



Demodulation of narrow-band FM by a phase-locked loop
by Michael D Berkoff

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY in Electrical Engineering
Montana State University
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Abstract:

The subject of the thesis is the comparison of performances of the first- and second-order loops at low carrier-to-noise ratios.

It was determined in this thesis that the first-order phase-locked loop is capable of suppressing more "clicks", or FM anomalies, than a second-order loop when demodulating a narrow-band FM signal and while subject to a constraint on the size of the phase error. The phase error constraint was chosen so that the loop would tend to operate in its linear region and the FM modulation was limited to very small modulation indexes (.2 and less). The ability of the loop to suppress "clicks" was computed in conjunction with Hess' "Click" model [8]. The phase error variance of the loop was estimated by an approximate quasi-stationary linear model. An extensive computer search over the fields of all first- and second-order loops led to the conclusion of the thesis about the superiority of the first-order loop under the above-mentioned operating conditions.

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MICHAEL D. BERKOFF

A thesis submitted to the Graduate Faculty in partial
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Electrical Engineering

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$E\left[\left[\phi(t) - \mu_0\right]^2 \mid a(t) > A_0\right]$ versus A_0 [a threshold of the normalized carrier amplitude $a(t)$].
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Abstract

The subject of the thesis is the comparison of performances of the first- and second-order loops at low carrier-to-noise ratios.

It was determined in this thesis that the first-order phase-locked loop is capable of suppressing more "clicks", or FM anomalies, than a second-order loop when demodulating a narrow-band FM signal and while subject to a constraint on the size of the phase error. The phase error constraint was chosen so that the loop would tend to operate in its linear region and the FM modulation was limited to very small modulation indexes (.2 and less). The ability of the loop to suppress "clicks" was computed in conjunction with Hess' "click" model [8]. The phase error variance of the loop was estimated by an approximate quasi-stationary linear model. An extensive computer search over the field of all first-and second-order loops led to the conclusion of the thesis about the superiority of the first-order loop under the above-mentioned operating conditions.

CHAPTER 1

INTRODUCTION

1.1 THRESHOLD EXTENSION BY PHASE-LOCKED LOOPS

The use of frequency modulation (FM) at low carrier-to-noise ratios (CNR) is limited by the threshold phenomenon. The linear relationship between input and output signal-to-noise ratios breaks down, and the output signal sharply deteriorates in quality. In the conventional FM detector the threshold, according to Rice [14], is caused by frequent sweeps of the carrier-plus-noise resultant vector around the origin of the phasor plane. (See Fig. 1.1). This may only happen when the noise is larger than the resultant magnitude of the carrier.

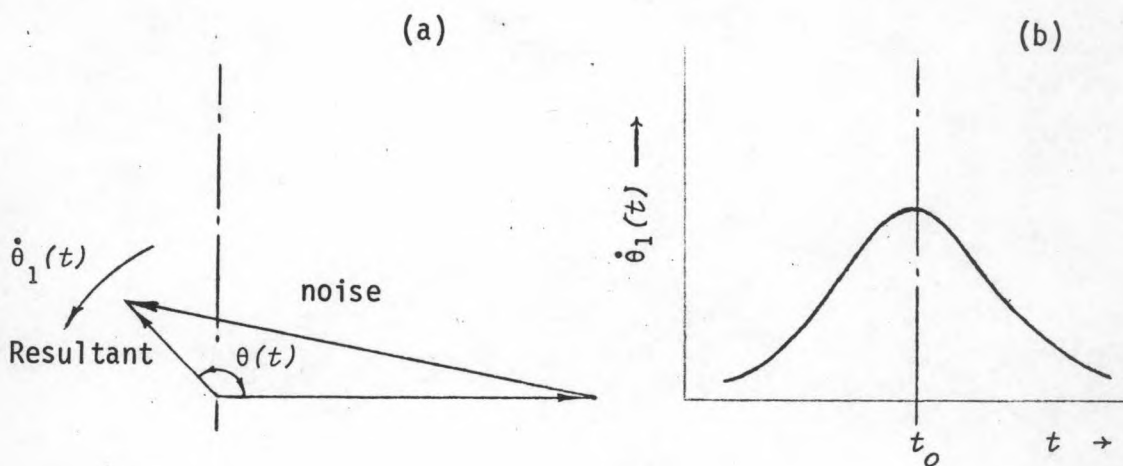


FIG. 1.1. (a) Phasor Diagram of a Sinusoid Plus Narrow-Band Noise Process During the "Click" Formation. (b) A Typical "Click".

Such a sweep around the origin produces a "click". A "click" is defined as the curve described by the angular velocity of the resultant phasor during the above mentioned sweep. The area of the "click" is equal to 2π or a multiple of it. Experimental results [8] indicate that they

are usually bell-shaped. The energy of a "click" lies mostly in the low-frequency range and thus cannot be filtered out from the demodulated signal. This threshold is greatly pronounced in cases of high modulation indexes. If FM is narrow-band (i.e., the modulation index is smaller than $\sqrt{2/3}$) the output signal-to-noise ratio (SNR) does not exhibit such a sharp break, but rather becomes proportional to CNR^2 instead of the linear relationship at higher values of CNR [11]. In either case, however, there is a substantial increase in the output noise and the lack of well-defined thresholds for narrow-band FM is explained by the large output bandwidth required to pass its demodulated signal. A large bandwidth admits more noise so that the "click" noise is somewhat masked. Conventionally the term threshold describes the sharp change in the SNR versus CNR curve. A more general meaning is given to this term in this thesis: threshold is interpreted here as the carrier-to-noise ratio below which the SNR decreases faster than the CNR, i.e., the point where the linear relationship between them ceases to exist. For a low-indexed FM signal demodulated by the FM detector, thresholds have been reported in the literature to lie in the vicinity of $\text{CNR} = 10 \text{ dB}$. [14]

There have been numerous attempts to delay the occurrence of the threshold phenomenon by means of different sorts of feedback demodulators. Phase-locked loops are often incorporated in these attempts.

There are two philosophies governing the use of the phase-locked loop (PLL) as a demodulator. One of them, widely accepted, advocates making the loop narrow in comparison to the input bandwidth. The PLL tracks the incoming carrier while the FM detector must simultaneously look at a fixed band of frequencies, thus requiring it to have a bandwidth wide enough to admit the largest possible deviation of the incoming modulated carrier. Therefore, a narrowband PLL will admit less noise than the FM detector. The phase error of a narrowband PLL, however, is large; and noise, even without the effect of "clicks", causes it to lose lock rather easily. These effects force the PLL into a nonlinear mode of operation which in turn causes distortions in the output of the loop when it is used as an FM demodulator.

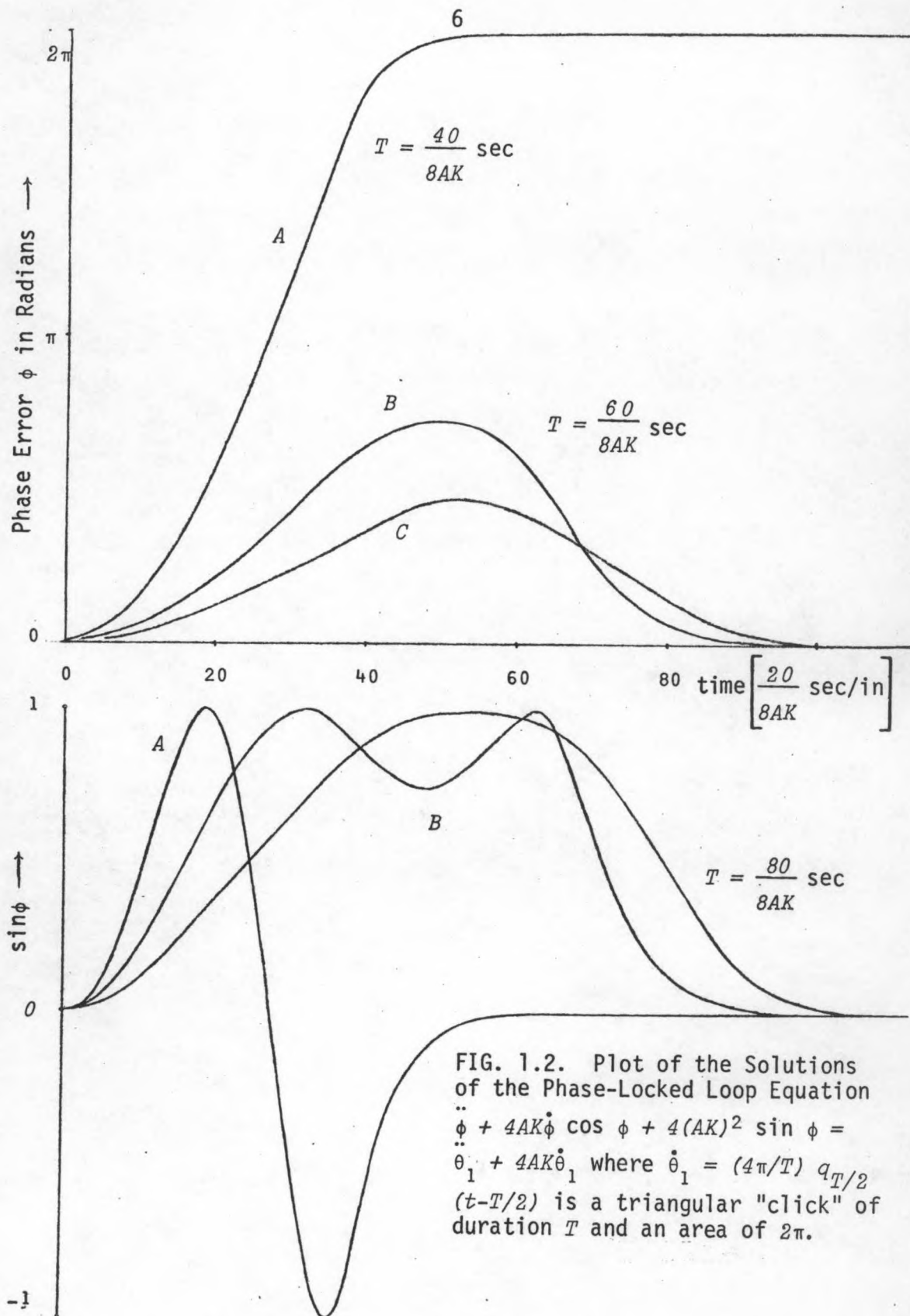
Another philosophy advocates prefiltering of the FM signal by a band-pass filter and then detecting it with a wide-band phase-locked loop. The width of the loop is determined by the desired fidelity of the demodulated signal. The band-pass filter is identical to the one which normally precedes the conventional FM detector. Therefore, one would expect this system to perform much the same as the FM detector. This indeed is the case for high carrier-to-noise ratios. At lower values of CNR, when "clicks" begin to appear in the input, some of them cause the PLL to lose lock and produce doublets in the output. A wide-band PLL has high gain and a small phase error; therefore, Gaussian noise will seldom cause loss of lock even at low carrier-to-noise ratios. When a "click" acts on a PLL, it may

or may not throw it out of lock depending on the loop parameters. Figure 1.2 depicts the effect that a "click" can have on the output of a PLL. This example assumes that there is no amplitude fading which generally is not true, but the following effects indicate the type of behavior to be expected:

- A. A sharp "click" makes the PLL skip a cycle; a doublet is created in the output whose mostly high-frequency energy, after post-PLL lowpass filtering, is reduced considerably.
- B. A lower and wider "click" makes the phase error exceed $\pi/2$ but is not large enough to cause the PLL to skip a cycle. The output is a large "spike".
- C. A very low "click" is reproduced by the PLL without much distortion.

Obviously, low frequency components account for most of the energies of outputs B and C. Such noise cannot be separated from the signal by linear filtering. Thus the wide-band PLL behaves like an FM detector except for conversion of some of the "clicks" into doublets, thus lowering the threshold.

The PLL output is usually filtered by a low-pass filter whose corner frequency is equal to the highest base-band (or "audio") frequency. In the case of narrow-band FM it may be quite high and may approach the highest frequency of the input band-pass filter f_o . A rough estimate of the spectral content of a "doublet" showed that its



energy is centered at frequency $.83 f_o^1$ and hence only approximately one-half of the "doublet" energy can be eliminated by post-detection filtering if the FM modulation index is, say, .2. Such a situation, however, is still better than detecting the "click" whose energy lies almost wholly within the input bandwidth $2f_o^1$. It is instructive to note that Hess [8] in his experiment² used a wide post-detection filter with the corner frequency set quite high; it would correspond to the base-band filter of the FM with the modulation index equal to .3, close to the upper limit on the modulation indexes (.2) used in this thesis. The filter was effective in filtering out some noise and the "doublets" so that the output "clicks" were readily recognized. An unfiltered "doublet" may be quite large and it is not always symmetric, as was observed by the author on many occasions, and can be easily confused with a "spike". Evidently most of the "doublet" energy lies outside the input bandwidth $2f_o$. Therefore, it is reasonable to assume that "click" suppression should be still useful in the cases of low-indexed FM. The process of "click" suppression for a high-indexed FM differs from the preceding case only by the width of the post-detection filter (it is narrow). The following listing of the threshold reductions shows the effectiveness

¹Hess' "click" model and the approximations described in chapter 2 were used to arrive at this number. The Fourier transform of the "click" is given in equation A1.1.

²The next section contains a brief description of the referenced paper.

of the phase-locked loop in signal demodulation at low carrier-to-noise ratios.

All the following reported threshold reductions by means of the PLL were done with post-detection filtering of the loop output. The bandwidth of the post-detection filter was equal to the modulating frequency, which means that the filter was very narrow when the modulation index was large. In some reported cases there was no modulation present but the output was nevertheless filtered as if the modulation were present. In all cases the modulation index was or was assumed to be 5 or larger. There has been no such work reported that is related to the narrowband FM; that is, there are no results showing what happens to the PLL output when the post-detection filter is wide. This is understandable, since the wideband FM is of more practical significance and the investigations tend to cover new subjects in the order of their importance.

As a rule, thresholds even with large modulation indexes fall below the $\text{CNR} = 10$ dB mark. For instance, Ridgeway and Carter [16] reported threshold locations at $\text{CNR} = 6$ dB for a modulation index of 9 using a second-order loop.

Work conducted by several investigators at the Polytechnic Institute of Brooklyn placed the PLL noise threshold (without modulation) between 0 and 4 dB. Noise thresholds at 2 dB with no modulation were obtained by Schilling et al [22]. Schilling and Billig [21] reported experimentally obtained thresholds for modulated carriers at about 1 dB; similar thresholds

were obtained by them in the absence of modulation, although they were not specifically cited in their report. While the author of this thesis was attending the Polytechnical Institute of Brooklyn, it was a regular procedure there to obtain the noise threshold for a given PLL without modulation in order to check whether it was operating properly. In 1967 a private communication from E. Hoffman (a co-worker of Schilling) placed the threshold for an unmodulated carrier at 3 or 4 dB of the carrier-to-noise ratio. Different results were obtained with different loops, but without exception cases without modulation produced low thresholds. This is perfectly natural, since modulation induces more "clicks" than an unmodulated carrier.

1.2 BRIEF LITERATURE SURVEY

There is an extensive literature dealing with phase-locked loops. Most of the works in this area concern themselves with finding the variance of the phase error as a function of the input carrier-to-noise ratio. Since the PLL differential equation has not been solved explicitly, approximations had to be used. A few representative papers will be briefly discussed.

Van Trees [27] applied a Volterra functional expansion to obtain approximate values of the variance of the phase error in a narrow-band phase-locked loop. This technique becomes cumbersome for loops of order greater than one and for low carrier-to-noise ratios, thus limiting its use to high carrier-to-noise ratios where only a few terms of the Volterra expansion suffice. For the reasons just mentioned it also is difficult to use the expansion for obtaining the frequency of cycle-skipping in a PLL. It is conceivable that the Volterra functional expansion may be used with wide-band loops as well as narrow-band loops, although no work has been done in this direction.

Tausworthe [25] assumed that the phase error is always distributed normally. This assumption allowed him to convert the differential equation of the phase-locked loop into an algebraic equation in terms of the spectrum of the phase error and to solve it. Results checked experimentally quite well in the case of the narrow-band PLL; it, too may conceivably be

used with the broad-band loop, since there is no built-in restriction as there is in the next case.

Tichonov [26], Viterbi [29], and Charles and Lindsey [7] used the Fokker-Planck stochastic differential equation to solve exactly for the phase-error variance and the probability of loss-of-lock in the first-order phase-locked loop and approximately in the second-order PLL. This technique is applicable only to narrow-band loops because the Fokker-Plank equation can be used only when the phase error is independent of the input noise.

Hess [8] solved approximately for the expected number of "spikes" ("spikes" are demodulated "clicks"; see curves B and C in Fig. 1.2) in the output of the first-order phase-locked loop driven by a carrier plus narrow-band noise. He assumed that all the incoming "clicks" are of the same exponential shape and that all "clicks" have identical height equal to the radius of gyration of the noise. By assuming a steady-state condition in case of a "spike" appearing in the output (i.e., he ignored the time derivative of the phase error in the phase-locked loop differential equation) he obtained an algebraic expression for the critical value of the loop gain required for cycle skipping. The expected number of spikes in the PLL output was easily obtained. Experimental and predicted results closely checked.

Shilling [20,21] departed considerably from the prevailing practice and proposed the use of a broad-band loop as an FM demodulator and a "click" suppressor. He linearized the loop equation in order to solve for an expected number of cycle skippings. Although the concept itself proved to be valid, the threshold location could not be predicted accurately.

Osborne and Schilling [13] considered cases of first-, second-, and third-order loops. Second- and third-order loops contained ideal proportional-plus-integral filters. The authors numerically solved the PLL differential equation when driven by a "click" whose shape was calculated as the conditional expectation of a function of three random parameters of the noise. This numerical solution provided them with a set of data from which, by means of a search technique, they found the boundaries of the three-dimensional "spike" region. (The random noise vector wandering into this region caused a "spike" in the PLL output.) The volume of the "spike" region was approximately evaluated by numerical integration. With these results they calculated the expected number of "spikes per second in the PLL output. In this paper, the FM sine-wave modulated carrier was approximated by the FM signal with a constant frequency-modulation offset. One important feature of this paper is that the loop parameters were constrained by the magnitude of the phase error due to signal modulation, and the effect of noise on the phase error was ignored. This feature immediately biased their results in favor of higher-order loops with an active filter. More will be said on this in chapter 4.

Lindsay [9] obtained an approximate solution of the Fokker-Planck equation describing a second-order loop. Comparing several loop configurations he concluded that the first-order loop yields the smallest phase-error variance when the loop is not deliberately offset in frequency from the center frequency of the incoming carrier. Due to the inherent

limitations of the Fokker-Plank equation he had to use a narrow-band loop (i.e., input noise had to be white) and no modulation. This paper is discussed more thoroughly in section 4.5.

1.3. THE OBJECTIVE OF THE THESIS

The objective of this thesis is to compare a wide-band first-order loop used as an FM demodulator with a second-order PLL when subject to a constraint on the size of the phase error. The performance criterion is taken to be the minimization of the expected number of "spikes" in the output of the PLL. The problem solution is approached in a semi-analytical manner. First, an approximate solution of the differential equation of the second-order loop with a forcing function represented by a "click" is derived, using Hess' empirical model of the "click". A Monte-Carlo search over the field of all first- and second-order loops is then described for the cases of narrow-band FM and no modulation. Loop parameters yielding suppression of the greatest portion of the incoming "clicks" define the optimum PLL. The only constraint in the search was an upper bound on the magnitude of the phase error variance which ensured that the demodulation was performed in the linear region of the loop. Since it was not possible to solve for the phase error variance, a conditional variance was used in lieu of it. The conditions on the variance were chosen so that it was close to the unconditional variance. Several different bounds were used. All the results of the search pointed to the first-order PLL as a more efficient "click" suppressor than the second-

order loop. Results were not exact, but the trend toward the first-order loop was unmistakable.

The search program had a built-in bias against the first-order loop configuration by consistently underestimating the phase error of second-order loops. The fact that the first-order loop outperformed other loops despite this bias implies that their performance difference must be considerable. In addition to that, an aspect which possibly has a significant bearing on the PLL behavior was omitted in this thesis: the loop behavior when recovering after a loss of lock due to a "click". It is conjectured that loops with "inertia" (second-order loops) would exhibit a transient phenomenon after the "click" occurrence, thus increasing the phase error variance. Omission of this effect had put the first-order loop at an additional disadvantage when comparing phase error variances in various loops.

It was not attempted in this thesis to conduct an impartial search for the optimum loop because of the computational difficulties in handling second-order loops. Some preliminary crude searches had indicated an apparent superiority of the first-order loop and the objective of the thesis was chosen to be a verification of this fact. It was successfully accomplished within the framework of constraints and assumptions that were mentioned above.

The outline of the thesis is as follows:

In chapter 2 the general second-order phase-locked loop differential equation is introduced and a mechanical analogue explained. An approximate solution of the differential equation driven by a "click"-forcing function is derived. A model of the "click" mechanism is discussed and justified.

Chapter 3 deals with the magnitude of the phase error of the loop caused by the incoming signal and noise. The phase-locked loop equation is justified for the non-deterministic input. Rice's approximation for the phase-error variance and a quasi-stationary method of obtaining the conditional phase-error variance are developed.

Chapter 4 describes the computer search program, presents results, and compares them with the results reported by Osborne and Schilling [13] and Lindsay [9].

Chapter 5 summarizes the results, discusses their significance, and points out possible extensions. Two appendices present numerical verification of the approximate solution of the PLL differential equation and the accuracy of the quasi-stationary method of obtaining the conditional phase-error variance.

CHAPTER 2

AN EFFECT OF A "CLICK" ON THE PHASE-LOCKED LOOP

2.1. THEORY OF PHASE-LOCKED LOOP

The phase-locked loop is a closed-loop control system which can be used as a coherent FM demodulator.¹ A diagram of the PLL is shown in Fig. 2.1.

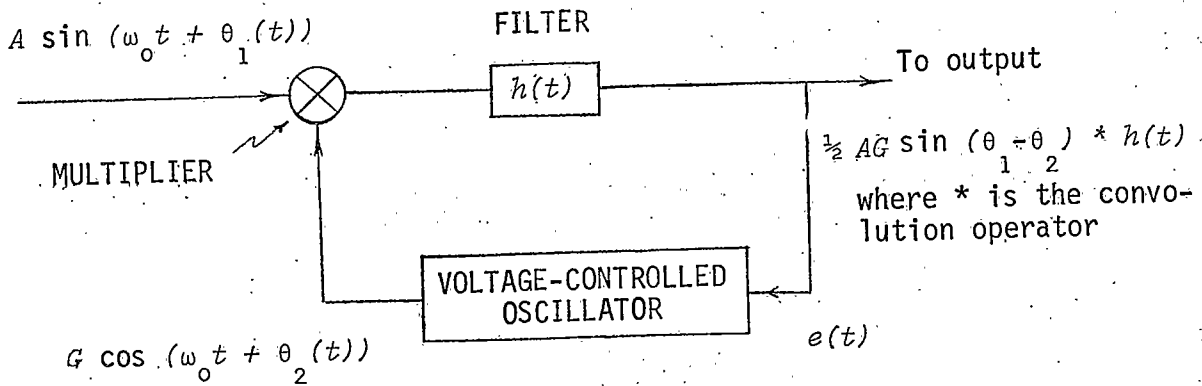


FIG. 2.1. Phase-Locked Loop Model.

