graphical limits are *both* counted as wrong.

Scoring for this problem is on a 0–3 scale based on the number of graphical limits to which the student's responses are consistently correct. The table on the top of the next page supplies the conversion from the number of correct responses to the score from 0–3.

# of Correct Responses	Score
8	3
6–7	2
4–5	1
0–3	0

<u>Translation Item 2.</u>

Students are asked to sketch the graph of a function which satisfies seven listed conditions. For grading purposes, the seven conditions come in four groups, each of which is conceptually different in some way than the other groups. These groups are shown by brackets on the answer key. A student's response to a group of conditions is correct if her graph satisfies *all* conditions in that group.

Scoring for this problem is on a 0–3 scale with points assigned equal to the number of condition groups satisfied by the student's graph *minus one*. So if a student's graph satisfies three of the condition groups, she receives a score of 2 out of 3. [*Exception:* If a student's graph does not satisfy any of the condition groups, she receives a score of 0, not –1.]

Some students define y as two or more functions of x by drawing a graph which does not pass the vertical line test. In such a case, the overall graph is judged as satisfying a particular limit condition if and only if *all* of the functions' graphs satisfy that condition.

For each infinite limit and for each limit at infinity, a student receives credit as long as it can be *reasonably* inferred that the condition is met.

In order to receive credit for the condition "$f(0) = -1$", a student *must* differentiate it from the limiting values of the function as x approaches zero from the left and from the right (e.g., by putting a filled-in circle at $(0, -1)$ and an "open" circle or no circle at the other points of interest).

<u>Interpretation Item 1.</u>

Students are asked to write two mathematical limit statements that model a "real-life" situation, one for question (i), the other for question (ii). A student's limit statement is correct if both the independent variable portion (the "$\lim_{p \to a}$" part) and the function portion (the "$\lim C(p) = L$" part) are written correctly. Her limit statement is only somewhat correct if she writes one of the two aforementioned portions correctly but not the other. Finally, a student's limit statement is incorrect if neither portion is written correctly.

Scoring for this problem is on a 0–3 scale. The table below provides a conversion from the "correct/somewhat correct/incorrect" evaluation to the score from 0–3.

Evaluation	**Score**
Both limit statements are correct *OR* One limit statement is correct, the other is somewhat correct	3
One limit statement is correct, the other is incorrect *OR* Both limit statements are only somewhat correct	2
One limit statement is somewhat correct, the other is incorrect	1
Both limit statements are incorrect	0

The most important elements to a correct limit statement are as follows:

- appropriate numerical values have been substituted in for a and L
- an attempt has been made to include both independent and dependent variables
- the "$\to$" and "=" signs have been used correctly

In particular, if a student's response contains neither "lim" nor the "$\to$" sign, her mathematical statement is counted as not being a limit statement at all and hence is incorrect.

Exceptions: Because they are not directly measuring understanding of the "most important elements" given above, the "errors" on the top of the next page are not to be counted against the student:

- writing ∞ in place of $+\infty$
- writing 5,000,000 (or even 5000) in place of 5
- writing x in place of p in one or more places
- writing f in place of C in one or more places
- writing just C instead of $C(p)$
- employing a two-sided limit instead of a one-sided limit
- failing to write "$p \to a$" *directly* below "lim"
- writing "$p \to 1$" (or some other real number that clearly represents all of the pollutant) on question (i) rather than writing "$p \to 100$"

- mixing up a and L *or* mixing up p and C with *no other mistakes* (other than the exceptions listed here) constitutes a somewhat correct limit statement

In addition, scoring for two particular cases needs elucidation. As one example, on question (i), a response of

$$\underset{p \to 100^-}{C(p)} = +\infty$$

is counted as being somewhat correct—specifically, the "$\lim\limits_{p \to a}$" portion is considered incorrect while the "$\lim C(p) = L$" portion is regarded as right. On the other hand, on question (ii), a response of

$$\lim_{p \to 0^+} = 5$$

or a response of

$$\lim_{p \to 0^+} 5$$

are both counted as being somewhat correct—specifically, the "$\lim C(p) = L$" portion is considered incorrect, while the "$\lim\limits_{p \to a}$" portion is regarded as right. Other similar examples fitting these cases are also scored this way. If both mistakes outlined above are made—with no "lim" and either no "$C(p)$" or no "=" sign—the statement is counted as being incorrect.

