MONTANA STATE UNIVERSITY

Optimal control of sampled-data and stochastic distributed-parameter systems
by Vichit Lorchirachoonkul

A thesis submitted to the Graduate Faculty in. partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Electrical Engineering
Montana State University
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Abstract:
The subject of this thesis is the development of concepts and techniques of optimal design for linear deterministic as well as stochastic distributed-parameter systems.

The primary purpose of the thesis project is the development of computationally feasible methods for optimal design of distributed-parameter systems under a linear or convex performance measure of the distributed error of the system.

The content of the thesis is summarized as follows: First, a new definition of distributed correlation function is introduced to facilitate the study of stochastic control of distributed-parameter systems. Relationships are derived which paralleled well-known relationships in the statistical analysis of lumped-parameter systems. With this definition of distributed correlation function, an optimal control problem is shown to be a Wiener-Hopf spectrum-factorization problem.

Second, a computationally feasible method is developed for optimal control of distributed-parameter systems. By use of the method, the optimization problem is reduced to one of linear programming for which well-known solution algorithms are available. Performance is optimized with respect to absolute values of distributed errors weighted both in time and in space. Magnitude constraints on both control and state-variable functions are incorporated. Both distributed and non-distributed inputs to the system are treated. Third, the optimal control of distributed-parameter systems is studied from dynamic programming point of view. A functional equation associated with a convex performance measure is derived with properties analoguous to those of the lumped-parameter case. A computationally feasible method is developed for minimizing the spatial-average-integral-square error of a class of distributed-parameter systems. With this method, feedback control of distributed-parameter systems can be achieved. Fourth, the theory of deadbeat response with minimal overshoot compromise is generalized to include systems enfolding time delay in the loop. The optimal trade-off between deadbeat-response time for a ramp input and step-response overshoot is derived for a single-loop sampled-data system with time delay.

In addition, sensitivity of the output response is studied as a function of parameter variations in the optimal controller, and stability limits are derived in terms of a change-in-gain parameter.

OPTIMAL CONTROL OF SAMPLED-DATA AND STOCHASTIC DISTRIBUTED-PARAMETER SYSTEMS

By

Vichit Lorchirachoonkul

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ABSTRACT

The subject of this thesis is the development of concepts and techniques of optimal design for linear deterministic as well as stochastic distributed-parameter systems.

The primary purpose of the thesis project is the development of computationally feasible methods for optimal design of distributed-parameter systems under a linear or convex performance measure of the distributed error of the system.

The content of the thesis is summarized as follows: First, a new definition of distributed correlation function is introduced to facilitate the study of stochastic control of distributed-parameter systems. Relationships are derived which paralleled well-known relationships in the statistical analysis of lumped-parameter systems. With this definition of distributed correlation function, an optimal control problem is shown to be a Wiener-Hopf spectrum-factorization problem. Second, a computationally feasible method is developed for optimal control of distributed-parameter systems. By use of the method, the optimization problem is reduced to one of linear programming for which well-known solution algorithms are available. Performance is optimized with respect to absolute values of distributed errors weighted both in time and in space. Magnitude constraints on both control and state-variable functions are incorporated. Both distributed and non-distributed inputs to the system are treated. Third, the optimal control of distributed-parameter systems is studied from dynamic programming point of view. A functional equation associated with a convex performance measure is derived with properties analoguous to those of the lumped-parameter case. A computationally feasible method is developed for minimizing the spatial-average-integral-square error of a class of distributed-parameter systems. With this method, feedback control of distributed-parameter systems can be achieved. Fourth, the theory of deadbeat response with minimal overshoot compromise is generalized to include systems enfolding time delay in the loop. The optimal trade-off between deadbeat-response time for a ramp input and step-response overshoot is derived for a single-loop sampled-data system with time delay. In addition, sensitivity of the output response is studied as a function of parameter variations in the optimal controller, and stability limits are derived in terms of a change-in-gain parameter.

CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1 PURPOSE OF THE THESIS PROJECT

The purpose of the thesis project is fivefold:

1. To review, evaluate and summarize the published work on optimization of distributed-parameter control systems.
2. To introduce a new definition of distributed correlation functions to facilitate the study of distributed-parameter control systems with stochastic forcing functions.
3. To develop an optimal design theory, based on linear programming techniques, for distributed-parameter systems with distributed outputs.
4. To develop a computational algorithm for optimal control functions of distributed-parameter systems based on dynamic programming techniques.
5. To generalize the theory of deadbeat responses with minimal overshoot compromise to apply to sampled-data systems enfolding time delay.

1.2 VALUE OF THE THESIS RESEARCH

In many situations, the energy of a system is spatially distributed throughout a medium. Examples of such systems are transmission lines, heat exchangers, nuclear reactors, chemical reactors, distributed RC networks, biological systems, et cetera. Numerous applications of distributed-parameter-systems theory are given in the literature (39, 60). Generally, the characteristics of a distributed-parameter system can be described by a set of partial differential equations with appropriate boundary conditions.

Recently, Kolb (39) noted that the development of a general body of theory for the analysis and synthesis of sampled-data, class-II-type,* distributed-parameter systems similar to that developed by Pierre (60) for class-I-type systems* would provide a worthy subject of investigation. Wang and Tung (86) suggest that a control theory for distributed systems may be developed in parallel with the existing theory for lumped-parameter systems. Also Pierre (60) states that a topic worthy of additional research is the development of a general body of theory for effecting the design of optimal distributed-parameter control systems, based on minimization of integral-square or mean-square error, subject to stated constraints.

Therefore, as will be evident in Chapters 3 and 4, the primary contributions of this thesis research consist of the development of a body of theory which satisfies these needs to a large extent. In addition, the theory of deadbeat response with minimal overshoot compromise (68) is generalized to sampled-data systems enfolding time delay. This enables one to apply this theory to cases where, for example, the speed of an on-line digital computer is not sufficiently fast and, thus, results in a pure time delay in the loop.

1.3 SCOPE OF THE THESIS

Chapter 2 enfolds the concept of distributed correlation functions in stochastic linear distributed-parameter systems. A new definition of

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*Pierre and Kolb (67) define class-I-type distributed-parameter systems as those which ostensibly contain no space-distributed inputs or outputs and class-II-type distributed-parameter systems as those containing space-distributed inputs or space-distributed outputs or both.

distributed correlation function is introduced and is shown to be appropriate for distributed-parameter systems with distributed outputs. An optimal stochastic control problem is solved with the aid of this distributed correlation function.

Chapter 3 is devoted to the development of linear programming techniques for use in the optimal design of linear distributed-parameter systems with distributed outputs. The time-and-space summation of absolute values of distributed error, with weighted coefficients, at discrete time-and-space points is minimized subject to linear constraints to obtain desirable control functions.

In Chapter 4, optimization of linear distributed-parameter systems is treated from a dynamic programming point of view. Properties of functionals are derived which are similar to those which apply to lumped-parameter systems. Also, a closed-form expression of the optimal control function is derived for the case of a quadratic performance measure of distributed error.

In Chapter 5, the theory of deadbeat response with minimal overshoot compromise is generalized to apply to both minimal and nonminimal phase plants enfolding time delay.

Finally, the concluding Chapter 6 contains a summary of the essential values of the thesis research and some aspects of the research which merit additional study.

1.4 LITERATURE REVIEW

The literature on distributed-parameter systems is extensively reviewed by Kolb (39) and Pierre (60). In this section, a survey of current literature

related to the general scope of this dissertation is made. The work directly related to the material in this dissertation is reviewed appropriately in the introductory sections of the chapters.

1.4.1 Optimization Techniques Applicable to Distributed-Parameter Systems

In 1960, the first major study of optimal control of distributed-parameter systems was made by Butkovskii and Lerner (12). They formulated three general optimal control problems associated with distributed-parameter systems. Later, Butkovskii (16) derived a maximum principle for distributed-parameter systems analogous to Pontryagin's maximum principle (69) for lumped-paramter systems. Solutions which satisfy the necessary conditions associated with this maximum principle are not obtainable in general without use of approximation (67). In reference (15), Butkovskii considered representations of distributed-parameter systems by corresponding spatially discretized models. By use of this approach, sets of partial differential equations characterizing such systems are reduced to sets of ordinary differential equations to which Pontryagin's maximum principle can be applied directly, but the usual difficulties of applying this principle to high-order systems is encountered. Also for this discretized model, Butkovskii (19) showed that a class of the optimal control problems associated with distributed-parameter systems can be solved by the method of moments (1). In reference (20), Butkovskii and Poltavskii solved the time-optimal control problem of the oscillating string with the aid of Krein's L-problem of moments (1) and variational methods.

The classical Krein's L-problem of moments (1) can be stated as follows: given n linearly independent vectors $h_1, h_2, \ldots, h_n$ of a normed linear

space E and n real numbers $c^1, c^2, \ldots, c^n$, not all zero, and the positive real number L, find the necessary and sufficient conditions on $c^1, c^2, \ldots, c^n$, and L for there to exist a linear functional f(h) which satisfies

$$f(h_i) = c^i \tag{1.1}$$

and

$$\|f\| = \sup_{\|h\|} \frac{f(h)}{\|h\|} \leq L \tag{1.2}$$

In 1957, Krassovskii (43) reduced the time-optimal control problem in lumped-parameter system theory to Krein's L-problem of moments. He used some properties derived by Krein (1) to obtain the expression for an optimal input which is bounded in amplitude. Two years later, Kulikowskii (45, 46, 47, 48) showed the Krein's results can be directly applied to derive time-optimal control with amplitude, energy, power, and fuel constraints. Also, Kulikowskii (48, 49) used results from functional analysis to treat certain problems associated with nonlinear plants and adaptive systems. The works by Krassovskii and Kulikowskii were generalized by Kranc and Sarachik (42) to apply to the case of a time-optimal n^{th}-order system with r inputs. Recently, Chaudhuri (22) further generalized the works by Kranc and Sarachik (42) to apply to the optimal design of linear distributed-parameter systems. But the difficulties in determining the optimal time of the system (22) are quite apparent. Swiger (80) showed that the problem of minimum-effort control of distributed-parameter systems can be solved by the theory of minimum normed operators (53). For an exact solution, this approach requires solution of an infinite number of linear algebraic equations with an infinite number of unknowns. But if only a finite number of points in the spatial

domain is used to represent the entire domain, the problem can be reduced to Krein's L-problem of moments. For a Mayer problem in distributed-parameter systems which contain nondistributed outputs, Katz (38) applied functional analysis and variational methods to formulate a general set of necessary conditions to be satisfied by an optimal system. Andreyer and Butkovskii (2) solved the time-optimal control problem for heating a massive body in a furnace by use of a spatial-harmonic-truncated model and functional analysis.

Of systems enfolding time delay, Newton, et al. (57), demonstrated the use of the Wiener-Hopf spectrum-factorization technique (64, 57) to effect minimum-integral-square or mean-square-error design. In order to effect the factorization, Pade approximations (84) were used to enable the application of the Wiener-Hopf technique in the usual manner. Pierre (62, 66) generalized the approach of Newton, et al. (57), to apply to linear distributed-parameter systems which contain no space-distributed inputs or outputs. By applying the Wiener-Hopf spectrum-factorization technique, Pierre derived necessary and sufficient conditions for the design of linear controllers to obtain a minimum-mean-square error. Kolb (39) and Kolb and Pierre (40) further generalized the Wiener-Hopf technique for the design of linear distributed-parameter systems with spatially distributed inputs or outputs or both. To effect the factorization, Pierre (62, 66), Kolb (39) and Kolb and Pierre (40) used rational fraction approximations (21, 41, 61) for the nonrational terms. Moreover, Kolb (39) discussed general classes of performance measures applicable in distributed-parameter systems. These performance measures paralleled those for lumped-parameter systems considered by Schultz and Rideout (75). Also, Kolb (39) clarified Senin's work (77) on the statistical design of distributed-parameter systems. In essense, the

concepts of correlation function analysis were extended to include distributed-parameter systems.

Goodson and Khatri (30) used the calculus of variations to obtain a necessary condition for optimality of a distributed system described by an integral equation. The output of the system is not spatially distributed. To obtain an explicit form of the optimal feedback control system, truncation of an infinite series is required. For another approach (30), the transcendental Laplace-transform transfer function of the distributed-parameter plant is approximated by a few terms in the infinite product approximations used by Oldenburger and Goodson (58); then Pontryagin's maximum principle is applied to obtain the optimal feedback control law for the approximate model.

Using dynamic programming techniques, Brogan (11) reduced the optimal control problems for a class of distributed-parameter systems to the solution of a pair of two-point boundary-value partial differential equations. The specific form of the boundary conditions depends upon the particular problem being considered and must be obtained by considering transversality conditions. Numerical algorithms for solving the two-point boundary-value problems are developed. Also Brogan (11) showed that requirements for controllability (37) of lumped-parameter systems guarantee only ε-controllability when applied to distributed-parameter systems. A system is said to be ε-controllable if the state of the system can be forced into an epsilon neighborhood of the desired state within a finite amount of time, but it is not necessary for the system to be maintained in this neighborhood. Miranker (56) also derived necessary and sufficient conditions for

ε-controllability of a certain class of distributed-parameter systems. To clarify the meaning of controllability of lumped-parameter systems Lemay (51, 52) defined the term reachability and maintainability. In essence, a system is said to have the reachable property if it can be driven from its initial state to the desired final state in finite time, and a system is said to have the maintainability property if controlling actions are capable of maintaining the system at this desired state throughout the remainder of the process. For distributed-parameter systems, Kolb (39) introduced the terms ε-reachability and ε-maintainability as follows: A system is said to be ε-reachable for a set of states if admissible control signals exist which will take the system from a given initial state to within an ε-neighborhood of any given desired state from the set of desired states in a finite time $T(\varepsilon)$; and a system is said to be ε-maintainable for a set of states if it is ε-reachable for the set of states and if, in addition, admissible control signals exist which will maintain any of the states in the set within a specified ε-neighborhood for all T greater than $T(\varepsilon)$. Also, Wang and Tung (86) investigated controllability and observability (37) of distributed-parameter systems.

McCausland (54, 55) developed three approximate methods, namely, a subdivision or finite difference method, a harmonic or Fourier series method, and a parabolic method, for synthesizing bang-bang controllers to obtain minimum-time performance for a linear diffusion process in which control inputs, applied at the spatial boundary, are subject to absolute-value constraints. In addition, McCausland (54, 55) investigated conditions for controllability of linear distributed-parameter systems.

Sakawa (74) derived necessary and sufficient conditions for optimality of one-dimensional linear stationary distributed-parameter systems which are controlled by boundary control functions. The performance measure used by Sakawa (74) is the spatial integral of the squared error at the terminal time, augmented by an energy measure of the control variables. When the control variables are not required to satisfy additional constraints, the optimal control is in the form of Fredholm integral equations of the second kind. But if the control functions are magnitude-limited, the optimal control is in the form of nonlinear integral equations. Egorov (25, 26) derived necessary and sufficient conditions, in the form of integral inequalities, for optimal, linear distributed-parameter systems. These same conditions are merely necessary conditions if the system is nonlinear.

Recently, several computational algorithms (3, 4, 11, 24 80) were developed to solve optimal control problems associated with distributed-parameter systems. Axelband (3, 4) applied function-space methods to effect the optimal control of distributed-parameter systems. In reference (4), Axelband suggested a convex programming algorithm (33) to solve the problem of minimizing the norm of the difference between desired system response and actual system response of linear distributed-parameter systems with bounded inputs. Denn, et al. (24), developed a computational algorithm based on the method of steppest ascent (33, 64) to solve an optimal control problem in conjunction with the necessary conditions for optimality which are derived by variational methods.

Erzberger and Kim (27) considered the problem of designing an optimum controller for boundary control of distributed-parameter systems based on a

quadratic error criterion. By applying dynamic programming, Erzberger and Kim derived a Hamilton-Jacobi equation for distributed-parameter systems, under a quadratic error criterion, the general solution of which is not obtainable by use of existing methods.

Recently Yeh and Tou (88) used Butkovskii's maximum principle (16, 18) to derive an optimal control function for a class of distributed-parameter systems in terms of integral equations. Uzgiris and D'Souza (83) used the properties developed by Krein (1) to obtain a time-optimal control function for a plant described by a linear partial differential equation of parabolic type with nonlinear boundary conditions. The model used in reference (83) is the spatially discretized one. However, Uzgiris and D'Souza did not justify application of the Krein's L-problem (1) to a class of nonlinear functional equations.

1.4.2 Modeling

In recent years, much attention has been devoted to modeling distributed-parameter systems. Pierre (60) and Pierre and Higgins (65) used sampled-data systems (35, 50, 72, 82) to represent distributed-parameter systems. Distributed-parameter elements can be isolated from the lumped-parameter elements by suitably selected sample-and-hold circuits. Selection of the sampling period is discussed in the literature (36, 65). Recently, Gravey and D'Souza (32) modified one of Pierre's sampled-data models (60, 65) by using only one sample-and-hold circuit. Gaither, et al., (28) proposed two techniques, namely, quantization and integral transformation, for modeling distributed-parameter systems. If the spatial variables are quantized, i.e., broken into "lumps" while the time variable is continuous, partial

differential equations are reduced to sets of ordinary differential equations. All the lumped-parameter techniques of modeling can then be applied. In the integral transformation method, only a finite number of spatial harmonics is considered in the model. Gaither, et al., found close agreement between the results in the frequency domain for both methods.

In his Ph.D. dissertation (39), Kolb used four different models for linear distributed-parameter systems. For distributed-parameter systems with boundary inputs and a distributed output, two models -- a series-form representation and a parallel-form representation -- are developed on the basis that only values of a distributed state variable at specific spatial points are of interest. In the series-form representation, the value of distributed state variable at a given point in the spatial domain is expressed in terms of the value of the distributed state variable at an adjoining point in the spatial domain. But in the parallel-form representation, the value of distributed state variable at each point of interest in the spatial domain is written in terms of the boundary inputs. The third model, the spatial-harmonic representation, is one in which a distributed-parameter system is represented by a harmonic series in the spatial variable. Generally, because of the presence of an infinite number of spatial harmonics, an infinite number of blocks is required to represent a distributed-parameter system exactly with this model. The fourth model developed by Kolb for linear distributed-parameter systems with distributed inputs and outputs involves implicit transfer functions between distributed input variables and outputs.

CHAPTER 2

CORRELATION FUNCTIONS AND OPTIMAL CONTROL OF DISTRIBUTED-PARAMETER SYSTEMS

2.1 INTRODUCTION

Methods of statistical analysis and synthesis are of importance for the design of controllers for systems which contain random parameters, measurement noises, or random forcing functions. A special form of correlation function for distributed-parameter systems was introduced by Itskovich (34) in 1963. Itskovich used a discrete form of correlation function to correlate output responses at two particular points in the space. Using this concept, he determined the optimum location of pickups in a distributed field. In 1964, Senin (76) extended Itskovich's ideas by defining a continuous form of correlation function. With his definition of correlation function, Senin (76,77) derived relationships between the correlation functions of the input and the output signals in a system characterized by the heat equation with several types of boundary conditions. Recently, Kolb (38) clarified and expanded the theory developed by Senin in reference (76).

In this chapter, a new definition of correlation function is introduced for use with distributed-paramter systems that are time and spatial in variant and that are excited by random signals which satisfy the ergodic hypothesis. In Section 2.2, relationships are derived which parallel well-known relationships in the statistical analysis of lumped-parameter systems. The optimal control problem of Section 2.3 is shown to be a Wiener-Hopf spectrum-factorization problem, the study of which is facilitated by the use of the new definition of distributed correlation function.

2.2 PROPERTIES OF THE DISTRIBUTED CORRELATION FUNCTION

In order to develop a method of statistical analysis in distributed-parameter systems in parallel to an existing method in lumped-parameter

systems, the autocorrelation function of a distributed signal $y_1(x,t)$ whose domain of definition is $[0,L] \times (-\infty,\infty)$ is defined as

$$R_{y_1y_1}(x,\tau) \triangleq \lim_{T\to\infty} \frac{1}{2TL} \int_{-T}^{T} \int_{0}^{L} y_1(\xi,t)y_1(x+\xi,t+\tau)\, d\xi dt \tag{2.1}$$

and the cross-correlation function of two distributed signals $y_1(x,t)$ and $y_2(x,t)$ is defined as

$$R_{y_1y_2}(x,\tau) \triangleq \lim_{T\to\infty} \frac{1}{2TL} \int_{-T}^{T} \int_{0}^{L} y_1(\xi,t)y_2(x+\xi,t+\tau)\, d\xi dt \tag{2.2}$$

The definitions of the distributed correlation functions in Equations 2.1 and 2.2 are somewhat different from those used by Senin (76, 77). Senin (76) defined the autocorrelation function of a distributed signal as

$$R_{y_1y_1}(x,z,\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y_1(x,t)y_1(z,t+\tau)\, dt \tag{2.3}$$

The advantage to the definition of the distributed autocorrelation function in Equation 2.1 is that when $x = 0$ and $\tau = 0$, the distributed autocorrelation function reduces to the time-and-space mean-square value of $y_1(x,t)$. This is not the case for the distributed autocorrelation function of Equation 2.3. When $x = z = x_1$ and $\tau = 0$, the distributed correlation function of Equation 2.3 yields only the time mean-square value of $y_1(x,t)$ at the point $x = x_1$. Therefore, the definition of the correlation function in Equation 2.3 is more appropriate if the output response of a system is not distributed; if the output is distributed, the definition of the correlation function in Equation 2.1 is more appropriate.

In conjunction with the definition of distributed cross-correlation function in Equation 2.2, the cross-correlation function associated with $y_1(x,t)$ and $y_2(x,t)$ can be written as

$$R_{y_2y_1}(x,\tau) = \lim_{T\to\infty} \frac{1}{2TL} \int_{-T}^{T} \int_{0}^{L} y_2(\xi,t)y_1(x+\xi,t+\tau)\, d\xi dt \tag{2.4}$$

By change of dummy variables of integration, it can be shown that

$$R_{y_1y_2}(x,\tau) = R_{y_2y_1}(-x,-\tau) \tag{2.5}$$

A correlation function between a distributed signal and a lumped signal is of use in statistical analysis when a boundary input of a distributed-parameter system is stochastic in nature. Therefore, the cross-correlation function associated with a distributed signal $y_1(x,t)$ and a lumped signal $y_3(t)$ is defined here as

$$R^M_{y_1y_3}(x,\tau) \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y_1(x,t)y_3(t+\tau)\, dt \tag{2.6}$$

The superscript M of R in Equation 2.6 denotes the mixture of the distributed and lumped signals in the correlation function. It can be shown that

$$R^M_{y_1y_3}(x,\tau) = R^M_{y_3y_1}(x,-\tau) \tag{2.7}$$

where

$$R^M_{y_3y_1}(x,\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y_3(t)y_1(x,t+\tau)\, dt \tag{2.8}$$

To use correlation functions in the analysis and synthesis of linear distributed-parameter systems, relationships between correlation functions

of distributed output signals and input signals of a system must be established. For simplicity, consider a distributed-parameter stable linear system, having a single stochastic input u(t) and a single distributed output y(x,t), which is characterized by

$$y(x,t) = \int_{-\infty}^{\infty} g(x,\sigma)u(t-\sigma)\, d\sigma \tag{2.9}$$

where g(x,t) is the Green's function or impulse response of the system.

The autocorrelation function associated with the distributed signal y(x,t) is given by the definition in Equation 2.1. After appropriate substitution of the right-hand member of Equation 2.9 into Equation 2.1, followed by an interchange in the order of integration,* it reads

$$R_{yy}(x,\tau) = \frac{1}{L} \int_0^L d\xi \int_{-\infty}^{\infty} d\sigma_1 g(\xi,\sigma_1) \int_{-\infty}^{\infty} d\sigma_2 g(x+\xi,\sigma_2) \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\, u(t-\sigma_1)u(t+\tau-\sigma_2)$$

$$= \frac{1}{L} \int_0^L d\xi \int_{-\infty}^{\infty} d\sigma_1 g(\xi,\sigma_1) \int_{-\infty}^{\infty} d\sigma_2 g(x+\xi,\sigma_2) R_{uu}(\tau+\sigma_1-\sigma_2) \tag{2.10}$$

where $R_{uu}(\tau)$ is the autocorrelation function associated with the lumped signal u(t).

Operationally, Equation 2.10 can be simplified by use of transform theory. Following a similar definition used for lumped-parameter systems,

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*In this chapter, the integrand is assumed to be integrable so that the interchange of the order of integration is valid.

the distributed spectral density $S_{yy}(x,s)$ is defined as the two-sided Laplace transform with respect to τ of $R_{yy}(x,\tau)$. $R_{yy}(x,\tau)$ is, of course, assumed to be Laplace transformable. Thus,

$$S_{yy}(x,s) = \int_{-\infty}^{\infty} d\tau R_{yy}(x,\tau)e^{-s\tau} \tag{2.11}$$

When the right-hand member of Equation 2.10 is substituted for $R_{yy}(x,\tau)$ in Equation 2.11, the result is

$$S_{yy}(x,s) = \frac{1}{L}\int_0^L d\xi \int_{-\infty}^{\infty} d\sigma_1 g(\xi,\sigma_1) \int_{-\infty}^{\infty} d\sigma_2 g(x+\xi,\sigma_2) \int_{-\infty}^{\infty} d\tau R_{uu}(\tau+\sigma_1-\sigma_2)e^{-s\tau}$$

$$= \frac{1}{L}\int_0^L d\xi \int_{-\infty}^{\infty} d\sigma_1 g(\xi,\sigma_1)e^{-s\sigma_1} \int_{-\infty}^{\infty} d\sigma_2 g(x+\xi,\sigma_2)e^{s\sigma_2}$$

$$\int_{-\infty}^{\infty} d\sigma_3 R_{uu}(\sigma_3)e^{-s\sigma_3}$$

$$= \frac{1}{L}\int_0^L d\xi G(\xi,s)G(x+\xi,-s)S_{uu}(s) \tag{2.12}$$

Another important property of the distributed correlation function merits consideration: the relationship of the autocorrelation function of a distributed error $e(x,t)$ in terms of correlations of an actual output $y(x,t)$ and a desired output $y^d(x,t)$. The distributed error $e(x,t)$ is defined as the difference between the desired response and the actual output. Mathematically,

$$e(x,t) = y^d(x,t) - y(x,t) \tag{2.13}$$

By definition,

$$R_{ee}(x,\tau) = \lim_{T\to\infty} \frac{1}{2TL} \int_{-T}^{T} dt \int_0^L d\xi e(\xi,t)e(x+\xi,t+\tau)$$

$$= \lim_{T\to\infty} \frac{1}{2TL} \int_{-T}^{T} dt \int_{0}^{L} d\xi [y^d(\xi,t) - y(\xi,t)] [y^d(x+\xi,t+\tau) - y(x+\xi,t+\tau)]$$

$$= R_{y^d y^d}(x,\tau) - R_{yy^d}(x,\tau) - R_{y^d y}(x,\tau) + R_{yy}(x,\tau) \qquad (2.14)$$

The corresponding distributed spectral density $S_{ee}(x,s)$ is

$$S_{ee}(x,s) = S_{y^d y^d}(x,s) - S_{yy^d}(x,s) - S_{y^d y}(x,s) + S_{yy}(x,s) \qquad (2.15)$$

The last two relationships to be derived are the relationship between $S_{yy^d}(x,s)$ and $S_{uy^d}(x,s)$ and that between $S_{y^d y}(x,s)$ and $S_{uy^d}(x,s)$. By definition in Equation 2.1,

$$R_{yy^d}(x,\tau) = \lim_{T\to\infty} \frac{1}{2TL} \int_{-T}^{T} dt \int_{0}^{L} d\xi y(\xi,t) y^d(x+\xi,t+\tau)$$

$$= \lim_{T\to\infty} \frac{1}{2TL} \int_{-T}^{T} dt \int_{0}^{L} d\xi y^d(x+\xi,t+\tau) \int_{-\infty}^{\infty} d\sigma u(t-\sigma) g(\xi,\sigma)$$

$$= \frac{1}{L} \int_{0}^{L} d\xi \int_{-\infty}^{\infty} d\sigma g(\xi,\sigma) R^M_{uy^d}(x+\xi,\tau+\sigma) \qquad (2.16)$$

And by use of Equation 2.5, it can be shown that

$$R_{y^d y}(x,\tau) = \frac{1}{L} \int_{0}^{L} d\xi \int_{-\infty}^{\infty} d\sigma g(x+\xi,\sigma) R^M_{y^d u}(\xi,\tau-\sigma) \qquad (2.17)$$

The cross-spectral density $S_{yy^d}(x,s)$ is obtained by Laplace transformation of both members of Equation 2.16, with the result that

$$S_{yy^d}(x,s) = \frac{1}{L} \int_{0}^{L} d\xi G(\xi,s) S^M_{uy^d}(x+\xi,s) \qquad (2.18)$$

and similarly,

$$S_{y^d y}(x,s) = \frac{1}{L} \int_{0}^{L} d\xi G(x+\xi,-s) S^M_{y^d u}(\xi,s) \qquad (2.19)$$

The relationship between $S^M_{uy^d}(x,s)$ and $S^M_{y^d u}(x,s)$ is readily established on the basis of the corresponding correlation function relationship in Equation 2.7; thus,

$$S^M_{y^d u}(x,s) = S^M_{uy^d}(x,-s) \tag{2.20}$$

With the result of Equation 2.20, Equation 2.19 becomes

$$S_{y^d y}(x,s) = \frac{1}{L}\int_0^L d\xi G(x+\xi,-s) S^M_{uy^d}(\xi,-s) \tag{2.21}$$

2.3 OPTIMAL STOCHASTIC CONTROL

For lumped-parameter systems, a basic statistical design criterion is the mean-square-error performance measure. By analogy, statistical design of distributed-parameter systems with distributed outputs may be effected with respect to time-and-space mean-square error; namely,

$$P = \lim_{T\to\infty} \frac{1}{2TL} \int_{-T}^{T} \int_0^L \{[e(x,t)]^2\}\, dxdt \tag{2.22}$$

An augmented version of the mean-square-error criterion is

$$P_A = \lim_{T\to\infty} \frac{1}{2TL} \int_{-T}^{T} \int_0^L \{[e(x,t)]^2 + \lambda u^2(t)\}\, dxdt \tag{2.23}$$

where λ is a "cost factor" associated with the input $u(t)$.

Use of Parseval's theorem gives the frequency-domain version of Equation 2.23:

$$P_A = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} [S_{ee}(0,s) + \lambda S_{uu}(s)]\, ds \tag{2.24}$$

where $S_{ee}(x,s)$ is the distributed spectral-density function of $e(x,t)$, and $S_{uu}(s)$ is the lumped spectral-density function of $u(t)$.

Figure 2.1 depicts a general linear stable distributed-parameter system with a stochastic boundary control signal. It is assumed that the external input $v(t)$ is a stochastic signal whose spectral density $S_{vv}(s)$ is known. Also, the cross-correlation function $R^{M}_{vy^d}(x,\tau)$ is assumed to be known, where $R^{M}_{vy^d}(x,\tau)$ is the cross-correlation function of the external input $v(t)$ and the desired response $y^d(x,\tau)$. $R^{M}_{vy^d}(x,\tau)$ can be determined experimentally, if necessary, by application of well-established techiques (5,79).

The problem now is to determine the transfer function $G_c(s)$ which yields the minimum of the augmented performance measure P_A in Equation 2.23.