Because of integrating action of the voltage-controlled oscillator (VCO),

$$\frac{d\theta_2'(t)}{dt} = \omega_0 + K_1 e(t) \quad (2.1)$$

where $\theta_1'(t) = \omega_0 t + \theta_1(t)$, $\theta_2'(t) = \omega_0 t + \theta_2(t)$ and K_1 is called the sensitivity of the VCO. The output of the multiplier is

$$\frac{AG}{2} \left\{ \sin [\theta_1'(t) - \theta_2'(t)] + \sin [\phi_1'(t) + \theta_2'(t)] \right\} \quad (2.2)$$

¹This derivation of PLL equations follows closely that of Viterbi's [29].

The sum-frequency term is eliminated by the loop. Therefore,

$$e(t) = \frac{AG}{2} \left\{ h(t) * \sin \left[\theta_1'(t) - \theta_2'(t) \right] \right\} \quad (2.3)$$

Substitution of equation (2.3) into (2.1) yields

$$\frac{d\theta_2'}{dt} = \omega_0 + \frac{AGK_1}{2} \int_0^t h(t-\tau) \sin \phi(\tau) d\tau \quad (2.4)$$

where

$$\phi(t) \triangleq \theta_1'(t) - \theta_2'(t) \quad (2.5)$$

is the phase error of the loop. Finally, let $AGK_1/2$ equal AK . Identities of (2.1) and (2.5) are used in (2.4) to obtain

$$\frac{d\phi}{dt} = \frac{d\theta_1}{dt} - AK \int_0^t h(t-\tau) \sin \phi(\tau) d\tau$$

where AK is defined as the loop gain.

The transfer function of the low-pass filter of any first- or second-order phase-locked loop can be described by the general expression $H(s) = (\alpha/\beta) \cdot (s+\beta)/(s+\alpha)$. In order for the filter to be low-pass, β must be larger than α . When $\alpha=\beta$, $H(s)$ becomes unity and the order of the loop is one. If β is very large, $H(s)$ becomes $\alpha/(s+\alpha)$ - the transfer function of an RC filter. If α is very small, $H(s)$ is approximately equal to $\alpha(s+\beta)/\beta s$ which is the transfer function of the ideal active proportional-plus-integral filter. (A lag network).

The differential equation of the second-order PLL with the above-mentioned general form of filter is

$$\ddot{\phi} + \dot{\phi} \left(\frac{\alpha AK}{\beta} \cos \phi + \alpha \right) + \alpha AK \sin \phi = \ddot{\theta}_1 + \alpha \dot{\theta}_1 \frac{\Delta}{\omega_1} D(t) \quad (2.6)$$

This is the equation of the pendulum under the influence of gravity, driving force $D(t)$, friction, and wind.¹

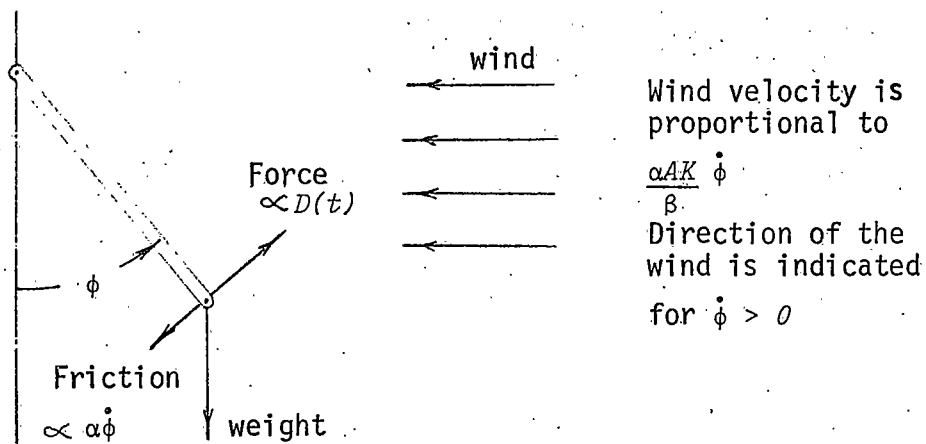


FIG. 2.2. Mechanical Analogue of the Second-Order PLL.

Wind in this model is generated by the motion of the pendulum. The pendulum consists of a point-mass connected by a rod to the hub, which in turn is coupled to a fan through a power amplifier. Thus, wind is produced only if the pendulum is in motion. When $\beta = \infty$ there is no wind present. This case corresponds to a low-pass RC filter. It is obvious from the figure that ϕ can increase up to π and come back without

¹This particular mechanical analogue was proposed by Charles and Lindsay [7].

sweeping a full circle. A revolving pendulum corresponds to the cycle-skipping phenomenon in PLL.

If the PLL is underdamped, the pendulum will tend to overshoot the 2π point, and additional input noise may cause several revolutions in a row.

The phase-locked loop differential equation may be expressed in terms of the elements of its mechanical analogue

$$\ddot{\phi} + \frac{f_2}{ML} \dot{\phi} + \frac{f_3}{ML} \dot{\phi} \cos \phi + \frac{Mg}{ML} \sin \phi = \frac{1}{ML^2} (\ddot{\theta}_1 + \alpha \dot{\theta}_1) \quad (2.7)$$

where M = mass of the pendulum

L = length of the arm of the pendulum

$g = 9.8 \text{ m/sec}^2$

f_2 = frictional constant

f_3 = windage-torque constant. Torque caused by the wind is equal to $f_3 L \dot{\phi} \cos \phi$.

If the loop gain AK is large, the situation corresponds to small L , or equivalently to a small rotational moment of inertia.

2.2 FM DEMODULATION IN THE ABSENCE OF NOISE

A high-gain PLL has a high phase error; therefore $\sin \phi \approx \phi$.

This is required for good detection of FM signals. The modulated carrier driving the PLL of Fig. 2.1 is $A \sin [\omega_0 t + \theta_1(t)]$ where $\frac{d}{dt} \theta_1(t)$ is the signal which the PLL has to detect. The linearized PLL differential equation is obtained by setting $\sin \phi = \phi$ and $\cos \phi = 1$ in 2.6:

$$\ddot{\phi} + \alpha\phi \left(1 + \frac{AK}{\beta}\right) + \alpha AK\phi = \ddot{\theta}_1 + \alpha\dot{\theta}_1 \quad (2.8)$$

After the Laplace transformation, the phase error can be expressed as

$$\Phi(s) = \frac{(s^2 + \alpha s) \theta_1(s)}{s^2 + \alpha s \left(1 + \frac{AK}{\beta}\right) + \alpha AK} \quad (2.9)$$

Since $s\theta_1(s)$ is the demodulated FM signal which is to be recovered, the PLL output $[AH(s)\Phi(s)]$ needs additional filtering by a system whose transfer function is

$$G(s) = \left(\frac{\beta}{\alpha A}\right) \frac{s^2 + \alpha s(1 + AK/\beta) + \alpha AK}{s + \beta} \quad (2.10)$$

If the loop is wide, often the whole filter can be ignored. In case of the first-order loop (when $\alpha = \beta$) $G(s)$ becomes a high-pass filter $[H(s) \propto s + AK]$. Thus, while the loop is high gain the output may be approximated by $\dot{\phi}(t)$. But when a "click" comes, the chances are good that the loop will not be able to handle it in the linear mode of operation and then the output will be proportional to $\sin \phi(t)$.

2.3 APPROXIMATE SOLUTION OF THE PLL DIFFERENTIAL EQUATION WITH A CONSTANT LOOP GAIN

Since "clicks" are responsible for the onset of threshold, it is necessary to be able to calculate whether the PLL will skip cycles under the influence of a given "click". Consider the case where the "click" is

symmetric and barely large enough to cause the PLL to jump a cycle.

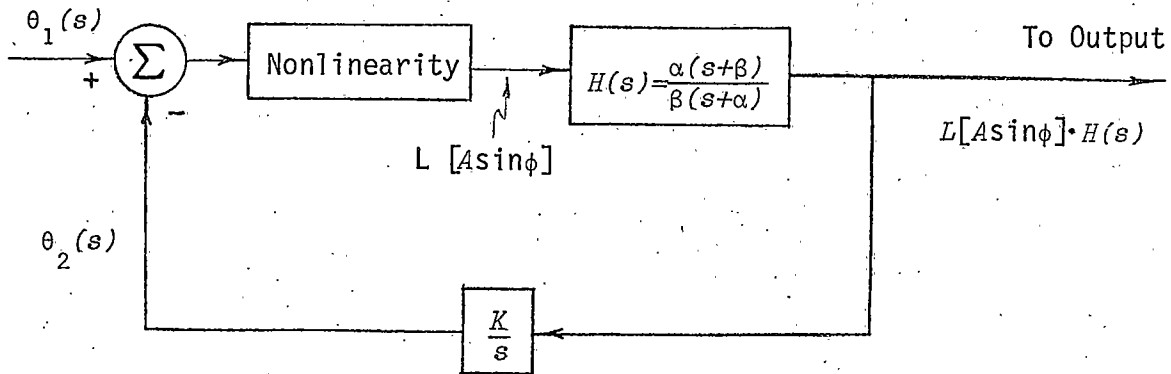


FIG. 2.3. Block Diagram of the Low-Pass Analogue of the Phase-Locked Loop With a Constant Loop Gain.

An equation describing the operation of the PLL is easily obtained from Fig. 2.3.

$$\frac{\alpha AK}{\beta} \left(\frac{s+\beta}{s+\alpha} \right) L[\sin \phi] = s [\theta_1(s) - \phi(s)] \quad (2.11)$$

It is assumed that no change in the loop gain AK takes place during the "click" occurrence. Since Hess' model of the "click" [8] is used in this thesis and that model assumes a very slow carrier amplitude A fading, there was no need to solve the exact time-varying differential equation.

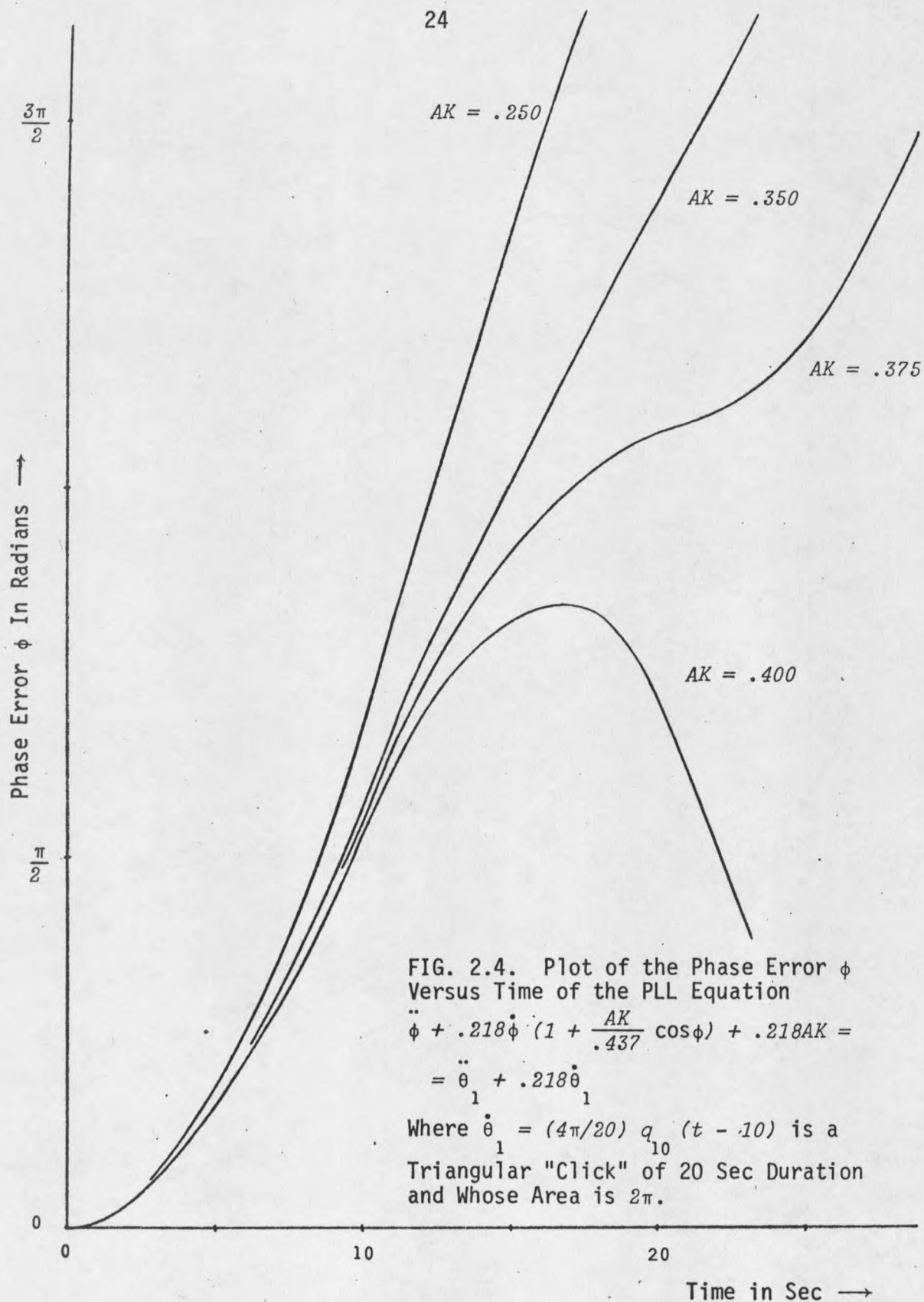
Since exact analytical solutions are not obtainable for this non-linear equation, approximation techniques are used. Analogue computer simulations of the response of the loop to a "click" gave a clue to one such technique. The phase error ϕ was observed to be very nearly linear as a function of time during a large portion of the duration of the "clicks"

driving the PLL. Figure 2.4 clearly shows such a phenomenon. The linearity of ϕ is particularly good in the range of $[\pi/2, 3\pi/2]$ where the phase-locked loop, due to the nature of the nonlinearity $\sin \phi$, is unable to track the driving signal.

The near linearity of ϕ versus time may be explained in the following way. If the "click" acting on the PLL is very large, the differential equation of the loop reduces to $\ddot{\phi} + \alpha\dot{\phi} = \ddot{\theta}_1 + \alpha\dot{\theta}_1$ or $\phi = \theta_1$.

Interestingly enough, this states that if $\dot{\theta}_1$ is a "click" whose area is equal to 2π , ϕ will jump only one cycle. This also implies that the PLL does not make any attempt to track $\theta_1(t)$, i.e., that the bandwidth of the transfer function of ϕ is infinite. If a smaller "click" is applied to the PLL and the cycle skipping again takes place, the effective bandwidth of the transfer function of the phase error is not as large as it was before because the loop tries to track $\theta_1(t)$, and ϕ is no longer a replica of $\theta_1(t)$. As $\dot{\theta}_1(t)$ (the "click") is chosen to be lower and wider, the PLL does a progressively better job of reducing the phase error. So, in the limiting case in which the "click" is just large enough to cause cycle skipping, the effective bandwidth of the transfer function of ϕ is small. Therefore, one may expect ϕ to be a low-pass filtered replica of $\theta_1(t)$. Low frequency corresponds to slow response in the time domain. Since the DC and the very low frequency part of the Fourier integral of $\theta_1(t)$ are presumably preserved by ϕ ,

$$\int_0^T \phi(t) dt = \int_0^T \dot{\theta}_1(t) dt = 2\pi. \quad \text{Hence, } \phi(t) \approx (\text{constant}) \times t. \quad \text{Numerous}$$



analogue computer solutions of the PLL equation repeatedly exhibited this linearity. At the beginning of the cycle skipping (when $\phi \ll \pi/2$), the phase error was not observed to be a linear function of time. But this deviation is explainable by the transients in the system.

The smaller the "click", the more significant the gravity, friction, and wind losses in the PLL mechanical analogue become. Thus, when the "click" is just big enough to cause cycle skipping, the transients are significant. As the loop gain is decreased, the effect of the transients rapidly diminishes. Figure 2.4 illustrates this point. Again, the linearity of the phase error becomes very pronounced as soon as the loop gain is slightly decreased from the critical value (the smallest value which is sufficient to prevent cycle skipping). Note that even in the marginal case, ϕ is almost linear up to $\phi = \pi$.

Since a large portion of $\phi(t)$ is linear in the case of cycle skipping caused by a "click", the following approximation in equation (2.11) is made: $\sin \phi \approx \sin \omega_0 t$ where $\omega_0 = \pi/\text{passage time}$. This is only a temporary assumption. After several manipulations of this equation the assumption will be dropped. The approximation allows one to introduce second-order corrections to the exact solution of the first-order loop.

Expansion of the left-hand side of equation (2.11) with an assumption that $\phi \approx \omega_0 t$ yields

$$\frac{\alpha AK}{\beta} \left(1 + \frac{\beta - \alpha}{s + \alpha} \right) \left(\frac{\omega_0}{s^2 + \omega_0^2} \right) = \frac{\alpha AK}{\beta} \left[\frac{\omega_0}{s^2 + \omega_0^2} + \frac{(\beta - \alpha)\omega_0}{\alpha^2 + \omega_0^2} \left(\frac{1}{s + \alpha} + \frac{\alpha - s}{s^2 + \alpha^2} \right) \right] \quad (2.12)$$

the inverse transform of which is

$$\frac{\alpha AK}{\beta} \left[\left(1 + \frac{\alpha(\beta - \alpha)}{\alpha^2 + \omega_0^2} \right) \sin \phi + \frac{(\beta - \alpha)\omega_0}{\alpha^2 + \omega_0^2} \left(e^{-\alpha t} - \cos \phi \right) \right] \quad (2.13)$$

Expression (2.12), in conjunction with equation (2.11), yields

$$\dot{\phi} + \frac{\alpha AK}{\beta} \left[\left(1 + \frac{\alpha(\beta - \alpha)}{\alpha^2 + \omega_0^2} \right) \sin \phi + \frac{(\beta - \alpha)\omega_0}{\alpha^2 + \omega_0^2} \left(e^{-\alpha t} - \cos \phi \right) \right] = \dot{\theta}_1 \quad (2.14)$$

This expression cannot be solved for ϕ in a closed form if the term

$e^{-\alpha t}$ is present. If $e^{-\alpha t}$ is replaced by a constant plus sine and cosine terms, then the closed form solution of the equation exists. Therefore, $e^{-\alpha t} \approx B_0 + B_1 \cos \phi$ where B_0, B_1 are determined by the least mean square error fit technique over the range of ϕ equal to $[0, \pi]$. The sine term is absent because its coefficient is equal to zero. Again, it is assumed here that $\phi \approx \omega_0 t$. The mean square error of the approximation for the exponential is

$$\epsilon^2 = \int_0^\pi \left(e^{-\frac{\alpha \phi}{\omega_0}} - B_0 - B_1 \cos \phi \right)^2 d\phi \quad (2.15)$$

In order to minimize the error, both sides of expression (2.15) are differentiated with respect to B_0 (B_1) and then $\partial \epsilon / \partial B_0$ ($\partial \epsilon / \partial B_1$) is set to zero. The right-hand side then provides the expression for B_0 or B_1 in terms of α , ω_0 , $\cos \phi$. It was found that

$$e^{-\frac{\alpha \phi}{\omega_0}} \approx \frac{\omega_0}{\alpha \pi} \left(1 - e^{-\frac{\alpha \pi}{\omega_0}} \right) + \frac{4\alpha \omega_0}{\pi(\alpha^2 + \omega_0^2)} \left(1 + e^{-\frac{\alpha \pi}{\omega_0}} \right) \cos \phi. \quad (2.16)$$

Finally, the approximation to the second-order PLL equation may be written

$$\begin{aligned} \dot{\phi}(t) + \frac{\alpha AK}{\beta} \left\{ \left(1 + \frac{\alpha(\beta - \alpha)}{\alpha^2 + \omega_0^2} \right) \sin \phi(t) + \frac{(\beta - \alpha) AK \omega_0^2}{(\alpha^2 + \omega_0^2) \beta \pi} \left(1 - e^{-\frac{\alpha \pi}{\omega_0}} \right) + \right. \\ \left. + \frac{\alpha AK(\beta - \alpha) \omega_0}{\beta \pi (\alpha^2 + \omega_0^2)^2} \left[4\alpha \omega_0 \left(e^{-\frac{\alpha \pi}{\omega_0}} + 1 \right) - \pi (\alpha^2 + \omega_0^2) \right] \cos \phi(t) \right\} \\ = \dot{\theta}_1(t) \end{aligned} \quad (2.17)$$

If $\alpha = \beta$, it reduces to $\dot{\phi} + AK \sin \phi = \dot{\theta}_1$ which is the exact equation of the first-order loop.

Expression (2.17) thus is exact when $\alpha = \beta$. If the damping coefficient of the second-order system is large, it behaves like the first-order loop and (2.17) is again very accurate. Finally, the larger $\dot{\theta}_1$ (the "click"), the less significant the sine term becomes, and (2.17) again approaches the exact equation.

2.4 APPROXIMATION OF THE DRIVING FUNCTION

The driving function of expression (2.17) is assumed to be a square pulse $k_1 p_\tau(t-\tau)$. Since, as was argued earlier in this chapter, ϕ is a filtered version of θ_1 , only the low-frequency part of the Fourier transform of $\dot{\theta}_1$ need be taken into consideration. The Fourier transform $[\dot{\theta}_1(t)]$ is expanded in a MacLaurin's series, and only the first two terms of this expansion are retained. The first term is equal to the area of $\dot{\theta}_1$, which is 2π . The second term is equal to $-\frac{m_2 \omega^2}{2!}$ where m_2 is the second moment of $\dot{\theta}_1(t)$ and is equal to $\int_{-\infty}^{\infty} t^2 \dot{\theta}_1(t) dt$. The terms containing odd powers of ω are equal to zero when $\dot{\theta}_1(t)$ is a symmetric function.

Constants k_1 and τ in $k_1 p_\tau(t-\tau)$ can be chosen so that the area and the second moments of the "click" and its pulse approximation are equal. Analogue computer solutions of the PLL driven by a triangular "click" and the digital computer solutions of the approximate equation (2.17) driven by the pulse agreed very well.

A further simplification can be made; namely, $k_1 p_\tau(t-\tau)$ can be replaced by $k_1 u(t)$ where $u(t)$ is the unit step. Indeed, if the PLL is driven by a square pulse and ϕ reaches the value of π *before* the arrival of the trailing edge of the pulse, the above replacement is absolutely

valid as long as the solutions of the PLL equation are limited to $t <$ (duration of the "click") and $|\phi| < \pi$.

In order to show that the passage time is always shorter than the "click" duration, a limiting case of the PLL equation will be considered. The general PLL equation with a constant gain is:

$$\ddot{\phi}(t) + \alpha \dot{\phi}(t) + \frac{\alpha AK}{\beta} \phi(t) \cos \phi(t) + \alpha AK \sin \phi(t) = \ddot{\theta}_1(t) + \alpha \dot{\theta}_1(t) \quad (2.18)$$

If AK is large and α is very small, then the above equation reduces to the equation of the PLL with an ideal proportional-plus-integral filter:

$$\ddot{\phi}(t) + \frac{\alpha AK}{\beta} \dot{\phi}(t) \cos \phi(t) + \alpha AK \sin \phi(t) = \ddot{\theta}_1(t) \quad (2.19)$$

Suppose a positive "click" ($\dot{\theta}_1$) acted on the loop and then expired but ϕ did not reach π while the "click" lasted. It will be shown below that ϕ does not reach π under these conditions.