<u>Interpretation Item 2.</u>

Students are asked to state in real-life terms what two mathematical limit statements mean. Hence, for credit, a student should interpret variables in English. A student's interpretation of a limit statement is correct if she clearly demonstrates specific (i.e., context dependent) understanding of the relationship between the independent and dependent variables as modeled by the given limit statement. A student's interpretation of a limit statement is only somewhat correct if she demonstrates specific understanding of the role of one of the numbers but is too vague with the other. A student's interpretation of a limit statement is incorrect if: (1) she does not appropriately interpret either one of the numbers' meaning in real-life terms; or (2) she appropriately interprets one of the numbers but interprets the other as having the *same* meaning (e.g., interprets both numbers as time or both numbers as money).

Scoring for this problem is on a 0–3 scale. The table below provides a conversion from the "correct/somewhat correct/incorrect" evaluation to the score from 0–3.

Evaluation	**Score**
Both limit statements correct	3
One limit statement correct, the other statement only somewhat correct	2
One limit statement correct, the other statement incorrect *OR* Both limit statements only somewhat correct	1
One limit statement only somewhat correct, the other statement incorrect *OR* Both limit statements incorrect	0

On part (i), possible correct responses include those on the top of the next page:

- No matter how much time Bob spends selling, he will make no more than $2000.
- As the time Bob spends selling increases, his sales approach $2000.

Note the number 2000 must be interpreted as a maximum amount or limiting amount of money, not just as money, to count.

Because they demonstrate understanding of what $+\infty$ represents in context but not what 2000 represents in context (or vice versa), the following sample part (i) answers are only somewhat correct:

- Bob can only make so much money no matter how long he sells door-to-door.
- An infinite amount of selling can only increase sales to a certain point.
- The maximum amount Bob can make is $2000, no matter what. (Note in this example time spent selling is *not* mentioned.)

On part (ii), possible correct responses include:

- Bob made a sale worth $30 at precisely the 750th minute of selling.
- Bob's sales jumped up $30 because he sold his product for 750 minutes rather than just 749 minutes.

The numbers to translate into real-life context on part (ii) are 750 and 30. Because they demonstrate understanding of what one number represents in context but not the other, the following sample part (ii) answers are only somewhat correct:

- The limit shows the amount of revenue made after 750 minutes selling.
- Bob sold $30 worth of product.

Exceptions: In evaluating the "correctness" of a student's response, certain exceptions are made so as to not overly penalize students demonstrating decent understanding. These are as follows:

- Reversing the roles of $S(t)$ and t or the roles of $A(t)$ and t while still employing context specific values, assuming no other mistakes are present, constitutes a somewhat correct limit statement.
- Writing an incorrect label such as hours instead of minutes will not be counted against a student.
- Writing in present or future tense instead of past tense on part (ii) will not be counted against a student.
- On part (ii), interpreting the 30 as a *cost* or *setback* at time $t = 750$ rather than a gain in *revenue* at that moment constitutes a somewhat correct limit statement (assuming the 750 is interpreted correctly).
- On part (ii), interpreting 750 as a dollar amount and 30 as the time a sale is made constitutes a somewhat correct limit statement.

Familiarization with the Formal Item.

Students are asked to respond to several graphical problems related to the formal ε-δ definition of limit. Although the problem has an overall scale of 0–4, half points are possible; in other words, a student could receive a score of 2.5 out of 4 on this problem. Each part of the problem is graded independently from the other parts of the problem. Finally, the student's scores from each part are added together to obtain an overall score.

Part (i)

This is counted as correct $(+\frac{1}{2}$ pt.) or incorrect (+0). A student earns the half point if she circles δ_1 and nothing else. Otherwise, she receives no points.

Part (ii)

This is scored on a $\{0, \frac{1}{2}, 1\}$ scale. A student earns the full point if he circles *both* ε_2 and ε_3 (but *not* ε_1.)

A student receives exactly $\frac{1}{2}$ point if either of the following occurs:

- ε_3 alone is circled
- ε_2 alone is circled *and* a *pair* of horizontal and/or vertical lines or a *pair* of points appear on the graph that illustrate a connection on *both* sides between δ and ε_2—even if it's not the "correct" connection

A student receives zero points for any response not mentioned above.

Part (iii)

This is scored on a $\{0, \frac{1}{2}, 1\}$ scale. A student earns the full point if she computes correctly that $\delta = \frac{1}{11}$.

A student receives exactly $\frac{1}{2}$ point if she claims the correct δ is $\frac{1}{9}, \frac{10}{11},$ or $-\frac{1}{11}$.

If a student claims the correct δ is $0.09\,(\approx \frac{1}{11}),$ $0.11\,(\approx \frac{1}{9}),$ $0.91\,(\approx \frac{10}{11}),$ or $-0.09\,(\approx -\frac{1}{11}),$ her work is checked to see if she got the corresponding exact answer and just rounded it. If so, the appropriate points are still awarded. Furthermore, if a student claims the correct δ is $1-\frac{10}{11}$ (or something similar where only the *arithmetic* has not been completed), the student is not penalized, and appropriate points are still awarded.