Before proceeding with the minimization, the relationship between $R^{M}_{uy^d}(x,\tau)$ and $R^{M}_{vy^d}(x,\tau)$ must be derived. By the convolution integral, the characteristic equation of the lumped-parameter compensator in Figure 2.1 is

$$u(t) = \int_{-\infty}^{\infty} g_c(\sigma)v(t-\sigma)\, d\sigma \tag{2.25}$$

Based on Equation 2.6, the cross-correlation function $R^{M}_{uy^d}(x,\tau)$ can be written as

$$\begin{aligned} R^{M}_{uy^d}(x,\tau) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\, u(t)y^d(x,t+\tau) \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\, y^d(x,t+\tau) \int_{-\infty}^{\infty} d\sigma\, g_c(\sigma)v(t-\sigma) \\ &= \int_{-\infty}^{\infty} d\sigma\, g_c(\sigma)\left[\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\, y^d(x,t+\tau)v(t-\sigma)\right] \\ &= \int_{-\infty}^{\infty} d\sigma\, g_c(\sigma)R^{M}_{vy^d}(x,\tau+\sigma) \end{aligned} \tag{2.26}$$

The corresponding spectral density $S^M_{uy_d}(x,s)$ is

$$S^M_{uy_d}(x,s) = G_c(s)S^M_{vy_d}(x,s) \tag{2.27}$$

From statistical analysis methods of lumped-parameter theory, the relationship between $S_{uu}(s)$ and $S_{vv}(s)$ can be written as

$$S_{uu}(s) = G_c(s)G_c(-s)S_{vv}(s) \tag{2.28}$$

With the results in Equations 2.12, 2.15, 2.18, 2.21, 2.27, and 2.28, the expression for $S_{ee}(0,s)$ can be written as

$$S_{ee}(0,s) = S_{y_d y_d}(0,s) - \frac{1}{L}\int_0^L d\xi G(\xi,s)S^M_{vy_d}(\xi,s)G_c(s) +$$

$$- \frac{1}{L}\int_0^L d\xi G(\xi,-s)S^M_{vy_d}(\xi,-s)G_c(-s) +$$

$$+ \frac{1}{L}\int_0^L d\xi G(\xi,s)G(\xi,-s)S_{vv}(s)G_c(s)G_c(-s) \tag{2.29}$$

With the results in Equations 2.28 and 2.29, Equation 2.24 can be arranged as

$$P_A = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} [A_4(s) + A_3(s)G_c(-s) + A_2(s)G_c(s) + A_1(s)G_c(s)G_c(-s)]\, ds \tag{2.30}$$

where

$$A_1(s) = [\lambda + \frac{1}{L}\int_0^L d\xi G(\xi,s)G(\xi,-s)]S_{vv}(s) \tag{2.31}$$

$$A_2(s) = -\frac{1}{L}\int_0^L d\xi G(\xi,s)S^M_{vy_d}(\xi,s) \tag{2.32}$$

$$A_3(s) = -\frac{1}{L}\int_0^L d\xi G(\xi,-s)S^M_{vy_d}(\xi,-s) \tag{2.33}$$

$$A_4(s) = S_{y_d y_d}(0,s) \tag{2.34}$$

It is shown by Pierre (64), with calculus of variations and the Wiener-Hopf spectrum-factorization technique, that if $g_c(t)$ is to be zero for t less than zero and not both of $A_2(s)$ and $A_3(s)$ are zero, the optimal expression $G_c^*(s)$ of $G_c(s)$ is given by

$$G_c^*(s) = \frac{-\left\{\dfrac{[A_2(-s) + A_3(s)]}{[A_1(s) + A_1(-s)]^-}\right\}_+}{[A_1(s) + A_1(-s)]^+} \tag{2.35}$$

where, in this case,

$$A_2(-s) + A_3(s) = -\frac{2}{L}\int_0^L d\xi G(\xi,-s)S^M_{vy_d}(\xi,-s) \tag{2.36}$$

and

$$A_1(s) + A_1(-s) = 2\left[\lambda + \frac{1}{L}\int_0^L d\xi G(\xi,s)G(\xi,-s)\right]S_{vv}(s) \tag{2.37}$$

The interpretation of the + and - superscripts and subscripts is as follows:

$$A(s) = A(s)^+A(s)^- \tag{2.38}$$

and

$$A(s) = A(s)_+ + A(s)_- \tag{2.39}$$

where the inverse Laplace transforms $L^{-1}[A(s)^+]$ and $L^{-1}[A(s)_+]$ equal zero for t less than zero, and where $L^{-1}[A(s)^-]$ and $L^{-1}[A(s)_-]$ equal zero for t greater than zero. In addition, for $G_c^*(s)$ to be physically realizable, $1/A(s)^+$ must be analytic in the right half-plane of the s domain.

When all A_i's are rational fraction functions, this process of spectrum factorization is straightforward (57, 64). When nonrational fraction terms appear in the A_i's, rational fraction approximations may be used to replace the nonrational terms (21, 41, 58, 61 and 84), and then the process of spectrum factorization can be effected to obtain an approximate solution. Examples of such approximate solutions can be found in the literature (38, 62).

2.4 CONCLUSION

A correlation-function concept for use with distributed-parameter systems is introduced to parallel the existing concept which applies to lumped-parameter systems. Relationships between correlation functions of distributed output signals and input signals of a system are established. With the use of the new definition of distributed correlation functions, an optimal control problem associated with distributed-parameter systems with random signals is shown to be solvable by the Wiener-Hopf spectrum-factorization technique. Though the spatial domain of the system considered in this chapter is of one dimension, the general approach is applicable to systems of higher dimensions, in which case the autocorrelation function is defined as

$$R_{yy}(X,\tau) = \frac{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{v} y(\xi,t)y(X+\xi,t+\tau)\, d\xi dt}{\int_{v} d\xi}$$

where v is a bounded domain of interest in an n-dimensional Euclidean space with spatial coordinate vector $X = [x_1 \; x_2 \; \ldots \; x_n]'$ and ξ is a spatial dummy vector of integration. As in lumped-parameter networks, the concept of distributed correlation function is applicable to the prediction and filtering problems in distributed-parameter networks. Details of the prediction and filtering problems in distributed-parameter networks, however, require additional study.

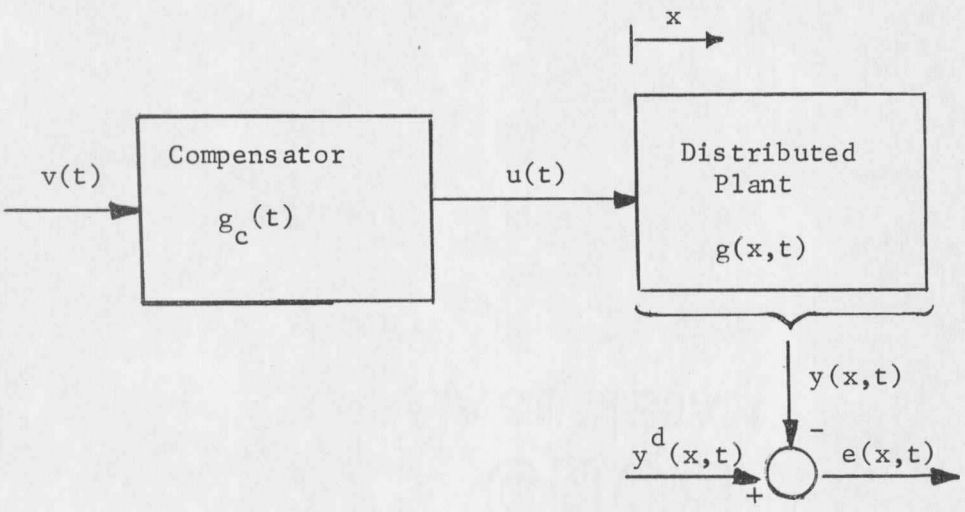


Figure 2.1. A representation of stochastic boundary control of distributed-parameter systems.

CHAPTER 3

OPTIMAL DESIGN OF DISTRIBUTED-PARAMETER SYSTEMS THROUGH LINEAR PROGRAMMING

3.1 INTRODUCTION

Considerable attention has been given in recent years to the optimal design of control systems by applying linear programming techniques. The advantage of the linear programming approach is that large-scale problems can be treated with relative ease. For linear, discrete, lumped-parameter systems, linear programming has proved to be a very efficient tool for system optimization (70, 71, 81, 89). For linear distributed-parameter systems, Sakawa (73) used linear programming techniques to obtain a control policy which minimizes the integral of the absolute value of a distributed error at a specified time; Sakawa considered a particular system characterized by a one-dimensional diffusion equation. Such techniques have not been developed for the design of sampled-data distributed-parameter systems. This chapter is therefore devoted to the development of linear programming techniques for use in the design of sampled-data distributed-parameter systems.

In Section 3.2, a general performance measure is developed for use as a design criterion for controlling a class of linear distributed-parameter systems. In Section 3.2.1, the multiple controlling action case is studied. In Section 3.3, the performance measure developed in Section 3.2 is extended to take intersampled-ripple effects into account. And in Section 3.5, an example is presented to illustrate the theories developed in the preceding sections.

3.2 DEVELOPMENT OF THE METHOD

In this section a class of performance measures is developed for linear distributed-parameter systems. The performance measures are of a form to which linear programming techniques can be applied.

Initially, consider the sampled-data model shown in Figure 3.1; therein, $H_0(s)$ is the transfer function of a zero-order hold*, $y(x,t)$ is representative of the output of the system, $y^d(x,t)$ is the desired distributed output, $e(x,t)$ is the distributed error of the system, $m(t)$ is the controlling action, and $u(t)$ is the boundary input to the distributed-parameter system.

It is assumed that the linear distributed-parameter plant of the system in Figure 3.1 is characterized by an equation of the form

$$y(x,t) = k_0(x,t) + \int_0^t g(x,t,\tau)u(\tau)\, d\tau \qquad (3.1)$$

where $g(x,t,\tau)$ is the Green's function or impulse response of the system and where $k_0(x,t)$ is the contribution of the initial condition $y(x,0)$ to the response of the system. Values of $y(x,t)$ of interest are assumed to be defined over the following domain: $0 \leq x \leq L$ and $0 \leq t \leq n_1 T$ where L is a constant, n_1 is a specified integer and T is the fixed sampling period.

Because of the sampler and the zero-order hold, the relation between the controlling action and the input to the system is

$$u(t) = m(kT) \triangleq m_k \quad \text{for} \quad kT \leq t < (k+1)T \qquad (3.2)$$

where $k = 0, 1, 2, \ldots$. The term m_k in Equation 3.2 denotes the value of the controlling action $m(t)$ at the k^{th} sampling instant.

Suppose that the values of $y(x,t)$ is to be found in the following range of t:

$$(j - 1)T \leq t < jT \qquad (3.3)$$

---

*The zero-order hold in Figure 3.1 is characterized by the following equation: $u(t) = m(kT)$ for $kT \leq t < (k+1)T$ where k assumes integer values and T is the fixed sampling period.

The effect of Equation 3.2 on Equation 3.1 is as follows:

$$y(x,t) = k_0(x,t) + \sum_{k=1}^{j-1} m_{k-1} \int_{(k-1)T}^{kT} g(x,t,\tau)\, d\tau + m_{j-1} \int_{(j-1)T}^{t} g(x,t,\tau)\, d\tau \tag{3.4}$$

which applies for t in the interval defined by Equation 3.3. At the end of the j^{th} sampling period, that is at $t = jT^- = \lim_{\varepsilon \to 0+} jT - \varepsilon$, Equation 3.4 reduces to

$$y(x,jT^-) = k_0(x,jT^-) + \sum_{k=1}^{j} m_{k-1} h_j(x,kT^-) \tag{3.5}$$

where

$$h_j(x,kT^-) = \int_0^T g[x,jT^-,(k-1)T+\tau]\, d\tau \tag{3.6}$$

It is convenient to write the first n_1 values of the output distribution at the end of each sampling period in matrix form as follows:

$$Y(x) = K_0(x) + H(x)M \tag{3.7}$$

where Y(x) is an $n_1 \times 1$ output matrix with j^{th} element $y(x,jT^-)$, M an $n_1 \times 1$ control matrix with j^{th} element m_{j-1}, $K_0(x)$ is an $n_1 \times 1$ matrix with j^{th} element $k_0(x,jT^-)$, and H(x) an $n_1 \times n_1$ matrix; namely,

$$H(x) = \begin{bmatrix} h_1(x,T^-) & 0 & \cdots & 0 \\ h_2(x,T^-) & h_2(x,2T^-) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{n_1}(x,T^-) & h_{n_1}(x,2T^-) & \cdots & h_{n_1}(x,n_1T^-) \end{bmatrix} \tag{3.8}$$

In the class of performance measures which are to be developed, only n_2 discrete values in the spatial domain of the distributed output are considered at each particular time. If n_2 is large and if intersampling ripple in the distributed output is essentially eliminated by the nature of the system, these n_2 spatially discrete points can be considered as representative of the entire spatial domain. Intersampling ripple may be salient in some cases, however, and methods of reducing this ripple are considered in Section 3.3.

In practice, the controlling action is usually bounded above and below by some functions of time. Such functions depend, of course, on the given system and the available source of input. Thus at each sampling instant, say at k^{th} sampling instant, the controlling action is bounded by

$$(m_k)_{min} \leq m_k \leq (m_k)_{max} \tag{3.9}$$

where $(m_k)_{max}$ and $(m_k)_{min}$ are the upper bound and the lower bound of the controlling action at $t = kT$. For the sake of notational simplicity, the lower bound on the controlling action is assumed to be zero, i.e.,

$$0 \leq m_k \leq (m_k)_{max} \tag{3.10}$$

If the lower bound $(m_k)_{min}$ is negative, however, the following approach can be readily applied by introducing two more nonnegative variables m_k^+ and m_k^- as follows:

$$m_k = m_k^+ - m_k^- \tag{3.11}$$

where

$$m_k^+ = 0, \quad \text{if} \quad m_k \leq 0$$

and

$$m_k^- = 0, \quad \text{if} \quad m_k \geq 0$$

Based on Equation 3.9, the bounds on m_k^+ and m_k^- are

$$0 \leq m_k^+ \leq (m_k)_{max} \tag{3.12}$$

and

$$0 \leq m_k^- \leq -(m_k)_{min} \tag{3.13}$$

Consider the performance measure P:

$$P = \min_{\substack{m_k \\ 0 \leq k \leq n_1 - 1}} \sum_{i=1}^{n_2} \sum_{j=1}^{n_1} a_{ij} \left| e(x_i, jT^-) \right| \tag{3.14}$$

The term $e(x_i, jT^-)$ is the error between the desired distributed output and the actual output distribution at $x = x_i$ and $t = jT^-$, namely,

$$e(x_i, jT^-) = y^d(x_i, jT^-) - y(x_i, jT^-) \tag{3.15}$$

The error at the end of each sampling period is considered because it is the error that exists just prior to the application of a new controlling action. The constraint on the controlling action is given by Equation 3.10. The term a_{ij} appearing in the performance measure is an appropriate weighting factor; the designer may choose the a_{ij}'s to reflect the seriousness of certain errors in space and time in a given system. For example, if all

a_{ij}'s except for $j = n_1$ are set equal to zero, the problem reduces to a terminal-time problem for which the errors at the terminal time are the only errors of interest.

In order to arrange the performance measure in Equation 3.14 into a standard form (29, 64) for linear programming, additional variables v_{ij}'s and z_{ij}'s are introduced as follows:

$$v_{ij} - z_{ij} = e(x_i, jT^-) \qquad 0 \leq i \leq n_2,\ 1 \leq j \leq n_1 \tag{3.16}$$

where

$$v_{ij} = 0, \quad \text{if} \quad e(x_i, jT^-) \leq 0 \tag{3.17}$$

and

$$z_{ij} = 0, \quad \text{if} \quad e(x_i, jT^-) \geq 0 \tag{3.18}$$

It can be seen from the relations in Equations 3.17 and 3.18 that in the final solution, at least one of v_{ij} and z_{ij} will be equal to zero for each ij combination, depending on whether the optimal $e(x_i, jT^-)$ is positive or negative. With Equation 3.16, the performance measure in Equation 3.14 becomes

$$P = \min_{\substack{m_k \\ 0 \leq k \leq n_1 - 1}} \sum_{i=1}^{n_2} \sum_{j=1}^{n_1} a_{ij}(v_{ij} + z_{ij}) \tag{3.19}$$

which is in the form to which the linear programming techniques can be readily applied. The linear constraint equations associated with the performance measure 3.19 are given in Equations 3.7, 3.10, 3.15, 3.16, 3.17 and 3.18.

3.2.1 Multiple Controlling Actions

The method of the preceding section is extended in this section to the case that r controlling actions are applied. In this case the characteristic equation of the system is

$$y(x,t) = k_0(x,t) + \int_0^t G(x,t,\tau)U(\tau)\, d\tau \tag{3.20}$$

where $U(t)$ is a $r \times 1$ input matrix whose k^{th} element is $u^k(t)$*, and $G(x,t,\tau)$ is a $1 \times r$ matrix relating the controlling actions to the output distribution $y(x,t)$. Following the same approach as in the one controlling action case, the output distribution at the end of the j^{th} sampling period is

$$y(x,jT^-) = k_0(x,jT^-) + \sum_{k=1}^{j} H_j(x,kT^-)M_{k-1} \tag{3.21}$$

where the $1 \times r$ vector $H_j(x,kT^-)$ is

$$H_j(x,kT^-) = \int_0^T G[x,jT^-,(k-1)T+\tau]\, d\tau \tag{3.22}$$

and where

$$M_{k-1} = \begin{Bmatrix} m^1_{k-1} \\ m^2_{k-1} \\ \cdot \\ \cdot \\ \cdot \\ m^r_{k-1} \end{Bmatrix} \tag{3.23}$$

*This is the same notation that one would use to denote $u(t)$ raised to the k^{th} power. Since the latter meaning is not used in this chapter, no ambiguity is encountered.

As in the preceding section, the first n_1 values of the output distribution at the end of each sampling period is conveniently written in matrix form:

$$Y(x) = K_0(x) + \underline{H}(x)\underline{M} \tag{3.24}$$

$Y(x)$ and $K_0(x)$ are the same as in one controlling action case and $\underline{H}(x)$ and $\underline{M}$ are the $n_1 \times n_1 r$ matrix and $n_1 r \times 1$ matrix, respectively, which follow:

$$\underline{H}(x) = \begin{bmatrix} H_1(x,T^-) & \underline{0} & \cdots & \underline{0} \\ H_2(x,T^-) & H_2(x,2T^-) & \cdots & \underline{0} \\ \vdots & \vdots & \cdots & \vdots \\ H_{n_1}(x,T^-) & H_{n_1}(x,2T^-) & \cdots & H_{n_1}(x,n_1T^-) \end{bmatrix} \tag{3.25}$$

and

$$\underline{M} = \begin{bmatrix} M_0 \\ M_1 \\ \vdots \\ M_{n_1-1} \end{bmatrix} \tag{3.26}$$

The terms denoted by $\underline{0}$ in Equation 3.25 are $1 \times r$ matrices all of whose elements are zero.

The performance measure in Equation 3.14 becomes

$$P = \min_{\substack{m_k^\alpha \\ 0\leq k\leq n_1-1 \\ 1\leq\alpha\leq r}} \sum_{i=1}^{n_2} \sum_{j=1}^{n_1} a_{ij}\, e(x_i, jT^-) \tag{3.27}$$

subject to

$$0 \leq m_k^\alpha \leq (m_k^\alpha)_{max} \tag{3.28}$$

By introducing additional variables v_{ij}'s and z_{ij}'s as before, the performance measure in Equation 3.27 is

$$P = \min_{\substack{m_k^\alpha \\ 0\leq k\leq n_1-1 \\ 1\leq\alpha\leq r}} \sum_{i=1}^{n_2} \sum_{j=1}^{n_1} a_{ij}(v_{ij} + z_{ij}) \tag{3.29}$$

The characteristic equation, Equation 3.20, may be viewed as an approximate model for the distributed input case. In such case, each element in $G(x,t)$ will be the Green's function associated with the corresponding input m^α along the spatial domain. If the spatial dimension is N rather than unity, the term x is simply replaced by the vector $X = [x^1\ x^2\ \ldots\ x^N]'$.

3.3 RIPPLE CONSIDERATION

As seen in the performance measure of Equation 3.14, time-and-space intersampling error is excluded from consideration in the optimization process. Under certain circumstances, deleterious ripple may appear in the "optimal" distribution. To restrain the intersampling ripple, both in time and space, the absolute values of the first α_1 time-derivatives and the first α_2 space-derivatives of the error distribution at the specified spatial points and at the sampling instants can be constrainted to be less than or

equal to some preassigned values. By proper selection of the bounds on the derivatives of the error, the ripple can be restrained to any order desired.

Partial differentiating with respect to time on both sides of Equation 3.1 yields

$$\frac{\partial y(x,t)}{\partial t} = \frac{\partial k_0(x,t)}{\partial t} + \int_0^t \frac{\partial g(x,t,\tau)}{\partial t} u(\tau)\, d\tau + g(x,t,t)u(t) \tag{3.30}$$

in which the differentiation and integration operations have been interchanged under the assumption that the partial derivative of the impulse response with respect to t is bounded. In general, if the initial condition response and the impulse response are continuously differentiable, the j^{th} partial derivative with respect to t of the output distribution can be written as

$$\frac{\partial^j y(x,t)}{\partial t^j} = \frac{\partial^j k_0(x,t)}{\partial t^j} + \int_0^t \frac{\partial^j g(x,t,\tau)}{\partial t^j} u(\tau)\, d\tau +$$

$$+ \sum_{k=1}^{j} \frac{\partial^{j-k}}{\partial t^{j-k}} \left[u(t) \frac{\partial^{k-1} g(x,t,\tau)}{\partial t^{k-1}} \bigg|_{\tau=t} \right] \tag{3.31}$$

Since u(t) is discontinuous at $t = kT$, $k = 1, 2, \ldots$, but not at $t = kT^-$, and is equal to a constant during each sampling, all the derivatives of u(t) at $t = kT^-$ are zero. Thus,

$$\frac{\partial^j y(x,kT^-)}{\partial t^j} = \frac{\partial^j k_0(x,kT^-)}{\partial t^j} + \int_0^{kT} \frac{\partial^j g(x,kT^-,\tau)}{\partial t^j} u(\tau)\, d\tau +$$

$$+ \left\{ u(t) \sum_{k=1}^{j} \frac{\partial^{j-k}}{\partial t^{j-k}} \left[\frac{\partial^{k-1} g(x,t,\tau)}{\partial t^{k-1}} \bigg|_{\tau=t} \right] \right\} \Bigg|_{t=kT^-} \tag{3.32}$$

Similarly, the j^{th} partial derivative with respect to x of the output distribution is

$$\frac{\partial^j y(x,t)}{\partial x^j} = \frac{\partial^j k_0(x,t)}{\partial x^j} + \int_0^t \frac{\partial^j g(x,t,\tau)}{\partial x^j}\, u(\tau)\, d\tau \tag{3.33}$$

The j^{th} partial derivatives with respect to t and x of the error response evaluated at $x = x_i$ and $t = kT^-$ are defined as follows:

$$e_t^j(x_i,kT^-) = \frac{\partial^j y^d(x_i,kT^-)}{\partial t^j} - \frac{\partial^j y(x_i,kT^-)}{\partial t^j} \tag{3.34}$$

$$e_x^j(x_i,kT^-) = \frac{\partial^j y^d(x_i,kT^-)}{\partial x^j} - \frac{\partial^j y(x_i,kT^-)}{\partial x^j} \tag{3.35}$$

To restrain the intersampling ripple, the absolute values of both time and spatial derivatives of the error response are bounded by some constants. These constants are to be determined by the designer on the basis of the feasibility of the solution and the degree of ripple that can be tolerated in the optimal system. Thus,

$$|e_t^j(x_i,kT^-)| \leq \varepsilon_{t_{ik}}^j \tag{3.36}$$

$$|e_x^j(x_i,kT^-)| \leq \varepsilon_{x_{ik}}^j \tag{3.37}$$

To arrange the constraint equations into a standard form for linear programming, more variables are introduced:

$$v_{t_{ik}}^j - z_{t_{ik}}^j = e_t^j(x_i,kT^-) \quad 1 \leq k \leq n_1;\ 1 \leq i \leq n_2;\ 1 \leq j \leq \alpha_1 \tag{3.38}$$

$$v_{x_{ik}}^j - z_{x_{ik}}^j = e_x^j(x_i,kT^-) \quad 1 \leq k \leq n_1;\ 1 \leq i \leq n_2;\ 1 \leq j \leq \alpha_2 \tag{3.39}$$

where

$$v^j_{t^{ik}} = 0 \quad \text{if} \quad e^j_t(x_i, kT^-) \leq 0 \tag{3.40}$$

$$z^j_{t^{ik}} = 0 \quad \text{if} \quad e^j_t(x_i, kT^-) \geq 0 \tag{3.41}$$

$$v^j_{x^{ik}} = 0 \quad \text{if} \quad e^j_x(x_i, kT^-) \leq 0 \tag{3.42}$$

$$z^j_{x^{ik}} = 0 \quad \text{if} \quad e^j_x(x_i, kT^-) \geq 0 \tag{3.43}$$

The performance measure in Equation 3.19 is now subject to $n_1 n_2(2\alpha_1 + 2\alpha_2)$ additional linear constraint equations, namely,

$$v^j_{t^{ik}} \leq \varepsilon^j_{t^{ik}} \tag{3.44}$$

$$z^j_{t^{ik}} \leq \varepsilon^j_{t^{ik}} \tag{3.45}$$

$$v^j_{x^{ik}} \leq \varepsilon^j_{x^{ik}} \tag{3.46}$$

$$z^j_{x^{ik}} \leq \varepsilon^j_{x^{ik}} \tag{3.47}$$

Clearly, from the definitions of $v^j_{t^{ik}}$ and $z^j_{t^{ik}}$ in Equations 3.40 and 3.41, at least one of them must assume the value of zero in the optimal solution for each combination of i, j and k. Similarly, at least one of $v^j_{x^{ik}}$ and $z^j_{x^{ik}}$ is zero in the optimal solution.

The same discussions are still valid in the multiple controlling action case except that the partial derivatives of the output distribution are given by

$$\frac{\partial^j y(x,kT^-)}{\partial t^j} = \frac{\partial^j k_0(x,t)}{\partial t^j} + \int_0^{kT} \frac{\partial^j G(x,kT^-,\tau)}{\partial t^j} U(\tau)\, d\tau +$$

$$+ \{U(t) \sum_{k=1}^{j} \frac{\partial^{j-k}}{\partial t^{j-k}} [\frac{\partial^{k-1} G(x,t,\tau)}{\partial t^{k-1}}\Big|_{\tau=t}]\}\Big|_{t=kT^-} \qquad (3.48)$$

for $1 \leq j \leq \alpha_1$ and

$$\frac{\partial^j y(x,t)}{\partial x^j} = \frac{\partial^j k_0(x,t)}{\partial x^j} + \int_0^t \frac{\partial^j G(x,t,\tau)}{\partial x^j} U(\tau)\, d\tau \qquad (3.49)$$

for $1 \leq j \leq \alpha_2$.

3.4 ILLUSTRATIVE EXAMPLE

To illustrate the method described above, the system characterized by

$$\frac{\partial y(x,t)}{\partial t} = \frac{\partial^2 y(x,t)}{\partial x^2} \qquad t > 0;\ 0 < x < 1 \qquad (3.50)$$

with the boundary conditions

$$y(0,t) = y(1,t) = u(t) \qquad (3.51)$$

and

$$y(x,0) = 0 \qquad (3.52)$$

is used. The boundary input u(t) is to be so determined that the performance measure, with all a_{ik}'s equal to unity, in Equation 3.14 is minimized. The

desired distribution $y^d(x,t)$ equals 1, and the constraint on the controlling action is

$$0 \leq m_k \leq 2 \tag{3.53}$$

Initial steps in the solution are as follows: First, the finite sine transform and the Laplace transform with respect to x and t, respectively, are taken of both sides of Equation 3.50; second, appropriate substitution of the boundary conditions is made on the basis of Equations 3.51 and 3.52; and third, inversion of the finite sine transform on both sides yields

$$y(x,s) = 4U(s) \sum_{n=1}^{\infty} [w_{2n-1}\sin(w_{2n-1}x)/(s + w_{2n-1}^2)] \tag{3.54}$$

where

$$w_{2n-1} = (2n-1)\pi \tag{3.55}$$

and where s is the Laplace transform variable.

Next, the characteristic equation in the form of Equation 3.1 is obtained by taking the inversed Laplace transform of both sides of Equation 3.54:

$$y(x,t) = 4 \int_0^t \sum_{n=1}^{\infty} w_{2n-1}\sin(w_{2n-1}x)\exp[-w_{2n-1}^2(t-\tau)]u(\tau)\, d\tau \tag{3.56}$$

From the theory of Fourier's series, it is known that at $x = 0$ and $x = 1$, the infinite series in the right-hand side of Equation 3.56 does not converge to $y(x,t)$. However, from the boundary conditions in Equation 3.51, the values of the output distribution at the boundaries are known to equal the input $u(t)$.

After comparing Equation 3.56 with Equation 3.1, the impulse response $g(x,t,\tau)$ initiated at $t = \tau$ is

$$g(x,t,\tau) = 4 \sum_{n=1}^{\infty} w_{2n-1}\sin(w_{2n-1}x)\exp[-w_{2n-1}^2(t-\tau)] \tag{3.57}$$

Then, on the basis of Equation 3.6, each nonzero element of $H(x)$ is

$$h_j(x,kT^-) = 4\int_0^T \sum_{n=1}^{\infty} w_{2n-1}\sin(w_{2n-1}x)\exp\{-w_{2n-1}^2[(j-k+1)T - \tau]\}\, d\tau$$

$$= 4 \sum_{n=1}^{\infty} \frac{\sin(w_{2n-1}x)}{w_{2n-1}} \exp[-w_{2n-1}^2(j-k+1)T][\exp(w_{2n-1}^2T) - 1] \tag{3.58}$$

The integration and summation operations are interchangeable in Equation 3.58 because the series $\sum_{n=1}^{\infty} w_{2n-1}\exp\{-w_{2n-1}^2[(j-k+1)T - \tau]\}$ is uniformly convergent.

From Equation 3.5, the distributed output is

$$y(x,jT^-) = \sum_{k=1}^{j} m_{k-1}h_j(x,kT^-) \tag{3.59}$$

The general term in the error sequence is then, from Equation 3.15,

$$e(x_i,jT^-) = 1 - \sum_{k=1}^{j} h_j(x_i,kT^-)m_{k-1} \tag{3.60}$$

For each pair of values x_i and kT^-, the numerical value of $h_j(x_i,kT^-)$ is evaluated by summing terms in the series of Equation 3.58 on a digital computer. The series is truncated on the basis that $\exp[-w_{2n-1}^2(j-k+1)T][\exp(w_{2n-1}^2T) - 1]/w_{2n-1}$ is less than 10^{-7} for $j \neq k$. For $j = k$, Equation 3.58 reduces to

$$h_j(x,jT^-) = 4 \sum_{n=1}^{\infty} \frac{\sin(w_{2n-1}x)}{w_{2n-1}} [1 - \exp(-w_{2n-1}^2T)]$$

$$= 1 - 4 \sum_{n=1}^{\infty} \frac{\sin(w_{2n-1}x)}{w_{2n-1}} \exp(-w_{2n-1}^2 T) \qquad (3.61)$$

The infinite series in the right-hand member of Equation 3.61 is evaluated on the same basis as mentioned above.

