



Distribution of the sample range for parent populations associated with Pearsons differential equation
by Glenn R Ingram

A THESIS Submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree
of Master of Science in Applied Mathematics

Montana State University

© Copyright by Glenn R Ingram (1954)

Abstract:

The distribution of the range in samples from Pearson-type populations is considered in this thesis. Explicit density functions of the range, together with the cumulative distributions and certain moments, are given for four types. The difficulties precluding exact distributions have been pointed out in the other cases. An asymptotic distribution is suggested as a means of approximating the distribution of the range for large samples from particular populations.

DISTRIBUTION OF THE SAMPLE RANGE FOR PARENT POPULATIONS
ASSOCIATED WITH PEARSON'S DIFFERENTIAL EQUATION

by

GLENN R. INGRAM

A THESIS

Submitted to the Graduate Faculty

in

partial fulfillment of the requirements

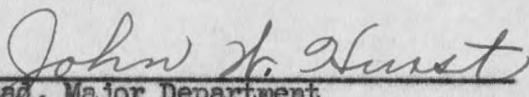
for the degree of

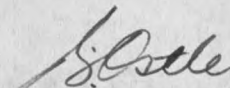
Master of Science in Applied Mathematics

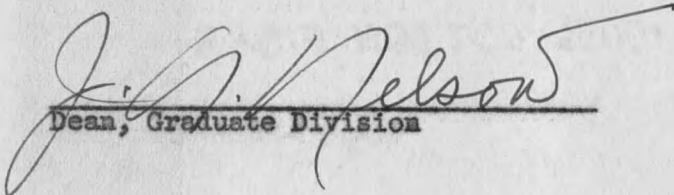
at

Montana State College

Approved:


Head, Major Department


Chairman, Examining Committee


Dean, Graduate Division

Bozeman, Montana
June, 1954

N1378

In4d

cop 2

	Page
Acknowledgment	3
Abstract	4
I. Introduction and Statement of the Problem	5
II. The Formal Solution for Distribution of the Range	7
III. The Distribution of Range for Members of the Pearson System	11
A. Type I	11
B. Type II	12
C. Type III	14
D. Type IV	17
E. Type V	17
F. Type VI	19
G. Type VII	20
H. Type VIII	21
I. Type IX	26
J. Type X	30
K. Type XI	33
L. Type XII	39
IV. Summary and Conclusions	41
V. Appendix	43
VI. Literature Cited	46

1098-B

ACKNOWLEDGMENT

The writer wishes to express his thanks to Dr. Bernard Ostle for suggesting this problem and for his help and encouragement in the completion of this thesis. Thanks are also due Dr. Robert Lowmeyer for his helpful suggestions.

ABSTRACT

The distribution of the range in samples from Pearson-type populations is considered in this thesis. Explicit density functions of the range, together with the cumulative distributions and certain moments, are given for four types. The difficulties precluding exact distributions have been pointed out in the other cases. An asymptotic distribution is suggested as a means of approximating the distribution of the range for large samples from particular populations.

I. INTRODUCTION AND STATEMENT OF THE PROBLEM

In a consideration of sample¹ statistics, one of the most obvious and easily obtained is the range; i.e., the difference between the largest and smallest sample values. Hence, if the distribution of this statistic could be obtained, it would be useful in situations where a minimum of calculation is expedient.

Formally, the density function of range is obtained as an integral from the joint distribution of largest and smallest sample values. However, evaluation of this integral for arbitrary sample size is difficult in most cases and impossible by exact methods in some.

Practical applications of the range suggest that further study of its properties would be worthwhile. Statistical quality control utilizes the sample range to a considerable extent because of the ease with which it is computed, in contrast with the more time consuming calculation of other measures of variation.

Most of the research concerning the sample range has been concerned with parent normal populations because of their wide application. One other population, the rectangular, has also been exhaustively studied.

A broad class of probability density functions, including the two mentioned above as special cases, is the Pearson System. This system is generated by specializing constants in the differential equation

$$(1) \quad \frac{dy}{dx} = \frac{(x-a)y}{b_0 + b_1x + b_2x^2}$$

1. All samples referred to in this thesis will be random samples.

Many of the distributions important in sampling theory are special cases of different members of the family of solutions of the above differential equation. Among these distributions are: the normal, chi-square, Student's t, certain correlation coefficients, and F.

This thesis is concerned with the distribution of the range in samples of size n from parent populations that are members of the Pearson system. Each of the twelve types will be considered, and the difficulties pointed out in cases where an explicit result has not been obtained.

II. THE FORMAL SOLUTION FOR DISTRIBUTION OF THE RANGE

The distribution of range in integral form can be obtained at once from the joint distribution of the largest and smallest sample values. This frequency function is given by

$$(2) \quad h(u,v) = n(n-1) [F(v) - F(u)]^{n-2} f(v)f(u), \quad a \leq u < v \leq b,$$

where $f(x)$ is the probability density function specifying the population sampled, with $a \leq x \leq b$,

u is the smallest sample value

v is the largest sample value

n is the sample size

and $F(x) = \int_a^x f(x)dx$ is the cumulative distribution function.²

By the transformation $v = u + R$, where R is sample range, a joint function of u and R is obtained,

$$(3) \quad h(u,R) = n(n-1) [F(u+R) - F(u)]^{n-2} f(u+R)f(u).$$

Then by integrating over the range of u , the desired density function is obtained,

$$(4) \quad g(R) = n(n-1) \int_a^{b-R} [F(u+R) - F(u)]^{n-2} f(u+R)f(u)du, \quad 0 \leq R \leq b-a.$$

Gumbel (4) has shown that the cumulative distribution takes an elegant, if not particularly useful, form when the upper limit is independent of R ;

2. With appropriate changes of the argument, this notation will be followed throughout the thesis. That is, a lower case letter will denote a probability density function, and the corresponding upper case letter will denote the cumulative distribution.

i.e., if $b = \infty$. Then

$$(5) \quad G(R) = n(n-1) \int_0^R dR \int_a^\infty [F(u+R) - F(u)]^{n-2} f(u+R)f(u)du.$$

Now if the integral defining $g(R)$ converges uniformly, it is permissible to interchange order of integration, giving

$$(6) \quad G(R) = n(n-1) \int_a^\infty f(u)du \int_0^R [F(u+R) - F(u)]^{n-2} f(u+R)dR \\ = n \int_a^\infty [F(u+R) - F(u)]^{n-1} f(u)du.$$

Or, with $dF(u) = f(u)du$,

$$(7) \quad G(R) = n \int_0^1 [F(u+R) - F(u)]^{n-1} dF,$$

which is Gumbel's result. He calls attention to the fact that this is of little practical use, since in general $F(u+R)$ cannot be expressed in terms of $F(u)$.

Still another form of $G(R)$ which we will find useful can be developed from (25),

$$(8) \quad G(R) = n \int_a^\infty [F(u+R) - F(u)]^{n-1} f(u+R)du \\ - n \int_a^\infty [F(u+R) - F(u)]^{n-1} [f(u+R) - f(u)] du \\ = n \int_a^\infty [F(u+R) - F(u)]^{n-1} f(u+R)du - \left\{ [F(u+R) - F(u)]^n \right\}_a^\infty \\ = n \int_a^\infty [F(u+R) - F(u)]^{n-1} f(u+R)du + [F(a+R)]^n.$$

If $a = -\infty$, (27) reduces to

$$(9) \quad G(R) = n \int_{-\infty}^{\infty} [F(u+R) - F(u)]^{n-1} f(u+R) du.$$

If we consider the asymptotic distribution of the range, the problem is simplified somewhat if the parent population is unlimited in one direction. Gumbel (4) states that if the population to be sampled is bounded to the left (right), and unlimited to the right (left), the asymptotic distribution of range becomes identical with the asymptotic distribution of the largest (smallest) sample value. These conditions are met by five members of the Pearson family.

The distribution of the largest sample value being

$$(10) \quad g(v) = n [1 - F(v)]^{n-1} f(v),$$

we see that if we make the transformation $w = (n-1)F(v)$, then $f(v)dv = dw/(n-1)$

and $\varphi_{(n)}(w) = n [1 - w/(n-1)]^{n-1}/(n-1)$.