A student receives zero points for any response not mentioned above. If a student's answer is not *clearly* "set apart" (e.g., not on the line or boxed), she receives zero points.

Part (iv)

This is scored on a $\{0, \frac{1}{2}, 1, 1\frac{1}{2}\}$ scale. First of all, the student receives $\frac{1}{2}$ point if he does *not* circle (A) or (D) *and* if he circles *two* (or *three*) choices from the following list: (B), (C), (E). [*Exception:* A student also receives this $\frac{1}{2}$ point if he circles (E) alone]. If a student circles a group of graphs not mentioned above, he receives zero points *no matter what other work appears on his paper*.

Assuming the student earns the first $\frac{1}{2}$ point, he receives an *additional* half point (for a total of $1,$ so far) if he also draws in the vertical lines of $x = 3$ and $x = 7$ as well as the horizontal lines of $y = 2$ and $y = 4$ on one or more of the circled graphs. Alternatively, a student may earn this second $\frac{1}{2}$ point by providing sufficient evidence that he believes $\lim_{x\to 5} f(x) = 3.$** Some ways to provide evidence of the belief that $\lim_{x\to 5} f(x) = 3$ include the following:

- drawing a single horizontal and a single vertical line on one or more of the circled graphs that intersect at the point $(5, 3)$ or drawing a pair of such intersecting lines on one or more of the circled graphs which are "centered" around the point $(5, 3),$ even if the lines imply the student has incorrectly determined δ and/or ε
- drawing a dot on one or more of the circled graphs at $(5, 3)$
- writing "labels" of c and L in appropriate locations on one or more of the circled graphs or in the "if-then" statement

****** This belief is not actually warranted, but it still demonstrates the student's ability to interpret the "if-then" statement in terms of an associated limit.

- simply writing $\lim_{x\to 5} f(x) = 3$ somewhere on the back page

Drawing vertical lines without corresponding horizontal ones, or vice versa, does *not* constitute sufficient evidence to award this $\frac{1}{2}$ point.

If a student clearly demonstrates understanding as described in the previous paragraph in just one of the required places but not anywhere else, the response is still scored as evincing meaningful understanding and the student is given this $\frac{1}{2}$ point. (Even if he writes nonsense elsewhere on his sheet, as long as he demonstrates understanding once, the student is awarded the $\frac{1}{2}$ point.)

The last available $\frac{1}{2}$ point is awarded *only* to those students who answer the question right with no errors (i.e., as shown on the answer key). To be specific, such a student circles (B) and (E) (nothing else) and draws in the vertical lines of $x = 3$ and $x = 7$ as well as the horizontal lines of $y = 2$ and $y = 4$ on *one* or *both* of the circled graphs.

<u>Instantiation Item 1.</u>

Students are asked to circle "True" or "False" in response to eight statements about limits. For grading purposes, the eight statements come in four groups of two statements, with the two statements testing similar concepts. These groups are shown by brackets on the answer key. A student's response to a group of statements is correct if his responses are correct to *both* statements in that group.

Scoring for this problem is on a 0–3 scale with points assigned equal to the number of statement groups to which the student's responses are consistently correct. So if a student's responses are consistently correct for two of the statement groups, for example, he receives a score of 2 out of 3. [*Exception:* If a student correctly answers all statement groups, he receives a score of only 3, not 4.]

<u>Instantiation Item 2.</u>

Students are asked to explain why they agree with Chad's "computation" or why they disagree with it. For that reason, simply stating that $\lim_{x\to 0}(1+x)^{1/x} = e$ or writing "no" without any corresponding reasoning is worth zero points. Scoring for this problem is on a 0–3 scale, and a student's score depends on the *category* under which his response falls. If a student's response appears to fit two different categories, the student receives points based on the "better" of the two categories. The table below provides a conversion from the category of response to the score from 0–3.

Category	Score
A	3
B, C	2
D, E	1
F, G, H	0

Category A: A student whose response falls under this category does these two things: (a) he disagrees with Chad's answer; (b) he claims that the exponent is unbounded as $x \to 0$. Note that although a student may couch this idea in technically incorrect terms like "the limit equals 1^{∞}," his basic understanding of the limit is arguably sound, so his answer falls under the best category.

Category B: A student whose response falls under this category does these two things: (a) he agrees with Chad's answer; (b) he claims that the exponent is unbounded as $x \to 0$. A student whose response fits this category understands the exponent is getting absolutely large, but supposes this does not change the fact that the limit is still just 1.