From the boundary conditions given in Equations 3.51 and 3.52, the output distribution of the system is known to be symmetric; thus, characteristic values of x need be specified either in the interval defined by $0 \leq x \leq 0.5$ or in the interval defined by $0.5 \leq x \leq 1$, but not in both. In this example, n_2 is set at 3 with $x_1 = 0.1$, $x_2 = 0.3$, and $x_3 = 0.5$. Finally, the sampling period T is assumed to be 0.01 second, and the total number of samples of interest is 30.

The values of $h_j(x,kT)$ for $x = 0.1$, $x = 0.3$ and $x = 0.5$, $T = 0.01$ and $1 \leq j \leq 30$ are tabulated in Table 3.1. The optimal input, computed by use of the IBM 1620-1311 Linear Programming System, is shown in Figure 3.2 and listed in Table 3.2. The optimal output is computed by use of Equation 3.59, is plotted in Figure 3.3, and tabulated in Table 3.3. The desired distribution, as seen from Figure 3.3, can be obtained in less than 0.15 second for all practical purposes. As compared with the output distribution of the system with the input u(t) equal unity for all t (Figure 3.5), there is a significant improvement in the output distribution of the optimal system. Though constraints on the derivatives of the distribution are not included in this example, the ripple in the optimal distribution is insignificant.

In cases where spatial ripple is not likely to appear in the output distribution, a different performance measure may be of use; namely, a

performance measure based on the absolute value of the error between a desired uniform distribution and the spatial average of the actual distribution, as follows:

$$P = \min_{\substack{m_k \\ 0 \leq k \leq n_1 - 1}} \sum_{j=1}^{n_1} \overline{e_a(x, jT^-)} \tag{3.62}$$

where

$$\overline{e_a(x, jT^-)} = y^d - \frac{1}{L} \int_0^L y(x, jT^-)\, dx \tag{3.63}$$

Constraint Equation 3.10 applies here also.

Following the same approach as in Section 3.2, the performance measure of Equation 3.62 becomes

$$P = \min_{\substack{m_k \\ 0 \leq k \leq n_1 - 1}} \sum_{j=1}^{n_1} (v_j + z_j) \tag{3.64}$$

The optimal control sequence for the system described by Equations 3.50, 3.51 and 3.52, with reference to the performance measure of Equation 3.62, is computed by the same standard program. The results are shown in Figures 3.4 and 3.5 and listed in Tables 3.4 and 3.5.

3.5 REMARKS

This chapter contains a detailed presentation of the optimal design of sampled-data distributed-parameter systems by application of linear programming techniques. The impulse-response concept is developed in the design

of digital control for linear time-invariant or time-variant distributed-parameter systems. With this approach, the design problem is reduced to the determination of a set of discrete values of the controlling actions.

The multiple controlling actions in Section 3.2.1 may be viewed as an approximation to distributed controlling actions in certain cases. In Section 3.3, a detailed discussion of both time and space ripple is given.

For systems of spatial dimension greater than one, the discussions are still valid provided the scalar x is replaced by the vector X. This sampled-data model can also be viewed as an approximate model for continuous systems provided the sampling period is made sufficiently small. The sampling frequency may be selected such that the amplitude associated with frequencies higher than one-half of the sampling frequency is below -30 db (36). Other details concerning the choice of the sampling period for discrete approximations are given in references (36, 65). For continuous systems, the performance measure in Equation 3.14 is an approximate form of

$$P = \min_{\substack{m(t) \\ 0 \le t \le T_1}} \int_0^L \int_0^{T_1} a(x,t) \, |e(x,t)| \, dt \, dx \tag{3.65}$$

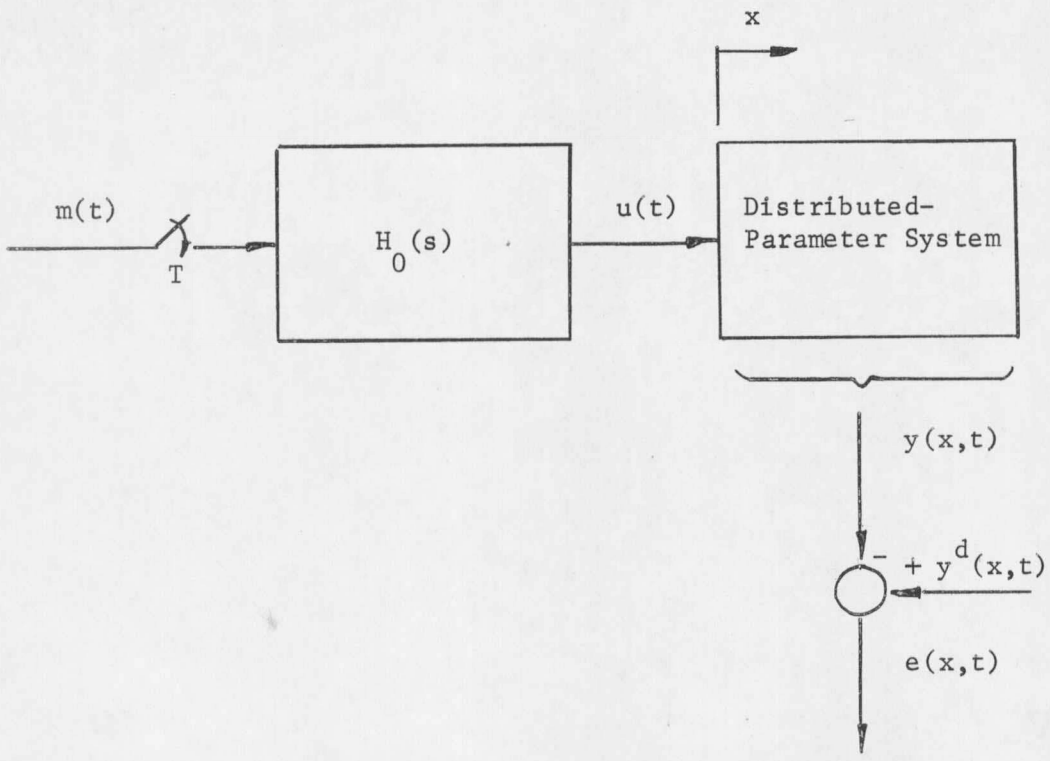


Figure 3.1. Sampled-data representation for distributed-parameter systems.

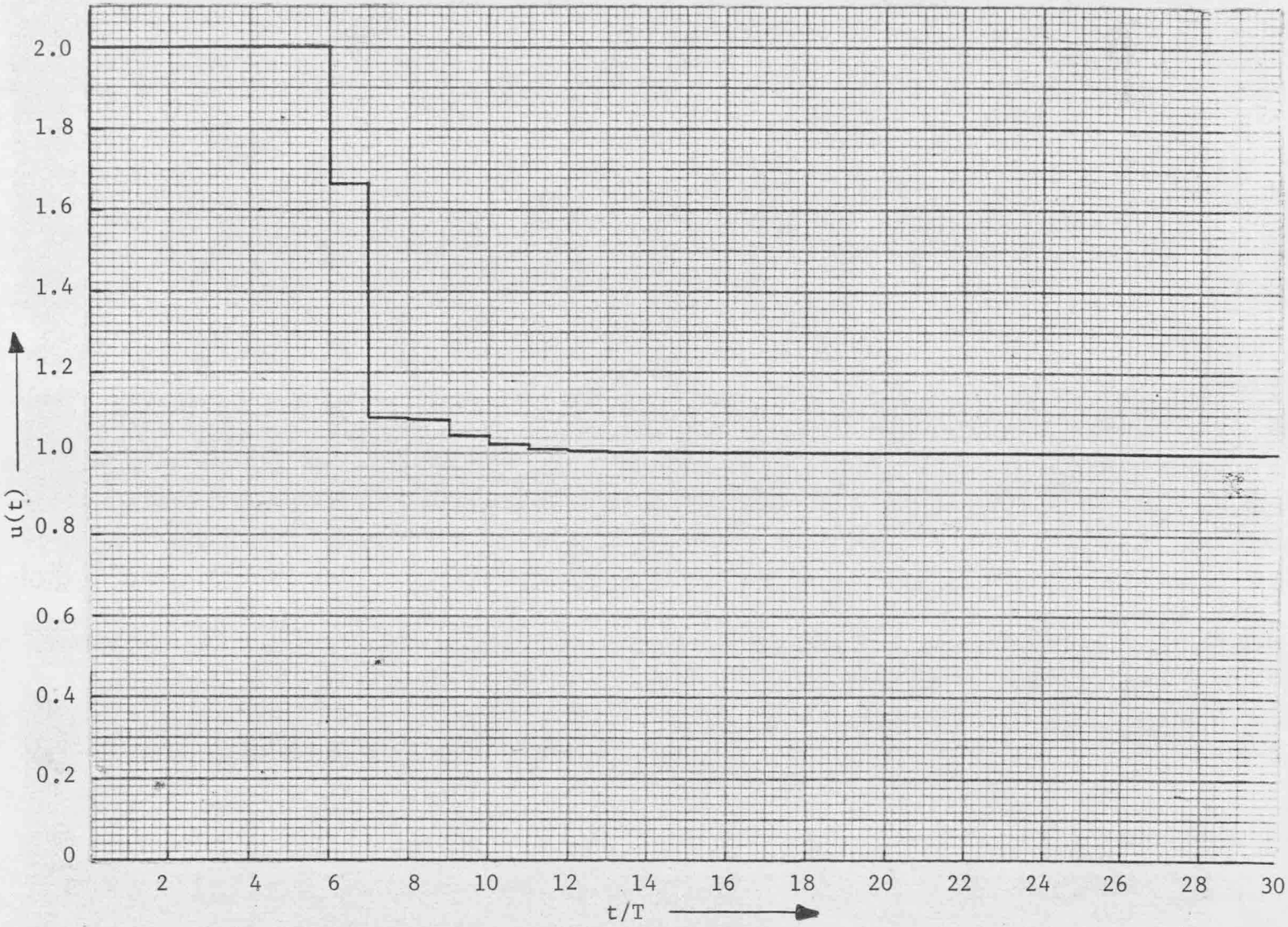


Figure 3.2. Optimal input which results in the minimization of the performance measure in Equation 3.4 with all a_{ik}'s = 1, $(m_k)_{max}$ = 2.0 and T = 0.01 second.

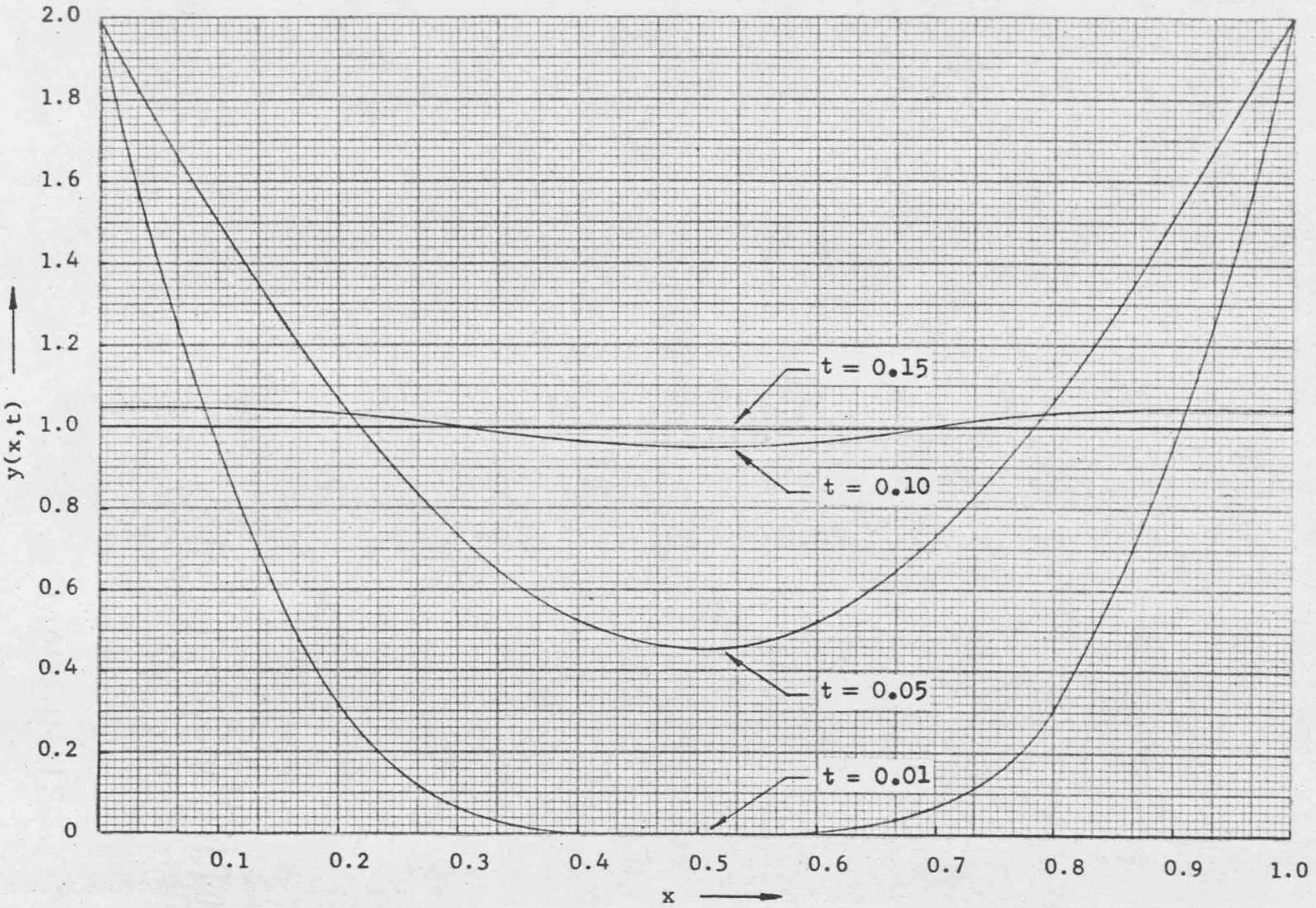


Figure 3.3. Output response of the system associated with the optimal input in Table 3.2.

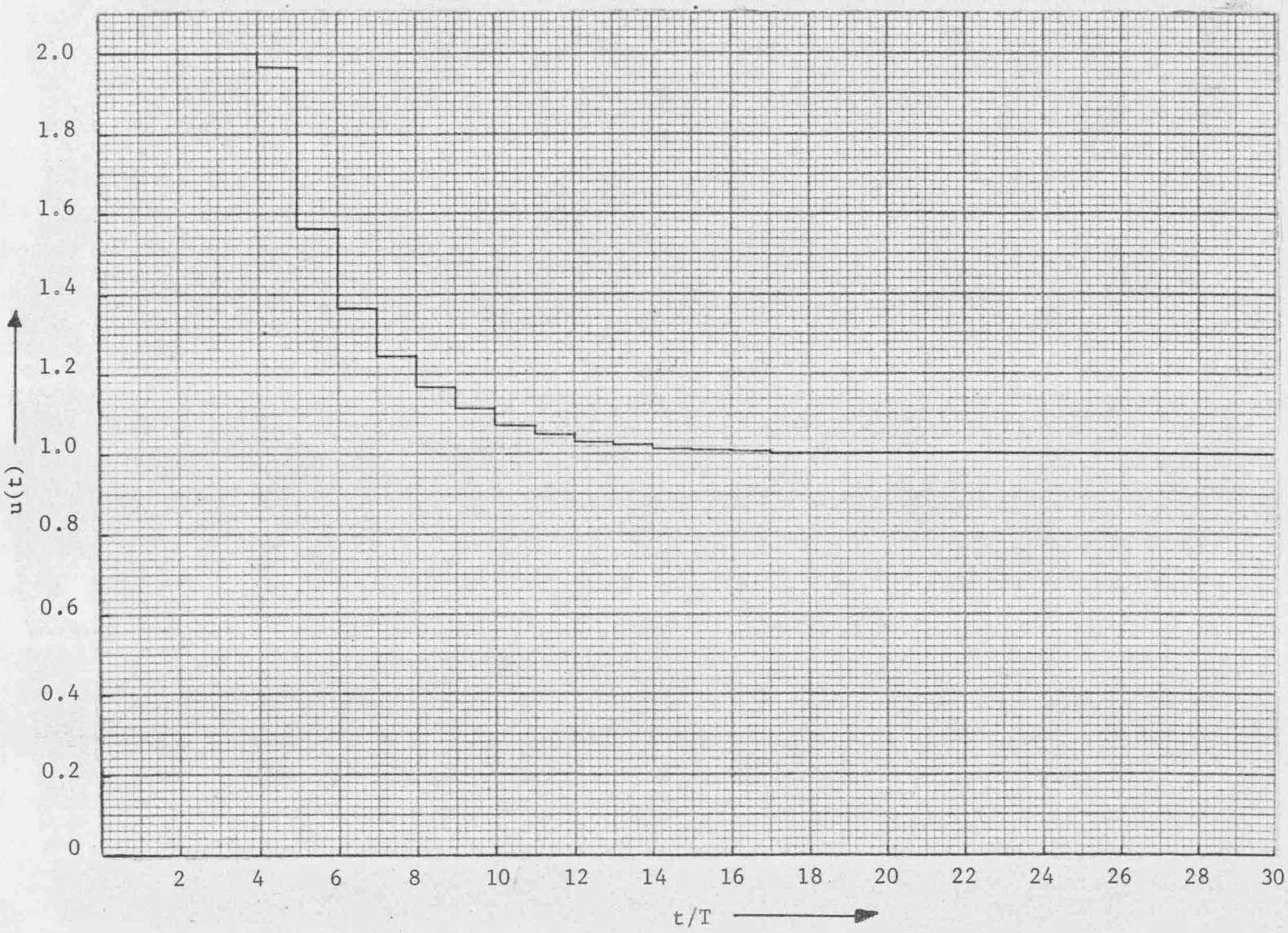


Figure 3.4. Optimal input which results in the minimization of the performance measure in Equation 3.62 with $(m_k)_{max} = 2.0$ and T = 0.01 second.

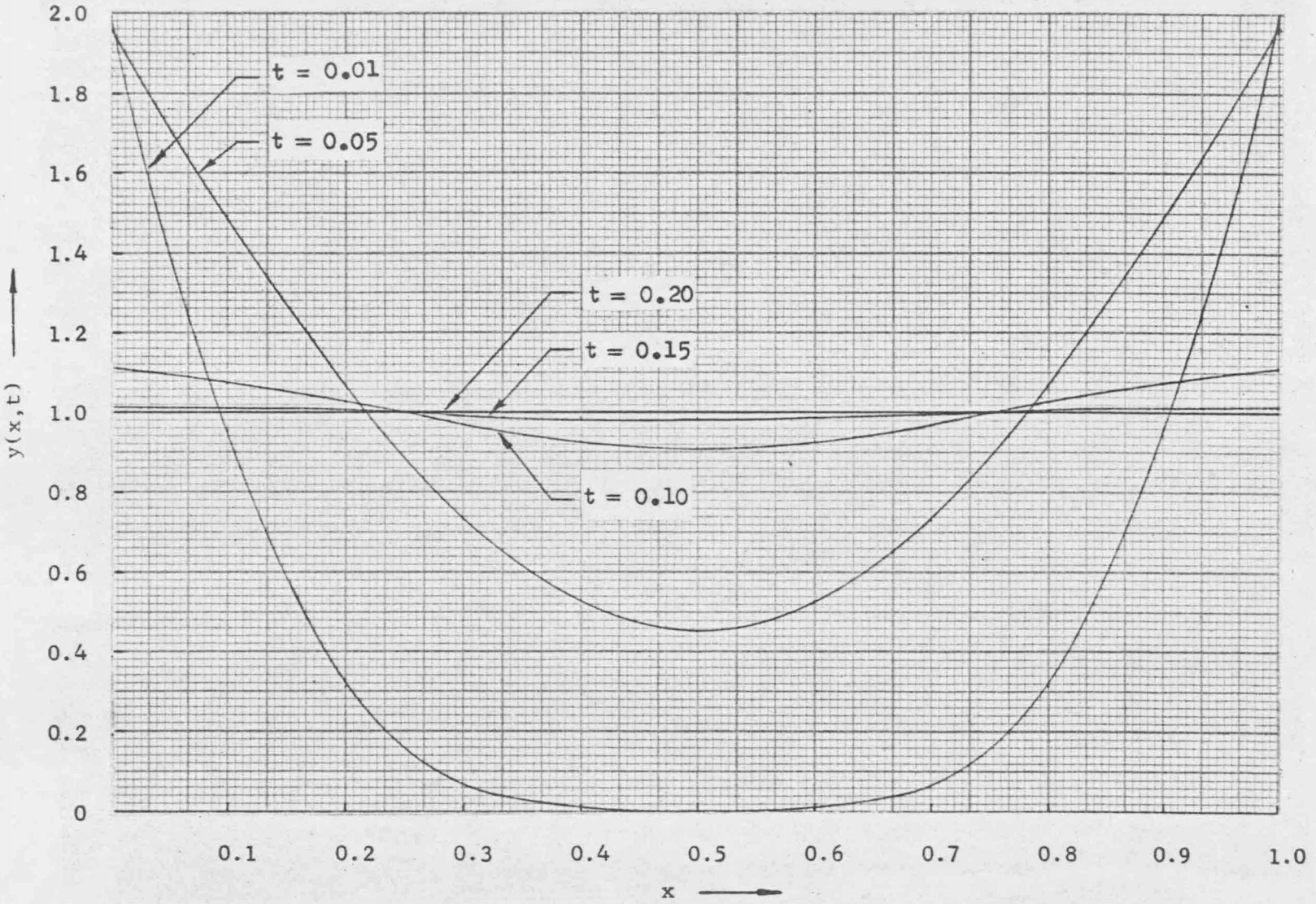


Figure 3.5. Output response of the system associated with the optimal input in Table 3.4.

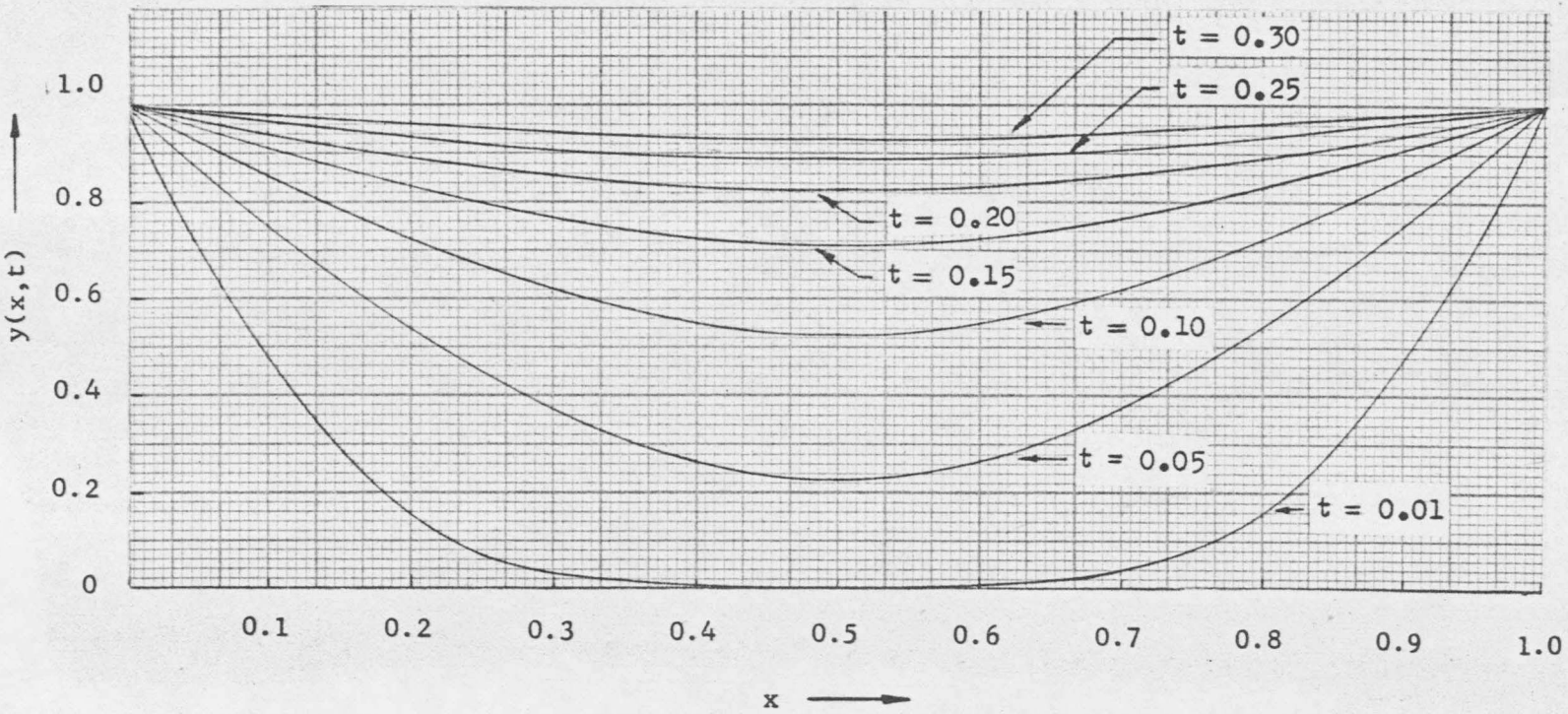


Figure 3.6. Output response of the system with input equal unity.

Table 3.1. Values of $h_j(x,kT^-)$, Equation 3.58.

k	$h_1(x,kT^-)$				
	x = 0.1	x = 0.2	x = 0.3	x = 0.4	x = 0.5
0.01	.479500	.157299	.033895	.004700	.000814
0.02	.137582	.160075	.100184	.043500	.024025
0.03	.066241	.097932	.090858	.068576	.057615
0.04	.041713	.068850	.077230	.074417	.071746
0.05	.030716	.054198	.067430	.072479	.073489
0.06	.024957	.045738	.060009	.067748	.070111
0.07	.021432	.040055	.053921	.062236	.064977
0.08	.018934	.035721	.048668	.056739	.059469
0.09	.016955	.032130	.044018	.051552	.054126
0.10	.015279	.029014	.039850	.046766	.049141
0.11	.013810	.026247	.036092	.042396	.044564
0.12	.012498	.023764	.032695	.038421	.040393
0.13	.011318	.021524	.029620	.034814	.036604
0.14	.010252	.019499	.026835	.031544	.033167
0.15	.009287	.017665	.024313	.028580	.030051
0.16	.008414	.016004	.022028	.025895	.027227
0.17	.007623	.014500	.019957	.023461	.024668
0.18	.006907	.013137	.018082	.021256	.022350
0.19	.006257	.011902	.016382	.019258	.020250
0.20	.005669	.010784	.014843	.017449	.018346
0.21	.005136	.009770	.013448	.015809	.016622
0.22	.004634	.008852	.012184	.014323	.015060
0.23	.004216	.008020	.011039	.012977	.013645
0.24	.003820	.007266	.010001	.011757	.012362
0.25	.003461	.006583	.009061	.010652	.011201
0.26	.003136	.005965	.008210	.009651	.010148
0.27	.002841	.005404	.007438	.008744	.009194
0.28	.002574	.004896	.006740	.007922	.008330
0.29	.002332	.004436	.006160	.007178	.007547
0.30	.002113	.004019	.005532	.006503	.006838

Note: $h_j(x,kT^-) = h_{j-1}(x,(k-1)T^-)$ for all j's and k's.

Table 3.2. Optimal control sequence which results in the minimization of the performance measure, Equation 3.14, with all a_{ik}'s = 1, $(m_k)_{max}$ = 2.0 and T = 0.01 second.

Time in seconds	Control sequence
0.00	2.000
0.01	2.000
0.02	2.000
0.03	2.000
0.04	2.000
0.05	2.000
0.06	1.665
0.07	1.091
0.08	1.089
0.09	1.045
0.10	1.025
0.11	1.013
0.12	1.007
0.13	1.004
0.14	1.002
0.15	1.001
0.16	1.001
0.17	0.999
0.18	1.000
0.19	1.000
0.20	1.000
0.21	1.000
0.22	1.000
0.23	1.000
0.24	1.000
0.25	1.000
0.26	1.000
0.27	1.000
0.28	1.000
0.29	1.000

Table 3.3 Output response of the system associated with the optimal input in Table 3.2.

k	Distributed output y(x,t)				
	x = 0.1	x = 0.2	x = 0.3	x = 0.4	x = 0.5
0.01	.9590	.3146	.0678	.0094	.0016
0.02	1.2342	.6347	.2682	.0964	.0497
0.03	1.3666	.8306	.4499	.2335	.1649
0.04	1.4501	.9683	.6043	.3824	.3084
0.05	1.5115	1.0767	.7392	.5273	.4554
0.06	1.5614	1.1682	.8592	.6628	.5956
0.07	1.4436	1.1956	.9557	.7857	.7253
0.08	1.1602	1.1231	1.0000	.8819	.8357
0.09	1.0920	1.0624	1.0000	.9371	.9109
0.10	1.0492	1.0339	1.0000	.9660	.9519
0.11	1.0268	1.0183	1.0000	.9817	.9741
0.12	1.0143	1.0099	1.0000	.9901	.9860
0.13	1.0077	1.0053	1.0000	.9947	.9925
0.14	1.0042	1.0029	1.0000	.9971	.9959
0.15	1.0022	1.0016	1.0000	.9985	.9978
0.16	1.0011	1.0008	1.0000	.9992	.9988
0.17	1.0008	1.0005	1.0000	.9995	.9993
0.18	.9998	1.0001	1.0000	.9998	.9997
0.19	.9999	.9999	.9999	.9998	.9998
0.20	.9999	.9999	.9999	.9998	.9998
0.21	.9999	.9999	.9999	.9999	.9999
0.22	.9999	.9999	.9999	.9999	.9999
0.23	.9999	.9999	.9999	.9999	.9999
0.24	.9999	.9999	.9999	.9999	.9999
0.25	.9999	.9999	.9999	.9999	.9999
0.26	.9999	.9999	.9999	.9999	.9999
0.27	.9999	.9999	.9999	.9999	.9999
0.28	.9999	.9999	.9999	.9999	.9999
0.29	.9999	.9999	1.0000	.9999	.9999
0.30	.9999	.9999	1.0000	.9999	.9999

Table 3.4. Optimal control sequence which results in the minimization of the performance measure, Equation 3.62, with $(m_k)_{max} = 2.0$ and T = 0.01 second.