Integrating both sides of the above equation with respect to time and with all the initial conditions equal to zero yields:

$$\dot{\phi}(t) + \frac{\alpha AK}{\beta} \sin \phi(t) + \alpha AK \int_0^t \sin \phi(t) dt = \dot{\theta}_1(t) \quad (2.20)$$

Since $|\phi| < \pi$ and hence $\sin \phi > 0$, the above expression may be written as

$$\dot{\phi}(t) \leq \dot{\theta}_1(t) \quad (2.21)$$

When the "click" has ceased to exist, the inequality becomes $\dot{\phi}(t) < 0$.

The phase error is on its way back to zero, the equilibrium position.

Thus, the passage time is always smaller than the duration of the "click" in the above case.

The equation with the ideal filter (2.19) has two terms missing in comparison to the general PLL equation (2.18). The terms are $\alpha\dot{\phi}$ on the left and $\alpha\dot{\theta}_1$ on the right side of the equality sign. In order to understand the effect of their presence or absence on the loop behavior, it is convenient to consider the mechanical analogue introduced in Section 1. $\alpha\dot{\phi}$ is proportional to the frictional torque and $\alpha\dot{\theta}_1$ is a part of the expression which is proportional to the driving torque of the pendulum. The exact expressions for the frictional torque and the contribution of $\dot{\theta}_1$ to the driving torque are $(LMg/AK\dot{\phi})$ and $(LMg/AK\dot{\theta}_1)$, respectively. These expressions are easily obtained from the differential equation of the pendulum. The ratios of the frictional torque to the $\dot{\theta}_1$ torque is just $\dot{\phi}/\dot{\theta}_1$. As α is gradually increased from its initial small value to infinity, the loop becomes increasingly overdamped and that in turn makes $\dot{\phi}$ and the ratio smaller. The fact that the frictional torque gets smaller while the driving torque caused by $\dot{\theta}_1$ does not change results in a steady erosion of the bound on the passage time. When α finally reaches infinity, the PLL equation becomes

$$\ddot{\phi}(t) + AK \sin \phi(t) = \dot{\theta}_1(t) \quad (2.22)$$

The $\ddot{\phi}$, $\ddot{\theta}_1$, and $\frac{\alpha AK}{\beta} \dot{\phi} \cos \phi$ terms dropped out of the general PLL equation because they did not contain α or, in case of the $\dot{\phi} \cos \phi$ term, because of the inequality restriction on β , ($\beta > \alpha$) which effectively eliminated α from that term.

The above differential equation represents a first-order phase-locked loop. Such a PLL does not have "inertia" or a "wind-force". When, for instance, a positive "click" ($\dot{\theta}_1$) has expired but ϕ is still less than π , $\dot{\phi}$ must become negative. This means that ϕ cannot reach π and thus skip a cycle.

Although the restriction on the passage-time was being steadily eroded as α was decreasing, the limiting cases where $\alpha = 0$ still barred ϕ from reaching π after the end of the "click". Therefore, the following conclusion can be made:

Whenever a "click" driving a second-order phase-locked loop has expired while the phase error of the loop is smaller than π , the loop will not skip cycles but will rather settle down, provided enough time is allowed for that. This justifies the replacement of the square-pulse driving force by a step-function. Now the approximate PLL equation (2.17) can be solved.

2.5 SOLUTION OF THE SECOND-ORDER TIME-INVARIANT PLL EQUATION DRIVEN BY A "CLICK"

Multiplying both sides of equation (2.10) by dt and separating functions of ϕ and t allows us to solve for the passage time t_0 :

$$t_o = \int_0^{\pi} \frac{d\phi}{k_2 - A_1 \sin \phi - A_2 \cos \phi} \quad (2.24)$$

where

$$k_2 = k_1 - \frac{(\beta - \alpha)AK\omega_o^2}{(\alpha^2 + \omega_o^2)\beta\pi} \left(1 - e^{-\frac{\alpha\pi}{\omega_o}} \right)$$

$$A_1 = \frac{\alpha AK}{\beta} \left[1 + \frac{\alpha(\beta - \alpha)}{\alpha^2 + \omega_o^2} \right]$$

$$A_2 = \frac{\alpha AK (\beta - \alpha) \omega_o}{\beta\pi (\alpha^2 + \omega_o^2)^2} \left[4\alpha\omega_o \left(e^{-\frac{\alpha\pi}{\omega_o}} + 1 \right) - \pi (\alpha^2 + \omega_o^2) \right]$$

The answer is

$$t_o = \frac{2}{\sqrt{k_2^2 - r^2}} \left[\tan^{-1} \left(\frac{k_2 \tan \frac{\chi}{2} + r}{\sqrt{k_2^2 - r^2}} \right) \right]_{\psi}^{\pi + \psi} \quad (2.25)$$

where $r^2 = A_1^2 + A_2^2$ and $\psi = \tan^{-1} \frac{A_1}{A_2}$ such that ψ lies in the second

quadrant. Finally

$$t_o = \frac{2}{\sqrt{k_2^2 - r^2}} \left[\pi + \tan^{-1} \frac{k_2 \tan \frac{\pi + \psi}{2} + r}{\sqrt{k_2^2 - r^2}} - \tan^{-1} \frac{k \tan \frac{\psi}{2} + r}{\sqrt{k_2^2 - r^2}} \right] \quad (2.26)$$

The extra π in the parenthesis came from the fact that the two integral limits are lying on adjacent (but disconnected) tangent branches. Similarly 2π would have been added if the limits on the integral were 0 and 2π .

The largest possible value of t_0 is $2\pi/k_1$, the "click" duration, and this value is used in the solution below. Equation (2.26) now can be solved by iterative computer techniques for the critical loop gain AK . The answer is not the exact value of AK , but rather an underestimate on the minimum loop gain required to cause the PLL to skip a cycle, given a "click" of a certain height k_1 . Appendix I contains data which shows the very good accuracy of the above solution.

2.6 THE "CLICK" MODEL

The driving function of equation (2.18) is a "click", briefly described in Chapter 1. In actuality, a "click" is not a deterministic function and may assume any shape. The majority of the "clicks" in nature are, however, bell shaped. Hess proposed an empirically-derived expression describing a typical "click" which he then successfully used to predict, with a high degree of accuracy, the expected number of cycle-skippings by the first-order loop. The same "click" model is used in this thesis.

The mechanism of a "click" formation is shown in Fig. 1.1. Modulated carrier and noise add up to form the resultant phasor. This phasor

occasionally sweeps about the origin and thus changes the reference angle θ_1 by $\pm 2\pi$. The angular velocity of the resultant phasor during such a sweep is the "click" in which we are interested - a bell-shaped function of time.

Hess proposed the following model of the "click":

$$\text{"click"} = \dot{\theta}_1(t) = \pm \dot{\theta}_1(t_o) \exp \left\{ - (t-t_o)^2 [\dot{\theta}_1(t_o)]^2 / 4\pi \right\} \quad (2.27)$$

Its time integral is equal to 2π . Actually, there are many possible descriptions of a bell-shaped function: cosine to an even power and polynomials can be used equally well. The important feature of Hess' model is the assumption that all "clicks" have the same peak height which is equal to two radii of gyration of the input narrowband noise.

$$\dot{\theta}_1(t_o) = 2 \left[\int_{-\infty}^{\infty} \omega^2 G_L(\omega) d\omega / \int_{-\infty}^{\infty} G_L(\omega) d\omega \right]^{\frac{1}{2}} \quad (2.28)$$

where $G_L(\omega)$ is the low-pass equivalent power spectrum of the narrow band noise centered at ω_o . The input noise is assumed to be white and Gaussian. The magnitude of the resultant phasor is presumed to remain constant during the "click" occurrence.¹ Actually, the amplitude of the carrier changes but much more slowly than $\dot{\theta}_1(t)$.

The proposed model is attractive because it allows all the "clicks" to be of the same height. This is not quite so in nature; but nevertheless, the hypothetical "clicks" of Hess provided him results which were

¹Rice [15], Hess [8]

in close agreement with experimental data. Allowing for variations in height from one "click" to another brings another degree of freedom to this process, doubles the complexity of calculations, and, according to Hess, does not improve the results. The assumption that the phasor magnitude stays constant during the "click" occurrence considerably simplifies the solution, as well.

2.7 INITIAL CONDITIONS

It is assumed that the "click" arrives at the input of the phase-locked loop when the loop is undisturbed (zero initial conditions). This was one of the assumptions in the Hess' modeling of the "click" suppression by the first order loop; this assumption is actually a corollary from Hess' "click" model which was discussed in the preceding section. The argument goes as follows: the loop gain (AK if the carrier amplitude stays constant, $a(t)K$ if it varies) changes very little while the "click" is taking place; the "click" is described by a Gaussian-like function which has extended low tails. Therefore, long before the arrival of the peak of the "click", the driving force of the loop is essentially zero and the gain is constant. Since the arrival of the peak of the "click" is preceded by a period of almost complete absence of the driving force, the loop should have enough time to come to rest before the main portion of the "click" arrives. Note that the assumption concerning the zero initial conditions is not valid when modulation

is present. In the latter case there remains a driving force (modulation) even when the noise disappears. The amplitude of the carrier $a(t)$ dips when the "click" occurs. Since, however, $a(t)$ varies more slowly than

$\dot{\theta}_1(t)^1$, it may be assumed that most of the change in it occurs when

$\dot{\theta}_1(t)$ is very small (when $\dot{\theta}_1(t)$ is still the leading tail of the "click").

Therefore, evidently Hess concluded that $a(t)K$ can be considered to be a constant during the "click" occurrence even when solving a PLL equation. He reported very accurate results based on the zero initial conditions assumption for the PLL as narrow as half the bandwidth of the input noise.

Computer simulations of some cases of a "click" action on a phase-locked loop (see Appendix I) indicate that with $a(t)K$ changing about half as fast as $\dot{\theta}_1(t)$, the phase-locked loop behavior is close to the case where $a(t)K$ is a constant.

¹Hess [8]

CHAPTER 3

ESTIMATION OF A PHASE ERROR VARIANCE

3.1 INTRODUCTION

The "optimality" of the phase-locked loop FM demodulator is governed by several criteria, one of which is the magnitude of the mean-square phase error. In the presence of noise the phase-locked loop is a randomly time-varying nonlinear system. The fidelity requirement of the demodulated signal, however, limits the operating range of the phase-locked loop to its linear region. Still, the exact evaluation of the mean-square phase error is not always possible. For the first-order linear stochastic time-varying system the output variance can be found exactly; variances of some second-order systems may be obtained as well, but no general solution is known at present for second and higher order systems. Nevertheless, in conjunction with certain approximations, the conditional mean-square phase error may be determined via the quasi-stationary approach, or its lower bound may be estimated (the variance of the output of the time-varying stochastic system is larger than the variance of the output of the time-invariant system obtained by replacing all the time-varying parameters of the former system by their expected values). Chapter 3 justifies certain approximations and outlines a method for determining the mean-square PLL phase error. Section 2 contains the derivation of the PLL equation in the presence of noise; section 3 introduces the approximate expressions for the phase and the amplitude of a modulated carrier and shows that they are approximately independent; in section 4, a lower bound on the phase-error variance

is found; in section 5 the quasi-stationary model and the method of obtaining a conditional phase-error variance are explained.

3.2 PHASE-LOCKED LOOP EQUATION IN THE PRESENCE OF NOISE

In Chapter 2, the phase-locked loop equation was derived for the noiseless condition. In nature, input signals to the PLL are usually corrupted by noise. Since only cases in which the PLL is preceded by a narrow band-pass filter are considered in this thesis, one may assume that the corrupting noise is white and Gaussian. Even if the noise is colored and non-Gaussian (but wideband), a narrow band-pass filter almost always makes the output noise Gaussian. In this light the above assumption seems to be reasonable.

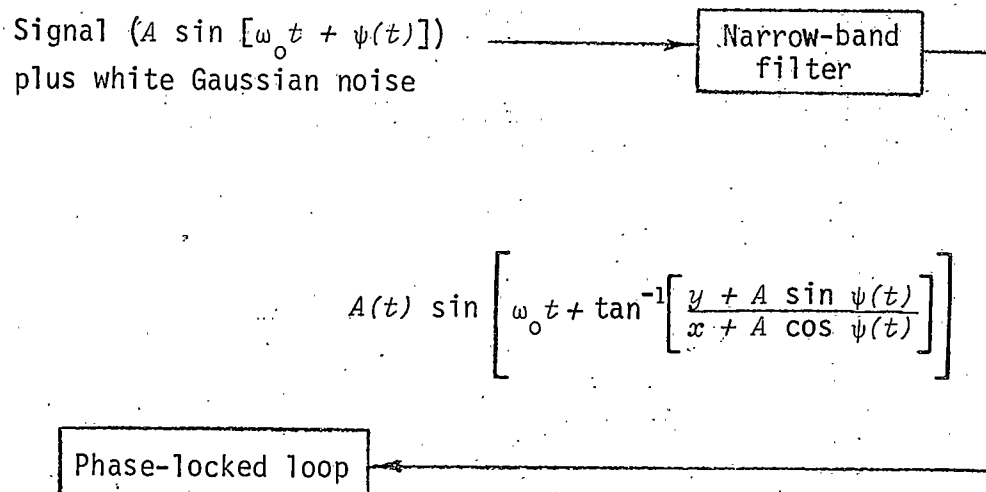


FIG. 3.1. Pre-detection Filtering of the PLL Input.

As indicated in Fig. 3.1, the modulated carrier and narrowband noise are expressable as:

$$\sqrt{[A \cos \psi(t) + x(t)]^2 + [A \sin \psi(t) + y(t)]^2} \times \sin \left[\omega_0 t + \tan^{-1} \left[\frac{y + A \sin \psi(t)}{x + A \cos \psi(t)} \right] \right] \quad (3.1)$$

where

$$\sqrt{[A \cos \psi(t) + x(t)]^2 + [A \sin \psi(t) + y(t)]^2} \triangleq a(t) = \text{Amplitude} \quad (3.2)$$

$$\tan^{-1} \left[\frac{y + A \sin \psi(t)}{x + A \cos \psi(t)} \right] \triangleq \theta_1(t) = \text{phase of the PLL input} \quad (3.3)$$

and $x(t)$ and $y(t)$ are respectively the cophase and quadrature functions of the narrowband Gaussian process. Note that the PLL input is modulated in both amplitude and angle by noise.

In chapter 2, it was shown that in the noiseless environment the low-frequency output of the multiplier is $A \sin \phi(t)$. Since the VCO output (see Fig. 2.1) has a constant amplitude, the derivation of the output of the mixer proceeds as in section 2.1, except that A is replaced by $a(t)$. Thus, the mixer output here is $a(t) \sin \phi(t)$.

Using the general low-pass filter of section 2.1, with voltage transfer function $H(s) = (\alpha/\beta) (s+\beta/s+\alpha)$, the phase-locked loop differential equation in the presence of noise is derived below. The impulse response of the above-mentioned filter is

$$h(t) = \frac{\alpha}{\beta} [\delta(t) + (\beta - \alpha) e^{-\alpha t} u(t)] \quad (3.4)$$

and one of the versions of the phase-locked loop equation is

$$\frac{d\phi}{dt} = \frac{d\theta_1}{dt} - \frac{\alpha K}{\beta} \int_0^t a(\tau) \sin \phi(\tau) [\delta(t-\tau) + (\beta - \alpha) e^{-\alpha(t-\tau)}] d\tau \quad (3.5)$$

Terms multiplied by the delta function can be integrated immediately:

$$\frac{d\phi}{dt} = \frac{d\theta_1}{dt} - \frac{\alpha K}{\beta} a(t) \sin \phi - \frac{\alpha(\beta - \alpha)K}{\beta} \int_0^t e^{-\alpha(t-\tau)} a(\tau) \sin \phi(\tau) d\tau \quad (3.6)$$

Differentiation of both sides with respect to time yields:

$$\begin{aligned} \ddot{\phi}(t) = \ddot{\theta}_1(t) - \frac{\alpha K}{\beta} \frac{d}{dt} [a(t) \sin \phi(t)] - \frac{\alpha(\beta - \alpha)K}{\beta} a(t) \sin \phi(t) + \\ + \frac{\alpha^2(\beta - \alpha)K}{\beta} \int_0^t e^{-\alpha(t-\tau)} a(\tau) \sin \phi(\tau) d\tau \end{aligned} \quad (3.7)$$

But,

$$\begin{aligned} \frac{\alpha^2(\beta - \alpha)}{\beta} K \int_0^t e^{-\alpha(t-\tau)} a(\tau) \sin \phi(\tau) d\tau = \\ = \alpha \left[\dot{\theta}_1(t) - \frac{\alpha K}{\beta} a(t) \sin \phi(t) - \phi(t) \right] \end{aligned} \quad (3.8)$$

Finally, the PLL equation in the presence of noise may be written:

$$\ddot{\phi}(t) + \alpha \dot{\phi}(t) + \frac{\alpha K}{\beta} \frac{d}{dt} [\alpha(t) \sin \phi(t)] + \alpha K \alpha(t) \sin \phi(t) = \ddot{\theta}_1(t) + \alpha \dot{\theta}_1(t) \quad (3.9)$$

3.3 APPROXIMATE EXPRESSIONS OF THE AMPLITUDE AND PHASE OF A MODULATED SIGNAL.

Rice, in his classical paper [14], proposed an approximate expression for the spectrum of the demodulated FM signal. This expression is vastly simpler than the exact one previously derived by Middleton, Rice, Stumpers, and others. In this thesis $\dot{\theta}_1(t)$ is such a demodulated signal (see equation 2.11). According to Rice, if the input to the FM detector is $A \cos \omega_0 t + x(t) \cos \omega_0 t + y(t) \sin \omega_0 t$ (x and y are quadrature functions of narrowband Gaussian noise), then the detector output is equal to

$$\frac{d}{dt} \tan^{-1} \frac{\dot{y}(t)}{A + x(t)} \approx \frac{\dot{y}(t)}{A} \text{ plus "clicks"} \quad (3.10)$$

Thus aside from the "clicks", the demodulated FM signal is equal to the first term of Taylor's series expansion of $\dot{\theta}_1(t)$. This approximation is accurate down to and including a carrier-to-noise ratio of 3 dB. Even

below this value the approximation holds after a fashion [23]. The spectrum of $\dot{\theta}_1(t)$ is therefore equal to the spectrum of $\frac{\dot{y}(t)}{A}$ plus the spectrum of the "clicks" (shot noise). The latter is very broad and, by Rice's assumption, independent of the other components of the total spectrum. The phase of the driving signal, in the presence of the sine-wave FM modulation, was given in expression (3.1). Here it is repeated again:

$$\theta_1(t) = \tan^{-1} \frac{y(t) + A \sin \psi(t)}{x(t) + A \cos \psi(t)}$$

where $\psi(t) = \Omega \sin \omega_m t$ and Ω is the FM modulation index. The spectrum of $\dot{\theta}_1(t)$ now includes noise, signal, and the cross-products of noise and signal. Again, using only the first term in Taylor's series expansion of $\theta_1(t)$, Rice suggested the following expression for its spectrum:

$$S(\omega) = \frac{\omega^2}{A^2} \sum_{n=-\infty}^{\infty} G_L(\omega + n\omega_m) J_n^2(\Omega) + \text{signal spectrum} \quad (3.11)$$

where $J_n(\Omega)$ is the n -th order Bessel function of Ω and $G_L(\omega)$ is the familiar equivalent low-pass spectrum of the input narrowband Gaussian noise. When the modulation index is small, $J_n(\Omega) \approx 1$ and the noise spectrum is unaffected by the signal presence. Thus the spectrum of

the phase of a weakly-modulated carrier is that of $\dot{y}(t)/A$ plus the spectrum of the signal and plus the spectrum of "clicks". Neglecting the "clicks" for a moment, the one-sided low-pass equivalent of the signal-plus-noise spectrum is

$$\sigma^2 \frac{\omega^2}{A^2} G_L(\omega) + \frac{\Omega^2}{2A^2} \delta(\omega_m) \quad (3.12)$$

In the absence of modulation the carrier amplitude may be approximated as follows:

$$a(t) \approx A + x(t) \quad (3.13)$$

This step seems to be logical after assigning $\dot{y}(t)/A$ to $\theta_1(t)$, both being sums of zeroeth and first-order terms of their respective Taylor's series expansion. The complete expression describing the amplitude of the PLL input was given before (see equation 3.2). It is reproduced here again:

$$\sqrt{[A \cos \psi(t) + x(t)]^2 + [A \sin \psi(t) + y(t)]^2} \quad (3.2)$$

In the absence of modulation it becomes

$$a(t) = \sqrt{[A + x(t)]^2 + y(t)^2} \quad (3.14)$$

and is approximately equal to expression (3.13). The validity of the approximation may be deduced from the probability density function of $a(t)$; the closer it is to being normal, the better the approximation holds. The probability density function of $a(t)$ when there is no

modulation is equal to:

$$p[\alpha(t)] = \frac{\alpha(t)}{\sigma^2} e^{-(\alpha^2(t) + A^2)/2\sigma^2} I_0\left(\frac{A\alpha(t)}{\sigma^2}\right) \quad (3.15)$$

This is the so-called Rician distribution. σ^2 is the variance of the input narrowband Gaussian process; I_0 is the modified Bessel function of the zeroeth order. When $\alpha(t)$ is large or σ^2 is small, I_0 has a large argument and

$$\lim_{\frac{A\alpha(t)}{\sigma^2} \rightarrow \infty} I_0\left(\frac{A\alpha(t)}{\sigma^2}\right) = \frac{1}{\sqrt{2\pi \frac{A\alpha(t)}{\sigma^2}}} \exp\left(-\frac{A\alpha(t)}{\sigma^2}\right) \quad (3.16)$$

Thus, $p[\alpha(t)]$ approaches

$$\frac{1}{\sqrt{2\pi\sigma^2}} \sqrt{\frac{\alpha(t)}{A}} \exp\left[-\frac{(\alpha(t) - A)^2}{2\sigma^2}\right] \quad (3.17)$$

When $\alpha(t)$ is far out on the tail of the curve, the exponential term dominates and causes $p[\alpha(t)]$ to behave like a Gaussian distribution.

When A is large (a case of large carrier-to-noise ratio), then $\alpha(t)/A \approx 1$ and again $p[\alpha(t)]$ is nearly normal. (See Middleton [10]). In all cases the standard deviation is that of $x(t)$ or $y(t)$. Therefore, the approximation 3.13 applies in these cases.

When there is no signal, $\theta_1(t)$ and $\alpha(t)$ are statistically independent in the range where the above-mentioned approximations are valid. (This follows from $x(t)$ and $y(t)$ being orthogonal and Gaussian).

When modulation is present the phase of the modulated carrier, according to Rice [14], is approximately equal to:

$$\theta_1(t) = \frac{1}{A} [y(t) \cos \psi(t) + x(t) \sin \psi(t)] \quad (3.18)$$

where $\psi(t) = \Omega \sin \omega_m t$ is the modulation. This expression was offered by Rice for moderate modulation indexes. Obviously, it should hold even better for small modulation indexes. Again, applying the similar degree of approximation to the carrier $a(t)$ yields:

$$a(t) = A + x(t) \cos \psi(t) + y(t) \sin \psi(t) \quad (3.19)$$

This is the sum of two terms of Taylor's series expansion of the exact expression 3.2. Note that, again, $a(t)$ and $\theta_1(t)$ are mutually orthogonal. (Compare expressions 3.18 and 3.19.) Since ψ is independent of $x(t)$ and $y(t)$, $a(t)$ and $\theta_1(t)$ are narrowband Gaussian processes. Their orthogonality is a sufficient condition for their statistical independence.