Category C: A student whose response falls under this category does these two things: (a) he disagrees with Chad's answer; (b) he claims that Chad cannot "plug in" zero for x because x is approaching zero, not equal to it. A student whose response fits this category understands that this limit is about what happens *near* $x = 0$ rather than what happens *at* $x = 0$, but he does not recognize the importance of the exponent's unbounded behavior to the limit evaluation.

Category D: A student whose response falls under this category does these two things: (a) he disagrees with Chad's answer; (b) he claims that Chad cannot "plug in" zero for x because $\frac{1}{0}$ is undefined. Note that a student may state this in a different way (e.g., $\sqrt[0]{1}$ does not exist, the exponent is undefined, etc.) A student whose response fits this category understands that Chad cannot "plug in" zero in for x in the base and not do so for x in the exponent, but he is not apparently cognizant of the fact that the limit is *not* actually about what happens *at* $x = 0$.

Category E: A student whose response falls under this category does these two things: (a) he agrees with Chad's answer; (b) he admits that the $\frac{1}{0}$ "problem" exists but does not ultimately matter. It is probable that such a student believes that the x in the base somehow goes to zero before the x in the exponent does.

Category F: A student whose response falls under this category disagrees with Chad's answer and does *one* of the following things: (a) he argues that Chad must simplify the expression before evaluating the limit; (b) he makes one correct computation of the expression for some *x*-value near zero and concludes the limit is not 1 based on that computation. A student whose response fits this category demonstrates some "knowledge" of algebra, but no real understanding of the limit concept of calculus.

Category G: A student whose response falls under this category either agrees or disagrees with Chad's answer, but his reasoning is nonsensical, nonexistent, or it does not fit one of the basic "logical" themes from the previous categories. With such a response, the student demonstrates no real understanding of the limit concept.

Category H: A student who fails to write any response down whatsoever has that "response" placed in this category.

<u>Instantiation Item 3.</u>

Students are asked to give the formula of a single function which satisfies four listed conditions. Scoring for this problem is on a 0–3 scale based on the number of listed conditions which are satisfied by the student's response. The following table supplies the conversion from the number of satisfied conditions to the score from 0–3.

# of Satisfied Conditions	Score
4	3
3	2
1–2	1
0	0

If a student's response contains formulas for several different functions instead of just one, then a listed condition is counted as being satisfied only if *all* the different functions in the student's response satisfy it. This rule is also followed for any student's response containing a "double definition" which cannot be easily rectified (for details, see the "Exception" below). For example, if a student's response is

$$f(x) = \begin{cases} x & x = 0 \\ x+3 & x \le 0 \\ x^2 & x < 0 \\ 2 & x > 0 \end{cases}$$

the student does not receive credit for any of the first three conditions, because none of those conditions is satisfied by *all* of the formulas with applicable domains that are listed. However, if the second formula above was written as $-x+3$ (for $x \le 0$), then the student receives credit for satisfying the third condition of $\lim_{x \to -\infty} f(x) = +\infty$ because both applicable formulas satisfy this condition. [*Exception:* If a student's "double definition" can be partially or fully repaired *without* generating another "double definition" by simply changing the *directionality* of an arrow or two (e.g., "$\le$" in place of "$\ge$"), then the problem is scored as if these "bad" arrows are pointing the right way. However, "triple definitions" are not to be considered repairable.]

If a student fails to provide a domain declaration for part of his formula, the domain is assumed to be the "natural domain" of that formula minus all the *x*-values which the student has clearly identified with another formula. For example, if a student's response is as follows:

$$f(x) = \begin{cases} x & x < 1 \\ \sqrt{x} & \end{cases}$$

then the domain for the second formula is assumed to be $x \geq 1$.

In Math 181, students consider only functions $f : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$. However, a student may declare domains that makes his function undefined for certain *x*-values or such that his function takes on nonreal values for certain *x*-values. Additionally, a student may write the "∞" symbol in one or more formulas (which is nonsensical in the context of Math 181). In such cases, the overall function is to be considered undefined at any such *x*-value as described in this paragraph. [*Exception:* Any formula corresponding to a declared domain with a *single* infinite "endpoint" of $\pm\infty$ (e.g., a domain of $x < -\infty$ or $x > \infty$) is to be ignored, because a student employing such a domain is arguably just trying to make the limits as $x \to \pm\infty$ work out (unsuccessfully, of course—but points for the other three conditions could still be suitably earned).]

Note that the two "Exceptions" above may *both* apply to the same student's response—for example, the arrow on $x < \infty$ can be reversed using the first "Exception," then the corresponding formula ignored using the second "Exception."