Time in seconds	Control sequence
0.00	2.000
0.01	2.000
0.02	2.000
0.03	2.000
0.04	1.964
0.05	1.561
0.06	1.364
0.07	1.244
0.08	1.166
0.09	1.113
0.10	1.077
0.11	1.053
0.12	1.036
0.13	1.025
0.14	1.017
0.15	1.011
0.16	1.008
0.17	1.005
0.18	1.004
0.19	1.003
0.20	1.002
0.21	1.001
0.22	1.001
0.23	1.001
0.24	1.000
0.25	1.000
0.26	1.000
0.27	1.000
0.28	1.000
0.29	1.000

Table 3.5. Output response of the system associated with the optimal input in Table 3.4.

k	Distributed output y(x,t)				
	x = 0.1	x = 0.2	x = 0.3	x = 0.4	x = 0.5
0.01	.9590	.3146	.0678	.0094	.0016
0.02	1.2342	.6347	.2682	.0964	.0497
0.03	1.3667	.8306	.4499	.2335	.1649
0.04	1.4501	.9683	.6043	.3824	.3084
0.05	1.4942	1.0710	.7380	.5272	.4553
0.06	1.3460	1.0934	.8407	.6592	.5944
0.07	1.2365	1.0744	.8982	.7628	.7124
0.08	1.1616	1.0535	.9324	.8368	.8007
0.09	1.1106	1.0373	.9543	.8882	.8631
0.10	1.0756	1.0257	.9689	.9235	.9063
0.11	1.0516	1.0176	.9788	.9478	.9359
0.12	1.0354	1.0121	.9856	.9643	.9562
0.13	1.0242	1.0083	.9902	.9756	.9701
0.14	1.0167	1.0057	.9933	.9834	.9796
0.15	1.0114	1.0040	.9955	.9887	.9861
0.16	1.0075	1.0027	.9969	.9923	.9905
0.17	1.0053	1.0018	.9979	.9948	.9936
0.18	1.0034	1.0012	.9986	.9964	.9956
0.19	1.0026	1.0009	.9990	.9976	.9970
0.20	1.0020	1.0007	.9994	.9984	.9980
0.21	1.0014	1.0006	.9996	.9989	.9986
0.22	1.0007	1.0004	.9998	.9993	.9991
0.23	1.0006	1.0003	.9999	.9996	.9994
0.24	1.0005	1.0003	1.0000	.9997	.9996
0.25	1.0001	1.0002	1.0000	.9999	.9998
0.26	1.0000	1.0001	1.0000	.9999	.9999
0.27	1.0000	1.0000	1.0000	1.0000	.9999
0.28	1.0000	1.0000	1.0000	1.0000	1.0000
0.29	1.0000	1.0000	1.0001	1.0000	1.0000
0.30	1.0000	1.0000	1.0001	1.0000	1.0000

Table 3.6. Output response of the system associated with the input equal unity.

k	Distributed output y(x;t)				
	x = 0.1	x = 0.2	x = 0.3	x = 0.4	x = 0.5
0.01	.4795	.1573	.0339	.0047	.0008
0.02	.6171	.3174	.1341	.0482	.0248
0.03	.6833	.4153	.2249	.1168	.0824
0.04	.7250	.4842	.3022	.1912	.1542
0.05	.7557	.5383	.3696	.2637	.2277
0.06	.7807	.5841	.4296	.3314	.2978
0.07	.8021	.6241	.4835	.3937	.3628
0.08	.8211	.6599	.5322	.4504	.4222
0.09	.8380	.6920	.5762	.5019	.4764
0.10	.8533	.7210	.6161	.5487	.5255
0.11	.8671	.7473	.6522	.5911	.5701
0.12	.8796	.7710	.6848	.6295	.6105
0.13	.8909	.7925	.7145	.6643	.6471
0.14	.9012	.8120	.7413	.6959	.6802
0.15	.9105	.8297	.7656	.7245	.7103
0.16	.9189	.8457	.7876	.7504	.7375
0.17	.9265	.8602	.8076	.7738	.7622
0.18	.9334	.8733	.8257	.7951	.7845
0.19	.9397	.8852	.8421	.8143	.8048
0.20	.9453	.8960	.8569	.8318	.8231
0.21	.9505	.9058	.8704	.8476	.8398
0.22	.9511	.9147	.8825	.8619	.8548
0.23	.9593	.9227	.8936	.8749	.8685
0.24	.9631	.9299	.9036	.8866	.8808
0.25	.9666	.9365	.9126	.8973	.8920
0.26	.9697	.9425	.9209	.9070	.9022
0.27	.9726	.9479	.9283	.9157	.9114
0.28	.9752	.9528	.9350	.9236	.9197
0.29	.9775	.9572	.9412	.9308	.9272
0.30	.9796	.9612	.9467	.9373	.9341

CHAPTER 4

OPTIMAL DESIGN OF DISTRIBUTED-PARAMETER SYSTEMS THROUGH DYNAMIC PROGRAMMING

4.1 INTRODUCTION

In the preceding chapter attention is focused on the optimal design of linear distributed-parameter systems by use of linear programming techniques. Here, the optimal control of distributed-parameter systems is studied from the dynamic programming point of view (6, 7, 8). The advantage of the dynamic programming approach is that both nonlinear performance measures and nonlinear systems may be treated.

In 1963, Wang and Tung (86), using the principle of optimality (6), derived a maximum principle for a general terminal control problem, i.e., a control problem for which the distributed error at the terminal time only is considered in the performance measure. Instead of the usual form of Pontryagin's maximum principle (69) for lumped-parameter systems, the results in Wang and Tung's maximum principle are in functional partial differential equation forms for which there is no analytical method of solution at the present time (67).

In Section 4.2, a general convex performance measure is treated from the dynamic programming point of view. A functional equation is derived with properties analogous to those of the lumped-parameter case. The system considered here is the same as that considered in Chapter 3. In Section 4.3, a square-error performance measure with an energy constraint on the input is studied. An analytical solution to the corresponding optimization problem is obtained. In Section 4.4, an example is used to illustrate the theories developed in the preceding sections.

4.2 DEVELOPMENT OF THE METHOD

4.2.1 Continuous Spatial Variable

In this section, a general convex performance measure of a continuously distributed error is minimized from the dynamic programming point of view. Special properties of the functional and the optimal solution are derived to reduce computational requirements. For convenience, the characteristic equation of the system from Equation 3.4 is rewritten here

$$y(x,t) = k_0(x,t) + \sum_{k=1}^{j-1} m_{k-1} \int_{(k-1)T}^{kT} g(x,t,\tau)\, d\tau + m_{j-1} \int_{(j-1)T}^{t} g(x,t,\tau)\, d\tau \tag{4.1}$$

for $(j-1)T \leq t < jT$. The first two terms of the right-hand member of Equation 4.1 are the contribution of the initial distribution and past controlling actions, respectively. It is appropriate for later use to express $y(x,t)$ in the form

$$y(x,t) = y(x,(j-1)T,t) + m_{j-1} \int_{(j-1)T}^{t} g(x,t,\tau)\, d\tau \tag{4.2}$$

for $(j-1)T \leq t < jT$, where the term $y(x,(j-1)T,t)$ is the contribution due to the initial distribution and to past controlling actions and is defined by

$$y(x,(j-1)T,t) \triangleq k_0(x,t) + \sum_{k=1}^{j-1} m_{k-1} \int_{(k-1)T}^{kT} g(x,t,\tau)\, d\tau \tag{4.3}$$

It is evident from Equation 4.2 that

$$y(x,(j-1)T) = y(x,(j-1)T,(j-1)T) \tag{4.4}$$

It is also clear from Equation 4.2 that the optimal selection, with respect to the performance measure to be described, of m_{j-1} depends on $y(x,(j-1)T,t)$ which in turn depends on the distribution $y(x,(j-1)T)$. Since m_{j-1} is not a function of x, the selection of m_{j-1} does not affect the form of the function of x but, of course, affects the value of the distribution. If m_k can assume a finite number a_k of distinct values, the number of distinct distributions which $y(x,(j-1)T)$ may assume is $\prod_{k=0}^{j-2} a_k$. If m_k can assume a continuum of values, the same can be said of $y(x,(j-1)T)$. For purposes of solution, however, only a finite number of discrete values of m_k is considered in this section, and interpolation between the tabulated values of m_k can be made if necessary. Conditions under which such interpolation is valid are established later in this section.

Now consider the return function* $I_{n_1}[y(x,0)]$:

$$I_{n_1}[y(x,0)] = \min_{\substack{m_j \in \Omega_j \\ 0 \leqslant j \leqslant n_1-1}} \sum_{j=0}^{n_1-1} \gamma_j[m_j, y(x,jT)] \tag{4.5}$$

where $\gamma_j[m_j, y(x,jT)]$ is a convex functional of $y(x,jT)$ for a fixed m_j and a convex function of m_j for a fixed $y(x,jT)$, and Ω_j is the set of admissible values for m_j. Generally, $\gamma_j[m_j, y(x,jT)]$ is a convex measure of the distributed error between the desired output $y^d(x,t)$ and the actual output $y(x,t)$ at $t = jT^-$ together with a convex measure of m_j.

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* The phrase "return function" is standard dynamic programming terminology (6, 7, 8).

For example,

$$\gamma_j[m_j, y(x,jT)] = \int_0^L [y^d(x,jT) - y(x,jT)]^2 \, dx + \lambda(m_j - m^s)^2$$

is a possible form for $\gamma_j[m_j, y(x,jT)]$ where m^s is the steady-state value of the control signal and where λ is a "cost" function associated with control action.

The subscript n_1 of I denotes the number of stages in the return function, i.e., the number of terms in the summation. The argument of the functional I_{n_1} represents the function on which the minimization process depends.

Since $\gamma_0[m_0, y(x,0)]$ is not a function of m_j, $1 \leq j \leq n_1-1$, Equation 4.5 can be written as

$$I_{n_1}[y(x,0)] = \min_{m_0 \in \Omega_0} \{\gamma_0[m_0, y(x,0)] + \min_{\substack{m_j \in \Omega_j \\ 1 \leq j \leq n_1-1}} \sum_{j=1}^{n_1-1} \gamma_j[m_j, y(x,jT)]\} \tag{4.6}$$

Descriptively, the second term in the right-hand member of Equation 4.6 is associated with the problem of optimal selection of the controlling actions from the second stage to the n_1^{th} stage. It is evident that such selection depends on the distribution at the end of the first stage. Thus, by definition in Equation 4.5, the second term in the right-hand member of Equation 4.6 can be represented as

$$I_{n_1-1}[y(x,T)] = \min_{\substack{m_j \in \Omega_j \\ 1 \leq j \leq n_1-1}} \sum_{j=1}^{n_1-1} \gamma_j[m_j, y(x,jT)] \tag{4.7}$$

By iteration, a recursive relation can be established, as follows,

$$I_{n_1-i}[y(x,iT)] = \min_{m_i \varepsilon \Omega_i} \{\gamma_i[m_i, y(x,iT)] + I_{n_1-i-1}[y(x,(i+1)T]\} \tag{4.8}$$

for $0 \leq i \leq n_1-2$, where

$$I_{n_1-i-1}[y(x,(i+1)T)] = \min_{\substack{m_j \varepsilon \Omega_j \\ i+1 \leq j \leq n_1-1}} \sum_{j=i+1}^{n_1-1} \gamma_j[m_j, y(x,jT)] \tag{4.9}$$

For $i = n_1-2$, Equation 4.9 simply reads

$$I_1[y(x,(n_1-1)T)] = \min_{m_{n_1-1} \varepsilon \Omega_{n_1-1}} \gamma_{n_1-1}[m_{n_1-1}, y(x,(n_1-1)T)] \tag{4.10}$$

The convexity of the functional I_{n_1-i} with respect to its argument is next to be derived. This property validates the interpolation of the value of the functional I_{n_1-i} about its argument. The proof of convexity of the functional I_{n_1-i} for all i's between 0 and n_1-1 is accomplished by mathematical induction. Let $y(x,(n_1-1)T)$ be expressed by

$$y(x,(n_1-1)T) = \alpha y'(x,(n_1-1)T) + (1-\alpha)y''(x,(n_1-1)T) \tag{4.11}$$

where α is a real number in the closed interval $[0,1]$, and where $y'(x,(n_1-1)T)$ and $y''(x,(n_1-1)T)$ are distributions at the end of the $(n_1-1)^{th}$ sampling period. With the assumption in Equation 4.11 and the convex property of γ_{n_1}, Equation 4.10 can be written as

$$I_1[y(x,(n_1-1)T)] \leq \min_{m_{n_1-1} \in \Omega_{n_1-1}} \{\alpha\gamma_{n_1-1}[m_{n_1-1}, y'(x,(n_1-1)T)] +$$

$$+ (1-\alpha)\ \gamma_{n_1-1}[m_{n_1-1}, y''(x,(n_1-1)T)]\}$$

$$\leq \alpha I_1[y'(x,(n_1-1)T)] + (1-\alpha)I_1[y''(x,(n_1-1)T)] \quad (4.12)$$

Thus, $I_1[y(x,(n_1-1)T)]$ is a convex functional of $y(x,(n_1-1)T)$.

Next, assume that $I_{n_1-i-1}[y(x,(i+1)T)]$ is a convex functional of $y(x,(i+1)T)$. From Equation 4.2, the expressions for $y(x,iT)$ and $y(x,(i+1)T)$ can be written as

$$y(x,iT) = k_0(x,iT) + \sum_{k=1}^{i} m_{k-1}h_i(x,kT^-) \quad (4.13)$$

and

$$y(x,(i+1)T) = k_0(x,(i+1)T) + \sum_{k=1}^{i+1} m_{k-1}h_{i+1}(x,kT^-) \quad (4.14)$$

It can be seen from Equations 4.13 and 4.14 that $y(x,(i+1)T)$ is dependent on $y(x,iT)$ and m_i. Therefore, if

$$y(x,iT) = \alpha y'(x,iT) + (1-\alpha)y''(x,iT) \quad (4.15)$$

then the corresponding $y(x,(i+1)T)$ can be written as

$$y(x,(i+1)T) = \alpha y'(x,(i+1)T) + (1-\alpha)y''(x,(i+1)T) \quad (4.16)$$

Because γ_i and I_{n_1-i-1} are convex functionals of $y(x,iT)$ and $y(x,(i+1)T)$, respectively, it follows from Equation 4.8 that

$$I_{n_1-i}[y(x,iT)]$$

$$\leq \min_{m_i \in \Omega_i} [\alpha\{\gamma_i[m_i, y'(x,iT)] + I_{n_1-i-1}[y'(x,(i+1)T]\} +$$

$$+ (1-\alpha)\{\gamma_i[m_i, y''(x,iT)] + I_{n_1-i-1}[y''(x,(i+1)T)]\}]$$

$$\leq \alpha I_{n_1-i}[y'(x,iT)] + (1-\alpha) I_{n_1-i}[y''(x,iT)] \tag{4.17}$$

Thus, it is concluded that $I_{n_1-i}[y(x,iT)]$ is a convex functional of $y(x,iT)$ for $0 \leq i \leq n_1-1$.

For fixed $y(x,iT)$, the convexity with respect to m_i of

$$\Gamma_i[m_i, y(x,iT)] = \gamma_i[m_i, y(x,iT)] + I_{n_1-i-1}[y(x,(i+1)T)] \tag{4.18}$$

is next to be derived. With this property, any relative minimum of $\Gamma_i[m_i, y(x,iT)]$ is also the absolute minimum provided that Ω_i is a closed convex set. This simplifies considerably the problem of searching for the minimum of $\Gamma_i[m_i, y(x,iT)]$. If $\Gamma_i[m_i, y(x,iT)]$ is strictly convex, it is known that the optimizing m_i is unique.

The proof of convexity of $\Gamma_i[m_i, y(x,iT)]$ is readily established by observing that, with $y(x,iT)$ being given, the only independent variable in the expression of $y(x,(i+1)T)$ is m_i, and $y(x,(i+1)T)$ is linear in m_i. Since $I_{n_1-i-1}[y(x,(i+1)T)]$ is a convex functional of its argument, it is also a convex function of m_i. Therefore, from the fact that the sum of two convex functions is also convex, it can be concluded that $\Gamma_i[m_i, y(x,iT)]$ is a convex function of m_i for a fixed $y(x,iT)$. This conclusion holds for all i's.

The continuity of the optimizing $m_i[y(x,iT)]$ is last to be proven. Continuity of the optimizing $m_i[y(x,iT)]$ assures that the value of the optimizing $m_i[y(x,iT)] + \Delta y(x,iT)]$ is close to that of the optimizing $m_i[y(x,iT)]$ if $\Delta y(x,iT)$ is a sufficiently small variation. This suggests that the search for the minimum of $\Gamma_i[m_i, y(x,iT) + \Delta y(x,iT)]$ can be performed in the neighborhood of the optimizing $m_i[y(x,iT)]$.

Suppose that $\hat{m}_i$ and $\hat{\hat{m}}_i$ minimize $\Gamma_i[m_i, y(x,iT)]$ and $\Gamma_i[m_i, y'(x,iT)]$ respectively. Let

$$m_i = \alpha\hat{m}_i + (1-\alpha)\hat{\hat{m}}_i \quad , \quad 0 \leq \alpha \leq 1 \tag{4.19}$$

By convexity of $\Gamma_i[m_i, y(x,iT)]$, it follows that

$$\Gamma_i[m_i, y(x,iT)] \leq \alpha\Gamma_i[\hat{m}_i, y(x,iT)] + (1-\alpha)\Gamma_i[\hat{\hat{m}}_i, y(x,iT)] \tag{4.20}$$

By definition of $\hat{m}_i$, Inequality 4.20 becomes

$$\Gamma_i[m_i, y(x,iT)] \leq \Gamma_i[\hat{\hat{m}}_i, y(x,iT)] \tag{4.21}$$

Similarly it can be shown that

$$\Gamma_i[m_i, y'(x,iT)] \leq \Gamma_i[\hat{m}_i, y'(x,iT)] \tag{4.22}$$

Since $\Gamma_i[m_i, y(x,iT)]$ is uniformly continuous, it follows that given $\varepsilon_1 > 0$, there exists $\delta_1 > 0$ such that

$$\left|\Gamma_i[m_i, y(x,iT)] - \Gamma_i[m_i', y'(x,iT)]\right| < \varepsilon_1 \tag{4.23}$$

for $\{(m_i - m_i')^2 + [y(x,iT) - y'(x,iT)]^2\}^{\frac{1}{2}} < \delta_1$ for all $x \in [0,L]$ and $m_i \in \Omega_i$.

In the special case that $m_i = m_i' = \hat{m}_i$, Inequality 4.23 reads

$$\Gamma_i[\hat{m}_i, y(x,iT)] - \Gamma_i[\hat{m}_i, y'(x,iT)] < \varepsilon_1 \tag{4.24}$$

Addition of corresponding members of Inequalities 4.21 and 4.24 yields

$$\Gamma_i[m_i, y(x,iT)] - \Gamma_i[\hat{m}_i, y'(x,iT)] < \varepsilon_1 \tag{4.25}$$

When $m_i' = m_i$, Inequality 4.23 can be written as

$$\Gamma_i[m_i, y'(x,iT)] - \Gamma_i[m_i, y(x,iT)] < \varepsilon_1 \tag{4.26}$$

By definition of $\hat{\hat{m}}_i$, it follows that

$$\Gamma_i[\hat{\hat{m}}_i, y'(x,iT)] - \Gamma_i[m_i, y(x,iT)] < \varepsilon_1 \tag{4.27}$$

It can be concluded from the results in Inequalities 4.25 and 4.27 that

$$|\Gamma_i[\hat{\hat{m}}_i, y'(x,iT)] - \Gamma_i[m_i, y(x,iT)]| < \varepsilon_1 \tag{4.28}$$

Again, for $m_i = m_i' = \hat{m}_i$, Inequality 4.23 can be written as

$$\Gamma_i[\hat{m}_i, y'(x,iT)] - \Gamma_i[\hat{m}_i, y(x,iT)] < \varepsilon_1 \tag{4.29}$$

Addition of corresponding members of Inequalities 4.22 and 4.29 yields

$$\Gamma_i[m_i, y'(x,iT)] - \Gamma_i[\hat{m}_i, y(x,iT)] < \varepsilon_1 \tag{4.30}$$

When $m_i' = m_i$, Inequality 4.23 reads, after the absolute-value signs are removed,

$$\Gamma_i[m_i, y(x,iT)] - \Gamma_i[m_i, y'(x,iT)] < \varepsilon_1 \tag{4.31}$$

Then it can be concluded from the results in Inequalities 4.30 and 4.31 together with the definition of $\hat{m}_i$ that

$$|\Gamma_i[m_i, y(x,iT)] - \Gamma_i[\hat{m}_i, y(x,iT)]| < 2\varepsilon_1 \tag{4.32}$$

By the property of the triangular inequality and the results in Inequalities 4.28 and 4.32, it follows that

$$|\Gamma_i[\hat{\hat{m}}_i, y'(x,iT)] - \Gamma_i[\hat{m}_i, y(x,iT)]| < 3\varepsilon_1 \tag{4.33}$$

for $\{(\hat{\hat{m}}_i - \hat{m}_i)^2 + [y(x,iT) - y'(x,iT)]^2\}^{\frac{1}{2}} < \delta_1$ for all $x \in [0,L]$. If

$$|y(x,iT) - y'(x,iT)| < \delta_2 \tag{4.34}$$

for all $x \in [0,L]$, then

$$|\hat{\hat{m}}_i - \hat{m}_i| < (\delta_1^2 - \delta_2^2)^{\frac{1}{2}} \tag{4.35}$$

Descriptively, the optimizing $m_i[y(x,iT)]$ is continuous if $\Gamma_i[m_i, y(x,iT)]$ is convex. It should be noticed that the continuity of $\Gamma_i[m_i, y(x,iT)]$ alone does not assure the continuity of the optimizing $m_i[y(x,iT)]$, but the convexity of $\Gamma_i[m_i, y(x,iT)]$ does.

4.2.2 Discrete Spatial Variable

It is evident that the method in Section 4.2.1 suffers severely from the dimensionality problem because of the continuous nature of the spatial variable in the formulation of the problem. In an attempt to ease this dimensionality problem, the spatial variable is discretized in this subsection. Generally, information from a finite number of points in the

spatial domain, say n_2 points, is sufficient to represent essentially all the information in the spatial domain.

From Equation 4.2, the distribution at n_2 discrete points in the spatial domain can be written in matrix form as

$$Y(jT) = Y_{j-1}(jT) + m_{j-1}G_{j-1}(jT) \tag{4.36}$$

where $Y(jT)$, $Y_{j-1}(jT)$ and $G_{j-1}(jT)$ are n_2 x 1 matrices, which have i^{th} elements $y(x_i,jT)$, $y(x_i,(j-1)T,jT)$ and $\int_0^T g(x_i,jT,(j-1)T+\tau)\, d\tau$, respectively.

The return function in the discrete spatial variable case, which is the counterpart to that in Equation 4.5, is

$$I_{n_1}[Y(0)] = \min_{\substack{m_j \in \Omega_j \\ 0 \leq j \leq n_1-1}} \sum_{j=0}^{n_1-1} \gamma_j[m_j, Y(jT)] \tag{4.37}$$

The equivalent recursive relation can be written as

$$I_{n_1-i}[Y(iT)] = \min_{m_i \in \Omega_i} [\gamma_i[m_i, Y(iT)] + I_{n_1-i-1}\{Y[(i+1)T]\}] \tag{4.38}$$

and

$$I_1\{Y[(n_1-1)T]\} = \min_{m_{n_1-1} \in \Omega_{n_1-1}} \gamma_{n_1-1}\{m_{n_1-1}, Y[(n_1-1)T]\} \tag{4.39}$$

By following the same approach as in Section 4.2.1, it can be shown that the corresponding properties of the functionals are still valid in the discrete spatial variable case.

If a_j^i is the number of values which $y(x_i, jT)$ can assume, the number of minimizations which must be carrid out at $t = jT$ is

$$\prod_{i=1}^{n_2} a_j^i$$

The total storage requirement for a particular problem is

$$3\left(\sum_{j=1}^{n_1} \prod_{i=1}^{n_2} a_j^i\right)$$

It is evident that the feasibility of the solution depends on the choice of n_2. In the heat conduction problem, for example, Paschkin and Baker (59) suggest that the range of appropriate values of n_2 is between 5 and 12.

4.3 SPATIAL-AVERAGE-SQUARE ERROR

In this section, an analytical expression of the optimal control is derived under the spatial-average-square-error criterion. A spatial-average-square-error return function is obtained by using a special form of $\gamma_j[m_j, y(x,jT)]$ in Equation 4.5; namely,

$$\gamma_j[m_j, y(x,jT)] = \int_0^L [y^d(x,jT) - y(x,jT)]^2 \, dx + \lambda(m_j - m^s)^2 \qquad (4.40)$$

where $y^d(x,jT)$ is the desired distribution at $t = jT$, m^s is the steady-state value of the controlling action, and λ is the "cost factor" associated with deviation of m_j away from m^s.

The general expression of the optimal control which is associated with the return function described by Equation 4.5 together with Equation 4.40 is shown in the following paragraphs to be

$$m_i^*[y(x,iT)] = \frac{Nm_i^*[y(x,iT)]}{Dm_i^*} \tag{4.41}$$

for $0 \leq i \leq n_1-2$, where

$$Nm_i^*[y(x,iT)]$$

$$\triangleq \sum_{j=1}^{n_1-i-1} \int_0^L [y^d(x,(n_1-j)T) - y(x,iT,(n_1-j)T)]h_{n_1-j}(x,(i+1)T^-)\,dx +$$

$$+ \sum_{j=1}^{n_1-i-2} m_{n_1-1-j}^*[y(x,iT)] \frac{\delta Nm_{n_1-1-j}^*}{\delta m_i} + \lambda m^s \tag{4.42}$$

and

$$Dm_i^* \triangleq \lambda + \sum_{j=1}^{n_1-i-1} \int_0^L h_{n_1-j}^2(x,(i+1)T^-)\,dx - \sum_{j=1}^{n_1-i-2} \left(\frac{\delta Nm_{n_1-1-j}^*}{\delta m_i}\right)^2 / Dm_{n_1-1-j}^* \tag{4.43}$$

and the last controlling action is

$$m_{n_1-1}^* = m^s \tag{4.44}$$

The term $\frac{\delta Nm_{n_1-1-j}^*}{\delta m_i}$ denotes the functional partial derivative (86) with respect to m_i. The argument of $Nm_{n_1-1-j}^*$ is understood to be $y(x,(n_1-1-j)T)$. The notation $y(x,iT,(n_1-j)T)$ represents the distribution at $t = (n_1-j)T$ by letting the distribution $y(x,iT)$ decay with zero input applied after $t = iT$. Mathematically,

$$y(x,iT,(n_1-j)T) = k_0(x,(n_1-j)T) + \sum_{k=1}^{i} m_{k-1}h_{n_1-j}(x,kT^-) \tag{4.45}$$

For $i = n_1-2$ the second summation in both Equation 4.42 and 4.43 is zero. In this case, the optimal control is

$$m^*_{n_1-2}[y(x,(n_1-2)T)]$$

$$= \frac{\int_0^L [y^d(x,(n_1-1)T) - y(x,(n_1-2)T,(n_1-1)T)]h_{n_1-1}(x,(n_1-1)T^-)\,dx + \lambda m^s}{\lambda + \int_0^L h^2_{n_1-1}(x,(n_1-1)T^-)\,dx} \qquad (4.46)$$

The result in Equation 4.44 is readily established by observing that at the n_1^{th} stage of the system, the return function is

$$I_1[y(x,(n_1-1)T)] = \min_{m_{n_1-1}} \{\int_0^L [y^d(x,(n_1-1)T) - y(x,(n_1-1)T)]^2\,dx + \lambda(m_{n_1-1} - m^s)^2\} \qquad (4.47)$$

and that $y(x,(n_1-1)T)$ is independent of m_{n_1-1}. The result in Equation 4.41 can be proven by mathematical induction. First, consider the $(n_1-1)^{th}$ stage of the system: At this stage, the return function is

$$I_2[y(x,(n_1-2)T)] = \min_{m_{n_1-2}} \{\int_0^L [y^d(x,(n_1-2)T) - y(x,(n_1-2)T)]^2\,dx + \lambda(m_{n_1-2} - m^s)^2 + I_1[y(x,(n_1-1)T)]\} \qquad (4.48)$$

The distribution at $t = (n_1-1)T$, obtained from Equation 4.2, is

$$y(x,(n_1-1)T) = y(x,(n_1-2)T,(n_1-1)T) + m_{n_1-2}h_{n_1-1}(x,(n_1-1)T^-) \qquad (4.49)$$

The necessary condition for the minimizing m_{n_1-2} is

$$2\lambda(m_{n_1-2} - m^s) + \frac{\delta I_1[y(x,(n_1-1)T)]}{\delta m_{n_1-2}} = 0 \qquad (4.50)$$

where $\frac{\delta I_1[y(x,(n_1-1)T)]}{\delta m_{n_1-2}}$ denotes the functional partial derivative.

On the basis of Equations 4.44 and 4.47, it follows that

$$I_1[y(x,(n_1-1)T)] = \int_0^L [y^d(x,(n_1-1)T) - y(x,(n_1-1)T)]^2 \, dx \qquad (4.51)$$

and the required functional partial derivative is

$$\frac{\delta I_1[y(x,(n_1-1)T)]}{\delta m_{n_1-2}}$$

$$= -2 \int_0^L [y^d(x,(n_1-1)T) - y(x,(n_1-2)T,(n_1-1)T)]h_{n_1-1}(x,(n_1-1)T^-) \, dx +$$

$$+ 2m_{n_1-2} \int_0^L h^2_{n_1-1}(x,(n_1-1)T^-) \, dx \qquad (4.52)$$

Solving for m_{n_1-2} in Equation 4.50 in conjunction with the result in Equation 4.52 leads to the expression for $m^*_{n_1-2}[y(x,(n_1-2)T)]$ as shown in Equation 4.46. With this result and those of Equations 4.49 and 4.51, $I_2[y(x,(n_1-2)T)]$ of Equation 4.48 can be expressed as follows:

$$I_2[y(x,(n_1-2)T)] = \int_0^L [y^d(x,(n_1-2)T) - y(x,(n_1-2)T)]^2\,dx +$$

$$+ \int_0^L [y^d(x,(n_1-1)T) - y(x,(n_1-2)T),(n_1-1)T)]^2\,dx +$$

$$- \frac{(Nm^*_{n_1-2}[y(x,(n_1-2)T)])^2}{Dm^*_{n_1-2}} + \lambda m^{s\,2} \qquad (4.53)$$

Next consider the $(n_1-2)^{th}$ stage of the system. At this stage, $\Gamma_{n_1-3}[m_{n_1-3}, y(x,(n_1-3)T)]$ from Equation 4.18 becomes

$$\Gamma_{n_1-3}[m_{n_1-3}, y(x,(n_1-3)T)] = \int_0^L [y^d(x,(n_1-3)T) - y(x,(n_1-3)T)]^2\,dx +$$

$$+ \lambda(m_{n_1-3} - m^s)^2 + I_2[y(x,(n_1-2)T)] \qquad (4.54)$$

From Equation 4.2, the distribution at $t = (n_1-2)T$ is

$$y(x,(n_1-2)T) = y(x,(n_1-3)T,(n_1-2)T) + m_{n_1-3}h_{n_1-2}(x,(n_1-2)T^-) \qquad (4.55)$$

Then the necessary condition for the minimum of $\Gamma_{n_1-3}[m_{n_1-3}, y(x,(n_1-3)T)]$ with respect to m_{n_1-3} is

$$2\lambda(m_{n_1-3} - m^s) + \frac{\delta I_2[y(x,(n_1-2)T)]}{\delta m_{n_1-3}} = 0 \qquad (4.56)$$

When $t = (n_1-1)T$ and $j = n_1-1$, Equation 4.3 becomes

$$y(x,(n_1-2)T,(n_1-1)T) = k_0(x,(n_1-1)T) + \sum_{k=1}^{n_1-3} m_{k-1}h_{n_1-1}(x,kT^-) +$$

$$+ m_{n_1-3}h_{n_1-1}(x,(n_1-2)T^-)$$

$$= y(x,(n_1-3)T,(n_1-1)T) + m_{n_1-3}h_{n_1-1}(x,(n_1-2)T^-) \tag{4.57}$$

With the expressions in Equations 4.55 and 4.57, the partial derivative of $I_2[y(x,(n_1-2)T)]$ can be written as

$$\frac{\delta I_2[y(x,(n_1-2)T)]}{\delta m_{n_1-3}}$$

$$= -2 \sum_{j=1}^{2} \int_0^L [y^d(x,(n_1-j)T) - y(x,(n_1-3)T,(n_1-j)T)]h_{n_1-j}(x,(n_1-2)T^-)\,dx +$$

$$+ 2m_{n_1-3} \sum_{j=1}^{2} \int_0^L h^2_{n_1-j}(x,(n_1-2)T^-)\,dx - 2m^*_{n_1-2}[y(x,(n_1-2)T)] \frac{\delta Nm^*_{n_1-2}}{\delta m_{n_1-3}} \tag{4.58}$$

The expression of the numerator of $m^*_{n_1-2}[y(x,(n_1-2)T)]$ in Equation 4.46 can be written explicitly in terms of m_{n_1-3} as

$$Nm^*_{n_1-2}[y(x,(n_1-2)T)]$$

$$= \int_0^L [y^d(x,(n_1-1)T) - y(x,(n_1-3)T,(n_1-1)T)]h_{n_1-1}(x,(n_1-1)T^-)\,dx +$$

$$+ \lambda m^s - m_{n_1-3} \int_0^L h_{n_1-1}(x,(n_1-2)T^-)h_{n_1-1}(x,(n_1-1)T^-)\,dx$$

$$= Nm^*_{n_1-2}[y(x,(n_1-3)T)] + m_{n_1-3} \frac{\delta Nm^*_{n_1-2}}{\delta m_{n_1-3}} \tag{4.59}$$

After appropriate substitution of the identities expressed by Equations 4.58 and 4.59 in Equation 4.56, Equation 4.56 is solved for the minimizing $m^*_{n_1-3}$ which can be expressed as

$$m^*_{n_1-3}[y(x,(n_1-3)T)] = \frac{Nm^*_{n_1-3}[y(x,(n_1-3)T)]}{Dm^*_{n_1-3}} \tag{4.60}$$

where

$$Nm^*_{n_1-3}[y(x,(n_1-3)T)]$$

$$= \sum_{j=1}^{2} \int_0^L [y^d(x,(n_1-j)T) - y(x,(n_1-3)T,(n_1-j)T)]h_{n_1-j}(x,(n_1-2)T^-)\,dx +$$

$$+ m^*_{n_1-2}[y(x,(n_1-3)T)] \frac{\delta Nm^*_{n_1-2}}{\delta m_{n_1-3}} + \lambda m^s \tag{4.61}$$

and

$$Dm^*_{n_1-3} = \lambda + \sum_{j=1}^{2} \int_0^L h^2_{n_1-j}(x,(n_1-2)T^-)\,dx - \left(\frac{\delta Nm^*_{n_1-2}}{\delta m_{n_1-3}}\right)^2 / Dm^*_{n_1-2} \tag{4.62}$$

The expression for $m^*_{n_1-3}[y(x,(n_1-3)T)]$ in Equations 4.60, 4.61, and 4.62 corresponds to the result in Equations 4.41, 4.42, and 4.43 for $i = n_1-3$.