The limiting distribution is then

$$(11) \quad \varphi(w) = \lim_{n \rightarrow \infty} \frac{n}{n-1} \left[1 - \frac{w}{n-1} \right]^{n-1} = e^{-w}, \quad w > 0.$$

The cumulative distribution is

$$(12) \quad \Phi(w) = \int_0^w e^{-w} dw = 1 - e^{-w}.$$

This approach may be compared with that used by Carlton (1) in investigating the rectangular population. This will be discussed later.

The disadvantage of this particular transformation is that it is non-linear. However, it might be useful in tabulating ordinates or percentage points approximately, for large samples, with $w = (n-1)F(R)$.

In summary, there is one integral form for the probability density

function of the range, and in certain cases, two for the cumulative distribution of the range:

$$(4 \text{ bis}) \quad g(R) = n(n-1) \int_a^{b-R} [F(u+R) - F(u)]^{n-2} f(u+R)f(u)du$$

$$(6 \text{ bis}) \quad G(R) = n \int_a^\infty [F(u+R) - F(u)]^{n-1} f(u)du$$

$$(8 \text{ bis}) \quad G(R) = n \int_a^\infty [F(u+R) - F(u)]^{n-1} f(u+R)du + [F(a+R)]^n.$$

III. THE DISTRIBUTION OF RANGE FOR MEMBERS OF THE PEARSON SYSTEM

A. Type I.

For the Type I density function, specified by

$$(13) \quad f(x) = \frac{1}{B(p+1, q+1)} x^p (1-x)^q, \quad 0 \leq x \leq 1,$$

the distribution of the range is

$$(14) \quad g(R) = \frac{n(n-1)}{B(p+1, q+1)^n} \int_0^{1-R} \left[\int_u^{u+R} x^p (1-x)^q dx \right]^{n-2} \cdot u^p (1-u)^q (u+R)^p (1-u-R)^q du \quad 0 \leq R \leq 1.$$

With no restrictions on p and q other than those required for convergence of $F(x)$, it would be necessary to consider evaluation of this integral by tedious numerical methods. The inner integral is the difference of two incomplete beta functions and, while values of the incomplete beta function have been extensively tabled by Karl Pearson (15), an exact solution for $g(R)$ is beyond our reach.

If p and q are integral, the cumulative distribution may be found by repeated integrations by parts to be

$$(15) \quad F(x) = \frac{1}{B(p+1, q+1)} \sum_{k=0}^q \frac{p!}{(p+k+1)!} \frac{q!}{(q-k)!} x^{p+k+1} (1-x)^{q-k} \\ = (p+q+1)! \sum_{k=0}^q \frac{1}{(p+k+1)!} \frac{1}{(q-k)!} x^{p+k+1} (1-x)^{q-k}.$$

However, even this form for $F(x)$ is of little use for our purpose except

for highly restrictive values of p , q , and n .

No account of approximate solutions for $g(R)$ nor of empirical determination of range distribution constants appears in the literature.

B. Type II.

For a Type II function,

$$(16) \quad f(x) = \frac{1}{aB(\frac{1}{2}, m+1)} \left(1 - \frac{x^2}{a^2}\right)^m, \quad -a \leq x \leq a,$$

the integral representation of $g(R)$ is

$$(17) \quad g(R) = \frac{n(n-1)}{a^n B(\frac{1}{2}, m+1)} \int_{-a}^{a-R} \left[\int_u^{u+R} \left(1 - \frac{x^2}{a^2}\right)^m dx \right]^{n-2} \left(1 - \frac{u+R}{a}\right)^m \left(1 - \frac{u^2}{a^2}\right)^m du.$$

For a general case, $m \neq 0$, the difficulties here are identical with those of Type I. Considering the evaluation of the cumulative distribution of the Type II function, we have

$$(18) \quad F(x) = \frac{1}{aB(\frac{1}{2}, m+1)} \int_{-a}^x \left(1 - \frac{x^2}{a^2}\right)^m dx,$$

and with the transformation $y = \frac{1}{2}(1 + \frac{x}{a})$, the integral becomes

$$(19) \quad F(x) = \frac{2^{2m+1}}{B(\frac{1}{2}, m+1)} \int_0^{\frac{a+x}{2a}} y^m (1-y)^m dy.$$

So the inner integral is again the difference of two incomplete beta functions.

If m is a positive integer, $F(x)$ may be evaluated by a binomial expansion of $(1 - \frac{x^2}{a^2})^m$. Then,

$$\begin{aligned}
 (20) \quad F(x) &= \frac{1}{aB(\frac{1}{2}, m+1)} \sum_{k=0}^m (-1)^k \binom{m}{k} \int_{-a}^x \left(\frac{x}{a}\right)^{2k} dx \\
 &= \frac{1}{B(\frac{1}{2}, m+1)} \sum_{k=0}^m \frac{(-1)^k}{2k+1} \binom{m}{k} \left[\left(\frac{x}{a}\right)^{2k+1} + 1 \right].
 \end{aligned}$$

And again for general values of m and n , this form is not useful.

Since an exact or approximate function of the range is not readily obtained, the most useful information available is the result of extensive experimental sampling. The results of sampling carried out by E. S. Pearson and N. K. Adyanthaya are summarized in Tables for Statisticians and Biometricians, Part II (14). Empirical distribution constants listed are: the mean, standard error, β_1 and β_2 , where β_1 is a measure of skewness ($\beta_1 = 0$ for a symmetrical distribution) and β_2 is a measure of peakedness which may be compared with $\beta_2 = 3$ for a normal distribution. These constants are given for samples of size 2, 5, 10 and 20, and they suggest a distribution very nearly symmetrical, and slightly less peaked than a normal distribution.

The rectangular population, which is a special case of Type II, has been probed in some detail and, with it as a parent population, the exact and limiting forms of the distribution of the range are available. Carlton (1) has given both. In his notation, the population is

$$(21) \quad f(x) = \frac{1}{L}, \quad \theta - \frac{L}{2} \leq x \leq \theta + \frac{L}{2}$$

Then the distribution of sample range is

$$(22) \quad f(R) = n(n-1) R^{n-2} (1-R), \quad 0 \leq R \leq L.$$

To obtain the limiting distribution, he considered the transformation $R' = n(1 - R)$, and the limiting distribution of R' is

$$(23) \quad f(R') = R'e^{-R'}, \quad R' > 0.$$

C. Type III.

The Type III density function can be expressed in the form

$$(24) \quad f(x) = \frac{1}{\Gamma(a+1)b^{a+1}} x^a e^{-\frac{x}{b}}, \quad x \geq 0.$$

The distribution of the range in samples for a Type III population is given by

$$(25) \quad g(R) = \frac{n(n-1)}{[\Gamma(a+1)b^{a+1}]^n} \int_0^\infty \left[\int_u^{u+R} x^a e^{-\frac{x}{b}} dx \right]^{n-2} u^a (u+R)^a e^{-\frac{u}{b}} e^{-\frac{u+R}{b}} du$$

That part of the integrand within square brackets is the difference between two incomplete gamma functions, and this points out the difficulty in obtaining an exact expression for $g(R)$.

In this case, also, research has taken the form of experimental determination of range constants. Repeated sampling by E. S. Pearson and N. K. Adyanthaya has lead to results tabulated in Tables for Statisticians and Biometricians, Part II (14). Values for the mean and standard error are listed as multiples of the population standard deviation, and these are compared with the theoretical values obtained for the range in samples from a normal population. The mean ranges are in fairly close agreement, but the standard error of range is in each case slightly larger

in samples from the Type III distribution.

For large values of n , the asymptotic distribution obtained earlier could be used to approximate the distribution of the range. We had

$$(11 \text{ bis}) \quad \varphi(w) = e^{-w}$$

and

$$(12 \text{ bis}) \quad \Phi(w) = 1 - e^{-w}$$

$$\text{where now } w = \frac{n-1}{\Gamma(a+1)b^{a+1}} \int_0^R x^a e^{-\frac{x}{b}} dx.$$

For any fixed value of R , the integral can be evaluated from tables of the incomplete gamma function, and then the distribution of w could be used to approximate the ordinates or probability levels of the distribution range.