When the modulation index is very small, $\theta_1(t)$ and $a(t)$ are equal to $y(t)/A$ and $A + x(t)$, respectively. The statistics of $a(t)$, however, do not appreciably change when the modulation index is not vanishingly small but is equal to, say, .2. The magnitude of $a(t)$ is still essentially equal to that of $a(t)$ when no modulation is present. Indeed:

$$\begin{aligned} |a(t)| &\approx \sqrt{(A + x(t) \cos \psi(t))^2 + (y(t) \sin \psi(t))^2} \\ &\leq \sqrt{(A + .980 x(t))^2 + (.199 y(t))^2} \\ &\approx A + x(t) \text{ with a very small error.} \end{aligned} \quad (3.20)$$

Therefore, once it has been established that $\theta_1(t)$ and $\alpha(t)$ are approximately independent, the statistics of $\alpha(t)$ are assumed to be those of an unmodulated carrier, i.e., a Rician distribution. Since Rice and Schwartz showed that the approximation for the phase $\theta_1(t)$ is valid at $\text{CNR} = 3 \text{ dB}$ [14] and marginal at $\text{CNR} = 1.75 \text{ dB}$ [23], it appears to be reasonable to expect similar accuracies from the simplified amplitude expression and, hence, to expect only a very low degree of interdependence between $\theta_1(t)$ and $\alpha(t)$.

3.4 A LOWER BOUND ON THE VARIANCE OF THE PHASE ERROR

If the phase-locked loop is used as an FM demodulator, no distortion should be present in the output, and thus $\sin \phi \approx \phi$ must hold. The linearized equation of the PLL disturbed by noise was derived in section 3.2; it is reproduced here again:

$$\ddot{\phi}(t) + \alpha \dot{\phi}(t) + \frac{d}{dt} \left[\frac{K\alpha(t)}{\beta} \phi(t) \right] + \alpha K\alpha(t) \phi(t) = \ddot{\theta}_1(t) + \alpha \dot{\theta}_1(t) \quad (3.21)$$

It describes a linear random system, i.e., a linear system whose driving function and some of the parameters are random functions. Such systems have been extensively studied and closed-form solutions derived for some isolated cases. This particular equation does not have a known solution,

but some bounds on the phase error may be deduced. The following development is due to Samuels [17,18,19]. Equation (3.22) is expressed in an alternative form:

$$\sum_{k=0}^2 \frac{d^k \phi(t)}{dt^k} b_k(t) = P(t) \quad (3.22)$$

where $P(t) = \ddot{\theta}_1(t) + \alpha \dot{\theta}_1(t)$

$$b_0(t) = \alpha K \alpha(t) + \alpha \dot{K} \alpha(t) / \beta$$

$$b_1(t) = \frac{\alpha K}{\beta} \alpha(t) + \alpha$$

$$b_2(t) = 0$$

$$\text{Let } b_k(t) = \bar{b}_k + \epsilon_k(t) \quad (3.23)$$

where $\bar{b}_k$ is the expected value of $b_k(t)$ and $\epsilon(t)$ is its variable part.

Substituting expression (3.23) into (3.22) results in

$$L\phi(t) = \sum_{k=0}^2 \frac{d^k \phi(t)}{dt^k} \bar{b}_k = P(t) - \sum_{k=0}^2 \frac{d^k \phi(t)}{dt^k} \epsilon_k(t) \quad (3.24)$$

where L is a constant-coefficient linear non-random differential operator.

An integral equation is obtained from the above expression:

$$\phi(t) = \int_{t_0}^t H(t;\zeta) P(\zeta) d\zeta - \int_{t_0}^t H(t;\zeta) \sum_{k=0}^2 \frac{d^k \phi(\zeta)}{d\zeta^k} \epsilon_k(\zeta) d\zeta + \sum_{k=0}^2 c_k u_k(t) \quad (3.25)$$

$H(t;\zeta)$ is the impulse response of the system associated with the L operator (i.e., the impulse response of the deterministic portion of the system).

Since the expected values of $\{b_k(t)\}$ are constant, $H(t;\zeta) = H(t-\zeta)$ in

this case. $\{u_k(t)\}$ is the fundamental set of solutions of the equation

$L\phi(t) = 0$; the c_k 's are constants and the integrals are convolution operations. Expression (3.25) can be modified further into

$$\phi(t) = F(t) - \int_{t_0}^t K(t-\zeta) \phi(\zeta) d\zeta \quad (3.26)$$

$$\text{where } K(t-\zeta) = \sum_{k=0}^2 (-1)^k \frac{\partial^k}{\partial \zeta^k} \left[\epsilon_k(t) H(t-\zeta) \right]$$

$$\text{and } F(t) = \int_{t_0}^t H(t-\zeta) P(\zeta) d\zeta + \sum_{k=1}^2 c_k u_k(t)$$

Note that $H(t) * \dot{K}(t) = \dot{H}(t) * K(t)$

Expression (3.26) states that the phase error $\phi(t)$ consists of two parts: the response of the time-invariant portion of the system and the contribution due to time-varying elements in the system. $\bar{b}_k$ of (3.23) thus belongs to the time-invariant part of the system (the bar over b_k denotes its expected value) and $\varepsilon_k(t)$ is responsible for the integral part of (3.26). The development up to this point is standard and has been used by several authors.

Since expression (3.21) cannot be solved exactly for the variance of $\phi(t)$, a lower bound for $\phi^2(t)$ is conjectured to be equal to the variance of the time-invariant portion of the system. In other words, random variations (independent of the driving function) of the parameters of a linear system disturb the system's output so that its variance increases when compared to the variance of the output of the undisturbed system. Little has been published that deals directly with this subject, but some experimental results were mentioned by Coughney et al [5,6]. The paper stated that simulations of dynamic stochastic systems showed that random variations of system parameters only unstabilized it. This was one of the papers published in reply to Samuels' erroneous claim [18] that some higher-order linear systems can be stabilized by means of random disturbance of its parameters.

Intuitively it is obvious that a random disturbance of a linear system's parameters will not reduce its mean-square error. There are no explicit statements in the literature proving or disproving this point;

perhaps because it is obvious, and also perhaps because of the difficulty involved in solving stochastic differential equations. Some results, though, can be deduced from the available publications; later in this section it will be shown that a linear system driven by any band-limited signal and whose parameters are disturbed by a white noise suffers an increase in its output variance as compared to the same system with undisturbed parameters.

The assertion that the time-varying system is "worse" than the invariant system can be argued by attempting to prove the contrary point. Suppose that the time-varying system is "better" than the invariant system. This means that the variance of the system output can be reduced by adding some random disturbance to the parameters of the system. A dubious possibility. Note that Samuels [18] tried to prove a somewhat different thing; namely, that when the time invariant linear system is unstable to start with, a random disturbance of its parameters would drive the system's poles (if one can talk about poles of a time-varying system) to the left-hand plane for the periods long enough to stabilize it. On the other hand, the assumption of the thesis concerning time-varying systems says that given a *stable* time-invariant system, the addition of noise to its parameters can only make the AC components of the output larger. That seems to be a reasonable assumption.

It was shown in section 3.2 that at large-to-moderate values of CNR $\alpha(t)$ and $\theta_1(t)$ are statistically independent. Therefore, the parameter variations and the driving function in the linearized PLL equation are statistically independent.

$\alpha(t)$ and $\dot{\theta}_1(t)$ are not always independent as suggested above.

Whenever $\alpha(t)$ becomes small, although it does not happen often, $\dot{\theta}_1(t)$ is very likely to become large, even infinitely large when $\alpha(t) = 0$. (This is easy to see from Fig. 1.1, the FM phasor diagram). Since $\alpha(t)$ is proportional to the loop gain, it is also, at least asymptotically, inversely proportional to the steady-state phase error. A small gain clearly corresponds to a large error; coupled with a sharp increase in the driving function, this leads to the conclusion that the false assumption of statistical independence of $\alpha(t)$ and $\dot{\theta}_1(t)$ would lead to an underestimate of the phase error.

A special case of the stochastic dynamic system occurs when the process disturbing the system parameters is uncorrelated with the driving function and when the spectra of the driving function and the disturbing process are drastically different. For instance, one process may be a narrowband noise and the driving function white noise. The general expression for the variance of $\phi(t)$ is easily obtained from equation (3.27):

$$E[\phi^2(t)] = E[F^2(t)] - E \left\{ F(t) \int_{t_0}^t K(t-\tau)\phi(\tau)d\tau \right\} + \quad (3.27)$$

$$+ E \left\{ \left[\int_{t_0}^t K(t-\tau)\phi(\tau)d\tau \right]^2 \right\}$$

Consider the second term of the right-hand side of the equality:

$$E[F(t) \int_{t_0}^t K(t-\tau)\phi(\tau)d\tau]$$

Since $\epsilon(t)$ is a narrowband noise, it varies much more slowly than $F(t)$ and $\phi(t)$. $K(t)$ (see equation 3.26) is proportioned to $\epsilon(t)$. Therefore, the expectation operation may be performed over the regrouped expression in the following manner:

$$E[F(t) \int_{t_0}^t K(t-\tau)\phi(\tau)d\tau] = \int_{t_0}^t E[F(t)K(t-\tau)\phi(\tau)]d\tau \quad (3.28)^1$$

$$= \int_{t_0}^t E[F(t)\phi(\tau)]E[K(t-\tau)]d\tau$$

¹From Samuels [17]: if f_1, f_2 are ergodic and their rates of change are far apart, then $\frac{1}{2x} \int_{-x}^x f_1(y)f_2(y)dy = f_1 \cdot f_2$

This can be proved through the integration by parts: the derivative of one of the functions is assumed to be negligible with respect to the other function proper.

Since $\overline{K(t)} = 0$, the integral is equal to zero and

$$\overline{\phi^2(t)} > \overline{F^2(t)}$$

(3.29)

where, as before, $F(t)$ is the response of the "expected" time-invariant system, and the bar over a function signifies an expectation operation.

In this special case no approximations were made in order to derive the inequality (3.29). In the actual phase-locked loop operation, the driving function $\alpha\ddot{\theta}_1(t) + \ddot{\theta}_1(t)$ does vary faster than $a(t)$ [or $\epsilon(t)$ in the notation of the above example], although not to the degree where it begins to look like white noise in comparison to $a(t)$. In the worst case when α is very large, $\dot{\theta}_1(t)$ is still significantly faster varying than $a(t)$. Middleton [11] had spectra calculated for $a(t)$ and $\dot{\theta}_1(t)$ (when the PLL driving function is a carrier plus a narrowband noise). The spectrum of $\dot{\theta}_1(t)$ is wider than that of $a(t)$ but certainly not by as much as an order of magnitude. Of course, when there is FM modulation present, the rate of change of $\dot{\theta}_1(t)$ is increased. Thus, for the cases where modulation is present, and especially where $\ddot{\theta}_1(t)$ is significant when compared with $\alpha\ddot{\theta}_1(t)$, the derivation of the inequality (3.29) is fully applicable to the PLL operation.

Further evidence for the validity of the assumption about the superiority of the time-invariant system can be demonstrated by the method of successive substitution applied to equation (3.26). Equation (3.26) is a Volterra integral equation which may be solved. A first approximation to $\phi(t)$ is

$\phi(t) = F(t)$; a second approximation is

$$\phi(t) = F(t) - \int_{t_0}^t F(\tau)K(t-\tau)d\tau \quad (3.30)$$

Further approximations may be obtained by inserting (3.30) into the original equation (3.26) resulting in

$$\phi(t) = F(t) - \int_{t_0}^t f(\tau)K(t-\tau)d\tau + \int_{t_0}^t \int_{t_0}^{\tau} K(t-\tau)K(\tau-\eta)F(\eta)d\tau d\eta \quad (3.31)$$

The pattern is established. The series converges to the true solution of (3.26) provided the free term $[F(t)]$ is absolutely integrable in the interval of the solution.

$$\text{Required: } \int_{t_0}^t |F(t)|dt \leq \text{Constant} \quad (3.32)$$

This condition is satisfied if a finite integration period is chosen.

According to Samuels [17], $\phi(t)$, $K(t)$, and $F(t)$ are stationary if the

$\bar{b}_k$'s are constants. Since this condition is satisfied and $\overline{\phi(t)} = 0$, the phase-error variance may be obtained by time-averaging $\phi^2(t)$ over a large time interval. The larger the time interval, the better the estimate of $\text{Var}[\phi(t)]$ is. Therefore, the finite time interval requirement can be satisfied within a small error which approaches zero as the interval approaches infinity.

Another requirement for the convergence of the series is that $K(t)$ must be finite. Since during any measurement period noise does not reach infinity, (due to the phase-locked loop component limitations) and since extremely large noise deviations occur very seldom, it may be assumed with a probability approaching unity that $K(t) \leq C$ for $t \in [t_0, T]$ where C and T are some large constants. Thus, by making the testing period finite and by taking enough terms of the series, an arbitrarily good estimate of $\text{Var}[\phi(t)]$ may be obtained. If only the first two terms of the series are used, (expression 3.30) the variance of $\phi(t)$ is equal to

$$\begin{aligned} \text{Var}[\phi(t)] = E[\phi^2(t)] &= E[F^2(t)] + E\left\{ \left[\int_{t_0}^t F(\tau)K(t-\tau)d\tau \right]^2 \right\} - \\ &- 2E\left[F(t) \int_{t_0}^t F(\tau)K(t-\tau)d\tau \right] \end{aligned} \quad (3.33)$$

where E signifies expectation operation. Because $\alpha(t)$ and $\theta_1(t)$ are nearly independent, the last term approaches zero. Indeed,

$$\begin{aligned} E[F(t) \int_{t_0}^t F(\zeta) K(t-\zeta) d\zeta] \\ = E_{\alpha(t)} \left\{ F(t) \int_{t_0}^t F(\zeta) E_{\theta_1(t)} [K(t-\zeta)] d\zeta \right\} \approx 0 \end{aligned} \quad (3.34)$$

because $E_{\theta_1(t)} [K(t)] \approx 0$. Subscripts of E signify the variable with respect to which the expectation is performed. Statistical independence of $\alpha(t)$ and $\theta_1(t)$ allowed the shown manipulations of expectation operators.

Therefore,

$$\text{Var}[\phi(t)] = E[F^2(t)] + E \left\{ \left[\int_{t_0}^t F(\zeta) K(t-\zeta) d\zeta \right]^2 \right\} \quad (3.35)$$

The first term is equal to the variance of the output of the time-invariant system; the second term is non-negative.

Within the accuracy of the two-term expansion, therefore, the linear time invariant system output yields an underestimate of $\phi^2(t)$. At high values of carrier-to-noise ratio the accuracy of the two-term expansion improves. Also, when the time constant of the phase-locked

loop is small (wide-band system) the first two terms of the series give a good approximation to $\phi(t)$. The phase-locked loop does not always operate in the linear mode. When the forcing function is large, or when the carrier amplitude $\alpha(t)$ dips to low values, the phase error builds up so that $\sin \phi(t) \neq \phi(t)$. It is well known that the PLL suffers a degradation in its tracking ability when the loop operates in the nonlinear mode. A testimonial to that fact is a host of feedback devices [2, 3, 25] that attempt to keep the phase-locked loop operations to its linear region as long as possible. The degradation in the size of the phase error is readily seen from the expression for the steady state phase error:

$$\phi_{s.s.} = \sin^{-1} \frac{\ddot{\theta}_1(t) + \alpha \dot{\theta}_1(t)}{\alpha(t)K} \quad (3.36)$$

The expression is obtained from the loop differential equation by dropping all the derivatives of $\phi(t)$. Obviously, there are no bounds for $\phi_{s.s.}$ if the inverse sine cannot be approximated by its argument. Therefore, a linearized phase-locked loop will have a smaller phase-error variance than the PLL with the sinusoidal nonlinearity in it. Hence, the assumption that the phase-locked loop is a time-invariant linear system would underestimate its phase error on two accounts: first, on account of assuming that the loop is a linear system; second, on account of further simplifying it into a time-invariant linear system. The search program of

chapter 4 uses this concept of the linear time-invariant underestimate of the phase-error variance as the basis of rejection of some low-gain narrowband loops unsuitable for the task of FM demodulation.

3.5 MEAN-SQUARE PHASE ERROR

In the region where the linearized model of the PLL is valid, the following approximation holds true:

$$\frac{d}{dt}[a(t)\phi(t)] \approx a(t)\dot{\phi}(t) \quad (3.37)$$

It was not really needed in the search program of chapter 4 because the search was based, among other things, on computing estimates of the phase-error variance of various loops and the estimates were actually underestimates in all cases except the first-order loop.

As it is later shown in this section, this approximation makes the phase error appear smaller than it is. It was used in all cases by the search program of chapter 4 because the approximation does not enter into the picture at all when solving the first-order loop equation. As the following development shows, the approximation is quite accurate in most other cases. $a(t)$ has most of its energy concentrated at zero frequency. The total energy of $a(t)$ and its DC part is governed by the following two expressions[12]:

$$W_T = 2\sigma^2(1+P)$$

$$W_{DC} = \frac{\pi\sigma^2}{2} e^{-P} \left[(1+P) I_0\left(\frac{P}{2}\right) + P I_1\left(\frac{P}{2}\right) \right] \quad (3.38)$$

where σ^2 = variance of input narrowband Gaussian noise

$$P = A^2/2\sigma^2$$

I_0 , I_1 are modified Bessel functions of the zeroeth and the first order, respectively.

At a carrier-to-noise ratio of 3 dB, the DC component accounts for 89 percent of the total $\alpha(t)$ (envelope) energy. The spectrum of $\alpha(t)$ has a very large impulse at zero frequency. By remaining small, $\phi(t)$ must necessarily be a good replica of the driving function $\alpha\dot{\theta}_1(t) + \ddot{\theta}_1(t)$. In section 3.2 it was shown that at high to moderate carrier-to-noise ratios $\alpha(t) \approx A + x(t)$ and $\theta_1(t) \approx \frac{y(t)}{A}$ where $x(t)$, $y(t)$ are cophase and quadrature components of the input noise. It follows then that $\phi(t)$ consists of a linear combination of first and second derivatives of $y(t)$, thus possessing a wider spectrum than $\alpha(t)$. When the resultant in Figure 3 ventures close to the origin, $\alpha(t)$ is small, but $\dot{\theta}_1(t)$ is likely to become extremely large and the approximation $\frac{d}{dt}[\alpha(t)\phi(t)] \approx \alpha(t)\dot{\phi}(t)$ holds even better (in the mean sense).

Consequently, the linearized phase-locked loop equation is equal to:

$$\ddot{\phi}(t) + \alpha[1 + \frac{K}{\beta}a(t)]\dot{\phi}(t) + \alpha Ka(t)\phi(t) = \ddot{\theta}_1(t) + \alpha\dot{\theta}_1(t) \quad (3.39)$$

The exact loop differential equation is

$$\ddot{\phi}(t) + \alpha[1 + \frac{K}{\beta}a(t)]\dot{\phi}(t) + \alpha K\phi(t)[a(t) + \frac{\dot{a}(t)}{\beta}] = \ddot{\theta}_1(t) + \alpha\dot{\theta}_1(t) \quad (3.40)$$

$a(t)$ and $\dot{a}(t)$ are orthogonal processes and because of their near-Gaussian statistics [especially when $a(t)$ is large] they are statistically independent, or at least tend to be so. By removing $\phi(t)\dot{a}(t)/\beta$ from expression (3.39) the number of independent noise sources in the system is reduced. According to the arguments presented in section 3.4, this development should result in the reduced output variance due to noise. In other words, evaluation of the system described by expression (3.38) would tend to underestimate $\phi^2(t)$. When values of α and β are near each other, or when α or β are very large with respect to AK , the term containing $\frac{\dot{a}(t)}{\beta}$ becomes negligible. This is readily seen by manipulating α and β in expression (2.13).

If the PLL is fast responding, then a quasi-stationary method of solving the loop equation for the phase-error variance may be used. In the quasi-stationary model, $a(t)$ is equated to a constant A_i [an instantaneous value of $a(t)$]. If the loop is fast responding, its phase error bears very little dependence on the past history of $a(t)$, it only depends on its present value and such an approximation may be quite accurate.

Replacing $a(t)$ with A_i in equation (3.39), taking its Laplace transform and then solving it for the phase error yields:

$$\Phi_1(s) = \left[\frac{(s^2 + \alpha s)}{s^2 + \alpha s(1 + KA_i/\beta) + \alpha KA_i} \right] \theta_1(s) \quad (3.41)$$

where the expression in brackets is the PLL transfer function $H_i(s)$. The conditional second moment of the phase error, given an instantaneous value of $a(t)$, can be obtained by the regular technique of linear systems

$$E [\phi_i^2(t) | a(t) = A_i] = \int_0^\infty |H_i(j\omega)|^2 S(\omega) d\omega \quad (3.42)$$

where $S(\omega)$ is the one-sided spectrum of $\theta_1(t)$ without the "clicks".

Since $\theta_1(t)$ and $a(t)$ are approximately independent for $\text{CNR} = 3$ dB and larger, it is quite proper to use the same spectrum $S(\omega)$ with the transfer function $H_i(s)$ evaluated at different values of $a(t)$. If $\theta_1(t)$ and $a(t)$ are not independent, then a different spectrum must be used with different instantaneous values of $a(t)$ lest an optimistic value of $\phi_i(t)$ is obtained.

Since the phase-locked loop is a time-varying system (the carrier amplitude $a(t)$ assumes values ranging from zero to plus infinity), the lower values of $a(t)$, whenever they occur, are apt to cause the loop to lose lock, or at least to create a large phase error. The latter condition means that the techniques of the linear system should not be used

when $a(t)$ assumes small values. The quasi-stationary assumption also breaks down in such situations, since the loop becomes sluggish. Consequently, the quasi-stationary method cannot be used to find the unconditional phase-error variance which, of course, encompasses all the possible values of $a(t)$.

Although the phase-error variance cannot be found, the method still can be used to calculate a conditional phase-error variance. Expression (3.42) describes a conditional variance for all instantaneous values of $a(t)$ exceeding the amplitude threshold A_o . The expression is accurate as long as the quasi-stationary assumption applies. The major obstacle to the use of the method - small values of $a(t)$ - is circumvented

$$E[\phi^2(t) | a(t) > A_o] = \frac{\int_{A_o}^{\infty} \int_0^{\infty} |H_i(j\omega)|^2 S(\omega) p(A_i) d\omega dA_i}{\int_{A_o}^{\infty} p(A_i) dA_i} \quad (3.43)$$

As already mentioned in the preceding section, when $a(t)$ assumes small values, one should not expect to find $a(t)$ and $\theta_1(t)$ to be independent.

In fact, the derivative of $\theta_1(t)$ may assume large values as $a(t)$ sweeps close to the origin of the phasor diagram (Fig. 1.1), some of those sweeps result in "clicks". M. Schwartz [23] suggested that when $a(t)$ is equal to approximately one third of its nominal value A , then a sharp increase in the number of "clicks" is taking place. Therefore, using the amplitude

thresholds $A_0 \geq .5A$ should keep the computation in the region where the quasi-independence of $\alpha(t)$ and $\theta_1(t)$ is not invalidated.

A physical mechanization of the quasi-stationary solution could be considered to be a time-sampling of various block lengths of $\phi(t)$. Duration and timing of such sampling are determined by the statistics of $\alpha(t)$: whenever $\alpha(t)$ is greater than a certain assigned threshold A_0 , the variance of $\phi(t)$ is computed; when $\alpha(t)$ drops below the threshold, then $\phi(t)$ is ignored and the computation stops until $\alpha(t)$ again climbs above the threshold. If the recovery time of the linearized PLL is shorter than the rate of change of $\alpha(t)$ and as long as $\alpha(t)$ is independent of the driving function $\phi(t)$, the quasi-stationary solution is applicable.