Appropriate substitution of the results in Equations 4.53, 4.55, 4.57, 4.59 and 4.60 into Equation 4.54, followed by rearrangement, yields the value of $I_3[y(x,(n_1-3)T)]$:

$$I_3[y(x,(n_1-3)T)]$$

$$= \int_0^L [y^d(x,(n_1-3)T) - y(x,(n_1-3)T)]^2 \, dx +$$

$$+ \sum_{j=1}^{2} \{ \int_0^L [y^d(x,(n_1-j)T) - y(x,(n_1-3)T,(n_1-j)T)]^2 \, dx +$$

$$- \frac{(Nm^*_{n_1-1-j}[y(x,(n_1-3)T)])^2}{Dm^*_{n_1-1-j}} + \lambda m^{s2} \} \tag{4.63}$$

Assume that at the $(i+2)^{th}$ stage of the system the optimal controlling action is given by

$$m^*_{i+1}[y(x,(i+1)T)] = \frac{Nm^*_{i+1}[y(x,(i+1)T)]}{Dm^*_{i+1}} \tag{4.64}$$

where

$$Nm^*_{i+1}[y(x,(i+1)T)]$$

$$= \sum_{j=1}^{n_1-i-2} \int_0^L [y^d(x,(n_1-j)T) - y(x,(i+1)T,(n_1-j)T)] h_{n_1-j}(x,(i+2)T^-) \, dx +$$

$$+ \sum_{j=1}^{n_1-i-3} m^*_{n_1-1-j}[y(x,(i+1)T)] \frac{\delta Nm^*_{n_1-1-j}}{\delta m_{i+1}} + \lambda m^s \tag{4.65}$$

and

$$Dm^*_{i+1} = \lambda + \sum_{j=1}^{n_1-i-2} \int_0^L h^2_{n_1-j}(x,(i+2)T^-) \, dx - \sum_{j=1}^{n_1-i-3} \left(\frac{\delta Nm^*_{n_1-1-j}}{\delta m_{i+1}}\right)^2 / Dm^*_{n_1-1-j} \tag{4.66}$$

and the value of $I_{n_1-i-1}[y(x,(i+1)T)]$ is given by

$$I_{n_1-i-1}[y(x,(i+1)T)] = \int_0^L [y^d(x,(i+1)T) - y(x,(i+1)T)]^2 \, dx +$$

$$+ \sum_{j=1}^{n_1-i-2} \{ \int_0^L [y^d(x,(n_1-j)T) - y(x,(i+1)T,(n_1-j)T)]^2 \, dx +$$

$$- \frac{(Nm^*_{n_1-1-j}[y(x,(i+1)T)])^2}{Dm^*_{n_1-1-j}} + \lambda m^{s2} \qquad (4.67)$$

To complete the inductive proof, it is to be shown that $m_i^*[y(x,iT)]$ is given by Equations 4.41, 4.42, and 4.43 and that $I_{n_1-i}[y(x,iT)]$ is also given by Equation 4.67 by replacing (i+1) by i.

In terms of the expression in Equation 4.40, the special case of Equation 4.18 is

$$\Gamma_i[m_i, y(x,iT)] = \int_0^L [y^d(x,iT) - y(x,iT)]^2 \, dx + \lambda(m_i - m^s)^2 +$$

$$+ I_{n_1-i-1}[y(x,(i+1)T)] \qquad (4.68)$$

The distribution at t = (i+1)T can be expressed from Equation 4.2 as

$$y(x,(i+1)T) = y(x;iT,(i+1)T) + m_i h_{i+1}(x,(i+1)T^-) \qquad (4.69)$$

Then the necessary condition for the minimum of $\Gamma_i[m_i, y(x,iT)]$ with respect to m_i is

$$2\lambda(m_i - m^s) + \frac{\delta I_{n_1-i-1}[y(x,(i+1)T)]}{\delta m_i} = 0 \qquad (4.70)$$

By use of Equation 4.45, $y(x,(i+1)T,(n_1-j)T)$ can be expressed as

$$y(x,(i+1)T,(n_1-j)T) = y(x,iT,(n_1-j)T) + m_i h_{n_1-j}(x,(i+1)T^-) \tag{4.71}$$

Since $y(x,(i+1)T,(n_1-j)T)$ is linear in m_i, $m^*_{n_1-2}[y(x,(i+1)T)]$ and $m^*_{n_1-3}[y(x,(i+1)T)]$, as seen from Equations 4.46 and 4.60, 4.61, and 4.62 after appropriate substitution of $y(x,(n_1-2)T,(n_1-1)T)$ by $y(x,(i+1)T,(n_1-1)T)$ in Equation 4.46 and $y(x,(n_1-3)T,(n_1-j)T)$ by $y(x,(i+1)T,(n_1-j)T)$ in Equation 4.61, are linear in m_i and can be written as

$$m^*_{n_1-2}[y(x,(i+1)T)] = m^*_{n_1-2}[y(x,iT)] + m_i \frac{\delta N m^*_{n_1-2}}{\delta m_i} / D m^*_{n_1-2}$$

and

$$m^*_{n_1-3}[y(x,(i+1)T)] = m^*_{n_1-3}[y(x,iT)] + m_i \frac{\delta N m^*_{n_1-3}}{\delta m_i} / D m^*_{n_1-3}$$

The optimal control $m^*_{n_1-1-j}[y(x,(i+1)T)]$ is assumed to take on the same form as $m^*_{n_1-2}$ and $m^*_{n_1-3}$ for $1 \leq j \leq n_1-i-2$ and thus can be written as

$$m^*_{n_1-1-j}[y(x,(i+1)T)] = m^*_{n_1-1-j}[y(x,iT)] + m_i \frac{\delta N m^*_{n_1-1-j}}{\delta m_i} / D m^*_{n_1-1-j} \tag{4.72}$$

Then, by use of the results in Equations 4.69, 4.71, and 4.72, the partial derivative of $I_{n_1-i-1}[y(x,(i+1)T)]$ in Equation 4.67 with respect to m_i can be written as

$$\frac{\delta I_{n_1-i-1}[y(x,(i+1)T)]}{\delta m_i}$$

$$= -2 \sum_{j=1}^{n_1-i-1} \{ \int_0^L [y^d(x,(n_1-j)T) - y(x,iT;(n_1-j)T)] h_{n_1-j}(x,(i+1)T^-)\, dx +$$

$$+ m^*_{n_1-1-j}[y(x,iT)] \frac{\delta N m^*_{n_1-1-j}}{\delta m_i} - m_i \int_0^L h^2_{n_1-j}(x,(i+1)T^-)\, dx \} +$$

$$+ \sum_{j=1}^{n_1-i-2} m_i \left(\frac{\delta N m^*_{n_1-1-j}}{\delta m_i}\right)^2 / D m^*_{n_1-1-j} \qquad (4.73)$$

After the partial derivative of $I_{n_1-i-1}[y(x,(i+1)T)]$ with respect to m_i in Equation 4.70 is replaced by the expression in Equation 4.73, Equation 4.70 is then solved for the minimizing m_i, denoted by $m^*_i[y(x,iT)]$, which agrees with the expression in Equations 4.41, 4.42, and 4.43. By mathematical induction, it is concluded that the optimal control sequence given by the set of Equations 4.41, 4.42, and 4.43 is valid for all i's from 0 to n_1-1.

Substitution of $y(x,(i+1)T)$ from Equation 4.69, $y(x,(i+1)T,(n_1-j)T)$ from Equation 4.71 and $m^*_{n_1-1-j}[y(x,(i+1)T)]$ from Equation 4.72 into Equation 4.67, followed by grouping the coefficients of m_i and m_i^2, yields

$$I_{n_1-i-1}[y(x,(i+1)T)]$$

$$= \sum_{j=1}^{n_1-i-1} \int_0^L [y^d(x,(n_1-j)T) - y(x,iT,(n_1-j)T)]^2 \, dx +$$

$$+ \sum_{j=1}^{n_1-i-2} [\lambda m^{s^2} - \frac{(Nm^*_{n_1-1-j}[y(x,iT)])^2}{Dm^*_{n_1-1-j}}] +$$

$$- 2m_i \{ \sum_{j=1}^{n_1-i-1} \int_0^L [y^d(x,(n_1-j)T) - y(x,iT,(n_1-j)T)] h_{n_1-j}(x,(i+1)T^-) \, dx +$$

$$- \sum_{j=1}^{n_1-i-2} m^*_{n_1-1-j}[y(x,iT)] \frac{\delta Nm^*_{n_1-1-j}}{\delta m_i} \} +$$

$$+ m_i^2 \sum_{j=1}^{n_1-i-1} \int_0^L h^2_{n_1-j}(x,(i+1)T^-) \, dx - \sum_{j=1}^{n_1-i-2} (\frac{\delta Nm^*_{n_1-1-j}}{\delta m_i})^2 / Dm^*_{n_1-1-j} \quad (4.74)$$

Then, with the results in Equations 4.41 and 4.74, the minimum of $\Gamma_i[m_i, y(x,iT)]$ in Equation 4.68 with respect to m_i is

$$I_{n_1-i}[y(x,iT)] = \int_0^L [y^d(x,iT) - y(x,iT)]^2 \, dx +$$

$$+ \sum_{j=1}^{n_1-i-1} \{ \int_0^L [y^d(x,(n_1-j)T) - y(x,iT,(n_1-j)T)]^2 \, dx +$$

$$- \frac{(Nm^*_{n_1-1-j}[y(x,iT)])^2}{Dm^*_{n_1-1-j}} + \lambda m^{s^2} \} \quad (4.75)$$

which is also obtainable from Equation 4.67 by simply replacing (i+1) by i. By mathematical induction, it can be concluded that the expression of $I_{n_1-i}[y(x,iT)]$ in Equation 4.75 is valid for all i's from 0 to n_1-1.

With the analytical result for the optimal control sequence in Equations 4.41, 4.42, and 4.43, the computational difficulties as appeared in the previous sections are greatly reduced. Numerical computation is feasible even in a small digital computer. The scheme of computation follows the same order as in a backward dynamic-programming solution except that determination of $m_i^*[y(x,iT)]$ requires only one computation of each $m^*_{n_1-1-j}[y(x,iT)]$, $1 \leqslant j \leqslant n_1-i-2$. Because of this property, the dimensionality of the problem is greatly reduced.

As in lumped-parameter systems, the optimal feedback control for a quadratic performance measure of linear distributed-parameter systems is obtained. If $y(x,iT)$ is measured and the decay of $y(x,iT)$ without any input, $y(x,iT,(n_1-j)T)$, can be computed, the optimal controlling action $m_1^*[y(x,iT)]$ can be determined as shown in Equations 4.41, 4.42, and 4.43. A form of feedback control of linear distributed-parameter systems is depicted in Figure 4.1.

4.4 ILLUSTRATIVE EXAMPLE

The system of Section 3.5 which is described by Equations 3.50, 3.51, and 3.52 is used in this section also. The performance measure in this example is described by Equation 4.5 in conjunction with Equation 4.40. The desired distribution $y^d(x,t)$ is still unity.

The steady-state value of the input can be determined from Equation 3.56. Let $u(t) = u^s$. Then, after interchanging the order of summation and integration, Equation 3.56 becomes

$$y(x,t) = 4u^s \sum_{n=1}^{\infty} \frac{\sin(w_{2n-1}x)}{w_{2n-1}}[1 - \exp(-w_{2n-1}^2 t)] \tag{4.76}$$

The steady-state value of $y(x,t)$ is then

$$y(x,\infty) = 4u^s \sum_{n=1}^{\infty} \frac{\sin(w_{2n-1}x)}{w_{2n-1}} \tag{4.77}$$

By recognizing from the theory of Fourier's series that

$$\sum_{n=1}^{\infty} \frac{\sin(w_{2n-1}x)}{w_{2n-1}} = 1/4, \text{ for } 0 < x < 1 \tag{4.78}$$

Equation 4.77 becomes

$$y(x,\infty) = u^s \tag{4.79}$$

Therefore, the steady-state value m^s of the control sequence is 1. Since the initial distribution as given in Equation 3.51 is zero, Equation 4.45 becomes

$$y(x,iT,(n_1-j)T) = \sum_{k=1}^{i} m_{k-1} h_{n_1-j}(x,kT^-) \tag{4.80}$$

in which, from Equation 3.58, the expression for $h_{n_1-j}(x,kT^-)$ can be written as

$$h_{n_1-j}(x,kT^-) = 4 \sum_{n=1}^{\infty} \frac{\sin(w_{2n-1}x)}{w_{2n-1}}[\exp\{-w_{2n-1}^2(n_1-j-k-1)T\}] \; [\exp(w_{2n-1}^2 T)-1] \tag{4.81}$$

The numerator of $m_i^*[y(x,iT)]$ in this example can be written from Equation 4.42 as

$$Nm_i^*[y(x,iT)] = \sum_{j=1}^{n_1-i-1} \int_0^L h_{n_1-j}(x,(i+1)T^-)\,dx +$$

$$- \sum_{j=1}^{n_1-i-1} \sum_{k=1}^{i} m_{k-1} \int_0^L h_{n_1-j}(x,kT^-)h_{n_1-j}(x,(i+1)T^-)\,dx +$$

$$+ \sum_{j=1}^{n_1-i-2} m^*_{n_1-1-j}[y(x,iT)] \frac{\delta Nm^*_{n_1-1-j}}{\delta m_i} + \lambda m^s \qquad (4.82)$$

The partial derivative of Nm_i^* with respect to m_α is then

$$\frac{\delta Nm_i^*}{\delta m_\alpha} = - \sum_{j=1}^{n_1-i-1} \int_0^L h_{n_1-j}(x,(\alpha+1)T^-)h_{n_1-j}(x,(i+1)T^-)\,dx +$$

$$+ \sum_{j=1}^{n_1-i-2} \left(\frac{\delta m^*_{n_1-1-j}}{\delta m_\alpha}\right) \left(\frac{\delta Nm^*_{n_1-1-j}}{\delta m_i}\right) \qquad (4.83)$$

The scheme of computation starts with computation of $m^*_{n_1-j}(0)$ from Equations 4.41, 4.42, and 4.43 with the results in Equations 4.81, 4.82, and 4.83 for $j = 2, 3, \ldots, n_1$. The optimal control at the first sampling instant is given by $m_0^*(0)$. Next $m^*_{n_1-j}[y(x,T)]$ is computed from Equations 4.41, 4.42, and 4.43, for $j = 2, 3, \ldots, n_1-1$, in conjunction with Equations 4.81, 4.82, and 4.83 and

$$y(x,T,(n_1-j+1)T) = m_0^*(0)h_{n_1-j+1}(x,T) \qquad (4.84)$$

Then the optimal control at the second sampling instant is given by $m^*_1[y(x,T)]$. By continuing in this manner, all the controlling actions can be determined.

For $n_1 = 31$ and $T = 0.01$ second, the optimal control sequences for $\lambda = 0$, $\lambda = 1.0$ and $\lambda = 2.0$ are tabulated in Table 4.1. The corresponding output distributions are displayed in Figures 4.3, 4.4, and 4.5 and are also tabulated in Tables 4.2, 4.3, and 4.4.

4.5 CONCLUSION

In this chapter the optimal design of distributed-parameter systems under a general convex performance measure is studied from the dynamic programming point of view. The convexity of the functional $I_{n_1-i}[y(x,iT)]$ and the continuity of the minimizing m_i are derived in Section 4.2.1. The feasibility of computation in the discrete spatial variable case is discussed in Section 4.2.2. In Section 4.3, a closed-form solution for the optimal control sequence under the spatial-average-square-error performance measure is obtained. By use of this approach, computational and core-storage requirements are not critical --- the computations are feasible even in a small digital computer.

The characteristic of the optimal system under the square-error criterion is quite evident in the results of the example in Section 4.4. The system is penalized much more severely for large errors than for small errors. This results in an optimal control of the "long tail" type. This is not the type of control obtained in Chapter 3 where a measure of the magnitude of the error is minimized. Advantages and disadvantages are associated with both approaches --- the designer must choose his own performance measure to fit the physical specification and requirements of the system.

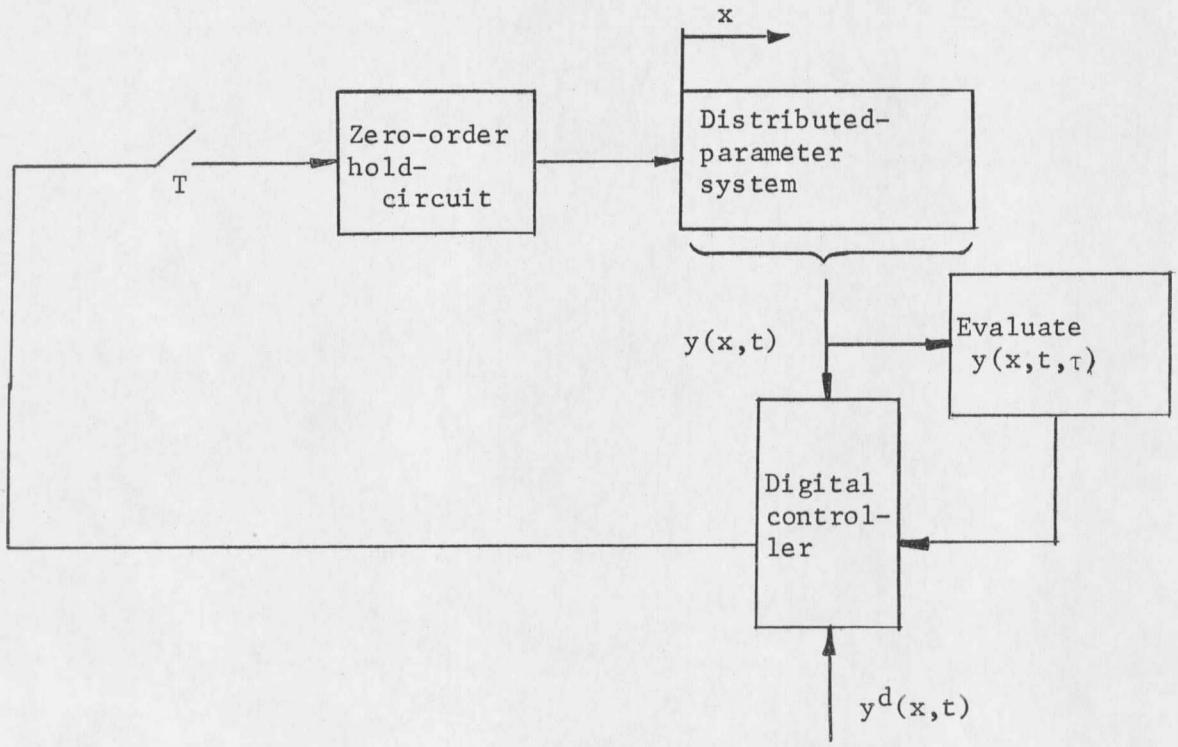


Figure 4.1. Feedback control of linear distributed-parameter systems.

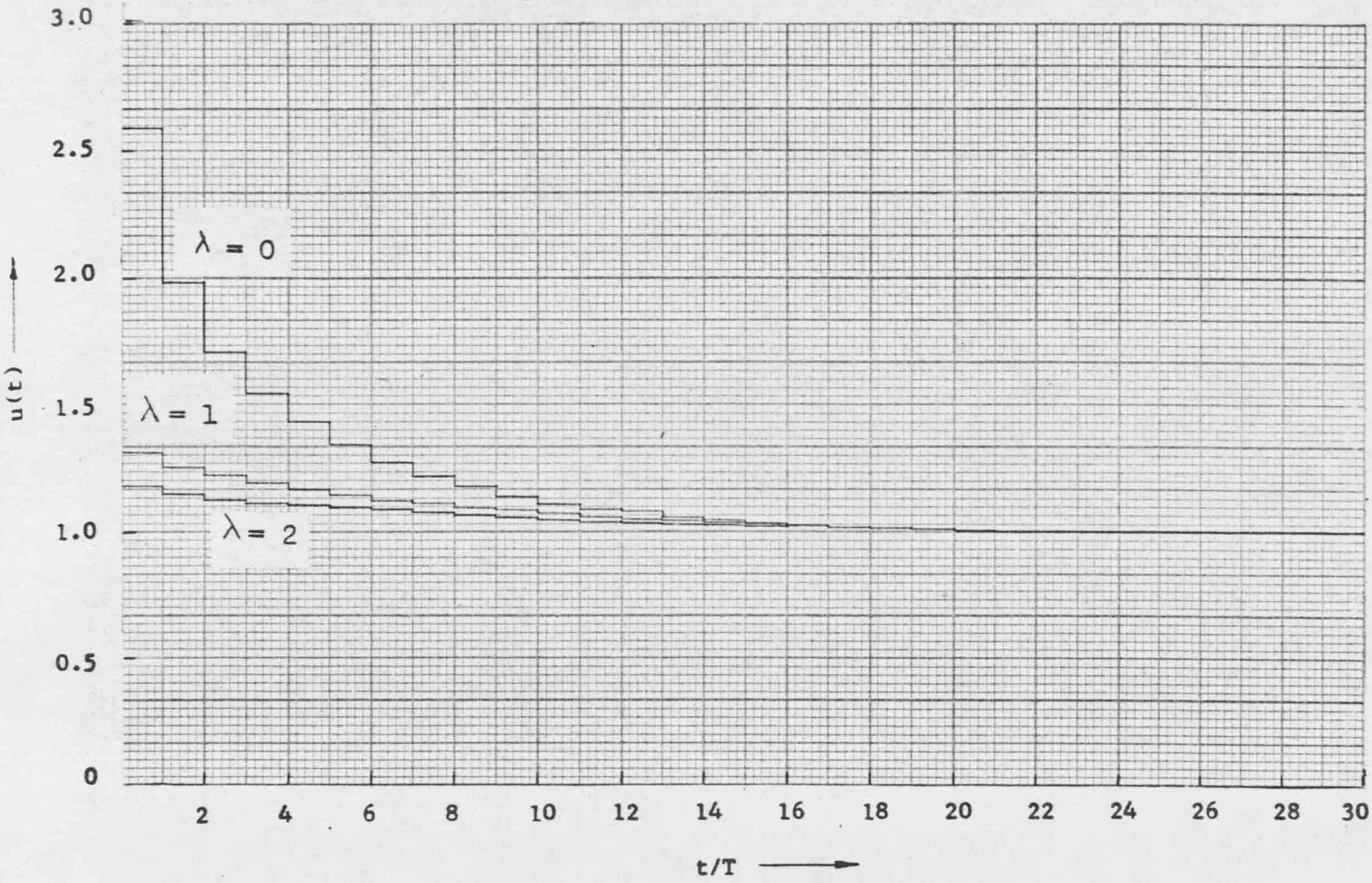


Figure 4.2. Optimal controlling actions.

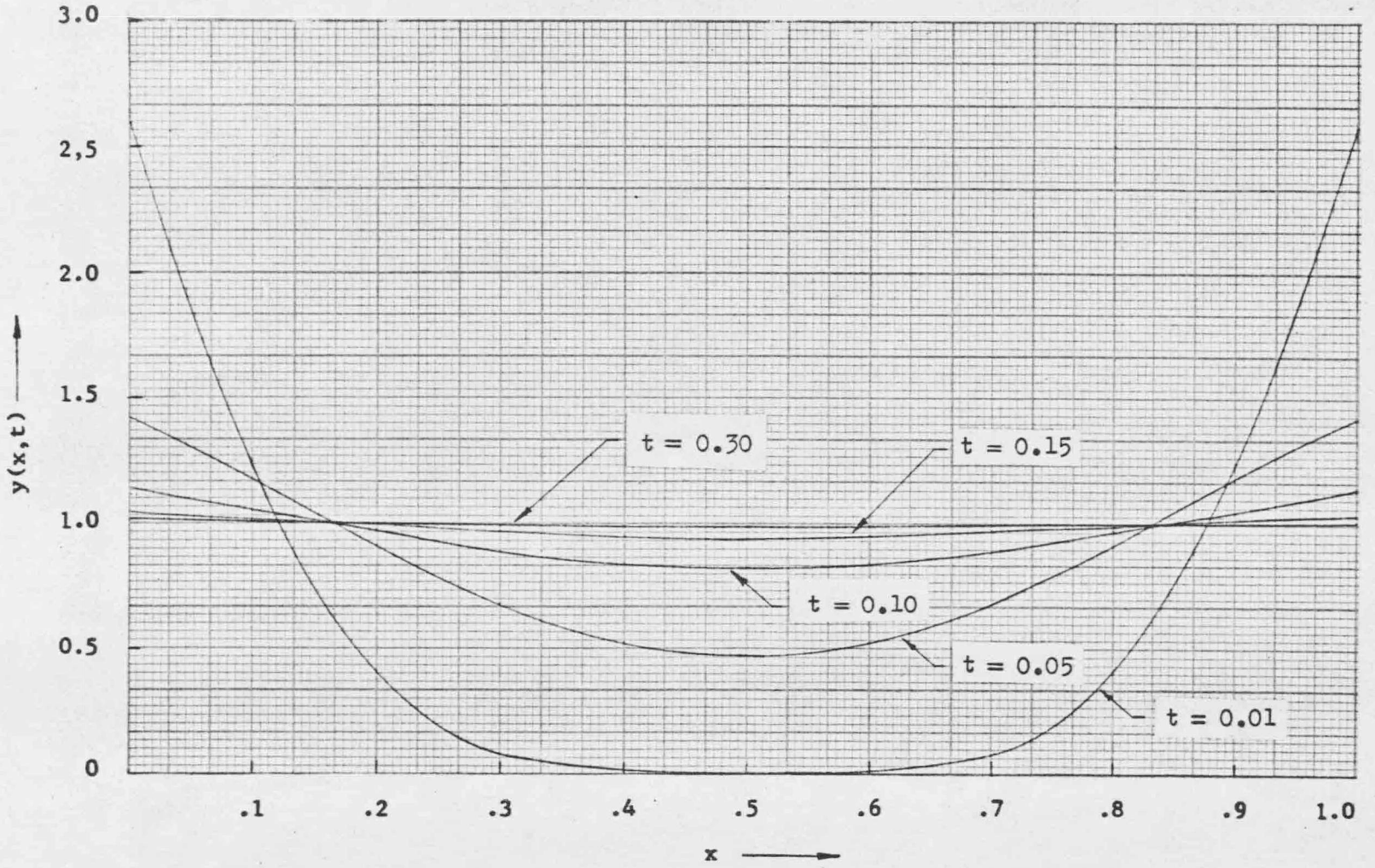


Figure 4.3. Output response $y(x,t)$ for $\lambda = 0$.

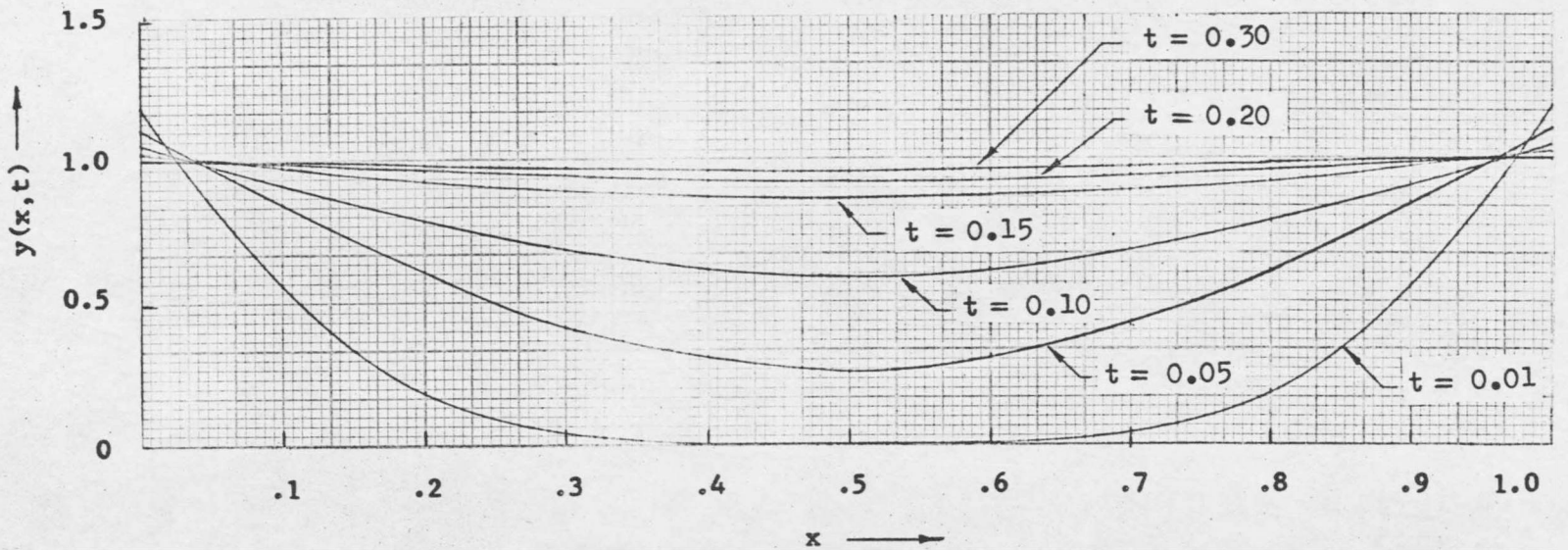


Figure 4.4. Output response y(x,t) for $\lambda = 1.0$.