The normal distribution

$$(26) \quad f(x) = \frac{1}{\sqrt{2\pi} \sigma} \exp. \left\{ - (x-\mu)^2 / 2\sigma^2 \right\}, \quad -\infty < x < \infty$$

can be considered a special case of Pearson's Type III distribution. This particular density function has commanded most of the interest in research regarding the distribution of the range.

Among the major contributors to this research has been L. H. C. Tippett, E. S. Pearson, and "Student,"³ with much of their work being summarized in Tables for Statisticians and Biometricians, Part II (14).

Tippett has given theoretical values of the moment coefficients of the distribution of the range, and has carried out the following:

3. W. S. Gosset wrote many papers under the pseudonym "Student."

1. Computed the mean range in terms of the population standard deviation for samples of sizes 2 to 1000.
2. Calculated the standard deviation of range for certain sample sizes.
3. Given approximations to the values of β_1 and β_2 for larger samples.

E. S. Pearson obtained the first four moments of the distribution of range for samples of size 2,3,4,5 and 6.

"Student" utilized these constants of the distribution and extended them to samples of 10,20 and 60. He then obtained equations of Pearson curves with the correct first four moments to fit these distributions.

Asymptotic distributions of the range when sampling from normal populations have been developed by Elfving (3) and Gumbel (4). Elfving has used a non-linear transformation of the range to obtain a distribution in terms of Bessel functions, while Gumbel made use of a linear transformation to obtain the distribution of range in terms of Hankel functions of order zero.

Approximate distributions of the range have been given by Cox (2), using a gamma distribution with the correct first two moments, and by Patnaik (10) through the distribution of chi.

Gumbel (5) has tabulated percentage levels of the range, while Pearson and Hartley (11) have prepared tables of the "Studentized" range; i.e., the range divided by the sample standard deviation, for samples of size 2 to 20. The latter tables were extended in computations performed by May (9).

D. Type IV.

For the function of Type IV given by

$$(27) \quad f(x) = k \left(1 + \frac{x^2}{a^2}\right)^{-m} e^{-c \tan^{-1} \frac{x}{a}}, \quad -\infty < x < \infty,$$

where

$$(28) \quad 1/k = \int_{-\infty}^{\infty} \left(1 + x^2/a^2\right)^{-m} e^{-c \tan^{-1}(x/a)} dx,$$

the distribution of the range is

$$(29) \quad g(R) = n(n-1)k^n \int_{-\infty}^{\infty} \left[\int_u^{u+R} \left(1 + \frac{x^2}{a^2}\right)^{-m} e^{-c \tan^{-1} \frac{x}{a}} dx \right]^{n-2} \\ \cdot \left(1 + \frac{u^2}{a^2}\right)^{-m} \left(1 + \frac{u+R^2}{a^2}\right)^{-m} e^{-c \tan^{-1} \frac{u}{a} + \tan^{-1} \frac{u+R}{a}} du.$$

The inner integral here must be evaluated by approximate methods, or from integrals tabulated in Tables for Statisticians and Biometricians, Part II (14). The difficulties encountered in this evaluation preclude an exact treatment of the distribution of the range.

E. Type V.

The Type V distribution is given by

$$(30) \quad f(x) = \frac{a^{p-1}}{\Gamma(p-1)} x^{-p} e^{-\frac{a}{x}}, \quad 0 < x < \infty.$$

The density function of the range in samples from a Type V population is given by

$$(31) \quad g(R) = \frac{n(n-1)a^{n(p-1)}}{[\Gamma(p-1)]^n} \int_0^{\infty} \left[\int_u^{u+R} x^{-p} e^{-\frac{a}{x}} dx \right]^{n-2} u^{-p} e^{-\frac{a}{u}} (u+R)^{-p} e^{-\frac{a}{u+R}} du.$$

The connection between the Type V and the Type III distributions points out the difficulties to be encountered in this case. That is, in general, the cumulative distribution of the Type V function cannot be evaluated exactly.

If p is an integer, the cumulative distribution is

$$(32) \quad F(x) = 1 - \left[1 + \frac{a}{x} + \frac{1}{2!} \left(\frac{a}{x} \right)^2 + \dots + \frac{1}{(p-2)!} \left(\frac{a}{x} \right)^{p-2} \right] e^{-\frac{a}{x}}.$$

The distribution of the range is then

$$(33) \quad g(R) = \frac{n(n-1)a^{2(p-1)}}{[(p-2)!]^2} \int_0^\infty \left[e^{-\frac{a}{u}} \sum_{k=0}^{p-2} \frac{1}{k!} \left(\frac{a}{x} \right)^k - e^{-\frac{a}{u+R}} \sum_{k=0}^{p-2} \frac{1}{k!} \left(\frac{a}{u+R} \right)^k \right]^{n-2} \cdot u^{-p} e^{-\frac{a}{u}} (u+R)^{-p} e^{-\frac{a}{u+R}} du.$$

So, the assumption of integral p does not greatly simplify the problem.

The asymptotic distribution given by equation (12) might be of benefit in establishing probability levels of the range for large values of n . Here,

$$(34) \quad w = \frac{(n-1)a^{p-1}}{\Gamma(p-1)} \int_0^R e^{-\frac{a}{x}} x^{-p} dx,$$

or using the transformation $z = \frac{1}{x}$,

$$(35) \quad w = \frac{(n-1)a^{p-1}}{\Gamma(p-1)} \int_{1/R}^\infty e^{-az} z^{p-2} dz \\ = (n-1) \left\{ 1 - \frac{a^{p-1}}{\Gamma(p-1)} \int_0^{1/R} e^{-az} z^{p-2} dz \right\}.$$

Now for fixed R , the value of this integral can be found from tables of the

incomplete gamma function. In this way, a set of values, w_i , corresponding to values, R_i , can be found, and these could be used to tabulate probability levels. For example,

$$(36) \quad P \{ w < w_0 \} = P \{ F(R) < F(R_0) \} = 1 - e^{-w_0}.$$

For a specified value of $F(R_0)$, R_0 can be found if the parameters of the original distribution are known. Hence, the above corresponds to the probability that R is less than R_0 .

F. Type VI.

When considering the Type VI distribution,

$$(37) \quad f(x) = \frac{a^{p-q-1}}{B(p-q-1, q+1)} x^{-p}(x-a)^q, \quad a \leq x < \infty, \quad p > q-1,$$

the integral expression for the distribution of the range is

$$(38) \quad g(R) = \frac{n(n-1)a^{n(p-q-1)}}{B(p-q-1, q+1)^n} \int_a^\infty \left[\int_u^{u+R} x^{-p}(x-a)^q dx \right]^{n-2} u^{-p}(u+R)^{-p}(u-a)^q(u+R-a)^q du.$$

Since the transformation $y = a/x$ transforms the Type VI density function into a beta, or Type I density function, the difficulties encountered in the consideration of Type I again confront us, and no solution is available.

This is, however, another distribution that is bounded to the left and unlimited to the right, so the asymptotic distribution specified by equation (12) may be invoked. In this case,

$$(39) \quad w = \frac{(n-1)a^{p-q-1}}{B(p-q-1, q+1)} \int_a^{a+R} x^{-p}(x-a)^q dx,$$

or with the transformation $y = \frac{a}{x}$,

$$\begin{aligned}
 (40) \quad w &= \frac{n-1}{B(p-q-1, q+1)} \int_{\frac{a}{a+R}}^1 y^{p-q-2} (1-y)^q dy \\
 &= (n-1) \left\{ 1 - \frac{1}{B(p+1, q+1)} \int_0^{\frac{a}{a+R}} y^{p-q-2} (1-y)^q dy \right\}.
 \end{aligned}$$

If the parameters of the initial distribution are known, the integral can be evaluated for fixed $\underline{R}$, using tables of the incomplete beta function. In this way corresponding values of $\underline{w}$ and $\underline{R}$ can be obtained. As in the case of Type V, these paired values could be utilized to approximate probability levels for the distribution of $\underline{R}$. Here,

$$(41) \quad P \{ w < w_0 \} = P \{ F(R) < F(R_0) \} = 1 - e^{-w_0},$$

and since $F(R_0)$ will determine $\underline{R}_0$, this will correspond to $P \{ R < R_0 \}$.