There is no well-defined rule determining the circumstances in which the quasi-stationary solution can be used; therefore, the following rule was postulated: A_0 , the smallest admissible instantaneous carrier amplitude, is chosen so that whenever $\alpha(t) > A_0$ is true, then $\alpha(t)$ must not change more than a certain fraction within the time interval equal to one time constant of the linearized loop equation.¹ An additional restriction is, of course, that the phase error must remain small so that the phase-locked loop behavior is described by a linear differential equation and the techniques of linear systems may be used to solve for the conditional phase-error variance. The definition of the time constant is reserved for

¹ More will be said about this subject further on.

first-order systems only. In lieu of it, the reciprocal of the real part of the dominant pole of the transfer function is used with second-order systems. The first-order system transfer function has only one pole; thus, the definition of the time constant is a special case of the latter concept. As an example, a few values of the threshold A_0 are estimated for the case of a first-order system.

Let the one-sided bandwidth of $\theta_1(t)$ be one rad/sec and the CNR be equal to 3 dB.¹ Then the standard deviation of $\dot{a}(t)$ is approximately equal to .289 [the computation is straightforward after the assumption: $a(t) = A + x(t)$]. Since the order of the system considered in this example is one, the "instantaneous" time constant is equal to $\tau = 1/a(t)K$. Changes in $a(t)$ that take place within a time interval which is equal to one time constant of the linearized loop are to be restricted to a certain value, for instance, 2, 1, .5.² These numbers are arbitrary; they will be referred to as "tolerance thresholds". The above-mentioned restriction is expressed algebraically as follows:

¹More on the normalization of the PLL parameters is presented in chapter 4, section 4.2.

²Results of Monte Carlo simulations described in appendix 2 showed that the tolerance threshold values as high as 2.3 can be used and still obtain a satisfactory accuracy from the quasi-stationary method.

$$\frac{.289\tau K}{a(t)K} \leq \text{Tolerance threshold} \quad (3.43)$$

This is the general expression; the substitution of $\tau = 1/a(t)K$ into it results in $\frac{.289}{a^2(t)K} \leq \text{Tolerance threshold}$ (3.44)

which only applies to first-order loops. For simplicity's sake, let A (the nominal carrier amplitude) be unity. The loop gain AK still can assume any value by changing the constant K ; furthermore, this assumption does not restrict the range of the PLL parameters in any way, since A and K appear in the loop differential equation as a product, never separately. With these assumptions in hand as an example, the following minimum first-order loop gains were obtained for $A_0 = .5$ and the tolerance thresholds of 2, 1, .5, respectively: .58, 1.16, 2.31.

3.6 CONCLUSION

All of the tools are now available for a global search for the phase-locked loop parameters such that the PLL output contains the minimum number of demodulated "clicks" and has a given mean-square phase error. The search for the optimum phase-locked loop is conducted on the basis of

the quasi-stationary model whenever the loop's parameters are such that the PLL is high-gain and broadband. When the quasi-stationary model is not applicable, the "expected" time-invariant form of the PLL equation of section 3.3 is solved for $\phi^2(t)$. In the latter case, the answer is an underestimate of the true solution. Since, in all cases throughout the entire computer search this underestimate was larger than the one for the broadband PLL, the narrowband and/or low-gain loops could be discarded without further ado. Were there at least one case in which the underestimate of $\phi^2(t)$ turned out to be smaller than the result of the quasi-stationary model, the search program would have had to be complicated a great deal. But such was not the case. Chapter 4 is devoted to the computer search program and covers the above-mentioned details more thoroughly.

CHAPTER 4

THE SEARCH PROGRAM

4.1 INTRODUCTION

From the totality of all the first- and second-order loops the optimum loop was found by a creeping random search method. The optimum loop satisfied the following requirements:

1. With all the initial conditions set to zero, it would most readily lose its lock and skip a cycle whenever it was acted upon by a "click", or to be exact, a model of a "click" discussed in chapter 2.
2. When not acted upon by the "click", its phase error was small. This was implemented by a constraint in the search program: the conditional variance of the phase error had to be small enough to allow the loop to operate in its linear region. In addition to noise, an FM signal sinusoidally modulated with a very small modulation index was assumed to drive the PLL.

Several starting points of the search program yielded identical answers, namely the first-order loop is the phase-locked loop which would demodulate a narrow-band FM signal the best in the sense of suppressing the highest number of "clicks" while keeping the phase error below a given constraint.

4.2. SCALING OF THE PLL DIFFERENTIAL EQUATION

The search for the "optimum" loop was conducted over the field of all realizable first- and second-order loops preceded by a narrow band-pass filter with one-sided bandwidth of one radian-per-second. (See Fig. 3.1). If desired, the translation of these results to a loop with a different input bandwidth is readily accomplished by a familiar time-scaling technique often used in analogue computations.

Let the PLL differential equation be time-scaled by a factor of ω_b , which is the new desired bandwidth of the input signal. The old time t is equal to τ/ω_b where τ is the new time. If $\omega_b > 1$, then the new solution will be faster than the old. Letting $\frac{d}{dt} = \omega_b \frac{d}{d\tau}$ and $\phi'(\tau) \triangleq (\tau)\phi'(\tau)$ and substituting them in the PLL differential equation yields:

$$\begin{aligned} \omega_b^2 \phi''(\tau) + \alpha \omega_b \phi'(\tau) + \frac{\alpha K \alpha(\tau)}{\beta} \omega_b \phi'(\tau) \cos \phi'(\tau) + \\ + \left(\alpha K \alpha(\tau) + \frac{\omega_b \alpha \dot{\alpha}(\tau) K}{\beta} \right) \sin \phi(\tau) = \omega^2 \theta_1''(\tau) + \alpha \omega_b \theta_1'(\tau) \end{aligned} \quad (4.1)$$

Evidently, the old PLL equation can be obtained by dividing every parameter by ω_b , $\alpha(\tau)K$ is to be treated as a single parameter-loop gain. Thus, the frequency translation from the normalized computer equation to any other PLL equation is accomplished by simply dividing each loop parameter by the new desired one-sided bandwidth in radians per second.

4.3. CONSTRAINTS AND THE CRITERION OF OPTIMIZATION

The optimum PLL is supposed to skip a cycle readily whenever a "click" comes along. The probability of such an event taking place in conjunction with a given phase-locked loop is determined as follows:

$a(t)K$ is a Rician random variable and represents the instantaneous loop gain. The technique developed in sections 2.3, 2.4, and 2.5 is used to calculate the maximum loop gain which would allow the cycle-skipping to occur in response to the "click". The conditional probability of cycle skipping, given a "click", can be expressed as:

$$Pr \left\{ \text{cycle skipping} \mid \text{"click"} \right\} = \int_0^{a(t)_{cr}} p(a) da \quad (4.2)$$

where $p(a)$ is the conditional probability density function of $a(t)$ given that the "click" is occurring.

The same probability of cycle skipping may be expressed differently where instead of $a(t)_{cr}$ the ratio $R = a(t)_{cr}K/(AK)$ is used in the lower limit of the integral.¹

$$Pr \left\{ \text{cycle skipping} \mid \text{"click"} \right\} = \text{CONST} \cdot \int_{-(1+R)}^{-1} \exp(-x^2/2\sigma^2) dx \quad (4.3)$$

Since this integral is a monotonic function of the lower limit the conditional probability can be maximized

¹ Section 4.5 presents a detailed treatment of this point

by maximizing the ratio $\frac{\alpha(t)K}{AK}$ (instantaneous loop gain/nominal loop gain). Maximization of this ratio is a criterion of the "optimum" loop selection.

The search program also contains a constraint on the size of the phase error. Section 3.5 discusses the reason which prevents the exact computation of the phase-error variance. Therefore, the conditional variance is calculated in its place by means of the quasi-stationary method.

The admissibility of the quasi-stationary solution was determined by the value of the "tolerance threshold" computed for each loop configuration by the search program. The constraint on the size of the "tolerance threshold" means that the lowest admissible $\alpha(t)$ does not change by more than the factor equal to the "tolerance threshold" within the time interval equal to the time constant of the instantaneous PLL transfer function (i.e., the transfer function evaluated for AK equal to the lowest admissible value of $\alpha(t)K$).

During each iteration of the search program the conditional phase-error variance is computed provided the loop operates within its linear region and within the region where the quasi-stationary solution discussed in section 3.6 is applicable. The applicability of the quasi-stationary solution is determined by finding the dominant pole of the phase-error transfer function. The inverse of its real part is used as the time constant of the loop and the "tolerance threshold" is obtained from it.

Since it was not a purpose of this program to build a hierarchical structure based upon the goodness of various phase-locked loops, but rather, to find the optimum loop, the above procedure suffices. In other words, the search program rejected unsuitable loop parameters if the loop performance fell below certain threshold regardless of how much it fell short.

Below are outlined the steps of the search program:

1. Values for three loop parameters - α , β , and AK are selected randomly.
2. The phase-locked loop time invariant $[a(t) = A]$ differential equation is approximately solved for the critical loop gain; that is for the largest loop gain that would allow Hess' FM "click"¹ to cause the PLL to skip a cycle. The approximate solution of the time-invariant PLL differential equation driven by a "click" is covered in sections 2.5 and 2.6.

The time invariance of the carrier amplitude during the "click" occurrence, as well as zero initial conditions, are stipulated by the Hess' "click" model. This point is discussed in section 2.6. The approximate solution of the loop equation was found to be very accurate² once a constant bias of 10 % was removed from it.

¹See sections 2.3, 2.4.

²See Appendix 1.

3. Loop parameters are checked to see whether the "tolerance threshold" is exceeded. If it is, a linear underestimate of the phase-error variance is computed and the program goes to item 5.
4. The time-variable phase-locked loop equation is solved for the conditional variance of the phase error with the loop being driven by a narrow-band noise plus an FM signal. Very small indices of modulation (up to 0.2) were used in the search program in order not to invalidate an assumption of zero initial condition preceding an advent of a "click". The governing criterion on the maximum allowable modulation index is the magnitude of the frequency deviation ($\Delta\omega$) caused by the signal. If $\Delta\omega$ is much smaller than the peak of the Hess' "click", the initial condition of the loop before the advent of the "click" may be assumed to be zero. For an input band-pass filter with one-sided bandwidth equal to 1 radian per second, the peak of the "click" is equal to twice the radius of gyration = 1.1546. With the modulation index equal to, say 0.5, the frequency deviation ($\Delta\omega$), due to sinusoidal modulation of frequency $\Delta\omega$, is equal to 1/3. (This comes from the equations relating the bandwidth of the FM signal to the modulation index (Ω): $(1 + \Omega)\omega_m = (\text{one-sided bandwidth})$ and $\Omega = \frac{\Delta\omega}{\omega_m}$). Obviously, $\Delta\omega=1/3$ is not much smaller than 1.1546, $\Omega=0.2$ yields

$\Delta\omega = .167$ which may be acceptable. Note that if the conditional variance exceeds a certain limit ($\pi/10$, $\pi/15$, and $\pi/20$ were used on separate occasions), the loop under consideration is rejected and a new set of α , β , AK are picked. Numerical solutions of the PLL differential equation under an influence of a "click" and a constant frequency offset due to a modulation are reported in Appendix 1. Results indicate that modulation indexes as high as .46 could be used and still keep the resultant variations of the value of the critical loop gain linear with respect to the size of the frequency offset. When the modulation index is small, the sense of the "clicks" becomes independent of the modulation. On the other hand, when the modulation index is large, the "clicks" overwhelmingly tend to oppose the modulation. This occurs because the modulation shifts the reference frequency with respect to the narrow-band noise spectrum. The higher the modulation index, the larger is $\Delta\omega$, the further away from the center frequency of the noise spectrum the instantaneous carrier frequency finds itself. The result is that the instantaneous frequency may swing so far that no noise spectrum energy is located in the path of its swing and then, of course, all the "click" will be of the opposite sense to the modulation. Calculations showed that at $CNR = 3$ dB and the modulation index of .2, the ratio of the number of "clicks" opposing the modulation to the

number of "clicks" boosting the modulation is at most 1.15. At $\Omega = .1$ the ratio becomes less than 1.05. The numbers are small enough to assume that the sign of the "click" is independent of modulation

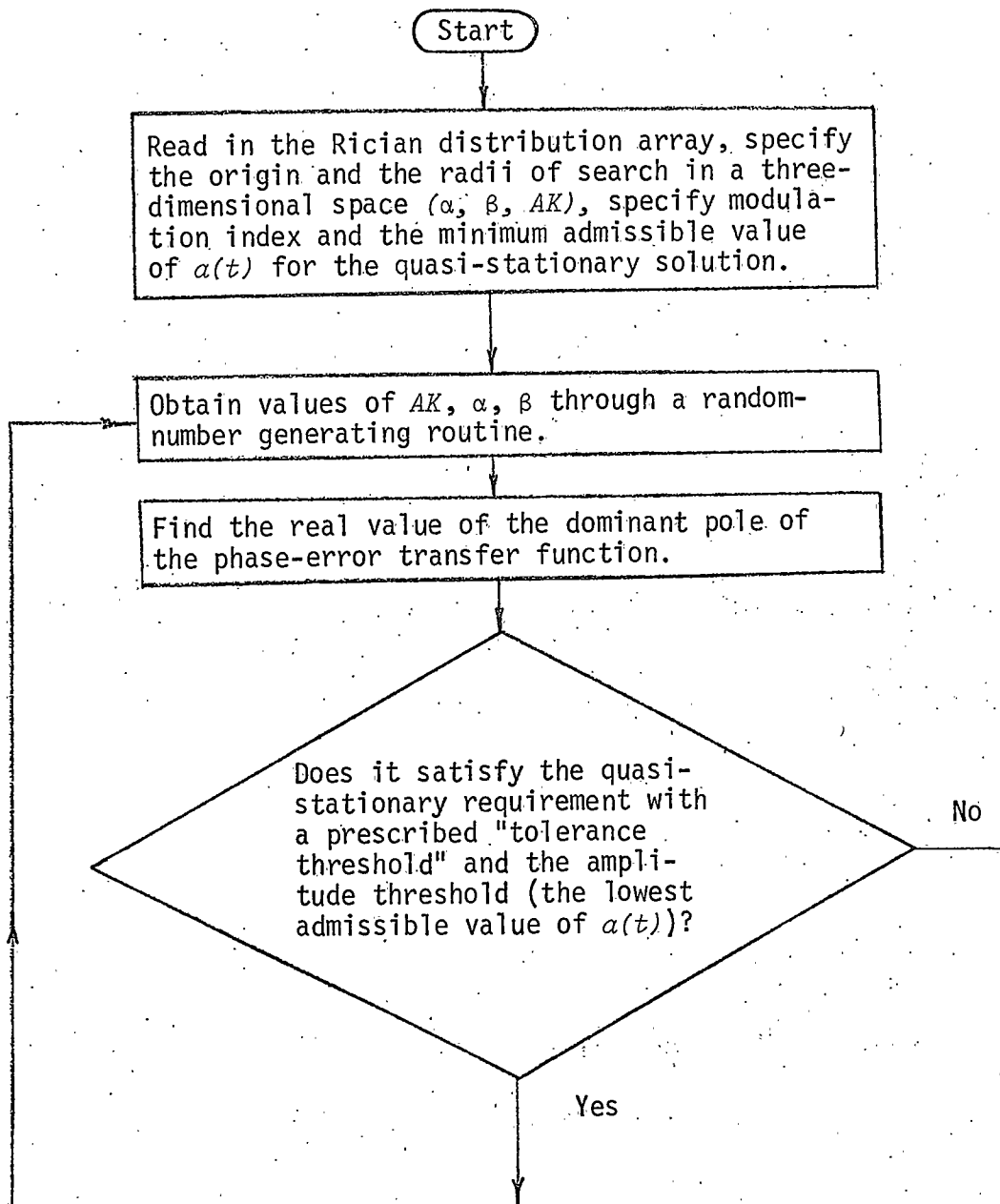
5. The "ratio" of section 4.3 (ratio of critical AK to the nominal AK) is computed and stored along with the loop parameters used in this particular iteration.

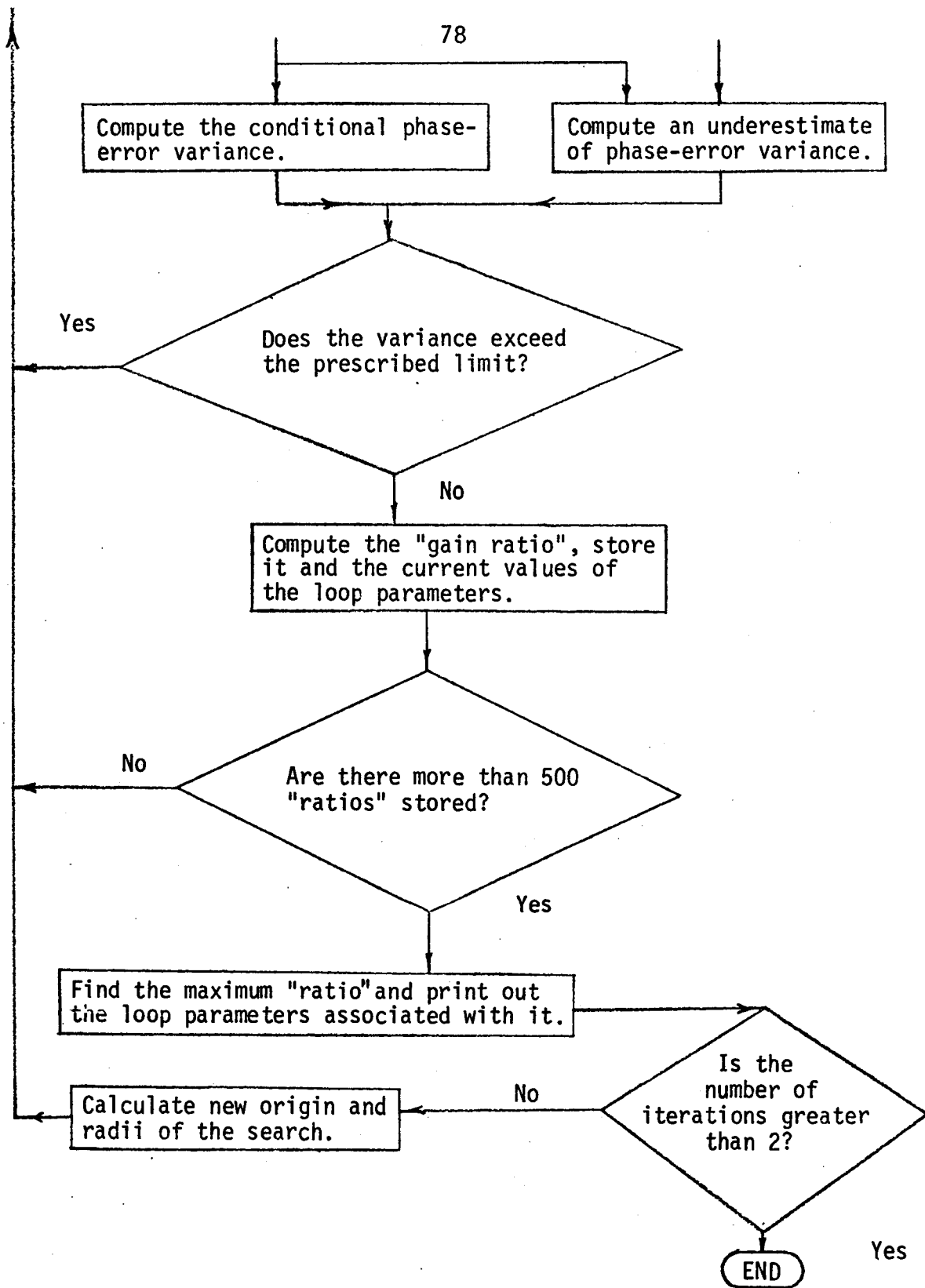
The four steps listed above are repeated a number of times (usually 500). Then the maximum "ratio" is chosen and the localized loop optimization is achieved.

The "creeping randomness" of the search program exists solely in order to save computer time. Each iteration of the search computes, say, 500 "gain ratios". Random values of loop parameters are generated about some arbitrary mean values (or the centers). The optimum ratio then provides the distance between the old center and the new optimum point in α , β , AK coordinates. This optimum point is then picked as the center for the next round of 500 "ratios" and the newly acquired radius of search is equal to the old radius decreased by a factor which is proportional to the distance between the new and the old search radii. This procedure is repeated several times. It allows one to search in the most promising region of the parameter space, thus making the search more efficient.

The search program was started from several different points. They all led to a PLL whose filter configuration was such that $\alpha \approx \beta$, i.e., the first-order loop. A flow chart of the search program is presented in table 4.1.

TABLE 4.1. Flow Chart of the Search Program.





4.4. RESULTS OF THE SEARCH PROGRAM.

Table 4.2 contains the data obtained by the search program. Most of the search was conducted at the carrier-to-noise ratio of 3 dB with the modulation indexes .2 and .1. It was not necessary to conduct extensive searches at higher values of CNR, since the data obtained at CNR = 3 dB were constrained by three different limits on the conditional phase-error variance. More stringent limits at CNR = 3 dB correspond to loose limits at higher values of CNR and thus the reported results are quite general. The majority of starting points of each search were chosen to be a loop configuration which is closer to the second-order type II loop than to the first-order loop. In all cases the final loop configuration has α and β nearly equal which, of course, corresponds to the first-order PLL. Changes did not affect the character of the point of arrival of the search program - the first-order loop was the answer in all cases.

Data contained in table 4.2 show that although α approaches β , they are never equal. This was done deliberately by the search program where β was chosen to be larger than α in order to avoid division by zero in the "Gain" subroutine which finds the critical loop gain. Note that the smaller α is, the more sensitive the loop is to the difference between α and β . The number of iterations in each search was preset, i.e., it was independent of the final outcome of the search.

TABLE 4.2. Search Results.

Search starting points			Search end points			Variance constr.	Tolerance threshold constraint	CNR (dB)	Amplitude Threshold	Number of iterations
α	β	AK	α	β	AK				A_o	
.05	.20	2.00	.0729	.0734	1.834	$(\pi/20)^2$	1.00	3	.5	2
.05	.20	2.00	.0898	.0908	1.674	$(\pi/20)^2$	.70	3	.7	2
.50	1.00	1.50	.6309	.6334	1.826	$(\pi/20)^2$	1.20	3	.5	2
.05	.30	2.00	.0348	.0385	1.232	$(\pi/15)^2$	.80	3	.6	4
2.00	6.00	2.00	2.0400	2.0650	.616	$(\pi/10)^2$	2.30	3	.5	2
.20	.60	2.00	.0337	.0340	.623	$(\pi/10)^2$	2.30	3	.5	3
1.00	3.00	1.00	.9725	.9779	.825	$(\pi/15)^2$	.90	5	.6	2
.02	.06	3.00	.0050	.0056	.928	$(\pi/15)^2$	.90	5	.6	3

4.5. COMPARISON OF THE RESULTS OF THE SEARCH PROGRAM TO SOME PUBLISHED DATA.

Schilling et al [1] obtained results which at a glance may seem to be contradictory to the conclusion of this thesis. Specifically, Schilling found that a wide-band second-order phase-locked loop with a low-pass RC filter in the loop would have the lowest threshold in demodulating an FM signal when the linearized loop differential equation is critically damped. Figure 4.1 is reproduced here from a report where these results were published.