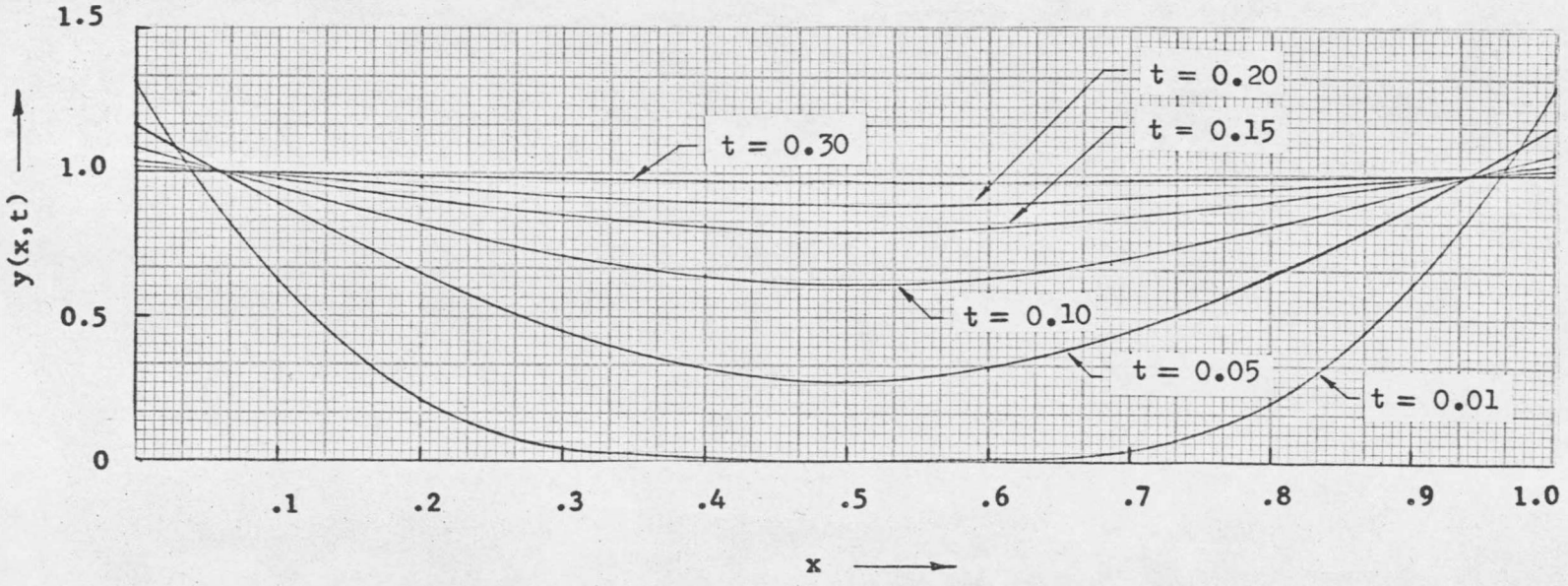


Figure 4.5. Output response y(x,t) for $\lambda = 2.0$.

Table 4.1. Sequence of the optimal input $m_i^*[y(x,iT)]$.

i	$m_i^* y(x,iT)$		
	$\lambda = 0$	$\lambda = 1.0$	$\lambda = 2.0$
0	2.5970	1.3100	1.1762
1	1.9700	1.2551	1.1486
2	1.7054	1.2183	1.1293
3	1.5440	1.1897	1.1140
4	1.4289	1.1659	1.1010
5	1.3411	1.1456	1.0898
6	1.2722	1.1279	1.0798
7	1.2175	1.1124	1.0710
8	1.1739	1.0988	1.0632
9	1.1390	1.0868	1.0562
10	1.1111	1.0763	1.0500
11	1.0889	1.0670	1.0444
12	1.0710	1.0588	1.0394
13	1.0568	1.0516	1.0350
14	1.0454	1.0452	1.0310
15	1.0363	1.0396	1.0274
16	1.0290	1.0346	1.0242
17	1.0232	1.0302	1.0214
18	1.0185	1.0263	1.0188
19	1.0147	1.0228	1.0164
20	1.0117	1.0197	1.0143
21	1.0093	1.0169	1.0124
22	1.0074	1.0144	1.0106
23	1.0058	1.0121	1.0090
24	1.0045	1.0100	1.0075
25	1.0035	1.0081	1.0061
26	1.0026	1.0064	1.0048
27	1.0018	1.0047	1.0035
28	1.0012	1.0031	1.0023
29	1.0006	1.0015	1.0011
30	1.0000	1.0000	1.0000

Table 4.2. Output response $y(x,t)$ for $\lambda = 0$.

t	y(x,t)				
	x = 0.1	x = 0.2	x = 0.3	x = 0.4	x = 0.5
0.01	1.2453	.4085	.0880	.0122	.0021
0.02	1.3019	.7256	.3269	.1222	.0640
0.03	1.2608	.8379	.4911	.2718	.1983
0.04	1.2137	.8876	.6027	.4098	.3420
0.05	1.1725	.9153	.6853	.5256	.4687
0.06	1.1384	.9338	.7493	.6200	.5736
0.07	1.1108	.9476	.7998	.6959	.6585
0.08	1.0886	.9582	.8400	.7567	.7268
0.09	1.0709	.9666	.8721	.8055	.7815
0.10	1.0567	.9733	.8977	.8444	.8253
0.11	1.0453	.9787	.9182	.8756	.8603
0.12	1.0362	.9829	.9346	.9005	.8883
0.13	1.0290	.9864	.9477	.9204	.9106
0.14	1.0232	.9891	.9582	.9364	.9285
0.15	1.0185	.9913	.9665	.9491	.9429
0.16	1.1048	.9930	.9732	.9593	.9543
0.17	1.0118	.9944	.9786	.9675	.9635
0.18	1.0094	.9955	.9829	.9740	.9708
0.19	1.0075	.9964	.9863	.9792	.9766
0.20	1.0060	.9971	.9890	.9833	.9813
0.21	1.0047	.9976	.9912	.9867	.9850
0.22	1.0037	.9981	.9929	.9893	.9880
0.23	1.0029	.9984	.9943	.9914	.9904
0.24	1.0022	.9987	.9954	.9931	.9923
0.25	1.0017	.9989	.9963	.9945	.9938
0.26	1.0013	.9991	.9970	.9955	.9950
0.27	1.0009	.9992	.9975	.9964	.9960
0.28	1.0005	.9992	.9980	.9971	.9967
0.29	1.0002	.9992	.9984	.9976	.9973
0.30	.9999	.9992	.9986	.9980	.9978

Table 4.3. Output response $y(x,t)$ for $\lambda = 1.0$.

t	y(x,t)				
	x = 0.1	x = 0.2	x = 0.3	x = 0.4	x = 0.5
0.01	.6282	.2061	.0444	.0062	.0011
0.02	.7821	.4071	.1738	.0629	.0325
0.03	.8436	.5208	.2861	.1502	.1066
0.04	.8758	.5953	.3776	.2421	.1965
0.05	.8960	.6506	.4547	.3291	.2860
0.06	.9106	.6952	.5211	.4081	.3690
0.07	.9221	.7328	.5789	.4785	.4436
0.08	.9317	.7652	.6296	.5408	.5099
0.09	.9400	.7936	.6741	.5958	.5686
0.10	.9472	.8184	.7132	.6442	.6203
0.11	.9535	.8401	.7476	.6869	.6658
0.12	.9590	.8593	.7778	.7245	.7059
0.13	.9638	.8761	.8045	.7575	.7411
0.14	.9680	.8909	.8279	.7865	.7722
0.15	.9717	.9039	.8485	.8121	.7995
0.16	.9750	.9153	.8666	.8346	.8235
0.17	.9778	.9254	.8825	.8544	.8446
0.18	.9803	.9342	.8965	.8718	.8632
0.19	.9824	.9419	.9088	.8870	.8795
0.20	.9843	.9487	.9196	.9005	.8938
0.21	.9859	.9546	.9291	.9123	.9064
0.22	.9872	.9598	.9374	.9227	.9175
0.23	.9884	.9644	.9447	.9318	.9273
0.24	.9894	.9684	.9511	.9398	.9358
0.25	.9902	.9718	.9567	.9468	.9433
0.26	.9908	.9748	.9616	.9529	.9499
0.27	.9913	.9773	.9658	.9582	.9556
0.28	.9916	.9795	.9695	.9629	.9606
0.29	.9919	.9814	.9728	.9670	.9650
0.30	.9920	.9830	.9756	.9705	.9688

Table 4.4. Output response y(x,t) for $\lambda = 2.0$.

t	y(x,t)				
	x = 0.1	x = 0.2	x = 0.3	x = 0.4	x = 0.5
0.01	.5640	.1850	.0399	.0055	.0010
0.02	.7126	.3690	.1568	.0566	.0292
0.03	.7775	.4767	.2602	.1359	.0963
0.04	.8147	.5495	.3461	.2207	.1786
0.05	.8401	.6049	.4196	.3018	.2616
0.06	.8596	.6506	.4837	.3764	.3394
0.07	.9757	.6897	.5404	.4438	.4104
0.08	.8895	.7239	.5906	.5041	.4742
0.09	.9017	.7541	.6353	.5581	.5314
0.10	.9124	.7810	.6750	.6062	.5824
0.11	.9219	.8048	.7104	.6491	.6279
0.12	.9303	.8261	.7420	.6873	.6684
0.13	.9378	.8450	.7701	.7214	.7045
0.14	.9445	.8618	.7951	.7517	.7367
0.15	.9504	.8768	.8174	.7787	.7654
0.16	.9557	.8901	.8372	.8028	.7909
0.17	.9604	.9020	.8549	.8242	.8136
0.18	.9645	.9125	.8706	.8433	.8339
0.19	.9682	.9219	.8846	.8603	.8519
0.20	.9714	.9303	.8970	.8754	.8679
0.21	.9743	.9377	.9081	.8889	.8822
0.22	.9768	.9443	.9180	.9009	.8949
0.23	.9790	.9501	.9267	.9115	.9062
0.24	.9809	.9553	.9345	.9210	.9163
0.25	.9826	.9598	.9414	.9294	.9252
0.26	.9841	.9639	.9475	.9369	.9332
0.27	.9853	.9674	.9530	.9435	.9402
0.28	.9863	.9706	.9577	.9494	.9465
0.29	.9872	.9733	.9620	.9546	.9520
0.30	.9879	.9757	.9658	.9591	.9569

CHAPTER 5

DEADBEAT RESPONSE OF TIME-DELAYED SYSTEMS WITH MINIMAL OVERSHOOT COMPROMISE

5.1 INTRODUCTION

With a ramp input, a sampled-data system which is completely controllable can be so designed that the output follows the input exactly after a minimal number of sampling periods. Such a system is said to exhibit deadbeat response. Unfortunately, a system designed to provide deadbeat response for one type of input may not provide satisfactory performance when subjected to other inputs. In particular, the step response of a linear sampled-data system exhibits a large overshoot if the system is designed to give minimal deadbeat response for a ramp input. To offset this, Bertram (10) suggested in 1956 the following criterion: for a given deadbeat ramp-response time in excess of the minimum, the coefficients of the closed-loop z-transform transfer function be selected to minimize the sum of the sampled step-response errors squared.

$$I = \sum_{k=0}^{\infty} [e(kT)]^2 \qquad (5.1)$$

In 1965, the work by Bertram (10) was extended by Pierre, Ross and the author (68). Four significant results obtained by Pierre, et al., are: firstly, the overshoot obtained by Bertram's approach is shown to be the minimum possible under the assumed deadbeat response constraints; secondly, the sensitivity of the response of the optimal system is studies as a function of a change-in-gain parameter; thirdly, stability limits due to the change-in-gain parameter are derived; and fourthly, Bertram's approach is extended to apply to certain nonminimum-phase plants.

In this chapter, the above-mentioned works are extended to account for a pure time delay in the system. In Section 5.2, the optimal trade-off between deadbeat-response time for a ramp input and step-response overshoot is derived for a single-loop sampled-data system with time delay. In Section 5.3, sensitivity of the output response is studied as a function of parameter variations in the optimal controller. Stability limits are derived (Section 5.4) in terms of a change-in-gain parameter. And in Section 5.5, examples are used to illustrate the theories of the preceding sections.

5.2 DEVELOPMENT OF THE METHOD

Pierre, Ross and this author (68) consider systems of the form shown in Figure 5.1, wherein $G_p(s)$ is the Laplace-transform transfer function of the plant, and D(s) is the transfer function of the controller to be determined. They show that, for any physically realizable D(s), a limit on the performance of such a system is

$$(n-1)B \geq 100 \tag{5.2}$$

where B is the percentage of overshoot resulting from a step input, and n is the number of sampling periods used for deadbeat response to a ramp input. For linear sampled-data systems enfolding pure time delay (Figure 5.2), a similar limit on system performance exists and is derived in this section. Consideration is also given to the case that some poles and zeros of the plant lie outside the unit circle of the z-plane --- z-transform theory (36, 50, 72) is utilized extensively in this chapter.

By definition, the error e(kT) between output and input* of the deadbeat system is identically zero at the sampling instants for $k \geq n$, where nT is the specified deadbeat response time, and T is the sampling period. In the system with the pure time delay of duration D, as shown in Figure 5.2, the first m sampled values of the error, $D/T + 1 \leq m \leq D/T + 2$, are uncontrollable and, therefore, can be omitted from the performance measure. Thus, an appropriate squared-error performance measure for the system of Figure 5.2 is

$$I = \sum_{k=m}^{n-1} [e(kT)]^2 \qquad n > m \tag{5.3}$$

To be physically realizable and to exhibit the desired deadbeat response, the overall z-transform function M(z) of the system in Figure 5.2 must be of the form

$$M(z) = b_m z^{-m} + b_{m+1} z^{-(m+1)} + \ldots + b_n z^{-n} \tag{5.4}$$

where the b_j's are constants. In order for the system to feature deadbeat response to a ramp input, it is well-known that 1 - M(z) must be of the form

$$1 - M(z) = (1 - z^{-1})^2 (1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_{n-2} z^{-n+2}) \tag{5.5}$$

No additional constraints need be placed on either M(z) or 1 - M(z) when $G_p(z)$ is minimal phase. If $G_p(z)$ contains zeros or poles outside the unit circle of the z plane, however, and is therefore nonminimal phase,

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* Unless otherwise stated, the input is either a step function or a ramp function initiated a time t = 0.

additional constraints on M(z) are required. Nonminimum-phase zeros of $G_p(z)$ must appear as roots of M(z), and nonminimum-phase poles of $G_p(z)$ must appear as roots of 1 - M(z) in order to avoid inexact cancellation of poles and zeros between D(z) and $G_p(z)$ outside the unit circle. In practice, such cancellation attempts would result in the instability of the overall system.

5.2.1 Minimum Phase $G_p(z)$

The minimal phase case of $G_p(z)$ is treated first. For Equations 5.4 and 5.5 to be simultaneously satisfied, it is necessary that the following equations be satisfied.

$$\begin{aligned}
&a_1 - 2 = 0 \\
&a_2 - 2a_1 + 1 = 0 \\
&a_3 - 2a_2 + a_1 = 0 \\
&\quad\vdots \qquad\quad \vdots \\
&a_{m-1} - 2a_{m-2} + a_{m-3} = 0
\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}
&b_m = -(a_m - 2a_{m-1} + a_{m-2}) \\
&b_{m+1} = -(a_{m+1} - 2a_m + a_{m-1}) \\
&b_{m+2} = -(a_{m+2} - 2a_{m+1} + a_m) \\
&\quad\vdots \qquad\quad \vdots \\
&b_{n-1} = -(-2a_{n-2} + a_{n-3}) \\
&b_n = -a_{n-2}
\end{aligned} \tag{5.7}$$

The set (5.6) of m-1 equations is solved for a_i, $1 \leq i \leq m-1$, to obtain

$$a_i = i + 1 \qquad , \quad 1 \leq i \leq m-1 \tag{5.8}$$

In view of Equation 5.4, the error sequence between the step input and the output can be written as

$$e_i = \begin{cases} 1 & 0 \leq i \leq m-1 \\ 1 - \sum_{k=m}^{i} b_k & m \leq i \leq n-1 \\ 0 & i \geq n \end{cases} \tag{5.9}$$

And in view of the known relationships (Equations 5.7 and 5.8) between the a_i's and the b_i's, it follows that

$$e_i = \begin{cases} 1 & 0 \leq i \leq m-1 \\ a_m - m & i = m \\ a_i - a_{i-1} & m+1 \leq i \leq n-2 \\ -a_{n-2} & i = n-1 \\ 0 & i \geq n \end{cases} \tag{5.10}$$

Now the minimization of I in Equation 5.3 can be performed with respect to the set of a_i's, $m \leq i \leq n-2$, by elementary calculus. Thus,

$$\frac{\partial I}{\partial a_i} = 2(a_i - a_{i-1}) - 2(a_{i+1} - a_i) = 0 \qquad m \leq i \leq n-3 \tag{5.11}$$

$$\frac{\partial I}{\partial a_{n-2}} = 2(a_{n-2} - a_{n-3}) + 2a_{n-2} = 0 \tag{5.12}$$

This set of n-m-2 equations is solved for a_i's, $m \leq i \leq n-2$, after making use of $a_{m-1} = m$ from Equation 5.8, to obtain

$$a_i = (n-1-i)\, a_{n-2} \qquad m \leq i \leq n-3 \qquad (5.13)$$

where

$$a_{n-2} = m/(n-m) \qquad (5.14)$$

It follows directly from Equations 5.7, 5.8, 5.13, and 5.14 that

$$b_m = n/(n-m) \qquad (5.15)$$

$$b_i = 0 \qquad m+1 \leq i \leq n-1 \qquad (5.16)$$

and

$$b_n = -m/(n-m) \qquad (5.17)$$

Use of these optimal coefficients in Equation 5.4 enables the determination of the z-transform $C_s(z)$ of the unit-step response $c_s(t)$ of the optimal system, as follows:

$$C_s(z) = \frac{1}{1-z^{-1}} M_{opt}(z)$$

$$= \frac{1}{1-z^{-1}} \left(\frac{n}{n-m} z^{-m} - \frac{m}{n-m} z^{-n} \right)$$

$$= \frac{n}{n-m} \sum_{k=m}^{n-1} z^{-k} + \sum_{k=n}^{\infty} z^{-k} \qquad (5.18)$$

Descriptively, the sampled unit-step response of the optimal system equals zero for the first m-1 sampling periods (this is due to the time-delay in the system), stays at n/(n-m) for the next n-m-1 sampling periods and then settles down to the desired value. The percent overshoot B of the unit-step response at the sampling instants is

$$B = \frac{100m}{n-m} \tag{5.19}$$

By following the same approach as in reference (32), it can be shown that this overshoot is the minimum possible for the present formulation of the problem. In other words, the value of B(n-m)/m is always greater than, or at best equal to 100 for the systems of the type depicted in Figure 5.2. The optimal controller in this case is

$$D_{opt}(z) = \frac{1}{G_p(z)} \cdot \frac{M_{opt}(z)}{1 - M_{opt}(z)}$$

$$= \frac{1}{G_p(z)} \cdot \frac{nz^{-m} - mz^{-n}}{(n-m) - (nz^{-m} - mz^{-n})} \tag{5.20}$$

5.2.2 Nonminimum Phase $G_p(z)$

In this subsection, the case that finite poles and zeros of $G_p(z)$ lie outside the unit circle is considered. Let q_p be the number of nonminimum-phase poles of $G_p(z)$ and q_z be that of nonminimum-phase zeros. Now the minimization of I in Equation 5.3 is subject to q_p plus q_z constraints, namely,

$$M(z)\Big|_{z=z_i} = 0, \qquad 1 \leq i \leq q_z \tag{5.21}$$

and

$$(1 - M(z))\Big|_{z=p_i} = 0, \qquad 1 \leq i \leq q_p \tag{5.22}$$

where z_i's and p_i's are the distinct zeros and poles of $G_p(z)$ outside the unit circle in the z-plane, respectively. In the case that some of the nonminimum-phase zeros and poles are repeated, the following constraints are necessarily placed on M(z) and 1 - M(z):

$$\frac{d^j M(z)}{dz^j}\Big|_{z=z_r} = 0, \qquad 0 \leq j \leq r_z - 1 \tag{5.23}$$

and

$$\frac{d^j}{dz^j}[1 - M(z)]_{z=p_r} = 0 \qquad 0 \leq j \leq r_p - 1 \tag{5.24}$$

where r_z is the order of the zero of $G_p(z)$ at $z = z_r$ and r_p is that of the pole of $G_p(z)$ at $z = p_r$. For the sake of notational simplicity, only the case that the nonminimum-phase poles and zeros of $G_p(z)$ are distinct is considered in detail in the following paragraphs. Essentially the same approach can be applied to the case in which nonminimum-phase poles and zeros are not distinct.

In terms of the specified form of M(z) in Equation 5.5, constraint Equations 5.21 and 5.22 reduce to

$$\frac{1}{(1-z_i^{-1})^2} = (1 + a_1 z_i^{-1} + a_2 z_i^{-2} + \ldots + a_{n-2} z_i^{-n+2}),\ 1 \leq i \leq q_z \qquad (5.25)$$

and

$$1 + a_1 p_i^{-1} + a_2 p_i^{-2} + \ldots + a_{n-2} p_i^{-n+2} = 0,\ 1 \leq i \leq q_p \qquad (5.26)$$

Let $1 + a_1 z^{-1} + \ldots + a_{n-2} z^{-n+2}$ be denoted by $f(z)$ and let h_i's and λ_i's be sets of Lagrange multipliers associated with the constraints in Equation 5.25 and those in Equation 5.26, respectively. Following the classical Lagrange multiplier approach, the augmented performance measure H is defined:

$$H = I + \sum_{i=1}^{q_z} h_i f(z_i) + \sum_{i=1}^{q_p} \lambda_i f(p_i) \qquad (5.27)$$

Stationary points of H must satisfy the following conditions:

$$\frac{\partial H}{\partial a_m} = 2(a_m - m) - 2(a_{m+1} - a_m) + \sum_{i=1}^{q_z} h_i z_i^{-m} + \sum_{i=1}^{q_p} \lambda_i p_i^{-m} = 0 \qquad (5.28)$$

$$\frac{\partial H}{\partial a_j} = 2(a_j - a_{j-1}) - 2(a_{j+1} - a_j) + \sum_{i=1}^{q_z} h_i z_i^{-j} + \sum_{i=1}^{q_p} \lambda_i p_i^{-j} = 0 \qquad (5.29)$$

for j in the range $m+1 \leq j \leq n-3$, and

$$\frac{\partial H}{\partial a_{n-2}} = 2(a_{n-2} - a_{n-3}) + 2\, a_{n-2} + \sum_{i=1}^{q_z} h_i z_i^{-n+2} + \sum_{i=1}^{q_p} \lambda_i p_i^{-n+2} = 0 \qquad (5.30)$$

By use of the identities given in Equation 5.7, the above set of necessary conditions reduces to

$$b_{j+1} = - \sum_{i=1}^{q_z} \frac{h_i}{2} z_i^{-j} - \sum_{i=1}^{q_p} \frac{\lambda_i}{2} p_i^{-j} , \quad m \leq j \leq n-2 \tag{5.31}$$

which gives all b_j's, except b_m and b_n, in terms of Lagrange multipliers and the z_i's and p_i's. By appropriate replacement of the b_i's in Equation 5.4 by the identities in Equation 5.31, M(z) is expressed in terms of the z_i's and p_i's, which are known, and in terms of h_i's, λ_i's, b_m and b_n, a total of $q_z + q_p + 2$ values to be determined. The values are determined, in general, through the solution of a set of $q_z + q_p + 2$ linear equations. Equations 5.21 and 5.22 supply $q_z + q_p$ of these, and the two remaining stem from the steady-state-error constraint (zero steady-state error to a ramp input) which requires that a second-order zero of $1 - M(z)$ exists at $z = 1$, namely,

$$[1 - M(z)]_{z=1} = 0 \tag{5.32}$$

and

$$\frac{d}{dz} [1 - M(z)]_{z=1} = 0 \tag{5.33}$$

Uniqueness of the solution is guaranteed provided that the fundamental determinant of the set of linear equations is not zero. If the determinant is equal to zero, either no solution to the set of linear equations exists or there exists infinitely many solutions. In the former case, it simply

means that deadbeat response cannot be obtained in the stipulated number of sampling periods. In the latter case, overshoot and smoothness of the response are factors which should be considered in the final selection of the b_i's.

Now consider the special case in which only one zero and no poles of $G_p(z)$ lie outside the unit circle. In this case, $q_z = 1$ and $q_p = 0$, Equation 5.31 reduces to

$$b_{j+1} = -\frac{h}{2} z_i^{-j} , \qquad m \leqslant j \leqslant n-2 \tag{5.34}$$

The constraint equations in Equations 5.21, 5.32, 5.33, together with the relations in Equations 5.4 and 5.34 yield

$$b_m z_1^{-m} - \frac{h}{2} \sum_{j=m}^{n-2} z_1^{-(2j+1)} + b_n z_1^{-n} = 0 \tag{5.35}$$

$$b_m - \frac{h}{2} \sum_{j=m}^{n-2} z_1^{-j} + b_n = 1 \tag{5.36}$$

and

$$mb_m - \frac{h}{2} \sum_{j=m}^{n-2} (j+1) z_1^{-j} + nb_n = 0 \tag{5.37}$$

Solving for b_m, h, and b_n from Equations 5.35, 5.36, and 5.37 gives

$$b_m = \frac{(ns_1 - z_1^{-n} s_2)}{\Delta} \tag{5.38}$$

$$h = \frac{2(nz_1^{-m} - mz_1^{-n})}{\Delta} \tag{5.39}$$

$$b_n = \frac{z_1^{-m} s_2 - m s_1}{\Delta} \tag{5.40}$$

where

$$s_1 \triangleq \frac{z_1^{-(2m+1)} - z_1^{-(2n-1)}}{1 - z_1^{-2}} \tag{5.41}$$

$$s_2 \triangleq \frac{m z_1^{-m} - n z_1^{-n+1}}{1 - z_1^{-1}} + \frac{z_1^{-m} - z_1^{-n}}{(1 - z_1^{-1})^2} \tag{5.42}$$

and

$$\Delta \triangleq \left(\frac{z_1^{-m} - z_1^{-n}}{1 - z_1^{-1}}\right)^2 - \frac{n - m}{(1 - z_1^{-2})} (z_1^{-2m} - z_1^{-2n}) \tag{5.43}$$

5.3 SENSITIVITY ANALYSIS IN THE MINIMUM-PHASE $G_p(z)$ CASE

In reference (32), a measure of sensitivity for deadbeat response systems is posed; namely, sensitivity of the transient response of the system with respect to system-parameter changes; detailed results are obtained for response sensitivity as a function of open-loop gain. In this section, a measure of sensitivity for deadbeat response systems is considered which is a function of open-loop gain and of poles and zeros of the controller.

From Equation 5.20, it can be seen that deadbeat-response design of sampled-data systems is achieved through cancellation of the minimum-phase zeros and poles of $G_p(z)$ by corresponding poles and zeros of $D_{opt}(z)$. The effect of these "canceled" poles and zeros on the closed-loop response is insignificant as long as the pole-zero dipoles which result from

inevitable erroneous cancellation are well within the unit circle in the z-plane. The main contribution of $D_{opt}(z)$ to the output response is generally due both to the open-loop gain and to the locations of poles and zeros in the second factor in the right-hand member of Equation 5.20. Therefore, the sensitivity of the step response of the optimal system obtained in Section 5.2 is investigated in this section as a function of loop-gain and of such poles and zeros.

Consider the case in which the controller D(z) in Figure 5.2 equals $D(z,x_1,x_2,x_3)$:

$$D(z,x_1,x_2,x_3) = \frac{x_3}{G_p(z)} \cdot \frac{x_1 z^{-m} - x_2 z^{-n}}{(x_1 - x_2) + (x_1 z^{-m} - x_2 z^{-n})} \tag{5.44}$$

Note that $D(z,x_1,x_2,x_3)$ equals $D_{opt}(z)$, Equations 4.20, when x_1 equals n, x_2 equals m, and x_3 equals 1. From Figure 4.2, the closed-loop transfer function can be written as

$$M(z,x_1,x_2,x_3) = \frac{D(z,x_1,x_2,x_3)\ G_p(z)}{1 + D(z,x_1,x_2,x_3)\ G_p(z)}$$

$$= \frac{x_3(x_1 z^{-m} - x_2 z^{-n})}{x_1 - x_2 + (x_3 - 1)(x_1 z^{-m} - x_2 z^{-n})} \tag{5.45}$$

Note that variations in parameters x_1, x_2 and x_3 do not affect the steady-state performance of the system, provided that x_1, x_2 and x_3 result in a stable system. This is confirmed by substituting $z = 1$ in Equation 5.45 to obtain $M(1) = 1$ in accordance with the final value theorem.

For the purposes of this section, it is assumed that the values x_1, x_2 and x_3 are known to lie in small neighborhoods around their optimal values, i.e.,

$$|x_1 - n| < \varepsilon_1 << 1 \tag{5.46a}$$

$$|x_2 - m| < \varepsilon_2 << 1 \tag{5.46b}$$

and

$$|x_3 - 1| < \varepsilon_3 << 1 \tag{5.46c}$$

Where ε_1, ε_2 and ε_3 are known constants. The objective of the following study is to determine the greatest change (from the optimum) in the unit-step response that can exist when x_1, x_2 and x_3 are constrainted as in Equation 5.46.