G. Type VII.

For a Type VII function,

$$(42) \quad f(x) = \frac{1}{aB(\frac{1}{2}, m-\frac{1}{2})} \left(1 + \frac{x^2}{a^2}\right)^{-m}, \quad -\infty < x < \infty,$$

the distribution of the range is

$$(43) \quad g(R) = \frac{1}{a^n B(\frac{1}{2}, m-\frac{1}{2})^n} \int_{-\infty}^{\infty} \left[\int_u^{u+R} \left(1 + \frac{x^2}{a^2}\right)^{-m} dx \right]^{n-2} \left(1 + \frac{u^2}{a^2}\right)^{-m} \left(1 + \frac{u+R}{a^2}\right)^{-m} du$$

This integral presents difficulties similar to those of Type I and Type VI.

Since an explicit expression for $g(R)$ is not readily obtained, sampling experiments have been performed to give an empirical approximation to the

distribution. The results, due to E. S. Pearson and N. K. Adyanthaya, are available in Tables for Statisticians and Biometricians, Part II (14).

Two distinct Type VII populations, with different degrees of peakedness, were sampled. The measures of peakedness were $\beta_2 = 4.12$, and $\beta_2 = 7.07$. The same distribution constants given for the distributions of the range when sampling from Type II and Type III populations are tabled, again for samples of size 2, 5, 10 and 20. The results indicate a distribution considerably different than the distribution of range in samples from a normal population. The mean range does not differ greatly from that of a normal population but the standard error increases with the value of β_2 . The values of β_1 and β_2 show a distribution much more skewed and peaked than is the distribution of range in normal samples.

H. Type VIII.

The distribution of the range in samples of size n from the Type VIII population,

$$(44) \quad f(x) = \frac{1-m}{a} \left(1 + \frac{x}{a}\right)^{-m}, \quad 0 < m < 1, \quad -a \leq x \leq 0,$$

is given by

$$(45) \quad g(R) = n(n-1) \left(\frac{1-m}{a}\right)^2 \int_{-a}^{-R} \left[\left(1 + \frac{u+R}{a}\right)^{1-m} - \left(1 + \frac{u}{a}\right)^{1-m} \right]^{n-2} \cdot$$

$$\cdot \left(1 + \frac{u}{a}\right)^{-m} \left(1 + \frac{u+R}{a}\right)^{-m} du, \quad 0 \leq R \leq a, \quad 0 < m < 1.$$

Or, using a binomial expansion,

$$\begin{aligned}
 (46) \quad g(R) &= \frac{n(n-1)(1-m)^2}{a^{n(1-m)}} \int_{-a}^{-R} \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} (a+u+R)^{(1-m)(n-2-k)-m} (a+u)^{k(1-m)-m} du \\
 &= \frac{n(n-1)(1-m)^2}{a^{n(1-m)}} \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \int_{-a}^{-R} (a+R+u)^{(1-m)(n-2-k)-m} (a+u)^{k(1-m)-m} du.
 \end{aligned}$$

Since m is not integral, successive integrations by parts will not yield a finite number of terms. However, it will lead to a convergent infinite series. Repeated integrations by parts gives

$$\begin{aligned}
 (47) \quad \int_{-a}^{-R} \varphi(u, R) du &= \int_{-a}^{-R} (a+R+u)^{(1-m)(n-2-k)-m} (a+u)^{k(1-m)-m} du \\
 &= \left\{ (a+R+u)^{(1-m)(n-2-k)-m} \frac{(a+u)^{(k+1)(1-m)}}{(k+1)(1-m)} \right. \\
 &\quad - [(1-m)(n-2-k)-m] (a+R+u)^{(1-m)(n-2-k)-m-1} \\
 &\quad \cdot \frac{(a+u)^{(k+1)(1-m)+1}}{[(k+1)(1-m)][(k+1)(1-m)+1]} + \dots + \\
 &\quad + (-1)^j \frac{\Gamma[(1-m)(n-1-k)]}{\Gamma[(1-m)(n-1-k)-j]} \cdot \frac{\Gamma[(k+1)(1-m)]}{\Gamma[(k+1)(1-m)+j+1]} \\
 &\quad \cdot (a+R+u)^{(1-m)(n-1-k)-j-1} (a+u)^{(k+1)(1-m)+j} \pm \\
 &\quad \left. \dots \right\}_{-a}^{-R}
 \end{aligned}$$

This expansion is one of $n-2$, dependent upon the value of $\underline{k}$. Now for certain rational $\underline{m}$, one or more of these expansions will terminate for a finite $\underline{j}$. For, if the denominator of $\underline{m}$ (when $\underline{m}$ is reduced) is not larger than $n-1$, the exponent of $(a+R+u)$ may vanish for certain $\underline{k}$. If we set this exponent equal to zero, we have

$$(48) \quad n-2-k-m(n-1-k)-j = 0$$

$$(49) \quad m = \frac{n-2-k-j}{n-1-k}, \quad k = 0, 1, 2, \dots, n-2, \quad j = 0, 1, 2, \dots$$

As an example, if $m = 2/3$, and samples of size 12 are taken, we will have a finite number of terms when $k = 2$, $k = 5$, and $k = 8$. For these values of $\underline{k}$, $\underline{j}$ -values will be 2, 1, and 0, and the number of terms will be 3, 2, and 1, respectively.

However, if $\underline{m}$ is irrational, or if it is rational, but with a denominator greater than $n-1$, then for each value of $\underline{k}$, we will have a non-terminating series. The integral is then

$$(50) \quad \int_{-a}^{-R} \psi(u, R) du = \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma[(1-m)(n-1-k)]}{\Gamma[(1-m)(n-1-k)-j]} \cdot \frac{\Gamma[(k+1)(1-m)]}{\Gamma[(k+1)(1-m)+j+1]} \cdot \left. (a+R+u)^{(1-m)(n-1-k)-j-1} (a+u)^{(k+1)(1-m)+j} \right\}_{u=-a}^{u=-R}$$

$$= \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma[(1-m)(n-1-k)]}{\Gamma[(1-m)(n-1-k)-j]} \cdot \frac{\Gamma[(k+1)(1-m)]}{\Gamma[(k+1)(1-m)+j+1]} \cdot a^{(1-m)(n-1-k)-j-1} (a-R)^{(k+1)(1-m)+j}$$

The distribution of the range is then

$$(51) \quad g(R) = \frac{n(n-1)(1-m)^2}{a} \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \left(1 - \frac{R}{a}\right)^{(k+1)(1-m)} \cdot \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma[(1-m)(n-1-k)]}{\Gamma[(1-m)(n-1-k)-j]} \cdot \frac{\Gamma[(k+1)(1-m)]}{\Gamma[(k+1)(1-m)+j+1]} \left(1 - \frac{R}{a}\right)^j, \quad 0 \leq R \leq a.$$

We can also obtain the cumulative distribution.

$$(52) \quad G(R) = n(n-1)(1-m)^2 \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma[(1-m)(n-1-k)]}{\Gamma[(1-m)(n-1-k)-j]} \cdot \frac{\Gamma[(k+1)(1-m)]}{\Gamma[(k+1)(1-m)+j+1]} \int_0^R \left(1 - \frac{R}{a}\right)^{(k+1)(1-m)+j} \frac{dR}{a} \\ = n(n-1)(1-m)^2 \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma[(1-m)(n-1-k)]}{\Gamma[(1-m)(n-1-k)-j]} \cdot \frac{\Gamma[(k+1)(1-m)]}{\Gamma[(k+1)(1-m)+j+1]} \left\{ \frac{1 - \left(1 - \frac{R}{a}\right)^{(k+1)(1-m)+j+1}}{(k+1)(1-m)+j+1} \right\} \\ = n(n-1)(1-m)^2 \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma[(1-m)(n-1-k)]}{\Gamma[(1-m)(n-1-k)-j]} \cdot \frac{\Gamma[(k+1)(1-m)]}{\Gamma[(k+1)(1-m)+j+2]} \left\{ 1 - \left(1 - \frac{R}{a}\right)^{(k+1)(1-m)+j+1} \right\}$$