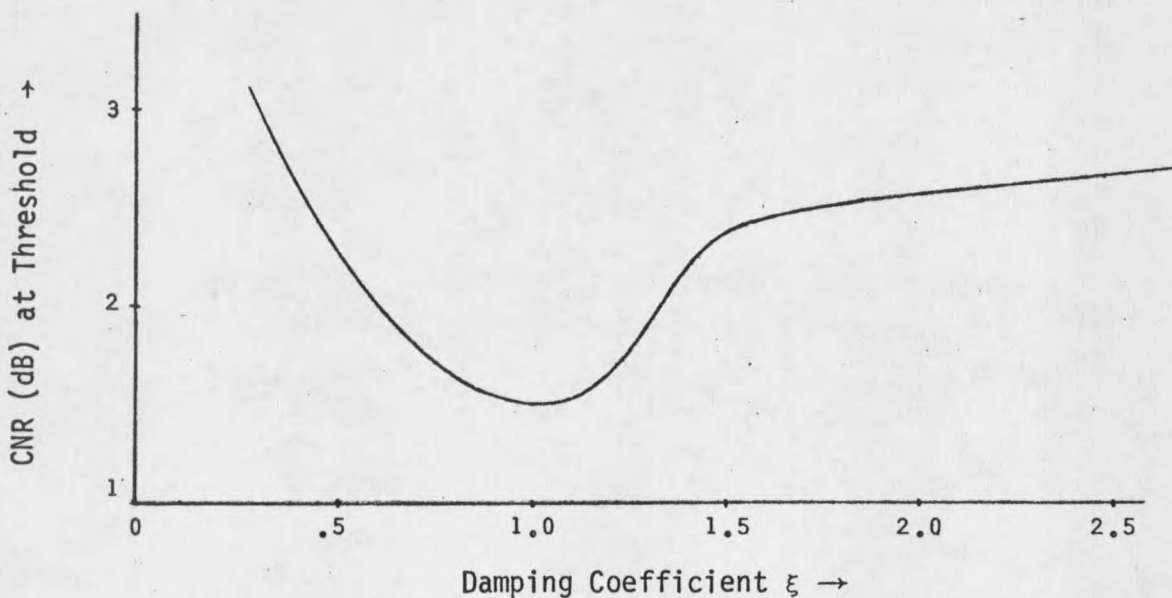


FIG. 4.1. Effect of Loop Damping on Threshold from Reference [1].

In the reported experiment the loop gain was kept constant while the damping coefficient (ξ) was varied. No modulation was present and the threshold was determined on the basis of the output noise power.

A few calculations are required in order to explain the shape of the curve in Fig. 4.1. The phase-locked loop used to obtain it may be scaled down by the method of section 4.2 to the standard loop used in this thesis with a one-sided input bandwidth of 1 rad/sec. The scaled-down parameters are $AK = 2.594$, $\alpha = 10.38$, and $\beta = \infty$, and the damping coefficient is equal to $\alpha/2K$. An underestimate of the phase-error variance can be computed by integrating the product of the spectrum of the driving noise (flat) and the square of the absolute value of the linearized transfer function of the PLL. This procedure was explained in section 3.5.

$$\overline{\phi^2(t)} \geq \sigma^2 \int_0^1 |H(j\omega)|^2 d\omega$$

$$\text{where } H(j\omega) = \frac{s^2 + \alpha s}{s^2 + \alpha s + \alpha AK} \quad (4.4)$$

The transfer function has no terms containing β , since this particular loop has only a simple R-C low-pass filter whose voltage transfer function is equal to $\alpha/(s + \alpha)$. σ^2 is the variance of the phase of the carrier plus narrowband noise process. Thus, at the carrier-to-noise ratio of 3 dB and carrier amplitude normalized to unity, $\sigma^2 = 0.25$. As explained in section 3.5, $\phi^2(t)$ calculated in the above-mentioned fashion will be an underestimate of the true variance; that is why there is an inequality sign in expression (4.4). Another calculation that needs to be made is to determine what fraction of the total number of the incoming

"clicks" was suppressed by this phase-locked loop. The "click" occurs when the phase of the input narrowband Gaussian process abruptly changes by $\pm 2\pi$.

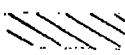
Details of the "click" formation were explained in Chapter 2. It is sufficient to say that at the time of the occurrence of the peak of the "click", $x(t)$ - the component of the Gaussian noise in-phase with the carrier - is negative, and its magnitude is larger than that of the carrier. $y(t)$ - the quadrature component of the Gaussian noise - is rapidly sweeping around the origin, so that the sign of $\dot{y}(t_0)$ is the sign of the "click". The necessary conditions for the occurrence of a "click" are, therefore, $-x(t_0) > A$ and $\dot{y}(t_0) \neq 0$. The loop gain, according to Hess¹, stays relatively constant for the duration of the "click" and is equal to $|A + x(t)| \cdot K$. Since $x(t)$ is Gaussian, it is easy to calculate the probability of $|A + x(t)|$ exceeding a certain level.

$$P(|A + x(t)| > A_0 \mid \text{"click" is taking place}) =$$

$$\frac{\frac{1}{\sigma\sqrt{2\pi}} \int_{-(A_0+A)}^{-A} e^{-x^2/2\sigma^2} dx}{\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{-A} e^{-x^2/2\sigma^2} dx} = \text{Fraction of the suppressed "clicks" out of the total number of incoming "clicks"}. \quad (4.5)$$

Figure 4.2 shows the domains of this expression.

 - Area equal to the numerator of equation (4.3).

 - Area equal to the denominator of equation (4.3).

¹See section 2.5.

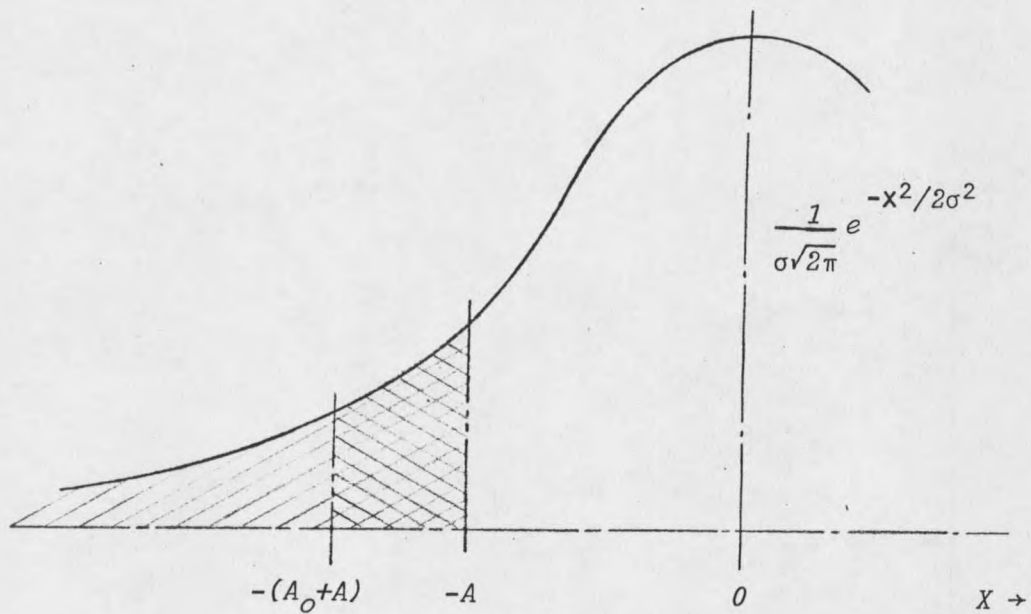


FIG. 4.2. Domains of Equation 4.

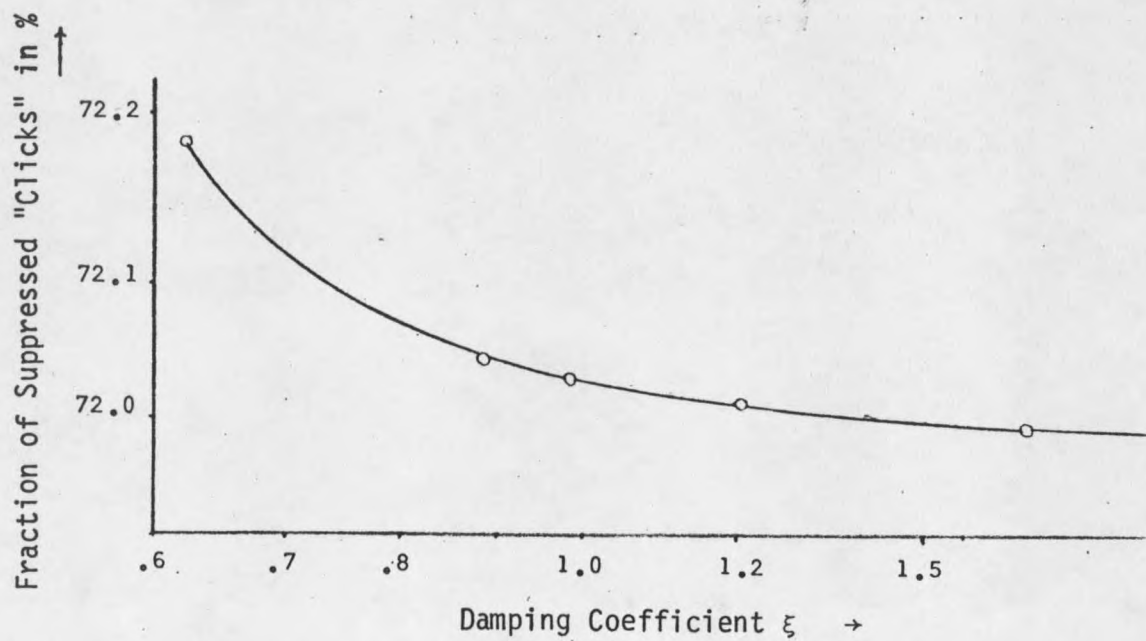


FIG. 4.3. Plot of Percentage of Suppressed "Clicks" Out of the Total Number of Incoming "Clicks" Versus the Damping Coefficient.

Several values of the critical loop gain were computed for the given loop parameters. The driving force was assumed to be Hess' "click". From these critical gains the fraction of the incoming "clicks" that were suppressed by this PLL was computed using the parameters of Schilling's loop and the results plotted in Fig. 4.3.

The plot shows that with high damping the loop's ability to suppress "clicks" deteriorates very slowly. Note that the slope of the threshold location versus ξ curve in Fig. 4.1 also has a very shallow slope at high values of ξ . Figure 4.3 and the data in Appendix I indicate that decreased damping enhances "click" suppression. However, as Fig. 4.4 demonstrates, the phase-error also increases with decreased damping. This effect apparently nullifies the benefit of more "clicks" suppressed in an under-damped loop.

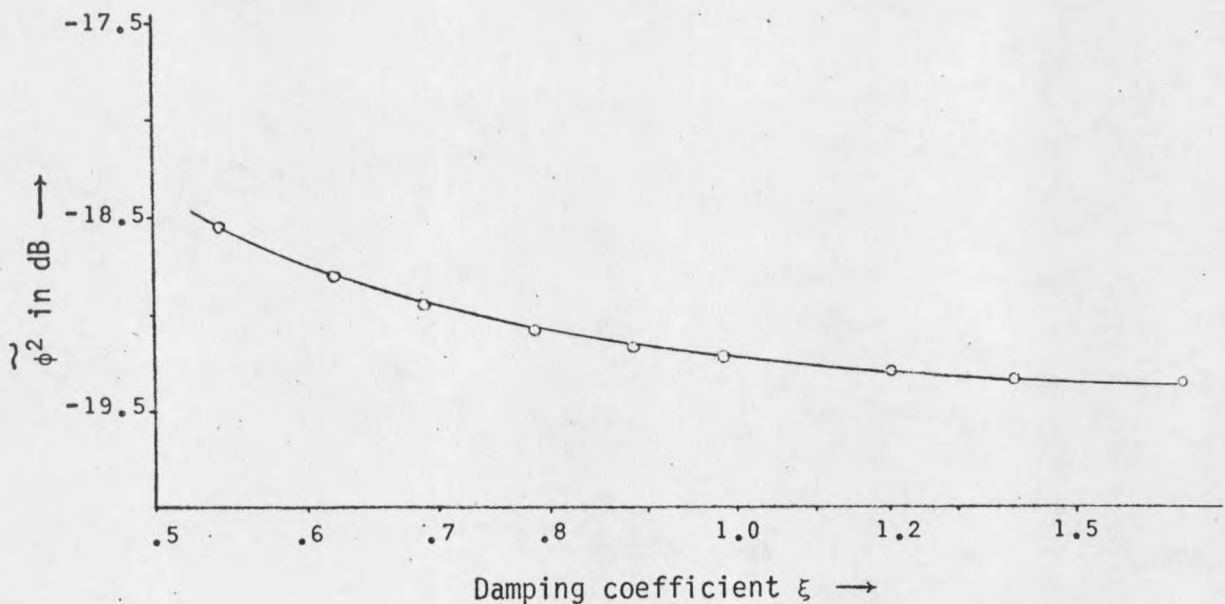


FIG. 4.4. Plot of an Underestimate of the Phase-Error Variance of the PLL.

Another additional possible cause of the worsened threshold performance at low damping coefficient is the transients causing an overshoot when the loop recovers after skipping a cycle. As was discussed in Chapter I, this phenomenon was not taken into account in search for the optimum loop. The fact that the underdamped second-order loops lost out to the first-order loop despite such a bias in their favor means that in real life the difference in their threshold performances is very significant.

The results presented in Fig. 4.3 and 4.4 rationalize the shape of the curve in Fig. 4.1 by means of the techniques used in this thesis. Although a quantitative analysis of Fig. 4.1 was not possible due to the inability to calculate exactly the noise and "spike" power put out by the PLL, superposition of curves from Fig. 4.3 and 4.4 gives qualitative interpretation to the behavior of the loop in question. The superiority of the first-order over the second-order loop was deduced by this thesis through techniques similar to ones used in obtaining Fig. 4.3 and 4.4 with the exception of using a more precise quasi-stationary solution instead of the expression (4.4). While the successful qualitative interpretation of the shape of the curve in Fig. 4.1 is not to be construed as proof of the correctness of the findings of this thesis, it nevertheless shows that there is no contradiction between the two results. Thus, in light of this thesis' findings, the curve in Fig. 4.1 could have a continuously negative slope if the loop gain were decreased while increasing ξ in such a way that the phase error remained constant. In obtaining the data for the plot in Fig. 4.1, the noise-induced phase-error was apparently allowed to shrink (it was not monitored as the damping coefficient of the loop was

stepped up). (See Fig. 4.3). This potential excess in the loop gain was not exploited and, therefore, the authors concluded that the second-order critically damped loop is the optimum one.

Another pertinent reference relating to the findings of this thesis is the paper published by P. Osborne and D. Schilling [13]. A brief discussion of this article was included in section 1.2. Here a somewhat expanded version of that discussion is given.

The authors considered cases of the first, second, and third-order loops. Higher-order loops contained ideal proportional-plus-integral filters (the transfer function of such a filter is equal to $K_1 + K_2/s$). The authors numerically solved the PLL differential equation driven by a "click" which was represented by a deterministic function equal to a conditional expectation of the quadrature components of the noise, given their values and their derivatives at the time of the peak of the "click". In other words, the authors varied the height of the "click" and allowed for variation in the loop gain during the occurrence of the "click" - a much more complex model than that of Hess. The numerical solution provided them with a set of data from which they constructed a three-dimensional spike region. The random noise vector wandering into this region would cause the loop to demodulate the "click" and cause a spike in the output. The volume of the spike region was then evaluated and the expected number of spikes per second was obtained. With the modulation present they found the second-order loop able to suppress

more "clicks" than the first-order loop by a factor of 100. (This does not contradict the results of the thesis: second-order loops, especially under-damped loops, were found to be better "click" suppressors than first-order loops ¹ as long as the constraint on the size of the phase-error is ignored.)

The main difference between the work reported in this thesis and the work done by Osborne and Schilling is that they studied cases with a broad-band FM, while the thesis studies were limited to a narrow-band FM and there is no contradiction between the two results because they cannot be compared.

Although the results of the article could not be compared to the findings of the thesis, it brought about an interesting point: if the first-order loop is optimum at low modulation indexes and the second-order loop with particular filter configuration is superior to it at high modulation indexes, there is a transition region where the configuration of the optimum loop filter must depend on the modulation index.

There are a few other comments that are to be made about this article.

1. Osborne and Schilling used a deterministic "click" in their simulation program. While their function is a perfectly logical choice for the "click" model and is intellectually more satisfying than Hess' model, it did not verify experimentally as well as Hess' model. The

¹See Appendix 1.

number of their predicted "spikes" varied from the experimental results by a factor of up to 3.

2. The constraints on the loop gain were determined by the magnitude of the phase error caused by the modulation; this phase error is referred to as ϕ_m and $|\phi_m| \ll \pi/2$ is then the constraint used by the authors. The maximally-flat second-order PLL with a proportional-plus-integral filter when operated in its linear region requires then that

$$\frac{\Omega \omega_m^2}{\sqrt{G^4 + \omega_m^4}} \ll \frac{\pi}{2} \quad (4.6)$$

where Ω is the modulation index, G is the loop gain and is equal to αAK in the notation of this thesis. Equation (4.6) corresponds to expression (12a) in [13]. The input signal bandwidth was fixed (recall that the signal was passed through a narrow-band filter before entering the loop). The 3-dB bandwidth of the modulated carrier, according to Carson's rule, is equal to $2(1 + \Omega)\omega_m$. Then, clearly, a large Ω would correspond to a small ω_m and phase error ϕ_m caused by modulation will approach zero. But so far the noise has been ignored. A linear underestimate of the phase error due to noise is equal to

$$\overline{\phi_n^2} = \int_0^\infty |H(j\omega)|^2 S(\omega) d\omega$$

where $S(\omega)$ is the spectrum of the input phase noise. This expression is independent of modulation and hence must at some values of Ω be the

dominant factor determining the magnitude of the phase error caused by both modulation and noise. Note that when the loop operates in the linear region, superposition holds and the total phase error is equal to the sum of ϕ_m and ϕ_n . Osborne and Schilling's computer program was not equipped to calculate noise. The authors chose the loop's gain so that $\phi_m \ll \pi/2$ also in the case when there was no modulation present. This naturally resulted in a poor first-order loop performance. Were they to take the noise into account and utilize the absence of modulation, they could have reduced the loop gain of the first-order loop considerably and it would have been the winner. The gain of the second-order loop could be reduced too, but less.

Finally, despite the danger of being repetitious, it is emphasized that there is no discrepancy between the conclusion of the thesis and the results of Osborne and Schilling, since they report on the loop performance under entirely different conditions.

The above two articles were analyzed in some detail in this section because, at first glance, their results seemed to be disputing the conclusion of the thesis. The following paper, by W. C. Lindsay [9], gives an indirect confirmation to the thesis' claim. Lindsay approximately solved the Fokker-Plank equation for the second-order phase-locked loop

which has a general filter of the form $H(s) = \frac{1 + \tau_1 s}{1 + \tau_2 s}$. Since the exist-

ence of the Fokker-Planck equation requires that the noise in the loop be a Markoff process, Lindsay was not able to prefilter the incoming signal, which in turn resulted in the loop noise being white. Also, Lindsay considered only cases without modulation for the very same reason. He was able to solve for $\phi^2(t)$ among other things. Note that this phase-error variance now also contains both suppressed "clicks" (or doublets) and unsuppressed "clicks" (or spikes). He stated that the first-order loop produces a smaller phase-error variance than a second-order loop, provided there is no frequency offset in tuning. Figures 7 and 10 in reference [9] illustrate this information. Lindsay investigated regular phase-locked loops with a sine-wave detector nonlinearity, as well as with the "optimum" nonlinearity equal to $\text{sgn}(\sin \phi)$ which amounts to a hard limiter. The latter proved to be more advantageous than the regular PLL in terms of minimizing $\phi^2(t)$. In both cases, however, the first-order loop came out ahead. The "optimum" PLL, unfortunately, seldom can be used as an FM demodulator, since it has no linear region of operation and cannot easily accomplish an undistorted demodulation.

Lindsay's conclusion about the superiority of the first-order loop over the second is important, since his PLL closely resembles the PLL used in this thesis, with the only difference being that his loop admitted more noise than the loop in the thesis.

CHAPTER 5

SUMMARY AND SUGGESTED FUTURE RESEARCH

5.1. INTRODUCTION

In this chapter, the results of the work reported in the thesis are summarized together with the methods of obtaining the results and the approximations that were made. The significance of the findings of this thesis is discussed and possible avenues for the extension of the work are mentioned.

5.2. SUMMARY AND CONCLUSIONS

It is the conclusion of this thesis that the first-order phase-locked loop is superior to the second-order loop for the task of undistorted demodulation of a small-indexed (.2 and less) frequency modulated (FM) signal. It is assumed that the PLL is tuned to the center frequency of the incoming modulated carrier and that the signal may be pre-filtered by a narrow-band filter.

The optimality of the loop is gauged by its ability to suppress (i.e., not to demodulate) "clicks" - fast changes of $\pm 2\pi$ radians of the phase in the incoming signal which are induced by noise.¹ Since "clicks" cause the threshold phenomenon in FM demodulations [14], this result implies that the first-order loop can operate at a lower carrier-to-noise ratio than the second-order loop, subject to the constraint that the phase-error variance remains small so as to keep the output signal from

¹See section 1.1 for a more detailed description of "clicks" and the threshold phenomenon.

being distorted and so that the detected signal can be separated from most of the accompanying noise by linear filtering. If the PLL operates in the non-linear region, it limits the signal and noise, thus creating harmonics and subharmonics of the signal which distort the signal. Furthermore, the loop becomes stressed and may lose lock due to the noise alone without encountering a "click" - an undesirable event in this case.

The conclusion about the superiority of the first-order loop was reached as the result of the computer search program over the field of all possible first- and second-order loops.¹ Steps leading to the construction of the search program and the approximations that were made are outlined below.

It was assumed that the derivative of the phase of an unmodulated or a small-indexed FM modulated carrier consists of "clicks" and Gaussian-like noise [14]. This assumption is, according to some published results, good down to and including carrier-to-noise ratios of 3 dB [4,23]. In this region the phase noise (not the "clicks") can be closely approximated by Gaussian noise.¹ Using the same order of approximation on the amplitude of the contaminated carrier, it was shown in chapter 3 that the amplitude and the phase of the contaminated carrier are statistically independent.

It was argued in chapter 3 that the variance of the output of a time-varying linear system, driven by a forcing function which is statistically independent of the time-varying parameters of the system, is

¹Sections 4.2, 4.3, 4.4 contain a detailed description of the search program.

larger than the variance of the similar linear time-invariant linear system where all the variable parameters are replaced by their expected values. It was not possible to prove this statement in general, and no such proof is known at the present time. However, it was shown to be true for two special cases. This method of obtaining an underestimate of the output variance was used in the thesis because of it being plausible, because of the proofs of two special cases and because of a few numerical verifications of it in appendix 2. It was further argued in chapter 3 that this method may be used in cases when the loop gain is low and the PLL operates part of the time in the non-linear mode. Since the sine nonlinearity inhibits the loop's tracking ability the resultant phase error is even larger than what it would be were the loop to operate in the linear mode only.¹ The underestimate of the phase-error variance was referred to as a linear underestimate in this thesis in order to emphasize the method of its computation.

In order to get better accuracy, a quasi-stationary method of estimating the conditional phase-error variance was developed in sections 3.6 and 3.7. It was to be used when the loop gain is high so that the frequency response of the PLL is faster than the changes taking place in the carrier amplitude $a(t)$. In such an event the phase-error variance can be computed as a conditional expectation of phase-error variances obtained for each value of $a(t)$ by standard technique used with time-

¹See section 3.4 for details.

invariant linear systems. It is not computed for every value of $a(t)$ because when $a(t)$ becomes small the quasi-stationary principle does not apply and the system is not linear any more. An arbitrary rule based on heuristic considerations for the decision whether the quasi-stationary principle is applicable was proposed in section 3.6. Results of a Monte Carlo simulation of the PLL driven by noise¹ were used to assign a numerical value for the minimum "tolerance threshold"² required for the application of the quasi-stationary method.