The Taylor series of $M(z,x_1,x_2,x_3)$ is expanded about the optimal values of X^*, as follows:

$$M(z,x_1,x_2,x_3) = M_{opt}(z) + (X - X^*)' \nabla M(z,x_1^*,x_2^*,x_3^*) + \ldots$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{5.47}$$

and

$$\nabla M(z,x_1^*,x_2^*,x_3^*) = \begin{bmatrix} \frac{-m}{(n-m)^2}(z^{-m} - z^{-n}) \\ \\ \frac{n}{(n-m)^2}(z^{-m} - z^{-n}) \\ \\ \frac{(n-m)(nz^{-m} - mz^{-n}) - (nz^{-m} - mz^{-n})^2}{(n-m)^2} \end{bmatrix} \tag{5.48}$$

The first variation of $M(z,x_1,x_2,x_3)$ with respect to x_1, x_2, and x_3 is defined as

$$\Delta M(z,x_1^*,x_2^*,x_3^*) = (X - X^*)' \nabla M(z,x_1^*,x_2^*,x_3^*) \tag{5.49}$$

Substituting the expression of $\nabla M(z,x_1^*,x_2^*,x_3^*)$ from Equation 5.48 into Equation 5.49, followed by a rearrangement of terms, yields

$$\begin{aligned} \Delta M(z,x_1^*,x_2^*,x_3^*) = (X - X^*)' [& \nabla M_m(x_1^*,x_2^*,x_3^*)z^{-m} + \nabla M_n(x_1^*,x_2^*,x_3^*)z^{-n} + \\ & + \nabla M_{2m}(x_1^*,x_2^*,x_3^*)z^{-2m} + \nabla M_{m+n}(x_1^*,x_2^*,x_3^*)z^{-m-n} + \\ & + \nabla M_{2n}(x_1^*,x_2^*,x_3^*)z^{-2n}] \end{aligned} \tag{5.50}$$

where

$$\nabla M_m(x_1^*, x_2^*, x_3^*) = \begin{bmatrix} \frac{-m}{(n-m)^2} \\ \\ \frac{n}{(n-m)^2} \\ \\ \frac{n}{(n-m)} \end{bmatrix} \tag{5.51}$$

$$\nabla M_n(x_1^*, x_2^*, x_3^*) = \begin{bmatrix} \frac{m}{(n-m)^2} \\ \\ \frac{-n}{(n-m)^2} \\ \\ \frac{-m}{(n-m)} \end{bmatrix} \tag{5.52}$$

$$\nabla M_{2m}(x_1^*, x_2^*, x_3^*) = \begin{bmatrix} 0 \\ \\ 0 \\ \\ \frac{-n^2}{(n-m)^2} \end{bmatrix} \tag{5.53}$$

$$\nabla M_{m+n}(x_1^*, x_2^*, x_3^*) = \begin{bmatrix} 0 \\ \\ 0 \\ \\ \frac{2mn}{(n-m)^2} \end{bmatrix} \tag{5.54}$$

$$\nabla M_{2n}(x_1^*, x_2^*, x_3^*) = \begin{bmatrix} 0 \\ \\ 0 \\ \\ \frac{-m^2}{(n-m)^2} \end{bmatrix} \tag{5.55}$$

Before proceeding further, two additional terms are introduced for convenience:

$$\Omega \triangleq \left[\frac{x_1-n}{n} \qquad \frac{x_2-m}{m} \qquad x_3-1\right]' \tag{5.56}$$

which represents the fractional deviation from the optimum of x_1, x_2 and x_3, and

$$\Lambda \triangleq \begin{bmatrix} n & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.57}$$

In conformity with a definition given by Pierre (33), the column sensitivity matrix S_j^M is given by

$$S_j^M \triangleq \Lambda \nabla M_j (x_1^*, x_2^*, x_3^*) \tag{5.58}$$

In words, S_j^M is the variation associated with the coefficient of z^{-j} in the closed-loop transfer function due to small parameter variations in x_1, x_2 and x_3. In terms of these column sensitivity matrices, Equation 5.50 becomes

$$\Delta M(z, x_1^*, x_2^*, x_3^*) = \Omega' [S_m^M z^{-m} + S_n^M z^{-n} + S_{2m}^M z^{-2m} + S_{m+n}^M z^{-m-n} + S_{2n}^M z^{-2n}] \tag{5.59}$$

where

$$S_m^M = \begin{bmatrix} \frac{-mn}{(n-m)^2} \\ \\ \frac{mn}{(n-m)^2} \\ \\ \frac{n}{(n-m)} \end{bmatrix} \tag{5.60}$$

$$S_n^M = \begin{bmatrix} \frac{mn}{(n-m)^2} \\ \\ \frac{-mn}{(n-m)^2} \\ \\ \frac{-m}{(n-m)} \end{bmatrix} \tag{5.61}$$

$$S_{2m}^M = \begin{bmatrix} 0 \\ 0 \\ \frac{-n^2}{(n-m)^2} \end{bmatrix} \tag{5.62}$$

$$S_{m+n}^M = \begin{bmatrix} 0 \\ 0 \\ \frac{2mn}{(n-m)^2} \end{bmatrix} \tag{5.63}$$

$$S_{2n}^M = \begin{bmatrix} 0 \\ 0 \\ \frac{-m^2}{(n-m)^2} \end{bmatrix} \tag{5.64}$$

The worst transient performance of the system corresponding to small changes (Equation 4.46) in parameters x_1, x_2 and x_3 can be determined on the basis of Equation 5.59. First, consider the case of $2m > n$. Then, making use of Equation 5.59, the first variation of the z-transform of the unit-step response of the system is determined:

$$\Delta C(z) = \frac{\Delta M(z, x_1^*, x_2^*, x_3^*)}{1-z^{-1}}$$

$$= \Omega' S_m^M \sum_{k=m}^{n-1} z^{-k} + \Omega' (S_m^M + S_n^M) \sum_{k=n}^{2m-1} z^{-k} - \Omega' (S_{m+n}^M + S_{2n}^M) \sum_{k=2m}^{m+n-1} z^{-k} +$$

$$- \Omega' S_{2n}^M \sum_{k=m+n}^{2n-1} z^{-k} \qquad (5.65)$$

The output response from m^{th} sampling instant to $(n-1)^{th}$ sampling instant, which is the transient period of the optimal system, is changed by the amount of $\Omega' S_m^M$. From Equation 5.60, it can be concluded that the worst transient performance condition (greatest deviation from the optimal step response) occurs when

$$\Omega' = \pm[-\varepsilon_1/n \quad \varepsilon_2/m \quad \varepsilon_3] \qquad (5.66)$$

Next, if $2m < n$, the first variation of the unit-step response becomes

$$\Delta C(z) = \Omega' S_m^M \sum_{k=m}^{2m-1} z^{-k} + \Omega' (S_m^M + S_{2m}^M) \sum_{k=2m}^{n-1} z^{-k} +$$

$$- \Omega' (S_{m+n}^M + S_{2n}^M) \sum_{k=n}^{m+n-1} z^{-k} - \Omega' S_{2n}^M \sum_{k=m+n}^{2n-1} z^{-k} \qquad (5.67)$$

One can readily verify that the absolute value of each element of S_m^M is greater than or equal to the corresponding element of $(S_m^M + S_{2m}^M)$. Therefore, the worst transient performance is again associated with the condition in Equation 5.66.

Finally, for $2m = n$, the first variation of the unit-step response is

$$\Delta C(z) = \Omega' S_m^M \sum_{k=m}^{n-1} z^{-k} - \Omega' (S_{m+n}^M + S_{2n}^M) \sum_{k=n}^{m+n-1} z^{-k} - \Omega' S_{2n}^M \sum_{k=m+n}^{2n-1} z^{-k} \qquad (5.68)$$

By comparing the first $(n-m-1)$ sampled values of the first variation of the unit-step response in Equation 5.65 with corresponding values in Equation 5.68, it can be concluded that the worst transient performance is yet associated with the condition in Equation 5.66.

5.4 STABILITY ANALYSIS IN THE MINIMUM-PHASE $G_p(z)$ CASE

In Section 5.3, the effect on the output response due to small changes in system parameters is investigated. At this point, it seems logical to inquire how changes in system parameters affect the stability of the system. Stability limits are derived here as a function of open-loop gain only.

Let $x_1 = n$, $x_2 = m$ and $x_3 = 1 + \varepsilon$ in Equation 5.45; the closed-loop transfer function becomes

$$M(z,n,m,1+\varepsilon) = \frac{(1+\varepsilon)\ (nz^{-m} - mz^{-n})}{(n-m) + \varepsilon(nz^{-m} - mz^{-n})} \qquad (5.69)$$

The quantity ε in the above equation is a change-in-gain parameter, and the case that $\varepsilon = 0$ is associated with the optimal system. From the pole-zero

point of view, any value of ε which renders the system on the boundary of stability is one for which closed-loop poles (or at least one such pole) lie on the unit circle in the z-plane. On the verge of stability, the characteristic equation associated with Equation 5.69 is

$$(n-m) + \varepsilon(ne^{-jm\theta} - me^{-jn\theta}) = 0 \tag{5.70}$$

Setting the real and imaginary parts equal to zero independently yields

$$(n-m) + n\varepsilon(\cos m\theta) - m\varepsilon(\cos n\theta) = 0 \tag{5.71}$$

and

$$-n(\sin m\theta) + m(\sin n\theta) = 0 \tag{5.72}$$

For given values of n and m, critical values of θ are determined by solving the transcendental Equation 5.72. A general closed-form solution for θ does not exist, but numerical solutions are rather easy to obtain on either an analog or a digital computer (see Appendix B). Of course, θ can be limited to the interval between 0 and 2π, and n and m are integers with $n > m$.

In general, there exists a finite set S_{nm} of values of θ which satisfy Equation 5.72 for given values of n and m. Values of ε which correspond to the values of θ in S_{nm} are determined on the basis of Equation 5.71; that is

$$\varepsilon = \left.\frac{n-m}{m(\cos n\theta) - n(\cos m\theta)}\right|_{\theta \epsilon S_{nm}} \tag{5.73}$$

Because ε is a change-in-gain parameter (ε=0 corresponds to the optimum), the system is stable if ε is confined to an open interval, namely

$$\varepsilon_{min} < \varepsilon < \varepsilon_{max}$$

where ε_{min} and ε_{max} are generated on the basis of Equation 5.73, as follows:

$$\varepsilon_{max} = \min_{\theta \varepsilon S_{nm}} \frac{n - m}{m(\cos n\theta) - n(\cos m\theta)} \tag{5.74}$$

$$\text{subject to } \frac{n - m}{m(\cos n\theta) - n(\cos m\theta)} > 0, \text{ and}$$

$$\varepsilon_{min} = \max_{\theta \varepsilon S_{nm}} \frac{n - m}{m(\cos n\theta) - n(\cos m\theta)} \tag{5.75}$$

$$\text{subject to } \frac{n - m}{m(\cos n) - n(\cos m)} < 0$$

Stability limits for n = 3, 4, 5, ... are tabulated in Table 5.1. The flow diagrams of the computer program and an error analysis are given in Appendix B.

As a check on the computer solution, special cases of Equation 5.72 are solved analytically in the remainder of this section --- the reader may omit this without loss of continuity. In the first place, it is shown that when n is an integral multiple of m, the solution of Equation 5.72 for θ is

$$\theta = \frac{k\pi}{m} \qquad k = 1, 2, \ldots, 2m$$

This solution is established by arranging Equation 5.72 as

$$\frac{1}{m} \sin m\theta = \frac{1}{n} \sin n\theta \tag{5.76}$$

and observing that both sin $m\theta$ and sin $n\theta$ are zero when θ equals $k\pi/m$ and n is an integral multiple of m. In order to rule out the possibility that $(\sin n\theta)/n$ equals $(\sin m\theta)/m$ for other values of θ, namely those in the open interval $k\pi/m<\theta<(k+1)\pi/m$, $k = 0, 1, 2, \ldots$, observe that $(\sin m\theta)/(m\theta)$ is greater than $(\sin n\theta)/(n\theta)$ for $0<\theta\leqslant\pi/n$ since $m\theta$ is less than $n\theta$. This implies that $(\sin m\theta)/m$ is greater than $(\sin n\theta)/n$ for $0<\theta\leqslant\pi/2m$. In the case that n/m is odd, it can be concluded from the symmetry of both $(\sin n\theta)/n$ and $(\sin m\theta)/m$ about $\theta=\pi/2m$ that $(\sin n\theta)/n$ never intercepts $(\sin m\theta)/m$ for $0<\theta<\pi/m$. When n/m is even, the same conclusion stems from the facts that $(\sin m\theta)/m$ is also greater than $-(\sin n\theta)/n$ for $0<\theta<\pi/2m$ and that $\sin n\theta$ is an odd function for $0\leqslant\theta\leqslant\pi/m$ about $\theta=\pi/2m$. It is evident, therefore, that for any θ lying between $k\pi/m$ and $(k+1)\pi/m$, $k = 0, 1, 2, \ldots$, the absolute value of $(\sin m\theta)/m$ is always greater than that of $(\sin n\theta)/n$. In other words, Equation 5.76 holds only when θ equals $k\pi/m$ where k is an integer. It is important to recall that the above conclusion is based on the assumption that n is an integral multiple of m. Therefore under this assumption, limiting values of ε are obtained by use of Equations 5.74 and 5.75:

$$\varepsilon_{max} = \begin{cases} \dfrac{n-m}{n+m}, & \text{if } n/m \text{ is even} \\ 1, & \text{if } n/m \text{ is odd} \end{cases} \tag{5.77}$$

and

$$\varepsilon_{min} = -1, \text{ if } n/m \text{ is an integer} \tag{5.78}$$

For m = 1, i.e., for D = 0 in Figure 5.2, the limiting values of ε in Equations 5.77 and 5.78 are the same as given in reference (32). If the settling time nT is extended to infinity for any finite value of m, it can be concluded, from Equation 5.77 and 5.78 that the optimal system remains stable for changes in gain up to $\pm$ 100%.

A differenct approach to the solution of Equations 5.71 and 5.72 is as follows: First, Equation 5.72 is solved for sin $m\theta$

$$\sin m\theta = (m/n)\sin n\theta \tag{5.79}$$

which is equivalent to

$$\cos m\theta = \pm[1 - m^2(\sin^2 n\theta)/n^2]^{\frac{1}{2}} \tag{5.80}$$

The right-hand member of Equation 5.80 is substituted for cos $m\theta$ in Equation 5.71, and the resulting equation is manipulated to obtain

$$\cos n\theta = \frac{(n-m) - \varepsilon^2(n+m)}{2\ m\varepsilon} \triangleq \gamma(\varepsilon) \tag{5.81}$$

A typical curve of $\gamma(\varepsilon)$ versus ε is shown in Figure 5.3. Note that $\gamma(\varepsilon)$ is greater than 1 for ε in the interval $0<\varepsilon<\varepsilon_+$ and that $\gamma(\varepsilon)$ is less than -1 for ε in the interval $\varepsilon_-<\varepsilon<0$. When values of $|\gamma(\varepsilon)|$ are greater than one, real values of θ cannot satisfy Equation 5.81, and consequently poles of the system cannot lie on the unit circle of the z-plane. The conclusion to be drawn from these observations is that a sufficient condition for stability is

$$\varepsilon_- < \varepsilon < \varepsilon_+ \qquad n > m \tag{5.82}$$

where ε_+ and ε_- are functions of n and m obtained by solving Equation 5.81 for ε when $\cos n\theta = 1$ and $\cos n\theta = -1$, respectively. The result of these solutions is

$$\varepsilon_+ = -\varepsilon_- = (n-m)/(n+m) \tag{5.83}$$

To investigate the conditions on n and m under which Equation 5.82 is both necessary and sufficient for stability, the identity for $\cos n\theta$ in Equation 5.81 is substituted appropriately into Equations 5.71, and the resulting equation is arranged as follows:

$$\cos m\theta = -\frac{(n-m) + \varepsilon^2 (n+m)}{2\varepsilon n} \tag{5.84}$$

At $\varepsilon = \varepsilon_+$, Equation 5.84 reduces to

$$\cos m\theta = -1 \tag{5.85}$$

Since $\cos n\theta = 1$ at $\varepsilon = \varepsilon_+$,

$$\theta = (2k\pi)/n \qquad k = 0, 1, 2, \ldots \tag{5.86}$$

Substitution of this θ into Equation 5.85 yields

$$\cos [(2mk\pi)/n] = -1 \tag{5.87}$$

If n is even and m is odd, the value of n/2 for k results in both $\cos n\theta = 1$ and $\cos m\theta = -1$ and consequently $\varepsilon_+ = \varepsilon_{max}$. If n/m is an even integer, k = n/2m results in satisfaction of Equation 5.87, and again

$\varepsilon_+ = \varepsilon_{max}$. This result agrees with that in Equation 5.77. But if n/m is an odd integer, no integer value of k satisfies Equation 5.87, in which case $\varepsilon_+ \neq \varepsilon_{max}$. In this case, it was shown that the value of ε_{max} is 1. The net result of this analytical investigation to determine ε_{max} is summarized below:

$$\varepsilon_{max} = \begin{cases} \frac{n-m}{n+m} & \text{, if m is odd and n is even} \\ \frac{n-m}{n+m} & \text{, if n/m is even} \\ 1 & \text{, if n/m is odd} \end{cases} \tag{5.88}$$

For $\varepsilon = \varepsilon_- = -\frac{n-m}{n+m}$, Equation 5.84 reduces to

$$\cos m\theta = 1 \tag{5.89}$$

Since $\cos n\theta = -1$ at $\varepsilon = \varepsilon_-$,

$$\theta = \frac{(2k-1)\pi}{n} \qquad k = 1, 2, \ldots \tag{5.90}$$

With this, Equation 5.89 becomes

$$\cos \frac{m(2k-1)\pi}{n} = 1 \tag{5.91}$$

If n is odd and m is even, k = (n+1)/2 satisfies Equation 5.91 in which case ε_{min} equals ε_-. For other combinations of n and m, it is evident that no integer k satisfies Equation 5.91.

In summary,

$$\epsilon_{min} = \begin{cases} -\frac{n-m}{n+m} & \text{, if n is odd and m is even} \\ -1 & \text{, if n/m is an integer} \end{cases} \qquad (5.92)$$

For combinations of n and m other than those listed in Equations 5.88 and 5.92, stability limits can be determined on a digital computer (Appendix B). A partial list of stability limits is given in Table 5.1.

5.5 EXAMPLES

Given m = 3 and $z_1 = -2$. The optimal closed loop transfer function $M_{opt}(z)$ is given by Equations 5.34, 5.38, 5.39 and 5.40.

$$M_{opt}(z) = .971z^{-3} + 1.384z^{-4} - .6921z^{-5} + .3460z^{-6} - 1.009z^{-7} \quad (n = 7)$$

$$M_{opt}(z) = .797z^{-3} + 1.183z^{-4} - .592z^{-5} + .296z^{-6} - .148z^{-7} + .074z^{-8} + \\ - .611z^{-9} \quad (n = 9)$$

$$M_{opt}(z) = .672z^{-3} + 1.008z^{-4} - .504z^{-5} + .252z^{-6} - .126z^{-7} + .063z^{-8} + \\ - .031z^{-9} + .016z^{-10} - .009z^{-11} + .004z^{-12} - .346z^{-13} \quad (n = 13)$$

$$M_{opt}(z) = .622z^{-3} + .933z^{-4} - .466z^{-5} + .233z^{-6} - .117z^{-7} + .058z^{-8} + \\ - .029z^{-9} + .014z^{-10} - .007z^{-11} + .004z^{-12} - .002z^{-13} + .001z^{-14} + \\ - .0004z^{-15} + .0002z^{-16} - .244z^{-17} \quad (n = 17) \qquad (5.93)$$

For $m = 3$ and $z_1 = 2$,

$$M_{opt}(z) = -1.089z^{-3} + 1.641z^{-4} + .820z^{-5} + .410z^{-6} + .205z^{-7} + .103z^{-8} + \\ + .051z^{-9} + .026z^{-10} - 1.166z^{-11} \qquad (n = 11)$$

$$M_{opt}(z) = -.833z^{-3} + 1.25z^{-4} + .625z^{-5} + .312z^{-6} + .156z^{-7} + .078z^{-8} \\ + .039z^{-9} + .019z^{-10} - .010z^{-11} + .005z^{-12} + .002z^{-13} + \\ + .001z^{-14} - .665z^{-15} \qquad (n = 15)$$

$$M_{opt}(z) = -.731z^{-3} + 1.096z^{-4} + .548z^{-5} + .274z^{-6} + .137z^{-7} + .068z^{-8} + \\ + .034z^{-9} + .017z^{-10} + .009z^{-11} + .004z^{-12} + .002z^{-13} + \\ + .001z^{-14} + .0005z^{-15} + .0003z^{-16} + .0001z^{-17} + .00007z^{-18} + \\ - .461z^{-19} \qquad (n = 19) \qquad (5.94)$$

The unit step responses in the case $z_1 = -2$ and the case $z_1 = 2$ are plotted in Figures 5.4 and 5.5, respectively. The optimal response in the case $z_1 = 2$ and $m = 3$ goes negative during the third sampling period. This unexpected phenomenon also occurs without pure time delay in the loop (32). To eliminate the negative response, $n - m$ constraints in addition to those in Equations 5.21 and 5.22 could be incorporated in the minimization process; namely,

$$\sum_{k=0}^{j} b_{m+k} \geq 0 \qquad j = 0, 1, 2, .., n-m \qquad (5.95)$$

In other words, the unit step response at the sampling instants could be restrained to be greater than or equal to zero.

In that case, the problem can be reduced to a problem of quadratic programming (39) by introducing additional variables:

$$x_j = \sum_{k=0}^{j} b_{m+k} \qquad j = 0, 1, \ldots, n-m \tag{5.96}$$

This phase of the problem is beyond the scope of the present research, however, and consideration of it is terminated here.

The transfer function of the optimal controller is given by Equation 5.20a. Corresponding to n = 7 in Equation 5.93, the optimal controller is

$$D_{opt}(z) = \frac{1}{G_p(z)} \cdot \frac{.971z^{-3} + 1.384z^{-4} - .6921z^{-5} + .346z^{-6} - 1.009z^{-7}}{1 - .971z^{-3} - 1.384z^{-4} + .6921z^{-5} - .346z^{-6} + 1.009z^{-7}} \tag{5.97}$$

It is important to realize that before $D_{opt}(z)$ is implemented physically, the assumed zero at z = -2 in $G_p(z)$ in the above equation must be cancelled with the corresponding zero in the numerator polynomial of Equation 5.97. To illustrate this point, let $\frac{s+b}{s(s+a)}$ be the transfer function of the plant $G_p(s)$ where a = 1/(2T) with T = 1 second. The value of b = 3.805 results in a zero of the plant at z = -2 in the z-plane. The corresponding z-transform of $G_p(s)$ is

$$G_p(z) = \frac{1 + 2z^{-1}}{(1-z^{-1})(1-.606z^{-1})} \tag{5.98}$$

Then the optimal controller in Equation 5.97 becomes

$$D_{opt}(z) = \frac{(.971z^{-1} - .558z^{-4} + .4239z^{-5} - .5018z^{-6})(1-z^{-1})(1-.606z^{-1})}{(1-.971z^{-3}-1.384z^{-4}+.6921z^{-5}-.346z^{-6}+1.009z^{-7})} \tag{5.99}$$

From Figure 5.2, the z-transform of v(t), with m = 3 is

$$V(z) = \frac{D_{opt}(z)\ R\ (z)}{1 + z^{-3} D_{opt}(z)\ G_p(z)} \tag{5.100}$$

With a unit-step input, the above equation becomes

$$V(z) = (.971z^{-3} - .558z^{-4} + .4239z^{-5} - .5z^{-6})(1 - .606z^{-1}) \tag{5.101}$$

and the modified z-transform* of the output response is given by

$$C(z,m) = V(z)\ G_p(z,m) \tag{5.102}$$

where

$$G_p(z,m) = \frac{z^{-1}\ (7.61 - 6.61\ e^{-m/2}) - (4.62 - 6.61\ e^{-m/2})z^{-1}}{(1-z^{-1})(1-.606z^{-1})} \tag{5.103}$$

For m = .2,

$$C(z,.2) = \frac{z^{-1}}{1-z^{-1}}\ (1.584z^{-3} + .41z^{-4} - .068z^{-5} - .239z^{-6} - .68z^{-7}) \tag{5.104a}$$

For m = .5,

$$C(z,.5) = \frac{z^{-1}}{1-z^{-1}}(2.39z^{-3} - .859z^{-4} + .745z^{-5} - 1.006z^{-6} - .265z^{-7}) \tag{5.104b}$$

---

*The modified z-transform of c(t) is defined as $C(z,m) = z^{-1} \sum_{n=0}^{\infty} c[(n+m)T]z^{-n}$ where $0 \leq m \leq 1$.

For m = .7,

$$C(z,.7) = \frac{z^{-1}}{1-z^{-1}} (2.865z^{-3} - 1.609z^{-4} + 1.228z^{-5} - 1.457z^{-6} - .02z^{-7}) \tag{5.104c}$$

The continuous response corresponding to that in Equation 5.102 is displayed in Figure 5.6

Now consider the minimum-phase-plant $G_p(s)$ case. Let m = 3, n = 9, and $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0.1$. Under the worst transient performance conditions, the variation in parameters, from Equation 5.66, is

$$\Omega' = \pm (-\frac{.1}{9} \frac{.1}{3} .1) \tag{5.105}$$

From Equation 5.45, the output responses under these conditions are

$$C(z,8.9,3.1,1.1) = \frac{9.79z^{-3} - 3.41z^{-9}}{5.8 - 5.8z^{-1} + .89z^{-3} - .89z^{-4} - .31z^{-9} + .31z^{-10}} \tag{5.106}$$

$$C(z,9.1,2.9,.9) = \frac{8.19z^{-3} - 2.61z^{-9}}{6.2 - 6.2z^{-1} - .91z^{-3} + .91z^{-4} + .29z^{-9} - .29z^{-10}} \tag{5.107}$$

The responses corresponding to the above equations are plotten in Figure 5.7.

5.6 CONCLUSIONS

In this chapter, the works by Bertram (31) and by Pierre, Ross, and this author (32) are extended to systems with time delay. The coefficients of the closed-loop transfer function of the system which is designed for deadbeat response to a ramp input are selected by minimizing the sum of the square of the errors, at sampling instants, for a step input. The

model used in reference (32) is reduced to be the special case, D = 0, of Figure 5.2. All the results in (32) can, therefore, be obtained here by letting m = 1.

In Section 5.2.1, the minimum possible overshoot, at the sampling instants for a minimum-phase plant $G_p(s)$ is derived in terms of n and m. The step response of the optimal system stays constant during the settling period. This implies the possibility of using a digital controller of variable gain type to eliminate such overshoot. The nonminimum-phase $G_p(s)$ case is examined in Section 5.2.2. Under certain conditions, the optimal step response goes negative during some periods of time. A method to eliminate such negative response is posed in Section 5.5. Variations in transient performance due to parameter variations in the optimal controller are derived for the minimum-phase $G_p(s)$ case. Under these conditions, the parameter variations are given by Equation 5.66, and the exact output response is then determined from Equation 5.45. In Section 5.4, stability limits as a function of open-loop gain are derived in the minimum-phase $G_p(s)$ case. As the number of sampling periods, n, for dead-beat response is allowed to approach infinity, the stability limits approach those of the lumped-parameter case.

Future research might well be concerned with the unusual negative response in the nonminimum-phase $G_p(s)$ case.

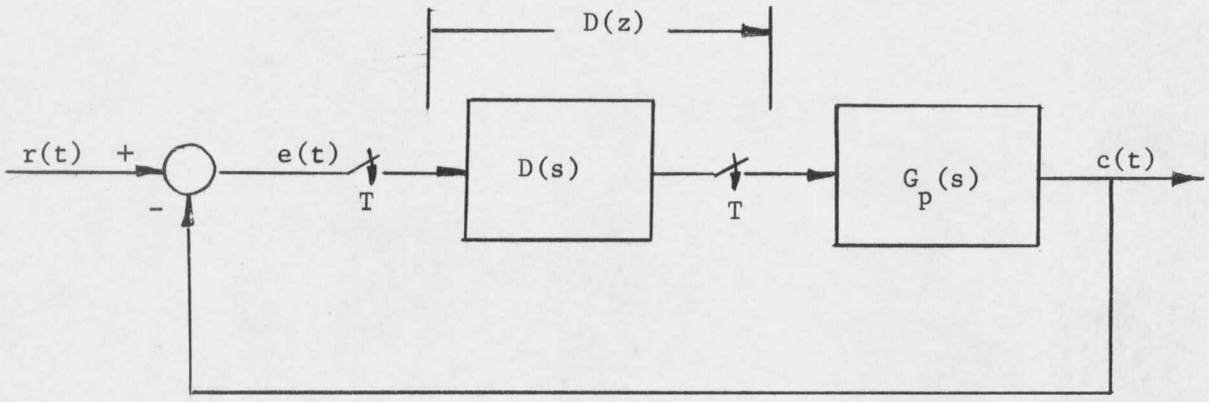


Figure 5.1. Error-sampled feedback control system with cascade digital controller.

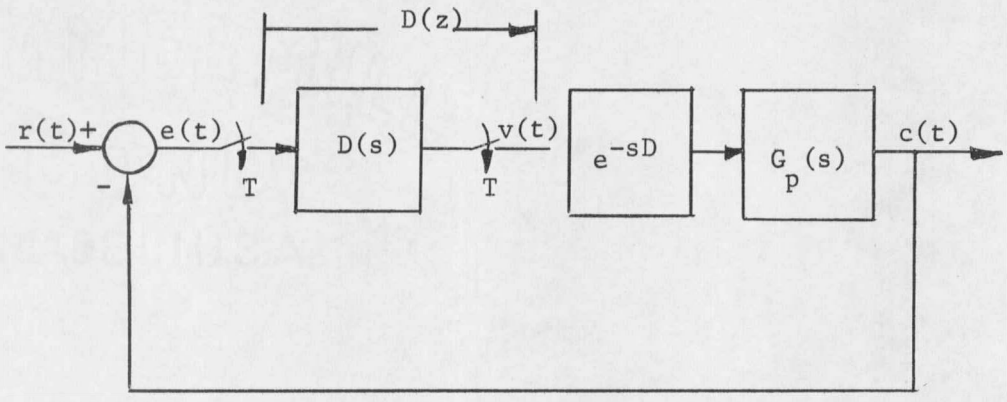


Figure 5.2. Error-sampled feedback time-delayed control system with cascade digital controller.