The moments about the origin for the distribution of range can be obtained directly

$$(53) \quad \mu_r' = \int_0^a R^r g(R) dR, \quad \underline{r} \text{ a positive integer}$$

$$(54) \quad \mu_r' = n(n-1)(1-m)^2 \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma[(1-m)(n-1-k)]}{\Gamma[(1-m)(n-1-k)-j]} \cdot \frac{\Gamma[(k+1)(1-m)]}{\Gamma[(k+1)(1-m)+j+1]} \int_0^a R^r \left(1 - \frac{R}{a}\right)^{(k+1)(1-m)+j} \frac{dR}{a}.$$

With the substitution $R = at$,

$$(55) \quad \mu_r' = n(n-1)(1-m)^2 \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma[(1-m)(n-1-k)]}{\Gamma[(1-m)(n-1-k)-j]} \cdot \frac{\Gamma[(k+1)(1-m)]}{\Gamma[(k+1)(1-m)+j+1]} a^r \int_0^1 t^r (1-t)^{(k+1)(1-m)+j} dt$$

$$= n(n-1)(1-m)^2 a^r \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma[(1-m)(n-1-k)]}{\Gamma[(1-m)(n-1-k)-j]} \cdot \frac{\Gamma[(k+1)(1-m)]}{\Gamma[(k+1)(1-m)+j+1]} \cdot B[r+1, (k+1)(1-m)+j+1].$$

I. Type IX.

For samples of size n from a Type IX population,

$$(56) \quad f(x) = \frac{m+1}{a} \left(1 + \frac{x}{a}\right)^m, \quad m > 0, \quad -a \leq x \leq 0,$$

the distribution of the range is

$$(57) \quad g(R) = n(n-1) \left(\frac{m+1}{a}\right)^2 \int_{-a}^{-R} \left[\left(1 + \frac{u+R}{a}\right)^{m+1} - \left(1 + \frac{u}{a}\right)^{m+1} \right]^{n-2} \\ \cdot \left(1 + \frac{u}{a}\right)^m \left(1 + \frac{u+R}{a}\right)^m du \\ = n(n-1) \left(\frac{m+1}{a}\right)^2 \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \int_{-a}^{-R} \left(1 + \frac{u}{a} + \frac{R}{a}\right)^{(m+1)(n-2-k)+m} \\ \cdot \left(1 + \frac{u}{a}\right)^{k(m+1)+m} du, \quad 0 \leq R \leq a.$$

If m is integral, this can be evaluated exactly. Expanding the first quantity under the integral sign as a binomial, we have

$$\begin{aligned}
 (58) \quad g(R) &= n(n-1) \frac{(m+1)^2}{a} \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \sum_{j=0}^{(m+1)(n-2-k)+m} \binom{(m+1)(n-2-k)+m}{j} \\
 &\quad \cdot \left(\frac{R}{a}\right)^j \int_{-a}^{-R} \left(1 + \frac{u}{a}\right)^{n(m+1)-2-j} \frac{du}{a} \\
 &= n(n-1) \frac{(m+1)^2}{a} \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \sum_{j=0}^{(m+1)(n-2-k)+m} \binom{(m+1)(n-2-k)+m}{j} \\
 &\quad \cdot \left(\frac{R}{a}\right)^j \frac{\left(1 - \frac{R}{a}\right)^{n(m+1)-1-j}}{n(m+1)-1-j} \\
 &= \frac{n(n-1)(m+1)^2}{a} \left(1 - \frac{R}{a}\right)^{n(m+1)-1} \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \cdot \\
 &\quad \sum_{j=0}^{(m+1)(n-2-k)+m} \binom{(m+1)(n-2-k)+m}{j} \left(\frac{R}{a-R}\right)^j \cdot \frac{1}{n(m+1)-1-j} .
 \end{aligned}$$

We can also find the cumulative distribution of the range, for

$$\begin{aligned}
 (59) \quad G(R) &= n(n-1)(m+1) \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \sum_{j=0}^{(m+1)(n-2-k)+m} \binom{(m+1)(n-2-k)+m}{j} \\
 &\quad \cdot \int_0^R \left(\frac{R}{a}\right)^j \frac{\left(1 - \frac{R}{a}\right)^{n(m+1)-1-j}}{n(m+1)-1-j} \frac{dR}{a} .
 \end{aligned}$$

Now let $t = \frac{R}{a}$. Then

$$(60) \quad G(R) = n(n-1)(m+1)^2 \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \sum_{j=0}^{(m+1)(n-2-k)+m} \binom{(m+1)(n-2-k)+m}{j} \frac{1}{n(m+1)-1-j} \cdot$$

$$\cdot \int_0^{R/a} t^j (1-t)^{n(m+1)-1-j} dt$$

This integral is an incomplete beta function, and can be evaluated as such for any particular value of R . Or, since n , m , and j are all integers, it may be evaluated by successive integrations by parts. Then,

$$(61) \quad G(R) = n(n-1)(m+1)^2 \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \sum_{j=0}^{(m+1)(n-2-k)+m} \binom{(m+1)(n-2-k)+m}{j} \cdot$$

$$\cdot \frac{1}{n(m+1)-1-j} \sum_{p=0}^{n(m+1)-1-j} \frac{j!}{(j+1+p)!} \frac{[n(m+1)-1-j]!}{[n(m+1)-1-j-p]!} \left(\frac{R}{a}\right)^{j+1+p} \cdot$$

$$\cdot \left(1 - \frac{R}{a}\right)^{n(m+1)-1-j-p}$$

The moments for the distribution of the range can be found directly:

$$(62) \mu_r' = \frac{n(n-1)(m+1)^2}{a} \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \sum_{j=0}^{(m+1)(n-2-k)+m} \binom{(m+1)(n-2-k)+m}{j} \frac{1}{n(m+1)-1-j}.$$

$$\cdot \int_0^a R^r \left(\frac{R}{a}\right)^j \left(1 - \frac{R}{a}\right)^{n(m+1)-1-j} dR$$

$$= n(n-1)(m+1)^2 \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \sum_{j=0}^{(m+1)(n-2-k)+m} \binom{(m+1)(n-2-k)+m}{j} \frac{a^r}{n(m+1)-1-j}.$$

$$\cdot \int_0^a \left(\frac{R}{a}\right)^{r+j} \left(1 - \frac{R}{a}\right)^{n(m+1)-1-j} \frac{dR}{a}$$

$$= n(n-1)(m+1)^2 \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \sum_{j=0}^{(m+1)(n-2-k)+m} \binom{(m+1)(n-2-k)+m}{j} \frac{a^r}{n(m+1)-1-j}.$$

$$\cdot B [r+j+1, n(m+1)-j].$$

Since $\underline{r}$, $\underline{j}$, $\underline{n}$, and $\underline{m}$ are integers, the beta function can be replaced by factorials, and the $\underline{r}^{\text{th}}$ movement about the origin is

$$\begin{aligned}
 (63) \mu'_r &= n(n-1)(m+1)^2 \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \sum_{j=0}^{(m+1)(n-2-k)+m} \frac{\binom{(m+1)(n-2-k)+m}{j} a^r}{n(m+1)-1-j} \\
 &\quad \cdot \frac{(r+j)! [n(m+1)-1-j]!}{[n(m+1)+r]!} \\
 &= n(n-1)(m+1)^2 a^r \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \sum_{j=0}^{(m+1)(n-2-k)+m} \frac{\binom{(m+1)(n-2-k)+m}{j}}{[n(m+1)+r]!} \\
 &\quad \cdot \frac{(r+j)! [n(m+1)-2-j]!}{[n(m+1)+r]!}
 \end{aligned}$$

J. Type X.

The Type X density function is

$$(64) \quad f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad \theta > 0, \quad x \geq 0.$$