A student's formula declaration of "$y = x^2$" instead of just "x^2" or even "$x = 0$" instead of just "0" is not to be counted against him, as long as the desired formula is clear.

If a student misinterprets this problem to mean "which of the following conditions is satisfied by $g(x) = \begin{cases} x+1 & x \leq 10 \\ \sqrt{x} & x > 10 \end{cases}$," he receives zero points. If a student produces a graph with no accompanying formula, he receives zero points.

APPENDIX D:

BASIC PCA AND LUA ADMINISTRATION PROTOCOLS

<u>Basic PCA Administration Protocol</u>

Preparation

(1) Write the following on the board: “Please pick up a test booklet and an answer sheet. Do NOT open the test booklet before being told to do so. Place the test face-up on your desk and wait for further instructions.”

(2) Take all of the test booklets and all of the answer sheets out of the appropriate envelopes and place them in the front where students can come get them.

(3) As students enter the classroom, direct them to take a test booklet and an answer sheet. Make sure no one begins the test early or looks inside the test booklet before you tell them to do so.

Administration

(1) At precisely the time class normally begins, tell students that the testing period is about to start and to take their seats (if they have not done so).

(2) State the following: “Please fill out the requested demographic information on both the test booklet and on the answer sheet, including your first and last initial on the top right corner of the answer sheet. Do *not* write any other identifying information on the booklet or on the answer sheet except for the information that is specifically requested. I will now give you one minute to accomplish this task.” Look to see when students are finished filling out the demographic information portion on the test booklet cover and on the answer sheet. When everyone is done, move on to the next step.

(3) State the following: “You will have 50 minutes to complete this quiz, which contains 25 multiple-choice items. It counts for 15 points toward your overall grade. The responses you record on your answer sheet will also be used as research data for a doctoral dissertation. Calculators are *not* allowed on the quiz. Use pencil and feel free to do your work directly on the test booklet. Please note, however, that only your circled answers on the separate answer sheet will be scored. There is a misprint on #18, which has been corrected here on the whiteboard. You may now begin the quiz. Good luck!”

(4) Count the number of students who are taking the test. Write “students =” along with the number of students on the envelope from which you took the booklets. Then repeat this step with the answer sheets.

(5) Give students 50 minutes to complete the quiz <u>from the time you finish administration step #3.</u>

Post-Administration

Place all of the students’ test booklets, along with the unused test booklets, back into the appropriate folder. Then repeat this step with the answer sheets. Close the envelopes if possible. The envelopes should be returned to Taylor Jensen.

<u>Basic LUA Administration Protocol</u>

Preparation

(1) Write the following on the board: “Please pick up a test booklet and a handout. Do NOT open the test booklet before being told to do so. Place the test face-up on your desk and wait for further instructions.”
(2) Take all of the test booklets and all of the accompanying handouts out of the appropriate envelopes and place them in the front where students can come get them.
(3) As students enter the classroom, direct them to take a test booklet and a handout. Make sure everyone places their test booklets face-up on their desks and that no one begins the test early or looks inside the test booklet before you tell them to do so.

Administration

(1) At precisely the time class normally begins, tell students that the testing period is about to start and to take their seats (if they have not done so).
(2) State the following: “Please fill out the requested demographic information on the test booklet. In addition, please write your first and last initial on the top right corner of the test booklet. Do *not* write any other identifying information on the booklet except for your initials, last 4 digits of your student ID number, and your instructor’s name. I will now give you one minute to accomplish this task.” Look to see when students are finished filling out the demographic information portion on the test booklet cover. When everyone is done, move on to the next step.
(3) State the following: “You will have 50 minutes to complete this quiz, which contains items designed to test your understanding of the limit concept from calculus. Although there are 8 quiz problems and most of the problems will be graded on a 0–3 scale, your score will be rescaled so as to count out of 15 points possible toward your overall grade in Math 181. The responses you record on your test booklet will also be used as research data for a doctoral dissertation. Calculators are *not* allowed on the quiz. Use pencil and feel free to do your work directly on the test booklet. If you need more space on a problem than the space provided, please use the back of the cover sheet to finish it. You may now begin the quiz. Good luck!”
(4) Count the number of students taking the test. Then write “students = [# of students taking the test]” on the envelope from which you took the test booklets.
(5) Give students 50 minutes to complete the quiz <u>from the time you finish administration step #3.</u>

Post-Administration

Place all of the students’ test booklets, along with the unused test booklets, back into the appropriate envelope and close it. All envelopes containing test booklets should be returned to Taylor Jensen. Students may keep the LUA handouts or throw them away as they see fit.