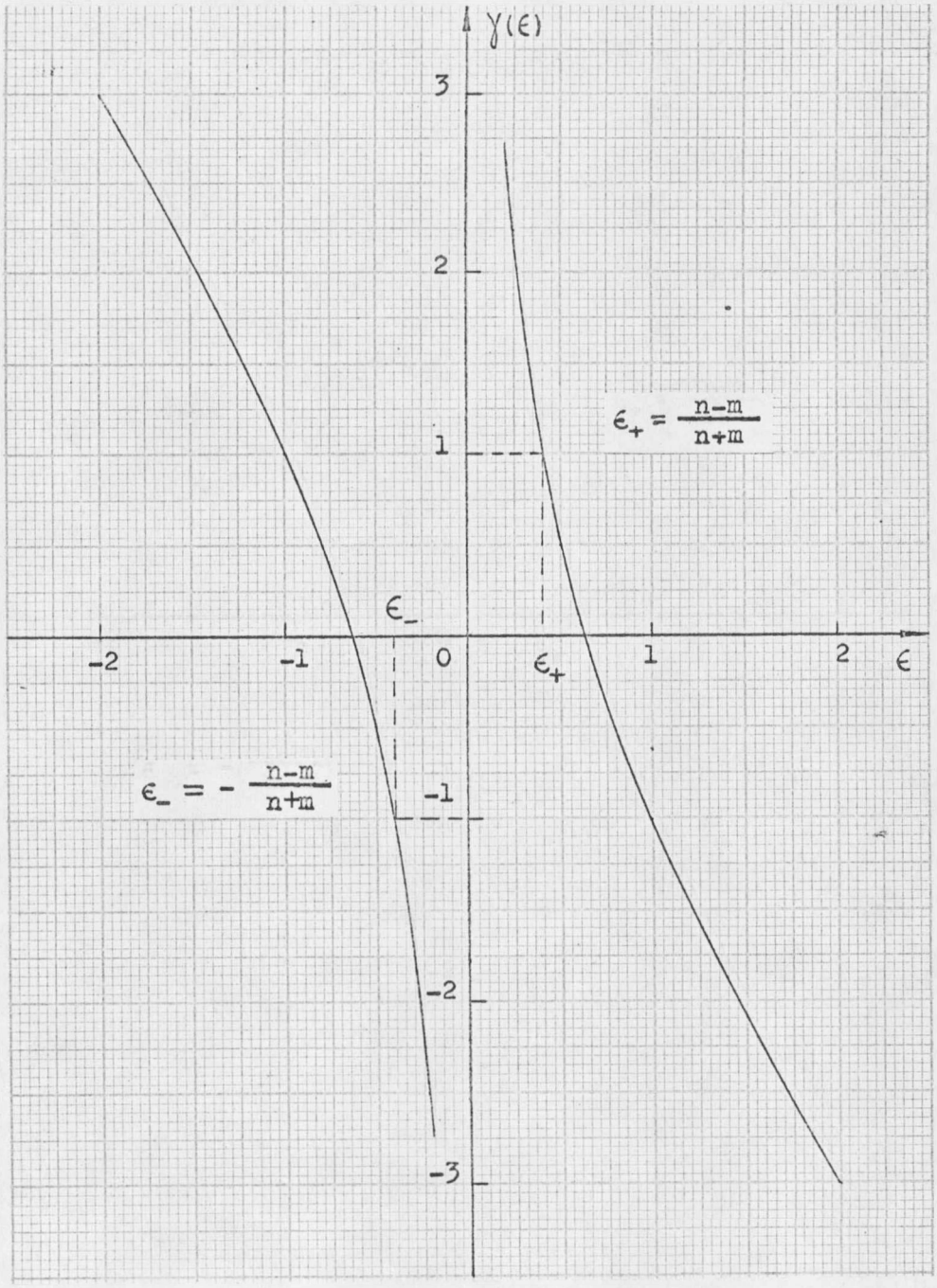


Figure 5.3. $\gamma(\epsilon)$ vs. ϵ from Equation 5.81 for $n = 7$ and $m = 3$

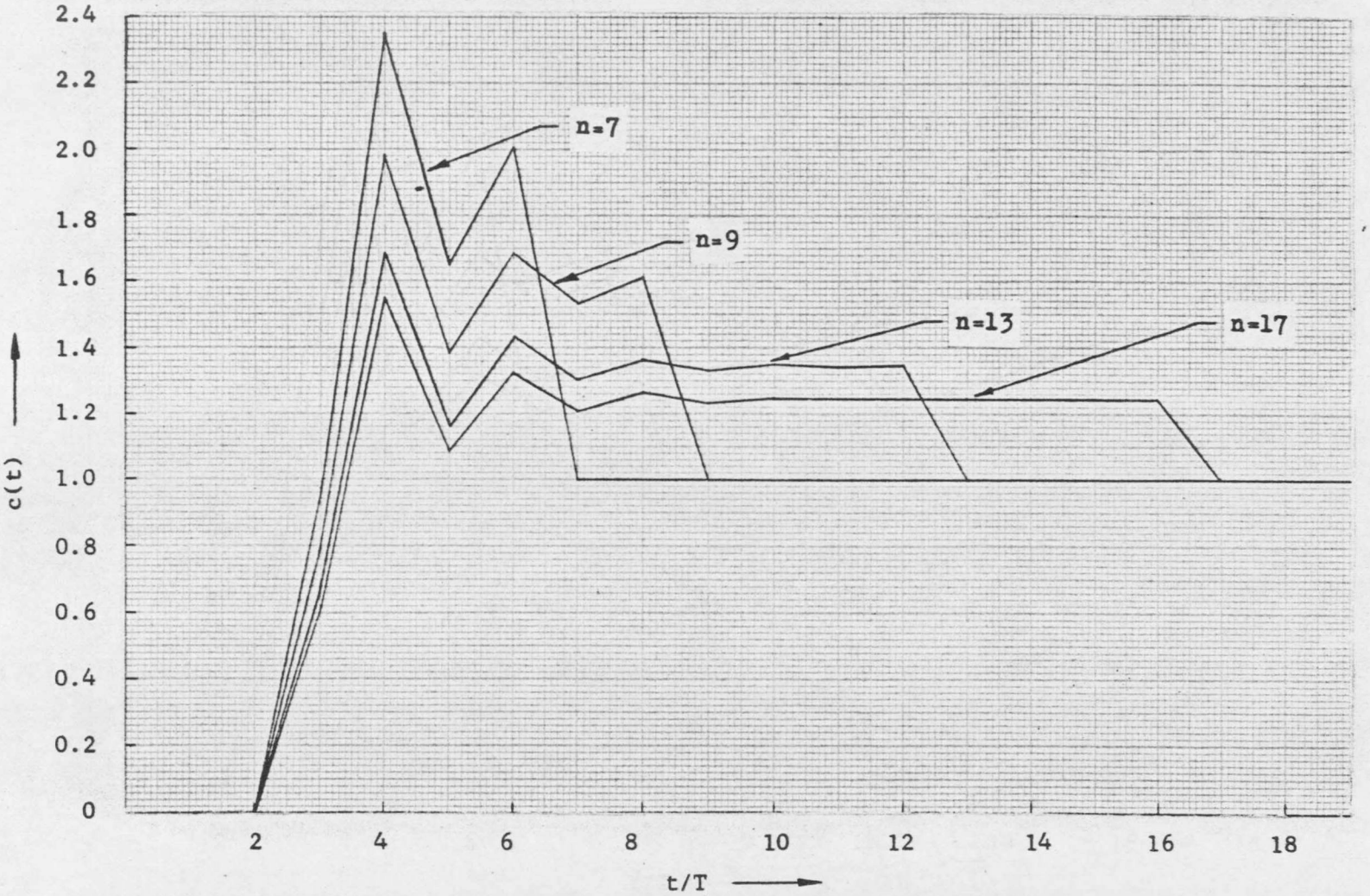


Figure 5.4. Optimal unit step response for $z_1 = -2$ and $m = 3$.

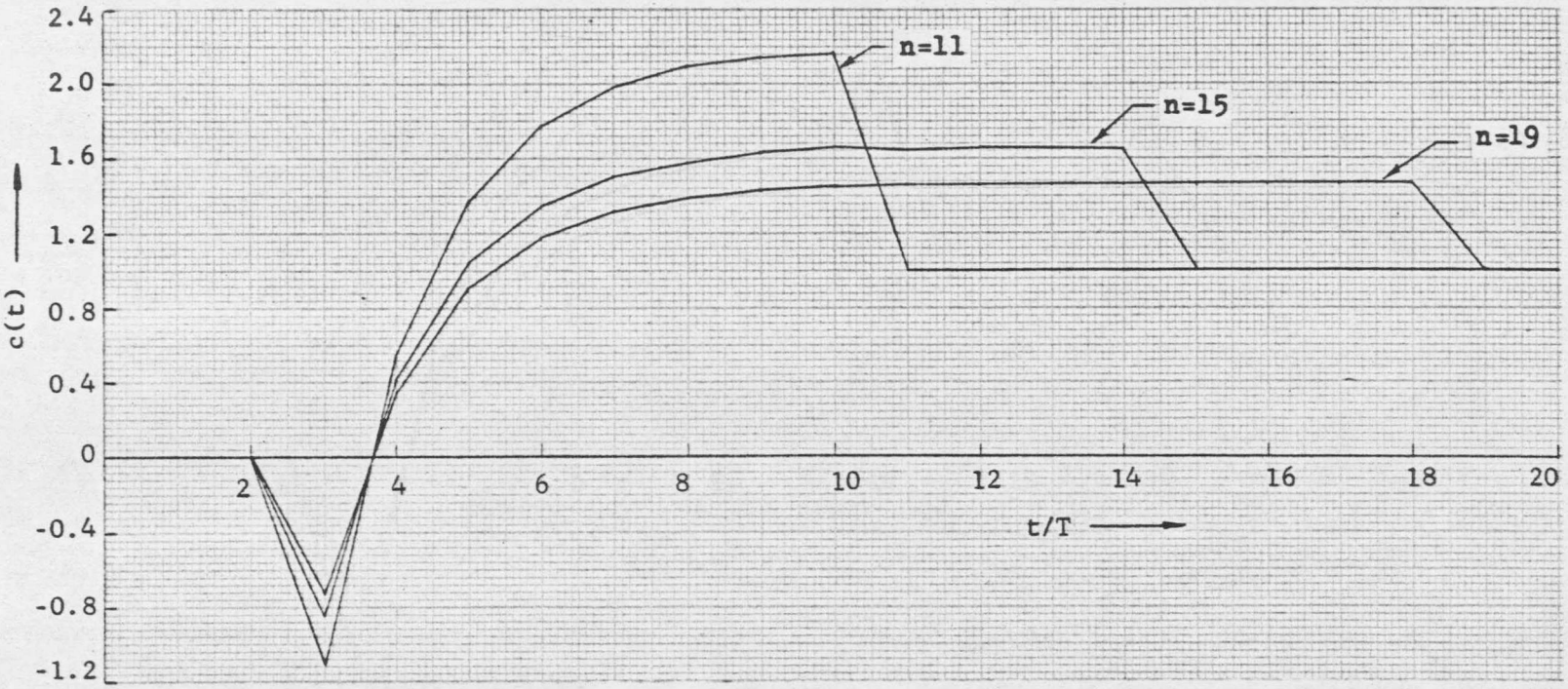


Figure 5.5. Optimal unit step responses for $z_1 = 2$ and $m = 3$.

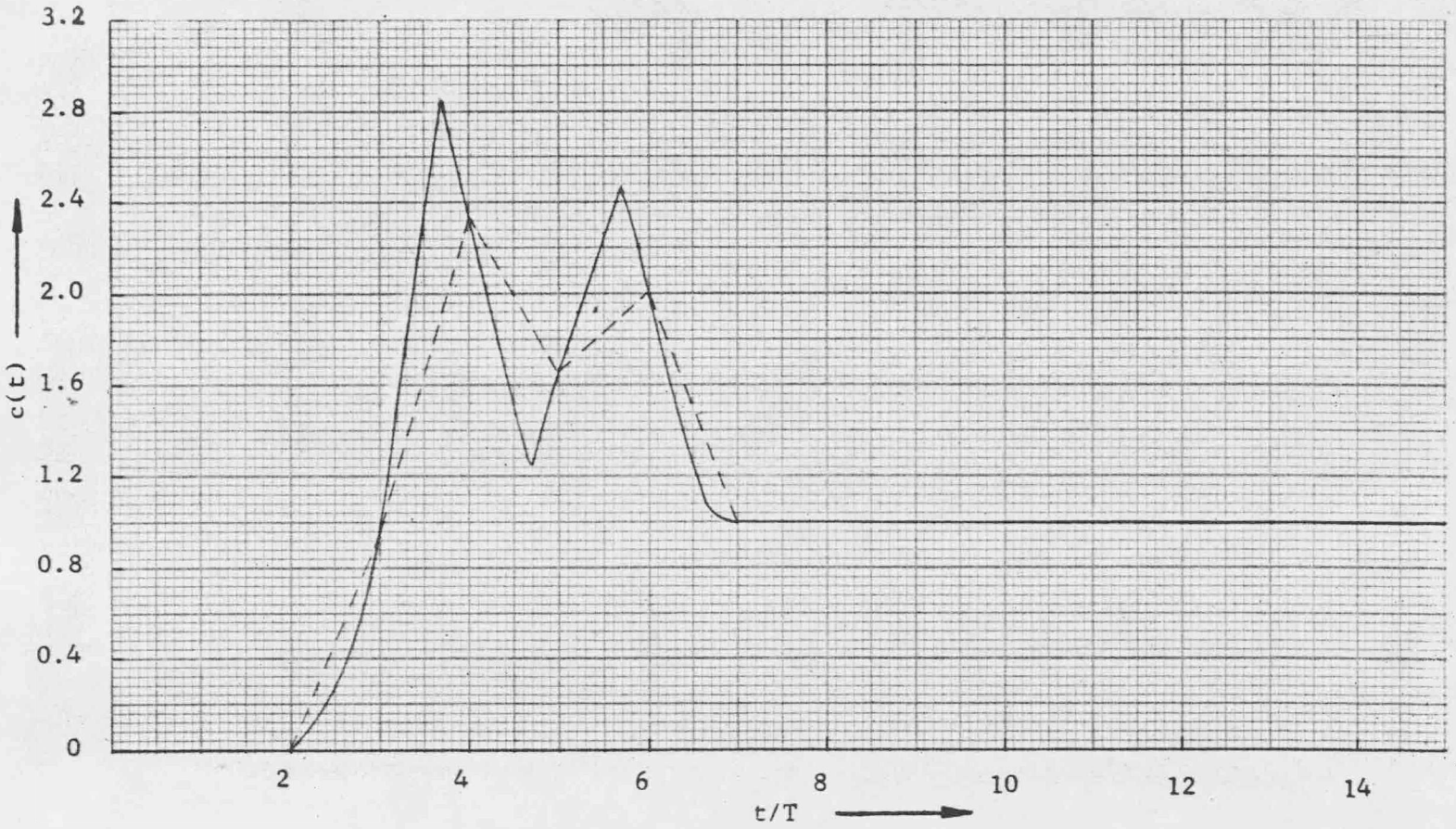


Figure 5.6. Optimal continuous unit step response for $z_1 = -2$, $n = 7$ and $m = 3$

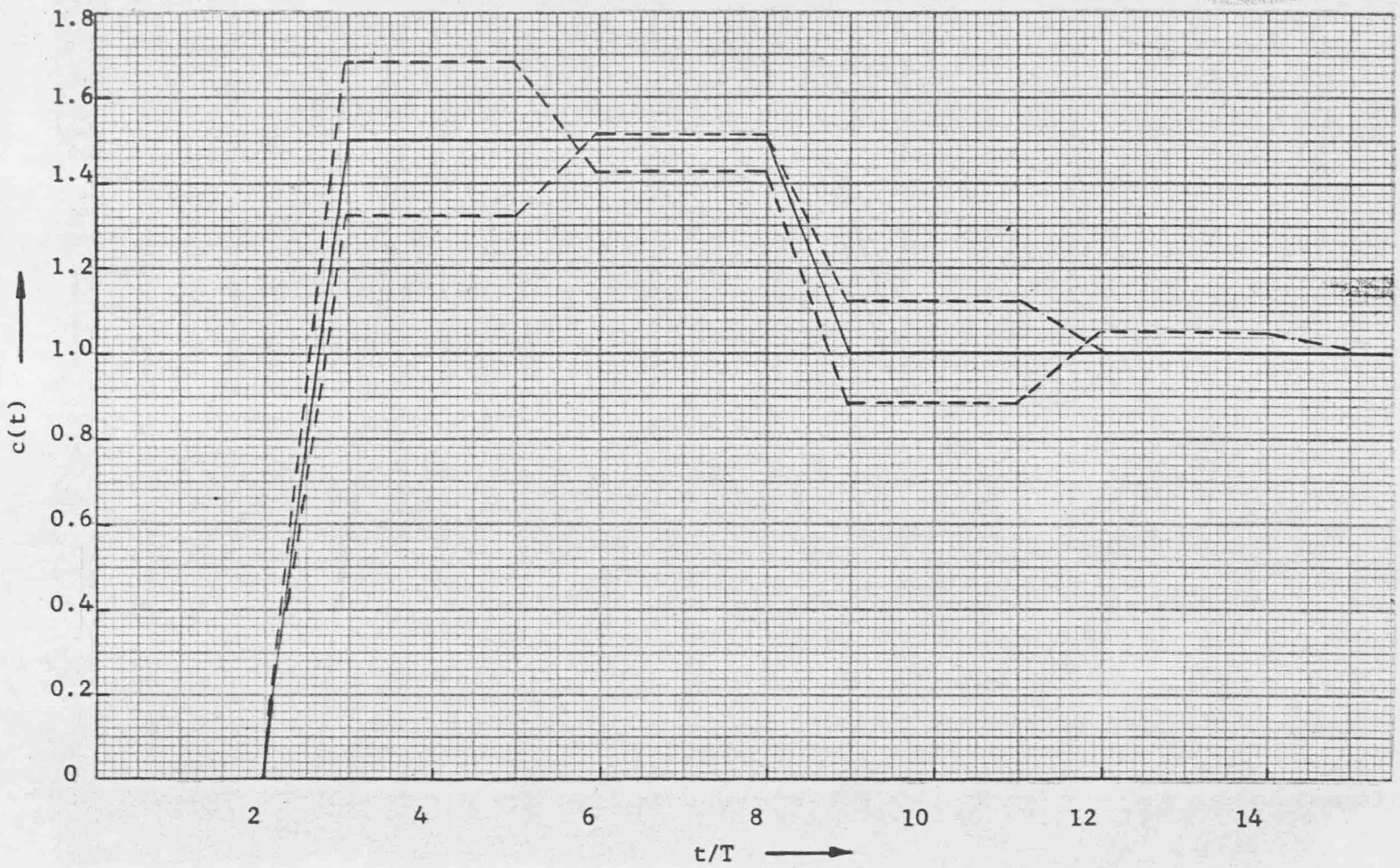


Figure 5.7. Deviation of the optimal step response due to $\pm$ 10 %. change in parameters of the optimal digital controller.

Table 5.1. Stability limits.

	m	ε_{max}	ε_{min}
n = 3	1	1.0000	-1.0000
n = 4	1	.6000	-1.0000
	2	.3333	-1.0000
n = 5	1	1.0000	-1.0000
	2	.4889	-.4286
	3	.2722	-.2722
n = 6	1	.7143	-1.0000
	2	1.0000	-1.0000
	3	.3333	-1.0000
	4	.2500	-.2000
n = 7	1	1.0000	-1.0000
	2	.6101	-.5556
	3	.4249	-.4249
	4	.2852	-.2727
	5	.1727	-.1727
n = 8	1	.7778	-1.0000
	2	.6000	-1.0000
	3	.4545	-.4792
	4	.3333	-1.0000
	5	.2308	-.2381
	6	.1428	-.1589
n = 9	1	1.0000	-1.0000
	2	.6844	-.6364
	3	1.0000	-1.0000
	4	.3982	-.3846
	5	.2937	-.2937
	6	.2500	-.2000
	7	.1275	-.1275
n = 10	1	.8182	-1.0000
	2	1.0000	-1.0000
	3	.5385	-.5617
	4	.4889	-.4216
	5	.3333	-1.0000
	6	.2722	-.2722
	7	.1765	-.1796
	8	.1183	-.1111

CHAPTER 6

SUMMARY AND SUGGESTED FUTURE RESEARCH

6.1 SUMMARY OF ESSENTIAL VALUES OF THE THESIS RESEARCH

The optimization of both stochastic and deterministic distributed-parameter control systems is examined in this thesis. In Chapter 2, a new definition of distributed correlation function is introduced and is shown to be more appropriate to use than Senin's definition (76, 77, 78) in the case of distributed outputs. Necessary properties of correlation functions between distributed output signals and input signals are derived. A problem of optimal stochastic control of distributed-parameter systems is shown to be a Wiener-Hopf spectrum-factorization problem.

The optimization of distributed-parameter control systems is approached from both linear programming and dynamic programming points of view. It is shown in Chapter 3 that, for a linear performance measure subject to linear constraints, optimization of linear distributed-parameter systems can be accomplished by linear programming techniques. The advantage of the linear programming approach is that large-scale problems can be treated with relative ease. In Chapter 4, the general convex performance measure of the distributed error function is minimized by use of the prinicple of optimality (6, 7, 8). The closed-form expression of the optimal control function is derived by use of the principle of optimality together with mathematical induction under the quadratic performance measure of the distributed error (see Equations 4.5 and 4.40). With this closed-form expression of the optimal control function, determination of the value of the optimal control function at each sampling instant requires only one computation of the value of the optimal control function at the following sampling instants.

The theory of deadbeat response with minimal overshoot compromise is extended to apply to both minimum and nonminimum phase systems enfolding time delay. The optimal tradeoff between step-response overshoot and ramp-input deadbeat-response time is derived. The sensitivity of the optimal system is examined as a function of parameter variations of the controller. For a given set of allowed parameter variations, the worst transient-behavior condition is derived. Also, stability limits are obtained for the optimal system in terms of open-loop gain.

6.2 ASPECTS MERITING ADDITIONAL STUDY

Four topics worthy of further investigation are evident as a consequence of this dissertation.

1. Methods of offseting the dimensionality limitations of present-day dynamic programming techniques constitute a topic of much importance. In this regard, the particular method used in Section 4.3 may be applied to a more general class of problems, but is by no means a panacea.

2. As seen in Chapter 5 and in reference (68), under certain conditions there exists a negative response or "undershoot" in the step response of a class of deadbeat-response systems. The cause and the cure of such undershoot in the deadbeat performance would be a worthy subject of investigation.

3. It would appear that theories similar to that in Chapter 5 could be developed for other types of input signals. Specifically, inputs with an acceleration-type nature would appear to merit attention because of the occurrence of such signals in practice.

4. A study of prediction and filtering problems which are associated with distributed-parameter systems would be of value. The concepts of distributed correlation functions (Chapter 2) would aid in such a study.

APPENDIX A

LINEAR PROGRAMMING

A.1 THE PROBLEM OF LINEAR PROGRAMMING

The general linear programming problem can be described as follows: Given a set of m linear equations in n variables, find nonnegative values of these variables which are a solution to the given set of equations and maximize some linear function of the variables.

Mathematically, the above statement can be written as: Find the column matrix $X = [x_1 \quad x_2 \quad . \quad . \quad . \quad x_n]'$ which maximizes the performance measure

$$P = C'X \tag{A.1}$$

where $C' = [c_1 \quad c_2 \quad . \quad . \quad . \quad c_n]$ is a row matrix subject to

$$X \geq \underline{0} \tag{A.2}$$

and

$$AX = B \tag{A.3}$$

where $\underline{0}$ is an n x 1 matrix, all elements of which are identically zero, A is an m x n matrix, $m < n$, and B is an m x 1 matrix of constants.

The problem of finding where a given linear function of several variables takes on an extreme value when the variables are required to satisfy constraints of the form of linear inequalities in addition to linear equalities can be reduced to the standard form in Equations A.1, A.2 and A.3 (see reference 64, for example).

A.2 PROPERTIES OF A SOLUTION

In this section, standard definitions of linear programming concepts are given. In addition, two important theorems are cited without proof.

Definition A.1 A solution is any set of x_i's which satisfies Equation A.3.

Definition A.2 A basic solution is the solution which is obtained by setting $n - m$ of x_i's equal to zero.

Definition A.3 A nondegenerate basic solution is a basic solution which contains n nonzero x_i's.

Definition A.4 A feasible solution is a solution which satisfies the inequality in Equation A.2.

Definition A.5 A basic feasible solution is a basic solution which satisfies the inequality in Equation A.2.

Definition A.6 A convex combination of the points $X_1, X_2, \ldots, X_n$ is a point

$$X = \alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_n X_n \tag{A.4}$$

where the α_i's satisfy

$$\alpha_i \geq 0 \tag{A.5}$$

for $i = 1, 2, \ldots, n$ and

$$\sum_{i=1}^{n} \alpha_i = 1$$

Definition A.7 A set R is convex if for any elements X_a and X_b in R, X_c which is any convex combination of X_a and X_b is also an element of R.

Definition A.8 A point X_v of a convex set R is a vertex (also referred to as an extreme point) if X_v cannot be expressed as a convex combination of any two distrinct points in R. In other words, no line segment completely in R of finite length contains X_v internally.

It follows immediately from definition A.7 that a hyperplane is a convex set. Furthermore the intersection of two hyperplanes is still a hyperplane and thus is a convex set. Let R_c be the set of points satisfying both Equations A.2 and A.3. Then it can be concluded that R_c is a convex set.

Theorem A.1 The vertices of R_c have at least n-m zero components

Theorem A.2 The performance measure in Equation A.1 assumes its maximum at a vertex of R_c. If it assumes its maximum at more than one vertex, then it takes on the same maximum value for every convex combination of those particular points.

The proofs of the above two theorems can be found in any standard work on linear programming (23, 29, 64).

One could proceed, in theory, by evaluating P at all the vertices of R_c. Still, for large n and m, it would not be feasible to test every vertex in order to find the optimal solution. To overcome this difficulty, Dantzig (23) developed an effective algorithm, the simplex algorithm, which enables one to progress in a systematic fashion to the next basic feasible solution from the present basic feasible solution such that the next basic feasible solution results in a greater value of P. Details about the simplex algorithm can be found in standard references (23, 29, 64).

APPENDIX-B

SOLVING $n\sin(m\theta) = m\sin(n\theta)$

The computer program consists of two phases. During the first phase, a rough initial serch is made using a grid whose width is determined by Shannon's sampling theorem. In the second phase, the Newton-Raphson method is applied in each likely neighborhood which is determined in the first phase.

The first phase is straightforward and consists of the evaluation of TEST which represents

$$TEST = m\sin(n\theta) - n\sin(m\theta) \tag{B.1}$$

The highest frequency f_h in TEST is

$$f_h = n/(2\pi) \tag{B.2}$$

A sufficient DELTA according to Shannon's sampling theorem is

$$(DELTA)_{ideal} = \pi/n \tag{B.3}$$

In the program, DELTA is set equal to 1/n in order to compensate for numerical error. Another advantage to having DELTA = 1/n is the better initial estimation of roots for the second phase. With this DELTA, it is theoretically assured that none of the information contained in TEST is neglected. The search is terminated when θ exceeds 2π. A block diagram for the first phase is shown in Figure B.1.

In the second phase of the search, the Newton-Raphson method is used. Let θ^0 be the initial estimation of θ^*, one of the roots of TEST, resulting from the first phase. The next estimation is the point of intersection of the line tangent to TEST at $\theta = \theta^0$ and the abscissa. Therefore,

$$\theta^0 - \theta^1 = \frac{TEST(\theta^0)}{DTEST(\theta^0)} \tag{B.4}$$

where $DTEST(\theta^0)$ is the derivative of TEST evaluated at $\theta = \theta^0$. In general, the $(k+1)^{th}$ estimation is

$$\theta^{k+1} = \theta^k - \frac{TEST(\theta^k)}{DTEST(\theta^k)} \tag{B.5}$$

The search is terminated if the absolute value of TEST is less than ACRE, a specified value. Assume that the search is terminated at the N^{th} iteration. Then

$$\theta^N = \theta^* + \Delta\theta \tag{B.6}$$

where $\Delta\theta$ is the error in θ^N. It is known from the terminating condition that

$$|m\sin[n(\theta^* + \Delta\theta)] - n\sin[m(\theta^* + \Delta\theta)]| < ACRE \tag{B.7}$$

In expansion of sine terms in Equation B.7 as functions of the arguments, together with the assumption that $n\Delta\theta$ is sufficiently small to use the first-order approximations of cosine and sine functions, a bound on $\Delta\theta$ is

$$\Delta\theta < \left|\frac{ACRE}{mn[\cos(n\theta^*) - \cos(m\theta^*)]}\right| \tag{B.8}$$

under the assumption that $\cos(n\theta^*) - \cos(m\theta^*)$ is not zero. In the actual computation, the bound on $\Delta\theta$ is determined from Equation B.8 by replacing θ^* by θ^N. The block diagram is shown in Figure B.2.

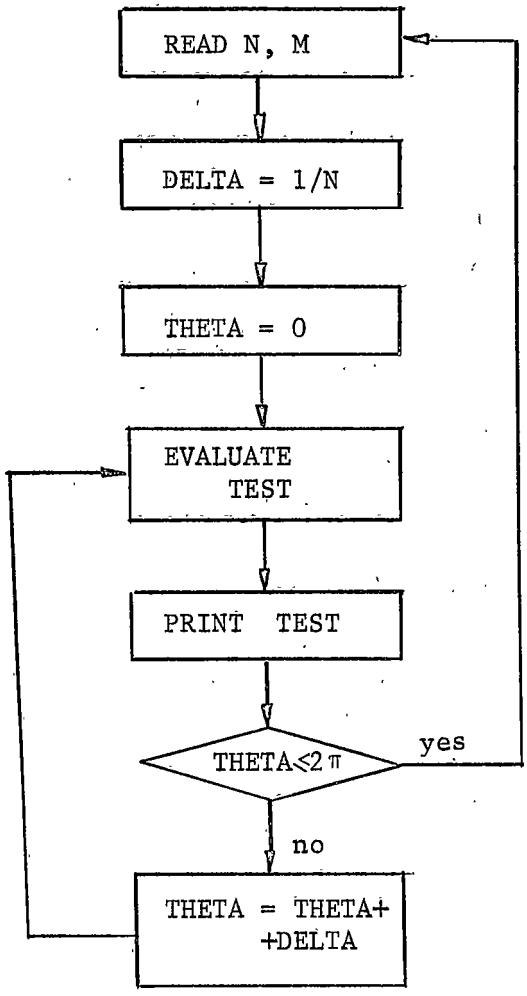


Figure B.1. Block diagram in the first phase of the method.

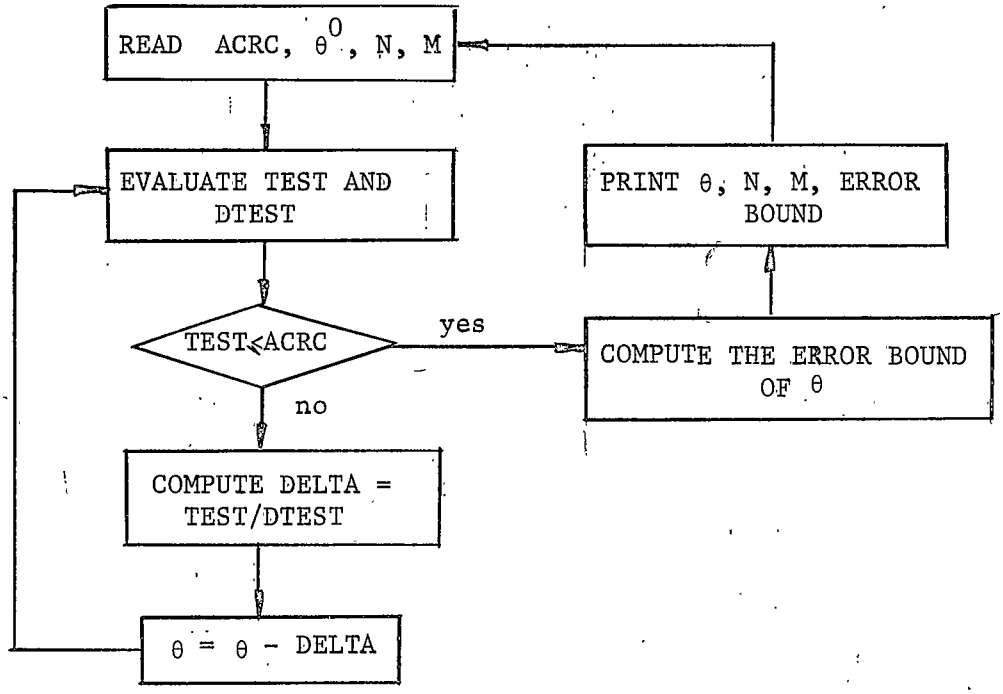


Figure B.2. Block diagram in the second phase of the method.

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