The distribution of range in samples from the Type X population has been investigated, and Link (8) has given the explicit distribution of the ratio of two ranges from the standardized form; i.e., $f(x) = e^{-x}$. So, obviously the distribution of the range has been previously obtained, but seemingly is not mentioned in the literature. It can be found by direct evaluation of the integral representation of $g(R)$.

$$\begin{aligned}
 (65) \quad g(R) &= \frac{n(n-1)}{\theta^2} \int_0^{\infty} \left[e^{-\frac{u}{\theta}} - e^{-\frac{u+R}{\theta}} \right]^{n-2} e^{-\frac{u}{\theta}} e^{-\frac{u+R}{\theta}} du, \quad 0 \leq R < \infty \\
 &= \frac{n(n-1)}{\theta^2} e^{-\frac{R}{\theta}} \left[1 - e^{-\frac{R}{\theta}} \right]^{n-2} \int_0^{\infty} e^{-\frac{nu}{\theta}} d\theta \\
 &= \frac{n-1}{\theta} e^{-\frac{R}{\theta}} \left[1 - e^{-\frac{R}{\theta}} \right]^{n-2}.
 \end{aligned}$$

The cumulative distribution of the range is

$$\begin{aligned}
 (66) \quad G(R) &= \frac{n-1}{\theta} \int_0^R \left[1 - e^{-\frac{R}{\theta}} \right]^{n-2} e^{-\frac{R}{\theta}} d\theta \\
 &= \left[1 - e^{-\frac{R}{\theta}} \right]^{n-1}.
 \end{aligned}$$

The simplest method of obtaining the moments of the distribution is by the use of the moment-generating function.

$$\begin{aligned}
(67) \quad m_R(t) &= E(e^{tR}) = \frac{n-1}{\theta} \int_0^\infty \left[1 - e^{-\frac{R}{\theta}}\right]^{n-2} e^{-\frac{R}{\theta}} e^{tR} dR \\
&= \frac{n-1}{\theta} \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \int_0^\infty e^{-\frac{kR}{\theta}} e^{-\frac{R}{\theta}} e^{tR} dR \\
&= \frac{n-1}{\theta} \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \int_0^\infty e^{-\frac{R}{\theta}(k+1-\theta t)} dR \\
&= (n-1) \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \frac{1}{k+1-\theta t} \left[-e^{-\frac{R}{\theta}(k+1-\theta t)} \right]_0^\infty \\
&= (n-1) \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{(-1)^k}{k+1-\theta t}, \quad t < \frac{k+1}{\theta}.
\end{aligned}$$

Then,

$$\begin{aligned}
(68) \quad \mu_r' &= \frac{d^r}{dt^r} m_R(0) = \frac{d^r}{dt^r} \left\{ (n-1) \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{(-1)^k}{k+1-\theta t} \right\}_{t=0} \\
&= (n-1) \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{(-1)^k r! \theta^r}{(k+1)^r} \\
&= (n-1) r! \theta^r \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{(-1)^k}{(k+1)^r}
\end{aligned}$$

K. Type XI.

The functional form for a Type XI distribution is

$$(69) \quad f(x) = b^{m-1}(m-1)x^{-m}, \quad b \leq x < \infty, \quad m > 1.$$

The density function for range in samples from a Type XI population is

$$(70) \quad g(R) = n(n-1)b^{2m-2}(m-1)^2 \int_b^\infty \left[\left(\frac{b}{u} \right)^{m-1} - \left(\frac{b}{u+R} \right)^{m-1} \right]^{n-2} \frac{du}{u^m(u+R)^m}, \quad R \geq 0,$$

$$= n(n-1)b^{n(m-1)}(m-1)^2 \int_b^\infty \left[\frac{1}{u^{m-1}} - \frac{1}{(u+R)^{m-1}} \right]^{n-2} \frac{du}{u^m(u+R)^m}.$$

In this integral, we make the transformation $y = \frac{R}{u}$. Then

$$(71) \quad g(R) = n(n-1)b^{n(m-1)}(m-1)^2 \int_0^{R/b} \left[\left(\frac{y}{R} \right)^{m-1} - \frac{y^{m-1}}{R^{m-1}(1+y)^{m-1}} \right]^{n-2} \cdot \frac{y^m}{R^m} \cdot \frac{y^m}{R^m(1+y)^m} \cdot \frac{Rdy}{y^2}$$

$$= \frac{n(n-1)b^{n(m-1)}(m-1)^2}{R^{n(m-1)+1}} \int_0^{R/b} \left[1 - \frac{1}{(1+y)^{m-1}} \right]^{n-2} \frac{y^{n(m-1)}}{(1+y)^m} dy$$

$$= \frac{n(n-1)b^{n(m-1)}(m-1)^2}{R^{n(m-1)+1}} \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \int_0^{R/b} \frac{y^{n(m-1)}}{(1+y)^{k(m-1)+m}} dy.$$

Recalling that $B(r,s) = \int_0^{\infty} \frac{t^{r-1}}{(1+t)^{r+s}} dt$, we see that the integral above

may be evaluated through the use of tables of the incomplete beta function.

To put this integral in the more familiar form, we employ the change of

variable $y = \frac{t}{1+t}$. Then

$$(72) \quad g(R) = \frac{n(n-1)b^{n(m-1)}(m-1)^2}{R^{n(m-1)+1}} \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \int_0^{\frac{R}{b+R}} t^{n(m-1)} (1-t)^{-(n-k-1)(m-1)-1} dt$$

Now if $(n-k-1)(m-1)$ is not an integer, the integral is the incomplete $B [n(m-1)-1, -(n-k-1)(m-1)-2]$, and with the use of tables we can calculate ordinates of the curve. The condition that $(n-k-1)(m-1)$ must not be an integer is certainly fulfilled if m is irrational, or if rational that its denominator is necessarily greater than $n-2$.

Percentage points of the cumulative distribution may be found in the same manner. Using Gumbel's form for the cumulative distribution,

$$(73) \quad G(R) = nb^{m-1}(m-1) \int_b^{\infty} \left[\left(\frac{b}{u} \right)^{m-1} - \left(\frac{b}{u+R} \right)^{m-1} \right]^{n-1} \frac{du}{u^m}.$$

Using the substitution previously applied, $y = \frac{R}{u}$, we have

$$\begin{aligned}
 (74) \quad G(R) &= \frac{n b^{n(m-1)} (m-1)}{R^{n(m-1)}} \int_0^{R/b} \left[1 - \frac{1}{(1+y)^{m-1}} \right]^{n-1} y^{n(m-1)-1} dy \\
 &= \frac{n(m-1)b^{n(m-1)}}{R^{n(m-1)}} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \int_0^{R/b} \frac{y^{n(m-1)-1}}{(1+y)^{k(m-1)}} dy.
 \end{aligned}$$

Again, this integral is an incomplete beta function, and may be put in the standard form. Then,

$$(75) \quad G(R) = \frac{n(m-1)b^{n(m-1)}}{R^{n(m-1)}} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \int_0^{\frac{R}{b+R}} t^{n(m-1)-1} (1-t)^{-(n-k)(m-1)-1} dt.$$

The moments of the distribution can be found in certain cases.

$$\begin{aligned}
 (76) \quad \mu'_r &= n(n-1)b^{n(m-1)} (m-1)^2 \int_0^\infty R^r dR \int_b^\infty \left[\frac{1}{u^{m-1}} - \frac{1}{(u+R)^{m-1}} \right]^{n-2} \frac{du}{u^m (u+R)^m} \\
 &= n(n-1)b^{n(m-1)} (m-1)^2 \int_0^\infty R^r dR \int_b^\infty \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \cdot \\
 &\quad \cdot \frac{du}{u^{(n-2-k)(m-1)+m} (u+R)^{k(m-1)+m}}
 \end{aligned}$$

Since the inner integral converges uniformly, we may interchange order of

integration. Then,

$$(77) \mu_r' = n(n-1)b^{n(m-1)}(m-1)^2 \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \int_b^{\infty} \frac{du}{u^{(n-2-k)(m-1)+m}} \cdot \int_0^{\infty} \frac{R^r dR}{(u+R)^{k(m-1)+m}}$$