With the two above-mentioned techniques in hand it was possible, in the main search program, to compute the conditional phase-error variance or an underestimate of the absolute variance, depending on loop parameters. The search program was set up so that whenever it computed a conditional variance it would also compute a linear underestimate of the variance and then choose the largest as the better estimate of the true $\phi^2(t)$. This was used as a guard against possible optimistic values obtained by the quasi-stationary method for the conditional variances.

Along with the estimate of the phase-error variance the search program required an estimate of the ability of the loop to suppress the "clicks". For this purpose Hess' [8] model of the "click" was used. Briefly, the model amounts to the description of the "click" by a deterministic Gaussian-shaped function and an assumption of the invariance of the carrier amplitude during the "click" occurrence.

¹See appendix 2.

²See section 3.6 for the definition of the "tolerance threshold".

An approximate solution of the PLL differential equation driven by a "click" was developed in chapter 2. The accuracy of the solution was verified in appendix 1 and was found to be good (better than 10% with a bias correction). The purpose of going to the approximate solution was to avoid a time-consuming numerical solution of the loop equation in the main search program where thousands of such equations had to be solved. The resultant saving in computer time was very significant. During the search for the optimum loop the approximate solution would yield the maximum loop gain (critical gain) for the given PLL whose phase error would be allowed to undergo a $\pm 2\pi$ phase shift under the influence of a "click". Hess developed the "click" model only for the case of a carrier-plus-noise. Introduction of a very small-indexed modulation (.2 and less) causes a small additional constant force acting along with the "click" on the loop. Appendix 1 contains a numerical verification of the fact that a small modulation would affect the size of the critical gain in a linear fashion so that an assumption that the forcing function due to the modulation is equal to its expected value (zero) has introduced no error. In other words, it was found that it is permissible to solve for the critical loop gain in the presence of a small-indexed modulation without bothering to take its effects into account. The ability of the given PLL to suppress "clicks" was measured by the magnitude of the critical gain. In order to assign a numerical value to this ability, the "gain ratio" was defined in section 4.3. It is equal to the nominal loop gain divided by the

critical loop gain. The larger the ratio, the more "clicks" can be suppressed by the loop.

The computer search program generated loop parameters randomly. Each combination defined a PLL which then would be solved for the "gain ratio" and an estimate of the phase error variance (a linear underestimate of $\phi^2(t)$ or a conditional variance if the latter may be found by the quasi-stationary method). The estimate of $\phi^2(t)$ had to be small, so that the loop operated in its linear region in the mean-square sense. If the estimate of $\phi^2(t)$ exceeded the constraint $[(\pi/20)^2]$ and $(\pi/20)^2$ were used on separate occasions], a new set of parameters was generated. If the $\phi^2(t)$ constraint were satisfied the "gain ratio" and loop parameters were stored and the next set of parameters was generated. After a number of successful calculations of the "gain ratios" the maximum "gain ratio" would be sorted out and that in turn would point out the optimum loop parameters. The search was conducted for cases of carrier-to-noise ratio equal to 3 and 5 dB. In all cases the first-order loop was found to be the optimum one. The results of the search can be generalized for higher values of CNR because the search runs were conducted for several constraints on the value of the conditional phase-error variance due to "Gaussian" noise.

5.3. THE SIGNIFICANCE OF THE RESULTS AND FURTHER POSSIBLE RESEARCH.

One aspect of the findings is obvious: whenever one has to use a small-indexed high fidelity frequency modulation communication over a long distance (small carrier-to-noise ratio), the first-order loop would demodulate the signal better than the second-order loop. This parallels what Lindsay [9] found for the case of narrow-band loops without modulation; namely that the first-order loop would yield the smallest phase-error variance.¹ It is well known that small-indexed FM has much less practical significance than the wide-band large-indexed FM; however, there may be contingencies when there is no choice but to use narrow-band FM. For instance, with the existing communication FM link one may temporarily have to increase the information rate without modifying the equipment. Then it would be useful to have a switch on a phase-locked loop receiver which would convert it into the first-order PLL. There are many similar hypothetical cases that can be mentioned but that would be redundant.

The other and most important point brought up by the thesis result is a proof that there exists a distinct optimum loop for the demodulation of each FM signal depending on its modulation index. This thesis showed that a very small-indexed FM is best demodulated by the first-order loop. Osborne and Schilling [13] showed by a different technique that the type II, second-order PLL is better (not the best) than the first-order loop for the demodulation of a very high-indexed FM.

¹See section 4.5 for a detailed discussion of this.

Apparently, in the intermediate region the optimum loop parameters will depend on the size of the FM modulation index.

The technique of this thesis may be used to investigate the ability of the PLL to suppress "clicks" and to maintain a small phase error in the presence of large-indexed FM modulation. For this, however, a new model of the "click" is required, since Hess' "click" was verified only without modulation. Therefore, some experimentation needs to be done first before proceeding with the work. All other assumptions and approximations that were made in this thesis would be unaffected by the introduction of large-indexed FM modulation. It is worthwhile to note that as of now there is still no definite knowledge about which loop configuration would demodulate any FM signal with the lowest threshold and not distort the output. Lindsay's results do not account for modulation and are applicable only to the loops that are tracking a carrier which is mixed with white noise. All practical uses of PLL receivers include a narrow-band filter before the receiver in order to get rid of some of the incoming noise. Thus, the Fokker-Plank technique gives a pessimistic performance estimate for a practical PLL receiver.

APPENDIX 1

NUMERICAL VERIFICATION OF THE ACCURACY OF SOME
APPROXIMATIONS MADE WITH RESPECT TO
THE EFFECT OF A "CLICK" ON THE LOOP

A1.1 INTRODUCTION

This appendix contains experimental evidence for the validity of the several approximations made when computing the PLL response to a "click". The body of the appendix may be divided into three parts:

1. A verification of the approximate solution of the PLL differential equation driven by a "click" with all the initial conditions set to zero and no modulation present;
2. Computer simulation data showing the loop's insensitivity to the gain variations during the "click" occurrence;
3. Computer simulation data demonstrating the insensitivity of the loop's "click" suppression ability to the presence of a small-indexed FM modulation.

A1.2 APPROXIMATE SOLUTION OF THE PLL DIFFERENTIAL EQUATION

The approximate solution of the loop equation with the forcing function equal to Hess' "click" was developed in sections 2.3 - 2.6 of the main body of the thesis. All the initial conditions in the phase-locked loop before the advent of the "click" were taken to be zero. The underlying assumptions of the approximate solution are: during the cycle skipping the phase error $\phi(t)$ is a linear function of time; a "click" can be approximated by a square pulse without any significant loss of accuracy in the solution; the maximum loop gain (critical loop gain) allowing cycle skipping occurs when the passage time t_o (see section 2.4) is equal to the duration of the square pulse approximation to the "click". A good agreement

between the approximate solution and the exact numerical solution for the critical loop gain vindicates these assumptions.

The following is a brief description of the approximate solution of the PLL equation. The loop differential equation with the constant gain was derived in Chapter II and is repeated here:

$$\ddot{\phi}(t) + \alpha \dot{\phi}(t) = \frac{\alpha AK}{\beta} \phi(t) \cos \phi(t) + \alpha AK \sin \phi(t) = \frac{d}{dt} + \alpha \dot{\theta}_1(t) \quad (2.19)$$

An assumption that $\phi(t) = \omega_0 t$ during the cycle skipping allowed one to solve the equation for the passage time t_0 [time required for $\phi(t)$ to go from zero to π under an influence of a "click" $\dot{\theta}_1(t)$]:

$$t_0 = \frac{2}{\sqrt{k_2^2 - r^2}} \left[\pi + \tan^{-1} \frac{k_2 \tan\left(\frac{\pi + \psi}{2}\right) + r}{\sqrt{k_2^2 - r^2}} - \tan^{-1} \frac{k_2 \tan\left(\frac{\psi}{2}\right) + r}{\sqrt{k_2^2 - r^2}} \right] \quad (2.26)$$

$$\text{where } k_2 = k_1 - \frac{(\beta - \alpha) AK \omega_0^2}{(\alpha^2 - \omega_0^2) \beta \pi}$$

k_1 = height of the square pulse approximation
for the "click" $\dot{\theta}_1(t)$

$$r^2 = A_1^2 + A_2^2$$

$$\psi = \tan^{-1} (A_1/A_2)$$

$$A_1 = \frac{\alpha AK}{\beta} \left[1 + \frac{\alpha(\beta - \alpha)}{\alpha^2 + \omega_0^2} \right]$$

$$A_2 = \frac{\alpha AK}{\beta \pi} \frac{(\beta - \alpha) \omega_0}{(\alpha^2 + \omega_0^2)} \left[4\alpha \omega_0 \left(e^{-\alpha \pi / \omega_0} + 1 \right) - \pi (\alpha^2 + \omega_0^2) \right]$$

and finally, $\omega_0 = \pi/t_0$.

Since the area under a "click" is equal to 2π , equation (2.26) (if all assumptions listed above are good) may be numerically solved for the critical loop gain by making t_0 to be equal to the duration of the square plus $\frac{2\pi}{k_1}$.

The numerical value of k_1 was found as follows: The carrier-to-noise ratio (CNR) was assumed to be 3 dB. Then the radius of gyration¹ of narrow-band noise whose bandwidth is 2 rad/sec is equal to $1/\sqrt{3}$. The peak value of Hess' exponential "click" is equal to $2/\sqrt{3}$ (twice the radius of gyration).

The equation of Hess' "click" was given as

$$\dot{\theta}_1(t) = \dot{\theta}_1(\tau_0) \exp \left\{ -(t-\tau_0)^2 [\dot{\theta}_1(\tau_0)]^2 / 4\pi \right\} \quad (2.27)$$

where $\dot{\theta}_1(\tau_0)$ now is equal to $2/\sqrt{3}$.

The Fourier transform of $\dot{\theta}_1(t)$ is equal to

$$F \{ \dot{\theta}_1(t) \} = 2\pi \exp (-\omega^2 \pi / [\dot{\theta}_1(\tau_0)]^2) \quad (A1.1)$$

and its Taylor's series expansion is

$$2\pi(1 - \pi\omega^2/1.333 + \pi^2\omega^4/2 \times 1.333^2 - \dots) \quad (A1.2)$$

where $[\dot{\theta}_1(\tau_0)]^2 = 1.333$

The Fourier transform of a square pulse $\frac{\pi}{\alpha} p_\alpha(t)$ of the duration 2α and the area of 2π is equal to

¹See section 2.5.

$$\frac{\pi}{\alpha} \frac{2 \sin \alpha \omega}{\omega} = 2\pi \left(1 - \frac{\alpha^2 \omega^2}{6} + \frac{\alpha^4 \omega^4}{120} - \dots \right) \quad (\text{A1.3})$$

Matching the first two terms of (A1.2) and (A1.3) yields $\alpha = \sqrt{6\pi/1.333}$ and the height of the pulse is $k_1 = \frac{\pi}{\alpha} = .8355$.

Equation (2.26) was solved for the critical gain using the values of $k_1 = .8355$, $\omega_0 = k_1/2$ and a set of widely-scattered values for α and β .

Simultaneously with the approximate solution, the exact numerical solution of the PLL differential equation driven by Hess' exponential "click" was solved for the exact value of the critical loop gain. The loop equation was normalized for the 2 rad/sec bandwidth of the input signal, same as the loop in the approximate solution. The differential equation was repeatedly solved for different values of α , β , AK by Adams' second-order prediction method with the sampling interval sufficiently fine to insure good accuracy. The critical value of AK was found by a Fibonacci-type search. The variant of the binomial search consisted of at first finding crude upper and lower limits on the value of the critical AK and then repeatedly dissecting this range thus refining the limits on the estimate of the value of the critical AK . The exact and approximate values of the critical loop gain are tabulated below for each given pair of α and β . The table shows that the approximate solution has a consistent bias of about 10%. Multiplying the approximate results by 1.1 would eliminate the bias and make the approximate method a very accurate one. Values of α and β were chosen to cover a very wide range of possible loop parameters. Note that when values of α and β are

close to each other such a configuration corresponds to the first-order loop; very large values of α also cause the loop to become first order.

TABLE A1.1. Comparison of Exact and Approximate Solutions for the Critical Loop Gain of a Phase-Locked Loop Driven by Hess' "Click".

α	β	Exact gain	Approximate gain	Error %
.001	.00125	.795	.706	-11.2
.001	.0125	7.634	6.833	-10.5
.01	.0125	.789	.701	-11.1
.01	.125	5.570	5.184	- 6.9
.1	.125	.741	.665	-10.3
.1	1.25	1.691	1.530	- 9.5
1.	1.25	.648	.575	-11.3
1.	12.5	.686	.596	-13.1
10.	12.5	.637	.565	-11.2
10.	125.	.637	.565	-11.3

In every case all the initial conditions were set to be zero and the loop gain AK was kept invariant throughout the duration of the "click". The exact solutions were started 6 seconds before the peak of the "click" and continued until 7 seconds after the peak.

Note that second-order loops ($\beta \gg \alpha$, small α) as a rule have a large critical gain, thus suggesting that they are better "click" suppressors than the first-order loop.¹

¹More on this subject is contained in section 4.5.

A1.3. EFFECTS OF CARRIER-AMPLITUDE VARIATIONS ON THE CRITICAL GAIN

Since the assumption on the invariability of the loop gain during the "click" occurrence is only an approximation, though a good one as demonstrated by Hess [8], it was attempted to spot-check how much variation in AK can be ignored by pretending that the loop-gain is constant and using the approximate technique of Chapter II to solve for the critical gain. A model was devised to represent the fading amplitude $\alpha(t)$. As before, the carrier-to-noise (CNR) was assumed to be 3 dB. Arbitrarily, the normalized carrier amplitude $\alpha(t)$ (called A when it does not vary) was defined in the following manner:

$$\alpha(t) = [1 - .8 \exp(-(t - \tau_0)^2 / 2\sigma^2)] \quad (\text{A1.4})$$

There are no claims made about the validity of this model. It was made just to introduce some variation in the loop gain. $.8$, the coefficient of the exponential term in (A1.4) was, however, chosen with some underlying heuristic reasoning.

Let $\alpha(t) = A + x(t)$, where $x(t)$ is the inphase and $y(t)$ the quadrature components of the input narrowband noise. The peak of a "click" occurs when $A + x(t) < 0$, $y(t) = 0$, and $\dot{y}(t) \neq 0$. $\dot{y}(t)$ is a Gaussian variable; therefore $|\dot{y}(t)|$ is Rayleigh distributed and its expected value is equal to $\sigma_y \sqrt{\pi/2}$. At CNR = 3 dB and 1 rad/sec one-sided bandwidth, it is equal to $.231$. The peak of the "click" is equal to $\dot{y}/[A - x(t)]$ at the moment when the denominator is negative and $y(t) = 0$. The peak of Hess' "click" is equal to 1.154 . Therefore, the expected value of the dip in the amplitude is equal to $.231/1.154 = .200$ of the nominal value of the

carrier amplitude A . In the normalized loop $A = 1$ while K takes up the slack for the gain. Equation (A1.4) makes $a(t)$ go to $.2A$ at some $t = \tau_0$. The standard deviation σ of (A1.4) was chosen to be a multiple of the standard deviation of Hess' "click" at $\text{CNR} = 3$ dB, which was computed to be 2.7933. Thus, σ was selected to be 1.5×2.7933 so that $a(t)$ varies somewhat slower than the "click". Theoretically, it should be much slower than the "click". The results demonstrate that even with such a severe fading the assumption that $a(t) = \text{constant}$ and is equal to its minimum value gives a fair agreement with the exact solution of the differential equation with time-varying parameters. The solution is similar in

TABLE A1.2. Comparison of Values of Critical Gain Obtained for Time-Varying and Time Invariant Loops.

α	β	EXACT CRITICAL GAIN Exact Solution for Critical Value of $[a(t)K]_{\min}$	APPROXIMATE CRITICAL GAIN Exact Solution from Table A1-1 for Constant Amplitude	Error %
.001	.00125	.670	.795	18.7
.001	.0125	6.458	7.634	18.2
.01	.0125	.667	.789	18.3
.01	.125	4.700	5.570	18.5
.1	.125	.632	.741	17.2
.1	1.25	1.338	1.691	26.4
1.	1.25	.554	.648	17.0
1.	12.5	.575	.686	19.3
10.	12.5	.545	.637	16.9
10.	125.	.545	.637	16.9

nature to the one used in the preceding section. The only difference in this case is that a fourth-order Runge-Kutta method was used instead of Adams' method to solve the differential equation.

Table A1.2 demonstrates that rather severe variations in the loop gain during the "click" occurrence did not produce a critical gain much different from the one obtained with a constant loop gain. According to Hess [8] the variations in $a(t)$ are much slower than in the "click", therefore, the results reported in table A1.2 could be considered an unlikely event. Slower variation in $a(t)$ would make the constant gain approximation better.

The above-mentioned results of computer simulations were not intended to serve as proof of Hess' contention that one may ignore changes in $a(t)$ during the "click" occurrence. A reference to his article [8] alone would have served the objective. This computer simulation was performed for the sole purpose of illustrating the point.

A1.4 AN EFFECT OF FM MODULATION ON THE CRITICAL GAIN

An effect of a constant-frequency FM modulation on the critical gain was spot-checked by means of computer simulation. A constant frequency modulation was used in lieu of sinusoidal modulation. Since the peak excursion of a sinusoidal frequency modulation puts the most stress on the loop, the constant-frequency offset thus may be considered the worst case situation. As in section A1.2, the critical gain was obtained for a set of given values of the loop parameters α and β . As before, the "click" was normalized for the input bandwidth of 2 rad/sec, thus making its peak value equal to 1.154. The forcing function in this case, however, consisted of

Hess' "click" plus a constant. Since the "click" represents the time derivative of the phase, a constant frequency offset would be manifested by a constant added to the "click" and it is assumed that Hess' model of a "click" is not affected by a small-indexed modulation, i.e., the frequency offset due to modulation remains much smaller than the peak value of the "click". "Exact" values of critical gains were obtained here, that is, the PLL differential equation was numerically solved rather than using the approximate solution discussed in section A1.2. It was assumed in this simulation that the loop gain remained constant during the "click" occurrence.

The results indicate that a frequency offset less than .314 rad/sec (it corresponds to the modulation index of .46) would cause the critical gain to vary linearly with the offset. Since the positive and negative "clicks" are almost equally likely when the modulation index is small, the average critical gain in the presence of small-indexed modulation will be equal to the value of critical gain obtained when there is no modulation present.

Values in this simulation were chosen so that one set of parameters ($\alpha = .1$, $\beta = .125$) resulted in a nearly first-order loop and the other set ($\alpha = .1$, $\beta = 1.25$) yielded a highly underdamped second-order loop. Note that the second-order loop has a higher critical gain and thus is better

able to suppress "clicks" than the first-order loop. However, this does not mean that the second-order loop is the optimum one.

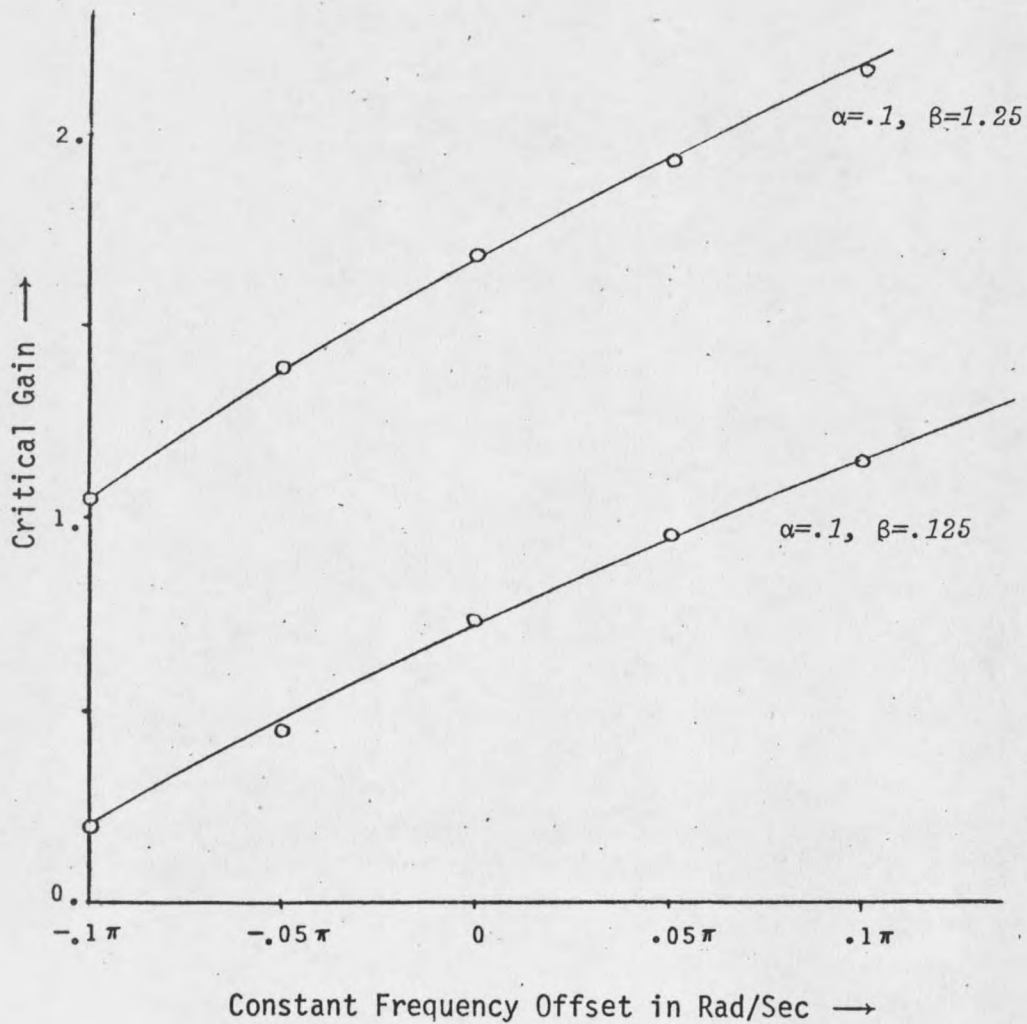


FIG. A1.1 Critical Gain of the PLL Response to a "Click" Plus Constant-Frequency Force Due to a Modulation.

APPENDIX 2

NUMERICAL VERIFICATION OF THE ACCURACY OF
THE APPROXIMATE METHODS OF OBTAINING
ESTIMATES OF THE PHASE ERROR VARIANCE

A2.1 INTRODUCTION

Appendix 2 contains the results of a computer Monte Carlo simulation which verified the accuracy of the quasi-stationary solution for the conditional phase error variance. As a by-product of this simulation it was also possible to compare the conditional variance as well as the linear underestimate of the unconditional phase error variance to the measured phase error variance. The results of the simulation showed that the conditional variance was quite close to the unconditional variance when the amplitude threshold was low enough and that the linear underestimate of the variance¹ is indeed an underestimate.