If $r < k(m-1)+m-1$, the second integral will exist. But since k can take on the value zero, this implies $r < m-1$. Assuming this condition is fulfilled, we may evaluate this integral through repeated integrations by parts:

$$(78) \int_0^{\infty} \frac{R^r dR}{(u+R)^{k(m-1)+m}} = \left\{ R^r \frac{(u+R)^{-k(m-1)-m+1}}{-k(m-1)-m+1} + r R^{r-1} \cdot \frac{(u+R)^{-(k+1)(m-1)-m+1}}{[(k+1)(m-1)] [(k+1)(m-1)-1]} + \dots \right. \\ \left. + (-1)^r r! \frac{(u+R)^{-(k+1)(m-1)+r}}{(-1)^{r+1} [(k+1)(m-1)] [(k+1)(m-1)-1] \dots [(k+1)(m-1)-r]} \right\} \Bigg|_{R=0}^{\infty}$$

Due to our restriction on r , every term vanishes at the upper limit. At the lower limit, every term vanishes except the last, and we have

$$(79) \int_0^{\infty} \frac{R^r dR}{(u+R)^{k(m-1)+m}} = \frac{r! \Gamma[(k+1)(m-1)-r]}{\Gamma[(k+1)(m-1)+1]} \cdot \frac{1}{u^{(k+1)(m-1)-r}}$$

Then,

$$\begin{aligned}
 (80) \quad \mu_r' &= n(n-1)b^{n(m-1)}(m-1)^2 \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \frac{r! \Gamma[(k+1)(m-1)-r]}{\Gamma[(k+1)(m-1)+1]} \\
 &\quad \cdot \int_b^{\infty} \frac{du}{u^{(n-2-k)(m-1)+m+(k+1)(m-1)-r}} \\
 &= n(n-1)b^{n(m-1)}(m-1)^2 r! \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \frac{\Gamma[(k+1)(m-1)-r]}{\Gamma[(k+1)(m-1)+1]} \\
 &\quad \cdot \left\{ \frac{-1}{[n(m-1)-r] u^{(m-1)n-r}} \right\}_{u=b}^{\infty} \\
 &= n(n-1)b^{n(m-1)}(m-1)^2 r! \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \frac{\Gamma[(k+1)(m-1)-r]}{\Gamma[(k+1)(m-1)+1]} \\
 &\quad \cdot \frac{1}{[n(m-1)-r] b^{n(m-1)-r}} \\
 &= \frac{n(n-1) b^r (m-1)^2 r!}{n(m-1)-r} \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \frac{\Gamma[(k+1)(m-1)-r]}{\Gamma[(k+1)(m-1)+1]} .
 \end{aligned}$$

An interesting feature of this result is that it holds true if $\underline{m}$ is integral, so long as the original condition on $\underline{r}$ is maintained. Thus, although we were unable to obtain the distribution for this case, we have the moments, and for particular values of $\underline{m}$ and $\underline{n}$, we could approximate the distribution with the first four correct moments.

One rather trivial situation leads to an exact distribution. If $\underline{m}$ is a rational number of the form $\frac{p+1}{p}$, with p an integer greater than 1, we can utilize equation (8) to advantage. The cumulative distribution is

$$(81) \quad G(R) = n(m-1)b^{n(m-1)} \int_b^{\infty} \left[\frac{1}{u^{m-1}} - \frac{1}{(u+R)^{m-1}} \right]^{n-1} \cdot \frac{du}{(u+R)^m} + \left[1 - \left(\frac{b}{b+R} \right)^{m-1} \right]^n.$$

Using the transformation $y = \frac{R}{u}$,

$$(82) \quad G(R) = \frac{n(m-1)b^{n(m-1)}}{R^{n(m-1)}} \int_0^{R/b} \left[1 - \frac{1}{(y+1)^{m-1}} \right]^{n-1} \cdot \frac{y^{n(m-1)-1}}{(y+1)^m} dy + \left[1 - \left(\frac{b}{b+R} \right)^{m-1} \right]^n.$$

Now if $m = \frac{p+1}{p}$, and we choose $n = p = \frac{1}{m-1}$, the integral can be evaluated directly:

$$\begin{aligned}
 (83) \quad G(R) &= \frac{b}{R} \int_0^{R/b} \left[1 - \frac{1}{(y+1)^{\frac{1}{n}}} \right]^{n-1} \frac{dy}{(y+1)^{\frac{1}{n}+1}} + \left[1 - \left(\frac{b}{b+R} \right)^{\frac{1}{n}} \right]^n \\
 &= \frac{b}{R} \left\{ \left[1 - \frac{1}{(y+1)^{\frac{1}{n}}} \right]^n \right\}_0^{R/b} + \left[1 - \left(\frac{b}{b+R} \right)^{\frac{1}{n}} \right]^n \\
 &= \left(\frac{b}{R} + 1 \right) \left[1 - \left(\frac{b}{b+R} \right)^{\frac{1}{n}} \right]^n = \left[\left(\frac{b+R}{R} \right)^{\frac{1}{n}} - \left(\frac{b}{R} \right)^{\frac{1}{n}} \right]^n.
 \end{aligned}$$

The density function can be found now by differentiation:

$$\begin{aligned}
 (84) \quad g(R) &= n \left[\left(\frac{b+R}{R} \right)^{\frac{1}{n}} - \left(\frac{b}{R} \right)^{\frac{1}{n}} \right]^{n-1} \left[\frac{1}{n} \left(\frac{b+R}{R} \right)^{\frac{1}{n}-1} \left(-\frac{b}{R^2} \right) - \frac{1}{n} \left(\frac{b}{R} \right)^{\frac{1}{n}-1} \left(-\frac{b}{R^2} \right) \right] \\
 &= \left[\left(\frac{b+R}{R} \right)^{\frac{1}{n}} - \left(\frac{b}{R} \right)^{\frac{1}{n}} \right] \left[\frac{\frac{1}{n}}{R^{\frac{1}{n}+1}} - \frac{b(b+R)^{\frac{1}{n}-1}}{R^{\frac{1}{n}+1}} \right].
 \end{aligned}$$

L. Type XII.

For the Type XII population given by

$$(85) \quad f(x) = \left(\frac{a}{b} \right)^m \frac{1}{(a+b)B(1+m, 1-m)} \left(\frac{1+x/a}{1-x/b} \right)^m, \quad -a \leq x \leq b \quad |m| < 1,$$

the distribution of sample range is

$$(86) \quad g(R) = \left(\frac{a}{b}\right)^{nm} \frac{n(n-1)}{(a+b)^n [B(1+m, 1-m)]^n} \cdot \int_a^{b-R} \left[\int_u^{u+R} \left(\frac{1 + \frac{x}{a}}{1 - \frac{x}{b}} \right)^m dx \right]^{n-2} \left(\frac{1 + \frac{u}{a}}{1 - \frac{u}{b}} \right)^m \left(\frac{1 + \frac{u+R}{a}}{1 - \frac{u+R}{b}} \right)^m du.$$

The Type XII distribution is a particular case of Type I. The specialization of parameters does not lead to any simplification of the difficulties encountered in the earlier case, and an explicit form for $g(R)$ has not been obtained.

IV. SUMMARY AND CONCLUSIONS

This thesis has been concerned with the distribution of the range in samples of size n from Pearson-type populations. Each of the twelve types has been considered, and, insofar as possible, prior research has been summarized briefly.

The following results have been obtained:

1. An asymptotic distribution of a simple function of the largest sample value has been derived. This distribution may be used to approximate probabilities of the range in large samples from a population bounded below and unlimited above. This result was considered in connection with Types III, V, and VI.
2. The distribution of the range in samples from a Type VIII population was developed as an infinite series. The cumulative distribution and the moments were also obtained in the form of infinite series.
3. The distribution of the range in samples from a Type IX population was obtained with a restriction on one of the original parameters. The cumulative function and the moments were given.
4. For a Type X population, the distribution of the sample range was obtained in a usable form, as was the cumulative distribution of the range. All moments of the distribution exist, and the moment-generating function was obtained.

5. With certain restrictions on one of the parameters, the distribution of the range in samples from a Type XI parent was developed as a sum of incomplete beta functions. The cumulative distribution and certain moments were found under similar restrictions. In a very special case, an explicit distribution was found.

In the cases where a distribution function of the range has been obtained, considerable numerical analysis would be necessary if the results were to be applied in a practical situation.

In cases where a density function for the range has not been obtained, numerical integration is indicated. Further research will probably take the form of extensive sampling to determine empirical distributions as has been done with Types II, III, and VII.

V. APPENDIX

In the consideration of Type VIII, it was necessary to assume uniform convergence of the infinite series to obtain the cumulative distribution of the range. To justify this assumption, consider Weierstrass' M-test for an infinite series:

$$1. |u_k(x)| \leq M_k \quad a \leq x \leq b, \quad k = 1, 2, \dots$$

$$2. \sum_{k=0}^{\infty} M_k \text{ converges}$$

$$\rightarrow \sum_{k=0}^{\infty} u_k(x) \text{ converges uniformly in } a \leq x \leq b.$$

For the series in question,

$$u_j(R) = (-1)^j \frac{\Gamma[(1-m)(n-1-k)]}{\Gamma[(1-m)(n-1-k)-j]} \cdot \frac{\Gamma[(k+1)(1-m)]}{\Gamma[(k+1)(1-m)+j+1]} \left(1 - \frac{R}{a}\right)^j, \quad 0 \leq R \leq a.$$

It is easily seen that the series converges uniformly in the interval

$0 < \epsilon \leq x \leq a$, regardless of how small ϵ may be. For, taking

$$M_j = \left| \frac{\Gamma[(1-m)(n-1-k)]}{\Gamma[(1-m)(n-1-k)-j]} \cdot \frac{\Gamma[(k+1)(1-m)]}{\Gamma[(k+1)(1-m)+j+1]} \cdot \left(1 - \frac{\epsilon}{a}\right)^j \right|$$

and applying the ratio test, we have

$$\lim_{j \rightarrow \infty} \frac{M_{j+1}}{M_j} = \lim_{j \rightarrow \infty} \left| \frac{(1-m)(n-1-k)-j-1}{(k+1)(1-m)+j+1} \cdot \left(1 - \frac{\epsilon}{a}\right) \right| = 1 - \frac{\epsilon}{a} < 1.$$

The ratio test fails at $R = 0$, however, for there the limiting ratio is one. To establish uniform convergence at $R = 0$, we require a more subtle

test, such as Raabe's. For Raabe's Test, if all u_k are positive, and

$$\lim_{k \rightarrow \infty} k \left(\frac{u_k}{u_{k+1}} - 1 \right) > 1, \text{ then the series converges.}$$

In this case, the M_j are all positive, and at $R = 0$ the ratio of the j^{th} to the $(j+1)$ st term is

$$\left| \frac{j+1+(k+1)(1-m)}{(1-m)(n-1-k)-j-1} \right| = \frac{j+1+(k+1)(1-m)}{j+1-(n-1-k)(1-m)} \quad \text{for large } j.$$

Then,

$$\begin{aligned} \lim_{j \rightarrow \infty} j \left(\frac{M_j}{M_{j+1}} - 1 \right) &= \lim_{j \rightarrow \infty} j \left[\frac{j+1+(k+1)(1-m)}{j+1-(n-1-k)(1-m)} - 1 \right] \\ &= \lim_{j \rightarrow \infty} j \left[\frac{(1-m)(k+1+n-1-k)}{j+1-(n-1-k)(1-m)} \right] \\ &= n(1-m). \end{aligned}$$

Now since $0 < m < 1$, this series will converge for

$$n(1-m) > 1,$$

and the original series will converge uniformly.

Interchanging the order of integration, as applied in the examination of Type XI is valid if

$$\int_b^{\infty} g(u, R) du \quad \text{is uniformly convergent.}$$

To establish this, we may apply the Weierstrass M-Test for integrals.

The conditions and conclusion of the test are:

1. $k(u, R) \in C$
2. $M(u) \in C$
3. $|k(u, R)| \leq M(u)$

4. $\int_b^\infty M(u) du$ exists

$$\rightarrow \int_b^\infty k(u, R) du \text{ converges uniformly.}$$

$$\text{We have } k(u, R) = n(n-1)b^{n(n-1)}(n-1)^2 \left[\frac{1}{u^{n-1}} - \frac{1}{(u+R)^{n-1}} \right]^{n-2} \cdot \frac{1}{u^n(u+R)^n}$$

Now $(u+R)^n > b > 0$, so we may take

$$M(u) = n(n-1)b^{n(n-1)}(n-1)^2 \left[\frac{1}{u^{n-1}} \right]^{n-2} \cdot \frac{1}{b} \frac{1}{u^n}.$$

$$\text{Then } \int_b^\infty M(u) du = \frac{n(n-1)b^{n(n-1)}(n-1)^2}{b} \cdot \int_b^\infty \frac{du}{u^{(n-2)(n-1)+n}}$$

$$= \frac{n(n-1)b^{n(n-1)}(n-1)^2}{b^{(n-1)(n-1)+1}} = n(n-1)b^{n-2}(n-1)^2,$$

and sufficient conditions for interchange of order of integration are established.

VI. LITERATURE CITED

1. Carlton, A. G. Estimating the Parameters of a Rectangular Distribution. *Annals of Math. Stat.*, Vol. 17, pp. 355-358. 1946.
2. Cox, D. R. A Note on the Asymptotic Distribution of the Range. *Biometrika*, Vol. 35, pp. 310-315. 1948.
3. Elfving, G. The Asymptotical Distribution of Range in Samples from a Normal Population. *Biometrika*, Vol. 34, pp. 111-119. 1947.
4. Gumbel, E. J. The Distribution of the Range. *Annals of Math. Stat.*, Vol. 18, pp. 384-412. 1947.
5. ———. Probability Tables for the Range. *Biometrika*, Vol. 36, pp. 142-148. 1949.
6. Hartley, H. O. and Pearson, E. S. Moment Constants for the Distribution of the Range in Normal Samples. *Biometrika*, Vol. 38, pp. 463-464. 1951.
7. Kendall, M. G. *The Advanced Theory of Statistics*. London, Charles Griffin and Company, Ltd. 1947.
8. Link, R. F. The Sampling Distribution of the Ratio of Two Ranges from Independent Samples. *Annals of Math. Stat.*, Vol. 21, pp. 112-116. 1950.
9. May, J. M. Extended and Corrected Tables of the Upper Percentage Points of the Studentized Range. *Biometrika*, Vol. 39, pp. 192-193. 1952.
10. Patnaik, P. B. The Use of Mean Range as an Estimator of Variance in Statistical Tests. *Biometrika*, Vol. 37, pp. 78-87. 1950.
11. Pearson, E. S. and Hartley, H. O. Tables of the Probability Integral of the Studentized Range. *Biometrika*, Vol. 33, pp. 89-99. 1943.
12. Pearson, E. S. Comparison of Two Approximations to the Distribution of the Range in Small Samples from Normal Populations. *Biometrika*, Vol. 39, pp. 130-136. 1952.
13. Pearson, K. *Early Statistical Papers*. Cambridge University Press. 1948.
14. ———. Tables for Statisticians and Biometricians, Part II. Cambridge University Press.

15. _____ Tables of the Incomplete Beta Function. Cambridge
University Press. 1932.



3 1762 10014545 5

N378		112500	
In4d		cop.2	
Ingram, Glenn R			
Distribution of the sample range			
for parent populations associated			
with Pearson's differential			
equation			
NAME AND ADDRESS			
APR 16 '59	Ray Hitchcock	712 S. 7th	PHA
	R	MAY 4 '59	lg
		MAY 12 '59	MAY 19 '59
10-34-63	David Seibel		
1-31-64	120 S. Tenth		
	Gordon Chang		
	839 Custer St		
In Lib.	Frank Cesar		
	411 1st St		

N378
In4d
cop.2

112500