A2.2 VERIFICATION OF THE QUASI-STATIONARY SOLUTION

A quasi-stationary solution for the conditional phase error variance was compared to the results of a Monte Carlo simulation of the phase-locked loop driven by a phase noise. The quasi-stationary method was developed in sections 3.5 and 3.6. Briefly, this approach assumed that the PLL behaves like a time-invariant linear system whenever the instantaneous loop gain $a(t)K$ exceeds a certain minimum value. The conditional phase-error variance, given an instantaneous value of the loop gain, can be evaluated by integrating the product of the squared absolute value of the PLL transfer function times the noise spectrum. It is assumed that noise is Gaussian with a flat spectrum and is band limited. Since the statistics of $a(t)$ are known, one can find the conditional expectation of this integral. Expression (3.42) describes this process analytically:

¹See section 3.5 for the description of the calculation of the phase error variance.

$$E[\phi^2(t) | \alpha(t) > A_0] = \frac{\int_{A_0}^{\infty} \int_0^{\infty} |H_i(j\omega)|^2 S_{\theta}(\omega) p(A_i) d\omega dA_i}{\int_{A_0}^{\infty} p(A_i) dA_i} \quad (3.42)$$

Where $|H_i(j\omega)|$ is the absolute value of the PLL phase-error transfer function with the instantaneous loop gain equal to $A_i K$, $p(A_i)$ is the probability density function of $\alpha(t)$, (A_i is an instantaneous value of $\alpha(t)$), and $S_{\theta}(\omega)$ is a one-sided noise spectrum. The test for the applicability of the quasi-stationary solution was arbitrarily defined as follows: the so-called tolerance threshold determined by the instantaneous loop parameters had to exceed a certain percentage point. The definition of the tolerance threshold is repeated here for $\text{CNR} = 3$ dB.

$$\text{Tolerance threshold} = \frac{.289 \tau K}{A_0 K} \quad (3.43)$$

Where τ is the reciprocal of the real part of the dominant pole of the PLL transfer function, $\alpha(t)$ is normalized so that the nominal value of the carrier amplitude A is equal to unity, $A_0 K$ is the minimum admissible instantaneous loop gain, $.289 \approx$ standard deviation of $\dot{\alpha}(t)$ when the one-sided bandwidth of the input process is normalized to 1 rad/sec and $\text{CNR} = 3$ dB.

A computer Monte-Carlo method of verification of the accuracy of the quasi-stationary solution was employed. Two independent arrays of random numbers were generated. These arrays were narrow-band filtered by means of a Fast Fourier transform routine. This was done in such a manner

that the resultant bandwidth was equal to 1 rad/sec and the sampling rate in the time-domain was .25 seconds. The two arrays represented the cophase ($x(t)$) and the quadrature ($y(t)$) components of Gaussian noise. The arrays were normalized so that the noise power was 3 dB below the normalized carrier power whose peak voltage amplitude is unity. In other words, CNR was equal to 3 dB. An array of Rician variables equal to $\sqrt{(1+x(t))^2 + y^2(t)}$ was generated to represent the resultant amplitude of the contaminated carrier. A histogram of the Rician array was obtained to be used in lieu of its probability density function. The phase noise that drives the PLL in the absence of "clicks", without or with a very small modulation, is $\tan^{-1} y(t)/[1+x(t)]$. Because of the computational difficulties it was decided to use $y(t)$ instead of the complete expression since, after all, the approximation is good for carrier-to-noise ratios (CNR) as low as 3 dB. The Rician array and the Fourier transform of $y(t)$ were stored permanently in the form of a catalog file in the computer drum library so that they could be accessed upon demand.

Each experiment would start with an assignment of a set of threshold values for $a(t)$, for instance .6, .7, ..., 1.3, and an assignment of values to α , β , AK (nominal loop gain). The latter determined the value of K , since A could be assumed to be unity without a loss of generality. The arrays representing $a(t)$ and $Y(\omega)$ [the Fourier transform of Gaussian array $y(t)$] were brought out of the storage. The forcing function of the PLL was formed out of $Y(\omega)$ through its multiplication by $(j\omega)^2 + \alpha j\omega$ and the inverse-transformation of the product back into the time domain. The result of this operation was equal to $\ddot{y}(t) + \alpha \dot{y}(t)$.

The PLL differential equation¹ was then numerically solved for the phase error by means of the fourth-order predicting-correcting Adams formula. A phase error array was formed. Next the program would go through the $a(t)$ array (the amplitude of the carrier) and store the indexes of those members of the $a(t)$ array that exceed the first threshold value assigned in the beginning of the experiment (.6 in this example). Since the $\phi(t)$ array was similarly ordered, the program could compute the conditional phase error variance based on the condition that the carrier amplitude must exceed the current threshold. Once this was done, the program would go back and calculate the conditional phase error variance for the next value of amplitude threshold, and so on. For each exact evaluation of the conditional variance the program would also evaluate the expression (3.42), the quasi-stationary approximation, where the histogram of $a(t)$ array was used in place of its probability density function and the noise spectrum $S(\omega)$ was that of $y(t)$. Some of the results of this Monte Carlo simulation are reported in Fig. A2.1 (a) and (b) below. The Monte Carlo simulation program was circumstantially constrained in several ways. It required a considerable amount of computer memory and the length of the $y(t)$ array therefore was set at 2048 words. The numerical solution of the differential equation required fine sampling intervals, which in turn limited the number of independent samples of $y(t)$ that could be contained in the fixed array length. Hence, the simulation was performed with only 276 out of 2048 samples of forcing function being independent.

¹See expression (3.9).

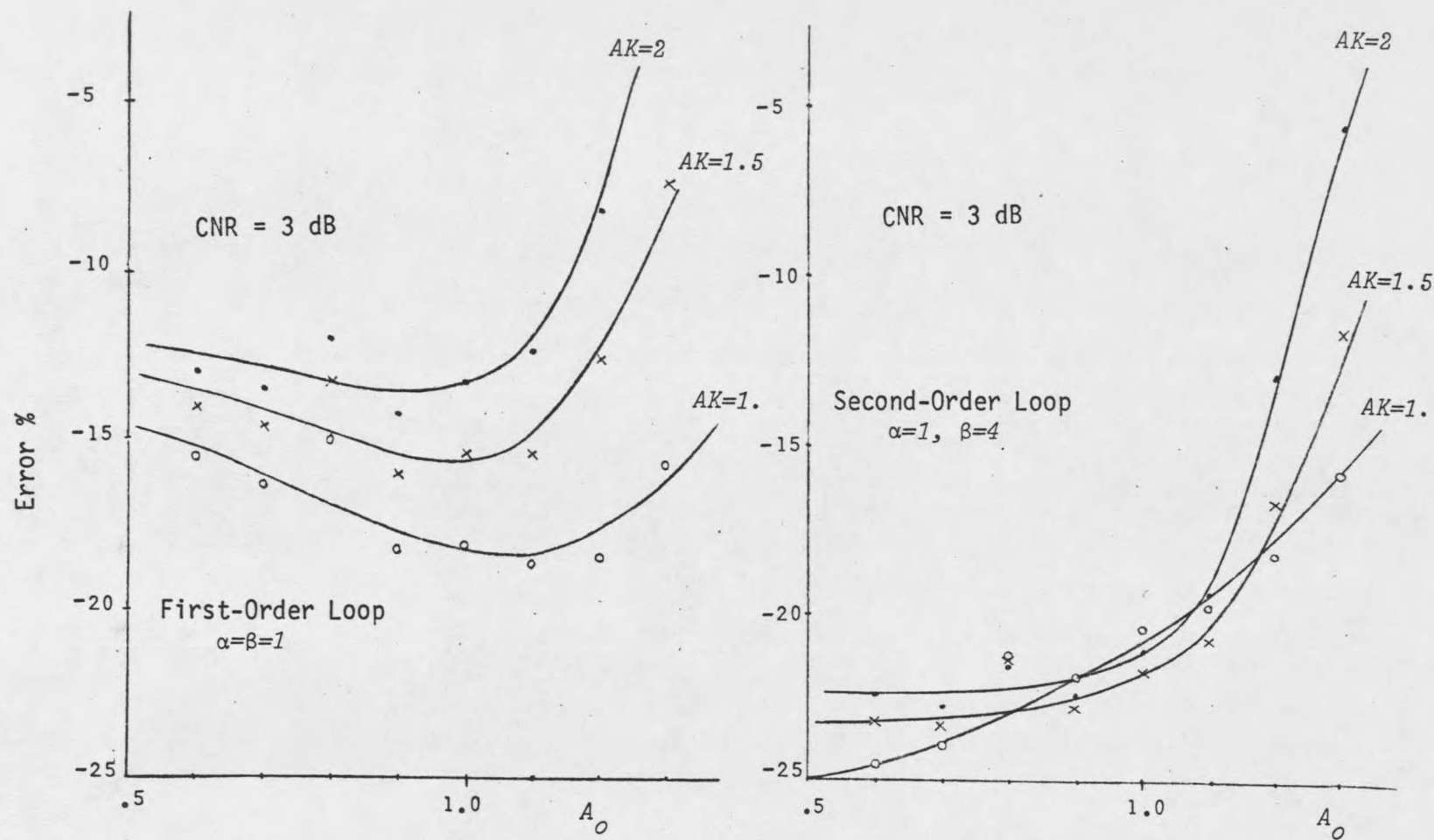


FIG. A2.1. Plots of the Error Occurring in Quasi-Stationary Estimation of the Conditional Variance $E[(\phi(t) - \mu_0)^2 | a(t) > A_0]$ Versus A_0 [a Threshold of the Normalized Carrier Amplitude $a(t)$]. Errors are Calculated with Reference to the Results of the Monte Carlo Computer Simulations.

Each curve was generated with exactly the same noise. Spacing between curves is due entirely to the differences in the nominal loop gain (the value of AK in the absence of noise). Because of the small number of independent samples in this simulation, points obtained with high threshold values showed a tendency to vary significantly when the simulation was repeated with a different noise array. This is not an unexpected development, since very few samples get to be used with higher thresholds. Therefore, the plots were limited to threshold values of 1.3.

It was also discovered in the course of this simulation that the means of the measured phase errors were not zeros: about 5% of the measured phase error power was a DC power. A larger sample size would have rectified this problem; after all, the loop is completely symmetric with respect to the sign of the phase error.

Since the expected value of the phase error is zero, and since the quazi-stationary method does not produce any DC (the loop transfer function has a zero at $j\omega = 0$), it was decided to compare the conditional variances in this appendix instead of the conditional second moments.

Figure 2A.1(a) shows the accuracies of about 80% can be achieved with a first-order loop gain of 1.0. It seems that even better accuracies could be achieved with much higher values of the loop gain but the overall results would not be much better. Some of the fairly lax constraints on the phase error variance in the main computer search program would preclude a participation by the quazi-stationary program in the final determination of the optimum loop configuration. By extrapolating some

of the points in Fig. A2.1 it seems that even first-order loop gains as low as .5 would yield results that are about 75% accurate for $A_0 \geq .5$. A simulation run was made to check this contention and the resulting worst case error turned out to be better than 30%. Therefore, it was decided that the main search program could use the quasi-stationary method whenever the tolerance threshold for the given loop would be less than 2.3 (number corresponding to $A_0 = .5$ and loop gain equal to .5).

Figure A2.1 indicates that in the case of the second-order loop [Fig. 2A.1(b)], the error is consistently larger than in the case of the first-order loop. Since the quasi-stationary solution ignores some of the effects of the amplitude variations by removing the term $\phi(t)\dot{a}(t)/\beta$ from the linearized PLL equation (3.38)¹ it computes an underestimate of the phase-error variance. Also note that the error does not necessarily decrease with larger values of AK . All the terms could have been included in the quasi-stationary solution by raising the dimension of the conditional variance integral to three. Since, however, the missing term does not appear in the first-order equation, and since its absence from the second-order equation does not bias the results in favor of the first-order loop, it was felt that added complexity of computations will not affect the results should the first-order loop become the winner.

¹See section 3.5 for details.

Figure A2.2 represents the plot of the results of application of the quasi-stationary method versus the tolerance threshold. Because of the wide scattering of the points, only the boundaries containing these points were sketched and no attempt to fit curves was made. It is evident that errors retain very little dependence on the tolerance threshold exceeding .2. Thus, the concept of the tolerance threshold would seem to be superfluous. It was retained, however, in the main search program in order to accommodate second-order loops which showed a greater degree of dependence upon it but with a lesser accuracy.

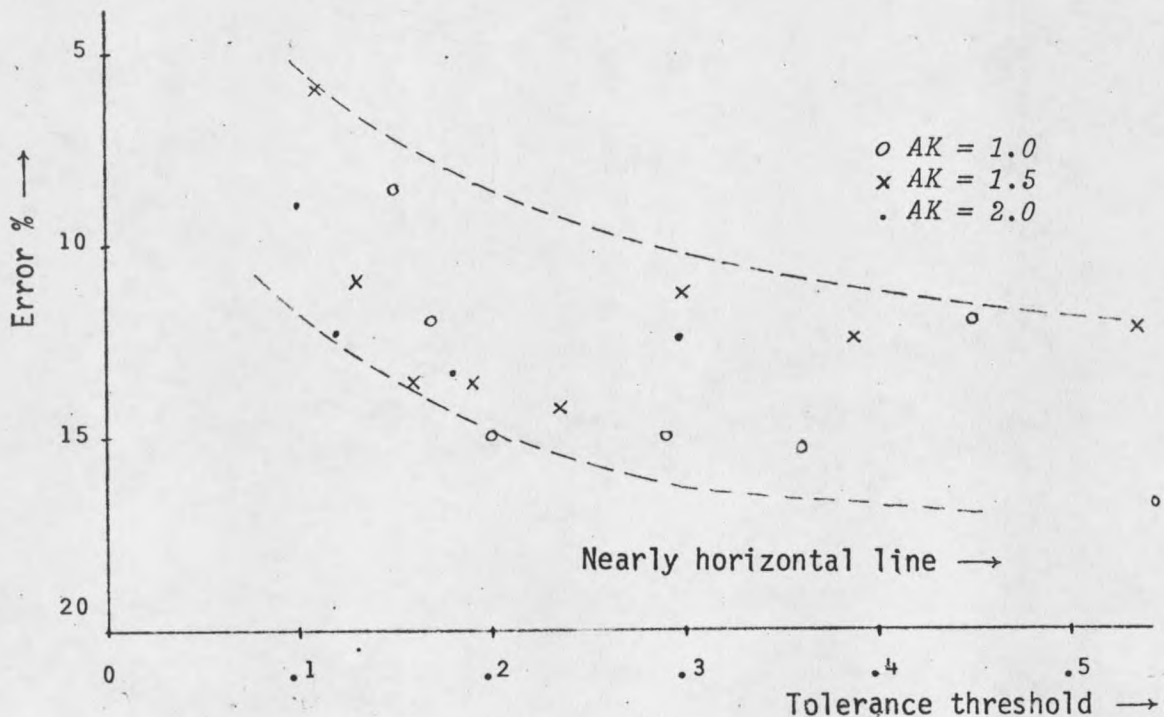


FIG. A2.2 Errors Introduced by the Quasi-Stationary Method of Estimation of the Conditional Phase Error Variance Plotted as a Function of the Tolerance Threshold in the First-Order Loop.

A2.3. RELATIVE MAGNITUDES OF CONDITIONAL, UNCONDITIONAL AND LINEAR UNDERESTIMATES OF PHASE ERROR VARIANCES.

The Monte Carlo simulation program generated data which allowed one to compare conditional variances and linear underestimates of variances to the appropriate unconditional phase error variances. The purpose of such a comparison was to show that conditional variances computed with low amplitude thresholds ($A_0 = .6A, .7A, \dots$) are not drastically different from the unconditional variance ($A_0 = 0$) and thereby give a fair estimate of the unconditional variance. Linear underestimates of variances were compared to the measured, or unconditional, variances in order to verify the contention of sections 3.4 and 3.5 that a linear dynamic system (i.e. a system with time-varying parameters) will produce an output whose variance would be larger than the one obtained from the same system but with all the varying parameters replaced with their expected values. The phase-locked loop is certainly not a linear system; but in this simulation the loop gain was kept high so that the phase error stayed small and the loop operated in its linear region.

The smaller the loop gain, the closer conditional variances are to the absolute variances. This is explained by high-gain loops having most of their errors occurring where the amplitude $a(t)$ dips close to zero. When $a(t)$ is large, the loop tracks the signal well and the errors are small in these regions. Small-gain loops do not track signals too well anywhere.

Underestimates of $\phi^2(t)$ (the phase error variance) depart further in the case of high-gain second-order loops than in the case of first-order loops. The difference reflects the fact that the missing term $\frac{\alpha \dot{\alpha}(t)K}{\beta}$ in simplified PLL equation (3.37)¹ was not negligibly small in the second-order loop, while it did not even appear in the first-order loop. In the computations of an underestimate of $\phi^2(t)$ all the time-varying parameters of the loop were replaced by their expected values. The first-order loop contains only one such parameter: $\alpha \dot{\alpha}(t)K \sin \phi(t)$. The second-order loop contains plus $\alpha \alpha(t)K/\beta \dot{\phi} \cos \phi(t)$ ¹. The mean value of the first term is zero and that means, according to section 3.4, that an additional independent noise source is removed in the calculation of the underestimate of $\phi^2(t)$. Hence, one should expect a greater departure in the underestimates of $\phi^2(t)$ from the true $\phi^2(t)$ in cases of second-order loops. Naturally, the higher the loop gain $\alpha(t)K$, the more of a departure should be observed, though the relationship is not a linear one.

Since the phase-error variance constraint in the main computer search program was between $(\pi/10)^2$ and $(\pi/20)^2$, the resultant normalized optimum loop gain turned out to be between .6 and 1.2. The table indicates that the true conditional variance of the first-order loop with $A_0 < .6AK$ would account for better than 95% of the total phase-error variance.

¹See equation 3.9.

TABLE A2.1. Percent Deviations of Conditional Variances and Underestimates of Variances from the Unconditional Phase Error Variances.

Loop Parameters			Percent deviation of the conditional variance from the unconditional one as a function of the amplitude threshold A_0 .			Percent deviation of the linear underestimate of the variance from the unconditional variance.
AK	α	β	$.6AK$	$.7AK$	$.8AK$	
1.0	1	1	- 5.18	- 9.29	-16.72	-20.54
1.5	1	1	- 9.61	-15.10	-23.49	-25.75
2.0	1	1	-13.15	-19.45	-28.12	-29.66
1.0	1	4	- 5.65	- 9.93	-17.82	-26.03
1.5	1	4	-10.17	-15.84	-24.71	-32.97
2.0	1	4	-14.16	-20.76	-29.82	-38.59

REFERENCES CITED

- 1 Abrams, B., J. Oberst, M. Berkoff, and D. Schilling, "Phase-Locked Loop Threshold Investigations," Polytechnic Institute of Brooklyn Electrical Engineering Department. Memorandum 73, June 1965.
- 2 Acampora, A., and A. Newton, "Use of Phase Subtraction to Extend the Range of a Phase-Locked Demodulator," RCA Review, vol. 27, no. 4, December 1966, pp. 577-599.
- 3 Biswas, B., and G. Datta, "Compound AFC-APC FM Demodulators," Proceedings IEEE, vol. 56, no. 2, February 1968, pp. 202-203.
- 4 Bogdanoff, J., and F. Kozin, "Moments of the Output of Linear Random Systems," Journal of the Acoustical Society of America, vol. 34, no. 8, August 1962, pp. 1063-1066.
- 5 Caughley, T., "Comments on *On the Stability of Random Systems*," Journal of Acoustical Society of America, vol. 32, no. 10, October 1960, p. 1356.
- 6 Caughley, T., and J. Diens, "The Behavior of Linear Systems with Random Parametric Excitation," Journal of Mathematics and Physics, vol. 41, no. 4, April 1962, pp. 300-318.
- 7 Charles, F., and W. Lindsay, "Some Analytical and Experimental Phase-Locked Loop Results for Low Signal-to-Noise Ratios," Proceedings IEEE, vol. 54, no. 9, September 1969, pp. 1152-1166.
- 8 Hess, D., "Cycle Skipping in a First-Order Phase-Locked Loop," IEEE Transactions on Communication Technology, vol. COM-16, no. 2, April 1968, pp. 255-260.
- 9 Lindsay, W., "Nonlinear Analysis of Generalized Tracking Systems," Proceedings IEEE, vol. 57, no. 10, October 1969, pp. 1705-1722.
- 10 Middleton, D., Introduction to Statistical Communication Theory. New York: McGraw-Hill, 1960, p. 414.
- 11 Ibid., p. 579 and p. 651.
- 12 Middleton, D., "Some General Results in the Theory of Noise Through Nonlinear Devices," Quarterly of Applied Mathematics, vol. 4, 1948, p. 445, equations 6.17 and 6.18.
- 13 Osborne, P., and D. Schilling, "Threshold Analysis of Phase-Locked Loop Demodulators Using Most Likely Noise," IEEE Transactions on Communication Technology, vol. COM-19, no. 1, February 1971, pp. 31-41.

- 14 Rice, S., "Noise in FM Receivers," Time Series Analysis. M. Rosenblatt, Editor. New York: Wiley, 1963, chapter 25.
- 15 Rice, S., "Statistical Properties of a Sine Wave Plus Random Noise," Bell Systems Technical Journal, vol. 27, no. 1, January 1948, p. 136.
- 16 Ridgeway, R., and J. Carter, "More Comments on Phase-Locked Loop Threshold," Proceedings IEEE, vol. 55, no. 8, August 1967, pp. 1531-1533.
- 17 Samuels, J., "On the Mean Stability of Random Linear Systems," IEEE Transactions on Circuit Theory, Special Supplement, vol. CT-6, May 1959, pp. 248-259.
- 18 Samuels, J., "On Stability of Random Systems and the Stabilization of Determined Systems with Random Noise," Journal of Acoustical Society of America, vol. 32, no. 5, May 1960, pp. 594-600.
- 19 Samuels, J., and A. Eringen, "On Stochastic Linear Systems," Journal of Mathematics and Physics, vol. 38, 1959, pp. 83-103.
- 20 Schilling, D., "The Response of an APC System to FM Signals and Noise," Proceedings IEEE, vol. 51, no. 10, October 1963, pp. 1306-1316.
- 21 Schilling, D., and J. Billig, "A Comparison of the Threshold Performance of the Frequency Demodulator Using Feedback and the Phase-Locked Loop," Records International Space Electronics Symposium. (Miami, Florida, November 1965), pp. 3-E1 - 3-E9.
- 22 Schilling, D., B. Abrams, J. Oberst, and M. Berkoff, "Phase-Locked Loop Threshold," Proceedings IEEE, vol. 53, no. 10, October 1965, p. 1673.
- 23 Schwartz, M., W. Bennett, and S. Stein, Communication Systems and Techniques. New York: McGraw-Hill, 1966, pp. 151-152.
- 24 Ibid., p. 123.
- 25 Splitt, G., "Design and Analysis of a Linear Phase-Locked Loop of Wide Dynamic Range," IEEE Transactions on Communication Technology, vol. COM-14, no. 4, August 1966, pp. 432-440.
- 26 Tausworthe, R., "A Method for Calculating Phase-Locked Loop Performance Near Threshold," IEEE Transactions on Communication Technology, pp. 502-506.

- 27 Tikhonov, V., "The Effect of Noise on Phase-Lock Oscillation Operation," Avtomatica i Telemekhanika, vol. 22, no. 9, September 1959. Also vol. 23, no. 3, 1960.
- 28 Van Trees, H., "Functional Techniques for the Analysis of the Non-linear Behavior of Phase-Locked Loops," Proceedings IEEE, vol. 52, no. 8, August 1964, p. 894.
- 29 Viterbi, A., Principles of Coherent Communication, New York: McGraw-Hill, 1966, pp. 10-24.
- 30 Ibid., pp. 76-120.

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